Polyhedral Study of the K-Median Problem

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I. Introduction

The past two decades have witnessed a tremendous growth in the literature on location problems. However, among the myriads of formulations the simple plant location problem and the k-median problem have played a central role. This phenomenon is due to the facts that both problems have a wide range of real-world applications and a mathematical formulation of these problems as an integer program has proven very fruitful in the derivation of solution methods.

Consider an index set $I=\{1, 2, \ldots, n\}$ of n points, and a positive integer $k \leq n$, and let c_{ij} be the shortest distance between two points $i,j \in I$. The k-median problem consists of identifying a subset S of I, $|S|=k$ so as to minimize$(\sum_{i=1}^{n} \underset{j \in S}{\mathrm{Min}}\ c_{ij})$. (Here $|S|$ denotes the cardinality of the set S). The k-median problem has the following combinatorial formulation.

Combinatorial Formulation:

$$\underset{\substack{S \subseteq I \\ |S|=k}}{\mathrm{Min}} \{\sum_{i \in I} \underset{j \in S}{\mathrm{Min}}\ c_{ij}\}$$

We introduce integer variables. Let $y_j=1$ if a point j is selected as a median, otherwise 0 and $x_{ij}=1$ if a point j is the closest median to point i, otherwise 0. With x, y variables the k-median problem is formulated as an integer program as follows.

Integer Program Formulation:

$$Z_{IP}=\mathrm{Min} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad (1)$$

subject to

$$\sum_{j=1}^{n} x_{ij}=1 \qquad i\in I \tag{2}$$

$$\sum_{j=1}^{n} y_j=k \tag{3}$$

$$0\leq x_{ij}\leq y_j\leq 1 \qquad i,j\in I \tag{4}$$

$$x_{ij},\ y_j \ \textit{integral} \qquad i,j\in I \tag{5}$$

A vast number of algorithms were proposed for the k-median problem. We refer readers to ReVelle [19], Francis and White [14], Christofides [7], Jacobsen and Pruzan [16], Handler and Mirchandani [15], Krarup and Pruzan [17], Cornuejols [9] [11] [12], Fisher and Hochbaum [13], Papadimitriou [18], Rosing [20], Beasley and Christofides [8], Boffey [5], Beasley [4].

Most of the successful algorithms for the k-median problem are based on the strong linear programming relaxation. In [1] [2] [3] we presented and explained why the strong linear programming relaxation provides a tight lower bound in the probablistic sense. In this paper we investigate the phenomenon with a polyhedral approach.

II. Polyhedral Analysis

In this section we investigate the polytope of the extreme solutions to the strong linear program relaxation of k-median problem constraints.

$$\sum_{j=1}^{n} x_{ij}=1 \qquad i\in I \tag{6}$$

$$\sum_{j=1}^{n} y_j=k \tag{7}$$

$$x_{ij}\leq y_j \qquad i,j\in I \tag{8}$$

$$y_j\leq 1 \qquad j\in I \tag{9}$$

$$x_{ij},\ y_j\geq 0 \qquad i,\ j\in I \tag{10}$$

Let P_n be the polytope defined by (6)$\sim$(10). We present properties of the fractional extreme points (x,y) to P_n below.

Lemma 1:

If (x,y) is a fractional extreme point of the polytope P_n,

then for each $i \in I$, there is at most one $j \in I$ with $0 < x_{ij} < y_j$.

Proof:

Let (x,y) be a fractional solution such that above condition does not hold.

Then there exist p, j_1, j_2 such that $x_{pj_1} < y_{j_1}$, $x_{pj_2} < y_{j_2}$.

Let $x^1_{pj_1} = x_{pj_1} + \epsilon$, $x^1_{pj_2} = x_{pj_2} - \epsilon$, $x^2_{pj_1} = x_{pj_1} - \epsilon$, $x^2_{pj_2} = x_{pj_2} + \epsilon$,

$x^1_{ij} = x^2_{ij} = x_{ij}$ for all other i, j and $y^1_j = y^2_j = y_j$ for all j.

where $\epsilon = \text{Min}\ [x_{pj_1},\ x_{pj_2},\ y_{j_1} - x_{pj_1},\ y_{j_2} - x_{pj_2}]$.

Then $(x, y) = (1/2)(x^1, y^1) + (1/2)(x^2, y^2)$. (x^1, y^1) and (x^2, y^2) both are feasible solutions to P_n. This contradicts the assumption that (x, y) is an extreme solution.//

A similar result is known for the simple plant location problem. In fact, Cornuejols et al [10] completely characterized the fractional extreme solutions to the simple plant location problem.

