A MODEL FOR OPTIMUM CASH BALANCE

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Ⅰ. INTRODUCTION

If the cash inflow exactly equaled the cash outflow at all times, there would be no necessity to hold cash for transactions purposes. However, the two are usually not perfectly synchronized. If the inflow lags the outflow over part of a period, the firm must carry funds to cover the deficiency period... transaction demand. So we need minimum desired cash balances.

Firms generally have minimum desired cash balances. There are various factors that influence cash holdings. As is the case with optimum inventory level, firms try to keep optimum cash balances. Accordingly, various lot-size models for the management of inventory have been applied to the management of cash. The purpose of this paper is to examine some of these models and to develop a new optimum model to the cash management problem.

Ⅱ. THE BAUMOL MODEL

The classic article on cash management by William J. Baumol applies the EOQ model to the cash management problem. Although Baumol's article emphasized the macroeconomic implications for monetary theory, he recognized the implications for business finance and set the stage for further work in

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this area, In essence, Baumol recognized the fundamental similarities of inventories and cash from a financial viewpoint.

In the case of inventories, there are ordering and stockout costs that make it expensive to keep inventories at a zero level by placing orders for immediate requirements only. But there are also costs involved with holding inventories, and an optimal policy balances off the opposing costs of ordering and holding inventory.

With cash and securities the situation is very similar. There are order costs in the form of clerical work and brokerage fees when making transfers between the cash account and investment portfolio. On the other side of the coin, there are holding costs consisting of interest foregone when large cash balances are held to avoid the costs of making transfers. Further, there are also costs associated with running out of cash, just as there are in the case of inventories. As with inventories, there is an optimal cash balance that minimizes these costs.

Let M=the total cash to be paid out over a period of time.

i=an opportunity cost (an interest rate) per period caused by borrowing cash or by withdrawing cash from investment,

Q=the optimal borrowing or withdrowal lot size

b=a constant transfer cost (fixed borrowing cost: brokers' fees).

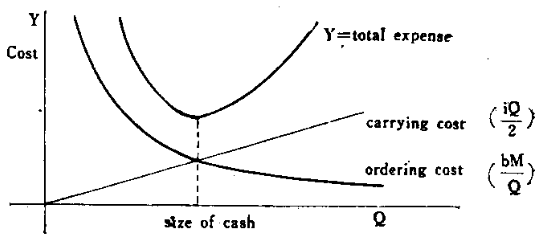


Figure 1. cost function of the Baumol model

As cash amount becomes larger, handling expenses lower the ordering costs, as shown in Figure 1. Thus, the total ordering cost per period $\left(\frac{bM}{Q}\right)$ becomes smaller as Q becomes greather. Conversely, as the cash amount (order size) increases, the carrying costs (opportunity cost) increase. Total carrying costs are given by multiplying the carrying cost per unit (i) times the average cash held $\left(\frac{Q}{2}\right)$.

Therefore, the total cost for transactions demand during the period (Y) is;

$$Y=\frac{bM}{Q}+\frac{iQ}{2}$$

Where the first term repesents total fixed costs associated with $\frac{M}{Q}$ borrowings or withdrawals spaced evenly over the period, and the second term is the opportunity cost of maintaining an average cash balance of $\frac{Q}{2}$. The purpose of the model is to determine the optimal borrowing or withdrawal lot size, Q. If we assume that the optimality criterion is mimimum cost per unit of time, the optimal value for Q is found by differentiating this function with respect to Q and setting the derivative equal to zero;

(1) $Y=\frac{bM}{Q}+\frac{iQ}{2}$

(2) $\frac{dY}{dQ}=-\frac{bM}{Q^2}+\frac{i}{2}$

(3) Set $\frac{dY}{dQ}=0$; $-\frac{bM}{Q^2}+\frac{i}{2}=0$

(4) $\frac{i}{2}=\frac{bM}{Q^2}$

(5) $Q^2i=2bM$

(6) $Q=\sqrt{\frac{2bM}{i}}$

Cash will be demanded in relation to the square root of the dollar volume of transactions. The higher the fixed cost of transfer, b, the higher the optimal borrowing or withdrawal size, Q, all other things held constant. The higher the opportunity cost of funds, i, however, the lower the cash balance that is desirable and the lower the optimal borrowing or withdrawal size, Q, all other things held constant.

