Polyhedral Study of the K-Median Problem

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I. Introduction

The past two decades have witnessed a tremendous growth in the literature on location problems. However, among the myriads of formulations the simple plant location problem and the k-median problem have played a central role. This phenomenon is due to the facts that both problems have a wide range of real-world applications and a mathematical formulation of these problems as an integer program has proven very fruitful in the derivation of solution methods.

Consider an index set \( I = \{1, 2, \ldots, n\} \) of \( n \) points, and a positive integer \( k \leq n \), and let \( c_{ij} \) be the shortest distance between two points \( i, j \in I \). The \( k \)-median problem consists of identifying a subset \( S \) of \( I \), \( |S| = k \) so as to minimize \( \sum_{i=1}^{n} \min_{j \in S} c_{ij} \). (Here \( |S| \) denotes the cardinality of the set \( S \)). The \( k \)-median problem has the following combinatorial formulation.

Combinatorial Formulation:

\[
\min \left\{ \sum_{i=1}^{n} \min_{j \in S} c_{ij} \right\}_{\substack{S \subseteq I \\ \ \ \ \ \ \ \ \ \ \ \ \ |S| = k}}
\]

We introduce integer variables. Let \( y_j = 1 \) if a point \( j \) is selected as a median, otherwise \( 0 \) and \( x_{ij} = 1 \) if a point \( j \) is the closest median to point \( i \), otherwise \( 0 \). With \( x, y \) variables the \( k \)-median problem is formulated as an integer program as follows.

Integer Program Formulation:

\[
Z_{LP} = \min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}
\]

subject to

(1)
\[ \sum_{j=1}^{n} x_{ij} = 1 \quad i \in I \] (2)

\[ \sum_{j=1}^{n} y_j = k \] (3)

\[ 0 \leq x_{ij} \leq y_j \leq 1 \quad i, j \in I \] (4)

\[ x_{ij}, y_j \text{ integral} \quad i, j \in I \] (5)

A vast number of algorithms were proposed for the $k$-median problem. We refer readers to ReVelle [19], Francis and White [14], Christofides [7], Jacobsen and Pruzan [16], Handler and Mirchandani [15], Krasnap and Pruzan [17], Cornuejols [9] [11] [12], Fisher and Hochbaum [13], Papadimitriou [18], Rosing [20], Beasley and Christofides [8], Boffey [5], Beasley [4].

Most of the successful algorithms for the $k$-median problem are based on the strong linear programming relaxation. In [1] [2] [3] we presented and explained why the strong linear programming relaxation provides a tight lower bound in the probabilistic sense. In this paper we investigate the phenomenon with a polyhedral approach.

II. Polyhedral Analysis

In this section we investigate the polytope of the extreme solutions to the strong linear program relaxation of $k$-median problem constraints.

\[ \sum_{j=1}^{n} x_{ij} = 1 \quad i \in I \] (6)

\[ \sum_{j=1}^{n} y_j = k \] (7)

\[ x_{ij} \leq y_j \quad i, j \in I \] (8)

\[ y_j \leq 1 \quad j \in I \] (9)

\[ x_{ij}, y_j \geq 0 \quad i, j \in I \] (10)

Let $P_*$ be the polytope defined by (6)–(10). We present properties of the fractional extreme points $(x, y)$ to $P_*$ below.
Lemma 1:
If \((x, y)\) is a fractional extreme point of the polytope \(P_n\),
then for each \(i \in I\), there is at most one \(j \in I\) with \(0 < x_{ij} < y_j\).

Proof:
Let \((x, y)\) be a fractional solution such that above condition does not hold.
Then there exist \(p, j_1, j_2\) such that \(x_{pj_1} < y_{j_1}, x_{pj_2} < y_{j_2}\).
Let \(x_{pj_1}^1 = x_{pj_1} + \epsilon, x_{pj_2}^1 = x_{pj_2} - \epsilon, x_{pj_1}^2 = x_{pj_1} - \epsilon, x_{pj_2}^2 = x_{pj_2} + \epsilon,\)
\(x_{ij}^i = x_{ij}^2 = x_{ij}\) for all other \(i, j\) and \(y_j^1 = y_j^2 = y_j\) for all \(j\).
where \(\epsilon = \text{Min} \{x_{pj_1}^1, x_{pj_2}^1, y_{j_1} - x_{pj_1}^1, y_{j_2} - x_{pj_2}^1\}\).
Then \((x, y) = (1/2)(x^1, y^1) + (1/2)(x^2, y^2)\). \((x^1, y^1)\) and \((x^2, y^2)\) both are feasible solutions to \(P_n\). This contradicts the assumption that \((x, y)\) is an extreme solution. \(\Box\)

A similar result is known for the simple plant location problem. In fact, Cornuejols et al [10] completely characterized the fractional extreme solutions to the simple plant location problem.