Suppose we are given the shortest distance matrix between all pairs of points. The optimal solution to the 1-median problem is reduced to find a column of the shortest distance matrix with smallest column sum. In fact, when $k=1$, all the extreme solutions to the polytope P_n are integral regardless of n. We show this in the next theorem.

Theorem 2:

The linear programming relaxation of the 1-median problem always has an integer optimal solution.

Proof:

Suppose there exists a fractional extreme solution to the linear programming relaxation of 1-median problem.

Let $J_1 = \{j \in I : 0 < y_j < 1\}$

Then for each $i \in I$, $x_{ij} = y_j$ for all $j \in J_1$ and $x_{ij} = 0$ for all $j \notin J_1$.

Choose any two points j_1, $j_2 \in J_1$, any $p \in I$ and let

$x^1_{pj_1} = x_{pj_1} + \epsilon$, $x^1_{pj_2} = x_{pj_2} - \epsilon$, $y^1_{j_1} = y_{j_1} + \epsilon$, $y^1_{j_2} = y_{j_2} - \epsilon$

$x^2_{pj_1}=x_{pj_1}-\epsilon,\ \ x^2_{pj_2}=x_{pj_2}+\epsilon,\ \ y^2_{j_1}=y_{j_1}-\epsilon,\ \ y^2_{j_2}=y_{j_2}+\epsilon$

All other x_{ij}, y_j remain unchanged.

Then $(x,y)=\frac{1}{2}(x^1,\ y^1)+\frac{1}{2}(x^2,\ y^2)$ and $(x^1,\ y^1)$, $(x^2,\ y^2)$ both are feasible solutions to P_n. This contradicts the assumption that (x,y) is an extreme solution.//

In the next theorem, we extend the above result to more general cases.

Theorem 3:

If (x,y) is an extreme solution to the polytope P_n, then $\sum_{j\in J_1} y_j\geq 2$.

Proof:

Suppose (x,y) is an extreme point to P_n with $\sum_{j\in J_1} y_j=1$.

Let $I_1=\{i\in I: 0<x_{ij}<1$ for some $j\in J_1\}$, $J_2=\{j\in J: y_j=1\}$.

We have two cases to consider here.

Case 1: for all $i\in I_1$, $\sum_{j\in J_1} x_{ij}=1$. (That is, $x_{ij}=0$ for all $j\in J_2$).

For this case we can derive contradiction in the same way as for Theorem 2.

Case 2: for some $i\in I_1$, $\sum_{j\in J_1} x_{ij}\neq 1$.

Here we have two subcases.

Case 2-1: for all $i\in I_1$, $x_{ij}=y_j$ for only one $j\in J_1$. That is, $x_{ip}=1-y_p$ for only one $p\in J_2$ due to Lemma 1.

Case 2-2: for some $i\in I_1$, $x_{ij}=y_j$ for several $j\in J_1$.

Proof:

Choose any two j_1, $j_2\in J_1$.

Let $I_3=\{i\in I_1 : x_{ij_1}=y_{j_1}\}$,

$I_4=\{i\in I_1 : x_{ij_2}=y_{j_2}\}$,

$J_3=\{j\in J_2 : x_{ij}=1-y_{j_1}$ and $i\in I_3\}$

$J_4=\{j\in J_2 : x_{ij}=1-y_{j_2}$ and $i\in I_4\}$

For case 2-1:

Then we can construct two feasible solutions (x_1, y_1) and (x_2, y_2) as follows.

$y^1_{j_1}=y_{j_1}+\epsilon,\ \ y^1_{j_2}=y_{j_2}-\epsilon,\ \ y^2_{j_1}=y_{j_1}-\epsilon,\ \ y^2_{j_2}=y_{j_2}+\epsilon,$

$y^1_j=y^2_j=y_j$ for all $j\neq j_1$ or j_2.

