

Towards quantum black hole microstates

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ABSTRACT: We study the cohomology of local BPS operators in $\mathcal{N} = 4$ Yang-Mills theory. The finite N cohomologies consist of the graviton part (subject to the stringy exclusion principle) and the rest which may describe black hole microstates in quantum AdS/CFT. We construct an infinite tower of non-graviton cohomologies in the SU(2) theory and study to what extent they simulate quantum black holes. We find signals for partial no-hair behaviors by showing that certain gravitons are forbidden to dress these cohomologies. This is in qualitative agreement with the perturbative hairs allowed around black holes, which also leads us to a natural setup to construct hairy BPS black holes. The cohomologies are simpler to study in the BMN matrix model truncation of the classical field theory.

KEYWORDS: AdS-CFT Correspondence, Black Holes in String Theory, Supersymmetric Gauge Theory

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1 Introduction

Black holes exhibit extreme properties with the information they carry. For instance, the Bekenstein-Hawking entropy [1, 2] of black holes is supposed to represent the maximal amount of information allowed per size [3]. So regarding this entropy as the most fine-grained information of Nature, we can learn from it the fundamental degrees of freedom of quantum gravity [4–6]. To explore other extreme behaviors of black holes, we want access to the individual black hole microstates which account for this entropy. In this paper, we ‘construct’ (in certain sense) and study the local BPS operators which can describe the microstates of BPS black holes in quantum AdS/CFT [7]. The entropies of BPS black holes in $AdS_{D>3}$ were recently computed from the dual field theories: see [8–10] and references thereof. We shall construct representations of some of these microstates and study their properties, hoping that we can better address and exactly solve interesting black hole information problems with this knowledge.

We shall study the 4d $\mathcal{N} = 4$ Yang-Mills theory dual to type IIB string theory in $AdS_5 \times S^5$. We are interested in BPS states that preserve 2 of the 32 supersymmetries [11, 12], called $\frac{1}{16}$ -BPS states. The spectrum of these BPS states may change as the coupling constant g_{YM} is varied, and in fact they are different between the free and 1-loop levels.

The 1-loop BPS states of the $SU(N)$ theory were studied in the past [12–16]. They can be reformulated as classical cohomologies with respect to a nilpotent supercharge Q . It has been conjectured that the spectrum of 1-loop BPS states remains unchanged at general nonzero coupling [12, 13, 17]. This conjecture is partly proved in perturbation theory [18] with certain assumptions. The discussions in this paper will often assume this conjecture. Although we shall construct representatives of new cohomologies, they are generally not equal to the BPS operators. The two in general differ by certain Q -exact terms. So what we achieve is weaker than ‘constructing’ the black hole microstates. It remains to be seen what kind of questions can be addressed just from cohomologies. (See section 3 for an example.)

Our precise goal is to construct cohomologies at finite N which are not of the graviton type. One may wonder what gravitons mean at finite N . First, single-particle graviton cohomologies are constructed with single trace operators only. Multi-gravitons are simply defined as products of the single-particle gravitons.¹ At large N , all these operators are independent because there are no trace relations of matrix fields. More precisely, an operator is free of trace relations when the number of fields appearing in the operator is no greater than N . When the number of fields is larger than N , trace relations may apply. On the gravity side, trace relations are realized by gravitons polarizing into D3-brane giant gravitons [19–21], after which fewer states are allowed than the naive estimate. This is called the ‘stringy exclusion principle.’ The physical mechanism of this principle is same at all $N \geq 2$. So it makes physical sense to define finite N graviton cohomologies as multiplications of single-trace cohomologies modulo trace relations. These cohomologies are fully understood.

Unless there are exotic microstates that neither qualify to be called gravitons nor black holes, we expect the remaining cohomologies to describe the black hole microstates in AdS. We shall therefore call them black hole cohomologies for simplicity of nomenclature, but also having in mind that ‘black hole’ could broadly mean all possible novelties beyond gravitons. There may be two viewpoints on these cohomologies at finite N . First, they are intermediate steps to the cohomologies at parametrically large N . Second, more progressively, one may regard the finite N Yang-Mills theory as a model of quantum AdS/CFT at finite Newton constant. The ‘black hole cohomologies’ at finite N may simulate quantum black holes therein. Not all interesting questions on semiclassical black holes survive in these finite N models, but some questions do. For instance, the Cardy limit [9, 22–25] exhibits universal deconfining behaviors at large charges, naturally generalizing the large black hole physics to all finite N .

The progress in this paper is all with the $SU(2)$ theory, the most quantum version of AdS/CFT. Already in this model, the new cohomologies exhibit some qualitative features reminiscent of black holes. Although this problem has been discussed since 2005, not a single black hole cohomology was found until last year. In [18], it was shown that the $SU(2)$ theory has the lowest black hole cohomology at energy $E = \frac{19}{2}$, R-charges $R_1 = R_2 = R_3 = \frac{3}{2}$ and angular momenta $J_1 = J_2 = \frac{5}{2}$. A simple representative of this

¹While products of BPS operators are generally not BPS, cohomologies multiply to yield cohomologies.

cohomology was constructed in [26]. This is the primary of a superconformal representation of $\text{PSU}(1, 2|3) \subset \text{PSU}(2, 2|4)$. There are infinitely many descendants that one can trivially construct from this primary.

We construct an infinite tower of new black hole primaries in the $\text{SU}(2)$ theory. For technical reasons, we often focus on the ‘BMN sector’ of the Yang-Mills theory on $S^3 \times \mathbb{R}$ [27, 28]. This is a consistent truncation of the classical theory, also yielding a consistent truncation of our cohomology problem. The superconformal index [11, 12] of the Yang-Mills theory can also be restricted to the BMN sector. For the $\text{SU}(2)$ theory in the BMN sector, we find infinitely many new cohomologies which saturate the index. More concretely, the energy $E_{(n)}$, R-charge $R_{(n)} (= R_1 = R_2 = R_3)$ and the angular momentum $J_{(n)} (= J_1 = J_2)$ of our n ’th ‘core’ black hole primary O_n are given by

$$(E_{(n)}, R_{(n)}, J_{(n)}) = \left(\frac{19}{2} + 4n, \frac{3}{2}, \frac{5}{2} + 2n \right), \quad n = 0, 1, 2, \dots \quad (1.1)$$

The operator found in [18] corresponds to $n = 0$. One can exactly compute the full index Z and the graviton index Z_{grav} in the BMN sector. Their difference is given by

$$Z - Z_{\text{grav}} = \left[-\frac{e^{-4(\Delta_1 + \Delta_2 + \Delta_3)}}{1 - e^{-2(\Delta_1 + \Delta_2 + \Delta_3)}} \right] \cdot \left[\prod_{I=1}^3 (1 - e^{-\Delta_I}) \right] \cdot \left[\prod_{I=1}^3 \frac{1}{1 - e^{-\Delta_I} e^{-\Delta_1 - \Delta_2 - \Delta_3}} \right], \quad (1.2)$$

where Δ_I is the chemical potential conjugate to the charge $R_I + J$. The three factors in the square brackets come respectively from: (left) the tower of core black hole primaries O_n , (middle) the Fock space of their superconformal descendants, (right) the hairs by a limited subset of gravitons. The BMN sector in the $\text{SU}(2)$ theory does not show large enough entropy of BPS states even at large charges. However, we expect that the BPS entropy in the BMN sector should exhibit a black hole like growth at large N .

Since cohomologies are multiplicative, we can consider the product cohomologies of gravitons and core black hole cohomologies. We find that a surprisingly large set of them does not appear in the index. The most natural interpretation of this phenomenon is that these product operators are Q -exact (i.e. absent in the BPS Hilbert space), which we prove explicitly at some low orders. We view this as a partial no-hair theorem of black holes in the $\text{SU}(2)$ model, in that our black hole cohomologies abhor the dressing by some gravitons. This phenomenon is most clearly visible for the conformal primary states of gravitons. In (1.2), the index only captures 3 out of 17 species of BMN graviton particles dressing our black holes. We also studied the general $\text{SU}(2)$ index for black holes up to 40’th order in the charge $j = 6(R + J)$. Till this order, the index captures only 3 out of 32 conformal primary gravitons dressing the core black holes. (See sections 3.2 and 4 for the conformal descendants.) Since AdS black holes are expected to allow certain graviton hairs [29–32], partial no-hair phenomenon seems to be the right behavior in AdS/CFT models. In section 4 we perturbatively study the black hole hairs in the BPS limit to clarify the similar behaviors in the gravity dual.

We also estimate when new black hole cohomologies should appear in the $\text{SU}(2)$ theory beyond O_n . By studying the index and all possibilities of finite N graviton hairs, we show that new black hole core primaries should appear at or before the $j = 39$ order.

Note that black holes in the BMN model were studied recently [33–35]. As emphasized in [34, 35], microscopic detection of the quasinormal modes is a signature of seeing black holes since these modes are falling into the horizon. In our stationary (BPS) setup, a direct consequence is the no-hair theorem for the corresponding modes: we try to perturb a black hole by multiplying certain gravitons and find that they do not exist in the BPS sector after a long time.

The remaining part of this paper is organized as follows. In section 2, we explain the cohomology problem and the graviton cohomologies at finite N . In section 3.1, we study the BMN sector and find an infinite class of new black hole cohomologies for $SU(2)$. In section 3.2, we study the general $SU(2)$ cohomologies up to $j \leq 40$ and find no-hair behaviors. In section 4, we study how perturbative black hole hairs behave in the BPS limit and comment on similarities with the results in section 3. In section 5, we conclude with remarks.

2 The cohomology problem

We review the problem of local BPS operators in $\mathcal{N} = 4$ Yang-Mills theory on \mathbb{R}^4 and its cohomological formulation. By operator-state map, they map to BPS states of the Yang-Mills theory on $S^3 \times \mathbb{R}$. We shall discuss the $SU(N)$ theory. We sketch the problem briefly before explaining the details, to emphasize the nature of the problem. We start by selecting two supercharges among 32 of them, which will annihilate our BPS operators. We first consider all gauge-invariant local BPS operators in the free limit. Then we turn on small coupling $g_{YM} \neq 0$ and see how many of the free BPS operators remain BPS, order by order in perturbation theory.

The $\mathcal{N} = 4$ Yang-Mills theory consists of the following fields in the adjoint representation of $SU(N)$ ($N \times N$ traceless matrices):

$$\begin{aligned}
 \text{vector} : A_\mu &\sim A_{\alpha\dot{\beta}}, & \mu &= 1, 2, 3, 4, & \alpha &= \pm, \dot{\beta} = \pm \\
 \text{scalar} : \Phi_{ij} & (= -\Phi_{ji}), & \bar{\Phi}^{ij} &\sim \frac{1}{2} \epsilon^{ijkl} \Phi_{kl}, & i, j, k, l &= 1, 2, 3, 4 \\
 \text{fermion} : \Psi_{i\alpha}, & \bar{\Psi}_{\dot{\alpha}}^i. & & & &
 \end{aligned} \tag{2.1}$$

We consider the Euclidean CFT on \mathbb{R}^4 , which is related to the CFT on $S^3 \times \mathbb{R}$ by radial quantization. $\alpha, \dot{\alpha}$ are the doublet indices of $SU(2)_L \times SU(2)_R \sim SO(4)$ which rotate the S^3 , and μ is the vector index. Uppercase i, j are the indices for the fundamental representation of $SU(4)$ R-symmetry, while the lowercase indices are for the anti-fundamental representation. Hermitian conjugations of these fields should be understood with a bit care in radial quantization. See, for instance, [15] for the details. It will be useful to decompose these fields in the $\mathcal{N} = 1$ language as follows, with manifest $SU(3) \times U(1) \subset SU(4)$ symmetry:

$$\begin{aligned}
 \text{vector multiplet} : A_{\alpha\dot{\beta}}, & \quad \lambda_\alpha = \Psi_{4\alpha}, \quad \bar{\lambda}_{\dot{\alpha}} = \bar{\Psi}_{\dot{\alpha}}^4 \\
 \text{chiral multiplets} : \phi_m = \Phi_{4m}, & \quad \bar{\phi}^m = \bar{\Phi}^{4m}, \quad \psi_{m\alpha} = -i\Psi_{m\alpha}, \quad \bar{\psi}_{\dot{\alpha}}^m = i\bar{\Psi}_{\dot{\alpha}}^m.
 \end{aligned} \tag{2.2}$$

$m = 1, 2, 3$ is the $SU(3)$ index, either for the fundamental/anti-fundamental representation. The system carries a continuous real coupling g_{YM} , and enjoys $\mathcal{N} = 4$ superconformal

symmetry at any value of g_{YM} . (The theta angle will not be relevant in our discussions.) The supercharges consist of 16 Poincare supercharges $Q_\alpha^i, \bar{Q}_{i\dot{\alpha}}$ and 16 conformal supercharges $S_{i\alpha}, \bar{S}_{\dot{\alpha}}^i$. In radial quantization, Q and S are Hermitian conjugate to each other: $(Q_\alpha^i)^\dagger = S_{i\alpha}^\dagger, (\bar{Q}_{i\dot{\alpha}})^\dagger = \bar{S}^{i\dot{\alpha}}$. The quantum supercharges may depend on the coupling g_{YM} , which is hard to write down in general. The supercharges of the interacting classical field theory contain terms up to half-loops, i.e. $\mathcal{O}(g_{\text{YM}}^1)$ order. The transformations for the classical Poincare supercharges are listed in appendix A.

The BPS operators of our interest are defined with $Q \equiv Q_-^4$ and $S \equiv S_4^- = Q^\dagger$. We are interested in gauge-invariant local operators O located at the origin $x^\mu = 0$ of \mathbb{R}^4 , satisfying

$$[Q, O] = 0, [Q^\dagger, O] = 0. \quad (2.3)$$

The part of the superconformal algebra which is important to us is

$$Q^2 = 0, (Q^\dagger)^2 = 0, \{Q, Q^\dagger\} \sim H - \sum_{I=1}^3 R_I - \sum_{i=1}^2 J_i. \quad (2.4)$$

H is the scaling dimension of the local operators, or the energy of the corresponding states times the radius of S^3 . R_I are the three Cartan charges of $\text{SO}(6) \sim \text{SU}(4)$ which rotate three orthogonal 2-planes in \mathbb{R}^6 . J_i are the two angular momenta on \mathbb{R}^4 rotating two orthogonal 2-planes. The eigenvalues of R_I and J_i are integers for bosons and half of odd integers for fermions. The BPS operators O annihilated by Q and Q^\dagger commute with $\{Q, Q^\dagger\}$. They can be arranged to be the eigenstates of H, R_I, J_i , whose eigenvalues E, R_I, J_i satisfy

$$E = \sum_I R_I + \sum_i J_i. \quad (2.5)$$

In the free limit, the operators satisfying the BPS relation (2.5) can be easily constructed using the BPS elementary fields satisfying the same relation. The charges for the fields and the derivative $\partial_\mu \sim \partial_{\alpha\dot{\beta}}$ are listed in table 1. The field strengths in the bispinor basis are defined by $f_{\alpha\beta} \sim (\sigma^{\mu\nu})_{\alpha\beta} F_{\mu\nu}$ and $f_{\dot{\alpha}\dot{\beta}} \sim (\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}} F_{\mu\nu}$. Among them, the gauge-covariant BPS fields and derivatives which satisfy the BPS relation are given by

$$\bar{\phi}^m, \psi_{m+}, f_{++}, \bar{\lambda}_{\dot{\alpha}}, \partial_{+\dot{\alpha}}. \quad (2.6)$$

With these, we construct independent ‘letters’ for the gauge invariant operators. Basically, acting derivatives $\partial_{+\dot{\alpha}}$ to a BPS field forms a letter. However, the equation of motion operator is zero and should not be included. The only equation of motion constructed using (2.6) is

$$\partial_{+\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} = 0 \Leftrightarrow \partial_{+[\dot{\alpha}} \bar{\lambda}_{\dot{\beta}]} = 0. \quad (2.7)$$

So the $\text{SU}(2)_R$ indices carried by the derivatives and the gaugino have to be symmetrized in a letter. Also, in the free theory, the derivatives $\partial_{+\dot{\alpha}}$ acting on the same field commute. Therefore, all $\text{SU}(2)_R$ indices appearing in a letter should be symmetrized. So we find the following independent letters (all at $x^\mu = 0$),

$$\begin{aligned} \partial_{+(\dot{\alpha}_1 \cdots \dot{\alpha}_n)} \varphi &\sim (\partial_{++})^{n_1} (\partial_{+\cdot})^{n_2} \varphi \quad (n_1, n_2 \geq 0, n \geq 0) \\ \partial_{+(\dot{\alpha}_1 \cdots \dot{\alpha}_{n-1} \bar{\lambda}_{\dot{\alpha}_n})} &\sim n_1 (\partial_{++})^{n_1-1} (\partial_{+\cdot})^{n_2} \bar{\lambda}_{\dot{\alpha}_n} + n_2 (\partial_{++})^{n_1} (\partial_{+\cdot})^{n_2-1} \bar{\lambda}_{\dot{\alpha}_n} \quad (n_1, n_2 \geq 0, n \geq 1) \end{aligned} \quad (2.8)$$

field	E	(R_1, R_2, R_3)	(J_1, J_2)
ϕ_m	1	$(-1, 0, 0), (0, -1, 0), (0, 0, -1)$	$(0, 0)$
$\bar{\phi}^m$	1	$(1, 0, 0), (0, 1, 0), (0, 0, 1)$	$(0, 0)$
$\psi_{m\pm}$	$\frac{3}{2}$	$(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$	$(\pm\frac{1}{2}, \pm\frac{1}{2})$
$\bar{\psi}_{\pm}^m$	$\frac{3}{2}$	$(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	$(\pm\frac{1}{2}, \mp\frac{1}{2})$
$A_{+\pm}$	1	$(0, 0, 0)$	$(1, 0), (0, 1)$
$A_{-\pm}$	1	$(0, 0, 0)$	$(-1, 0), (0, -1)$
f_{++}, f_{+-}, f_{--}	2	$(0, 0, 0)$	$(1, 1), (0, 0), (-1, -1)$
$f_{+\dot{+}}, f_{+\dot{-}}, f_{-\dot{-}}$	2	$(0, 0, 0)$	$(1, -1), (0, 0), (-1, 1)$
λ_{\pm}	$\frac{3}{2}$	$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$	$(\pm\frac{1}{2}, \pm\frac{1}{2})$
$\bar{\lambda}_{\pm}$	$\frac{3}{2}$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$(\pm\frac{1}{2}, \mp\frac{1}{2})$
$\partial_{+\dot{\alpha}}$	1	$(0, 0, 0)$	$(1, 0), (0, 1)$
$\partial_{-\dot{\alpha}}$	1	$(0, 0, 0)$	$(-1, 0), (0, -1)$

Table 1. The charges of elementary fields. Charges in the parenthesis are listed in the order of $m = 1, 2, 3$ or $\dot{\alpha} = \dot{+}, \dot{-}$ or in the order of the fields listed in the first column.

where $n_1 + n_2 = n$. φ denotes any field except $\bar{\lambda}_{\dot{\alpha}}$, i.e. chosen among $\bar{\phi}^m, \psi_{m+}, f_{++}$. Multiplying these letters and pairwise contracting the $SU(N)$ indices, one can construct general gauge-invariant BPS operators in the free theory.

Now we consider the 1-loop BPS operators, i.e. operators which diagonalize H and saturate the BPS relation till $\mathcal{O}(g_{\text{YM}}^2)$ order. For very small g_{YM} , we can start from the free BPS operators explained in the previous paragraph and study which of them remain BPS at the 1-loop level. The only modification needed is to replace $\partial_{+\dot{\alpha}}$ and f_{++} by the covariant expressions by adding $\mathcal{O}(g_{\text{YM}}^1)$ corrections. The covariant letters used in the free theory should be modified as follows:

$$D_{+(\dot{\alpha}_1 \cdots \dot{\alpha}_n)} \varphi, \quad D_{+(\dot{\alpha}_1 \cdots \dot{\alpha}_{n-1} \bar{\lambda}_{\dot{\alpha}_n})}. \quad (2.9)$$

Here $D_{+\dot{\alpha}}$ is the covariant derivative defined by $D_{+\dot{\alpha}} = \partial_{+\dot{\alpha}} - ig_{\text{YM}}[A_{+\dot{\alpha}}, \]$, related to f_{++} by $[D_{+\dot{\alpha}}, D_{+\dot{\alpha}}] \sim [D_{+[\dot{\alpha}}, D_{+\dot{\beta}}]] \sim g_{\text{YM}} f_{++}$. Although two different $D_{+\dot{\alpha}}$'s do not commute anymore, we still completely symmetrize the derivatives in a letter. This is to avoid introducing redundant letters: if some D 's are antisymmetrized in (2.9), it can be rewritten as a sum of products of letters involving f_{++} 's. We also symmetrize the $SU(2)_R$ indices in $D_{+\dot{\alpha}}$ and $\bar{\lambda}_{\dot{\alpha}}$ to avoid redundant letters, because the gaugino equation is given by $g_{\text{YM}}[\bar{\phi}^m, \psi_{m+}] \sim D_{+\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} \sim D_{+[\dot{\alpha}} \bar{\lambda}_{\dot{\beta}}]$. Among all possible gauge-invariant operators O constructed from (2.9), we would like to find the combinations satisfying

$$0 = \left[H - \sum_I R_I - \sum_i J_i, O \right] \sim [\{Q, Q^\dagger\}, O]. \quad (2.10)$$

The last equation holds if and only if $[Q, O] = 0$ and $[Q^\dagger, O] = 0$. Note that the coupling dependence is only in H, Q, Q^\dagger , since the quantized non-Abelian charges R_I, J_i cannot

depend on g_{YM} . At the 1-loop level, $H - \sum_I R_I - \sum_i J_i$ acting on the BPS letters (2.9) is at $\mathcal{O}(g_{\text{YM}}^2)$ order, while Q, Q^\dagger acting on (2.9) are at $\mathcal{O}(g_{\text{YM}}^1)$ order. Their complete forms are given in [36]. In particular, Q at the half-loop order is simply that of the classical field theory at nonzero g_{YM} , which transform the BPS fields by

$$\begin{aligned} [Q, \bar{\phi}^m] &= 0, & \{Q, \bar{\lambda}_{\dot{\alpha}}\} &= 0, & \{Q, \psi_{m+}\} &= -ig_{\text{YM}}\epsilon_{mnp}[\bar{\phi}^n, \bar{\phi}^p] \\ [Q, f_{++}] &= -ig_{\text{YM}}[\psi_{m+}, \bar{\phi}^m], & [Q, D_{+\dot{\alpha}}](\dots) &= -ig_{\text{YM}}[\bar{\lambda}_{\dot{\alpha}}, (\dots)] \end{aligned} \quad (2.11)$$

$S = Q^\dagger$ acts on pairs of letters [36]. From here, we absorb g_{YM} into the normalization of fields.

Instead of directly constructing the BPS operators, we study the cohomology classes which are in 1-to-1 map to the BPS operators. The cohomology class is defined by the set of operators O made of (2.9) that are closed under the action of Q , i.e. $[Q, O] = 0$, with the equivalence relation $O \sim O + [Q, \Lambda]$. Here, Λ is also an operator constructed with (2.9). We can call this a cohomology because of the nilpotency $Q^2 = 0$. This cohomology class is in 1-to-1 map to the BPS operators O_{BPS} satisfying $[Q, O_{\text{BPS}}] = 0$ and $[Q^\dagger, O_{\text{BPS}}] = 0$, because the latter is basically a harmonic form in this language [15]. The cohomology problem is defined using Q without referring to $S = Q^\dagger$. Note also that this problem is completely classical, since the transformation (2.11) is that of the classical field theory.

It has been conjectured implicitly/explicitly (especially in [17]) that the 1-loop BPS states remain BPS at general nonzero coupling. However, even if this conjecture is true, it does not mean that the form of the BPS operator O_{BPS} takes the same form at different values of g_{YM} . The conjecture just claims that the operators will survive to be BPS with g_{YM} -dependent deformations, being in 1-to-1 map with the 1-loop BPS states without any pairwise jumps. Recently, the validity of this conjecture was argued at all orders in perturbation theory [18].

We shall construct the representatives of new cohomology classes. Our interest is those which have chances to describe black holes rather than BPS gravitons. The BPS graviton cohomologies are completely understood, even at finite N subject to the stringy exclusion principle. We shall explain them in section 2.1. However, enumerating them without overcounting is technically quite tricky, due to the trace relations. We shall present a strategy which allows us to solve the trace relations in terms of diagonal matrix fields. The strategy applies to arbitrary N , but is particularly useful for $N = 2$. (At $N = 2$, this idea was employed in [15] in special subsectors.) We shall analytically/numerically implement this idea in section 3 (and appendix C) to efficiently count graviton-type cohomologies.

2.1 The finite N gravitons

We first explain the cohomologies for the BPS gravitons in the large N limit, and then explain how to define and understand the finite N graviton cohomologies.