Suppose we are given the shortest distance matrix between all pairs of points. The optimal solution to the 1-median problem is reduced to find a column of the shortest distance matrix with smallest column sum. In fact, when \(k=1\), all the extreme solutions to the polytope \(P_n\) are integral regardless of \(n\). We show this in the next theorem.

Theorem 2:
The linear programming relaxation of the 1-median problem always has an integer optimal solution.

Proof:
Suppose there exists a fractional extreme solution to the linear programming relaxation of 1-median problem.

Let \(J_1 = \{j \in I : 0 < y_j < 1\}\)
Then for each \(i \in I\), \(x_{ij} = y_j\) for all \(j \in J_1\) and \(x_{ij} = 0\) for all \(j \notin J_1\).
Choose any two points \(j_1, j_2 \in J_1\), any \(p \in I\) and let
\(x_{pj_1}^1 = x_{pj_1} + \epsilon, x_{pj_2}^1 = x_{pj_2} - \epsilon, y_{j_1}^1 = y_{j_1} + \epsilon, y_{j_2}^1 = y_{j_2} - \epsilon\)
\[ x_{p1}^2 - x_{p1} - \epsilon, \quad x_{p2}^2 - x_{p2} + \epsilon, \quad y_{j1}^2 = y_{j1} - \epsilon, \quad y_{j2}^2 = y_{j2} + \epsilon \]

All other \( x_{ij}, y_j \) remain unchanged.

Then \((x, y) = \frac{1}{2}(x^1, y^1) + \frac{1}{2}(x^2, y^2)\) and \((x^1, y^1), (x^2, y^2)\) both are feasible solutions to \(P_n\). This contradicts the assumption that \((x, y)\) is an extreme solution. \(/\)

In the next theorem, we extend the above result to more general cases.

**Theorem 3:**

If \((x, y)\) is an extreme solution to the polytope \(P_n\), then \(\sum_{i \in I_j} y_i \geq 2\).

**Proof:**

Suppose \((x, y)\) is an extreme point to \(P_n\) with \(\sum_{i \in I_j} y_i = 1\).

Let \(I_1 = \{i \in I : 0 < x_{ij} < 1\} \text{ for some } j \in J_1\}, \quad J_2 = \{j \in J : y_j = 1\}.

We have two cases to consider here.

**Case 1:** for all \(i \in I_1\), \(\sum_{i \in I_j} x_{ij} = 1\). (That is, \(x_{ij} = 0\) for all \(j \in J_2\)).

For this case we can derive contradiction in the same way as for Theorem 2.

**Case 2:** for some \(i \in I_1\), \(\sum_{i \in I_j} x_{ij} \neq 1\).

Here we have two subcases.

**Case 2-1:** for all \(i \in I_1\), \(x_{ij} = y_j\) for only one \(j \in J_1\). That is, \(x_{ip} = 1 - y_j\) for only one \(p \in J_2\) due to Lemma 1.

**Case 2-2:** for some \(i \in I_1\), \(x_{ij} = y_j\) for several \(j \in J_1\).

**Proof:**

Choose any two \(j_1, j_2 \in J_1\).