$x^1_{ij_1}=x_{ij_1}+\epsilon$, $x^1_{ij}=x_{ij}-\epsilon$ for all $i\in I_3$ & $j\in J_3$

$x^1_{ij_2}=x_{ij_2}-\epsilon$, $x^1_{ij}=x_{ij}+\epsilon$ for all $i\in I_4$ & $j\in J_4$

$x^2_{ij_1}=x_{ij_1}-\epsilon$, $x^2_{ij}=x_{ij}+\epsilon$ for all $i\in I_3$ & $j\in J_3$

$x^2_{ij_2}=x_{ij_2}+\epsilon$, $x^2_{ij}=x_{ij}-\epsilon$ for all $i\in I_4$ & $j\in J_4$

$x^1_{ij}=x^2_{ij}=x_{ij}$ for all $i\neq I_3$ or I_4, for all $j\neq J_3$ or J_4

Where $\epsilon=$Min $[y_{j_1}, y_{j_2}$, min $\{x_{ij}, (1-x_{ij}$ for all $i\in I_3$ & $j\in J_3\}$, min$\{x_{ij}$, $(1-x_{ij})$ for all $i\in I_4$ & $j\in J_4\}]$

$(x,y)=(1/2)(x^1,y^1)+(1/2)(x^2,y^2)$ contradicts the assumption of an extreme soltuion.

For case 2-2:

Let i^* be a index set such that $\sum_{j\in J_1} x_{i^*}=1$.

Then we can construct (x^1, y^1) and (x^2, y^2) as for (case 1) except for i^*. for i^*, let $x^1_{i^*j_1}=x_{i^*j_1}+\epsilon$, $x^1_{i^*j_2}=x_{i^*j_2}-\epsilon$.

$$x^2_{i^*j_1}=x_{i^*j_1}-\epsilon,\quad x^2_{i^*j_2}=x_{i^*j_2}+\epsilon.$$

We can express (x,y) as a convex combination of above two feasible solution the same way as we did for case 1. This completes proof.//

Let $Q_{m,n}$ be the polytope of the feasible solutions to (6) and (8)~(10): that is, the polytope of the feasible solutions to the strong linear programming relaxation of the simple plant location problem. When Min$(n,m)\leq 2$, it has

Table 1 : Case 1 of Theorem 3

(y_j Matrix & x_{ij} Matrix)

Case 1:

	1/3	1/3	1/3	0	1	0	1	←y_j
	1	2	3	4	5	6	7	
1	1/3	1/3	1/3	—	—	—	—	
2	1/3	1/3	1/3	—	—	—	—	
3	1/3	1/3	1/3	—	—	—	—	
4	1/3	1/3	1/3	—	—	—	—	←x_{ik}
5	—	—	—	—	1	—	—	
6	—	—	—	—	—	—	1	
7	—	—	—	—	—	—	1	

Case 2-1:

Table 2 : Case 2-1 of Theorem 3

(y_j Matrix & x_{ij} Matrix)

	1/3	1/3	1/3	0	1	0	1	←y_j
	1	2	3	4	5	6	7	
1	1/3	—	—	—	2/3	—	—	
2	—	1/3	—	—	2/3	—	—	←x_{ij}
3	—	—	1/3	—	—	—	2/3	$J_1=\{1,2,3\}$
4	1/3	—	—	—	—	—	2/3	$J_2=\{5,7\}$
5	—	—	—	—	1	—	—	$I_3=\{1,4\}$
6	—	—	1/3	—	2/3	—	—	$I_4=\{2\}$
7	—	—	—	—	—	—	1	$\epsilon=1/3$

Case 2-2:

Table 3 : Case 2-2 of Theorem 3

(y_j Matrix & x_{ij} Matrix)

	1/3	1/3	1/3	0	1	0	1	0	←y_j
	1	2	3	4	5	6	7	8	
1	1/3	1/3	—	—	1/3	—	—	—	←x_{ij}
2	—	1/3	—	—	2/3	—	—	—	$J_1=\{1,2,3\}$
3	—	—	1/3	—	—	—	2/3	—	$J_2=\{5,7\}$
4	1/3	—	1/3	—	—	—	1/3	—	$I_3=\{1,4\}$
5	—	—	—	—	1	—	—	—	$I_4=\{2,8\}$
6	1/3	1/3	1/3	—	—	—	—	—	$i^*=\{6\}$
7	—	—	—	—	—	—	1	—	$\epsilon=1/3$
8	—	1/3	1/3	—	—	—	1/3	—	

been shown by Cho, Padberg, and Rao [6], Krarup and Pruzan [17] that all the extreme points of $Q_{m,n}$ are integral. The constraint matrix, in fact, is totally unimodular in this case. However, for values as small as $m=n=3$, $Q_{m,n}$ has fractional extreme points. For example, when $c_{13}=c_{21}=c_{32}=1$, all other $c_{ij}=0$ and $f_j=1$ for $j=1,2,3$, the unique optimal solution of minimizing $\sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}x_{ij}+\sum_{j=1}^{m} f_j y_j$ is $y_j=1/2$ for $j=1,2,3$ and $x_{11}=x_{12}=x_{22}=x_{23}=x_{31}={}_{33}=1/2$, all other $x_{ij}=0$. The value of this fractional solution is 1.5.