We first take $N \gg 1$ and the number of fields in the cohomologies much smaller than N (say at order 1). In this limit, the mixing of single- and multi-trace operators by the dilatation operator H is suppressed by $\frac{1}{N}$. So one can first construct the BPS operators using single trace operators only, and then multiply them to obtain multi-trace

$(-1)^F E'$	J'	R'_1	R'_2	construction
n	0	n	0	$ n\rangle$
$-(n + \frac{1}{2})$	$\frac{1}{2}$	$n - 1$	0	$\bar{Q}_{m\dot{\alpha}} n\rangle$
$n + 1$	0	$n - 2$	0	$\bar{Q}_{m\dot{+}}\bar{Q}_{n\dot{-}} n\rangle$
$-(n + 1)$	0	$n - 1$	1	$Q_+^m n\rangle$
$n + \frac{3}{2}$	$\frac{1}{2}$	$n - 2$	1	$Q_+^m\bar{Q}_{n\dot{\alpha}} n\rangle$
$-(n + 2)$	0	$n - 3$	1	$Q_+^m\bar{Q}_{n\dot{+}}\bar{Q}_{p\dot{-}} n\rangle$
$n + 2$	0	$n - 1$	0	$Q_+^m Q_+^n n\rangle$
$-(n + \frac{5}{2})$	$\frac{1}{2}$	$n - 2$	0	$Q_+^m Q_+^n \bar{Q}_{p\dot{\alpha}} n\rangle$
$n + 3$	0	$n - 3$	0	$Q_+^m Q_+^n \bar{Q}_{p\dot{+}}\bar{Q}_{q\dot{-}} n\rangle$

Table 2. The state contents of the PSU(1,2|3) multiplet S_n . For low n 's, the rows with negative R'_1 are absent. $|n\rangle$ schematically denotes the superconformal primaries.

BPS operators. The former and latter are the single- and multi-particle graviton states, respectively.

We consider the single-trace BPS operators from the viewpoint of the Q -cohomology problem. The cohomology problem is well-defined in the single trace sector, since the action of Q does not change trace numbers. This problem is completely solved [14, 16]. The single trace cohomologies can be arranged into the supermultiplets of $\text{PSU}(1, 2|3) \subset \text{PSU}(2, 2|4)$ which commute with our Q, Q^\dagger . In [12], the short multiplets for the single trace cohomologies are called S_n , with $n = 2, 3, \dots$.² In this paper, we shall call the multiplets/representations of $\text{PSU}(1, 2|3)$ ‘ $\frac{1}{16}$ -BPS multiplets/representations’ and those of $\text{PSU}(2, 2|4)$ ‘ $\mathcal{N} = 4$ multiplets/representations,’ respectively. The ways in which $\frac{1}{16}$ -BPS representations sit in the $\mathcal{N} = 4$ representations are explained in the appendix B.3 of [12], which we review in appendix B. The $\frac{1}{16}$ -BPS multiplet S_n is contained in the short $\mathcal{N} = 4$ multiplet $B\bar{B}[0; 0]_n^{[0, n, 0]}$ in the notation of [37], where the superscripts/subscript are the SU(4) Dynkin labels and the scaling dimension of the primaries, respectively, and $[J_1 + J_2, J_1 - J_2] = [0, 0]$ denotes the spins of the primaries. (This is a multiplet of **bb** type in the notation of [12].) Following [12], we define $E' = E + \frac{J_1 + J_2}{2}$, $J' = \frac{J_1 - J_2}{2}$, and $[R'_1, R'_2]$ is the SU(3) highest weight. The states in S_n obtained by acting Q_+^m and $\bar{Q}_{m\dot{\alpha}}$ generators of PSU(1, 2|3) are listed in table 2. These are obtained by assuming that all supercharges anticommute. In fact some of the anticommutators yield $\{Q_+^m, \bar{Q}_{n\dot{\alpha}}\} \sim P_{+\dot{\alpha}}$, but they generate the conformal descendants can be generated later. In the gravity dual language, the table lists the Kaluza-Klein field contents in AdS₅. The conformal descendants are obtained by acting $P_{+\dot{\alpha}}$ on the ‘fields’ in the table, which generate the wavefunctions in AdS₅.

The superconformal primaries $|n\rangle$ of table 2 are given by the following operators,

$$\text{tr}[\bar{\phi}^{(m_1} \dots \bar{\phi}^{m_n)}]. \tag{2.12}$$

²The S_1 multiplet comes from the overall U(1) mode of the U(N) theory, which is absent for SU(N).

From the cohomology viewpoint, (2.12) can be understood as follows. All gauge-invariant operators made of BPS scalars are Q -closed since $Q\bar{\phi}^m = 0$. However, the presence of anti-symmetrized pairs of scalars will make the operator Q -exact because $[\bar{\phi}^m, \bar{\phi}^n] \sim \epsilon^{mnp} Q\psi_{p+}$ from (2.11).³ So among the single-trace operators made of n scalars, only (2.12) represent nontrivial cohomology classes. All the other states in the table are obtained by acting $Q_+^m, \bar{Q}_{m\dot{\alpha}}$'s on (2.12). Multiplying them yields independent multi-particle cohomologies at large N . Cohomologies in S_2 are given in appendix A.

Now we discuss the finite N cohomologies of graviton type. Finite N gravitons are nothing but the ‘graviton cohomologies’ that we have explained so far. Various steps that we explained above go through for finite N cohomologies. For instance, constructing cohomologies within single trace operators is consistent because Q does not change the trace number. Also, despite the existence of trace relations at finite N which may relate some single trace cohomologies to multi-trace ones, it is just a matter of using redundant basis and we can eliminate them later. (Of course, it will be convenient later to use only a subset of a such single trace cohomologies, as we will explain.) Next we consider the step of multiplying them to make multi-trace operators. Due to the Leibniz rule of Q , cohomologies multiply to yield a new cohomology. Therefore, even at finite N , multiplying the single trace cohomologies yields multi-trace cohomologies. Again due to trace relations, there might be some multiplications which yield a trivial cohomology, i.e. Q -exact operator. However, this is again a question of redundancy.

The only new issue to consider is that the number of independent cohomologies reduce because of the trace relation. Even at large N , one should take into account the trace relations if the number of fields becomes larger than N . This is the regime in which semi-classical giant gravitons can account for the trace relations as the stringy exclusion principle. Even at finite N , it makes good sense to regard these cohomologies as the graviton cohomologies in quantum AdS gravity. In other words, these BPS states can be completely understood by knowing the (giant) graviton physics.⁴

On the other hand, these operators cannot account for certain finite N physics. For instance, in the high-temperature Cardy limit [9], charges/energy are taken to be N^2 times a large number independent of N . The entropy in this limit scales like N^2 times a universal large number for the whole sequence of $U(N)$ gauge theories, from small to large N . Although this is not large N at all, such a deconfining behavior $\propto N^2$ is universal at large energy/temperature. This generalizes the large black hole physics straightforwardly to finite N . (The black hole like growth of the entropy at finite N was also confirmed by numerical studies [38, 39].) Such behaviors do not happen with the finite N graviton cohomologies. All these make it natural to study novel finite N cohomologies which are not of the graviton type as defined above.

³ $\bar{\phi}^{m_1} \bar{\phi}^{n_1} \dots \bar{\phi}^{n_i} \bar{\phi}^{m_2} - \bar{\phi}^{m_2} \bar{\phi}^{n_1} \dots \bar{\phi}^{n_i} \bar{\phi}^{m_1}$ is also a linear combination of terms involving commutators.

⁴Historically, giant gravitons are discovered as objects realizing the trace relations, reducing the number of states than the naive estimates. This is the context in which we quote giant gravitons here. However, note that it has been shown [40] recently that more complicated giant graviton states (with brane intersections and light open strings stretched between the branes) represent black hole microstates.

In section 3, an important technical step of identifying the black hole cohomologies will be counting finite N gravitons and subtracting them from the full degeneracy (index). Once we know the overcomplete set for gravitons as described above, it is in principle straightforward to identify the redundant operators due to trace relation. In practice, even if one uses a computer, this becomes quickly cumbersome as the charges and N grow. In the remaining part of this section, we will explain some structures which make this estimate easier. We explain two tricks, both related to the fact that the primaries (2.12) are given by symmetrized scalars.

The single-trace cohomologies in the S_n multiplets are the generators of the graviton cohomologies, i.e. we multiply them to generate multi-trace graviton cohomologies. If some of them decompose to products of other generators in S_n 's, clearly they need not be used as generators. For $SU(N)$, one can show that all elements of $S_{n \geq N+1}$ decompose to the those of $S_{n \leq N}$. To see this, we first show that any primary of the form (2.12) at $n \geq N+1$ decomposes to those at $n \leq N$. This can be shown using the Cayley-Hamilton equation,

$$M^N + c_{N-1}(M)M^{N-1} + \dots c_1(M)M + c_0(M)\mathbf{1}_{N \times N} = 0, \quad (2.13)$$

where M is an $N \times N$ matrix, and the coefficients $c_n(M)$'s are explicitly known in terms of $\text{tr}(M^k)$ with $k \leq N - n$. (In particular, $c_0(M) = (-1)^N \det(M)$.) Using this identity, one can express the matrix M^N (the first term) as a linear combination of various multi-trace operators times the matrix M^n , where $0 \leq n \leq N - 1$. Inserting $M = M_1 + M_2$, $M = M_1 + M_2 + M_3$, \dots , $M = M_1 + \dots + M_N$ with independent matrices and applying the Cayley-Hamilton equations repeatedly to various parts, one obtains a more generalized identity for the symmetrized product $M_{(1} \dots M_N)$ of N matrices M_i . For traceless M_i , some examples at low N are

$$\begin{aligned} N = 2 : M_{(1}M_2) &= \frac{1}{2}\text{tr}(M_1M_2)\mathbf{1}_{2 \times 2} & (2.14) \\ N = 3 : M_{(1}M_2M_3) &= \frac{1}{2}\text{tr}(M_{(1}M_2)M_3) + \frac{1}{3}\text{tr}(M_{(1}M_2M_3))\mathbf{1}_{3 \times 3} \\ N = 4 : M_{(1}M_2M_3M_4) &= \frac{1}{2}\text{tr}(M_{(1}M_2)M_3M_4) + \frac{1}{3}\text{tr}(M_{(1}M_2M_3)M_4) \\ &\quad + \frac{1}{4}\text{tr}(M_{(1}M_2M_3M_4))\mathbf{1}_{4 \times 4} - \frac{1}{2 \cdot 2 \cdot 2!}\text{tr}(M_{(1}M_2)\text{tr}(M_3M_4))\mathbf{1}_{4 \times 4} \\ N = 5 : M_{(1} \dots M_5) &= -\frac{1}{2 \cdot 2 \cdot 2!}\text{tr}(M_{(1}M_2)\text{tr}(M_3M_4)M_5) - \frac{1}{2 \cdot 3}\text{tr}(M_{(1}M_2)\text{tr}(M_3M_4M_5))\mathbf{1}_{5 \times 5} \\ &\quad + \frac{1}{2}\text{tr}(M_{(1}M_2)M_3M_4M_5) + \frac{1}{3}\text{tr}(M_{(1}M_2M_3)M_4M_5) \\ &\quad + \frac{1}{4}\text{tr}(M_{(1}M_2M_3M_4)M_5) + \frac{1}{5}\text{tr}(M_{(1} \dots M_5))\mathbf{1}_{5 \times 5} . \end{aligned}$$

The factors on the right-hand sides are the symmetry factors for the cyclicity of matrices inside a trace and also for exchanging identical single-trace operators. The sign is -1 if the term involves even number of single-trace operators. Inserting this expression into (2.12) with $n \geq N + 1$, many times if necessary, the single trace operators can be written as sums of products of operators of the form (2.12) with $n \leq N$. In other words, the chiral primaries of $S_{n \geq N+1}$ can be written in terms of those of $S_{n \leq N}$. Acting $Q_+^m, \bar{Q}_{m\dot{\alpha}}, P_{+\dot{\alpha}}$ to

these expressions, all cohomologies in $S_{n \geq N+1}$ can be written in terms of those in $S_{n \leq N}$. So finite N gravitons can be constructed with the single-trace generators in $S_{n \leq N}$.

Even if the single-trace generators $S_{n \leq N}$ are independent in the sense that none of them can be written as multi-trace operators, still there are new trace relations among multi-traces operators after multiplying them. For example, consider the $SU(2)$ theory and the following primaries of the S_2 multiplet:

$$\text{tr}(\bar{\phi}^{(m}\bar{\phi}^{n)}) \sim \text{tr}(X^2), \text{tr}(Y^2), \text{tr}(Z^2), \text{tr}(XY), \text{tr}(YZ), \text{tr}(ZX). \quad (2.15)$$

Here and below, we shall often use the notation $(X, Y, Z) \equiv (\bar{\phi}^1, \bar{\phi}^2, \bar{\phi}^3)$. Since we consider cohomologies, we study the trace relations up to Q -exact terms. One finds the following relations at the double trace order,

$$\begin{aligned} \text{tr}(X^2)\text{tr}(Y^2) - [\text{tr}(XY)]^2 &\sim \text{tr}([X, Y][X, Y]) \sim Q\text{tr}(\psi_3[X, Y]) \\ \text{tr}(X^2)\text{tr}(YZ) - \text{tr}(XY)\text{tr}(XZ) &\sim \text{tr}([X, Y][X, Z]) \sim Q\text{tr}(\psi_3[X, Z] - [X, Y]\psi_2), \end{aligned} \quad (2.16)$$

and 4 more relations obtained by permuting X, Y, Z . More covariantly, one can write them as

$$\epsilon_{mab}\epsilon_{ncd}\text{tr}(\bar{\phi}^{(a}\bar{\phi}^{c)})\text{tr}(\bar{\phi}^{(b}\bar{\phi}^{d)}) = Q\text{-exact}, \quad (2.17)$$

in the representation $\bar{\mathbf{6}}$, or $[0, 2]$, of $SU(3)$. They are the primaries of a constraint supermultiplet of $PSU(1, 2|3)$, contained in $B\bar{B}[0; 0]_4^{[2, 0, 2]}$ of $PSU(2, 2|4)$. These constraints are not independent in the sense that there are ‘relations of relations’ at higher orders. See, for instance, [15] on these relations of relations in special sectors. Unfortunately, we do not know a simple algorithm to identify all such trace relations in terms of the generators.

The strategy we explained so far, using single-trace generators, is analogous or dual to the counting of BPS states from giant gravitons (D3-branes expanded in S^5). There is another way of counting graviton-like BPS states using dual giant gravitons, which are D3-branes expanded in AdS_5 . In the field theory, this is dual to counting operators in terms of the eigenvalues of diagonal matrices, as we shall explain in a moment. For the chiral primaries (2.12), the two complementary methods are fully explored and yield the same result [41, 42]. For general graviton-like BPS states, neither method has been fully developed. In a sense, we shall now provide a hybrid method of the two.

The multiplet $B\bar{B}[0; 0]_n^{[0, n, 0]}$ for single-particle gravitons is absolutely protected, meaning that it cannot combine with other short multiplets to form a long multiplet. BPS operators in this multiplet never acquire anomalous dimensions as the coupling changes. Therefore, one may count the graviton cohomologies in the free limit $g_{YM} \rightarrow 0$.

We start by considering the chiral primaries (2.12). Since all the scalars are symmetrized in the trace, we can regard all the fields as diagonal matrices for the purpose of enumerating cohomologies. Each scalar contains $N - 1$ eigenvalues, or N eigenvalues whose sum is zero. The generators (2.12) are Weyl-invariant polynomials of the eigenvalues. Then we consider the superconformal descendants of (2.12) in the free theory limit. In free theory, supersymmetry transformation of diagonal fields only involves diagonal components of the superpartner fields. So we can restrict all the elementary fields appearing

in $S_{n \leq N}$ to be diagonal. The covariant derivatives on the fields also reduce to ordinary derivatives since $g_{YM} = 0$. In particular, trace relations are now relations of eigenvalue polynomials, including the ordinary derivatives acting on the eigenvalues. So we should count the polynomials which can be written as sums of products of the generators in $S_{n \leq N}$. The reduction to the polynomial counting will ease numerical studies, and sometimes will admit analytic counting.

This procedure becomes simpler in the SU(2) theory, because each elementary field is represented by one eigenvalue. We denote by $x, y, z, \psi_m, \lambda_\alpha, f$ the eigenvalues of the fields $X, Y, Z, \psi_{m+}, \bar{\lambda}_{\dot{\alpha}}, f_{++}$, respectively, e.g. $X = \text{diag}(x, -x)$. The BPS derivatives $D_{+\dot{\alpha}}$ can also be replaced by ordinary derivatives, which we call ∂_α . We should consider polynomials of these eigenvalues and derivatives acting on them,

$$\partial_+^{n_1} \partial_-^{n_2} (x, y, z, \psi_m, f), \quad n_1 \partial_+^{n_1-1} \partial_-^{n_2} \lambda_+ + n_2 \partial_+^{n_1} \partial_-^{n_2-1} \lambda_- \quad (2.18)$$

with $n_1, n_2 \geq 0$. In order for a polynomial to be a BPS graviton, it must be a sum of products of the generators in S_2 . In terms of eigenvalues, those generators are given by

$$\begin{aligned} & \partial_+^{n_1} \partial_-^{n_2} (x^2, y^2, z^2, xy, yz, zx) \quad (2.19) \\ & \partial_+^{n_1} \partial_-^{n_2} (y\psi_1, z\psi_1, x\psi_2, z\psi_2, x\psi_3, y\psi_3, x\psi_1 - y\psi_2, y\psi_2 - z\psi_3) \\ & \partial_+^{n_1} \partial_-^{n_2} (x\lambda_\pm, y\lambda_\pm, z\lambda_\pm), \quad \partial_+^{n_1} \partial_-^{n_2} \left(xf - \frac{1}{2}\psi_2\psi_3, yf - \frac{1}{2}\psi_3\psi_1, zf - \frac{1}{2}\psi_1\psi_2 \right) \\ & \partial_+^{n_1} \partial_-^{n_2} (\psi_1\lambda_\pm + 2y\partial_\pm z, \psi_2\lambda_\pm + 2z\partial_\pm x, \psi_3\lambda_\pm + 2x\partial_\pm y) \\ & \partial_+^{n_1} \partial_-^{n_2} (\lambda_+\lambda_-), \quad \partial_+^{n_1} \partial_-^{n_2} \left(\lambda_\pm f - \frac{2}{3}\psi_m \partial_\pm \phi^m + \frac{1}{3}\phi^m \partial_\pm \psi_m \right). \end{aligned}$$

If a polynomial is expressed as a sum of the products of these generators, it must be invariant under the Weyl reflection of the eigenvalues:

$$(x, y, z, \psi_m, \lambda_\alpha, f) \rightarrow -(x, y, z, \psi_m, \lambda_\alpha, f). \quad (2.20)$$

In other words it must be an even polynomial. We used this approach to numerically count the SU(2) graviton operators till certain order in the charge expansion. The studies of section 3.2 will be based on this calculation. In a simple subsector of the SU(2) theory, we can analytically count independent graviton cohomologies using this approach as explained in appendix C.

3 The black hole cohomologies for SU(2)

[18] systematically constructed all cohomologies till certain order. We shall employ a more streamlined approach in this section. We will first compute the index over the black hole cohomologies by subtracting the contributions from finite N gravitons. Then with some guesses if necessary, we shall explicitly construct the black hole cohomologies which account for this index. We find that this approach is much more efficient in detecting new cohomologies.

The index of the $\mathcal{N} = 4$ Yang-Mills theory is defined as [11, 12]

$$Z(\Delta_I, \omega_i) = \text{Tr} \left[(-1)^F e^{-\sum_{I=1}^3 \Delta_I R_I} e^{-\sum_{i=1}^2 \omega_i J_i} \right] \quad (3.1)$$

where the chemical potentials should satisfy $\sum_I \Delta_I - \sum_i \omega_i = 0$ for this quantity to be an index. For the $SU(N)$ theory, this is given by the following integral

$$Z(\Delta_I, \omega_i) = \frac{1}{N!} \int_0^{2\pi} \prod_{a=1}^N \frac{d\alpha_a}{2\pi} \exp \left[\sum_{a \neq b} \sum_{n=1}^{\infty} \frac{f(n\Delta_I, n\omega_i) - 1}{n} e^{in(\alpha_a - \alpha_b)} \right] \quad (3.2)$$

$$\times \exp \left[(N-1) \sum_{n=1}^{\infty} \frac{f(n\Delta_I, n\omega_i)}{n} \right],$$

where $f(\Delta_I, \omega_i) \equiv 1 - \frac{(1-e^{-\Delta_1})(1-e^{-\Delta_2})(1-e^{-\Delta_3})}{(1-e^{-\omega_1})(1-e^{-\omega_2})}$ is the single letter index. Since this index is independent of the coupling g_{YM} , we can regard it as the index over our 1-loop cohomologies.

3.1 The BMN sector

The radially quantized QFT lives on $S^3 \times \mathbb{R}$. The fields are expanded in spherical harmonics of $SO(4)$. It was shown in [28] that the *classical* $\mathcal{N} = 4$ Yang-Mills theory has a consistent truncation which keeps finite degrees of freedom, described by the BMN matrix model [27]. The modes kept after the truncation are given by: (1) s-wave modes $\phi_m(t), \bar{\phi}^m(t)$ of the scalars, (2) lowest spinor harmonics modes $\psi_{m\alpha}(t), \lambda_\alpha(t)$ (the spinor indices are defined using the labels of Killing spinor fields [28]), (3) vector potential 1-form restricted to $A = A_0(t)dt + A_i(t)\sigma_i$ where σ_i with $i = 1, 2, 3$ are the right-invariant 1-forms on S^3 in our convention. This is a consistent truncation of the nonlinear equations of motion, and not a quantum reduction in any sense. So the full quantum BMN theory is a priori unrelated to the 4d Yang-Mills theory. However, since our 1-loop cohomology problem uses classical supercharge Q only, it can be truncated to the BMN model. If the conjecture of [17] is true, the whole BPS cohomology problem would have a quantum truncation to this model.

In general, the BMN theory and the full Yang-Mills theory behave differently in many ways. The difference starts from the number of ground states. The Yang-Mills theory on $S^3 \times \mathbb{R}$ has a unique vacuum, while the BMN model has many ground states labeled semiclassically by the discrete values of A_i . In the quantum BMN theory, viewed as M-theory in the plane wave background, these ground states describe various M2/M5-brane configurations with zero lightcone energies [43]. In the Yang-Mills theory, however, there are large gauge transformations on S^3 which can gauge away these ground states to $A_i = 0$. So if one wishes to study the Yang-Mills theory using this matrix model, it suffices to consider the physics around $A_i = 0$.

Recall that our cohomology problem is completely classical, using the classical supercharge Q at half-loop order. Therefore, this problem should have a truncation to the BMN matrix model. This turns out to be the cohomology problem defined using

$$\bar{\phi}^m, \quad \psi_{m+}, \quad f_{++}, \quad (3.3)$$

without using any gauginos $\bar{\lambda}_{\dot{\alpha}}$ or derivatives $D_{+\dot{\alpha}}$. These operators close by the action of Q : $[Q, \bar{\phi}^m] = 0$, $\{Q, \psi_{m+}\} = -i\epsilon_{mnp}[\bar{\phi}^m, \bar{\phi}^n]$, $[Q, f_{++}] = -i[\psi_{m+}, \bar{\phi}^m]$. So it is possible to restrict the cohomology problem by using operators constructed using the letters (3.3). Note that the truncation is also applied to the operator Λ when one identifies two operators O_1 and O_2 related as $O_2 - O_1 = [Q, \Lambda]$. This is why the gauginos $\bar{\lambda}_{\dot{\alpha}}$ cannot be included in this truncation. Although it is Q -closed by itself, $\bar{\lambda}_{\dot{\alpha}}$ can be obtained by acting Q on the covariant derivative, $[Q, D_{+\dot{\alpha}}] = -i[\bar{\lambda}_{\dot{\alpha}}, \cdot]$. So if one had tried to include $\bar{\lambda}_{\dot{\alpha}}$ into the truncation and construct operators like O_1, O_2, Λ , one may incorrectly conclude that certain O_1 and O_2 are different by not including derivatives in Λ . This truncation of the cohomology problem was known in [18, 26], although the relation to the BMN truncation was not explicitly addressed.⁵ Notice also that this truncation is not kinematic, i.e. cannot be inferred without knowing the dynamical information of the classical theory.

The BMN truncation is the $SU(2)_R$ invariant truncation. In our cohomology problem, this means that no ingredients include the $\dot{\alpha}$ indices for $SU(2)_R$. This is why $\bar{\lambda}_{\dot{\alpha}}$ and $D_{+\dot{\alpha}}$ are excluded. Similarly, in the representation theory, only a small subset of $PSU(1, 2|3)$ generators can be used to generate a multiplet. Among the $PSU(1, 2|3)$ generators $Q_+^m, \bar{Q}_{m\dot{\alpha}}$ and $P_{+\dot{\alpha}}$, only the three supercharges Q_+^m which belong to $SU(1|3)$ act within BMN cohomologies.

We also consider the index over the BMN cohomologies. Keeping the letters (3.3) only, the letter index is given by

$$\tilde{f} = \sum_{I=1}^3 e^{-\Delta_I} - \sum_{I<J} e^{-\Delta_I - \Delta_J} + e^{-\Delta_1 - \Delta_2 - \Delta_3} = 1 - (1 - e^{-\Delta_1})(1 - e^{-\Delta_2})(1 - e^{-\Delta_3}). \quad (3.4)$$

The first three terms come from $\bar{\phi}^m$, the next three terms from ψ_{m+} , and the last term from f_{++} . Δ_I is the chemical potential conjugate to $R_I + \frac{J_1 + J_2}{2} \equiv R_I + J$. The fourth chemical potential $\omega_1 - \omega_2$ of (3.1) does not appear since BMN truncation is $SU(2)_R$ invariant. The matrix integral expression for the BMN index is

$$\begin{aligned} Z(\Delta_I) &= \frac{1}{N!} \int_0^{2\pi} \prod_{a=1}^N \frac{d\alpha_a}{2\pi} \exp \left[\sum_{a \neq b} \sum_{n=1}^{\infty} \frac{\tilde{f}(n\Delta_I) - 1}{n} e^{in(\alpha_a - \alpha_b)} \right] \exp \left[(N-1) \sum_{n=1}^{\infty} \frac{\tilde{f}(n\Delta_I)}{n} \right] \\ &= \frac{1}{N!} \int_0^{2\pi} \prod_{a=1}^N \frac{d\alpha_a}{2\pi} \cdot \prod_{a \neq b} \frac{(1 - e^{i\alpha_{ab}}) \prod_{I<J} (1 - e^{-\Delta_I - \Delta_J} e^{i\alpha_{ab}})}{(1 - e^{-\Delta_1 - \Delta_2 - \Delta_3} e^{i\alpha_{ab}}) \prod_{I=1}^3 (1 - e^{-\Delta_I} e^{i\alpha_{ab}})} \\ &\quad \cdot \left[\frac{\prod_{I<J} (1 - e^{-\Delta_I - \Delta_J})}{(1 - e^{-\Delta_1 - \Delta_2 - \Delta_3}) \prod_{I=1}^3 (1 - e^{-\Delta_I})} \right]^{N-1} \end{aligned} \quad (3.5)$$

where $\alpha_{ab} \equiv \alpha_a - \alpha_b$. This expression cannot be obtained from the original index (3.2) by taking limits of the chemical potentials, because the BMN truncation is not kinematic. To ease the calculations, one may replace the Haar measure part of the integrand by half of it [44],

$$\frac{1}{N!} \prod_{a \neq b} (1 - e^{i\alpha_{ab}}) \rightarrow \prod_{a < b} (1 - e^{i\alpha_{ab}}), \quad (3.6)$$

without changing the integral.

