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ON THE COMPLEXITY OF THE PRODUCTION-TRANSPORTATION PROBLEM*

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Abstract. The production-transportation problem (PTP) is a generalization of the transportation problem. In PTP, we decide not only the level of shipment from each source to each sink but also the level of supply at each source. A concave production cost function is associated with the assignment of supplies to sources. Thus the objective function of PTP is the sum of the linear transportation costs and the production costs. We show that this problem is generally NP-hard and present some polynomial classes. In particular, we propose a polynomial algorithm for the case in which the transportation cost matrix has the Monge property and the number of sources is fixed. The algorithm generalizes a polynomial algorithm of Tuy, Dan, and Ghannadan [*Oper. Res. Lett.*, 14 (1993), pp. 99–109] for the problem with two sources.

Key words. production-transportation problem, concave minimization, parametric linear programming, Monge sequence

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0. Introduction. It is known that the minimization problem over a polyhedron is polynomial when the objective function is *convex* [GLS88]. In contrast, many concave minimization problems are NP-hard. We consider here a concave minimization problem over transportation constraints called the production-transportation problem (PTP). PTP is a generalization of the transportation problem. In PTP we need to decide not only the level of shipment from each source to each sink but also the level of supply at each source. A concave production cost function is associated with the assignment of supplies to sources. The objective function is the sum of the linear transportation costs and the concave production costs.

A special class of this problem has been previously studied by Tuy, Dan, and Ghannadan [TDG93]. It was shown that when there are only two sources, the problem can be reduced to a problem of finding all breakpoints of a parametric two-source linear transportation problem with a parametrized supply level. The number of breakpoints of the parametric problem is bounded by the number of sinks and can be found in strongly polynomial time. For a two-source problem, this results in a strongly polynomial time algorithm.

We give here the proof of the NP-hardness of PTP by reducing the set cover problem to PTP. We also describe some subclasses that are solvable in polynomial time. One polynomial subclass is PTP with fixed number of sinks. Another polynomial subclass is the problem with fixed number of sources and a transportation cost matrix satisfying the *Monge* property [Hof63]. The Monge property of a matrix is recognizable in polynomial time [ACHS89]. Trivial examples of matrices with the Monge property are those with identical costs in the rows or columns of the transportation cost matrix. With such cost matrices the problem is solvable in linear time even with an arbitrary number of sources or sinks.

It was pointed out recently by Tuy et al. [TGMV93a] that a strongly polynomial algorithm was independently developed for the problem with a fixed number of sources. It was also shown that the algorithm can be used to solve the minimum concave cost network flow problem (MCCNFP) with a fixed number of sources and nonlinear arc costs in strongly polynomial time. Also, when the number of sources and nonlinear arc costs are fixed, this problem can be reduced to a PTP with a fixed number of sources in strongly polynomial time [TGMV93b].

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Thus the results in [TGMV93a] establish the strong polynomiality of the PTP with a fixed number of sources, which has a considerable potential of practical applications.

The algorithm of [TGMV93a], however, has no explicitly specified complexity. The algorithm presented in this paper shows that the problem is solved more efficiently when the problem additionally has the transportation cost matrix with the Monge property.

The paper is organized as follows. Section 1 presents the formulation of PTP. In §2, we prove that PTP is NP-hard and discuss some polynomial subclasses of PTP. Section 3 presents a polynomial algorithm for the problem with a transportation cost matrix satisfying the Monge property and a fixed number of sources. Finally, some open problems are presented in §4.

1. The PTP. Consider a transportation problem with a set of sources, $\{1, \ldots, m\}$ and a set of sinks, $\{1, \ldots, n\}$. Let c_{ij} be the cost of transporting a unit from i to j. The supply allocation to the sources is not prescribed but depends on a concave cost function $g(x_1, x_2, \ldots, x_m)$ for $x_1, x_2, \ldots, x_m$, the variables representing the supply levels at the m sources. The problem is to allocate the supplies to the sources and to send them to the sinks at the minimum total cost. Thus the PTP, which is formulated as follows, is a generalization of the transportation problem:

$$\text{(PTP)} \qquad \min\ g(x_1, x_2, \ldots, x_m) + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij},$$

$$\sum_{j=1}^{n} x_{ij} = x_i \quad i = 1, \ldots, m,$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad j = 1, \ldots, n,$$

$$x_i, x_{ij} \geq 0 \quad i = 1, \ldots, m,\ j = 1, \ldots, n.$$

It is reasonable to assume that the marginal cost of production decreases as the production level increases. The production cost function g which is concave reflects these cost economies of scale. Note that if the production system is homogeneous for all supply centers, the cost function g is symmetric with respect to the m variables, $x_1, x_2, \ldots, x_m$.

Let $B = b_1 + b_2 + \cdots + b_n$ be the total demand. PTP is feasible if and only if there exists $\mathbf{x}$ such that $x_1 + x_2 + \cdots + x_m = B$ and each x_i is nonnegative. A nonnegative vector, $\mathbf{x} \in \mathbf{R}^m$ is called a *feasible production plan* if $x_1 + x_2 + \cdots + x_m = B$.

Let $\mathbf{x}^1, \mathbf{x}^2 \in \mathbf{R}^m$ be feasible production plans. Consider the production plan, $\mathbf{x}^3 = \frac{1}{2}(\mathbf{x}^1 + \mathbf{x}^2)$. Then $\mathbf{x}^3$ is also a feasible production plan. Note that the concavity of g implies that the cost of the production plan $\mathbf{x}^3$ is at least as much as the sum of the halves of the costs of the production plan $\mathbf{x}^1$ and $\mathbf{x}^2$.