This is the well-known Baumol model. This model is so similar to inventory model: inflows are represented by "the orders" in inventory model; they come principally from receipts, borrowing and sale of securities, Out flows are represented by the inventory usage line. The primary carrying cost of cash is the opportunity cost of having funds tied up in nonearning assets, the principal ordering costs are brokerage costs assoicated with borrowing funds or converting marketable securities into cash.

The Baumol model assumes that a firm's cash balances behave, over time, in a saw-tooth manner, as shown in Figure 2. Receipts come in at periodic intervals, such as time 0, 1, 2, 3, and so forth; expenditure occur continuously throughout the periods.[2)]

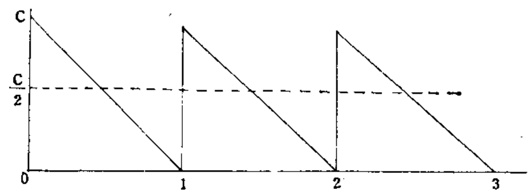


Figure 2. Baumol's pattern of Receipts & Expenditures

Here one point must be crystal clear. An optimum cash balance model to be valid must also predict the optimum cash balance under a net positive cash flow where cash must be invested as well as the more usual condition of model assumptions where a net negative cash flow requires borrowing or selling of investments to maintain optimum cash balance.[3] So, in the Baumol model, the M should be the net cash flow for the period of time, either positive or negative, and not as "total cash to be paid out" as defined in the model.

If one makes the impractical assumption of a common b and i for all sources of funds, from receipts, borrowing, sales or purchase of securities, the Baumol model for determining the optimum cash transfer size from or to cash balances under any given situation looks great especially if the flows of funds are as assumed. The major weakness of this model, however, is to ignore the precautionary and speculative motives for holding cash balances. The lack of attention Baumol gives to precautionary demands leaves the system incomplete.

Ⅲ. THE OTHER MODELS

The Beranek,[4] White and Norman Model[5]

Beranek, White and Norman, for the compensation of the weakness of the Baumol Model, developed their model given the probability distribution of net cash flow. Especially, the decision variable in Beranek's model is the allocation of funds between cash and investments at the begining of the period. Withdrawals from investment are assumed possible only at the end of each planning period. Thus, in Beranek's model, the financial manager is regarded as having total resources of K dollars available at the begining of a planning period. He expects his net cash drain (receipts less disbursements) at the end of the period to be Y dollars (either positive or negative), with a probability distribution. His objective of maximizing returns by investment in securities is constrained by transaction costs and the risk of being short of cash when funds are needed for expenditures. Beranek develops a cost function and differentiates it to find the optimal cash balance.

He states that additional reserves may be held by the firm for precautionary purposes but doesn't deal with this analysis. He just insists that under risk, the net cash drain for transactions purposes for the period form a probability distrbiution, as shown in Figure 3.

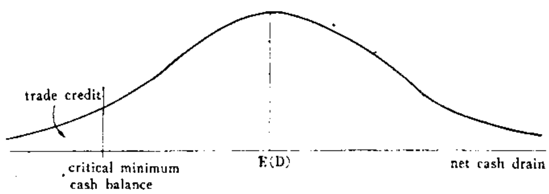


Figure 3. Beranek's probability distribution of net cash flows

He assumes some critical minimum cash balance which is never expected to be violated, If it is assumed to be absorbed by postponement of trade payments; this assumption may be acceptable in many cases but many managements would not rely upon this as an "escape valve" for excessive cash drains. Thus, this model ignores the alternative of liquidating investments to meet cash need.

The White and Norman Model, analagous to Beranek's short-cost function, also ignores transactions costs or implicitly considers them as the net rate of return on investments. In the Beranek and Norman-White versions, information must be fed into the model and a decision derived each time a transfer between cash and securities is being considered. This must be counted as a disadvantage of these models.