⁵We thank Nakwoo Kim for first pointing this out to us.

(3.5) can be computed easily for low N 's because the integrand is a finite product. One can evaluate it by a residue sum. For instance, for the $SU(2)$ index, the only nontrivial integral variable is $e^{i\alpha_{12}}$ along the unit circle. We evaluate the integral by the residue sum by shrinking this unit circle. After the replacement (3.6) in the integrand, we should sum over four residues at the following poles:

$$e^{i\alpha_{12}} = e^{-\Delta_1}, e^{-\Delta_2}, e^{-\Delta_3}, e^{-\Delta_1-\Delta_2-\Delta_3}. \quad (3.7)$$

We have a general result, but just to illustrate it at $e^{-\Delta_1} = e^{-\Delta_2} = e^{-\Delta_3} \equiv t^2$, one obtains

$$Z = \left[1 + 3t^2 + 12t^4 + 20t^6 + 42t^8 + 48t^{10} + 75t^{12} + 66t^{14} + 81t^{16} + 55t^{18} + 54t^{20} + 27t^{22} + 19t^{24} + 6t^{26} + 3t^{28} \right] \frac{(1-t^2)^3}{(1-t^{12})(1-t^8)^3}. \quad (3.8)$$

This index contains both black hole and graviton cohomologies made with (3.3).

To compute the index over black hole cohomologies in this sector, we need to enumerate the graviton cohomologies and subtract their index from the full index. This can be done analytically for the $SU(2)$ gravitons in the BMN sector, employing the eigenvalue counting explained in section 2. We need to consider even polynomials of the following seven eigenvalues:

$$x, y, z, f, \psi_1, \psi_2, \psi_3, \quad (3.9)$$

where one should remember that the last three are Grassmannian numbers. We can write all possible even monomials of these eigenvalues, and find the combinations which can be written as sums of products of the following BMN generators in S_2 :

$$\begin{aligned} \mathbf{6} : & x^2, y^2, z^2, xy, yz, zx \\ \mathbf{8} : & \psi_1 \cdot (y, z), \psi_2 \cdot (z, x), \psi_3 \cdot (x, y), \psi_1 x - \psi_2 y, \psi_2 y - \psi_3 z \\ \mathbf{3} : & xf - \frac{1}{2}\psi_2\psi_3, yf - \frac{1}{2}\psi_3\psi_1, zf - \frac{1}{2}\psi_1\psi_2. \end{aligned} \quad (3.10)$$

Counting these polynomials in the right order, one can count them basically at the monomial levels. The procedure is explained in appendix B. We can compute the full partition function for these cohomologies. Giving -1 weights to fermions and unrefining some fugacities, we can compute the index $Z_{\text{grav}}(\Delta_I)$ over gravitons.

Subtracting this from the full index, the difference between Z and Z_{grav} is given by

$$Z - Z_{\text{grav}} = -\frac{e^{-4(\Delta_1+\Delta_2+\Delta_3)}}{1 - e^{-2(\Delta_1+\Delta_2+\Delta_3)}} \cdot \prod_{I=1}^3 (1 - e^{-\Delta_I}) \cdot \prod_{I=1}^3 \frac{1}{(1 - e^{-\Delta_I} e^{-\Delta_1-\Delta_2-\Delta_3})}. \quad (3.11)$$

Unrefining as $e^{-\Delta_I} = t^2$, one obtains a series in t given by

$$Z - Z_{\text{grav}} = -t^{24} + 3t^{26} - 3t^{28} + t^{30} - 3t^{32} + \dots. \quad (3.12)$$

t is conjugate to $j \equiv 6(R + J)$, where $R \equiv \frac{R_1+R_2+R_3}{3}$, $J \equiv \frac{J_1+J_2}{2}$. From this formula, one finds the first black hole cohomology at $j = 24$. This ‘threshold’ black hole cohomology

was already identified in [18, 26], as we shall review and rewrite in a more compact form below. It may look like there are many black hole states beyond this threshold, but most of them are rather trivial. To make this point clear, we would like to first interpret various factors of (3.11), which will be extensively justified later.

(3.11) is a multiplication of three factors, divided by the \cdot products. We interpret the first factor as the ‘core’ black hole primary operators. Constructing this part of the cohomologies will be the goal of this subsection. The second factor comes from the SU(13) descendants obtained from the first factor by acting Q_+^m . The supercharge Q_+^m carries charges $R_I = \delta_{I,m} - \frac{1}{2}$ and $J = \frac{1}{2}$, so is weighted by $e^{-\Delta_I}$. So the second factor comes from the Fock space obtained by acting three Q_+^m 's. Finally, the third factor comes from the multiplications of certain multi-gravitons to the core black hole cohomologies. Among the 17 graviton states listed in (3.10), only 3 types on the third line can contribute. The remaining 14 gravitons of (3.10) multiplying the core black hole operators do not appear in the index. This aspect will be discussed further in section 3.2.

For SU(2), the BMN index (3.11) does not show very large entropy even at large charges. From an inspection of the integral (3.5) and having in mind applying the techniques developed in [45], we expect that at large enough N and charges one will be able to show that the entropies will scale like those for black holes.

Now we construct the cohomologies that will account for the first factor of (3.11),

$$-\frac{t^{24}}{1-t^{12}} = -t^{24} - t^{36} - t^{48} - t^{60} - \dots, \tag{3.13}$$

where $t^6 = e^{-\Delta_1 - \Delta_2 - \Delta_3}$. The index predicts unique fermionic cohomology at $j = 24 + 12n$ ($n = 0, 1, 2, \dots$), all singlets of $SU(3) \subset SU(4)$. For SU(2) gauge group, we use the 3-dimensional vector notation for the adjoint fields. In the remaining part of this subsection, $\phi^m = (X, Y, Z)$, ψ_m, f would mean 3 dimensional vectors, and inner/outer products will replace the trace/commutators. The Q -transformations of these 3-vectors are given by

$$Q\phi^m = 0, \quad Q\psi_m = \frac{1}{2}\epsilon_{mnp}\phi^n \times \phi^p, \quad Qf = \phi^m \times \psi_m. \tag{3.14}$$

See [26] and appendix A for our normalization.

O_0 operator at t^{24} . This operator has charges $E = \frac{19}{2}$, $R_1 = R_2 = R_3 = \frac{3}{2}$, $J_1 = J_2 = \frac{5}{2}$. A representative of this cohomology [26] is given by

$$\begin{aligned} O'_0 = & (X \cdot \psi_1 - Y \cdot \psi_2)(X \cdot \psi_3)(\psi_2 \cdot \psi_1 \times \psi_1) + (Y \cdot \psi_2 - Z \cdot \psi_3)(Y \cdot \psi_1)(\psi_3 \cdot \psi_2 \times \psi_2) \\ & + (Z \cdot \psi_3 - X \cdot \psi_1)(Z \cdot \psi_2)(\psi_1 \cdot \psi_3 \times \psi_3). \end{aligned} \tag{3.15}$$

Note that the second and third terms are obtained by making cyclic permutations of (X, ψ_1) , (Y, ψ_2) , (Z, ψ_3) on the first term. The cyclic permutations are part of the SU(3) symmetry, thus symmetries of the cohomology problem, On the other hand, odd permutations accompanied by the sign flips of all ψ_m 's and ϕ^m 's are part of $SU(4) \times SU(2)_L$ symmetry which leave Q invariant, thus being symmetries of the cohomology problem.

To construct a better representative of this cohomology, consider the following operator obtained by permuting $(X, \psi_1) \leftrightarrow (Y, \psi_2)$ and flipping signs of all ϕ^m, ψ_m on (3.15):

$$O_0'' = (X \cdot \psi_1 - Y \cdot \psi_2)(Y \cdot \psi_3)(\psi_1 \cdot \psi_2 \times \psi_2) + (Y \cdot \psi_2 - Z \cdot \psi_3)(Z \cdot \psi_1)(\psi_2 \cdot \psi_3 \times \psi_3) \\ + (Z \cdot \psi_3 - X \cdot \psi_1)(X \cdot \psi_2)(\psi_3 \cdot \psi_1 \times \psi_1) . \quad (3.16)$$

One can show

$$O_0' - O_0'' = -2Q[(\psi_1 \cdot \psi_2)(\psi_2 \cdot \psi_3)(\psi_3 \cdot \psi_1)] , \quad (3.17) \\ O_0 \equiv -5(O_0' + O_0'') = \epsilon^{p_1 p_2 p_3} v_{p_1}^m v_{p_2}^n (\psi_m \cdot \psi_n \times \psi_{p_3}) ,$$

where

$$v_n^m \equiv (\phi^m \cdot \psi_n) - \frac{1}{3} \delta_n^m (\phi^p \cdot \psi_p) \quad (3.18)$$

are graviton cohomologies in the S_2 multiplet. O_0 is manifestly an $SU(3)$ singlet. Note that the second term of v proportional to δ_n^m drops out when v is inserted into (3.17), because of the symmetry of $\psi_m \cdot \psi_n \times \psi_{p_3}$ and the antisymmetry of $\epsilon^{p_1 p_2 p_3}$. So we can write

$$O_0 = \epsilon^{p_1 p_2 p_3} (\phi^m \cdot \psi_{p_1})(\phi^n \cdot \psi_{p_2})(\psi_m \cdot \psi_n \times \psi_{p_3}) . \quad (3.19)$$

To show that O_0 is a black hole cohomology, one should check that it is Q -closed, not Q -exact, and not of graviton type. The first and third are trivial. O_0 is not graviton-like because it consists of seven (odd) letters: since $SU(2)$ gravitons are made of operators in S_2 , they always have an even number of letters. To check Q -closedness, first note that Q acts only on $\psi_m \cdot \psi_n \times \psi_{p_3}$ because v_n^m are Q -closed. One finds

$$Q(\psi_m \cdot \psi_n \times \psi_p) = \frac{3}{2} \epsilon_{(m|qr} (\phi^q \times \phi^r) \cdot (\psi_{|n} \times \psi_p)) = 3 \epsilon_{(m|qr} (\phi^q \cdot \psi_{|n})(\phi^r \cdot \psi_p) = 3 \epsilon_{(m|qr} v_{|n}^q v_{p)}^r . \quad (3.20)$$

At the last step, the second term of $\phi^q \cdot \psi_n = v_n^q + \delta_n^q(\dots)$ etc. does not survive after the index contractions. Inserting it to QO_0 and replacing the product of two ϵ 's by three δ 's, QO_0 is given by various row/column contractions of four 3×3 traceless matrices v_n^m . Possible terms are $\text{tr}(v^4)$ and $\text{tr}(v^2)\text{tr}(v^2)$, but the fermionic nature of v and the cyclicity of trace ensure that they are all zero. So QO_0 is zero because there are no nonzero terms that can contribute.

The non- Q -exactness was originally shown after a calculation using computer [18, 26]. Here we provide an analytic argument. We assume Q -exactness, narrow down the possible Q -exact terms and then show that no combination of them works. O_0 is at the $\mathcal{O}(\phi^2 \psi^5)$ order. If this is Q -exact, the schematic structure should be as follows:

$$\phi^2 \psi^5 = Q_f [f \phi \psi^4] + Q[\psi^6] . \quad (3.21)$$

Q_f means the part of Q acting on f . Q may also act on ψ in this term to produce a term at $\mathcal{O}(f \phi^3 \psi^3)$ order, and if O_0 is completely Q -exact, $Q_\psi(f \phi \psi^4)$ should cancel $Q_f[f^2 \phi^2 \psi^2]$. We shall only consider the Q -exactness of O_0 within the $\phi^2 \psi^5$ order and find a contradiction. The terms on the right hand side should respect all the $SU(3) \times SU(2)$ tensor structures

of the left hand side. There might be terms violating some of these structures separately on the first and second terms, but they should cancel by themselves and we do not care about this part. We consider the terms which respect them and the equation should hold within this sector separately if (3.21) is generally true. (3.19) is given by multiplying the following scalar and fermion factors,

$$\begin{aligned} \phi_{(i}^{(m} \phi_{j)}^{n)} : (\mathbf{6}, \mathbf{5} + \mathbf{1}) \in \text{SU}(3) \times \text{SU}(2) \quad (3.22) \\ \psi_{p_1(i} \psi_{p_2|j)} (\psi_m \cdot \psi_n \times \psi_{p_3}) : (\mathbf{3}, \mathbf{5} + \mathbf{1}) \otimes (\overline{\mathbf{10}}, \mathbf{1}), \end{aligned}$$

where $i, j = 1, 2, 3$ are $\text{SU}(2) \sim \text{SO}(3)$ indices. The operators on the right hand side of (3.21) should respect these structures.

We shall first write down all possible terms on the right hand side satisfying several consequences of (3.22) after contracting all the indices, obtaining only a small number of terms. Some useful requirements are: (1) $\text{SU}(3)$ singlet condition of O_0 , (2) exchange symmetry of the two $\text{SU}(3)$ indices carried by the scalars. We first consider the term $Q[\psi^6]$. Q acting on any ψ produces a term of the form $\phi^m \times \phi^n$, violating the condition (2). So there are no terms of the form $Q[\psi^6]$ that we can write down. Now we try to write down all the gauge-invariant operators at $f\phi\psi^4$ order which can appear inside Q_f in (3.21). Since it consists of six letters, we take three pairwise inner products. (Contractions by two ϵ tensors can also be written as three inner products.) The possible terms are

$$(f \cdot \phi^m)(\psi_{[n_1} \cdot \psi_{n_2]})(\psi_{[p_1} \cdot \psi_{p_2]}), \quad (\phi^m \cdot \psi_{n_1})(f \cdot \psi_{n_2})(\psi_{[p_1} \cdot \psi_{p_2]}). \quad (3.23)$$

Q_f transformation of the first term violates the condition (2) since $Q[f \cdot \phi^m] = (\phi^n \times \psi_n) \cdot \phi^m = (\phi^m \times \phi^n) \cdot \psi_n$. Now imposing the condition (1) on the second term, one should contract the $\text{SU}(3)$ indices to form singlets. One finds

$$(\mathbf{8} \oplus \mathbf{1}) \otimes \mathbf{3} \otimes \overline{\mathbf{3}} \rightarrow \mathbf{27} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1} \oplus \mathbf{1}$$

so there are two possible singlets. They are

$$(\phi^m \cdot \psi_m) \epsilon^{npq} (f \cdot \psi_n) (\psi_p \cdot \psi_q), \quad (\phi^m \cdot \psi_n) (f \cdot \psi_m) \epsilon^{npq} (\psi_p \cdot \psi_q). \quad (3.24)$$

Acting Q_f on them and separating the ϕ^2 and ψ^5 parts as we did in (3.22), we obtain a part consistent with (3.22) and the rest. Focusing on the former part, they are given by $\phi_{(i}^{(m} \phi_{j)}^{r)}$ times

$$\psi_{(m}^{(i} (\psi_r \times \psi_n)^j) \epsilon^{npq} (\psi_p \cdot \psi_q), \quad \psi_n^{(i} (\psi_r \times \psi_m)^j) \epsilon^{npq} (\psi_p \cdot \psi_q) \quad (3.25)$$

respectively. If O_0 is Q -exact, a suitable linear combination of these two terms should yield O_0 . The agreement should happen for every coefficient of $\phi_{(i}^{(m} \phi_{j)}^{r)}$ separately, demanding

$$\epsilon^{pqr} \psi_p^{(i} \psi_q^{j)} (\psi_m \cdot \psi_n \times \psi_r) = A \psi_{(m}^{(i} (\psi_n \times \psi_r)^j) \epsilon^{pqr} (\psi_p \cdot \psi_q) + B \psi_r^{(i} (\psi_n \times \psi_m)^j) \epsilon^{pqr} (\psi_p \cdot \psi_q) \quad (3.26)$$

for suitable A, B . Inserting two different sets of m, r, i, j , we found that there are no solutions for A and B . This proves that O_0 is not Q -exact.

One can also easily show the non- Q -exactness by studying the $SU(1|3)$ descendants obtained by acting $Q_+^a Q_+^b$. For instance, one obtains

$$\begin{aligned}
 Q_+^2 Q_+^1 O'_0 = & \tag{3.27} \\
 & -(Y \cdot f + \psi_3 \cdot \psi_1)^2 \psi_3 \cdot (\psi_2 \times \psi_2) - (X \cdot f + \psi_2 \cdot \psi_3)(Z \cdot f + \psi_1 \cdot \psi_2) \psi_1 \cdot (\psi_3 \times \psi_3) \\
 & -(X \cdot f + \psi_2 \cdot \psi_3)(X \cdot \psi_3) f \cdot (\psi_1 \times \psi_1) + 2(Y \cdot \psi_2 - Z \cdot \psi_3)(Y \cdot f + \psi_3 \cdot \psi_1) \psi_3 \cdot (\psi_2 \times f) \\
 & -2(Y \cdot f + \psi_3 \cdot \psi_1)(X \cdot \psi_3) \psi_2 \cdot (\psi_1 \times f) - (Z \cdot \psi_3 - X \cdot \psi_1)(Z \cdot f + \psi_1 \cdot \psi_2) f \cdot (\psi_3 \times \psi_3)
 \end{aligned}$$

which contains uncanceled $\phi^0 \psi^7$ terms on the second line. Since acting Q always creates one or more ϕ factors, these terms cannot be Q -exact. Since a descendant of O'_0 is not Q -exact, O_0 cannot be Q -exact either, providing a simpler proof. Or alternatively, one can prove non- Q -exactness by acting three Q_+ 's to O'_0 and check that it contains nonzero term at $f\psi^6$ order,

$$\begin{aligned}
 Q_+^1 Q_+^2 Q_+^3 O'_0 = Q_+^1 Q_+^2 Q_+^3 O''_0 = & \tag{3.28} \\
 & (X \cdot f + \psi_2 \cdot \psi_3)^2 f \cdot (\psi_1 \times \psi_1) + 2(X \cdot f + \psi_2 \cdot \psi_3)(Y \cdot f + \psi_3 \cdot \psi_1) f \cdot (\psi_1 \times \psi_2) \\
 & + (1, 2, 3 \rightarrow 2, 3, 1) + (1, 2, 3 \rightarrow 3, 1, 2) = G^m G^m f \cdot (\psi_m \times \psi_n)
 \end{aligned}$$

where $G^m \equiv \phi^m \cdot f + \frac{1}{2} \epsilon^{mnp} \psi_n \cdot \psi_p$. Proof of this sort will sometimes be useful later. For instance, one can show that $(Z \cdot f + \psi_1 \cdot \psi_2) O'_0$ is not Q -exact, since its descendant

$$Q_+^2 Q_+^1 [(Z \cdot f + \psi_1 \cdot \psi_2) O'_0] = (Z \cdot f + \psi_1 \cdot \psi_2) Q_+^2 Q_+^1 O'_0 \tag{3.29}$$

contains a term at $\phi^0 \psi^9$ order.

O_1 operator at t^{36} . Now we construct the cohomology which accounts for the $-t^{36}$ term of (3.13). It should be fermionic, has charge $j = 6(R + J) = 36$, and should be an $SU(2)_R \times SU(3)$ singlet because we expect unique cohomology (unless there is a cancellation at this order which obscures the true degeneracy). We call this operator O_1 . From the last condition, we set three R_I equal and two J_i equal. Still, we do not know the individual R and J so we should make a guess. Our first guess was to add extra $\Delta J = 2$ to the charges $R = \frac{3}{2}$, $J = \frac{5}{2}$ of O_0 . We listed all operators in this sector and found the cohomology by computer. Then we made several trials until we found the following $SU(3)$ -invariant representative:

$$\begin{aligned}
 O_1 = & \tag{3.30} \\
 & (f \cdot f) \epsilon^{c_1 c_2 c_3} (\phi^a \cdot \psi_{c_1}) (\phi^b \cdot \psi_{c_2}) (\psi_a \cdot \psi_b \times \psi_{c_3}) \\
 & + \epsilon^{b_1 b_2 b_3} \epsilon^{c_1 c_2 c_3} (f \cdot \psi_{b_1}) (\phi^a \cdot \psi_{c_1}) (\psi_{b_2} \cdot \psi_{c_2}) (\psi_a \cdot \psi_{b_3} \times \psi_{c_3}) \\
 & - \frac{1}{72} \epsilon^{a_1 a_2 a_3} \epsilon^{b_1 b_2 b_3} \epsilon^{c_1 c_2 c_3} (\psi_{a_1} \cdot \psi_{b_1} \times \psi_{c_1}) (\psi_{a_2} \cdot \psi_{b_2} \times \psi_{c_2}) (\psi_{a_3} \cdot \psi_{b_3} \times \psi_{c_3}) .
 \end{aligned}$$

It is not graviton type since it is made of nine (odd) letters. One can also easily check that it is not Q -exact. This is because the last term contains no scalars. Since Q transformations (3.14) always yield scalars, the last term cannot be made Q -exact. So O_1 is not Q -exact.

Now we discuss the Q -closedness. O_1 takes the form of

$$O_1 = (f \cdot f)O_0 + f \cdot \xi + \chi, \quad (3.31)$$

where the SU(2) triplet $\vec{\xi}$ and the singlet χ are given by

$$\begin{aligned} \vec{\xi} &= \epsilon^{b_1 b_2 b_3} \epsilon^{c_1 c_2 c_3} \vec{\psi}_{b_1} (\phi^a \cdot \psi_{c_1}) (\psi_{b_2} \cdot \psi_{c_2}) (\psi_a \cdot \psi_{b_3} \times \psi_{c_3}) \\ \chi &= -\frac{1}{72} \epsilon^{a_1 a_2 a_3} \epsilon^{b_1 b_2 b_3} \epsilon^{c_1 c_2 c_3} (\psi_{a_1} \cdot \psi_{b_1} \times \psi_{c_1}) (\psi_{a_2} \cdot \psi_{b_2} \times \psi_{c_2}) (\psi_{a_3} \cdot \psi_{b_3} \times \psi_{c_3}) \\ &= -120 \psi_1^1 \psi_1^2 \psi_1^3 \psi_2^1 \psi_2^2 \psi_2^3 \psi_3^1 \psi_3^2 \psi_3^3. \end{aligned} \quad (3.32)$$

Q -closedness is equivalent to the following equations:

$$2(\vec{\phi}^m \times \vec{\psi}_m)O_0 + Q_\psi \vec{\xi} = 0, \quad \vec{\phi}^m \cdot (\vec{\psi}_m \times \vec{\xi}) + Q_\psi \chi = 0. \quad (3.33)$$

Note that $\vec{\xi}$ is related to O_0 by

$$\vec{\xi} = -\frac{1}{2} \epsilon^{mnp} \vec{\psi}_m \psi_n \cdot \frac{\partial}{\partial \phi^p} O_0. \quad (3.34)$$

So the first equation can be written as the following equations of O_0 :

$$\begin{aligned} 4(\vec{\phi}^m \times \vec{\psi}_m)O_0 = \\ \left[(\vec{\phi}^a \times \vec{\phi}^b) \psi_a \cdot \frac{\partial}{\partial \phi^b} + \vec{\psi}_a (\phi^a \times \phi^b) \cdot \frac{\partial}{\partial \phi^b} - \vec{\psi}_a (\psi_b \times \phi^a) \cdot \frac{\partial}{\partial \psi_b} + \vec{\psi}_b (\psi_a \times \phi^a) \cdot \frac{\partial}{\partial \psi_b} \right] O_0. \end{aligned} \quad (3.35)$$

This is a property of O_0 . The second/third terms cancel due to $(\phi^b \times \frac{\partial}{\partial \phi^b} + \psi_b \times \frac{\partial}{\partial \psi_b}) O_0 = 0$, which holds because it is the SU(2) gauge transformation on a gauge invariant operator O_0 . One can further simplify (3.35) using various properties of O_0 . Obvious ones are

$$\begin{aligned} \phi^m \cdot \frac{\partial}{\partial \phi^m} O_0 = n_B O_0, \quad \psi_m \cdot \frac{\partial}{\partial \psi_m} O_0 = n_F O_0 \quad (n_B, n_F) = (2, 5) \\ \vec{\varepsilon}_a^b \left[\phi^a \cdot \frac{\partial}{\partial \phi^b} - \psi_b \cdot \frac{\partial}{\partial \psi_a} \right] O_0 = 0 \quad (\vec{\varepsilon}_a^a = 0). \end{aligned} \quad (3.36)$$

The first two equations count the numbers of bosonic/fermionic fields in O_0 . The last equation is the SU(3) invariance of O_0 , which holds for any ε . Equivalently, one obtains

$$\left[\phi^a \cdot \frac{\partial}{\partial \phi^b} - \psi_b \cdot \frac{\partial}{\partial \psi_a} \right] O_0 = \frac{1}{3} (n_B - n_F) \delta_b^a O_0. \quad (3.37)$$

Finally, note that δ_{ij} contracts the SU(2) gauge triplet indices only between boson-fermion pairs in O_0 , while fermion indices are contracted only with ε_{ijk} . This effectively promotes SU(2) \sim SO(3) to SL(3) within O_0 , where bosons/fermions transform in the fundamental and anti-fundamental representations, respectively. This leads to the following property:

$$\left[\phi_i^a \cdot \frac{\partial}{\partial \phi_j^a} - \psi_a^j \cdot \frac{\partial}{\partial \psi_a^i} \right] O_0 = \frac{1}{3} (n_B - n_F) \delta_i^j O_0. \quad (3.38)$$

Using these properties, (3.35) can be written as

$$(\vec{\psi}_a \times \vec{\phi}^b)(\phi^a \cdot \frac{\partial}{\partial \phi^b})O_0 = \left(4 - \frac{n_F + 2n_B}{3}\right)(\vec{\phi}^m \times \vec{\psi}_m)O_0 = (\vec{\phi}^m \times \vec{\psi}_m)O_0. \quad (3.39)$$

Both (3.39) and the second equation of (3.33) can be easily checked on a computer. We have no extra analytic insights on why (3.39) this holds, except that using complicated representation analysis of $SU(2) \times SU(3)$ should provide the analytic proof. (We tried to simplify the equation for O_0 as much as possible since they might provide insights on the generalization to higher N 's in the future.) On the other hand, one can easily prove the second equation of (3.33). First note that $\psi_m \times \xi$ is an $SU(2)$ vector involving 8 ψ 's. There are nine independent operators involving eight ψ 's, depending on which of the 9 components is lacking. So it is proportional to $\frac{\partial}{\partial \psi_m^i} \chi$. Since it has to form a gauge-invariant by contracting with two scalars ϕ_i^m, ϕ_j^a , one should be able to write $\frac{\delta}{\delta \psi_m^i} \chi$ as an object with two $SU(3)$ antifundamental and two $SU(2)$ triplet indices by multiplying invariant tensors. The only possible term is $\epsilon_{man} \epsilon_{ijk} \frac{\partial}{\partial \psi_n^k} \chi$. One can compute the proportionality constant by computing a term, e.g. at $m = 1, a = 2, i = 1, j = 2$, finding $-\frac{1}{2}$. So one obtains

$$\phi^m \cdot (\psi_m \times \xi) = -\frac{1}{2} \phi_i^m \phi_j^a \epsilon_{man} \epsilon_{ijk} \frac{\partial \chi}{\partial \psi_n^k} = -\frac{1}{2} \epsilon_{man} (\phi^m \times \phi^a) \cdot \frac{\partial \chi}{\partial \psi_n} = -Q_\psi \chi, \quad (3.40)$$

proving the second equation of (3.33).

