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Transaction Costs and the Arbitrage Pricing Theory

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Ⅰ. Introduction

The Arbitrage Pricing Theory(APT), originally developed by Ross(1976), has been subject to various extensions(e.g., Connor, 1984; Chamberlain and Rothschild, 1983). We can broadly classify the existing variants of the APT into two different classes: arbitrage-based APT and "equilibrium" APT. The arbitrage-based APT includes Ross(1976), Huberman(1982), Chamberlain (1983), Chamberlain and Rothschild(1983) and Ingersoll(1984). The equilibrium APT includes Connor(1984), Dybvig(1983), Grinblatt and Titman (1983), Wei(1988), Milne(1988) and Latham(1989) among others.

The existing arbitrage-based APT models have an approximate pricing relation under an uncorrelated-residuals or approximate factor structure. In the absence of arbitrage opportunities, there exist factor "prices" such that

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an infinite sum of squared deviations from the multifactor pricing equation is finite. This approximate pricing has been criticized on empirical grounds. Since the model prices "most" assets well but permits arbitrarily large deviations from exact pricing on a finite set of assets, it is difficult to conceive of any(finite) empirical procedure that could be used to refute the actual conclusion of the APT(see Shanken, 1985).

The existing "Equilibrium" APT models have an exact or approximate multi-factor pricing relation under an uncorrelated-residuals or approximate factor structure. These models restrict investors' preferences. Connor(1984) derives an exact pricing relation based on the assumption that the market portfolio is "well-diversified". Dybvig(1983), Grinblatt and Titman(1983) and Wei (1988) provide an explicit bound for the pricing error for individual securities. The equilibrium APT models have also been criticized on testability grounds, since testing these models requires a joint hypothesis about the true market portfolio. In particular, the equilibrium APT is subject to the same difficulties encountered in testing the Capital Asset Pricing Model as raised by the Roll's(1977) critique (see Shanken, 1982, 1985).

The pricing error bound has been derived only in the context of the equilibrium APT. In the existing literature, there is a presumption that the important and interesting pieces of the APT are based on equilibrium rather than arbitrage analysis(see Latham, 1989). This paper derives an APT model in the arbitrage framework which yields the exact pricing error from the APT pricing for individual securities. Thus, this paper contributes to the strand of literature associated with arbitrage-based APT models.

In the arbitrage framework, Chamberlain(1983) has demonstrated that an exact pricing APT model holds in the sense that idiosyncratic risk is not priced, if the mean-variance frontier is well-diversified. In Chamberlain (1983), a global "benchmark" portfolio should be diversified in order to have a well-diversified mean-variance frontier. This paper shows that an exact pricing APT holds if the efficient arbitrage portfolio is well-diversified. The efficient arbitrage portfolio is unique only up to a given subset of assets in the economy. We do not need to refer to the universe of assets.[1] Our model is consistent with the original APT which merely requires that a given subset of assets satisfies a factor structure(see Roll and Ross, 1980).

This paper explains how idiosyncratic risk is priced if the efficient arbitrage portfolio is not well-diversified. The efficient arbitrage portfolio is important in pricing and plays a role as an extra factor. Our model does not require an additional distribution assumption on idiosyncratic risk, unlike Wei (1988) who requires equilibrium analysis. Our model can be used to find the impact of omitting factors in an empirical investigation of the APT. If all of the factors are omitted, then our model becomes an exact APT pricing relation with one factor as the efficient arbitrage portfolio.

This paper also demonstrates the robustness of the original arbitrage-based APT to market imperfections. The APT has been examined in 'friction-free' markets. The arbitrage-based APT survives with market imperfections associated with transaction costs. We derive an APT model by imposing the no arbitrage condition in an economy where investors pay transaction costs in trading securities. In this model, there are two additional pricing factors, the efficient arbitrage portfolio and the transaction cost factor. Reinganum (1981) tests APT model with the data of the CRSP daily stock returns between 1962 and 1978. His test shows that the APT is not supported by the data. He finds that portfolios of small firms earn on average 20% per year more than portfolios of large firms, even after controlling for APT risk premium. Our model provides the implication that if transaction costs for trading a small firm stock are greater than a large firm stock, the expected

1) Ingersoll(1984) has shown that the pricing error for a well-diversified portfolio is zero. However, he has not shown the pricing error for a portfolio which is not well-diversified.

rate of return on a small firm stock should be greater than the expected rate of return on a large firm stock, given other things held constant. Thus, it may explain the size anomaly of the APT raised by Reinganum(1981). Stoll and Whaley(1983) also find a similar result using a similar data set except for 6% mean abnormal returns on the portfolio of the smallest firms. This theoretical analysis is consistent with the findings of Stoll and Whaley.