Here we first provide a similar result about P_n and extend it by one more dimension.

Proposition 4:

If (x,y) is an extreme point of P_n, then $|J_1|\geq 3$.

Proof:

Immediate consequence of Theorem 3.

That is, $|J_1|=2$ means $\sum_{j\in J_1} y_j=1$ and this directly contradicts Theorem 3. //

A direct consequence of the above proposition is that when $n\leq 2$, the k-median problem always has an integer optimal solution. In fact the constraint matrix of the k-median problem is totally unimodular when $n\leq 2$.

Now we extend above results to the case when $n\leq 3$.

Theorem 5:

If (x,y) is an extreme point of P_n, then $|J_1|\geq 4$.

Proof:

Assume (x,y) is a fractional extreme solution to P_n with $|J_1|=3$. Then we must have $\sum_{j\in J_1} y_j\neq 2$. For the case that $\sum_{j\in J_1} y_j=1$ is eliminated due to theorem 3. Let j_1, j_2, j_3 be the index such that $0<y_{j_1}<y_{j_2}<y_{j_3}<1$. We should examine 2 cases.

Case 1: For all $i\in I_1$, $\sum_{j\in J_1} x_{ij}=1$. That is, $x_{ij}=0$ for all $j\in J_2$. Note that for each $i\in I_1$, exactly two $x_{ij}\neq 0$ because the sum of any three y_j, $j\in J_1$ is larger than 1.

Case 2: For some $i\in I_1$, $\sum_{j\in J_1} x_{ij}\neq 1$.

Here we have two subcases.

Note that for each $i\in I_1$, exactly two $x_{ij}\neq 0$ due to Lemma 1.

(Case 2-1) For all $i\in I_1$, $x_{ij}=y_j$ for only one $j\in J_1$.

(Case 2-2) For some $i\in I_1$, $x_{ij}=y_j$ for two $j\in J_1$.

For case 1:

Let $J_1=\{j_1, j_2, j_3\}$ and $\epsilon=\text{Min}\ [x_{ij}, i\in I_1, \ \&\ j\in J_1]$.

Let $I_5=\{i\in I_1 : 0<x_{ij_1},\ x_{ij_2}<1\}$

$I_6=\{i\in I_1 : 0<x_{ij_1},\ x_{ij_3}<1\}$

$I_7=\{i\in I_1 : 0<x_{ij_2},\ x_{ij_3}<1\}$

$I_8=\{i\in I_1 : x_{ij}=\epsilon\}$

We construct two feasible solutions (x^1, y^1) and (x^2, y^2) as follows.

$y^1_{j_1}=y_{j_1}+\epsilon$, $y^1_{j_2}=y_{j_2}-\epsilon$, $y^2_{j_1}=y_{j_1}-\epsilon$, $y^2_{j_2}=y_{j_2}+\epsilon$,

$y^1_j=y^2_j=y_j$ for other j.

$x^1_{ij_1}=x_{ij_1}+\epsilon$, $x^1_{ij_2}=x_{ij_2}-\epsilon$, $x^2_{ij_1}=x_{ij_1}-\epsilon$, $x^2_{ij_2}=x_{ij_1}+\epsilon$, for all $i \in I_5/I_8$.

$x^1_{ij^1}=x_{ij_1}+\epsilon$, $x^1_{ij_3}=x_{ij_3}-\epsilon$, $x^2_{ij_1}=x_{ij_1}-\epsilon$, $x^2_{ij_3}=x_{ij_3}+\epsilon$, for all $i \in I_6/I_8$.

$x^1_{ij_2}=x_{ij_2}-\epsilon$, $x^1_{ij_3}=x_{ij_3}+\epsilon$, $x^2_{ij_2}=x_{ij_2}+\epsilon$, $x^2_{ij_3}=x_{ij_3}-\epsilon$, for all $i \in I_7/I_8$.

$x^1_{ij}=x^2_{ij}=x_{ij}$ for all other i, j.

The fact that $(x,y)=(1/2)(x^1,y^1)+(1/2)(x^2,y^2)$ contradicts the assumption of extreme solution.

For case 2-1:

Choose any two $j \in J_1$, for example j_1, j_2 and

let $\epsilon=1-y_{j_2}$. Note that $y_{j_1}<y_{j_2}$.