2. The complexity status of PTP. In this section, we show that PTP is NP-hard and present some polynomial subclasses. As mentioned in [TGMV93a], the NP-hardness can be observed from the fact that the minimum concave cost flow problem with a single source and m nonlinear arc costs, which is NP-hard [GP90], can be reduced to a PTP with $m + 1$ factories. Also, as pointed out in [TGMV93a], the plant location problem is a special case of PTP.

In this paper, we prove the NP-hardness of PTP more directly: we show that the set cover problem is reducible to PTP in strongly polynomial time. As a by-product of this particular reduction, we conclude that even if the problem has g as a separable function with identical component functions and the transportation cost matrix with each element zero or the same constant, PTP is still NP-hard.

2.1. PTP is NP-hard. We are unable to argue the membership of (the decision problem of) PTP in NP since g is assumed to be an arbitrary concave function and the input size of g is not well defined. We assume a computation model with a function evaluation oracle providing a single evaluation as unit operation.

We demonstrate the NP-hardness of PTP by showing that PTP is at least as hard as a known NP-hard problem—the optimization version of a known NP-complete decision problem. As g may be arbitrary, the polynomial reducibility of an NP-hard problem to PTP needs to be elaborated on. In this paper, we say that an NP-hard problem, (P), is polynomially reducible to PTP if,

(i) (P) is formulated as an instance of PTP, and

(ii) the sum of the input sizes of the numbers in the instance (excluding the concave function g) is polynomially bounded in the input size of (P).

With this definition, we prove that PTP is NP-hard by showing that the optimization problem SET COVER (see MINIMUM COVER of [GJ79]) is polynomially reducible to PTP.

SET COVER
Given a collection $K = \{S_1, S_2, \ldots, S_m\}$ of subsets of a finite set $S = \{1, 2, \ldots, n\}$. Find a subcollection K' of K of a minimum number of subsets such that the union of the subsets in K' is equal to S.

For a minimum cover K' of S, define a map from S to K' which assigns a set $S(j) \in K'$ with each element j of S so that $j \in S(j)$. Such a map exists since K' is a cover. Also the map is onto since otherwise the cover is not minimum.

SET COVER can be viewed as the problem of assigning with each element $j \in S$ a subset $S(j)$ containing j so that the total number of the subsets utilized in the assignment is minimum.

THEOREM 2.1. PTP *is* NP-*hard.*

Proof. We show that SET COVER is polynomially reducible to PTP.

For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, let

$$x_{ij} = \begin{cases} 1 & \text{if } S_i = S(j), \\ 0 & \text{otherwise.} \end{cases}$$

Since each element is assigned with exactly one set, $\sum_{i=1}^{m} x_{ij} = 1$ for each $j = 1, \ldots, n$.

Consider the PTP with transportation costs

$$c_{ij} = \begin{cases} 0 & \text{if } j \in S_i, \\ M & \text{otherwise,} \end{cases} \tag{2.1.1}$$

for M a constant greater than $1/n$ and a concave production cost function,

$$g(x_1, x_2, \ldots, x_m) = (x_1 + 1)^{1/n} + (x_2 + 1)^{1/n} + \cdots + (x_m + 1)^{1/n}.$$

Consider the following PTP formulation with these costs. In this proof, we show that an optimal solution of SET COVER can be recovered from an optimal solution of this PTP formulation:

$$\text{(P2.1)} \qquad \min (x_1+1)^{1/n} + (x_2+1)^{1/n} + \cdots + (x_m+1)^{1/n} + \sum_{\substack{i,j \text{ s.t.} \\ j \notin S_i}} M\, x_{ij},$$

$$\sum_{j=1}^{n} x_{ij} = x_i \quad i = 1, \ldots, m,$$

$$\sum_{i=1}^{m} x_{ij} = 1 \quad j = 1, \ldots, n,$$

$$x_{ij} \geq 0 \quad i = 1, \ldots, m,\ j = 1, \ldots, n.$$

For this purpose, we need the following two claims.

Let $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m; \bar{x}_{11}, \bar{x}_{12}, \ldots, \bar{x}_{mn})$ be an optimal solution of (P2.1).

CLAIM 1. *$\bar{x}_{ij} > 0$ only if $c_{ij} = 0$.*

Proof. Assume the contrary: suppose there exists a pair $1 \leq p \leq m$ and $1 \leq q \leq n$ such that $\bar{x}_{pq} > 0$ and $c_{pq} = M$. Since SET COVER is feasible, there is $1 \leq p' \leq m$ such that $c_{p'q} = 0$. Consider the solution $\hat{\mathbf{x}}$ obtained from $\bar{\mathbf{x}}$ by modifying the values of x_{pq} and $x_{p'q}$ as follows:

$$x_{pq} \leftarrow 0 \quad \text{and} \quad x_{p'q} \leftarrow \bar{x}_{p'q} + \bar{x}_{pq}.$$

Since $x_{p'q}$ is increased by $\bar{x}_{pq}$, $x_{p'}$ is also increased by $\bar{x}_{pq}$. Since $(x_{p'}+1)^{1/n}$ is strictly concave, the increase in the objective function due to the increase in $x_{p'q}$ is less than

$$\frac{1}{n}\,(\bar{x}_{p'q}+1)^{\frac{1-n}{n}}\,\bar{x}_{pq} \leq \frac{1}{n}\,\bar{x}_{pq},$$

where

$$\frac{1}{n}\,(\bar{x}_{p'}+1)^{\frac{1-n}{n}}$$

is the derivative of $(x_{p'}+1)^{1/n}$ at $x_{p'} = \bar{x}_{p'}$.

On the other hand, the decrease in the objective function due to the decreasing of $\bar{x}_{pq}$ is greater than $M\,\bar{x}_{pq}$. Thus the net change of the objective value is less than $(\frac{1}{n} - M)\bar{x}_{pq}$.