Transaction costs may vary depending on assets. A striking result from our model is that there may be multiple rates of return on zero-beta portfolios. These rates are dependent on transaction costs of the assets which are asset-specific. The analysis under market imperfections establishes a convenient framework for understanding pricing an asset in an international context. Thus far, international extensions of the APT are minimal, and they are conducted only under perfect market conditions characterized by exchange rate risks(e.g., Solnik, 1983; Cho, Eun and Senbet, 1986.) This is an issue that we shall pursue in the future, given that market imperfections vary not only across assets but also across national boundaries.

Section 2 derives an APT model by imposing the absence of arbitrage profits in 'friction-free' markets. It derives the exact pricing error for individual assets. It also examines a condition for an exact pricing APT model.

Section 3 derives an APT model in the arbitrage framework when transaction costs exist. It shows how transaction costs affect the exact pricing error for individual assets. Finally, section 4 provides conclusions.

Ⅱ. Arbitrage Pricing Theory

The APT assumes that returns are generated by a K-factor structure denoted by

$$R = E + Bf + e \tag{1}$$

where R = an N-dimensional vector of the random asset returns

E = an N-dimensional vector of the ex ante expected returns

B = an N x K matrix of factor loadings

f = an K-dimensional vector of mean zero factors, which are assumed to be uncorrelated with each other[2)]

e = an N-dimensional vector of mean zero idiosyncratic disturbances, which are assumed to be uncorrelated with the factors but correlated with each other.

Suppose we form an arbitrage portfolio with no systematic risk such that

$$w'1_N = 0 \text{ and } w'B = 0 \tag{2}$$

where w = an N-dimensional vector of portfolio weights and 1_N is an N-dimensional vector of ones.

Ross(1976) has shown that an approximate pricing APT model holds under a uncorrelated-residuals factor structure in an infinite economy. The sum of squared deviation from APT pricing is bounded as the number of assets approaches

infinity, i.e.

2) The assumption that factors are uncorrelated with each other is not necessary to derive our model. It can be easily relaxed but we follow the convention.

$$(E - \lambda_0 1_N - B\lambda)(E - \lambda_0 1_N - B\lambda)' < \infty \text{ as } N \rightarrow \infty . \quad (3)$$

where λ_0 is a constant and λ is a K-dimensional vector. In order to obtain the pricing relationship (3), Ross considers the following problem.

$$\begin{aligned} &\underset{w}{\text{Minimize}} \quad w'Vw \\ &\text{subject to} \quad w'1_N = 0 \\ &\qquad\qquad\quad\ w'B = 0 \\ &\text{and} \quad 0 < c \leq w'E . \end{aligned}$$

where V is the covariance matrix of e.[3]

He demonstrates that the minimum variance for the above problem is strictly positive. In other words, the variance of the efficient arbitrage portfolio with zero factor risk and zero net investment which provides a positive expected return must be bounded away from zero (i.e., positive). This result is intuitively appealing, since if a zero-investment, zero-loadings arbitrage portfolio with positive expected return has zero variance, it implies an arbitrage opportunity.

In what follows, we will derive an APT model by using the Ross result and duality. The Ross result implies that if

$$w^{*\prime}1_N = 0, \quad w^{*\prime}B = 0, \text{ and } w^{*\prime}E > 0 \text{ for } w^* \in R^N, \quad (4)$$

then we must have

$$w^{*\prime}Vw^* > 0 \quad (5)$$

where w^* is the efficient arbitrage portfolio, the solution of the Ross Problem. In other words, Ross shows that the variance of the efficient

3) This problem allows a correlated-residual structure so that V need not be diagonal.

arbitrage portfolio is strictly positive.

It follows from (4) and (5) that there is no solution, $w^* \in R^N$, which satisfies the following system,

$$w^{*\prime} 1_N = 0, \quad w^{*\prime} B = 0, \; w^{*\prime} V w^* = 0 \text{ and } w^{*\prime} E > 0. \tag{6}$$

The following lemma shows that if and only if there is no solution for the system (6), the other system has a solution.

Lemma 1. Exactly one of the following systems has a solution.

System A: $w^{*\prime} 1_N = 0$, $w^{*\prime} V w^* = 0$ and $w^{*\prime} E > 0$ for $w^* \in R^N$.

$w^{*\prime} B = 0$,

System B: $E = \lambda_0 1_N + B\lambda + \alpha V w^*$ for $\lambda \in R^K$,

where λ_0 and α are constants.

Proof: See Appendix.