One may wonder if O_1 is a descendant of O_0 , or a lower black hole operator times graviton operators appearing in (3.11). Since O_1 is at t^{36} order, the only possible way of getting operators at this order from O_0 is $(Q_+^m Q_+^n O_0)(\phi^p \cdot f + \frac{1}{2} \epsilon^{pqr} \psi_q \cdot \psi_r)$. However, during our numerical construction of the cohomologies at this order, we separately constructed the last operator which is not cohomologous to O_1 . See also the end of this subsection for an analytic proof (applicable to all O_n 's with $n \geq 1$).

O_n operator at t^{24+12n} ($n \geq 2$). We can use the structures of the operators O_0 and O_1 to analytically construct an infinite tower of cohomologies O_n accounting for (3.13). Consider

$$O_n \equiv (f \cdot f)^n O_0 + n(f \cdot f)^{n-1} f \cdot \xi + \frac{2n^2 + n}{3} (f \cdot f)^{n-1} \chi \quad (3.41)$$

for $n \geq 2$. At $n = 1$, this is just O_1 that we discussed above. We will now show that these are new black hole like cohomologies at t^{24+12n} order. It is again easy to show that these are not graviton type because they are made of odd letters. It is not Q -exact because the last term does not contain scalars.

Now we derive the Q -closedness. Its Q -action is given by

$$\begin{aligned} QO_n = (f \cdot f)^{n-1} & \left[\vec{f} \cdot \left(2n(\vec{\phi}^m \times \vec{\psi}_m)O_0 + nQ_\psi \vec{\xi} \right) + n(\vec{\phi}^m \times \vec{\psi}_m) \cdot \vec{\xi} + \frac{2n^2 + n}{3} Q_\psi \chi \right] \\ & + 2(n^2 - n)(f \cdot f)^{n-2} \vec{f} \cdot (\vec{\phi}^m \times \vec{\psi}_m)(f \cdot \xi) \\ & + \frac{2n(n-1)(2n+1)}{3} (f \cdot f)^{n-2} \vec{f} \cdot (\vec{\phi}^m \times \vec{\psi}_m) \chi. \end{aligned} \quad (3.42)$$

The first two terms on the first line cancel due to the first equation of (3.33). The last term on the second line is zero because it includes 10 fermions. Inserting the second equation of (3.33) to the last term on the first line, one obtains

$$QO_n = \frac{2(n^2 - n)}{3}(f \cdot f)^{n-2} [-(f \cdot f)(\phi^m \times \psi_m) \cdot \xi + 3(f \times \phi^m) \cdot \psi_m(f \cdot \xi)] . \quad (3.43)$$

The second term contains 8 fermions, where the fermions carry ma indices for $SU(3)$ and three $SU(2)$ triplet indices to be contracted with $(f \times \phi^m)_k, f_i, \phi_j^a$. From the contraction structures of ξ , one finds that b_1, c_1 are antisymmetric so the corresponding i, j indices should be symmetric. The only possible 8-fermion terms satisfying these conditions are

$$\epsilon_{man} \delta_{ij} \frac{\partial}{\partial \psi_n^k} \chi, \quad \epsilon_{man} \delta_{k(i} \frac{\partial}{\partial \psi_n^{j)}} \chi . \quad (3.44)$$

Explicitly computing two components in the second term of (3.43), one finds that the linear combination is

$$\epsilon_{man} \left[\delta_{ij} \frac{\partial}{\partial \psi_n^k} - \delta_{k(i} \frac{\partial}{\partial \psi_n^{j)}} \right] \chi . \quad (3.45)$$

Contracting this with $f_i, \phi_j^a, (f \times \phi^m)_k$, one obtains

$$\begin{aligned} & \epsilon_{man} \left[(f \cdot \phi^a)(f \times \phi^m) \cdot \frac{\partial}{\partial \psi_n} - \frac{1}{2} [(f \times \phi^m) \cdot \phi^a] f \cdot \frac{\partial}{\partial \psi_n} \right] \chi \\ &= \frac{1}{2} \epsilon_{man} \left[[f \times (f \times (\phi^m \times \phi^a))] \cdot \frac{\partial}{\partial \psi_n} - [(f \times \phi^m) \cdot \phi^a] f \cdot \frac{\partial}{\partial \psi_n} \right] \chi \\ &= -\frac{1}{2} \epsilon_{man} (f \cdot f)(\phi^m \times \phi^a) \cdot \frac{\partial}{\partial \psi_n} \chi = -(f \cdot f) Q_\psi \chi = (f \cdot f)(\phi^m \times \psi_m) \cdot \xi . \end{aligned} \quad (3.46)$$

So the second term of (3.43) cancels the first term, ensuring that O_n is Q -closed. So we have shown that the operator

$$\begin{aligned} O_n &= (f \cdot f)^n \epsilon^{c_1 c_2 c_3} (\phi^a \cdot \psi_{c_1}) (\phi^b \cdot \psi_{c_2}) (\psi_a \cdot \psi_b \times \psi_{c_3}) \\ &+ n (f \cdot f)^{n-1} \epsilon^{b_1 b_2 b_3} \epsilon^{c_1 c_2 c_3} (f \cdot \psi_{b_1}) (\phi^a \cdot \psi_{c_1}) (\psi_{b_2} \cdot \psi_{c_2}) (\psi_a \cdot \psi_{b_3} \times \psi_{c_3}) \\ &- \left(\frac{n}{72} + \frac{n^2 - n}{108} \right) (f \cdot f)^{n-1} \epsilon^{a_1 a_2 a_3} \epsilon^{b_1 b_2 b_3} \epsilon^{c_1 c_2 c_3} (\psi_{a_1} \cdot \psi_{b_1} \times \psi_{c_1}) (\psi_{a_2} \cdot \psi_{b_2} \times \psi_{c_2}) (\psi_{a_3} \cdot \psi_{b_3} \times \psi_{c_3}) \end{aligned} \quad (3.47)$$

at t^{24+12n} order is a black hole cohomology.

One may wonder if these are primaries captured in the first factor of (3.11), or if they are related to other $O_{n'}$ with $n' < n$ by acting some Q_+^m 's and/or gravitons on the third factor. One can show that the latter possibilities are all impossible. Suppose O_n is obtained by acting p Q 's on $O_{n'}$ and multiplying q gravitons. Then p, q should satisfy

$$2p + 8q = 12(n - n'), \quad p = 0, 1, 2, 3, \quad q \geq 0 . \quad (3.48)$$

Possible solutions are

$$(p, q, n - n') = (2, 1, 1), (0, 3, 2), (2, 4, 3), (0, 6, 4), (2, 7, 5), (0, 9, 6), \dots . \quad (3.49)$$

The cases with even $n - n'$ and $p = 0$ yield operators at t^{24+12n} order obtained by multiplying $O_{n'}$ and $\frac{3}{2}(n - n')$ graviton operators of the form $\phi^m \cdot f + \frac{1}{2}\epsilon^{mnp}\psi_n\psi_p$. However, these cannot be cohomologous to O_n because they do not have a term at $\mathcal{O}(f^{2n-2}\phi^0\psi^9)$ order that O_n has, which cannot be changed by adding Q -exact terms. Now we consider the cases with odd $n - n'$ and $p = 2$, $q = \frac{3}{2}(n - n') - \frac{1}{2}$, and again consider whether the operator $(Q_+^a Q_+^b O_{n'}) (\phi \cdot f + \psi \cdot \psi)^q$ has a term at $f^{2n-2}\phi^0\psi^9$ order. Let us first study how the actions of Q_+^a and Q_+^b on $O_{n'}$ can produce a term with no scalars. Q_+^a either act as $\phi \rightarrow \psi$ or $\psi \rightarrow f$, so there are following possibilities:

$$f^{2n'}\phi^2\psi^5 \rightarrow f^{2n'}\psi^7, \quad f^{2n'-1}\phi\psi^7 \rightarrow f^{2n'}\psi^7, \quad f^{2n'-2}\psi^9 \rightarrow f^{2n'}\psi^7. \quad (3.50)$$

In all three cases, we multiply gravitons of the form $(\phi \cdot f + \psi \cdot \psi)^q$ and see whether there can be a term at $f^{2n-2}\phi^0\psi^9$ order. This is possible only if $n = n' + 1$, $p = 2$, $q = 1$. That is, the only possible relations between different O_n 's are

$$O_n \stackrel{?}{\sim} \epsilon_{abc}(Q_+^a Q_+^b O_{n-1})(\phi^c \cdot f + \frac{1}{2}\epsilon^{cde}\psi_d \cdot \psi_e), \quad (3.51)$$

where \sim means up to a multiplicative factor and addition of Q -exact terms. We act three Q_+^a 's on (3.51) and show that this equation cannot hold. Acting $Q_+^1 Q_+^2 Q_+^3$ on the right hand side yields zero, so if this equation is true, $Q_+^1 Q_+^2 Q_+^3 O_n$ should be Q -exact. However, this cannot be the case since it contains a term at $f^{2n+1}\psi^6$ order, which does not contain scalars so cannot be Q -exact. More concretely, one starts from

$$O_n = (f \cdot f)^n O_0 + \frac{20n}{3}(f \cdot f)^{n-1} \sum_{\text{cyclic}} (f \cdot \psi_3)(\psi_3 \cdot \psi_2)(X \cdot \psi_2)(\psi_1 \cdot \psi_1 \times \psi_1) - \frac{10}{3} \left(\frac{n}{6} + \frac{n^2 - n}{9} \right) (f \cdot f)^{n-1} (\psi_1 \cdot \psi_1 \times \psi_1)(\psi_2 \cdot \psi_2 \times \psi_2)(\psi_3 \cdot \psi_3 \times \psi_3) \quad (3.52)$$

where \sum_{cyclic} means summation over the cyclic permutations of (X, ψ_1) , (Y, ψ_2) , (Z, ψ_3) . Acting $Q_+^1 Q_+^2 Q_+^3$, one obtains the following terms without scalars,

$$\begin{aligned} & Q_1^+ Q_2^+ Q_3^+ O_0 \quad (3.53) \\ &= -10(X \cdot f + \psi_2 \cdot \psi_3)^2 f \cdot (\psi_1 \times \psi_1) - 20(X \cdot f + \psi_2 \cdot \psi_3)(Y \cdot f + \psi_3 \cdot \psi_1) f \cdot (\psi_1 \times \psi_2) + \text{cyclic} \\ &\rightarrow -20(\psi_2 \cdot \psi_3)^2 (f \cdot \psi_1 \times \psi_1) + \text{cyclic} \\ & Q_1^+ Q_2^+ Q_3^+ (f \cdot \psi_3)(\psi_3 \cdot \psi_2)(X \cdot \psi_2)(\psi_1 \cdot \psi_1 \times \psi_1) + \text{cyclic} \\ &\rightarrow -3(f \cdot f)(\psi_2 \cdot \psi_3)^2 (f \cdot \psi_1 \times \psi_1) + 6(f \cdot \psi_2)(f \cdot \psi_3)(\psi_2 \cdot \psi_3)(f \cdot \psi_1 \times \psi_1) + \text{cyclic} \\ & Q_1^+ Q_2^+ Q_3^+ (\psi_1 \cdot \psi_1 \times \psi_1)(\psi_2 \cdot \psi_2 \times \psi_2)(\psi_3 \cdot \psi_3 \times \psi_3) = -27(f \cdot \psi_1 \times \psi_1)(f \cdot \psi_2 \times \psi_2)(f \cdot \psi_3 \times \psi_3) \\ &= 18((f \cdot f)(\psi_2 \cdot \psi_3)^2 (f \cdot \psi_1 \times \psi_1)) - 2(f \cdot \psi_2)(f \cdot \psi_3)(\psi_2 \cdot \psi_3)(f \cdot \psi_1 \times \psi_1) + \text{cyclic} . \end{aligned}$$

These terms at $f^{2n+1}\phi^0\psi^6$ order do not cancel, implying that $Q_+^1 Q_+^2 Q_+^3 O_0$ cannot be Q -exact. So at least among the possibilities visible in the index (3.11), we have checked that different O_n 's are not related in trivial manners.

Note also that the product of two O_n 's vanishes, $O_m O_n = 0$. This is because each operator includes 5 or more ψ 's, so the product involves 10 or more ψ 's which vanishes by Fermi statistics.

3.2 General sector and partial no-hair theorem

Counting the graviton cohomologies with a computer using the eigenvalue setup explained in section 2, we obtained its index Z_{grav} till t^{40} order. The remaining $SU(2)$ index is given by

$$\begin{aligned}
 Z - Z_{\text{grav}} = & \left[-t^{24} - \chi_{(1,3)} t^{32} - (\chi_{(1,\bar{3})} + \chi_{(3,6)}) t^{34} - \chi_{(2,3)} t^{35} + (\chi_{(3,1)} + \chi_{(3,8)}) t^{36} \right. \\
 & - (\chi_{(2,\bar{3})} + \chi_{(4,6)}) t^{37} + \chi_{(5,3)} t^{38} + (\chi_{(2,1)} + 2\chi_{(4,1)} + \chi_{(4,8)}) t^{39} \\
 & \left. - (2\chi_{(1,6)} + \chi_{(3,\bar{3})} + \chi_{(5,\bar{3})} + \chi_{(5,6)}) t^{40} \right] \chi_D + \mathcal{O}(t^{41}) .
 \end{aligned} \tag{3.54}$$

The $SU(2)_R \times SU(3)$ characters and the factor χ_D are given by

$$\begin{aligned}
 \chi_{(2J'+1,R)} & \equiv \chi_{J'}^{\text{SU}(2)_R}(p) \chi_R^{\text{SU}(3)}(x, y) \\
 \chi_D & \equiv \frac{(1-t^2 z_1)(1-\frac{t^2}{z_2})(1-\frac{t^2 z_2}{z_1}) \cdot (1-\frac{tp}{z_1})(1-\frac{t}{pz_1})(1-tz_2 p)(1-\frac{tz_2}{p})(1-\frac{tz_1 p}{z_2})(1-\frac{tz_1}{z_2 p})}{(1-t^3 p)(1-\frac{t^3}{p})}
 \end{aligned} \tag{3.55}$$

where $t^6 = e^{-\Delta_1 - \Delta_2 - \Delta_3} = e^{-\omega_1 - \omega_2}$, $z_1 = e^{-\frac{2\Delta_1 + \Delta_2 + \Delta_3}{3}}$, $z_2^{-1} = e^{\frac{\Delta_1 - 2\Delta_2 + \Delta_3}{3}}$, $p = e^{-\frac{\omega_1 + \omega_2}{2}}$. The function χ_D is factored out for later convenience (but we do not necessarily assume that only a particular type of representation appears).

Since descendant operators of $PSU(1, 2|3)$ are not really new, our interest is the new possible $\frac{1}{16}$ -BPS primaries contained in (3.54). Constructing all the cohomologies order by order as done in [18] will clarify this in principle. However, we shall not comprehensively do this job in this paper. Rather, we shall study the possible superconformal representation structures of the $\frac{1}{16}$ -BPS states compatible with this index, finding many illuminating structures. As emphasized, we may miss some BPS states in case their multiplets completely cancel in the index.

The index can be written as a sum over the short $\mathcal{N} = 4$ representations. Equivalently, it can be written as a sum over $\frac{1}{16}$ -BPS multiplets of $PSU(1, 2|3) \subset PSU(2, 2|4)$. The last multiplets are embedded in the short representations of $PSU(2, 2|4)$ in canonical manners: see appendix B. Knowing this representation sum is knowing the primary contents. We will study this expansion order by order in t . As already mentioned in section 3.1 (and in the introduction), there are two classes of black hole cohomologies: those which can be written as products of other black hole cohomologies and gravitons which we call ‘hairy’ and the rest which we call ‘core.’

We start by studying the black hole cohomologies that we identified in section 3.1. Among these, two of them O_0, O_1 appear within the t^{40} order. We can show that all O_n ’s are core black hole primaries of the $\frac{1}{16}$ -BPS multiplets. The core-ness of O_n is already shown in section 3.1, at least within the states visible in the index (3.11), since it suffices to show this within the BMN sector. We only need to show that they are $\frac{1}{16}$ -BPS primaries in their full $PSU(1, 2|3)$ representations. O_0 is clearly a $\frac{1}{16}$ -BPS primary since it is the lowest black hole cohomology. Since $j \equiv J_1 + J_2 = 5$ is too large, O_0 can only belong to the $\mathcal{N} = 4$ multiplet $A_1 \bar{L}[4; 0]_9^{[2,0,0]}$. The primary O_0 of the $\frac{1}{16}$ -BPS multiplet is obtained by acting $Q' \equiv Q_+^4$ on a primary of this $\mathcal{N} = 4$ multiplet. The index over this multiplet is

$$\chi_{24} \equiv -t^{24} \chi_D(t, x, y, p), \tag{3.56}$$

where χ_D is defined in (3.55) and in appendix B. So the first term $-t^{24}$ in the square bracket of (3.54) corresponds to the contribution of this multiplet. Next we consider other O_n 's. We can prove that they are also primaries by showing that acting any of the nine Q 's in PSU(1, 2|3) yields nontrivial and independent cohomologies. (This is because O_n does not contain derivatives and cannot be a conformal descendant.) We have shown in section 3.1 that the action of any Q_+^m on O_n is nontrivial and independent because acting all three of them yields a nontrivial cohomology. One can also show that $\bar{Q}_{m\dot{\alpha}}O_n$ are all nontrivial and independent. It suffices to show that the six $\bar{Q}_{m\dot{\alpha}}$'s acting on $Q_+^1Q_+^2Q_+^3O_n$ are independent. This is easily shown by studying the terms obtained by acting $\bar{Q}_{m\dot{\alpha}}$ on the $\mathcal{O}(f^{2n+1}\phi^0\psi^6)$ order terms of $Q_+^1Q_+^2Q_+^3O_n$ in (3.53). In particular, one obtains terms at $f^{2n}\phi^0\psi^6D\psi$ by acting $\bar{Q}_{m\dot{\alpha}}$ on f . These terms cannot be Q -exact since it involves neither $\bar{\phi}^m$ or $\bar{\lambda}_{\dot{\alpha}}$. This proves that all 6 operators $\bar{Q}_{m\dot{\alpha}}Q_+^1Q_+^2Q_+^3O_n$ are nontrivial. They are also independent since their SU(2)_R × SU(3) quantum numbers are different. This shows that $O_{n \geq 1}$ are $\frac{1}{16}$ -BPS primaries. O_n belongs to the $\mathcal{N} = 4$ multiplet $A_1\bar{L}[4 + 4n; 0]_{9+4n}^{[2,0,0]}$, which contributes to the index as $-t^{24+12n}\chi_D(t, x, y, p)$.

Now with the nature of O_n understood, we come back to study the series (3.54) till t^{40} order, trying to better characterize other cohomologies order by order in t . Once the lowest operator O_0 is identified, all the states in its $\frac{1}{16}$ -BPS multiplet are not really new operators. So we subtract χ_{24} from $Z - Z_{\text{grav}}$ and see what are left:

$$\begin{aligned}
 Z - Z_{\text{gra}} - \chi_{24} = & \left[-\chi_{(1,3)}t^{32} - (\chi_{(1,\bar{3})} + \chi_{(3,6)})t^{34} - \chi_{(2,3)}t^{35} + (\chi_{(3,1)} + \chi_{(3,8)})t^{36} \right. \\
 & - (\chi_{(2,\bar{3})} + \chi_{(4,6)})t^{37} + \chi_{(5,3)}t^{38} + (\chi_{(2,1)} + 2\chi_{(4,1)} + \chi_{(4,8)})t^{39} \\
 & \left. - (2\chi_{(1,6)} + \chi_{(3,\bar{3})} + \chi_{(5,\bar{3})} + \chi_{(5,6)})t^{40} \right] \chi_D + \mathcal{O}(t^{41}) . \tag{3.57}
 \end{aligned}$$

Somewhat surprisingly, after subtracting the multiplet of O_0 , one finds that the remaining index starts from t^{32} order. Namely, in the range $t^{25} \sim t^{31}$, the index does not capture any new black hole cohomologies except the trivial descendants of O_0 . At first sight this may look like a boring result, but the triviality of the index in this range has a nontrivial implication.

Recall that cohomologies multiply to yield new cohomologies. This is because of the Leibniz rule of the classical Q acting on product operators. So apparently, one can multiply light graviton cohomologies to O_0 or its descendants to obtain many new cohomologies in the range $t^{25} \sim t^{31}$. The possible product cohomologies of O_0 and gravitons below t^{32} order are

$$\begin{aligned}
 & O_0(\bar{\phi}^{(m} \cdot \bar{\phi}^{n)}), \quad O_0(\bar{\phi}^m \cdot \bar{\lambda}_{\dot{\alpha}}), \quad O_0(\bar{\lambda}_{\dot{\alpha}} \cdot \bar{\lambda}_{\dot{\beta}}), \\
 & O_0\left(\bar{\phi}^m \cdot \psi_{n+} - \frac{1}{3}\delta_n^m \bar{\phi}^p \cdot \psi_{p+}\right), \quad O_0\left(\bar{\lambda}_{\dot{\alpha}} \cdot \psi_{m+} - \frac{1}{2}\epsilon_{mnp}\phi^n \cdot D_{+\dot{\alpha}}\phi^p\right), \\
 & O_0\partial_{+\dot{\alpha}}(\bar{\phi}^{(m} \cdot \bar{\phi}^{n)}) . \tag{3.58}
 \end{aligned}$$

Other possible products below t^{32} involving the descendants of O_0 are

$$\begin{aligned}
 & \bar{Q}O_0 \times \left(\bar{\phi}^m \cdot \bar{\phi}^n, \bar{\phi}^m \cdot \bar{\lambda}_{\dot{\alpha}}, \bar{\lambda}_{\dot{+}} \cdot \bar{\lambda}_{\dot{-}}, \bar{\phi}^m \cdot \psi_{n+} - \frac{1}{3} \delta_n^m \bar{\phi}^p \cdot \psi_{p+} \right), \\
 & (Q, \bar{Q}\bar{Q})O_0 \times (\bar{\phi}^m \cdot \bar{\phi}^n, \bar{\phi}^m \cdot \bar{\lambda}_{\dot{\alpha}}), \\
 & (Q\bar{Q}, \bar{Q}\bar{Q}\bar{Q}, \partial)O_0 \times (\bar{\phi}^m \cdot \bar{\phi}^n).
 \end{aligned} \tag{3.59}$$

The triviality of the index (3.57) in this range implies two possibilities for these product cohomologies. The first possibility is that these product cohomologies are Q -exact, i.e. absent in the BPS spectrum. Another possibility is that these product cohomologies are nontrivial but there are cancellations in the index, either among themselves or with new core black hole cohomologies.⁶ Among (3.58) and (3.59), we have explicitly shown after very nontrivial numerical/analytic calculations that

$$O_0(\bar{\phi}^m \cdot \bar{\phi}^n), O_0(\bar{\phi}^m \cdot \bar{\lambda}_{\dot{\alpha}}), O_0\left(\bar{\phi}^m \cdot \psi_{n+} - \frac{1}{3} \delta_n^m \bar{\phi}^p \cdot \psi_{p+}\right) \tag{3.60}$$

are all Q -exact. See appendix D for some details. We did not manage to prove the Q -exactness of other operators. Since these operators do not appear at all in the index, all of them may be Q -exact till t^{31} order. More robustly/modestly, we can say that our index exhibits a no-hair behavior for O_0 till t^{31} order. It will be interesting to clarify this issue in the future.