Since $M > 1/n$ the new solution $\hat{\mathbf{x}}$ has an objective function value less than that of $\bar{\mathbf{x}}$. This contradicts the optimality of $\bar{\mathbf{x}}$ for (P2.1). Thus we conclude that $\bar{x}_{ij} > 0$ only if $c_{ij} = 0$. □

CLAIM 2. *$\bar{x}_i$ is integer for every $i = 1, \ldots, m$.*

Proof. Suppose there is $1 \leq r \leq m$ such that $\bar{x}_r$ is not an integer. Then there exists an $1 \leq s \leq n$ such that $\bar{x}_{rs}$ is a fraction. This implies that there is an $1 \leq r' \leq m$, $r' \neq r$ such that $\bar{x}_{r's}$ is a fraction.

Assume that $\bar{x}_{r'} \leq \bar{x}_r$. Consider the feasible solution $\mathbf{x}'$ obtained by setting $x'_{r's} = 0$ and $x'_{rs} = \bar{x}_{rs} + \bar{x}_{r's}$.

Since the separable terms for $x_{r'}$ and x_r in the concave function g are identical and strictly concave, the assumption, $\bar{x}_{r'} \leq \bar{x}_r$, implies that the objective function value of the new solution $\mathbf{x}'$ is less than that of the optimal solution $\bar{\mathbf{x}}$; a contradiction.

Similarly, when $\bar{x}_r \leq \bar{x}_{r'}$ we can derive a contradiction by showing that decreasing x_{rs} to zero and increasing $x_{r's}$ from $\bar{x}_{r's}$ to $\bar{x}_{r's} + \bar{x}_{rs}$ decreases the objective value. Hence $\bar{x}_i$ is an integer for every $i = 1, \ldots, m$. □

Now we are ready to show how to derive an integral optimal solution of (P2.1) given optimal solution $\bar{\mathbf{x}}$. Consider the following transportation problem:

$$\text{(P2.2)} \qquad \min \sum_{\substack{i,j \text{ s.t.} \\ j \notin S_i}} M\, x_{ij},$$

$$\sum_{j=1}^{n} x_{ij} = \bar{x}_i \quad i = 1, \ldots, m,$$

$$\sum_{i=1}^{m} x_{ij} = 1 \quad j = 1, \ldots, n,$$

$$x_{ij} \geq 0 \quad i = 1, \ldots, m,\ j = 1, \ldots, n.$$

Notice from Claim 1 that $(\bar{x}_{11}, \bar{x}_{12}, \ldots, \bar{x}_{mn})$ is also an optimal solution of (P2.2). $\bar{x}_i$ are integers by Claim 2 and hence we can find an integral optimal solution, $\mathbf{x}^* = (x^*_{11}, x^*_{12}, \ldots, x^*_{mn})$ in polynomial time by using, for example, the minimum cost flow algorithm of [Orlin93]. Furthermore, $\mathbf{x}^*$ and $\bar{\mathbf{x}}$ have the same objective value for (P2.1) as $x^*_i = \bar{x}_i$ for every $i = 1, \ldots, m$ and the value of the linear cost term is zero for both solutions. So $(x^*_1, x^*_2, \ldots, x^*_m, x^*_{11}, x^*_{12}, \ldots, x^*_{mn})$ is an integral optimal solution of (P2.1). Also by applying Claim 1 to $\mathbf{x}^*$ and from (2.1.1), $\{x^*_{ij}\}$ is a feasible solution of SET COVER.

So far we have shown that a feasible solution of SET COVER can be derived from any optimal solution of (P2.1) in (strongly) polynomial time. To complete the proof, it remains to show that an integral optimal solution of (P2.1) is also an optimal solution of SET COVER. To show this, it suffices to prove that g is a strictly monotonic increasing function in the size of a cover. Notice that the size of a cover is equal to the number of *positive* (integer) elements in $(x_1, x_2, \ldots, x_m)$.

LEMMA 2.2. *Let p be an integer satisfying $0 < p < \min\{m, n\}$. Let $(x_1, x_2, \ldots, x_m)$ be a feasible solution of* (P2.1) *with p positive elements and let $(x'_1, x'_2, \ldots, x'_m)$ be a feasible solution of* (P2.1) *with $p+1$ positive elements. Then,*

$$(x_1+1)^{1/n} + (x_2+1)^{1/n} + \cdots + (x_m+1)^{1/n} < (x'_1+1)^{1/n} + (x'_2+1)^{1/n} + \cdots + (x'_m+1)^{1/n}.$$

Proof. Consider the following two problems:

$$\text{(P2.3)} \qquad P = \max\, (y_1+1)^{1/n} + (y_2+1)^{1/n} + \cdots + (y_p+1)^{1/n},$$

$$y_1 + y_2 + \cdots + y_p = n,$$

$$y_i \geq 1 \text{ integer}, \quad i = 1, \ldots, p.$$

$$\text{(P2.4)} \qquad Q = \min\, (y_1+1)^{1/n} + (y_2+1)^{1/n} + \cdots + (y_{p+1}+1)^{1/n},$$

$$y_1 + y_2 + \cdots + y_{p+1} = n,$$

$$y_i \geq 1 \text{ integer}, \quad i = 1, \ldots, p+1.$$

The vector consisting of the positive elements of $(x_1, x_2, \ldots, x_m)$ is feasible for (P2.3). Hence,

$$(x_1+1)^{1/n} + (x_2+1)^{1/n} + \cdots + (x_m+1)^{1/n} \leq P.$$

Similarly,

$$(x_1' + 1)^{1/n} + (x_2' + 1)^{1/n} + \cdots + (x_m' + 1)^{1/n} \geq Q.$$

In order to complete the proof of the lemma, we need to show that $P < Q$.