The Archer Model

Giving critical emphasis to the precautionary and speculative motive, Archer developed cash balance computations based on means and standard deviations arrived at by an empirical study of past cash requirements.[6] By empirical analysis of cash needs day by day over a period of years, he determined an individual firm's expected variability in net cash flow for transaction purposes. However. he never developed a sophisticated formula for quantifying the optimal cash balances, even if he states that added to the daily amount needed for transaction balances is the precautionary balance which is found by selecting the risk management is willing to assume of a stockout (running out of cash).

As a matter of fact, most financial managers would get emergency funds from banks through their line of credit arrangements at very nearly the prime or going rate of interest. Therefore, the making provisions for precautionary cash balances is not necessarily required to them. Besides, his cash balance computations based on means and standard deviations arrived at by an empirical study of past cash requirements will not work in the area of the dynamic and rapidly growing firm.

The Mckernie-Belt Model

Mckernic-Belt suggested[7], giving critical emphasis to the good will balance required by the banks for minimum desired cash balance

$$C=B+T_{t+1}$$

where B=the compensating balance or minimum deposit required by individual bank policy in order for the company to maintain the proper relations and good will with its banks.

T=the total amount of cash required to meet the cash needs of the next immediate time period, $t+1$.

C=the right amount of cash for a firm to hold at any given period of time.

He indicated that this cash need is found by frequent and constant prjections of future cash needs.

This model doesn't give any fruther analysis for precautionary, speculative motive, carrying costs, and the like, and loses much of the sophistication of the more theoretical models for quantifying the optimal cash balances. One thing should be added to B in the equation which is attributed to obtaining bank goodwill by maintaining some minimum cash balance on deposit. The definition must be expanded to include, not just the bank, but all creditors including trade accounts. Most creditors, not just banks, are more willing to loan money if the prospective borrower has a healthy cash balance from a practical standpoint

The Miller-Orr Model

The Miller- Orr Model assuming that the net cash flows behave as if they were completely stochastic and random, is designed to determine the time and size of transfers between an investment account and the cash account according to a decision process illustrated in Figure 4.[8] Changes in cash balances are allowed to wander until they reach some level h at time t_1; they are then reduced to level z, the return point, by investing h--z dodollars in the investment portfolio. Again the cash balance wanders aimlessly until it reaches the minimum balance point, r, at t_2, at which time enough earning assets are sold to return the cash balance to its return point,z. The model is based on a cost function similar to Baumol 's, and it includes elements for the cost of making transfers to and from cash and for the opportunity cost of holding cash.

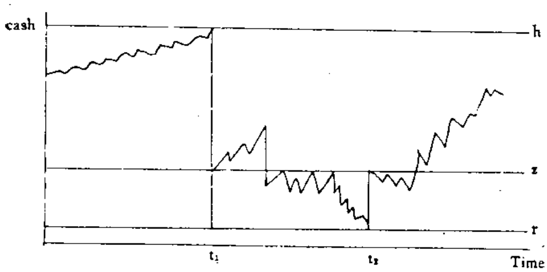


Figure 4 The Miller-Orr Cash Management Model

The cost function for the Miller-Orr model can be stated as

$$E(C)=bE(N)T/+iE(M),$$

where E(N) = the expected number of transfers between cash and the investment portfolio during the planning period;

b = the cost per transfer;

T = the number of days in the planning period;

E(M) = the expected average daily balance;

i = the daily rate of interest earned on the investments.

The objective is to minimize E(C) by choice of the variables h and z, the upper control limit and the return point, respectively.

The solution as derived by Miller and Orr becomes

$$Z=\left(\frac{3b\sigma^2}{4i}\right)^{1/3}$$

$$h=3z$$

for the special case where p (the probability that cash balances will increase) equals .5, and q (the probability that cash balances will decrese) equals .5. The variance of daily changes in the cash balance is represented by σ^2. As would be expected, a higher transfer cost, b, or variance, σ^2, would imply a greater spread between the upper and lower control limits. In the special case where $p=q=\frac{1}{2}$, the upper control limit will always be 3 times greater than the return point.