The Ross result (6) implies that there is no solution of $w^* \in R^N$ for System A. Lemma 1 states that if System A has no solution, there is a solution for System B. Thus, there is a solution of $\lambda_0 \in R$, $\lambda \in R^K$ and $\alpha \in R$ for System B. System B is an APT model

$$E = \lambda_0 1_N + B\lambda + \alpha V w^*. \tag{7}$$

We can identify α by premultiplying the efficient arbitrage portfolio, w^*, with both sides of (7). It is easy to see that α is the risk-return tradeoff for the efficient arbitrage portfolio. The Ross result implies that α is strictly positive.

Define a positive constant λ_β as $\alpha w^{*\prime} V w^* \equiv w^{*\prime} E$.

Theorem 1. Under no arbitrage and the factor structure (1), an APT model holds,

$$E = \lambda_0 1_N + B\lambda + \beta\lambda_\beta \tag{8}$$

where β is an N-dimensional vector of the efficient arbitrage portfolio betas,

i.e., $Vw^*/w^{*\prime}Vw^*$.

Note that the efficient arbitrage portfolio is important in pricing and acts as an additional factor. As long as the efficient arbitrage portfolio holds idiosyncratic risk, the idiosyncratic risk of the individual asset correlated to that of the efficient arbitrage portfolio is priced. However, the pure idiosyncratic risk of the individual asset is not priced when it is uncorrelated with the idiosyncratic risk of the efficient arbitrage portfolio. Namely, $var(\tilde{e}_i) - cov(\tilde{e}_i, \Sigma w_i^* \tilde{e}_i)$ is pure idiosyncratic risk, where $\tilde{e}_i$ and w_i^* are the ith elements of the vectors e and w^*, respectively.

Our model has a similar spirit to Wei (1988) in that residual risk can be priced. However, there are important differences. First, our model is derived in the arbitrage framework. On the other hand, Wei has used utility-based equilibrium arguments. Second, Wei has employed an assumption that the idiosyncratic risk follows a separating distribution of Ross (1978). Our result is immediate from Ross (1976). It does not require any additional distributional assumptions. Third, the efficient arbitrage portfolio plays a role in pricing residual risks here, whereas the market portfolio does in Wei (1988). Fourth, our pricing relationship is unique only up to a subset of assets. The pricing relationship in Wei (1988) includes all of the universe of assets. As argued earlier, the equilibrium APT has testability issues of Roll's critique variety. Finally, an exact error bound in the equilibrium context has already been established, but never in the arbitrage framework.

An explicit pricing error bound for the Ross APT for the individual asset can be easily examined by using (7) and (8). Dybvig (1984) and Grinblatt and Titman (1984) have shown in the utility-based framework, or equivalently, in an "Equilibrium" APT framework that under an uncorrelated-residuals structure (i.e., V is diagonal), the pricing error for asset i is bounded by $R\sigma_i^2\alpha_i$, where R is the measure of relative risk aversion, σ_i^2 is the asset i's residual variance and α_i is the proportion of total wealth represented by the asset. Assuming that V is diagonal, the exact pricing error is given by $\alpha\sigma_i^2 w_i^*$, where α is the risk-return tradeoff of the efficient arbitrage portfolio and w_i^* is the proportion of this efficient arbitrage portfolio represented by the asset.

Wei (1988) has derived an exact pricing error for the individual asset in the utility-based framework, which is given by $Rb_{im}var(e_m)$ where R is the measure of relative risk aversion, b_{im} is the market-residual beta andis the $var(e_m)$ is the residual variance of the market portfolio. His result requires an additional assumption that each residual $\tilde{e}_i$ and total wealth are bivariate normally distributed.

However, our model derives an exact pricing error in the arbitrage pricing context and needs no additional distribution assumptions to derive an exact pricing error. It follows from (7) and (8) that the exact pricing error is given by $\lambda_\beta\beta_i \equiv \alpha\beta_i var(w^{*\prime}e)$, where β_i is the efficient arbitrage portfolio beta of asset i and $var(w^{*\prime}e)$ is the idiosyncratic risk of the efficient arbitrage portfolio, i.e., $w^{*\prime}Vw$.

One interesting result in our analysis is that the exact pricing error is zero for all assets if the variance of the efficient arbitrage portfolio is zero, or alternatively, if the efficient arbitrage portfolio is well-diversified.[4] A

4) In order to have zero variance, the efficient arbitrage portfolio should have zero

similar observation is made by Chamberlain (1982) and Connor (1984). They have shown that an exact APT pricing is obtained if the mean-variance frontier or the market portfolio is well-diversified. Their models require that a global benchmark portfolio, which includes all of the universe of assets, needs to be well-diversified. On the other hand, our model does not require knowledge of the universe of assets but only the efficient arbitrage portfolio, which is unique up to the subset of assets.