The Q -exactness of these product operators implies that O_0 abhors the dressings by certain gravitons, reminiscent of the black hole no-hair theorem. Especially, $(\bar{\phi}^m \cdot \bar{\phi}^n) \sim \text{tr}(\bar{\phi}^m \bar{\phi}^n)$ multiplied to O_0 are Q -exact. This is interesting because these operators correspond to bulk scalar fields which have been discussed in the context of hairy AdS₅ black holes [30–32]. More precisely, it is the ‘s-wave’ modes of these scalars that have been used to construct hairy black holes, precisely dual to the conformal primary operator $\text{tr}(\bar{\phi}^m \bar{\phi}^n)$. Here, note that the BPS limits of the hairy black holes constructed this way all exhibit substantial back reactions to the core black holes, at least near the horizon, no matter how small the hair parameter is [31, 32]. In section 4, we shall revisit these aspects in the gravity dual and more carefully discuss the possible forms of hairy BPS black holes.

Now we consider the lowest term $-\chi_{(1,3)} t^{32}$ of (3.57). In fact, this term comes from the following product of O_0 and gravitons:

$$O_0 \left(\bar{\phi}^m \cdot f_{+++} + \frac{1}{2} \epsilon^{mnp} \psi_{n+} \cdot \psi_{p+} \right). \tag{3.61}$$

It is easy to show that this is not Q -exact, e.g. by acting two Q_+^m as shown in (3.29). These operators contain terms at $f^0 \phi^0 \psi^9$ order, which cannot be Q -exact. So the operators (3.61) themselves are not Q -exact either. Therefore, the no-hair interpretation that we made so

⁶We have checked that cancellations cannot happen within the product cohomologies listed above. It is logically possible (although a bit unnatural) that some new core black hole primaries appear in this range, precisely canceling with some of the product operators above if they are not Q -exact. Although in different contexts, certain black holes are known not to appear in the index. For instance, asymptotically flat multi-center BPS black holes or BPS black rings are not captured by the index [46].

far holds only for certain low-lying gravitons, at best. See section 4 for more detailed discussions on examples of ‘allowed’ hairs. Among the conformal primaries of S_2 listed in appendix A, these three gravitons are the only ones which explicitly appear in the index when multiplied to O_0 . At this stage, it may seem that two more gravitons $f_{++} \cdot \bar{\lambda}_{\dot{\alpha}} + \frac{2}{3}\psi_{m+} \cdot D_{+\dot{\alpha}}\bar{\phi}^m - \frac{1}{3}\bar{\phi}^m \cdot D_{+\dot{\alpha}}\psi_{m+}$ at $\mathcal{O}(t^9)$ might multiply O_0 to show up at t^{33} order, but we will see below that the index does not capture them. Therefore, out of the 32 particle species of conformal primary particles in the S_2 multiplet, 29 gravitons except $\bar{\phi}^m \cdot f_{++} + \frac{1}{2}\epsilon^{mnp}\psi_{n+} \cdot \psi_{p+}$ do not appear in the index when they multiply O_0 . In the BMN sector, our studies in section 3.1 imply a similar theorem for all O_n , at least as seen by the index. Among the 17 particle species of gravitons in the BMN sector, all 14 particles except $\bar{\phi}^m \cdot f_{++} + \frac{1}{2}\epsilon^{mnp}\psi_{n+} \cdot \psi_{p+}$ do not appear in the index when they multiply any O_n .

The 3 product cohomologies at t^{32} order violating the no-hair theorem should be the primaries of $\text{PSU}(1,2|3)$. This is again contained in a short multiplet of $A_1\bar{L}$ type, whose contribution to the index is given by $\chi_{32} = t^{32}\chi_{(1,3)}\chi_D$. We subtract this from $Z - Z_{\text{grav}} - \chi_{24}$, and study the remaining cohomologies. We can then try to interpret the lowest order term of the remainder and judge whether it comes from new core black hole primaries or products of already known core primaries and gravitons. If one can clarify the nature of the cohomologies at this lowest order, one can again subtract the characters of their supermultiplets and keep exploring even higher orders. Since it becomes more and more difficult to judge the Q -exactness of the possible product operators, we shall only make much simpler and structural studies till the t^{40} order. Namely, we shall try to see if the surviving index can be explained as the products of known gravitons and core primaries O_n , without the need of any new core black hole primaries. Studies we made so far showed that this is possible till t^{32} order. Namely, the index till this order is compatible with having no more new core primaries and only three more product cohomologies (3.61). We shall show that the graviton spectrum is such that new core black hole primaries should appear at t^{39} order at the latest. This not only proves from the index the existence of new core black hole primaries, but will also show scenarios of possible hairy black holes (to be further discussed in section 4).

After eliminating the contribution of the multiplet χ_{32} to the index, the remaining index vanishes at t^{33} order. In principle, there are two possible product operators which completely cancel in the index if they are not Q -exact, given by

$$O_0\partial_{+\dot{\alpha}}(\bar{\lambda}_{\dot{\beta}} \cdot \bar{\lambda}^{\dot{\beta}}), \quad O_0\left(f_{++} \cdot \bar{\lambda}_{\dot{\alpha}} + \frac{2}{3}\psi_{m+} \cdot D_{+\dot{\alpha}}\bar{\phi}^m - \frac{1}{3}\bar{\phi}^m \cdot D_{+\dot{\alpha}}\psi_{m+}\right). \quad (3.62)$$

So these product operators, even if they exist, do not appear in the index. The lowest nonzero term of $Z - Z_{\text{grav}} - \chi_{24} - \chi_{32}$ is $-(\chi_{(1,\bar{3})} + \chi_{(3,6)})t^{34}$. The only possible product operators which may account for this term, if they are not Q -exact, are

$$O_0\partial_{+\dot{\alpha}}\left(\bar{\lambda}_{\dot{\alpha}} \cdot \psi_{m+} - \frac{1}{2}\epsilon_{mnp}\bar{\phi}^n \cdot D_{+\dot{\alpha}}\bar{\phi}^p\right), \quad O_0\partial_{+\dot{\alpha}}\partial_{+\dot{\beta}}(\bar{\phi}^m \cdot \bar{\phi}^n). \quad (3.63)$$

If they are nontrivial, they are in the $\mathcal{N} = 4$ representations $A_1\bar{L}[6; 2]_{13}^{[2,2,0]}$ and $A_1\bar{L}[6; 0]_{13}^{[3,0,1]}$ representations, respectively. Assuming that they are both non- Q -exact, this order can be

accounted for by hairy black hole operators. Their multiplets will contribute $-(\chi_{(1,\bar{3})} + \chi_{(3,6)})t^{34}\chi_D$ to the index. Subtracting them, the leading term is $-\chi_{(2,3)}t^{35}$. The only possible product cohomologies which can account for this term are

$$O_0\partial_{+\dot{\alpha}}\left(f_{++}\cdot\bar{\phi}^m + \frac{1}{2}\epsilon^{mnp}\psi_{n+}\cdot\psi_{p+}\right), \quad (3.64)$$

if they are not Q -exact. In this case, its multiplet is again $A_1\bar{L}$ type and contributes $-\chi_{(2,3)}t^{35}\chi_D$ to the index. Subtracting this, the lowest term is $+(\chi_{(3,1)} + \chi_{(3,8)})t^{36}$. Since there is one fermionic black hole primary O_1 , we study whether the product cohomologies may account for $+(1 + \chi_{(3,1)} + \chi_{(3,8)})t^{36}$. The only possible set is

$$\begin{aligned} O_0\partial_{+\dot{\alpha}}\left(f_{++}\cdot\bar{\lambda}_{\dot{\beta}} + \frac{2}{3}\psi_{m+}\cdot D_{+\dot{\beta}}\bar{\phi}^m - \frac{1}{3}\bar{\phi}^m\cdot D_{+\dot{\beta}}\psi_{m+}\right), \\ O_0\partial_{+\dot{\alpha}}\partial_{+\dot{\beta}}\left(\bar{\phi}^m\cdot\psi_{n+} - \frac{1}{3}\delta_n^m\bar{\phi}^p\cdot\psi_{p+}\right). \end{aligned} \quad (3.65)$$

Subtracting the contributions of these multiplets if not Q -exact, again in the $A_1\bar{L}$ multiplets, the lowest term is $-(\chi_{(2,\bar{3})} + \chi_{(4,6)})t^{37}$. The only possible product cohomologies that can account for this term are

$$O_0\partial_{+\dot{\alpha}}\partial_{+\dot{\beta}}\left(\bar{\lambda}_{\dot{\beta}}\cdot\psi_{m+} - \frac{1}{2}\epsilon_{mnp}\bar{\phi}^n\cdot D_{+\dot{\beta}}\bar{\phi}^p\right), \quad O_0\partial_{+\dot{\alpha}}\partial_{+\dot{\beta}}\partial_{+\dot{\gamma}}(\bar{\phi}^m\cdot\bar{\phi}^n). \quad (3.66)$$

Further processing to subtract the contributions of their multiplets, the lowest term is $+\chi_{(5,3)}t^{38}$. one possible set of product cohomologies which can account for this is

$$O_0\partial_{+(\dot{\alpha}\dot{\beta}\dot{\gamma})}\partial_{+\dot{\delta}}(\bar{\lambda}_{\dot{\delta}}\cdot\bar{\phi}^m). \quad (3.67)$$

Apart from these, the following two sets of product cohomologies

$$O_0\partial_{+\dot{\alpha}}\partial_{+\dot{\beta}}\partial_{+\dot{\gamma}}(\bar{\lambda}_{\dot{\gamma}}\cdot\bar{\phi}^m), \quad O_0\partial_{+\dot{\alpha}}\partial_{+\dot{\beta}}\left(f_{++}\cdot\bar{\phi}^m + \frac{1}{2}\epsilon^{mnp}\psi_{n+}\cdot\psi_{p+}\right) \quad (3.68)$$

exactly cancel in the index, so there are two possible ways in which product hairy cohomologies can account for this order. In either case, they are all in the $A_1\bar{L}$ type multiplets.

Subtracting the last multiplets, the lowest term is $+(\chi_{(2,1)} + 2\chi_{(4,1)} + \chi_{(4,8)})t^{39}$. All possible product cohomologies at this order are

$$\begin{aligned} (\mathbf{4}, \mathbf{1})^F &: O_0\partial_{+\dot{\alpha}}\partial_{+\dot{\beta}}\partial_{+\dot{\gamma}}(\bar{\lambda}_{\dot{\delta}}\cdot\bar{\lambda}^{\dot{\delta}}), \\ (\mathbf{2}, \mathbf{1})^B &: O_0\partial_{+\dot{\alpha}}\partial_{+\dot{\beta}}\left(f_{++}\cdot\bar{\lambda}_{\dot{\beta}} + \frac{2}{3}\psi_{m+}\cdot D_{+\dot{\beta}}\bar{\phi}^m - \frac{1}{3}\bar{\phi}^m\cdot D_{+\dot{\beta}}\psi_{m+}\right), \\ (\mathbf{4}, \mathbf{1})^B &: O_0\partial_{+(\dot{\alpha}\dot{\beta})}\partial_{+\dot{\gamma}}\left(f_{++}\cdot\bar{\lambda}_{\dot{\gamma}} + \frac{2}{3}\psi_{m+}\cdot D_{+\dot{\gamma}}\bar{\phi}^m - \frac{1}{3}\bar{\phi}^m\cdot D_{+\dot{\gamma}}\bar{\psi}_m\right), \\ (\mathbf{4}, \mathbf{8})^B &: O_0\partial_{+\dot{\alpha}}\partial_{+\dot{\beta}}\partial_{+\dot{\gamma}}\left(\bar{\phi}^m\cdot\psi_{n+} - \frac{1}{3}\delta_n^m\bar{\phi}^p\cdot\psi_{p+}\right). \end{aligned} \quad (3.69)$$

We have shown the superscripts B/F for their bosonic/fermionic statistics, respectively. With these candidates, we find that the closest one can get to the index at this order is

the case in which all three classes of bosonic operators are nontrivial while the fermionic operators are Q -exact. In this case, their contribution at this order is maximal and becomes $+(\chi_{(2,1)} + \chi_{(4,1)} + \chi_{(4,8)})t^{39}$. Therefore, there should be at least 4 core black hole primaries to account for the remaining $+\chi_{(4,1)}t^{39}$. Of course this is the latest order in which new core black hole primaries should appear, and it may well appear at the lowest orders by some non- Q -exactness assumptions we made for product cohomologies being invalid.

So we have shown that, from the index data till t^{40} order, there should exist more core primary operators except O_n in the BMN sector. This conclusion is obtained by supposing otherwise, and trying to explain the index as product cohomologies of O_n and gravitons but finding a contradiction at t^{39} . We should also emphasize that the structure of the index admits natural explanations in terms of hairy product operators in a wide range $t^{33} \sim t^{38}$. Note also that most of the gravitons appearing in this range are conformal descendants in the S_2 multiplet. We shall find an interpretation of this in the next section.

4 Comments on hairy black holes

In section 3.2, we explored possible hairy black hole states in the $SU(2)$ theory. To better address this concept, we should understand the ways in which hairy BPS black holes can form in AdS. In this section, we study scalar hairs of the BPS black holes in both wave and probe particle perspectives and present a qualitative picture in the bulk gravity as to which scalar hairs may dress the BPS black holes.

The simplest form of hairy black holes has been explored in [29, 30], called the ‘non-interacting mix’ of small black holes and dilute graviton hairs. The idea is that the back-reaction of the black hole to the hair is weak when the black hole size q is small, and vice versa when the hair density ε is low. In the limit of small q and ε , the leading hairy black hole solution is a superposition of the non-hairy black hole and the graviton wavefunction in vacuum AdS. Systematic back-reactions to each other can be computed as a series in small q, ε , rendering a nonlinear superposition. Nonlinear superposition is naturally expected in the BPS sector, even at finite q, ε , where the cancellation of mutual forces is often ensured by supersymmetry. In fact, our product cohomologies realize such superpositions of BPS states in some sense. Although the true BPS states do not generally admit product forms, there are product representatives of the cohomologies. Such superposition representations were crucial in [15] for gaining a deeper understanding of multi-graviton states beyond the low-energy limit.

Meanwhile, some numerical solutions for hairy BPS black holes are known from [31, 32]. These solutions appear to be somewhat different from the non-interacting mix picture of [29, 30] in the following sense. No matter how small the hair density ε is, it is shown that the hair back-reacts substantially to the black hole near the horizon. In fact, the back-reaction is such that a (mild) singularity is created at the horizon. On the other hand, we would like to interpret our hairy black hole operators of section 3 as realizing the non-interacting mix picture of [29, 30], as superpositions are realized in the cohomology setup. In this section, we will try to better understand these two notions of hairy BPS black holes and explore a setup in which BPS solutions along the line of [29, 30] may exist.

One no-hair theorem proved in appendix D is the absence of $(\bar{\phi}^m \cdot \bar{\phi}^n)O_0$. On the other hand, in [30–32], the bulk scalar dual to $\text{tr}(\bar{\phi}^m \bar{\phi}^n)$ was discussed as a hair. We shall first review our current knowledge on hairy BPS black holes in $AdS_5 \times S^5$ with this scalar. We shall try to clarify the relations of these results with our findings in section 3 along the way.

Attempts made in the literature were to find continuous deformations of non-hairy black holes. The simplest non-hairy BPS black holes are those carrying equal R-charges and angular momenta, $R \equiv R_1 = R_2 = R_3$, $J \equiv J_1 = J_2$ [47]. They carry one free parameter q . (This is related to the three μ_I parameters of [48] by $\mu_1 = \mu_2 = \mu_3 = q$.) We shall follow the notation of [49] for later convenience. The metric and the gauge field are given by

$$\begin{aligned} ds^2 &= -f^2(dt + w\sigma_3)^2 + f^{-1}h_{mn}dx^m dx^n \\ &= -f^2(dt + w\sigma_3)^2 + f^{-1} \left[\frac{dx^2}{4xh} + \frac{x}{4} (\sigma_1^2 + \sigma_2^2 + h\sigma_3^2) \right] \\ A &= -f(dt + w\sigma_3) + U\sigma_3 \end{aligned} \tag{4.1}$$

where the 1-forms σ_i satisfying $d\sigma_1 = -\sigma_2 \wedge \sigma_3$, etc. are given by

$$\sigma_1 = \cos \psi d\theta + \sin \psi \sin \theta d\phi, \quad \sigma_2 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi, \quad \sigma_3 = d\psi + \cos \theta d\phi, \tag{4.2}$$

and various functions are given by

$$H \equiv f^{-1} = 1 + \frac{q}{x}, \quad U = \frac{x}{2\ell} + \frac{q}{\ell}, \quad h = 1 + \frac{x+3q}{\ell^2}, \quad w = \frac{x+3q}{2\ell} + \frac{3q^2}{4\ell x}. \tag{4.3}$$

ℓ is the radius of AdS_5 . The following 4d vierbeins

$$e^{\hat{x}} = \frac{dx}{2\sqrt{xh}}, \quad e^{\hat{1},\hat{2}} = \frac{\sqrt{x}}{2}\sigma_{1,2}, \quad e^{\hat{3}} = \frac{\sqrt{xh}}{2}\sigma_3. \tag{4.4}$$

for the metric h_{mn} will be useful below. The boundary $x \rightarrow \infty$ is asymptotically AdS_5 , whose canonical time and angle coordinates t', ψ' are related to those above by $t' = t$, $\psi' = \psi + 2t$. The unprimed coordinate system rotates with the event horizon, located at $x = 0$. The mass and charges are given by

$$M = \frac{3R+2J}{\ell}, \quad R = \frac{N^2}{2} \left(\frac{q}{\ell^2} + \frac{q^2}{2\ell^2} \right), \quad J = \frac{N^2}{2} \left(\frac{3q^2}{2\ell^4} + \frac{q^3}{\ell^6} \right). \tag{4.5}$$

The solution has one free parameter q but two distinct charges R, J . So R, J should satisfy a relation on the solution space, which is given by [9]

$$\mathcal{G}(R, J) \equiv R^3 + \frac{N^2}{2}J^2 - \left(3R + \frac{N^2}{2} \right) (3R^2 - N^2J) = 0. \tag{4.6}$$

Hairy BPS black holes have been studied as a 1-parameter deformation of the solution above. A real scalar field φ was added, only depending on the radial variable x and preserving the $SU(2)_R \times U(1) \subset SO(4)$ isometry. The functions $f(x), U(x), h(x), w(x), \varphi(x)$

now satisfy coupled ODE's in x . [31, 32] solved these equations numerically first in the non-BPS case and then reduced the mass to get close to the BPS limit. One important feature reported in [31, 32] is that the solutions seem to exhibit a singularity which replaces the event horizon. The limiting solution has an extra continuous parameter apart from q . We would like to first see whether BPS black holes in this setup can be understood from the non-interacting mix picture, in the sense of superposing the solution. In particular, we set the hair condensate $\varphi \sim \varepsilon$ to be small, keeping the black hole size general. We shall try to keep φ as a small perturbation around fixed black hole background (4.1), trying to realize the spirit of [29, 30]. (The failure of this procedure will be related to the unique nature of the solutions of [31, 32].)

We study the BPS equations for a real scalar φ in the notation of [49]. Their scalar is related to the scalar Φ of [30] by $\Phi = 2 \sinh \varphi$. The functions $f(x), U(x), w(x), h(x), \varphi(x)$ satisfy eq. (4.7) of [49]. For small $\varphi \sim \varepsilon$, f, U, w, h are given by (4.3) at the leading $\mathcal{O}(\varepsilon^0)$ order. At the next-to-leading $\mathcal{O}(\varepsilon^1)$ order, φ should satisfy

$$\varphi'(x) = -\frac{2U(x)}{\ell x h(x)} \sinh \varphi(x) . \tag{4.7}$$

At this order, we plug in the zeroth order solutions (4.3) for U, h . Since we keep the hair small, $\varphi \sim \varepsilon$, we take $\sinh \varphi \approx \varphi$ and consider the linear differential equation for φ . Normally, if we solve the general equation of motion in the AdS black hole background, one studies the second order differential equation with the normalizable boundary condition at the boundary $x \rightarrow \infty$ and the infalling boundary condition at the horizon $x \rightarrow 0$. This yields the quasinormal modes at discrete choices of frequency ω_n . These frequencies will typically have nonzero imaginary parts, either implying that the modes fall into the horizon ($\text{Im}(\omega_n) < 0$) or that they trigger superradiant instability ($\text{Im}(\omega_n) > 0$). However, in the BPS equation, we only expect stationary modes with $\text{Im}(\omega_n) = 0$. (In the time coordinate chosen above, φ will be t -independent and real.) (4.7) can be integrated with the integration constant being its overall amplitude ε . So one has no free parameter left to match either normalizability at $x = \infty$ or the infalling (or in the BPS case, stationary normalizable) condition at $x = 0$. We just find either acceptable or pathological solution.

Plugging in (4.3) to (4.7), the solution for φ is given by

$$\tanh \frac{\varphi(x)}{2} = \varepsilon x^{-\frac{2q/\ell^2}{1+3q/\ell^2}} \left(1 + \frac{3q}{\ell^2} + \frac{x}{\ell^2} \right)^{-\frac{1+q/\ell^2}{1+3q/\ell^2}} . \tag{4.8}$$

The left hand side should be understood as $\approx \frac{\varphi(x)}{2}$ in our small hair approximation. At the boundary $x \rightarrow \infty$, it behaves like the proper normalizable mode $\sim x^{-1}$ for the scalar dual to the Yang-Mills operator $\text{tr}(X^2 + Y^2 + Z^2)$ [30]. However, the solution diverges at $x = 0$ as

$$\varphi(x) \sim \varepsilon x^{-\frac{2q/\ell^2}{1+3q/\ell^2}} . \tag{4.9}$$

So our perturbative approach with small φ breaks down. Also, for small black hole size q , this solution provides the first correction to the graviton wavefunction in vacuum AdS ($q = 0$) due to the small black hole ($q \neq 0$), which again diverges. So the iteration procedure

of [29, 30] for the nonlinear superposition would not work even for small ε, q (say, without finite temperature regularization). The exact numerical results of [31, 32] report divergence not with the scalar but with certain tidal forces (geometry), very different from what we find above. Since the nature of the background black hole solution changes by turning on small ε , the superposition picture of [29, 30] does not seem to apply straightforwardly in this setup.

To improve the situation, we first show that φ beyond the s-wave ansatz may provide smooth solutions in the perturbative regime. To find these solutions, we should keep a complex scalar including φ and consider the BPS equations given by PDE's in AdS₅. We study a small complex scalar sitting in a hypermultiplet and identify the complex scalar which contains φ . The general BPS equation for the scalars in the hypermultiplet is given in [50], eq. (3.12) and (3.27). (3.12) demands t -independence of the scalar, while (3.27) is a spatial differential equation given by

$$D_m q^X = (\Phi^r)_m{}^n D_n q^Y (J^r)_Y{}^X . \quad (4.10)$$

Here, q^X is the scalars in the hypermultiplet, with $X, Y = 1, \dots, 4$, $(\Phi^r)_m{}^n$ with $r = 1, 2, 3$ are the three complex structures of the base space and $(J^r)_X{}^Y$ are the three complex structures of the scalar target space. They satisfy

$$(\Phi^1)_m{}^n (\Phi^2)_n{}^p = (\Phi^3)_m{}^p, \quad (\Phi^2)_m{}^n (\Phi^3)_n{}^p = (\Phi^1)_m{}^p, \quad (\Phi^3)_m{}^n (\Phi^1)_n{}^p = (\Phi^2)_m{}^p, \quad (4.11)$$

and similar relations hold for J^r . D_m is the covariant derivative involving our U(1) gauge field A_μ . Φ^r can be taken to be

$$\Phi^1 = e^{\hat{x}} \wedge e^{\hat{2}} - e^{\hat{3}} \wedge e^{\hat{1}}, \quad \Phi^2 = e^{\hat{2}} \wedge e^{\hat{3}} - e^{\hat{x}} \wedge e^{\hat{1}}, \quad \Phi^3 = e^{\hat{x}} \wedge e^{\hat{3}} - e^{\hat{1}} \wedge e^{\hat{2}} \quad (4.12)$$

on the black hole background (after some permutations of $r = 1, 2, 3$: e.g. Φ^3 given above is called $J = \Phi^2$ in [49] and J^1 in [47]). In this setup, introducing the following inverse-vierbeins $E_{\hat{x}}, E_{\hat{1}, \hat{2}, \hat{3}}$ for (4.4) on the 4d base space,

$$\begin{aligned} E_{\hat{x}} &= 2\sqrt{xh} \frac{\partial}{\partial x}, & E_{\hat{3}} &= \frac{2}{\sqrt{xh}} \frac{\partial}{\partial \psi} \\ E_{\hat{1}} &= \frac{2}{\sqrt{x}} \left(\cos \psi \frac{\partial}{\partial \theta} + \frac{\sin \psi}{\sin \theta} \left(\frac{\partial}{\partial \phi} - \cos \theta \frac{\partial}{\partial \psi} \right) \right), \\ E_{\hat{2}} &= \frac{2}{\sqrt{x}} \left(-\sin \psi \frac{\partial}{\partial \theta} + \frac{\cos \psi}{\sin \theta} \left(\frac{\partial}{\partial \phi} - \cos \theta \frac{\partial}{\partial \psi} \right) \right), \end{aligned} \quad (4.13)$$

one can show

$$\mathcal{E}_i^m (\Phi^r)_m{}^n = -i(\sigma^r)_i{}^j \mathcal{E}_j^n, \quad i, j = 1, 2 \quad (4.14)$$

with $\mathcal{E}_i^m \equiv ((E_{\hat{x}} + iE_{\hat{3}})^m, (E_{\hat{1}} + iE_{\hat{2}})^m)$, where σ^r are the Pauli matrices. Defining $\tilde{\mathcal{E}}_i^m \equiv ((E_{\hat{1}} - iE_{\hat{2}})^m, -(E_{\hat{x}} - iE_{\hat{3}})^m)$, it also satisfies

$$\tilde{\mathcal{E}}_i^m (\Phi^r)_m{}^n = -i(\sigma^r)_i{}^j \tilde{\mathcal{E}}_j^n . \quad (4.15)$$

The Φ^r in (4.12) are chosen such that the right hand side is proportional to σ^r . (The overall factor in front of σ^r should be $-i$ for the relation (4.11) to hold.) One can similarly introduce a complex basis in the field space. For small scalar fluctuations, we can approximate

the hypermultiplet target space to be flat. Combining four real scalars q^X to two complex scalars $Q^a = (Q, \tilde{Q}^*)$, with $a = 1, 2$, $q^Y (J^r)_Y{}^X$ can be rewritten as $-iQ^b (\sigma^r)_b{}^a$. As we shall see below, with the gauging $D_\mu Q^a \equiv (\partial_\mu - 2iA_\mu)Q^a$, Q will become the complex scalar which contains φ .

