More proofs for “Determination of Stability with respect to Positive Orthant for a Class of Positive Nonlinear Systems”

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Abstract: In the published paper “Determination of Stability with respect to Positive Orthant for a Class of Positive Nonlinear Systems,” IEEE Trans. on Automatic Control, vol. 53, no. 5, pp. 1329–1334, 2008, by the authors, some proofs are omitted due to the space limitation of the journal. In this note, we present those omitted proofs.

Proof of Claim 1: Define
\[ \bar{T} := \begin{bmatrix} \frac{1}{A_{22}}, A_{21} & 0 \\ 0 & I \end{bmatrix}, \]
then \( \bar{T}^{-1} = \begin{bmatrix} \frac{1}{A_{22}}, A_{21} & 0 \\ 0 & I \end{bmatrix}. \)
With \( \bar{T}^{-1} \) and (10), it is verified that
\[ (E_k A(0) E_k^{-1}) \bar{T}^{-1} = \bar{T}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{A_{22}} \end{bmatrix}. \]
With \( T_0 := \bar{T} E_k \), it follows that
\[ T_0 A(0) T_0^{-1} = (\bar{T} E_k) A(0) (\bar{T} E_k)^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{A_{22}} \end{bmatrix}. \]
Also, it holds that \( (T_0(1)) = (T E_k(1)) = e_2^T \) because the first and the \( k \)-th columns of \( T \) are exchanged in \( T E_k \). In addition, since \( E_k^{-1} = E_k \), we obtain that
\[ (T_0^{-1}(1)) = (E_k^{-1} \bar{T}^{-1}(1)) = E_k \left( -\frac{1}{A_{22}}, A_{21} \right). \]
Let \( [\bar{T}_1, \bar{T}_2] := T_0^{-1} \) where \( \bar{T}_1 \in \mathbb{R}^{n \times 1} \) and \( \bar{T}_2 \in \mathbb{R}^{n \times (n-1)} \), and let \( A_1(u) \in \mathbb{R} \) and \( t_1(u) \in \mathbb{R}^{n \times (n-1)} \) be \( C^2 \) functions of \( u \) such that \( A_1(0) = 0 \), \( t_1(0) = 0 \), and
\[ A(u) t_1(u) = A_1(u) t_1(u), \]
for each \( u \in [0, \bar{u}] \). (Then, \( A_1(0) \) and \( t_1(u) \) are an eigenvalue and the corresponding eigenvector of \( A(u) \), respectively.)

Let \( \mathcal{T}(u) := [t_1(u), \bar{T}_1] \). Then, because \( T(0) = T_0^{-1} \), there exists a positive \( \bar{u}_1 \leq \bar{u} \) such that \( T(u) \) is nonsingular for \( u \in [0, \bar{u}_1] \). Now let \( D(u) := T^{-1}(u) A(u) T(u) \). Since \( T^{-1}(u) A(u) T(u) = T^{-1}(0) t_1(0), \bar{T}_1 = I \), we have \( T^{-1}(u) t_1(u) = e_1 \), which leads to
\[ D(u) = [A_1(u) T^{-1}(u) t_1(u), A(u) T^{-1}(u) A(u) t_1(u) T(u) \bar{T}_1] \]
for some \( A_1(u) \times \bar{u}_1 \) and \( A(u) \times [0, \bar{u}_1] \rightarrow \mathbb{R}^{(n-1) \times (n-1)} \). It follows that
\[ D(0) = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{A_{22}} \end{bmatrix}, \]
which implies \( A_1(0) = [0, 0, \ldots, 0] \). Since \( A_1(0) = 0 \) and \( A_2(u) \) is Hurwitz for each \( u \in [0, \bar{u}_1] \), there exists a \( \bar{u}_1 \leq \bar{u} \) such that \( A_1(u) \) and \( -A_2(u) \) have distinct eigenvalues for \( u \in [0, \bar{u}] \). As a result, there exists a unique (continuous) solution \( S(u) : [0, \bar{u}] \rightarrow \mathbb{R}^{(n-1) \times (n-1)} \) to the Sylvester equation
\[ A_1(u) S(u) - S(u) A_2(u) = -A_2(u). \]

Here, it is also seen that \( S(0) = [0, 0, \ldots, 0] \). Let
\[ t_2(u) := \mathcal{T}(u) \begin{bmatrix} S(u) \\ I \end{bmatrix}. \]
Then, it follows that \( t_2(0) = \bar{T}_2 \). Therefore, we have \( [t_1(0), t_2(0)] = [\bar{T}_1, \bar{T}_2] = T_0^{-1} \). This implies that there exists an \( \bar{u}_1 \leq \bar{u}_1 \) such that \( [t_1(u), t_2(u)] \) is nonsingular for \( u \in [0, \bar{u}_1] \).