Since (P2.3) is symmetric with respect to its variables and the objective is concave, the optimal solution to the relaxed problem has all variables equal to n/p. Therefore,

$$P \leq p\left(\frac{n}{p} + 1\right)^{1/n}.$$

Equation (P2.4) is also symmetric but it involves minimization. Hence the optimal solution of the continuous version of (P2.4) is achieved at any vertex, i.e., any solution vector with p 1's and a single $n-p$ as its entry. Thus,

$$Q \geq p\,2^{1/n} + (n - p + 1)^{1/n}.$$

To show $P < Q$, it suffices to prove

$$p\left(\frac{n}{p} + 1\right)^{1/n} < p\,2^{1/n} + (n - p + 1)^{1/n}, \tag{2.1.2}$$

or, equivalently,

$$\left(p\left(\frac{n}{p} + 1\right)^{1/n}\right)^n < (p\,2^{1/n} + (n - p + 1)^{1/n})^n.$$

Since

$$\left(p\left(\frac{n}{p} + 1\right)^{1/n}\right)^n = p^n + n\,p^{n-1},$$

(2.1.2) is equivalent to

$$(p\,2^{1/n} + (n - p + 1)^{1/n})^n > p^n + n\,p^{n-1}. \tag{2.1.3}$$

But

$$\begin{aligned}
&(p\,2^{1/n} + (n - p + 1)^{1/n})^n \\
&\quad = 2p^n + \binom{n}{n-1} p^{n-1} 2^{(n-1)/n} (n - p + 1)^{1/n} + \cdots \\
&\qquad + \binom{n}{1} p\,2^{1/n} (n - p + 1)^{(n-1)/n} + (n - p + 1) \\
&\quad > p^n + \binom{n}{n-1} p^{n-1} 2^{(n-1)/n} (n - p + 1)^{1/n} \\
&\quad = p^n + n\,p^{n-1} 2^{(n-1)/n} (n - p + 1)^{1/n}.
\end{aligned}$$

Since $n - p + 1 > 1$,

$$p^n + n\,p^{n-1} 2^{(n-1)/n} (n - p + 1)^{1/n} \geq p^n + n\,p^{n-1}.$$

Hence (2.1.3) follows and the proof of Lemma 2.2, and hence of Theorem 2.1, is complete. □

From the above reduction it follows that PTP is NP-hard even when g is separable and symmetric (i.e., every component function is identical) and each element of the transportation cost matrix is zero or a same constant.

2.2. Two polynomial classes of PTP.

2.2.1. PTP with a fixed number of sinks. Consider the problem PTP as in §1. Since $g(x_1, x_2, \ldots, x_m)$ is a concave function and $x_i = \sum_{j=1}^{n} x_{ij}$ is a linear function, g is a concave function of $\{x_{ij}\}$. Hence the objective is a concave function of $\{x_{ij}\}$ and an optimal solution is attained at a vertex of the polyhedron defined by the last n constraints in $\{x_{ij}\}$ and the nonnegative restrictions on $\{x_{ij}\}$. Thus to solve the problem, it suffices to enumerate all vertices of the polyhedron and choose a vertex with the maximum objective value.

Each vertex of the polyhedron corresponds to a set of n independent columns of the matrix defined by the n constraints. So in a vertex, there is only one nonzero variable, say x_{i_j}, from m variables of jth constraint where $j = 1, 2, \ldots, n$. (In other words, the polyhedron is the product, $\prod_{j=1}^{n} S_j$, where S_j is the $m-1$ dimensional simplex determined by the intersection of the hyperplane, $\sum_{i=1}^{m} x_{ij} = b_j$ and the nonnegative orthant, $x_{ij} \geq 0$.) Hence there are $O(m^n)$ vertices and each vertex is of the form

$$x_{ij} = \begin{cases} b_j & \text{if } i = i_j, \\ 0 & \text{if } i \neq i_j \end{cases}$$

for some i_j for each $j = 1, \ldots, n$.

Thus, when the number of sinks, n is fixed, PTP is solvable in polynomial time.

2.2.2. PTP with identical costs in rows or columns. Consider the problem PTP with a transportation cost matrix in which the costs are identical in each row. That is, for each $i = 1, 2, \ldots, m$, there is a constant c_i such that $c_{ij} = c_i$ for all $j = 1, 2, \ldots, n$. The objective function is written as $g(x_1, x_2, \ldots, x_m) + \sum_{i=1}^{m} c_i x_i$, which is concave with respect to $(x_1, x_2, \ldots, x_n)$. So an optimal solution is attained at a vertex of the polyhedron of PTP. In this case, the polyhedron is the simplex which is the intersection of the hyperplane $\sum_{i=1}^{m} x_i = B\,(= b_1 + b_2 + \cdots + b_n)$ and the nonnegative orthant $x_i \geq 0$. Hence each vertex is of the form,

$$(*) \qquad x_i^* = \begin{cases} B & \text{if } i = i', \\ 0 & \text{if } i \neq i' \end{cases}$$

for some $1 \leq i' \leq m$. Note that there are $O(m)$ vertices and hence $O(m)$ function evaluations are enough to solve the problem.

A similar observation applies when the costs are identical along each column. Namely, for each $j = 1, 2, \ldots, n$, there is a constant c_j such that $c_{ij} = c_j$ for all $i = 1, 2, \ldots, m$. In this case the linear cost term is the constant

$$\sum_{j=1}^{n} c_j \left(\sum_{i=1}^{m} x_{ij} \right) = \sum_{j=1}^{n} c_j b_j$$

and hence the objective function is again concave in $(x_1, x_2, \ldots, x_n)$. The optimal solution is again of the form (*) and can be determined in linear time. A special case of the above polynomial class is when all c_{ij} are identical, $c_{ij} = c$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$.

Therefore the problem is easy when the linear term matrix has a special structure so that the objective function is concave with respect with the variables, $x_1, x_2, \ldots, x_m$. Thus the general structure of the linear term in the objective function appears to be an important factor in the "hardness" of the problem. For example, as we have seen in the previous subsection, even when all elements of the cost matrix are zero or one, if in arbitrary positions, it is enough to make the problem NP-hard.