Let \(I_3 = \{i \in I_1 : x_{ij_1} = y_{j_1}\}\),

\(I_4 = \{i \in I_1 : x_{ij_2} = y_{j_2}\}\),

\(J_3 = \{j \in J_2 : x_{ij_1} = 1 - y_{j_1} \text{ and } i \in I_3\}\)

\(J_4 = \{j \in J_2 : x_{ij_2} = 1 - y_{j_2} \text{ and } i \in I_4\}\)

For case 2-1:

Then we can construct two feasible solutions \((x_1, y_1)\) and \((x_2, y_2)\) as follows.

\[ y_{j1}^1 = y_{j1} + \epsilon, \quad y_{j2}^1 = y_{j2} - \epsilon, \quad y_{j1}^2 = y_{j1} - \epsilon, \quad y_{j2}^2 = y_{j2} + \epsilon, \]

\[ y_{j1}^i = y_{j2} = y_j \text{ for all } j \neq j_1 \text{ or } j_2. \]
\[ x_{ij} = x_{ij} + \epsilon, \quad x_{ij} = x_{ij} - \epsilon \text{ for all } i \in I_3 \text{ and } j \in J_3 \]
\[ x_{ij} = x_{ij} - \epsilon, \quad x_{ij} = x_{ij} + \epsilon \text{ for all } i \in I_4 \text{ and } j \in J_4 \]
\[ x_{ij} = x_{ij} - \epsilon, \quad x_{ij} = x_{ij} + \epsilon \text{ for all } i \in I_3 \text{ and } j \in J_3 \]
\[ x_{ij} = x_{ij} + \epsilon, \quad x_{ij} = x_{ij} - \epsilon \text{ for all } i \in I_4 \text{ and } j \in J_4 \]
\[ x_{ij} = x_{ij}^2 = x_{ij} \text{ for all } i \neq I_3 \text{ or } I_4, \text{ for all } j \neq J_3 \text{ or } J_4 \]

Where \( \epsilon = \min \{ y_{ij_1}, y_{ij_2}, \min \{ x_{ij_3}, (1 - x_{ij_4}) \text{ for all } i \in I_3 \text{ and } j \in J_3 \}, \min \{ x_{ij_5}, (1 - x_{ij_6}) \text{ for all } i \in I_4 \text{ and } j \in J_4 \} \} \]

\((x, y) = (1/2)(x^1, y^1) + (1/2)(x^2, y^2)\) contradicts the assumption of an extreme solution.

For case 2-2:

Let \( i^* \) be a index set such that \( \sum_{j \in J_1} x_{i^*} = 1. \)

Then we can construct \((x^1, y^1)\) and \((x^2, y^2)\) as for (case 1) except for \( i^* \).

for \( i^* \), let \( x_{i^*j_1} = x_{i^*j_1} + \epsilon, \quad x_{i^*j_1} = x_{i^*j_1} - \epsilon \).

\[ x_{i^*j_1} = x_{i^*j_1} + \epsilon, \quad x_{i^*j_1} = x_{i^*j_1} - \epsilon. \]

We can express \((x, y)\) as a convex combination of above two feasible solution the same way as we did for case 1. This completes proof. \( \blacksquare \)

Let \( Q_{m,n} \) be the polytope of the feasible solutions to (6) and (8)~(10): that is, the polytope of the feasible solutions to the strong linear programming relaxation of the simple plant location problem. When \( \text{Min}(n, m) \leq 2 \), it has

| Case 1: Table 1: Case 1 of Theorem 3 (y, Matrix & x, Matrix) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( y \)        | \( x \)        |
| 1/3            | 1/3            | 1/3            | 1/3            | 1/3            | 1/3            | 1/3            |                  |
| 1              | 1              | 2              | 3              | 4              | 5              | 6              | 7               |
| 1              | 1              | 1              | 1              | 1              | 1              | 1              | 1               |
| 2              | 1              | 1              | 1              | 1              | 1              | 1              | 1               |
| 3              | 1              | 1              | 1              | 1              | 1              | 1              | 1               |
| 4              | 1              | 1              | 1              | 1              | 1              | 1              | 1               |
| 5              | -              | -              | -              | -              | -              | 1              | -               |
| 6              | -              | -              | -              | -              | -              | -              | 1               |
| 7              | -              | -              | -              | -              | -              | -              | 1               |
been shown by Cho, Padberg, and Rao [6], Krarup and Pruzan [17] that all the extreme points of $Q_{m,n}$ are integral. The constraint matrix, in fact, is totally unimodular in this case. However, for values as small as $m=n=3$, $Q_{m,n}$ has fractional extreme points. For example, when $c_{13}=c_{21}=c_{32}=1$, all other $c_{ij}=0$ and $f_j=1$ for $j=1,2,3$, the unique optimal solution of minimizing $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{j=1}^{n} f_j y_j$ is $y_j=1/2$ for $j=1,2,3$ and $x_{11}=x_{12}=x_{22}=x_{32}=x_{33}=1/3$, all other $x_{ij}=0$. The value of this fractional solution is 1.5.