We finally define
\[ T(u) := [t_1(u), t_2(u)]^{-1} \]
for \( u \in [0, \bar{u}_1] \), and obtain from (32) and (33) that
\[ D(u) [S(u) I] = [S(u) I] A_2(u). \]

By virtue of (34), this leads to \( D(u) T^{-1}(u) t_2(u) = T^{-1}(u) t_2(u) A_2(u), \) which implies that \( A(u) t_2(u) = t_2(u) A_2(u). \)

(31)

With (31), we have
\[ A(u) [t_1(u), t_2(u)] = [t_1(u), t_2(u)] \begin{bmatrix} A_1(u) & 0 \\ 0 & A_2(u) \end{bmatrix}, \]
which, together with (35), completes the proof.

Proof of Claim 2: We begin with the idea taken from [3], where the existence of center manifold for a parameter-dependent system is proved. System (18) is regarded as
\[ \hat{z}_1 = g_1(z_1, z_2, u) \]
\[ \hat{z}_2 = \bar{A}_2 z_2 + g_2(z_1, z_2, u) \]
\[ u = 0. \]

Then the \((z_1, u)\)-dynamics has its Jacobian matrix at the origin whose eigenvalues have zero real parts. Therefore, the standard existence proof of center manifold gives a function \( (z_1, u) \rightarrow \tau(z_1, u) \) such that \( \tau(0, 0) = 0 \), \( \frac{\partial \tau}{\partial z_1}(0, 0) \), and (21) holds for all \( 0 \leq u \leq \bar{u}_2 \) and \( |z_1| \leq \bar{r}_1 \) with some \( \bar{r}_1 > 0 \) and \( 0 < \bar{u}_2 \leq \bar{u}_1 \). However, to complete the proof, we have to show that \( \tau(0, u) = 0 \) for \( 0 \leq u \leq \bar{u}_2 \). This is obtained by a slight modification of the standard proof. Instead of repeating the whole proof here, we refer to the proof of [6, Lemma C.6] with the following key ingredient.

Let \( S \) be the set of functions \( \tau(\cdot, \cdot) \) such that (1) it is continuously differentiable, (2) \( \tau(0, u) = 0 \), \( \frac{\partial \tau}{\partial u}(0, 0) = 0 \), (3) \( |\tau(z_1, u)| \leq c_1, (4) \frac{\partial \tau}{\partial z_1,u}(z_1, u) \leq c_2 \), and (5)
\[ |\frac{\partial \tau}{\partial z_1}(z_1, u) - \frac{\partial \tau}{\partial z_1}(z_1, u)| \leq c_3 |z_1 - (z_1, u)| \] with some positive constants \( c_1, c_2 \) and \( c_3 \). Now define a map on the set \( S \) as
\[ (P\tau)(z_1, u) := \int_{-\infty}^{0} \exp(-\bar{A}_2 s) \times g_2(\phi(s; (z_1, u), \pi), \pi(\phi(s; (z_1, u), \pi), u), ds, \]
where \( \phi(s; (\xi, u), \pi) \) is the solution\(^2\) of
\[ \dot{z}_1 = g_1(z_1, \pi(z_1, u), u), \quad z_1(0) = \xi. \]

Then, the existence of the function \( \tau \) of our purpose is proved by the contraction mapping theorem if we prove that those properties (1)–(5) also hold for the function \( (z_1, u) \rightarrow (P\tau)(z_1, u) \), i.e., the set \( S \) is mapped into \( S \) by (37), and that the map (37) is contracting so that it has a fixed point \( \tau \) (i.e., \( \tau = (P\tau) \)). In fact, the contracting property and the properties (1), (3), (4) and (5) are easily proved just as in [6]. In order to prove (2), that is, \( (P\tau)(0, u) = 0 \), we note that \( \phi(s; (0, u), \pi) = 0 \) for \( s \geq 0 \), that is, \( z_1 = 0 \) is an equilibrium of

\(^2\)In the original proof, the function \( g_1 \) and \( g_2 \) are modified to have zero values outside a local neighborhood so that the solution exists for all positive time and the integral (37) is well-defined. We do not get into details here, but assume just that \( g_1 \) and \( g_2 \) have such nice properties.
(38) since \( g_1(0,0,u) = 0 \) and \( \pi \in S \) so that \( \pi(0,u) = 0 \). Then, it is obvious that \( (P\pi)(0,u) = 0 \). 

Proof of Theorem 1: Since system (3) can be viewed as system (7) with \( u = 0 \), it directly follows from Theorem 2 that \( x^* \) is locally asymptotically stable w.r.t. \( \mathbb{R}^n_+ \) when \( \frac{\partial^2 \psi}{\partial s^2}(0) < 0 \).

In order to prove the unstability, note that system (22) with \( u = 0 \) (that is equivalent to (3)) is rewritten as

\[
\begin{align*}
\dot{z}_1 &= c_1(0)z_1^2 + o(z_1^2) + N_1(z_1, w, 0) \\
\dot{w} &= \bar{A}_{22}w + N_2(z_1, w, 0)
\end{align*}
\]

(39)
in which the origin corresponds to the equilibrium \( x^* \) of (3), and \( c_1(0) > 0 \) from the proof of Claim 3. Then, there exists a \( \tilde{z}_1 > 0 \) such that the solution of (39) initiated from \( (\tilde{z}_1, 0) \) diverges from the origin, which implies that \( x^* \) is unstable w.r.t. \( \mathbb{R}^n_+ \). 

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