3. A strongly polynomial algorithm for PTP with the Monge cost matrix and a fixed number of sources. In this section, we show that if the transportation cost matrix (c_{ij}) has the *Monge* property and m is fixed, then PTP is solvable in strongly polynomial time.

We show first that PTP is solvable by enumerating the breakpoints of a multiparametric transportation problem (§3.1). Section 3.2 has a discussion and definition of the *Monge* property. In §3.3, we show that the breakpoints of the multiparametric transportation defined in §3.1, can be enumerated in strongly polynomial time if the transportation cost matrix has the Monge property and m is fixed.

The approach here could be viewed as a generalization of the approach in [TDG92] for $m = 2$, as matrices with only two rows (or two columns) always satisfy the Monge property (see, e.g., [ACHS89]).

3.1. A parametric problem. Consider the parametric transportation problem

$$\text{(TP}(\boldsymbol{\alpha}))\qquad z(\boldsymbol{\alpha}) = \min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij},$$

$$\sum_{j=1}^{n} x_{ij} = \alpha_i \quad i = 1,\dots,m,$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad j = 1,\dots,n,$$

$$x_{ij} \geq 0 \quad i = 1,\dots,m,\ j = 1,\dots,n,$$

where each α_i is nonnegative and $\alpha_1 + \cdots + \alpha_m = B$ (where, $B = b_1 + \cdots + b_m$).

It is known that on the domain $D = \{\boldsymbol{\alpha} : \alpha_i \geq 0 \text{ for } i = 1, 2, \dots, m,\ \alpha_1 + \cdots + \alpha_m = B\}$, the optimal value of (TP($\boldsymbol{\alpha}$)), $z(\boldsymbol{\alpha})$, is a piecewise affine function (see, e.g., [Murty83]). D is the union of polyhedral subdomains, $D_1, D_2, \dots, D_M$, so that $z(\boldsymbol{\alpha})$ is linear on each of the subdomains. Note that the subdomains have disjoint relative interiors.

Let $w(\boldsymbol{\alpha}) = g(\boldsymbol{\alpha}) + z(\boldsymbol{\alpha})$ (where g is the concave cost production function of PTP). Then the optimal value of PTP is equal to

$$\min_{\boldsymbol{\alpha}\in D} w(\boldsymbol{\alpha}) = \min_{k=1,\dots,M} \min_{\boldsymbol{\alpha}\in D_k} w(\boldsymbol{\alpha}). \tag{3.1}$$

Note that D and its subdomains D_k are (bounded) polytopes. Hence there is a finite set of points of D, $L = \{\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2, \dots, \boldsymbol{\alpha}^{\mathbf{N}}\}$ such that each subdomain D_k is the convex hull of a subset L_k of L. Each element of L is called a breakpoint of the parametric transportation problem, (TP($\boldsymbol{\alpha}$)).

Notice that $w(\boldsymbol{\alpha})$ is a concave function on each D_k since $z(\boldsymbol{\alpha})$ is affine on each D_k. So $\min_{\boldsymbol{\alpha}\in D_k} w(\boldsymbol{\alpha})$ can be determined by evaluating function values of $w(\boldsymbol{\alpha})$ at the breakpoints $\boldsymbol{\alpha} \in L_k$ and choosing a breakpoint giving the smallest value. From (3.1), the optimal objective function value of PTP is obtained by evaluating function values of $w(\boldsymbol{\alpha})$ at all the breakpoints of L and choosing a breakpoint giving the smallest value.

Thus, to solve PTP, it suffices to find all breakpoints and the corresponding solution of (TP($\boldsymbol{\alpha}$)).

The parametric approach does not necessarily lead to an algorithm for the problem, unless it is known how to find all the breakpoints. Tuy, Dan, and Ghannadan [TDG92] showed that for $m = 2$, (TP($\boldsymbol{\alpha}$)) has at most $n + 1$ breakpoints that can be found in $O(n \log n)$ elementary

operations. As discussed in §2.1, PTP has 2^n vertices when $m = 2$. Thus the parametric approach represents a substantial improvement.

Although it is not explicitly observed in [TDG92], the reason for this efficiency is that every $2 \times n$ cost matrix has the *Monge* property [ACHS89] which is discussed in the following section.

Remark. In [TDG92], for $m = 2$ the parametric transportation problem is derived in the context of *rank* 2 *condition*. It is also possible to derive (TP($\boldsymbol{\alpha}$)) using *rank m condition*, which is a straightforward generalization of rank 2 condition. Using rank m condition, we can show that a more general class of concave minimization problem can be reduced into the problem enumerating the breakpoints of a parametric linear problem.

3.2. The Monge property. Consider the transportation problem (TP) defined with the cost matrix $C = (c_{ij})$:

$$\text{(TP)} \qquad \min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij},$$

$$\sum_{j=1}^{n} x_{ij} = a_i \quad i = 1, \ldots, m,$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad j = 1, \ldots, n,$$

$$x_{ij} \geq 0 \quad i = 1, \ldots, m,\ j = 1, \ldots, n.$$

C is said to have the *Monge property* [Hof63] if there exists a permutation $((i_1, j_1), (i_2, j_2), \ldots, (i_{mn}, j_{mn}))$ of indices of the cost matrix such that

(M) for every $1 \leq i, k \leq m$, $1 \leq j, l \leq n$, whenever (i, j) precedes both $(i, l)$and (k, j) the corresponding entries in matrix C are such that $c_{ij} + c_{kl} \leq c_{il} + c_{kj}$.

Any permutation satisfying (M) is called a *Monge sequence*. If C has the Monge property (and hence admits a Monge sequence), for any nonnegative integers, $a_1, \ldots, a_m$, $b_1, \ldots, b_n$ (with $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$), (TP) is solved efficiently by a greedy algorithm. The following theorem is due to Hoffman [Hof63].