Now contracting the equation (4.10) with \mathcal{E}_i^m or $\tilde{\mathcal{E}}_i^m$, and changing the real index X to the complex index a , we study the BPS equation for Q^a . Using $(\sigma^r)_i{}^j (\sigma^r)_b{}^a = \frac{3}{2} \delta_i^a \delta_b^j - \frac{1}{2} (\sigma^s)_i{}^a (\sigma^s)_b{}^j$, these equations reduce to

$$\mathcal{E}_i^m D_m Q^i = 0, \quad \tilde{\mathcal{E}}_i^m D_m Q^i = 0. \quad (4.16)$$

In terms of the component scalars, these equations are

$$(E_{\hat{x}} + iE_{\hat{3}})^m D_m Q = -(E_{\hat{1}} + iE_{\hat{2}})^m D_m \tilde{Q}^*, \quad (E_{\hat{1}} - iE_{\hat{2}})^m D_m Q = (E_{\hat{x}} - iE_{\hat{3}})^m D_m \tilde{Q}^*. \quad (4.17)$$

Keeping only one complex scalar Q while turning off the other $\tilde{Q} = 0$, one obtains

$$(E_{\hat{x}} + iE_{\hat{3}})^m D_m Q = 0, \quad (E_{\hat{1}} - iE_{\hat{2}})^m D_m Q = 0. \quad (4.18)$$

This is the BPS equation for the complex scalar Q that we wish to study.⁷ We have checked that (4.18) implies the following equation of motion

$$\left[\frac{1}{\sqrt{-g}} D_\mu (\sqrt{-g} g^{\mu\nu} D_\nu) + \frac{4}{\ell^2} \right] Q = 0. \quad (4.19)$$

Inserting the inverse-vierbeins, (4.18) can be written as

$$\left(xh \frac{\partial}{\partial x} + i \frac{\partial}{\partial \psi} \right) Q = -\frac{2}{\ell} U Q, \quad \left(\sin \theta \frac{\partial}{\partial \theta} - i \left(\frac{\partial}{\partial \phi} - \cos \theta \frac{\partial}{\partial \psi} \right) \right) Q = 0. \quad (4.20)$$

We believe that Q is the complex scalar of [30] appearing in their consistent truncation. (It carries the same U(1) charge as their scalar and satisfies the same linearized equation of motion.) When Q depends on x only, the first equation reduces to (4.7) with $Q = 2 \sinh \varphi \approx 2\varphi$.

Now we study all smooth solutions of (4.20) at $\mathcal{O}(\varepsilon^1)$. We take

$$Q(x, \theta, \phi, \psi) = \Phi(x) e^{im\psi} f(\theta, \phi) \quad (4.21)$$

where m is half-integral. The equations reduce to

$$\Phi' = \frac{m\ell^2 - x - 2q}{x(x + 3q + \ell^2)} \Phi, \quad z^* \frac{\partial f}{\partial z^*} = 2m \frac{1 - |z|^2}{1 + |z|^2} f \quad (4.22)$$

where $z \equiv \tan \frac{\theta}{2} e^{-i\phi}$. The solutions are given by

$$\begin{aligned} \Phi(x) &= \varepsilon x^{\frac{m-2q/\ell^2}{1+3q/\ell^2}} \left(1 + \frac{3q}{\ell^2} + \frac{x}{\ell^2} \right)^{-\frac{1+m+q/\ell^2}{1+3q/\ell^2}} \\ f(\theta, \phi) &= \left(\frac{z^*}{(1+|z|^2)^2} \right)^{\frac{m}{2}} z^\alpha = \left(\cos \frac{\theta}{2} \right)^{\frac{3m}{2} - \alpha} \left(\sin \frac{\theta}{2} \right)^{\frac{m}{2} + \alpha} e^{i(\frac{m}{2} - \alpha)\phi} \end{aligned} \quad (4.23)$$

⁷On the other hand, turning off $Q = 0$, \tilde{Q} satisfies similar equations with $D_\mu \tilde{Q} \equiv (\partial_\mu + 2iA_\mu)\tilde{Q}$. Solving these equations following our studies in the next paragraph, one finds that there are no normalizable solutions at $x \rightarrow \infty$. In general, we think the normalizability of the solutions of (4.17) at $x \rightarrow \infty$ should demand $\tilde{Q} = 0$.

where the factor z^α represents the integration constant for z^* given by arbitrary functions of z . One should take $\alpha = -\frac{m}{2}, -\frac{m}{2} + 1, \dots, \frac{3m}{2}$ to have regular solutions on $S^2(\theta, \phi)$,

$$f(\theta, \phi)e^{im\psi} = \left(\cos \frac{\theta}{2} e^{i\phi_1}\right)^{m_1} \left(\sin \frac{\theta}{2} e^{i\phi_2}\right)^{m_2}, \quad (\psi, \phi) = \phi_1 \pm \phi_2 \quad (4.24)$$

where $m_1 + m_2 = 2m$ and $m_1, m_2 = 0, 1, 2, \dots$. The angle-dependent part is the BPS spherical harmonics. It is just the usual ‘derivative’ factor for the BPS graviton wavefunctions in AdS₅, or in the dual QFT, $(\partial_{+\dot{+}})^{m_1} (\partial_{+\dot{-}})^{m_2}$ acting on the conformal primary operators. One finds the following behaviors at infinity and on the horizon:

$$\Phi(x) \sim \begin{cases} x^{-1} & \text{for } x \rightarrow \infty \\ x^{\frac{m-2q/\ell^2}{1+3q/\ell^2}} & \text{for } x \rightarrow 0 \end{cases}. \quad (4.25)$$

We get the expected normalizable boundary condition at $x \rightarrow \infty$. At the horizon $x \rightarrow 0$, the solution remains finite only for the modes satisfying

$$m = \frac{m_1 + m_2}{2} \geq \frac{2q}{\ell^2}. \quad (4.26)$$

This gives an intuitive understanding of the well-behaved perturbative BPS hairs. If the scalar field (\sim particle) rotates with large spherical harmonics quantum numbers $m_1, m_2 \gg 1$ in vacuum AdS, its average radial position will be far away from the center. On the other hand, the conformal primary state with $m_1, m_2 = 0$ prefers to be at the central region of AdS. This can be seen from $\Phi(x)$ of (4.23) at $q = 0$ (no black hole):

$$\Phi(x) \rightarrow \frac{\varepsilon x^m}{(1 + x/\ell^2)^{m+1}}. \quad (4.27)$$

The radial wavefunction $\Phi(x)$ is peaked at $x = x_\star \equiv m\ell^2$, but the peak is not sharp. This is basically because the damping after the peak x_\star is slow, related to its slow asymptotic decay $\Phi \sim x^{-1}$ for the mass $M^2\ell^2 = -4$. Anyway, the average radial position moves outward for larger $m = \frac{m_1+m_2}{2}$. Now imagine placing the same field (\sim particle) in the black hole background. The central region of AdS is now occupied by the black hole, behind the event horizon. Therefore, the fields (\sim particles) with lower m_1, m_2 quantum numbers will be more likely to be swallowed by the black hole, or to back-react more strongly to it. (4.23) shows that, below the threshold $\frac{2q}{\ell^2}$ for m , this expectation is manifested by the exterior wavefunction exhibiting a divergence at the event horizon. To have reasonable hairy black holes below this threshold, the hair should give a large back-reaction to the background black hole.

Our studies so far intrinsically used the wave picture. Let us also consider the particle picture. Note that $\Phi(x)$ in (4.23) is never peaked around its maximum at any choice of m, q , basically due to the slow fall-down $\sim x^{-1}$ at large x . However, if one considers a semi-classical Kaluza-Klein graviton with large mass and large orbital angular momentum quantum number m , one can have a sharply peaked wavefunction in AdS. We shall reconsider the perturbative hair as a particle probe in this case, assuming that it is realized by

a sharply peaked wavepacket of a field with large mass and angular momenta. This viewpoint will provide new understandings of (4.26). For instance, consider a spinless graviton particle dual to the chiral primary (2.12) with $n \gg 1$. The scaling dimension $\Delta = n$ and the charge $\delta R = \frac{n}{3}$ of this operator are dual to the following mass and charge for the graviton:

$$(M\ell)^2 = n(n-4) \approx n^2, \quad \delta R = \frac{n}{3}. \quad (4.28)$$

The bosonic probe particle action in the black hole background (4.1) is given by

$$S = \frac{n}{\ell} \int d\tau \left[-\sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} - \frac{1}{3} A_\mu \dot{x}^\mu \right]. \quad (4.29)$$

We try to find the rotating BPS solutions for the time-like worldline with $t(\tau) = \tau$, outside the horizon $x > 0$. Denoting by $\delta J_1, \delta J_2$ the Noether charges of the particle for the rotations on ϕ_1, ϕ_2 , respectively, one finds the following stationary radius x as a function of charges:

$$x = \ell^2 \left(\frac{\delta J_1 + \delta J_2}{3\delta R} - \frac{2q}{\ell^2} \right) \equiv \ell^2 \left(\frac{2\delta J}{3\delta R} - \frac{2q}{\ell^2} \right) > 0 \rightarrow \delta J > \frac{nq}{\ell^2}. \quad (4.30)$$

The particle can orbit outside the event horizon when the particle is rotating fast enough. This is morally in accordance with (4.26). In particular, blindly inserting $n = 2$ and $\delta J = m$ for the quantum particle of the field Φ , one happens to recover the bound (4.26).

When the bound (4.30) is saturated, the particle is precisely at the horizon. Then it will be ambiguous to distinguish whether this is a hair or part of the black hole, at least in the point particle approximation. Indeed one finds an interesting signature of this ambiguity. To explain this, recall that the background black hole satisfies a charge relation (4.6). Absorbing a probe hair into the black hole will shift its charges by $R \rightarrow R + \delta R$ and $J \rightarrow J + \delta J$, where $\delta R, \delta J$ from the probe should be much smaller than the background charges. Since we take the background charges R, J to be at order N^2 , the hair charges should also satisfy $\delta R \ll N^2, \delta J \ll N^2$. In this limit, the shift in polynomial \mathcal{G} in (4.6) is given approximately by

$$\delta\mathcal{G}(R, J) \approx \frac{N^4}{2} \left(1 + \frac{q}{\ell^2} \right)^3 \left(\delta J - \frac{3q}{\ell^2} \delta R \right). \quad (4.31)$$

If the probe exists outside the horizon, the hair violates the charge relation (4.6) in an over-rotating manner, $\mathcal{G} > 0$. If the probe particle is exactly at the horizon, one finds from (4.30) that the charge relation is respected. So adding such ‘hairs’ moves the black hole within the charge sector of the non-hairy black holes, as long as their charges are concerned. An outside observer unable to see such particles exactly at the horizon, only seeing the total charges, may regard the net system as a non-hairy black hole satisfying the charge relation. This is just a naive probe particle picture. In the wave picture, the solution (4.23) has finite nonzero value $\Phi(0)$ at the horizon. Perhaps it might be interesting to study its interaction with the strong-coupling modes of the near-horizon AdS_2 [51] and see if there are quantum lessons to be learned.⁸

⁸Also, since the divergence (4.9) at $m_1, m_2 = 0$ below the threshold is absent in the finite temperature regularization [31, 32] of AdS_2 , this may also be related to the coupling to the near-horizon modes. We thank Shiraz Minwalla and Gustavo J. Turiaci for pointing it out to us.

To conclude, we found a picture on why the conformal primary states (with $m = 0$ or $\delta J = 0$) of a graviton cannot be the BPS black hole hairs while respecting the superposition picture. This is in accordance with the QFT operator spectrum that we found in section 3 for $SU(2)$. Our studies in this section also imply that particles with fast enough orbital motions may provide black hole hairs along the strategy of [29, 30]. This is why we had in mind the hairs from conformal descendant gravitons in the range $t^{34} \sim t^{39}$, beyond just a logical possibility. Although the results in our sections 3 and 4 are in the opposite extreme, $N = 2$ vs. ∞ , we think this picture is natural. It will be exciting to further explore this issue by finding new black hole cohomologies at higher N , and also by exploring hairy black holes more comprehensively.

5 Conclusion and remarks

In this paper, we studied the classical cohomologies of local BPS operators in $\mathcal{N} = 4$ Yang-Mills theory with $SU(N)$ gauge group. Even for low N 's, we distinguish the cohomologies for the gravitons (subject to the stringy exclusion principle) and the rest. The latter have a chance to describe quantum black hole microstates in AdS/CFT at finite Newton constant if N is finite. In the $SU(2)$ theory, we constructed an infinite family O_n of non-graviton cohomologies. We studied the partial no-hair behavior of these cohomologies when graviton cohomologies are multiplied. We first showed that almost all conformal primary gravitons multiplied to our new cohomologies do not appear in the superconformal index, providing in a sense a partial no-hair theorem. One natural explanation is that the product cohomologies are trivial, i.e. Q -exact. For a selection of products involving O_0 and simple conformal primary gravitons, we explicitly showed the Q -exactness. On the other hand, the index allows a chance that conformal descendant gravitons may dress our new cohomologies. We provided a gravity interpretation of these phenomena by studying when the graviton hairs can be perturbatively superposed with non-hairy black holes.

We would like to discuss various issues and further interesting directions.

We should be able to construct new cohomologies at larger N and large charges. This will allow us to address dual black hole physics more quantitatively. Considering that this is a completely well defined and classical combinatorics problem, it is amazing to confront the computational complexity towards its solutions. In this paper we mostly relied on analytic insights to construct a small subset of these new cohomologies, and this strategy may continue to work to a certain extent for larger charges and N . However, to understand the full landscape of these cohomologies, probably one should rely on efficient computerized calculations. It will also be valuable to understand the time and resources required for this calculation, which by itself may shed light on the nature of the black hole microstates.

Since constructing all cohomologies is cumbersome, we employed a streamlined strategy. We computed the index over finite N gravitons and subtracted it from the full index to first notice which charge sectors host new cohomologies. Although finite N gravitons are completely well defined, enumerating them without overcounting is tricky due to the trace relations. In fact, computing the finite N graviton index has been a major bottle-

neck of this project. Since the basic ingredients are diagonal matrices, we hope that this combinatoric problem can be solved without too much difficulty.

To count the finite N gravitons more easily, we extensively studied the so-called BMN sector of the cohomology problem. Counting graviton cohomologies in this sector is much easier since one just needs to consider diagonal matrices without any derivatives. For $SU(2)$ theory, we could analytically count the BMN gravitons completely. After subtracting the graviton index from the full index, we constructed all the cohomologies that saturate the remaining index. We hope that similar calculations in the BMN sector will be not too difficult for higher $SU(N)$ groups. We expect the BMN spectrum to be much more interesting at higher N . For instance, having in mind applying the large N techniques of [45], we envision that the ‘small black hole’ like scaling of the entropy $S \sim \frac{j^{3/2}}{N}$ will be discovered when $\frac{j}{N^2}$ is given by a small number independent of N .

For higher N , it is important to find the ‘threshold’ cohomology, the lowest operator not of graviton type. For $N = 2$, it has scaling dimension $E = \frac{19}{2}$ which is quite larger than $N^2 = 4$. However, the threshold for large N should be much smaller than N^2 . This is because small black holes exist at charges given by ϵN^2 with an arbitrary small number ϵ independent of N . To accommodate such black holes, the threshold energy level should not scale like N^2 . The situation is in contrast to the black hole threshold in AdS_3 . BTZ black holes start to appear at an energy proportional to the central charge, which is a parameter analogous to N^2 , without having a small black hole branch with negative specific heat.

Perhaps the infinite tower of new cohomologies O_n in section 3.1 is the most surprising discovery of this paper. This strongly signals that the black hole cohomologies may have unexpected emergent structures. It will be interesting to see if similar towers can be found for higher $SU(N)$ theories. Also, it may be interesting to generalize these findings to other $\mathcal{N} = 1$ SCFTs with or without gravity duals. In particular, since the operator O_0 involves a baryon-like factor $\psi_{m+} \cdot (\psi_{n+} \times \psi_{p+})$ of the $SO(3)$ vector-valued fields, it may be possible to understand and generalize them from the viewpoint of Regge trajectories.

We also emphasize that we provided a rather operational criterion to distinguish graviton and black hole cohomologies, although we think it is natural. It should be valuable to establish more intrinsic (maybe information theoretic) criteria to distinguish the two.

We would like to comment on aspects of hairy black holes as seen by our cohomologies.

One feature of non-hairy BPS black holes is the charge relation. In AdS_5 , this relation is given by (4.6). Why this relation holds, if any, and how hairy black holes violate it are still not well understood microscopically. Here note that recent studies of BPS black holes were often made using the index which cannot see one charge. So the charge relation cannot be addressed with the index. On the other hand, once we construct the cohomologies, all charge information is available and the charge relation can be studied in principle.

We also try to classify the possible patterns of hairs. We find three possibilities. In a sense, all three possibilities are realized in our new cohomologies, at least morally.

- The first class of hairs is discussed in detail in this paper, ‘superposing’ already existing black holes and gravitons. In gravity, this generalizes the spirit of [29, 30] to the BPS sector, and the back-reaction to each other is controlled and suppressed as

one tune the parameters of the solutions. We probed this class of hairs by studying the perturbative scalar beyond s-waves in section 4. In the QFT dual, non- Q -exact product cohomologies of black holes and gravitons are interpreted as hairy black hole operators in this class.

- Second, there may be hairy black holes in which the back-reaction of the hair to the core black hole is essential. The numerical hairy black holes of [31, 32] seem to belong to this class, in that even very small hairs demand completely new core black holes with new near-horizon structures. Such cores are possible only by the back-reaction of the hair. Somewhat curiously, we know a black hole cohomology that behaves like this. It is our $SU(2)$ threshold operator O_0 (3.17). There, two on-shell (i.e. Q -closed) graviton cohomologies v_n^m are multiplied to an off-shell core of the form $\psi_{m+} \cdot (\psi_{n+} \times \psi_{p+})$. The latter could be made on-shell only with the aid, or back-reactions, of the gravitons.
- Third, it is a logical possibility that the back-reaction of the core black hole to the gravitons may allow qualitatively new hairs. This may have been realized in our black hole cohomologies $O_{n \geq 1}$, (3.41) or (3.47). The first term of this operator takes the form of an on-shell black hole operator O_0 times an off-shell matter $(f \cdot f)^n$. After adding two more terms as a result of the back-reactions, this could be promoted to an on-shell operator. It will be interesting to clarify whether this signals a more exotic form of hairs or otherwise this is better viewed as representing internal structures of black holes.

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A Supersymmetry and supergravitons

In this section, we list symmetry transformations for the classical Poincare supercharges in the $\mathcal{N} = 4$ SYM theory. The $\mathcal{N} = 4$ SYM fields comprise a gauge field $A_{\alpha\beta}$, six scalars Φ_{ij} and fermions $\Psi_{i\alpha}, \bar{\Psi}^{j\dot{\alpha}}$. The transformations of the fields generated by some Poincare supercharges are given by [15, 41]

$$\begin{aligned}
 [Q_\alpha^i, \Phi_{jk}] &= \delta_j^i \Psi_{k\alpha} - \delta_k^i \Psi_{j\alpha} \\
 \{Q_\alpha^i, \Psi_{j\beta}\} &= -2i\delta_j^i f_{\alpha\beta} - \epsilon_{\alpha\beta} [\bar{\Phi}^{ik}, \Phi_{kj}] \\
 \{Q_\alpha^i, \bar{\Psi}_\beta^j\} &= 2iD_{\alpha\beta} \bar{\Phi}^{ij} \\
 [Q_\alpha^i, A_{\beta\dot{\gamma}}] &= -\epsilon_{\alpha\beta} \bar{\Psi}_{\dot{\gamma}}^i,
 \end{aligned}
 \tag{A.1}$$

and the action of $\bar{Q}_{m\dot{\alpha}}$ is given similarly. Our focus will be on the $\frac{1}{16}$ -BPS operators that are annihilated by both $Q \equiv Q_-^4$ and $S \equiv S_4^-$. The BPS fields are given by

$$\bar{\phi}^n \equiv \bar{\Phi}^{4m}, \quad \psi_{n+} \equiv -i\Psi_{n+}, \quad n = 1, 2, 3, \quad \bar{\lambda}_{\dot{\alpha}} \equiv \bar{\Psi}_{\dot{\alpha}}^4, \quad f_{++}, \quad (\text{A.2})$$

together with the covariant derivatives

$$D_{+\dot{\alpha}} = \partial_{+\dot{\alpha}} - i[A_{+\dot{\alpha}}, \cdot], \quad (\text{A.3})$$

where $[D_{+\dot{\alpha}}, D_{+\dot{\beta}}] = \epsilon_{\dot{\alpha}\dot{\beta}} f_{++}$. The letters are also constrained by the equation of motion $D_{+[\dot{\alpha}} \bar{\lambda}_{\dot{\beta}]} = \epsilon_{\dot{\alpha}\dot{\beta}} [\psi_{n+}, \bar{\phi}^n]$.

The transformation rules of BPS fields under Q are given by

$$\begin{aligned} [Q, \bar{\phi}^n] &= 0 \\ \{Q, \psi_{n+}\} &= -i\epsilon_{nmp} [\bar{\phi}^m, \bar{\phi}^p] \\ \{Q, \bar{\lambda}_{\dot{\beta}}\} &= 0 \\ [Q, f_{++}] &= -i[\psi_{m+}, \bar{\phi}^m] \\ [Q, D_{+\dot{\alpha}}] &= -i[\bar{\lambda}_{\dot{\alpha}}, \cdot]. \end{aligned} \quad (\text{A.4})$$

The supercharges Q_+^m and $\bar{Q}_{m\dot{\alpha}}$ in PSU(1,2|3), which commute with Q , act on the BPS fields as follows

$$\begin{aligned} [Q_+^m, \bar{\phi}^n] &= i\epsilon^{mnp} \psi_{p+} \\ \{Q_+^m, \psi_{n+}\} &= -2i\delta_n^m f_{++} \\ \{Q_+^m, \bar{\lambda}_{\dot{\alpha}}\} &= -2iD_{+\dot{\alpha}} \bar{\phi}^m \\ [Q_+^m, D_{+\dot{\alpha}}] &= 0 \\ [Q_+^m, f_{++}] &= 0 \\ [\bar{Q}_{m\dot{\alpha}}, \bar{\phi}^n] &= -\delta_m^n \bar{\lambda}_{\dot{\alpha}} \\ \{\bar{Q}_{m\dot{\alpha}}, \psi_{n+}\} &= -2\epsilon_{mnp} D_{+\dot{\alpha}} \bar{\phi}^p \\ \{\bar{Q}_{m\dot{\alpha}}, \bar{\lambda}_{\dot{\beta}}\} &= \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon_{mnp} [\bar{\phi}^n, \bar{\phi}^p] \\ [\bar{Q}_{m\dot{\alpha}}, D_{+\dot{\beta}}] &= -\epsilon_{\dot{\alpha}\dot{\beta}} [\psi_{m+}, \cdot] \\ [\bar{Q}_{m\dot{\alpha}}, f] &= D_{+\dot{\alpha}} \psi_{m+}. \end{aligned} \quad (\text{A.5})$$

The cohomologies in S_2 can be taken to be

$$\begin{aligned} &\text{tr}(\bar{\phi}^m \bar{\phi}^n), \quad \text{tr}(\bar{\phi}^m \bar{\lambda}_{\dot{\alpha}}), \quad \text{tr}(\bar{\lambda}_{\dot{\alpha}} \bar{\lambda}_{\dot{\beta}}), \quad (\text{A.6}) \\ &\text{tr}(\bar{\phi}^m \psi_{n+}) - \frac{1}{3} \delta_n^m \text{tr}(\bar{\phi}^l \psi_{l+}), \quad \text{tr}(\bar{\lambda}_{\dot{\alpha}} \psi_{m+} - \epsilon_{mnp} \bar{\phi}^n D_{+\dot{\alpha}} \bar{\phi}^p), \\ &\text{tr} \left(\bar{\phi}^m f_{++} - \frac{1}{4} \epsilon^{mnp} \psi_{n+} \psi_{p+} \right), \quad \text{tr} \left(\bar{\lambda}_{\dot{\alpha}} f_{++} - \frac{2}{3} \psi_{m+} D_{+\dot{\alpha}} \bar{\phi}^m + \frac{1}{3} \bar{\phi}^m D_{+\dot{\alpha}} \psi_{m+} \right). \end{aligned}$$

as well as arbitrary number of derivatives $\partial_{+\dot{\alpha}}$ acting on them.