Wei (1988) has examined the impact of omitted factors from an econometric model. Since there is no consensus regarding the number of factors generating asset returns, it would be informative to find the effect of omitting factors. Let us investigate the impact of omitting factors in empirical tests of our APT model, when the true number of factors is K, but the first J factors (less than K) are included in empirical testing. Define $B \equiv [B_1 \vdots B_2]$, where B_1 is an N x J matrix of factor loadings and B_2 is an N x (K-J) matrix. Let $f' \equiv [f_1 \vdots f_2]'$, where f_1 is a J-dimensional vector of factors and f_2 is a (K-J)-dimensional vector. Let $\overline{V} = V + B_2' E(f_2 f_2') B_2$ be the matrix of residuals with omitted factors, where E is an expectation operator. Thus, the omitted factor structure is given by $R = E + B_1 f_1 + \overline{V}$. We can easily demonstrate using the technique deriving Theorem 1 that when K-J factors are omitted, an APT model with J+1 factors holds: $E = \lambda_0 1_N + B_1 \lambda_1 + \overline{\beta}\, \overline{\lambda}_\beta$, where λ_1 is a J-dimensional vector of factor risk premia, $\overline{\beta}$ is an N-dimensional vector of the efficient arbitrage portfolio betas and $\overline{\lambda}_\beta$ is a positive constant. If J is zero (i.e., all the factors are omitted), we have the model that $E = \lambda_0 1_N + \overline{\beta}\, \overline{\lambda}_\beta$. This is an exact pricing one factor model with the efficient arbitrage portfolio as the factor. Thus, when factors are omitted, the expected rate of return on an asset is a linear combination of non-omitted factors loadings and the

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expected return. If it has a positive expected return, it allows an arbitrage opportunity.

efficient arbitrage portfolio beta. In the extreme case, where all the factors are omitted, the risk premium of an asset is solely determined by the efficient arbitrage portfolio beta.

Ⅲ. The Model with Transaction Costs

There has been an extensive empirical investigation of variants of the APT and the associated extensions at the theoretical level. Surprisingly, virtually no work exists that looks at the effects of transaction costs on pricing in an APT framework. In this section, we introduce transaction costs into our APT model developed in the previous section. We examine the impact of transaction costs on the APT pricing. Understanding the impact of transaction costs may be insightful in understanding some important anomalies in empirical investigations. Further, a theoretical analysis in this context allows us to study international pricing of assets in an APT framework. Fortunately, as demonstrated below, the APT is robust to a simple but reasonable transaction cost structure.

We assume that there are two types of transaction costs. Let T_i^b denote transaction costs which investors pay for buying or selling asset i at the beginning of the period. Let $\tilde{T}_i^e$ denote transaction costs which investors pay for buying or selling asset i at the end of the period. Transaction costs, $\tilde{T}_i^e$, are unknown at the beginning of the period. Following Garman and Ohlson (1981), for simplicity, we assume that transaction costs are linear in terms of the number of shares traded.

We assume that pre-transaction cost returns are generated by a K-factor structure denoted as[5)]

5) We have made this assumption in order to examine the effect of transaction costs on the APT pricing.

$$R = E + Bf + e. \tag{1}$$

The ith element of the ex post return vector, $\tilde{R}_i$, is given by

$$\tilde{R}_i = \frac{\tilde{X}_i}{P_i} - 1 \tag{9}$$

Where $\tilde{X}_i$ is the observed payoff of asset i at the end of the period and is the pre-transaction costs price of asset i at the beginning of the period. The post-transaction costs (ex post) return on asset i, $\tilde{R}_i^T$, is

$$\tilde{R}_i = \frac{\tilde{X}_i - \tilde{T}_i^e}{P_i + T_i^b} - 1 \tag{10}$$

Our transaction cost structure is very similar to the structure of Garman and Ohlson(1983). Ours is different from theirs in that ours does not differentiate transaction costs in the case of buying from selling whereas theirs does. This simplification is made because it would not change the implications from the model. Define t_i^b as the transaction cost rate at the beginning of the period given by

$$t_i^b = \frac{T_i^b}{P_i + T_i^b} \tag{11}$$

and $\tilde{t}_i^e$ as the transaction cost rate at the end of the period given by

$$\tilde{t}_i^e = \frac{\tilde{T}_i^e}{P_i + T_i^b}. \tag{12}$$

Substituting X_i of (9) into (10) and using (1), (11) and (12) gives

$$\tilde{R}_i^T = (1-t_i^b)[E_i + B_i f + e_i] - \tilde{t}_i^e - t_i^b \tag{13}$$

where E_i and e_i are the ith elements of vectors E and e , respectively.