THEOREM 3.1. *A permutation* $((i_1, j_1), (i_2, j_2), \ldots, (i_{mn}, j_{mn}))$ *is a Monge sequence of* C *if and only if for any nonnegative integers,* $a_1, \ldots, a_m$, $b_1, \ldots, b_n$ *with* $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, *the solution obtained by Algorithm Greedy* (in Figure 1) *is an optimal solution of* (TP).

The Monge property can be characterized in polynomial time. The algorithm of [ACHS89] tests whether an $m \times n$ matrix has a Monge sequence in $O(m^2 n \log n)$.

Every $2 \times n$ matrix, C has a Monge sequence: renumber the columns of the matrix so that $c_{11} - c_{21} \leq c_{12} - c_{22} \leq \cdots \leq c_{1n} - c_{2n}$. Then $((1, 1), (1, 2), \ldots, (1, n), (2, 1), (2, 2), \ldots, (2, n))$ is a Monge sequence as can be easily checked. The parametric transportation problem for $m = 2$ follows Algorithm Greedy.

```
ALGORITHM GREEDY

begin
    u_i ← a_i for i = 1,...,m; v_j ← b_j for j = 1,...,n;
    for k = 1 to mn do
    begin
            x_{i_k j_k} ← min{u_{i_k}, v_{j_k}};
            u_{i_k} ← u_{i_k} − x_{i_k j_k};
            v_{i_k} ← v_{i_k} − x_{i_k j_k};
    end; {for}
end; {Algorithm}
```

FIG. 1. *Algorithm Greedy.*

$$(\mathrm{TP}_2(\beta)) \qquad \min \sum_{i=1}^{2}\sum_{j=1}^{n} c_{ij}x_{ij},$$

$$\sum_{j=1}^{n} x_{1j} = \beta,$$

$$\sum_{i=1}^{2} x_{ij} = b_j \quad j = 1,\ldots,n,$$

$$x_{ij} \geq 0 \quad i = 1,2,\ j = 1,\ldots,n,$$

for $0 \leq \beta \leq B$ (only a single parameter is required by deleting a redundant constraint).

THEOREM 3.2 (see [TDG92]). (i) *The breakpoints of the optimal value function in* $(\mathrm{TP}_2(\beta))$ *are in the set,* $\{\beta^0, \beta^2, \ldots, \beta^n\}$, *where*

$$\beta^0 = 0,\ \beta^{k+1} = \beta^k + b_{k+1},\ k = 0, 1, \ldots, n-1.$$

(ii) *If* $\sum_{j=0}^{l-1} b_j \leq \beta \leq \sum_{j=0}^{l} b_j$ *(with* $b_0 = 0$*) for some* $0 \leq l \leq n-1$, *then the following solution is optimal for* $(\mathrm{TP}_2(\beta))$:

$$x_{11} = b_1,\ x_{21} = 0; \ldots; x_{1l-1} = b_{l-1}, x_{2l-1} = 0;$$

$$x_{1l} = \beta - \sum_{j=0}^{l-1} b_j, x_{2l} = \sum_{j=0}^{l} b_j - \beta;$$

$$x_{1l+1} = 0, x_{2l+1} = b_{l+1}; \ldots; x_{1n} = 0, x_{2n} = b_n.$$

The algorithm presented in the following section reduces a parametric problem into a two-row problem. We then use the algorithm implied by Theorem 3.2 to solve the two-row problem.

ALGORITHM PTP

begin

1 set $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$;

2 $Q \leftarrow \{\alpha_1 \geq 0,\ \alpha_2 \geq 0, \ldots, \alpha_{m-1} \geq 0,\ \alpha_1 + \alpha_2 + \cdots + \alpha_{m-1} \leq B\}$;

3 set $u_i = \alpha_i$ for $i = 1, \ldots, m-1$, $u_m = B - (\alpha_1 + \alpha_2 + \cdots + \alpha_{m-1})$ and $v_j = b_j$ for $j = 1, \ldots, n$;

4 call Procedure Breakpoint-Finder (I,J);

5 evaluate the objective function value for the solution corresponding to each breakpoint;

6 choose a solution of minimum value as an optimal solution of PTP;

end;

FIG. 2. *Algorithm* PTP.

3.3. An algorithm. Consider Algorithm PTP presented in Figure 2. It consists of three main parts: initialization (lines 1–3), breakpoint enumeration for (TP($\boldsymbol{\alpha}$)) (line 4) which is done by the procedure Breakpoint-Finder described in Figure 3, and optimization (lines 5–6), which selects a breakpoint and a corresponding solution yielding the smallest objective value.

From the arguments of §3.1 the validity of Algorithm PTP immediately follows from the validity of the procedure Breakpoint-Finder.

The procedure Breakpoint-Finder applies Algorithm Greedy to the parametric problem (TP($\boldsymbol{\alpha}$)) using the Monge sequence. At each iteration, Algorithm Greedy needs to be combined with a branching step to enumerate the possible outcomes of optimal solution as the right-hand sides are parametrized in terms of $\boldsymbol{\alpha}$.

At each branching step, the current problem branches into two subproblems of smaller dimensions. This branching is recursively repeated until the current problem has only two rows. Then the two-row parametric problem is solved using Theorem 3.1.

THEOREM 3.3. *Procedure Breakpoint-Finder correctly returns all the breakpoints of* (TP($\boldsymbol{\alpha}$)).

Proof. At each iteration the procedure considers the cell (i, j) of the current problem, which is at the head of the Monge sequence (line 1). By Theorem 3.1, the greedy optimal solution is such that

$$x_{ij} = \begin{cases} u_i & \text{if } u_i \leq v_j, \\ v_j & \text{if } u_i \geq v_j. \end{cases} \tag{3.3.1}$$

Thus we need to consider two cases, $u_i \leq v_j$ and $u_i \geq v_j$. When $u_i \leq v_j$, we assign u_i to the cell (i, j) and the equation $u_i \leq v_j$ is added to Q to specify the subdomains on which the assignment is optimal (line 3).