We can represent the SU(2) adjoint fields in the basis of Pauli matrices:

$$\begin{aligned} \bar{\phi}^1 &= \frac{1}{\sqrt{2}}X \cdot \vec{\sigma}, & \bar{\phi}^2 &= \frac{1}{\sqrt{2}}Y \cdot \vec{\sigma}, & \bar{\phi}^3 &= \frac{1}{\sqrt{2}}Z \cdot \vec{\sigma}, & \bar{\lambda}_\alpha &= \lambda_\alpha \cdot \vec{\sigma} \\ \psi_{1+} &= \psi_1 \cdot \vec{\sigma}, & \psi_{2+} &= \psi_2 \cdot \vec{\sigma}, & \psi_{3+} &= \psi_3 \cdot \vec{\sigma}, & f_{++} &= -\frac{1}{\sqrt{2}}f \cdot \vec{\sigma} \end{aligned} \quad (\text{A.7})$$

where $\vec{\sigma} \equiv (\frac{\sigma_1}{2}, \frac{\sigma_2}{2}, \frac{\sigma_3}{2})$ and $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$. $\phi^m = (X, Y, Z)$, $\psi_m, \lambda_\alpha, f$ denote 3-dimensional vectors. We abbreviate bars and dots for vectors. In this notation, inner and outer products will be used instead of trace and commutators. The Q -transformations are given by

$$Q\phi^m = 0, \quad Q\psi_m = \frac{1}{2}\epsilon_{mnp}\phi^n \times \phi^p, \quad Qf = \phi^m \times \psi_m, \quad [Q, D_\alpha] = \lambda_\alpha \times \quad (\text{A.8})$$

where the field strength is written as $\epsilon_{\alpha\beta}f \times = -\sqrt{2}[D_\alpha, D_\beta]$ and the equation of motion is written as $-\sqrt{2}D_{[\alpha}\lambda_{\beta]} = \epsilon_{\alpha\beta}\phi^m \times \psi_m$. In the vector notation, the cohomologies in S_2 are written as

$$\begin{aligned} \phi^{(m} \cdot \phi^{n)}, & & \phi^m \cdot \lambda_\alpha, & & \lambda_+ \cdot \lambda_-, \\ \phi^m \cdot \psi_n - \frac{1}{3}\delta_n^m \phi^p \cdot \psi_p, & & \lambda_\alpha \cdot \psi_m - \frac{1}{2}\epsilon_{mnp}\phi^n \cdot D_\alpha\phi^p, & & \\ \phi^m \cdot f + \frac{1}{2}\epsilon^{mnp}\psi_n \cdot \psi_p, & & \lambda_\alpha \cdot f + \frac{2}{3}\psi_m \cdot D_\alpha\phi^m - \frac{1}{3}\phi^m \cdot D_\alpha\psi_m. & & \end{aligned} \quad (\text{A.9})$$

When we use vector notation to act Q_+^m on S_2 multiplet, a set of numerical factors emerges. For instance, $Q_+^m\bar{\phi}^n$ equals $\sqrt{2}i\epsilon^{mnp}\psi_p$. However, we can simplify matters by defining the action of Q_+^m in a way that eliminates these numerical factors:

$$\begin{aligned} Q_+^m\phi^n &\equiv \epsilon^{mnp}\psi_p, \\ Q_+^m\psi_n &\equiv \delta_n^m f, \\ Q_+^m\lambda_\alpha &\equiv -D_{+\alpha}\phi^m. \end{aligned} \quad (\text{A.10})$$

According to this definition,

$$Q_+^n(\phi^m \cdot \psi_n) = \phi^m \cdot f + \frac{1}{2}\epsilon^{mlp}\psi_l \cdot \psi_p \quad (m \neq n), \quad (\text{A.11})$$

where n is not subject to any summation.

B Representations of PSU(2,2|4) and $\frac{1}{16}$ -BPS states

In this section we review the structures of representations of the $\mathcal{N} = 4$ superconformal group PSU(2,2|4), and of its subgroup PSU(1,2|3) that commutes with Q . We refer the readers to [37] for more details.

Superconformal group of 4-dimensional $\mathcal{N} = 4$ theory is PSU(2,2|4), whose bosonic subgroup consists of the 4d conformal group SO(2,4) and the R-symmetry SU(4). States, or weights, of its representations are often labeled with scaling dimension E , Dynkin labels

j and \bar{j} for the Lorentz group $\text{SO}(4) = \text{SU}(2) \times \text{SU}(2)$, and Dynkin labels $[r_1, r_2, r_3]$ for the R-symmetry group $\text{SU}(4)$. (We hope that j appearing in the representation theory here, $j = J_1 + J_2$, is not confused with $j = 6(R + J)$ that often appears in the series expansion in the main part of this paper.) We often refer to scaling dimension E as energy with radial quantization in mind. A representation of the bosonic subalgebra is often expressed as

$$[j; \bar{j}]_E^{[r_1, r_2, r_3]}, \tag{B.1}$$

using its highest weights. It is also conventional to define angular momenta

$$J_1 = \frac{j + \bar{j}}{2}, \quad J_2 = \frac{j - \bar{j}}{2} \tag{B.2}$$

and R-symmetry charges

$$R_1 = r_2 + \frac{r_1 + r_3}{2}, \quad R_2 = \frac{r_1 + r_3}{2}, \quad R_3 = \frac{r_1 - r_3}{2}, \tag{B.3}$$

as charges for rotations on the orthogonal 2-planes. Dynkin labels j, \bar{j}, r_1, r_2 and r_3 are all integers so that $j \pm \bar{j}$ is even for bosons and odd for fermions. It follows that J_1 and J_2 are integers for bosons and half-integers for fermions.

A representation of the superconformal group $\text{PSU}(2, 2|4)$ is completely determined by its superconformal primary. Given a superconformal primary that is annihilated by conformal supercharges, superconformal descendants are obtained by the action of supercharges, of which there are 16 in the 4d $\mathcal{N} = 4$ theory. There are a finite number of such descendants due to fermion statistics, and following each descendant is an infinite number of conformal descendants, obtained by acting any number of P^μ .

By unitarity, the scaling dimension of a superconformal primary is bounded from below. Let $[j; \bar{j}]_E^{[r_1, r_2, r_3]}$ be the bosonic subalgebra representation of the primary. For generic j, E must be not smaller than certain value determined by its quantum numbers and in the exceptional case of $j = 0$, an isolated value of E which is exactly 2 below the bound is allowed. To summarize,

$$E \geq 2 + j + \frac{1}{2}(3r_1 + 2r_2 + r_3) = 2 + J_1 + J_2 + R_1 + R_2 + R_3, \\ \text{or } E = \frac{1}{2}(3r_1 + 2r_2 + r_3) = R_1 + R_2 + R_3 \quad \text{and} \quad j = 0. \tag{B.4}$$

If the inequality is not saturated, the superconformal representation that follows from the primary is *long*: the primary is not annihilated by any of supercharges. The representation is said to be of *L*-type. If the inequality is saturated, the representation is *short*. If further $j \neq 0$ then a supercharge annihilates the primary, and the representation is said to be of *A*₁-type. If instead $j = 0$ then successive action of two supercharges is needed to annihilate the primary, and the representation is said to be of *A*₂-type. Finally, in the exceptional case of the second line of (B.4), the primary is annihilated by some supercharges and the representation is said to be of *B*₁-type.

Similar unitarity bounds and classification of representations according to \bar{j} in place of j and $r_1 + 2r_2 + 3r_3$ in place of $3r_1 + 2r_2 + r_3$ apply independently, although this classification is

less relevant to the following discussion. So for instance, the superconformal representation with primary $[j; \bar{j}]_E^{[r_1, r_2, r_3]}$ that belongs to A_1 -type and \bar{L} -type with respective definitions above, will be denoted as

$$A_1 \bar{L}[j; \bar{j}]_E^{[r_1, r_2, r_3]}.$$

For more details, see [37].

We turn to a subject more relevant to the body of this paper: $\frac{1}{16}$ -BPS contents of these representations. We choose a $\frac{1}{16}$ -BPS sector that is annihilated by a supercharge $Q \equiv Q_-^4$ with quantum numbers

$$(E, j, \bar{j}, r_1, r_2, r_3) = \left(\frac{1}{2}, -1, 0, 1, 0, 0\right) \leftrightarrow (E, J_1, J_2, R_1, R_2, R_3) = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right). \quad (\text{B.5})$$

Subgroup of $\text{PSU}(2, 2|4)$ that commutes with the chosen supercharge Q is $\text{PSU}(1, 2|3)$. The Dynkin labels, or charges, associated with this subgroup have the same scaling dimension E and \bar{j} for the Lorentz group, as well as $[r_2, r_3]$ for the R-symmetry group. It is possible to decompose representations of the larger group $\text{PSU}(2, 2|4)$ into multiple representations of the subgroup $\text{PSU}(1, 2|3)$.

Unitarity puts a lower bound on the scaling dimension of any states in the theory:

$$E \geq j + \frac{1}{2}(3r_1 + 2r_2 + r_3) = J_1 + J_2 + R_1 + R_2 + R_3. \quad (\text{B.6})$$

A state that is annihilated by the chosen supercharge Q , in other words a $\frac{1}{16}$ -BPS state, is characterized by saturation of the unitarity bound (B.6).

The linearity of this relation has an important implication. Suppose there are two representations of the bosonic subalgebra of $\text{PSU}(2, 2|4)$, say $[j; \bar{j}]_{E_1}^{[r_1, r_2, r_3]}$ and $[k; \bar{k}]_{E_2}^{[s_1, s_2, s_3]}$, both of which saturate (B.6). The highest weights of both representations are $\frac{1}{16}$ -BPS. Now take the direct product of these two representations, and expand as a sum over irreducible representations. Due to the linearity of the unitarity bound, only the one where Dynkin labels simply add up, $[j+k; \bar{j}+\bar{k}]_{E_1+E_2}^{[r_1+s_1, r_2+s_2, r_3+s_3]}$, contains $\frac{1}{16}$ -BPS state.

Of 16 supercharges, there is a single supercharge $Q' \equiv Q_+^4$ whose quantum numbers satisfy $E - J_1 - J_2 - R_1 - R_2 - R_3 = -2$. For all other supercharges, $E - J_1 - J_2 - R_1 - R_2 - R_3 \geq 0$. In particular, 9 supercharges that belong to $\text{PSU}(1, 2|3)$ as well as our $\frac{1}{16}$ -BPS supercharge Q have quantum numbers such that $E - J_1 - J_2 - R_1 - R_2 - R_3 = 0$.

With this information, let us now examine the $\frac{1}{16}$ -BPS contents of superconformal representations.

First consider the B -type representations: when the superconformal primary $[j; \bar{j}]_E^{[r_1, r_2, r_3]}$ has $j = 0$ and satisfies $E = \frac{1}{2}(3r_1 + 2r_2 + r_3) = R_1 + R_2 + R_3$. Clearly the highest weight of this primary saturates (B.6), it is a $\frac{1}{16}$ -BPS state. It is annihilated by the supercharge Q (because it saturates the bound (B.6)) and Q' (otherwise it will create a state violating (B.6)), among others. States obtained from the highest weight using lowering operators of the commuting subgroup $\text{PSU}(1, 2|3)$ are also annihilated by Q , and are $\frac{1}{16}$ -BPS states. They belong to a representation $[\bar{j}]_E^{[r_2, r_3]}$ under bosonic subalgebra of $\text{PSU}(1, 2|3)$. All states in the $\text{PSU}(1, 2|3)$ representation that starts with the primary $[\bar{j}]_E^{[r_2, r_3]}$ are annihilated by Q as the primary is. Meanwhile, these are the only states in the

original representation of $\text{PSU}(2, 2|4)$ that are $\frac{1}{16}$ -BPS. This is because for a state in the $\text{PSU}(2, 2|4)$ representation to not belong to the $\text{PSU}(1, 2|3)$ representation, some operator that does not commute with Q must be applied to the primary. However, such an operator has $E - J_1 - J_2 - R_1 - R_2 - R_3 > 0$ so no weights in the product representation between the primary and such an operator can saturate the unitarity bound.

Next consider the A -type representations, either A_1 or A_2 : when the superconformal primary $[j; \bar{j}]_E^{[r_1, r_2, r_3]}$ satisfies $E = 2 + j + \frac{1}{2}(3r_1 + 2r_2 + r_3) = 2 + J_1 + J_2 + R_1 + R_2 + R_3$. No weights in the primary saturate the unitarity bound (B.6). For a state in the representation to be $\frac{1}{16}$ -BPS, it needs an agent that raises its charges more than its energy, and Q' is precisely the one. Among direct product between the primary $[j; \bar{j}]_E^{[r_1, r_2, r_3]}$ and $Q' = [1; 0]_{\frac{1}{2}}^{[1, 0, 0]}$, a single irreducible representation of the bosonic subalgebra $[j + 1; \bar{j}]_{E + \frac{1}{2}}^{[r_1 + 1, r_2, r_3]}$ contains weights that are $\frac{1}{16}$ -BPS. Such states are annihilated by Q because now they saturate the bound (B.6). From this point, the analysis is the same as for B -type multiplets. A $\text{PSU}(1, 2|3)$ representation that starts from the primary $[\bar{j}]_{E + \frac{1}{2}}^{[r_2, r_3]}$ of its bosonic subalgebra is the $\frac{1}{16}$ -BPS contents of the original representation.

L -type representations do not contain any $\frac{1}{16}$ -BPS states.

Before we conclude, we present some examples.

Free graviton multiplets are defined as a trace over an arbitrary number of symmetrized free scalars, and their descendants. From this principle, it is clear that the superconformal primary must be $[0; 0]_n^{[0, n, 0]}$. n is the number of scalars inside trace, and is restricted to $2 \leq n \leq N$ for the $\text{SU}(N)$ theory due to trace relations. This is of type $B\bar{B}$ according to the aforementioned classification. It was named S_n in [12]. Therefore,

$$S_n = B\bar{B}[0; 0]_n^{[0, n, 0]}. \tag{B.7}$$

Its $\frac{1}{16}$ -BPS contents have appeared in table 2.

Also appearing with importance are $A_1\bar{L}$ -type multiplets. For example, the black hole threshold operator with $(E, j, \bar{j}, r_1, r_2, r_3) = (\frac{19}{2}, 5, 0, 3, 0, 0)$ belongs to $A_1\bar{L}[4; 0]_9^{[2, 0, 0]}$. All the other BPS operators that we encounter in section 3 belong the $A_1\bar{L}$ type multiplets, except for the pure BPS graviton multiplets (in $B\bar{B}$).

Here let us first explain why A_1 -type multiplets appear so often in our discussions on new $\frac{1}{16}$ -BPS states. As explained above, the primary of an A_1 -type multiplet have $j \neq 0$. $\frac{1}{16}$ -BPS states within this multiplet form a $\text{PSU}(1, 2|3)$ multiplet. Its primary is obtained by acting Q' on the primary of the A_1 -type multiplet, it therefore carries $j > 1$. On the other hand, since the primary of an A_2 -type multiplet carries $j = 0$, the $\text{PSU}(1, 2|3)$ primary of the $\frac{1}{16}$ -BPS states should carry precisely $j = 1$. Finally, the primary of a B -type multiplet has $j = 0$ and is also the primary of an $\text{PSU}(1, 2|3)$ multiplet that consists of $\frac{1}{16}$ -BPS states. Therefore, all the $\text{PSU}(1, 2|3)$ primaries belonging to either A_2 or B type multiplet should have $j = 1$ or 0 . At this point, recall from the discussion below (B.6) that when taking a product of two representations that contain $\frac{1}{16}$ -BPS states, only the irreducible representation whose Dynkin labels are the sum of those of both representations remain $\frac{1}{16}$ -BPS. Therefore, when constructing a $\frac{1}{16}$ -BPS operator out of the BPS letters in

the free theory ($\bar{\phi}^m, \psi_{m+}, \bar{\lambda}_{\dot{\alpha}}, f_{++}$ and derivatives $D_{+\dot{\alpha}}$ on them), the quantum number $j \geq 0$ of these letters will only add and the cases with $j = 0$ or 1 will be rather rare.

First, the net quantum number j of a $\text{PSU}(1,2|3)$ primary can be zero only if the operator is made only of $\bar{\phi}^m, \bar{\lambda}_{\dot{\alpha}}$, without any derivatives. The cohomologies in this sector are strongly believed to be well understood [12]. They are the $\frac{1}{8}$ -BPS chiral rings made of commuting or anticommuting (in the case of two $\bar{\lambda}_{\dot{\alpha}}$'s) letters within the trace, annihilated by Q_+^4 as well as our $Q \equiv Q_-^4$. If they are made of scalars only, they are the primaries of $B\bar{B}$ for the gravitons, because in order for Dynkin labels to add up the scalars must appear symmetrized. If they contain $\bar{\lambda}_{\dot{\alpha}}$'s, they are descendants obtained by acting $\bar{Q}_{m\dot{\alpha}}$'s on the scalar primaries. So the case with $j = 0$ is the graviton multiplets.

The $\text{PSU}(1,2|3)$ multiplet with $j = 1$ for the primaries should belong to A_2 . The $\text{PSU}(1,2|3)$ primaries in this class can contain arbitrary numbers of $\bar{\phi}^m$ and $\bar{\lambda}_{\dot{\alpha}}$ fields, together with only one ψ_{m+} or $D_{+\dot{\alpha}}$ to meet $j = 1$.

With these structures in mind, we can understand some of the setups of section 3. Namely, we considered the core black hole primaries O_n which contain five or more ψ fields in section 3.1, and also considered the possible hairy black hole primaries which are products of gravitons and O_n . These operators have more than two ψ fields, having $j > 1$. For these, it suffices to consider the A_1 multiplets only.

We can also understand why the BPS operators in section 3 belong to $A_1\bar{L}$ -type multiplets, rather than $A_1\bar{B}$, $A_1\bar{A}_2$ or $A_1\bar{A}_1$. First, primaries of $A_1\bar{B}$ -type multiplets of $\text{PSU}(2,2|4)$ satisfy $J_1 = J_2, R_3 + J_2 + 1 = 0$ and $E = 1 + J_1 + R_1 + R_2$. (See table 21 of [37], knowing that their $R_{1,2,3}$ correspond to lowercase $r_{1,2,3}$ in (B.3).) The $\frac{1}{16}$ -BPS primaries obtained by acting Q' on them satisfy $J_1 = J_2, R_3 + J_2 = 0$ and $E = R_1 + R_2 + J_1$. These states are enhanced $\frac{1}{8}$ -BPS states, preserving extra supersymmetry \bar{Q}_{3-} . They are the so-called Schur operators. Second, primaries of $A_1\bar{A}_2$ - or $A_1\bar{A}_1$ -type multiplets satisfy $R_3 + J_2 = 0$. Acting Q' , the $\frac{1}{16}$ -BPS primaries in the multiplets satisfy $R_3 + J_2 = 1$. Thus, for all the $\frac{1}{16}$ -BPS states in these three classes, one finds that $R_3 + J_2 = 0$ or 1 . On the other hand, in section 3, we either considered the $\frac{1}{16}$ -BPS multiplets of O_n or graviton multiplied by O_n . For O_n , $R_3 + J_2 = 4 + 2n \geq 4$ so they cannot belong to these multiplets. Multiplying BPS gravitons to O_n never decreases the value of this charge, since all $\frac{1}{16}$ -BPS operators carry non-negative values of $R_3 + J_2$. This implies that all the possible short multiplets discussed in section 3 should belong to $A_1\bar{L}$.⁹

With the relevance of $A_1\bar{L}$ -type multiplets in our problems better understood, let us now explain their structures. Given an $A_1\bar{L}$ -type representation with primary $[j; \bar{j}]_E^{[r_1, r_2, r_3]}$, we have seen above that the first $\frac{1}{16}$ -BPS state in this representation is $[\bar{j}]_{E+\frac{1}{2}}^{[r_2, r_3]} \subset [j + 1; \bar{j}]_{E+\frac{1}{2}}^{[r_1+1, r_2, r_3]}$. The $A_1\bar{L}$ -type representation is generic enough, in some sense *minimally* $\frac{1}{16}$ -BPS, so that all 9 supercharges $\in \text{SU}(1,2|3)$ can be applied without annihilating. They

⁹This does not mean that the other three short multiplets discussed in this paragraph are irrelevant to the black hole microstates. For instance, the $\frac{1}{8}$ -BPS states in $A_1\bar{B}$ are believed to host exotic black hole like entropy growth, from the study of the Macdonald index [9].

are

$$\begin{aligned} Q_+^m &= [0]_{\frac{1}{2}}^{[1,0]} \quad \text{and } j = 1, \\ \bar{Q}_{m\pm} &= [1]_{\frac{1}{2}}^{[0,1]} \quad \text{and } j = 0. \end{aligned} \tag{B.8}$$

That is, there are $\mathbf{3}$ Q 's that raise j by 1 and E by $\frac{1}{2}$, and $\mathbf{2} \times \mathbf{3}$ \bar{Q} 's that only raise E by $\frac{1}{2}$. Note that during the construction of the descendants, it is convenient to regard all 9 supercharges as anti-commuting operators. Namely, despite some pairs yielding nontrivial anticommutators $\{Q_+^m, \bar{Q}_{n\dot{\alpha}}\} \sim \delta_n^m P_{+\dot{\alpha}}$, we dismiss the right-hand side for a moment since the conformal descendants will be supplemented later. Then each of the 9 supercharges can be applied only once. The states obtained this way forms the conformal primaries. Then, one can act arbitrary numbers of $P_{+\dot{\alpha}} \sim \partial_{+\dot{\alpha}}$ to construct the conformal descendants.

In terms of unrefined index $\text{Tr } t^{2E+j}$ that we often use, the $\mathbf{3}$ Q 's each contribute by a factor of $-t^2$, the $\mathbf{2} \times \mathbf{3}$ \bar{Q} 's by $-t$, and each of two derivatives by t^3 . Therefore, the $\frac{1}{16}$ -BPS character of the representation $A_1 \bar{L}[j; \bar{j}]_E^{[r_1, r_2, r_3]}$ is given by

$$\chi_{BPS} \left[A_1 \bar{L}[j; \bar{j}]_E^{[r_1, r_2, r_3]} \right] = (-1)^{2E+1} d_{\bar{j}} d_{[r_2, r_3]} t^{2E+j+2} \cdot \frac{(1-t)^6 (1-t^2)^3}{(1-t^3)^2}, \tag{B.9}$$

where $d_{\bar{j}} = \bar{j} + 1$ and $d_{[r_2, r_3]} = \frac{1}{2}(r_2 + 1)(r_3 + 1)(r_2 + r_3 + 2)$ multiply to dimension of the first $\frac{1}{16}$ -BPS states $[\bar{j}]_{E+\frac{1}{2}}^{[r_2, r_3]}$. The refined version where p, z_1 and z_2 are used as fugacities for $\text{SU}(2)_R \times \text{SU}(3) \subset \text{PSU}(1, 2|3)$ (see below (3.54) for convention), is

$$\chi_{BPS} \left[A_1 \bar{L}[j; \bar{j}]_E^{[r_1, r_2, r_3]} \right] = (-1)^{2E+1} t^{2E+j+2} \cdot \chi_{(\bar{j}+1, [r_2, r_3])} \cdot \chi_D, \tag{B.10}$$

with the characters defined as (3.55).

C Counting BMN gravitons for SU(2)

In this section, we count graviton-type cohomologies in the BMN sector of the SU(2) theory. In terms of eigenvalues, BPS graviton polynomials are arbitrary products of

$$\begin{aligned} \mathbf{6} &: x^2, y^2, z^2, xy, yz, zx \\ \mathbf{8} &: \psi_1 \cdot (y, z), \psi_2 \cdot (z, x), \psi_3 \cdot (x, y), \psi_1 x - \psi_2 y, \psi_2 y - \psi_3 z \\ \mathbf{3} &: xf - \frac{1}{2}\psi_2\psi_3, yf - \frac{1}{2}\psi_3\psi_1, zf - \frac{1}{2}\psi_1\psi_2. \end{aligned} \tag{C.1}$$

and the goal of this section is to count independent polynomials among them. $\psi_{1,2,3}$ are Grassmann variables while x, y, z, f are bosonic.

In the third line of (C.1), xf, yf, zf are accompanied by two-fermion terms, but for the purpose of counting independent graviton polynomials, these terms can be omitted, as we prove now.

Let \mathfrak{V} be the infinite set of all possible products of $6 + 8 = 14$ polynomials in the first two lines of (C.1). Define two series of vector spaces V_k and \tilde{V}_k as

$$\begin{aligned} V_k &= \text{span} \left\{ \mathbf{v} \times (xf)^a (yf)^b (zf)^c \mid \mathbf{v} \in \mathfrak{V}, a + b + c \leq k \right\}, \\ \tilde{V}_k &= \text{span} \left\{ \mathbf{v} \times \left(xf - \frac{1}{2}\psi_2\psi_3 \right)^a \left(yf - \frac{1}{2}\psi_3\psi_1 \right)^b \left(zf - \frac{1}{2}\psi_1\psi_2 \right)^c \mid \mathbf{v} \in \mathfrak{V}, a + b + c \leq k \right\}. \end{aligned} \tag{C.2}$$

We want to show that rank of V_∞ and rank of \tilde{V}_∞ are equal. We do this by induction. Clearly $\text{rank}(V_0) = \text{rank}(\tilde{V}_0)$. Now, suppose that $\text{rank}(V_{k-1}) = \text{rank}(\tilde{V}_{k-1})$ and let us show that $\text{rank}(V_k) = \text{rank}(\tilde{V}_k)$. The equivalent statement is the following:

- Consider a pair of polynomials

$$v = \sum_{i=1}^n r_i (xf)^{a_i} (yf)^{b_i} (zf)^{c_i},$$

$$\tilde{v} = \sum_{i=1}^n r_i \left(xf - \frac{1}{2}\psi_2\psi_3\right)^{a_i} \left(yf - \frac{1}{2}\psi_3\psi_1\right)^{b_i} \left(zf - \frac{1}{2}\psi_1\psi_2\right)^{c_i},$$

where $r_i \in V_0 = \tilde{V}_0$ and $a_i + b_i + c_i = k$ for all i so that $v \in V_k$ and $\tilde{v} \in \tilde{V}_k$.