Here we first provide a similar result about $P_n$ and extend it by one more dimension.
Proposition 4:
If \((x,y)\) is an extreme point of \(P_n\), then \(|J_1| \geq 3\).

Proof:
Immediate consequence of Theorem 3.

That is, \(|J_1| = 2\) means \(\sum_{j \in J_1} y_j = 1\) and this directly contradicts Theorem 3. 

A direct consequence of the above proposition is that when \(n \leq 2\), the \(k\)-median problem always has an integer optimal solution. In fact the constraint matrix of the \(k\)-median problem is totally unimodular when \(n \leq 2\).

Now we extend above results to the case when \(n \leq 3\).

Theorem 5:
If \((x,y)\) is an extreme point of \(P_n\), then \(|J_1| \geq 4\).

Proof:
Assume \((x,y)\) is a fractional extreme solution to \(P_n\) with \(|J_1| = 3\). Then we must have \(\sum_{j \in J_1} y_j \neq 2\). For the case that \(\sum_{j \in J_1} y_j = 1\) is eliminated due to theorem 3. Let \(j_1, j_2, j_3\) be the index such that \(0 < y_{j_1} < y_{j_2} < y_{j_3} < 1\). We should examine 2 cases.

Case 1: For all \(i \in I_1\), \(\sum_{j \in J_1} x_{ij} = 1\). That is, \(x_{ij} = 0\) for all \(j \in J_2\). Note that for each \(i \in I_1\), exactly two \(x_{ij} \neq 0\) because the sum of any three \(y_j, j \in J_1\) is larger than 1.

Case 2: For some \(i \in I_1\), \(\sum_{j \in J_1} x_{ij} \neq 1\).

Here we have two subcases.

Note that for each \(i \in I_1\), exactly two \(x_{ij} \neq 0\) due to Lemma 1.

(Case 2-1) For all \(i \in I_1\), \(x_{ij} = y_j\) for only one \(j \in J_1\).

(Case 2-2) For some \(i \in I_1\), \(x_{ij} = y_j\) for two \(j \in J_1\).

For case 1:
Let \(J_1 = \{j_1, j_2, j_3\}\) and \(\epsilon = \min [x_{ij}, i \in I_1, \& \ j \in J_1]\).

Let \(I_0 = \{i \in I_1 : 0 < x_{ij}, \ x_{ij} < 1\}\)
\(I_6 = \{i \in I_1 : 0 < x_{ij}, \ x_{ij} < 1\}\)
\(I_7 = \{i \in I_1 : 0 < x_{ij}, \ x_{ij} < 1\}\)
\(I_8 = \{i \in I : x_{ij} = \epsilon\}\)
We construct two feasible solutions \((x^1, y^1)\) and \((x^2, y^2)\) as follows.
\[
y_{ij}^1 = y_{ij} + \epsilon, \quad y_{ij}^2 = y_{ij} - \epsilon, \quad y_{ij}^3 = y_{ij} - \epsilon, \quad y_{ij}^4 = y_{ij} + \epsilon,
\]
\[
y_j^1 = y_j^2 = y_j \text{ for other } j.
\]
\[
x_{ij}^1 = x_{ij} + \epsilon, \quad x_{ij}^2 = x_{ij} - \epsilon, \quad x_{ij}^3 = x_{ij} - \epsilon, \quad x_{ij}^4 = x_{ij} + \epsilon, \text{ for all } i \in I_5/I_8.
\]
\[
x_{ij}^1 = x_{ij} + \epsilon, \quad x_{ij}^2 = x_{ij} - \epsilon, \quad x_{ij}^3 = x_{ij} - \epsilon, \quad x_{ij}^4 = x_{ij} + \epsilon, \text{ for all } i \in I_6/I_8.
\]
\[
x_{ij}^1 = x_{ij} - \epsilon, \quad x_{ij}^2 = x_{ij} + \epsilon, \quad x_{ij}^3 = x_{ij} + \epsilon, \quad x_{ij}^4 = x_{ij} - \epsilon, \text{ for all } i \in I_7/I_8.
\]
\[
x_j^1 = x_j^2 = x_j \text{ for all other } i, j.
\]
The fact that \((x, y) = (1/2)(x^1, y^1) + (1/2)(x^2, y^2)\) contradicts the assumption of extreme solution.