Also if $u_i \leq v_j$, all cells other than (i, j) of the row i will not be assigned any positive shipment in the greedy solution. Hence the row i is deleted from further consideration and the demand level of the column j needs to be decreased to $v_j - u_i$ (line 4).

Lines 6–8 describe the analog procedure for the case, $u_i \geq v_j$.

After the branching step, the current problem branches into two subproblems; the one with reduced rows, and the other with reduced columns if the number of columns is larger than 1. For each subproblem, the branching is recursively repeated if the number of rows is >2 (lines 5 and 8).

```
PROCEDURE BREAKPOINT-FINDER(I,J)

begin
    if |I| > 2 then do
    begin
1       select the cell (i,j) such that i ∈ I and j ∈ J, which is
        at the top of the Monge sequence;
2       delete (i,j) from the Monge sequence;
        {row deletion}
3       let x_ij = u_i; Q ← Q ∪ {u_i ≤ v_j};
4       I ← I − {i}; v_j ← v_j − u_i;
5       call Procedure Breakpoint-Finder (I,J);
        {column deletion}
        if |J| > 1, then do
        begin
6           let x_ij = v_j; Q ← Q ∪ {v_j ≤ u_i};
7           J ← J − {j} ; u_i ← u_i − v_j;
        end; {if}
8       else, call Procedure Breakpoint-Finder (I,J);
    end; {if}
    else, do
    begin {solving two-row problem}
9       say I = {1,2} and J = {1,2,...,k};
10      sort {c_11 − c_21, c_12 − c_22, ..., c_1k − c_2k} in an increasing order
        {we assume, for the simplicity, that the sequence is already sorted} ;
        for p = 1 to k do
        begin
11          set x_11 = v_11, x_21 = 0, ..., x_1p−1 = v_1p−1, x_2p−1 = 0, x_1p = u_1 − Σ_q≤p−1 v_q,
            x_2p = Σ_q≤p v_q − u_1, x_1p+1 = 0, x_2p+1 = v_p+1, ..., x_1k = 0, x_2k = v_k;
12          Q ← Q ∪ {Σ_q≤p−1 v_q ≤ u_1, u_1 ≤ Σ_q≤p v_q}, where v_0 = 0;
            {calculation of breakpoints}
13          solve every subset of m − 1 equations of Q;
14          accept a solution (α_1, α_2, ..., α_m−1) as a breakpoint if
            α_i ≥ 0 for all i and α_1 + α_2 + ··· + α_m−1 ≤ B;
15          return the accepted breakpoints and the
            corresponding solution {x_ij};
        end; {for}
    end; {else}
end; {Procedure}
```

FIG. 3. *Procedure Breakpoint-Finder.*

Note that u_i and v_j are affine functions of $\boldsymbol{\alpha}$. Initially $u_i = \alpha_i$ and $v_j = b_j$. By induction it is easy to see that at each iteration, $u_i = (\rho_{i1}\alpha_1 + \rho_{i2}\alpha_2 + \cdots + \rho_{im}\alpha_m) - \rho_{i0}$ and $v_j = \sigma_{j0} - (\sigma_{j1}\alpha_1 + \sigma_{j2}\sigma_2 + \cdots + \sigma_{jm}\alpha_m)$, where ρ_{ik} and σ_{jl} are nonnegative integers for $k, l = 0, \ldots, m$.

So the linear equation, $u_i = v_j$ defines a hyperplane that divides the domain $D = \{\boldsymbol{\alpha} : \alpha_i \geq 0 \text{ for } i = 1, 2, \ldots, m-1,\ \alpha_1 + \cdots + \alpha_{m-1} \leq B\}$ into two polyhedral subdomains.

Clearly on each subdomain the greedy solution of x_{ij} defined in (3.3.1) is an affine function of $\boldsymbol{\alpha}$. Hence the cost contribution of the greedy solution of x_{ij} to the objective value $z(\boldsymbol{\alpha})$ is also an affine function of $\boldsymbol{\alpha}$ on each subdomain. Thus by induction on the number of iterations, it follows that the cost of the currently assigned shipment levels (x_{ij}'s with $(i,j) \notin I \times J$) is an affine function of $(\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$ in the current subdomain defined by the equations in Q.

So far we have shown that at each branching iteration, the procedure assigns the shipment level to an additional cell in an optimal manner and generates the equations in $(\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$ that describe the region on which the additional assignment is optimal and the cost incurred is an affine function of $(\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$. The equations are then added to Q. Thus the equations in Q describe a polyhedral subset of $D = \{\boldsymbol{\alpha} : \alpha_i \geq 0$ for $i = 1, 2, \ldots, m-1\}$, on which the current greedy solution is optimal and the cost of the solution is an affine function.

This branching is repeated until the number of remaining rows becomes 2 (line 9). The optimal parametric solution can then be found using Theorem 3.2. The solution (line 11) and the equations describing corresponding ranges (line 12) are obtained from Theorem 3.2 (ii) by simply substituting $n = k$, $\beta = u_1$, and $b_l = v_l$ for $l = 1, \ldots, k$. If there are k columns in the problem, there are k corresponding ranges. Clearly, in each of the ranges the cost incurred by the shipment levels assigned to the cells in the two-row problem is an affine function of the $(\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$.

After line 11, we obtain a complete set of parametric solution $\{x_{ij} \mid i = 1, \ldots, m, j = 1, \ldots, n\}$ of (TP($\boldsymbol{\alpha}$)). The equations in Q after the execution of line 12 describe a complete subdomain on which the objective function value $z(\boldsymbol{\alpha})$ of the above solution is an affine function.