and B_i the ith row of the matrix B. We rewrite (13) in matrix notation

$$R^T = E_t + B_t f + e_t - t^e - t^b \tag{14}$$

where $E_t = [(1-t_1^b)E_1, \cdots, (1-t_N^b)E_N]'$, $B_t = [(1-t_1^b)B_1, \cdots, (1-t_N^b)B_N]'$,

$e_t = [(1-t_1^b)\tilde{e}_1, \cdots, (1-t_N^b)\tilde{e}_N]'$, $t^b = [t_1^b, \cdots, t_N^b]'$ and

$t^e = [\tilde{t}_1^e, \cdots, \tilde{t}_N^e]'$

We assume that an idiosyncratic disturbance is uncorrelated with the transaction cost rate at the end of period for any securities, i.e., for all i and $Cov(\tilde{e}_i, \tilde{t}_j^e) = 0$ for all j.[6] However, we allow factors to be correlated with transaction costs at the end of period for any securities., i.e., $Cov(\tilde{f}_i, \tilde{t}_j^e) \neq 0$ for all i and j. The covariance matrix of the residuals need not be diagonal. Factors are assumed to have unitary variances.

Post-transaction costs returns and risks should be considered in this analysis rather than pre-transaction costs counterparts, since investors are concerned about net receipts. Suppose that an investor forms a portfolio whose weights are given by w. The expected rate of post-transaction-costs return on the portfolio is given by

$$E(w'R^T) = w'E_t - E(w't^e) - w't^b \tag{15}$$

where E denotes an expectation operator.

The variance of post-transaction-costs return on the portfolio is given by

$$Var(w'R^T) = w' E(B_t B_t')w + w' V_t w + w'Var(t^e)w + 2w'B_t Cov(f, t^e)w \tag{16}$$

where V_t is defined as $[(1-t_1^b)1_N \cdots (1-t_N^b)1_N]V[(1-t_1^b)1_N \cdots (1-t_N^b)1_N]'$ and $Var(t^e)$ need not be diagonal (i.e., transaction cost for one asset at the end

6) This assumption can be relaxed. If an idiosyncratic disturbance is correlated with the transaction cost rate at the end of period, then the idiosyncratic risk is priced through its correlation with the efficient arbitrage portfolio's transaction cost rate at the end of period.

of period may be correlated with transaction costs for other assets).

We formulate a problem in accordance to Ross(1976). We assume that the arbitrage portfolio requires zero investment and zero transaction-cost-adjusted-factor loadings such that[7)]

$$w'1_N = 0 \text{ and } w'B_t = 0 \tag{17}$$

where B_t is defined in (14). Consider the following problem.

$$\begin{aligned} \text{Problem 1: } & \underset{w}{\text{Minimize}} \ \ \text{var}(w'R^T) \\ & \text{subject to } \ w'1_N = 0 \\ & \qquad\qquad\quad w'B_t = 0 \\ & \text{and} \quad 0 < c \leq E(w'R^T) = w'E_t - E(w't^e) - w't^b. \end{aligned}$$

The solution of the above problem is defined as the efficient arbitrage portfolio, w^*. Following Ross(1976), it can be shown that the minimum variance of Problem 1 is strictly positive. The arbitrage portfolio with a positive post-transaction-costs expected return should have a positive variance. Otherwise, there would be an arbitrage opportunity, in accordance with the Ross problem.

We derive an APT model under transaction costs from Problem 1. Problem 1 implies that if

$$w^{*\prime}1_N = 0,\ w^{*\prime}B_t = 0 \text{ and } E(w^{*\prime}R^T) > 0 \text{ for } w^* \in R^N, \tag{18}$$

then we must have

$$\text{var}(w^{*\prime}R^T) > 0 \tag{19}$$

7) If the arbitrage portfolio is allowed to have non-zero transaction-cost-adjusted- factor loadings, i.e., $w'B_t \neq 0$, then theoretically the existence of transaction costs affects the factor risk premia.

where w^* is the efficient arbitrage portfolio, the solution of Problem 1. It follows that there is no solution, $w^* \in R^N$, which satisfies the following system,

$$w^{*\prime}1_N = 0,\ w^{*\prime}B = 0,\ w^{*\prime}V_t w^* = 0,\ w^{*\prime}Var(t^e)w^* = 0 \text{ and}$$
$$w^{*\prime}E_t - E(w^{*\prime}t^e) - w^{*\prime}t^b > 0. \quad (20)$$

We can find the alternative system to (20) by using the following lemma.