We construct two feasible solutions (x^1, y^1) and (x^2, y^2) as follows.

$y^1_{j_1}=y_{j_1}+\epsilon$, $y^1_{j_2}=y_{j_2}-\epsilon$, $y^2_{j_1}=y_{j_1}-\epsilon$, $y^2_{j_2}=y_{j_2}+\epsilon$,

$y^1_j=y^2_j=y_j$ for all $j \neq j_1$ or j_2.

$x^1_{ij_1}=x_{ij_1}+\epsilon$, $x^1_{ij}=x_{ij}-\epsilon$ for all $i \in I_3$ & $j \in J_3$.

$x^1_{ij_2}=x_{ij_1}-\epsilon$, $x^1_{ij}=x_{ij}+\epsilon$ for all $i \in I_4$ & $j \in J_4$.

Table 4 : Case 1 of Theorem 5

Case 1:

(y_j Matrix & x_{ij} Matrix)

	7/12	8/12	9/12	0	0	0	1	1	←— y_j
	1	2	3	4	5	6	7	8	
1	7/12	—	5/12	—	—	—	—	—	←— x_{ij}
2	4/12	8/12	—	—	—	—	—	—	$I_5=\{2, 4\}$
3	—	3/12	9/12	—	—	—	—	—	$I_6=\{1\}$
4	7/12	5/12	—	—	—	—	—	—	$I_7=\{6\}$
5	3/12	9/12	—	—	—	—	—	—	$I_8=\{3, 5\}$
6	—	8/12	4/12	—	—	—	—	—	$\epsilon=3/12$
7	—	—	—	—	—	—	1	—	
8	—	—	—	—	—	—	—	1	

Case 2-1:

Table 5 : Case 2-1 of Theorem 5

(y_j Matrix & x_{ij} Matrix)

	7/12	8/12	9/12	0	1	0	1	←y_j
	1	2	3	4	5	6	7	
1	7/12	—	—	—	5/12	—	—	←x_{ij}
2	—	8/12	—	—	4/12	—	—	$j_1=1,\ j_2=2$
3	—	—	9/12	—	—	—	3/12	$I_3=\{1,4\}$
4	7/12	—	—	—	—	—	7/12	$I_4=\{2,6\}$
5	—	—	—	—	1	—	—	$J_3=\{5,7\}$
6	—	8/12	—	—	—	—	4/12	$J_4=\{5,7\}$
7	—	—	—	—	—	—	1	$\epsilon=4/12$

Case 2-2:

Table 6 : Case 2-2 of Theorem 5

(y_j Matrix & x_{ij} Matrix)

	7/12	8/12	9/12	0	1	0	1	0	←y_j
	1	2	3	4	5	6	7	8	
1	7/12	—	—	—	5/12	—	—	—	←x_{ij}
2	—	8/12	—	—	4/12	—	—	—	
3	—	—	9/12	—	—	—	3/12	—	$I_3=\{1\}$
4	4/12	8/12	—	—	—	—	—	—	$I_4=\{2\}$
5	—	—	—	—	1	—	—	—	$I_5=\{4\}$
6	4/12	—	8/12	—	—	—	—	—	$I_6=\{6\}$
7	—	—	—	—	—	—	1	—	$I_7=\{8\}$
8	3/12	9/12	—	—	—	—	—	—	$I_8=\{3\}$

$j_1=1,\ j_2=2.$ $J_3=J_4=\{5\}$ $\epsilon=3/12.$

$x^2_{ij_1}=x_{ij_1}-\epsilon,\ x^2_{ij}=x_{ij}+\epsilon$ for all $i\in I_3$ & $j\in J_3$.

$x^2_{ij_2}=x_{ij_1}+\epsilon,\ x^2_{ij}=x_{ij}-\epsilon$ for all $i\in I_4$ & $j\in J_4$.

$x^1_{ij}=x^2_{ij}=x_{ij}$ for all $i\neq I_3$ or I_4, $j\neq J_3$ or J_4.

Expression of $(x,\ y)=(1/2)(x^1,\ y^1)+(1/2)(x^2,\ y^2)$ means contradiction.

For case 2-2:

We can think of case 2-2 as a composite of case 1 and case 2-1, and we derive a contradiction as we did for case 1 and case 2-1. //

Corollary 6:

The k-mediian problem of $n\leq 3$ always has an integer optimal solution.

Proof: Immediate consequence of Theorem 5. //

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