This model is similar to the Baumol model in that it gives critical emphasis to the costs arising from transfers between the cash account and the investment portfolio. However, the assumption of random changes in cash balances is not particularly realistic, since near term cash flows are highly predictiblê for many fincncial managers.

Ⅳ. a New Model

We have found that the previous models we have briefly and roughly examined do not fully contribute to the management of cash. Therefore, a new model which is desirable for optimal cash balance should be developed. The new model, I guess, might be a revised Baumol model based upon the assumption that demand is probabilistically described, time is a continuous variable, and the ordering and holding costs are stationary. In other words, it might be a stochastic dynamic continuous review model.(9)

The purchase cost component of the objective function the cost of making transfers between the cash account and an investment portfolio is just like the Baumol model:

$$(1)\quad \frac{bM}{Q}+cM \quad \text{(average ordering cost)}$$

Where M=expected net cash to be paid out over a period of time;

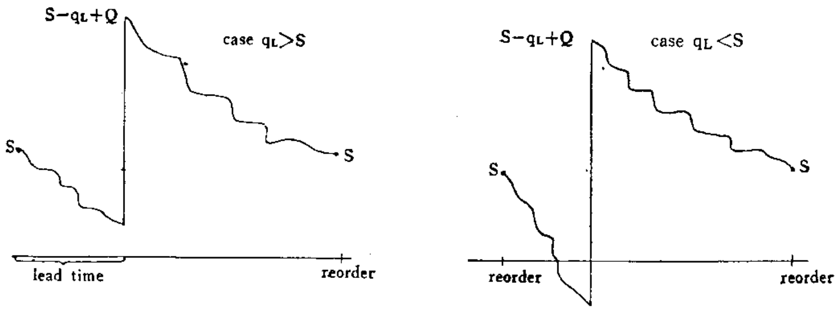
b=constant transfer cost;

Q=optimal borrowing or withdrawal lot size;

c=unit transfer cost.

Each term in (1) can be derived as follows: Since $\frac{M}{Q}$ is the average amout of setups per unit of time, $\frac{bM}{Q}$ is the average setup cost per unit of time; Since all demand must be met, cM is the average transfer cost per unit of time. We next obtain the expected holding and penalty cost.

We can assume the lead time (L) as in the economic lot-size model, which is the length of the interval when making transfers between an investment portfolio and the cash account. Let us consider the time interval between two successive reorder actions; two examples of what can happen are shown in Figure 5, one for actual demand during lead time (q_L) less than the reordeder point(S),q_L<S, and one for q_L>S.



Note: q demand during lead time
Q reorder quantity
s reorder point

Figure 5. Sawtooth Patterns for Probablistic Demand.

In the case $q_L>S$, taking into account that the cash balance is 0 before the replenishment arrives makes the holding cost formulas complicated. Therefore, as a mathematical approximation, assume that when q_L>S, cash becomes o just before the replenishment arrive. Next, let the probability distribution of demand during a lead time be $p_L(q_L)$. Then, the expected avereage cash balance during lead time is

$$(2)\ \sum_{q_l=o}^{S}\frac{1}{2}[S+(S-q_L)]P_L(q_L)+\sum_{q_l>s}\frac{1}{2}[S+0]P_L(q_L)$$

$$=\frac{1}{2}[S+\sum_{q_l=o}^{S}(S-q_L)P_L(q_L)].$$

Let M=expected cash amount demanded during an interval of L units of time. Then, the expected average cash balance after replenishment until next reorder is

(3) $\frac{1}{2}[(S-M_L+Q)+S]=\frac{1}{2}(2S-M_L+Q)$

The expression in (2) must be weighted by $\frac{M_L}{Q}$, the fraction of time the system is waiting for a replenishment, and correspondingly, the expression in (3) must be weighted by $\left(1-\frac{M_L}{Q}\right)$. We can verify that after applying these weights, adding the results, and rearranging terms, we get the expected average cash account per unit of time as follows:

(4) $\frac{M_L}{2Q}[-2S+M_L-Q+S+\sum_{ql=0}^{s}(S-q_L)P_L(q_L)]+\frac{1}{2}(2S-M_L+Q).$

The expression in (4) can be rewritten as

(5) $\frac{Q}{2}-M_L+S+\frac{M_L}{2Q}\sum_{q_l>s}(q_L-S)P_L(q_L)$, since $\sum_{q_l=0}^{s}(S-q_L)P_L(q_L)=S-M_L-\sum_{q_l>s}(S-q_L)P_L(q_L).$

Finally, the expected shortage during a lead time is

(6) $\sum_{q_l>s}(q_L-S)\ P_L(q_L),$

which must be weighted by $\frac{M}{Q}$ to obtain the expected shortage per unit of time. The expected cost equals the sum of the transfer cost, the expected holding cost, and the expected penalty cost. Accordingly, combining (1), (5) multiplied by the holding cost i, and (6) multiplied by $\frac{M}{Q}$ times the penalty cost π gives the expected average cost per unit of time as follows:

(7) $E(AG)=\frac{bM}{Q}+CM+i(\frac{Q}{2}-M_L+s)+\left(\frac{iM_L}{2Q}+\frac{M\pi}{Q}\right)\sum_{q_l>s}(q_L-s)P_L(q_L).$

Of course, the holding cost consists of interest foregone when large cash balances are held to avoid the costs of making transfers. The penalty cost also is the cost associated with running out of cash, just as there are in the case of inventories. The penalty cost $\pi>o$ is stationary and proportional to the size of a backlog just before a replenishment order arrives.

Partially differentiating E(AG) with respect to Q, gives the formula

(8) $Q=\sqrt{\frac{2bM}{i}+\left(M_L+\frac{2M\pi}{i}\right)\sum_{q_l>s}(q_L-S)P_L(q_L)}$ (determination of order quanity).

It should be noted that we assumed for the development of the above formula the following:

(i) The probability distribution of demand during a lead time does not depend on when the cash account reaches the reorder point.

(ii) The cash balance can be treated as a continuous variable.

(iii) After a replenishment order arrives, there exists a future moment in time when the cash balance reaches reorder point, and a reorder action occurs as a consequence.

(iv) In an optimal policy, the reorder point s>0 and during any lead time, actual demand does not exceed the order quantity ($q_L \leq Q$).

This model is to further develop the Baumol model: It is a revised Baumol model compansating the weakness of the Baumol model which ignores the precautionary and speculative motives for holding cash balances. The model might be an excellent approximation to the management of dyuamic cash flows despite its containing a few internal inconsistencies. However, the validity of this model will depend largely upon the accuracy of the parametric assumptions. Therefore, in the solutions the choice of value for parameters will be at stake.

V. CONCLSION

My solution of developing the new model to describe the elements and element relationship involved in cash balance equilibrium is probably of no more practical value than the existing models. Like models of the economy in economics, they must be simplistic, even if extremly complex, because the number of elemental variables and their interwoven relationships and the probabilities of certain movements of the elements are simply beyond the capabilities of any model to fully account for. Therefore, models should be relegated to their proper place of being an ivory tower exercise in math showing the relationships between a limited number of variable elements based upon certain major and minor assumptions regarding the elements, their relations hips, and the "real" world environment which may or may not be accurate with regard to that "real" world. Consequently, they can only be guides to solving the problem of a financal manager faced with determining an optimum cash balance, but not keys to it.

Of course, crust of the matter as shown in the various models is that the optimal cash balance winds up as a subjective desision made by the organization and possibly the financial manager in particular since none of the models or equations can give a definite mathematical answer because of the nebulus assumptions and management criteria with regard to the use of funds for creditor goodwill, precautionary and/or speculative purposes can not be quantified or modeled.(10)

〈*Foot-Notes*〉

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(10) Weston, J.F., & Brigham, E.F., *Essentials of Managerial Finance.* (Holt, Rinehart and Wisnston, Inc., 1971), p. 299.

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