For case 2-1:

Choose any two \(j \in J\), for example \(j_1, j_2\) and let \(\epsilon = 1 - y_{j_1}\). Note that \(y_{j_1} < y_{j_2}\).

We construct two feasible solutions \((x^1, y^1)\) and \((x^2, y^2)\) as follows.
\[
y_{ij}^1 = y_{ij} + \epsilon, \quad y_{ij}^2 = y_{ij} - \epsilon, \quad y_{ij}^3 = y_{ij} - \epsilon, \quad y_{ij}^4 = y_{ij} + \epsilon,
\]
\[
y_j^1 = y_j^2 = y_j \text{ for all } j \neq j_1 \text{ or } j_2.
\]
\[
x_{ij}^1 = x_{ij} + \epsilon, \quad x_{ij}^2 = x_{ij} - \epsilon \text{ for all } i \in I_3 \& j \in J_3.
\]
\[
x_{ij}^1 = x_{ij} - \epsilon, \quad x_{ij}^2 = x_{ij} + \epsilon \text{ for all } i \in I_4 \& j \in J_4.
\]

| Case 1: \((y_j, \text{ Matrix} \& x_{ij}, \text{ Matrix})\) |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 7/12    | 8/12    | 9/12    | 0       | 0       | 0       | 1       | 1       | \(\cdots\) |
| 1       | 2       | 3       | 4       | 5       | 6       | 7       | 8       |
| 1       | 7/12    | \(-\)   | 5/12    | \(-\)   | \(-\)   | \(-\)   | \(-\)   | \(-\)   |
| 2       | 4/12    | 8/12    | \(-\)   | \(-\)   | \(-\)   | \(-\)   | \(-\)   | \(-\)   |
| 3       | \(-\)   | 3/12    | 9/12    | \(-\)   | \(-\)   | \(-\)   | \(-\)   | \(-\)   |
| 4       | 7/12    | 5/12    | \(-\)   | \(-\)   | \(-\)   | \(-\)   | \(-\)   | \(-\)   |
| 5       | 3/12    | 9/12    | \(-\)   | \(-\)   | \(-\)   | \(-\)   | \(-\)   | \(-\)   |
| 6       | \(-\)   | 8/12    | 4/12    | \(-\)   | \(-\)   | \(-\)   | \(-\)   | \(-\)   |
| 7       | \(-\)   | \(-\)   | \(-\)   | \(-\)   | \(-\)   | 1       | \(-\)   | \(-\)   |
| 8       | \(-\)   | \(-\)   | \(-\)   | \(-\)   | \(-\)   | \(-\)   | 1       | \(-\)   |

\(I_5 = (2, 4)\)
\(I_6 = (1)\)
\(I_7 = (6)\)
\(I_8 = (3, 5)\)
\(\epsilon = 3/12\)
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<td>I₆=[3]</td>
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j₁=1, j₂=2, J₅=J₆=[5], ε=3/12.

\[ x_{i,j} = x_{i,j} - \epsilon, \quad x_{i,j} = x_{i,j} + \epsilon \] for all \( i \in I₅ \) & \( j \in J₃ \).

\[ x_{i,j} = x_{i,j} + \epsilon, \quad x_{i,j} = x_{i,j} - \epsilon \] for all \( i \in I₄ \) & \( j \in J₄ \).

\[ x_{i,j} = x_{i,j} \] for all \( i \notin I₃ \) or \( I₄ \), \( j \notin J₃ \) or \( J₄ \).

Expression of \( (x, y) = (1/2)(x^1, y^1) + (1/2)(x^2, y^2) \) means contradiction.

For case 2-2:

We can think of case 2-2 as a composite of case 1 and case 2-1, and we derive a contradiction as we did for case 1 and case 2-1. //

**Corollary 6:**

The \( k \)-mediam problem of \( n \leq 3 \) always has an integer optimal solution.
Proof: Immediate consequence of Theorem 5. \[ \]

References