Note that the total number of equations in Q is at most $m+n$. Thus each subdomain has at most $\binom{m+n}{m-1}$ vertices (breakpoints of (TP($\boldsymbol{\alpha}$)) which can be found by solving every subset of $m-1$ equations of Q as in line 13. In line 14, we choose only the solutions that satisfy the valid conditions, namely, each α_i is nonnegative and $\alpha_1 + \cdots + \alpha_{m-1} \leq B$.

Thus the procedure Breakpoint-Finder correctly generates the complete set of breakpoints of (TP($\boldsymbol{\alpha}$)) and the corresponding solutions. The theorem follows. □

From §3.1 and Theorem 3.3, we have Corollary 3.4.

COROLLARY 3.4. *Algorithm* PTP *is valid.*

Now we show that Algorithm PTP is strongly polynomial.

THEOREM 3.5. *Assume that there is a function evaluation oracle that provides a single evaluation as unit operation. Then the complexity of* PTP *is* $O(n^m)$.

Proof. The initialization (lines 1–3) of the algorithm can be done in $O(n)$ steps as can be easily verified. The number of unit operations required for the optimization (lines 5–6) is linear in the number of breakpoints generated by the procedure Breakpoint-Finder: the number of function evaluations (line 5) is the same as the number of breakpoints and the minimum value of line 6 can be found in linear time in the number of function values by using the algorithm in [BFPRT72].

The number of breakpoints generated by the procedure Breakpoint-Finder is bounded by the number of elementary arithmetic operations executed by Breakpoint-Finder.

So to prove the theorem, it suffices to show that the running time of Breakpoint-Finder is $O(n^m)$.

Let $T(m,n)$ be the number of elementary operations taken by the procedure Breakpoint-Finder when a parametric problem has m rows and n columns. Then by the recursive structure of the procedure, we have

$$T(m,n) \leq T(m-1,n) + T(m,n-1). \tag{3.3.2}$$

Applying (3.3.2) repeatedly, we get

$$\begin{aligned}
&T(m-1,n)+T(m,n-1)\\
&\quad\le T(m-1,n)+T(m-1,n-1)+T(m,n-2)\\
&\quad\le T(m-1,n)+T(m-1,n-1)+T(m-1,n-2)+T(m,n-3)\\
&\quad\le T(m-1,n)+T(m-1,n-1)+\cdots+T(m-1,2)+T(m,1)\\
&\quad\le (n-1)T(m-1,n)+T(m,1).
\end{aligned}$$

To evaluate $T(m,1)$, note that for a single column problem, the algorithm performs $O(m)$ unit operations. The initialization takes $O(m)$ operations. There are only $m-1$ row deletions from lines 1–8 of Procedure Breakpoint-Finder. The remaining steps of lines 9–10 require $O(m)$ operations. Thus there exists a constant $A>0$ such that,

$$T(m,n)\le (n-1)T(m-1,n)+Am. \tag{3.3.3}$$

We now show that the number of elementary operations required to solve the two-row problem is $T(2,n)=O(n^2)$. When the number of rows is 2 ($|I|=2$), after the initialization which takes $O(n)$ operations, the algorithm performs the steps of lines 9–15 in Procedure Breakpoint-Finder. The sorting in line 10 can be done in $O(n\log n)$ steps. For each $p=1,\ldots,k$, lines 11 and 12 can be done in $O(n)$ steps. The operations of lines 13 and 14 can be done in $O(n)$ steps since, as shown in the proof of Theorem 3.3, when m is fixed, the total number of the sets of equations to be solved is $\binom{m+n}{m-1}=O(n^{m-1})$. Each set of equations is solvable in $O(m^3)$ which is a constant as m is fixed.

Thus for each $p=1,\ldots,k$, the complexity is $O(n)$ and hence $T(2,n)=O(n^2)$ as $k\le n$.

Assume, by induction, that for every $k=2,\ldots,m-1$, $T(k,n)=B\,k^2n^k$, where B is a constant with $B>A$. Then by (3.3.3),

$$\begin{aligned}
T(m,n)&\le (n-1)T(m-1,n)+Am\\
&\le (n-1)B(m-1)^2n^{m-1}+Am\\
&\le n\,B(m-1)^2n^{m-1}+Am\\
&= B\,m^2n^m-2B\,m\,n^{m+}B\,n^m+A\,m\\
&< B\,m^2n^m-2B\,m\,n^{m+}B\,n^m+B\,m\ (\text{since } B>A)\\
&< B\,m^2n^m.
\end{aligned}$$

Hence the theorem follows. □

Remark. A careful look at the algorithm and its complexity analysis indicates that we may assume a weaker form of oracle. The function evaluations are required solely to find a solution yielding the smallest objective value. However a smallest valued solution can be found without calculating function values explicitly if only we can determine the ordinal sizes of the function values of given solutions used in comparisons (see, e.g., [BFPRT72]). Thus it suffices to assume an oracle to determine the relative size of the function values (rather than calculating the explicit function values) of given solutions.

4. Open problems. It is an interesting problem to improve the polynomial algorithm presented in this paper. It may be possible, for instance, to find $O(n)$ algorithms for the cases discussed in §§2.2 and 3.

The PTP with a fixed number of sources is an important subclass. It has applications in many practical problems that have a small number of sources (e.g., factories or warehouses) and a large number of sinks (e.g., retailers or consumers). As mentioned earlier, the algorithm of [TGMV93] establishes the strong polynomiality of this problem. The value of the algorithm seems, however, to be theoretical rather than practical. So it is desirable to find algorithms of improved complexities that are less dependent on the constant m.

Further research in a somewhat different direction is to develop efficient heuristics for the general case. No such approximations are known even if we make some additional assumptions on the concave production cost functions such as symmetry and separability.

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