Then $v \in V_{k-1}$ if and only if $\tilde{v} \in \tilde{V}_{k-1}$.

The \leftarrow part is easy. If $\tilde{v} \in \tilde{V}_{k-1}$, then \tilde{v} equals a linear combination of polynomials that are at most of degree $k-1$ in $xf - \frac{1}{2}\psi_2\psi_3$ and the likes. Collecting terms with degree k in f , the equality becomes $v = 0 \in V_{k-1}$.

To show the \rightarrow part, first note that $v \in V_{k-1}$ implies $v = 0$, since v is homogeneous in f with degree k . Now,

$$\tilde{v} = -\frac{1}{2} \sum_i^n r_i \left[\begin{aligned} & a_i \psi_2 \psi_3 \left(xf - \frac{1}{2}\psi_2\psi_3\right)^{a_i-1} \left(yf - \frac{1}{2}\psi_3\psi_1\right)^{b_i} \left(zf - \frac{1}{2}\psi_1\psi_2\right)^{c_i} \\ & + b_i \psi_3 \psi_1 \left(xf - \frac{1}{2}\psi_2\psi_3\right)^{a_i} \left(yf - \frac{1}{2}\psi_3\psi_1\right)^{b_i-1} \left(zf - \frac{1}{2}\psi_1\psi_2\right)^{c_i} \\ & + c_i \psi_1 \psi_2 \left(xf - \frac{1}{2}\psi_2\psi_3\right)^{a_i} \left(yf - \frac{1}{2}\psi_3\psi_1\right)^{b_i} \left(zf - \frac{1}{2}\psi_1\psi_2\right)^{c_i-1} \end{aligned} \right]. \quad (\text{C.3})$$

If $r_i \psi_j \psi_{j+1}$ for all i and $j = 1, 2, 3$ all belong to $V_0 = \tilde{V}_0$, it will establish $\tilde{v} = \tilde{V}_{k-1}$. Indeed, if r_i , which is a product of 14 polynomials in the first two lines of (C.1), contains any of the **6** in the first line, this factor can combine with two ψ 's and $r_i \psi_j \psi_{j+1} \in V_0$. For example, $y^2 \psi_2 \psi_3 = (\psi_2 y - \psi_3 z)(\psi_3 y)$. On the other hand, if r_i contains two or more factors of the **8** in the second line, after multiplication by two ψ 's it will vanish due to Grassmannian nature of ψ , so automatically $r_i \psi_j \psi_{j+1} = 0 \in V_0$.

Therefore the only possibility that remains in concern is when r_i is precisely one of the **8**. This leaves only a finite number of exceptions that one can explicitly work out. That is, if $v = 0$ with the eight r_i (they cannot mix with other r_i due to homogeneity) with appropriate numerical coefficients α_i :

$$r_1 = \alpha_1 \psi_1 y, \quad r_2 = \alpha_2 \psi_1 z, \quad \dots, \quad r_8 = \alpha_8 (\psi_2 y - \psi_3 z),$$

it follows that $\tilde{v} = 0$ as well. This completes the proof that $\text{rank}(V_k) = \text{rank}(\tilde{V}_k)$ given $\text{rank}(V_{k-1}) = \text{rank}(\tilde{V}_{k-1})$, and by induction the number of independent products of (C.1) is not affected by the $\psi\psi$ terms in the third line.

With this rule established, we now count the number of independent graviton polynomials in the BMN sector. This task is greatly simplified by the fact that all $6 + 8 + 3 = 17$

but only two single-graviton generators are monomials, because linear independence between monomials is rather transparent. Our strategy will be to order the counting problem carefully so that we can work with the monomial basis as far as possible, and treat the contribution from the two polynomial generators later.

Since there are 3 Grassmann variables, it is convenient to classify the graviton operators into $2^3 = 8$ sectors according to their Grassmannian contents.

0-fermion sector. We first focus on the 0-fermion sector: graviton operators that do not contain any ψ 's. It is clear that such operators are created by multiplying bosonic single-gravitons on the first and third lines of (C.1). Since all of them are monomials, we may simply write down a list of distinct monomials that can be obtained by multiplying bosonic single-gravitons, then their linear independence is guaranteed. The first six single-gravitons can be used to create any monomial $x^a y^b z^c$, where a, b, c are non-negative integers and $a + b + c$ is even. Including xf, yf, zf , an eligible monomial may contain any number of f as long as it is supported by at least as many x, y , or z . Therefore, multi-gravitons in the 0-fermion sector are precisely described as

$$G_0 = \{x^a y^b z^c f^d \mid a, b, c, d \in \mathbb{Z}^{\geq 0}, a + b + c \geq d, a + b + c + d = 0 \pmod{2}\}. \quad (\text{C.4})$$

Because we can attribute to each of x, y, z and f a unit of their own quantum numbers, the partition function for G_0 can be simply defined by the sum over monomials,

$$Z_0(x, y, z, f) = \sum_{g \in G_0} g. \quad (\text{C.5})$$

It can be computed as follows. If there were no restrictions to a, b, c, d except being non-negative integers, the generating function would be $\frac{1}{(1-x)(1-y)(1-z)(1-f)}$. From this, we subtract the sum of monomials for which $d > a + b + c$, which is

$$\frac{1}{(1-xf)(1-yf)(1-zf)} \cdot \frac{f}{1-f}. \quad (\text{C.6})$$

Then we project to the even part under $(x, y, z, f) \rightarrow (-x, -y, -z, -f)$, obtaining

$$\begin{aligned} Z_0 &= \left[\frac{1}{(1-x)(1-y)(1-z)(1-f)} - \frac{1}{(1-xf)(1-yf)(1-zf)} \cdot \frac{f}{1-f} \right]_{\text{even}} \\ &= \left[\frac{1 - f(xy + yz + zx - xyz) + f^2 xyz}{(1-x)(1-y)(1-z)(1-xf)(1-yf)(1-zf)} \right]_{\text{even}} \\ &= \frac{1 + \chi_2 + f(\chi_3 - \chi_1 \chi_2) + f^2(\chi_3^2 + \chi_1 \chi_3)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)}. \end{aligned} \quad (\text{C.7})$$

Abbreviations for cyclic polynomials

$$\begin{aligned} \chi_1 &= x + y + z, \\ \chi_2 &= xy + yz + zx, \\ \chi_3 &= xyz, \end{aligned} \quad (\text{C.8})$$

will be used from now on.

1-fermion sector. Now we list (independent) operators with one fermion, either ψ_1 , ψ_2 or ψ_3 . These are obtained by multiplying any operator in 0-fermion sector G_0 by a generator on the second line of (C.1). As mentioned earlier, the last two of these may create non-monomial operators, so let us first proceed without them.

Operators with one ψ_1 can only be obtained by multiplying operators in G_0 by either $y\psi_1$ or $z\psi_1$. As a result, the list of such operators is simply the following monomials:

$$\{x^a y^b z^c f^d \psi_1 \mid a, b, c, d \in \mathbb{Z}^{\geq 0}, b+c \geq 1, a+b+c-1 \geq d, a+b+c+d = 1 \pmod{2}\}. \quad (\text{C.9})$$

Operators containing one ψ_2 or one ψ_3 can be listed by cyclic permutations of letters.

Next, we ask what new operators arise when multiplying an operator in the 0-fermion sector G_0 by $x\psi_1 - y\psi_2$. If $x\psi_1 - y\psi_2$ multiplies $x^a y^b z^c f^d \in G_0$ such that (i) $c \geq 1$ or (ii) $a \geq 1$ and $b \geq 1$, both monomials $x^{a+1} y^b z^c f^d \psi_1$ and $x^a y^{b+1} z^c f^d \psi_2$ that appear in the product are already counted in (C.9) and corresponding ψ_2 sector respectively. So no new independent operators arise. Therefore, new operators that are obtained using $x\psi_1 - y\psi_2$ are classified as follows:

1. $(x^{a \geq 1} y^0 z^0 f^d) \cdot (x\psi_1 - y\psi_2)$: in this case, the second monomial $x^a y^1 z^0 f^d \psi_2$ is already counted in ψ_2 sector corresponding to (C.9), while the first monomial is not counted in the ψ_1 sector. Therefore, these can be regarded new monomials $x^{a+1} y^0 z^0 f^d \psi_1$ in ψ_1 sector.
2. $(x^0 y^{b \geq 1} z^0 f^d) \cdot (x\psi_1 - y\psi_2)$: in this case, the first monomial $x^1 y^b z^0 f^d \psi_1$ is already counted in ψ_1 sector (C.9), while the second monomial is not counted in the ψ_2 sector. Therefore, these can be regarded new monomials $x^0 y^{b+1} z^0 f^d \psi_2$ in ψ_2 sector.
3. $(1) \cdot (x\psi_1 - y\psi_2)$: in this case, both monomials $x\psi_1$ and $y\psi_2$ have not been counted in respective sectors. Therefore, this cannot be regarded as a new monomial in one of ψ_1 or ψ_2 sector. Instead, this should be understood as an exceptional non-monomial operator.

Similar arguments can be made for multiplication by $y\psi_2 - z\psi_3$.

As a result, the list of monomials in ψ_1 sector is now extended to

$$G_{\psi_1} = \{x^a y^b z^c f^d \psi_1 \mid a, b, c, d \in \mathbb{Z}^{\geq 0}, a+b+c-1 \geq d, a+b+c+d = 1 \pmod{2}\} \setminus \{x\psi_1\}. \quad (\text{C.10})$$

List of monomials G_{ψ_2} in ψ_2 sector and G_{ψ_3} in ψ_3 sector are defined by cyclicity. In addition, there are two exceptional operators $x\psi_1 - y\psi_2$ and $y\psi_2 - z\psi_3$ that are not monomials and do not belong to any of G_{ψ_m} . So the whole set G_1 of 1-fermion BPS gravitons is given by

$$G_1 = G_{\psi_1} \cup G_{\psi_2} \cup G_{\psi_3} \cup \{x\psi_1 - y\psi_2, y\psi_2 - z\psi_3\}. \quad (\text{C.11})$$

Alternatively, one can take G_{ψ_1} to *not* exclude $x\psi_1$, similarly G_{ψ_2} and G_{ψ_3} to *not* exclude $y\psi_2$ and $z\psi_3$ respectively, but instead exclude just $x\psi_1 + y\psi_2 + z\psi_3$ at the end.

The existence of such non-monomial operators forbids us from attributing individual quantum numbers to ψ 's. Instead, they carry a negative unit of respective scalar quantum

numbers, and a positive unit of overall ψ -number:

$$x \rightarrow [x], y \rightarrow [y], z \rightarrow [z], f \rightarrow [f], \psi_1 \rightarrow \frac{[\psi]}{[x]}, \psi_2 \rightarrow \frac{[\psi]}{[y]}, \psi_3 \rightarrow \frac{[\psi]}{[z]}. \quad (\text{C.12})$$

The partition function of the 1-fermion sector is given by a function of x, y, z, f and ψ . The partition function in ψ_1 sector (and of the rest of the 1-fermion sector) can be computed analogously to the 0-fermion sector. Starting from $\frac{1}{(1-x)(1-y)(1-z)(1-f)} \cdot \frac{\psi}{x}$, we implement the restriction $a + b + c - 1 \geq d$ by subtracting its complement, extract the odd part under $(x, y, z, f) \rightarrow (-x, -y, -z, -f)$, and further subtract $x\psi_1 \rightarrow \psi$.

$$\begin{aligned} Z_{\psi_1} &= \left[\frac{1}{(1-x)(1-y)(1-z)(1-f)} - \frac{1}{(1-xf)(1-yf)(1-zf)} \cdot \frac{1}{1-f} \right]_{\text{odd}} \cdot \frac{\psi}{x} - \psi \\ &= \left[\frac{x+y+z - (xy+yz+zx)(1+f) + xyz(1+f+f^2)}{(1-x)(1-y)(1-z)(1-xf)(1-yf)(1-zf)} \right]_{\text{odd}} \cdot \frac{\psi}{x} - \psi \\ &= \frac{\chi_1 + \chi_3 - f(\chi_2 + \chi_2^2 - \chi_1\chi_3 - \chi_3^2) + f^2\chi_3(1 + \chi_2)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)} \cdot \frac{\psi}{x} - \psi. \end{aligned} \quad (\text{C.13})$$

Note that Z_{ψ_2} and Z_{ψ_3} can be computed similarly. Further including $x\psi_1 - y\psi_2, y\psi_2 - z\psi_3$, one obtains the following partition function for G_1 :

$$Z_1 = \frac{\chi_1 + \chi_3 - f(\chi_2 + \chi_2^2 - \chi_1\chi_3 - \chi_3^2) + f^2\chi_3(1 + \chi_2)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)} \cdot \frac{\chi_2}{\chi_3} \cdot \psi - \psi. \quad (\text{C.14})$$

2-fermion sector. We consider operators that contain two of three ψ 's. These are obtained by multiplying a generator on the second line of (C.1) to an operator in G_1 . Focusing on the $\psi_1\psi_2$ sector, we first note there are three ways to obtain an operator in this sector.

1. Multiply either $x\psi_2$ or $z\psi_2$ to an operator in G_{ψ_1} (C.10). Such a set of operators are $\{x^a y^b z^c f^d \psi_1 \psi_2 \mid a, b, c, d \in \mathbb{Z}^{\geq 0}, a+c \geq 1, a+b+c-2 \geq d, a+b+c+d=0 \pmod{2}\} \setminus \{x^2 \psi_1 \psi_2\}$. (C.15)

2. Multiply either $y\psi_1$ or $z\psi_1$ to an operator in G_{ψ_2} , analogous to (C.10): $\{x^a y^b z^c f^d \psi_1 \psi_2 \mid a, b, c, d \in \mathbb{Z}^{\geq 0}, b+c \geq 1, a+b+c-2 \geq d, a+b+c+d=0 \pmod{2}\} \setminus \{y^2 \psi_1 \psi_2\}$. (C.16)

3. Multiply $x\psi_2, z\psi_2, y\psi_1$ or $z\psi_1$ to $x\psi_1 - y\psi_2$. These supplement $x^2\psi_1\psi_2$ and $y^2\psi_1\psi_2$ excluded in (C.15) and (C.16).

Taking the union of the three sets above, we arrive at

$$G_{\psi_1\psi_2} = \{x^a y^b z^c f^d \psi_1 \psi_2 \mid a, b, c, d \in \mathbb{Z}^{\geq 0}, a+b+c-2 \geq d, a+b+c+d=0 \pmod{2}\}, \quad (\text{C.17})$$

and similarly for $\psi_2\psi_3$ and $\psi_3\psi_1$ sectors.

Note that we have not explicitly considered multiplying, for example, $x\psi_3$ or $y\psi_3$ to $x\psi_1 - y\psi_2$. Both monomials obtained this way are already included in $G_{\psi_3\psi_1}$ and $G_{\psi_2\psi_3}$,

so they do not add any new independent operators. Furthermore, there is a possibility of multiplying $x\psi_1 - y\psi_2$ or $y\psi_2 - z\psi_3$ to the operators in the 1-fermion sector. These may give rise to

$$\begin{aligned} (x\psi_1 - y\psi_2)(x\psi_1 - y\psi_2) &\sim xy\psi_1\psi_2, \\ (y\psi_2 - z\psi_3)(y\psi_2 - z\psi_3) &\sim yz\psi_2\psi_3, \\ (x\psi_1 - y\psi_2)(y\psi_2 - z\psi_3) &\sim xy\psi_1\psi_2 + yz\psi_2\psi_3 + zx\psi_3\psi_1, \end{aligned} \tag{C.18}$$

but again, all of the monomials are already counted in respective 2-fermion sectors. Therefore, we conclude that the 2-fermion sectors can be written completely in monomial basis, by (C.17) and its cyclic versions:

$$G_2 = G_{\psi_1\psi_2} \cup G_{\psi_2\psi_3} \cup G_{\psi_3\psi_1}. \tag{C.19}$$

The partition function of 2-fermion sector can be computed as before. The result is:

$$Z_{\psi_1\psi_2} = \frac{\chi_1^2 - \chi_2 - \chi_2^2 + 2\chi_1\chi_3 + \chi_3^2 + f(\chi_3 - \chi_1\chi_2) + f^2\chi_3(\chi_1 + \chi_3)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)} \cdot \frac{\psi^2}{xy}, \tag{C.20}$$

for the individual sector, and

$$\begin{aligned} Z_2 &= Z_{\psi_1\psi_2} + Z_{\psi_2\psi_3} + Z_{\psi_3\psi_1} \\ &= \frac{\chi_1^2 - \chi_2 - \chi_2^2 + 2\chi_1\chi_3 + \chi_3^2 + f(\chi_3 - \chi_1\chi_2) + f^2\chi_3(\chi_1 + \chi_3)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)} \cdot \frac{\chi_1}{\chi_3} \cdot \psi^2, \end{aligned} \tag{C.21}$$

for the entire 2-fermion sector.

3-fermion sector. We finally investigate the 3-fermion sector, i.e. operators that contain all ψ_1, ψ_2 and ψ_3 . One way to obtain 3-fermion operators is to multiply $x\psi_3$ or $y\psi_3$ to the $\psi_1\psi_2$ -sector (C.17). Set of such operators is

$$\{x^a y^b z^c f^d \psi_1 \psi_2 \psi_3 \mid a, b, c, d \in \mathbb{Z}^{\geq 0}, a + b \geq 1, a + b + c - 3 \geq d, a + b + c + d = 1 \pmod{2}\}. \tag{C.22}$$

By cyclicity, there are two more sets of 3-fermion operators that are obtained by $x \rightarrow y \rightarrow z \rightarrow x$ from (C.22). Their union is,

$$G_{\psi_1\psi_2\psi_3} = \{x^a y^b z^c f^d \psi_1 \psi_2 \psi_3 \mid a, b, c, d \in \mathbb{Z}^{\geq 0}, a + b + c - 3 \geq d, a + b + c + d = 1 \pmod{2}\}. \tag{C.23}$$

One can easily check that multiplying non-monomial blocks $x\psi_1 - y\psi_2$ or $y\psi_2 - z\psi_3$ to 2-fermion sector does not produce any new operator.

Partition function of the 3-fermion sector (C.22) is

$$Z_3 = \left[\frac{-1 + \chi_1^2 - 2\chi_2 - \chi_2^2 + 2\chi_1\chi_3 + \chi_3^2 + f(\chi_1 + \chi_3) - f^2(\chi_2 + \chi_2^2 - \chi_1\chi_3 - \chi_3^2) + f^3\chi_3(1 + \chi_2)}{(1-x^2)(1-y^2)(1-z^2)(1-xf)(1-yf)(1-zf)} + 1 \right] \frac{\psi^3}{f\chi_3}. \tag{C.24}$$

The index. The complete list of BPS multi-graviton operators in BMN sector of the SU(2) theory is given by (C.4), (C.11), (C.19) and (C.23). Corresponding partition function is $Z_0 + Z_1 + Z_2 + Z_3$, each of which is presented in (C.7), (C.14), (C.21) and (C.24). Attributing minus sign to the fermion number ψ in the partition function and further setting $\psi, f \rightarrow xyz$ will yield the index, where $(x, y, z) = (e^{-\Delta_1}, e^{-\Delta_2}, e^{-\Delta_3})$.

To facilitate comparison with the other parts of this paper, we compute the unrefined index of the graviton partition function. This is obtained simply by substituting

$$x, y, z \rightarrow t^2, \quad f \rightarrow t^6, \quad \psi \rightarrow -t^6. \quad (\text{C.25})$$

into the partition function. The result is

$$Z_{\text{grav}} = \frac{1+3t^4-8t^6-6t^{10}+10t^{12}+9t^{14}-9t^{16}+16t^{18}-18t^{20}-3t^{22}+t^{24}-3t^{26}+9t^{28}-2t^{30}+3t^{32}-3t^{34}}{(1-t^4)^3(1-t^8)^3}. \quad (\text{C.26})$$

The difference $Z - Z_{\text{grav}}$ will be the index that counts non-graviton operators. We find a simple analytic formula for the difference:

$$Z - Z_{\text{grav}} = -\frac{t^{24}}{1-t^{12}} \cdot \frac{(1-t^2)^3}{(1-t^8)^3}. \quad (\text{C.27})$$

The fully refined index is given by (1.2).

D Q-exactness of some graviton hairs

In this appendix, we report the Q-exactness of some product operators.

Six operators $O_0(\bar{\phi}^{(m)} \cdot \bar{\phi}^{(n)})$ at t^{28} order are all Q-exact. An SU(3) covariant expression is

$$\begin{aligned} O_0(\bar{\phi}^{(m)} \cdot \bar{\phi}^{(n)}) = & -\frac{1}{14} Q [20\epsilon^{rs(m)}(\bar{\phi}^{(n)} \cdot \psi_{p+})(\bar{\phi}^{(p)} \cdot \psi_{r+})(\bar{\phi}^{(q)} \cdot \psi_{q+})(f_{++} \cdot \psi_{s+}) \\ & -20\epsilon^{prs}(\bar{\phi}^{(m)} \cdot \psi_{p+})(\bar{\phi}^{(n)} \cdot \psi_{r+})(\bar{\phi}^{(q)} \cdot \psi_{q+})(f_{++} \cdot \psi_{s+}) \\ & +30\epsilon^{prs}(\bar{\phi}^{(m)} \cdot \psi_{p+})(\bar{\phi}^{(n)} \cdot \psi_{r+})(\bar{\phi}^{(q)} \cdot \psi_{s+})(f_{++} \cdot \psi_{q+}) \\ & -7\epsilon^{a_1 a_2 p} \epsilon^{b_1 b_2} (m) (\bar{\phi}^{(n)} \cdot \psi_{p+})(\bar{\phi}^{(q)} \cdot \psi_{q+})(\psi_{a_1+} \cdot \psi_{a_2+})(\psi_{b_1+} \cdot \psi_{b_2+}) \\ & +18\epsilon^{a_1 a_2 p} \epsilon^{b_1 b_2} (m) (\bar{\phi}^{(n)} \cdot \psi_{q+})(\bar{\phi}^{(q)} \cdot \psi_{p+})(\psi_{a_1+} \cdot \psi_{a_2+})(\psi_{b_1+} \cdot \psi_{b_2+})] . \end{aligned} \quad (\text{D.1})$$

Six operators $O_0(\bar{\phi}^m \cdot \bar{\lambda}_{\dot{\alpha}})$ at t^{29} order are also all Q-exact. An $SU(2)_R \times SU(3)$ covariant expression is

$$\begin{aligned} O_0(\bar{\phi}^m \cdot \bar{\lambda}_{\dot{\alpha}}) = & \frac{1}{8} Q [40\epsilon^{mnp}(f_{++} \cdot \psi_{q+})(\bar{\lambda}_{\dot{\alpha}} \cdot \psi_{r+})(\bar{\phi}^q \cdot \psi_{n+})(\bar{\phi}^r \cdot \psi_{p+}) \\ & -4\epsilon^{ma_1 a_2} \epsilon^{nb_1 b_2} (\bar{\lambda}_{\dot{\alpha}} \cdot \psi_{n+})(\bar{\phi}^p \cdot \psi_{p+})(\psi_{a_1+} \cdot \psi_{a_2+})(\psi_{b_1+} \cdot \psi_{b_2+}) \\ & +6\epsilon^{ma_1 a_2} \epsilon^{nb_1 b_2} (\bar{\lambda}_{\dot{\alpha}} \cdot \psi_{p+})(\bar{\phi}^p \cdot \psi_{n+})(\psi_{a_1+} \cdot \psi_{a_2+})(\psi_{b_1+} \cdot \psi_{b_2+}) \\ & +\epsilon^{na_1 a_2} \epsilon^{pb_1 b_2} (\bar{\lambda}_{\dot{\alpha}} \cdot \psi_{n+})(\bar{\phi}^m \cdot \psi_{p+})(\psi_{a_1+} \cdot \psi_{a_2+})(\psi_{b_1+} \cdot \psi_{b_2+})] . \end{aligned} \quad (\text{D.2})$$

Eight operators $O_0 \left(\bar{\phi}^m \cdot \psi_{n+} - \frac{1}{3} \delta_n^m \bar{\phi}^p \cdot \psi_{p+} \right)$ at t^{30} order are all Q -exact. An $SU(3)$ covariant expression is

$$\begin{aligned}
 & O_0 \left(\bar{\phi}^m \cdot \psi_{n+} - \frac{1}{3} \delta_n^m \bar{\phi}^p \cdot \psi_{p+} \right) \\
 &= \frac{1}{4} Q \left[\epsilon_{npq} \epsilon^{ra_1 a_2} \epsilon^{qb_1 b_2} \epsilon^{mc_1 c_2} (\bar{\phi}^p \cdot \psi_{r+}) (\psi_{a_1+} \cdot \psi_{a_2+}) (\psi_{b_1+} \cdot \psi_{b_2+}) (\psi_{c_1+} \cdot \psi_{c_2+}) \right].
 \end{aligned}
 \tag{D.3}$$

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