Lemma 2. Exactly one of the following systems has a solution.[8)]

System A: $w^{*\prime}1_N = 0,\ w^{*\prime}B = 0,\ w^{*\prime}V_t w^* = 0,\ w^{*\prime}Var(t^e)w^* = 0$ and $w^{*\prime}E_t - E(w^{*\prime}t^e) - w^{*\prime}t^b > 0$ for $w^* \in R^N$.

System B: $E_t - E(t^e) - t^b = \widehat{\lambda_0}1_N + B_t\hat{\lambda} + \alpha_1 Var(t^e)w^* + \alpha_2 V_t w^*$ for $\hat{\lambda} \in R^K$, where $\widehat{\lambda_0}$, α_1 and α_2 are constants.

From Problem 1, there is no solution of $w^* \in R^N$ for System A. Thus, there a solution of $\widehat{\lambda_0} \in R, \hat{\lambda} \in R^K, \alpha_1 \in R$ and $\alpha_2 \in R$ for System B. Note that the notations $\widehat{\lambda_0}$ and $\hat{\lambda}$ and are used to indicate that they can differ from λ_0 and λ obtained in Section 2 without transaction costs. System B is an APT model under transaction costs

$$E_t = E(t^e) + t^b + \widehat{\lambda_0}1_N + B_t\hat{\lambda} + \alpha_1 Var(t^e)w^* + \alpha_2 V_t w^*. \quad (21)$$

By premultiplying both sides of (21) with w^*, we obtain

$$w^{*\prime}E_t - E(w^{*\prime}t^e) - w^{*\prime}t^b \equiv E(w^{*\prime}R^T) = \alpha_1 w^{*\prime}Var(t^e)w^* + \alpha_2 w^{*\prime}V_t w^* \quad (22)$$

8) The proof of this lemma is omitted since it is similar to Lemma 1.

Thus, we can easily see that α_1 and α_2 are the marginal risk premia from taking an additional one unit of transaction-cost risks and residual risks for the efficient arbitrage portfolio respectively. Rearranging (21), we obtain an APT pricing model under transaction costs given by the following theorem.

Theorem 2. Under no arbitrage, the factor structure (1) and the transaction costs structure (11) and (12), the pre-transaction-costs expected rate of return on asset i is given by

$$E_i = \widehat{\lambda_0^i} + B_i\hat{\lambda} + \beta_1^i\lambda_{\beta_1} + \beta_2^i\lambda_{\beta_2} \tag{23}$$

where

$$\widehat{\lambda_0^i} \equiv \frac{\widehat{\lambda_0} + t_i^b + E(t_i^e)}{1 - t_i^b}, \quad \beta_1^i \equiv \frac{\mathrm{Cov}(t_i^e, w^{*'}t^e)}{(1-t_i^b)\mathrm{Var}(w^{*'}t^e)}, \quad \beta_2^i \equiv \frac{\mathrm{Cov}(e_i, w^{*'}t^e)}{\mathrm{Var}(w^{*'}t^e)},$$

$\lambda_{\beta_1} \equiv \alpha_1 \mathrm{Var}(w^{*'}t^e)$, $\lambda_{\beta_2} \equiv \alpha_2 \mathrm{Var}(w^{*'}e_t)$, $\mathrm{Var}(w^{*'}t^e) \equiv w^* \mathrm{Var}(t^e) w^*$ and

$\mathrm{Var}(w^{*'}t^e) \equiv w^* V_t w^*$.

If there are no transaction costs, then (23) becomes (8). Thus, the pricing equation derived in Section 2 is a special case of (23).

Several explanations for (23) are warranted. First, our arbitrage based APT model is robust to market imperfections caused by a transaction cost structure examined here. The arbitrage argument survives in markets with frictions.

Second, the constant, $\widehat{\lambda_0^i}$, is not identical to all assets. It includes asset-specific transaction costs. A zero-beta portfolio may have different expected rates of returns, depending on transaction costs. The small firm effect of the APT exists as an empirical anomaly (see Reinganum (1982)).

Reportedly, the expected rate of return on portfolio of small firms is greater than that of large firms by average 6% per year, even after controlling for APT risks (see Stoll and Whaley (1983)). Perhaps, transaction cost rate of small firms' stocks may be higher than transaction cost rate of large firms' stocks. The expected rate of return on small firms should be higher than that on large firms, ceteris paribus. Our model with transaction costs is consistent with the empirical anomaly, although it may not explain its entirety. This explanation is consistent with an empirical study of Stoll and Whaley (1983) and a mean-variance theoretical analysis with imperfect information by Merton (1987). Merton's analysis is based on the notion that individual securities are held by a subset of the universe of investors. Thus, each security has its own followership.

Third, the factor risk premia under transaction costs can differ from that under no transaction costs. The factor risk premia under transaction costs indirectly depend on the structure of transaction costs. If a zero transaction-cost-adjusted-loadings portfolio is unavailable, it affects the value of the factor risk premia. However, the functional form would not be changed. The pricing equation (23) still holds. Even though frictions (i.e., transaction costs) are asset specific, there exist the factor risk premia whose elements do not depend on assets. This result is consistent with the previous literature on asset pricing with frictions (e.g., Garman and Ohlson (1981) and Prisman (1986)). Garman and Ohlson (1981) and Prisman (1986) using the state preference model have the implicit prices which are identical across securities.

Fourth, unless the efficient arbitrage portfolio completely eliminates transaction cost risks, namely, $w^{*'}Var(t^e)w^* \neq 0$, a transaction cost risk plays as a factor and is priced. The quantity of risk is captured by β_1, the efficient arbitrage portfolio beta with respect to transaction costs. As long as the efficient arbitrage portfolio includes residual risk, namely, $w^{*}V_t w^* \neq 0$, the risk is priced. The quantity of risk is captured by β_2, the efficient arbitrage portfolio beta with respect to residual risk.

We can examine the pricing error of the Ross APT under transaction costs. From (23), the exact pricing error for asset i can be written as

$$E_i - \widehat{\lambda_0} - B_i\widehat{\lambda} = \widehat{\lambda_1^i} + \lambda_{\beta_1}\beta_1^i + \lambda_{\beta_2}\beta_2^i \tag{24}$$

where $\widehat{\lambda_1^i} \equiv [(1+\widehat{\lambda_0})t_i^b + E(\widetilde{t_i^e})]/(1-t_i^b)$ and explicitly bounded by

$$E_i - \widehat{\lambda_0} - B_i\widehat{\lambda} \le |\widehat{\lambda_1^i}| + |\lambda_{\beta_1}|[\mathrm{var}(\widetilde{t_i^e})/(1-t_i^b)\mathrm{Var}(w^{*\prime}t^e)]^{1/2}$$
$$+ |\lambda_{\beta_2}|[\mathrm{var}(\widetilde{e_i})/\mathrm{var}(w^{*\prime}e_t)]^{1/2}. \tag{25}$$

If transaction costs at the end of the period are known at the beginning of the period, β_1 disappears.

If some factors are omitted in empirical testing our model with transaction costs, then we can analogously demonstrate that an APT model holds. If all the factors are omitted, then we have an exact pricing APT model with two factors, the efficient arbitrage portfolio and the transaction cost factor.

The preceding model can be used conveniently to analyze the pricing of assets internationally where transaction costs are not only firm-specific but vary across national boundaries. Indeed, the model suggests that there exist multiple zero-beta returns in the presence of transaction costs. This is consistent with the existence of multiple real rates of interest across countries. Thus far, the limited APT extensions into an international context have been under perfect market conditions and hence do not permit analysis of the effects of market segmentation on international pricing. This is an issue that we shall pursue further.

Ⅳ. Conclusion

This paper derives an APT pricing model in the arbitrage-free economy with and without transaction costs. This model does not require additional distributional assumptions and restrictions on investors' preferences. Nor does it require that the market portfolio as the global benchmark portfolio be well-diversified and efficient. The model is derived by using the nonlinear duality theory and the intermediate result of Ross (1976) that the efficient arbitrage portfolio with zero factor risk and zero net investment which earns a positive expected return must have a positive variance.

The model yields an exact pricing relation for individual securities. The efficient arbitrage portfolio plays an extra pricing factor if the efficient arbitrage portfolio is not well-diversified. The pricing error from APT pricing for individual asset is readily derived. An exact APT pricing model arises when the efficient arbitrage is well-diversified. This efficient arbitrage portfolio is not a global portfolio unlike the market portfolio in the literature of equilibrium APT. It is unique up to a given subset of assets. Thus, we do not need to refer to the universe of assets.

Our arbitrage-based model is robust to market frictions raised by transaction costs. The existence of transaction costs reduces a room for arbitrage. However, the absence of arbitrage profits under transaction costs yields the model. Transaction costs play as an extra pricing factor along with the efficient arbitrage portfolio. Our model gives the implication that the small firm effect of the APT can be partially explained through the effects of transaction costs (see Reinganum (1981)). In the future, we shall pursue an empirical test of this model with market data. Also the model establishes a convenient framework for the study of asset pricing in an international context through an APT framework. Despite the prevalence of market imperfections and possible segmentation of international capital markets (at least for a subset), there is very little work done in extending the APT into an imperfect international financial environment. Indeed, the current model itself, which delivers multiple zero beta returns, may be consistent with multiple international real interest rates. This is also a future research agenda.

APPENDIX

Proof of Lemma 1

i) Suppose system A has a solution $w^* \in R^N$. Then we have to show that system B has no solution. On the contrary, suppose that system B has a solution, λ_0, λ and α. Multiplying w^* on both sides of system B gives

$$w^{*\prime}E = \lambda_0 w^{*\prime}1_N + w^{*\prime}B\lambda + \alpha w^{*\prime}Vw^*.$$

From system A,

$$0 < w^{*\prime}E = \lambda_0 w^{*\prime}1_N + w^{*\prime}B\lambda + \alpha w^{*\prime}Vw^* = 0,$$

a contradiction. Hence, system B cannot have a solution.

ii) Suppose system A has no solution. Then we have to show that system B has a solution, λ_0, λ and α. Note that the efficient arbitrage portfolio w^* is the optimal solution of the Ross problem. This fact provides that the following system (which is referred as system A′) has no solution $w \in R^N$

$$w'B = 0, w'1_N = 0, w'Vw^* < 0 \text{ and } w'E = 0.$$

If there is a solution $w \in R^N$ to system A′, then we choose $w^* + \delta w$

where δ is arbitrarily small positive number. It follows from system A′ that

$$(w^* + \delta w)'B = w^{*\prime}B = 0,\ (w^* + \delta w)'1_N = w^{*\prime}1_N = 0,\ (w^* + \delta w)'E = w^{*\prime}E > 0 \text{ and}$$

$$(w^* + \delta w)'V(w^* + \delta w) < w^{*\prime}Vw^*$$

where the last inequality follows from $w'Vw^* < 0$. The last inequality contradicts the fact that w^* is the optimal solution of the Ross problem.

Consider the following sets:

$$C_1 = \{(u, x, y, z) : w'B = u,\ w'1_N = x,\ w'Vw^* = y \text{ and } w'E = z\}$$

$$C_2 = \{(u, x, y, z) : u = 0,\ x = 0,\ y < 0 \text{ and } z = 0\}$$

Since there is no solution for system A′, $C_1 \cap C_2 = \varnothing$. Then there exists a hyperplane that separates C_1 and C_2. That is, there exist non-zero vector γ_1 and non-zero scalars γ_2, γ_3 and γ_4 such that

$$w^{*\prime}B\gamma_1 + w^{*\prime}1_N\gamma_2 + w^{*\prime}Vw^*\gamma_3 + w^{*\prime}E\gamma_4 \geq u\gamma_1 + x\gamma_2 + y\gamma_3 + z\gamma_4$$

for u, x, y and $z \in$ cl C_2, $\gamma_1 \in R^K$, $\gamma_2 \in R$, $\gamma_3 \in R$ and $\gamma_4 \in R$.

Since y can be an arbitrarily large negative number, it follows that $\gamma_3 > 0$.

Let $u = 0$, $x = 0$, $y = 0$ and $z = 0$, since the origin is in the closure of C_2. Then we have that

$$w'(B\gamma_1 + 1_N\gamma_2 + Vw^*\gamma_3 + E\gamma_4) \geq 0.$$

By choosing $w = -(B\gamma_1 + 1_N\gamma_2 + Vw^*\gamma_3 + E\gamma_4)$, it follows that

$$\|B\gamma_1 + 1_N\gamma_2 + Vw^*\gamma_3 + E\gamma_4\|^2 \leq 0.$$

Thus $B\gamma_1 + 1_N\gamma_2 + Vw^*\gamma_3 + E\gamma_4 = 0.$

This equation is called the Fritz John optimality condition.

Suppose system A has no solution. Then we have that

$$\{ w'B=0,\ w'1_N=0,\ w^{*'}Vw^{*}\geq 0 \text{ and } w^{*'}E>0\} \Rightarrow \{w^{*'}Vw^{*}>0\}.$$

It follows from Corollary 18 (Mangasarian (1969, p26))) that there exist non-zero vector γ_1 and non-zero scalars γ_2, γ_3 and γ_4 such that

$$B\gamma_1 + 1_N\gamma_2 + Vw^*\gamma_3 + E\gamma_4 = 0$$

which is identical to the Fritz John optimality condition. It implies that

$$E = \lambda_0 1_N + B\lambda + \alpha Vw^*$$

where $\lambda_0 = -\gamma_2/\gamma_4, \lambda = -\gamma_1/\gamma_4$ and $\alpha = -\gamma_3/\gamma_4$. Hence, system B has a solution.

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