A Graphical Exposition of the Period of Oscilation in a Second Order Difference Equation

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I. The Isoperiod Map

The Smyth's β-γ diagram [3], or Baumol's similar diagram [1] provides a simple way for identification of qualitative behavior of dynamic system expressed in a second order difference equation,

(1) $Y_t+\beta Y_{t-1}+\gamma Y_{t-2}=Y_0$.

The diagram effectively separates regions in which the system is stable or unstable, and oscilatory or non-oscilatory. But neither Smyth nor Baumol utilizes the diagram to derive maximum information about the dynamic system.

In the diagram in Figure 1 if parameters of the dynamic system locate to the right of the parabola EAOBG, i.e.,

(2) $\beta^2<4\gamma$

then the system is oscilatory. If the parameters are within the triangle ABC, i.e.,

(3-a) $1+\beta+\gamma>0$

(3-b) $1-\beta+\gamma>0$

(3-c) and $\gamma<1$

the system is stable. In case of (long) oscilation, i.e., to the right of the parabola, the line segment AB separates the stable region and unstable region. But we know that an oscilation is characterized not only by the changes in amplitude but also by its period. In this paper I will show that the period of oscilation can be explained graphically in the β-γ diagram.

The roots of the characteristic equation of the difference equation (1) are given by

(4) $\lambda_1, \lambda_2 = \frac{-\beta}{2} \pm \frac{\sqrt{4\gamma-\beta^2}}{2} i$, for $\beta^2<4\gamma$.

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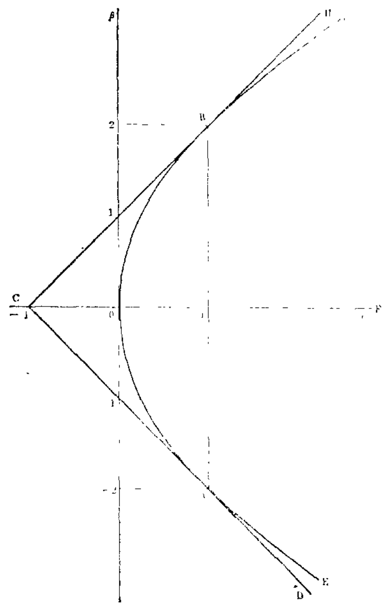

Figure 1: β-γ diagram

The modulus of the conjugate complex roots is

(5)　$\sqrt{(-\beta/2)^2+(4\gamma-\beta^2)/4}=\sqrt{\gamma}$

Therefore, if we define θ such that

(6)　$\cos\theta=(-\beta/2)/\sqrt{\gamma}$, $0\leqq 0\leqq\pi$

the period of oscilation T is given by

(7)　$T=\frac{2\pi}{\theta}$

Combining (6) and (7), we get

(8)　$\beta=\left(-2\cos\frac{2\pi}{T}\right)\sqrt{\gamma}.$

The equation (8) shows that for given T, the locus of points in β-γ diagram is half parabola. For example, if

T=2, then $\beta=2$, $\sqrt{\gamma}$, or the half of parabola OBG,

T=4, $\beta=o$, or the degenerated half parabola OF,

T=∞, $\beta=-2\sqrt{\gamma}$, or the half parabola OAE.

If we graph the curve for each T, $2\leqq T\leqq\infty$, we get an "isoperiod map", a collection of half parabolas. (Figure 2.)

Next, I will examine the properties of the map.

II. The Properties of the Isoperiod Map

Property 1: Given γ, and the condition (2), as β increases T decreases.

Proof: From (7) we get

$$\frac{dT}{d\theta}=-\frac{2\pi}{\theta^2}<0$$

From (6) we get

$$\frac{\partial\theta}{\partial\beta}=\frac{1}{2\sqrt{\gamma}\sin\theta}>0,\quad \frac{\partial\theta}{\partial\gamma}=-\frac{\beta}{4\sqrt{\gamma^3}\sin\theta}\gtreqless 0 \text{ as } \beta\lesseqgtr 0.$$

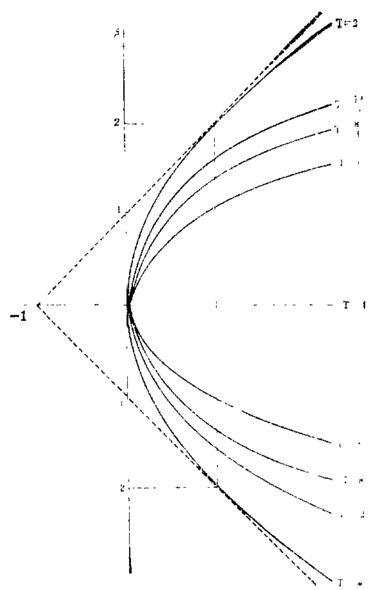

Figure 2: The isoperiod map

Therefore,

$$(9)\qquad \frac{\partial T}{\partial\beta}=\frac{dT}{d\theta}\,\frac{\partial\theta}{\partial\beta}<0.$$

Property 2: Given β, and the condition (2), as γ increases T increases when $\beta>0$, and decreases when $\beta<0$,

Proof: From the above we get immediately

$$(10)\qquad \frac{\partial T}{\partial\gamma}=\frac{dT}{d\theta}\,\frac{\partial\theta}{\partial\gamma}\lesseqgtr 0 \text{ as } \beta\gtreqless 0.$$

Graphically the two properties are obvious in Figure 2.

III. Some Applications

Consider the Metzler's [2] inventory cycle model,

$$(11)\qquad Y_t-2cY_{t-1}+cY_{t-2}=Y_0,\quad 0<c<1.$$

It is well known that the model is stable oscilatory. Graphically, as c increases from 0 to 1, the system moves from 0 to A along the line segment OA in Figure 3. If we put

$\beta=-2c$, and $\gamma=c$,

then we get from (6),

$$\cos\theta=\sqrt{c}\,, \tag{12}$$

$$\frac{d\theta}{dc}=-\frac{1}{2\sqrt{c}\sin\theta}<0. \tag{13}$$

Therefore we get

$$\frac{dT}{dc}=\frac{dT}{d\theta}\cdot\frac{d\theta}{dc}>0. \tag{14}$$

From this it follows that in model (11) as c increases from 0 to 1, i.e., as the system moves from 0 to A along the line segment OA in Figure 3, the period of oscilation of the system increases monotonically.

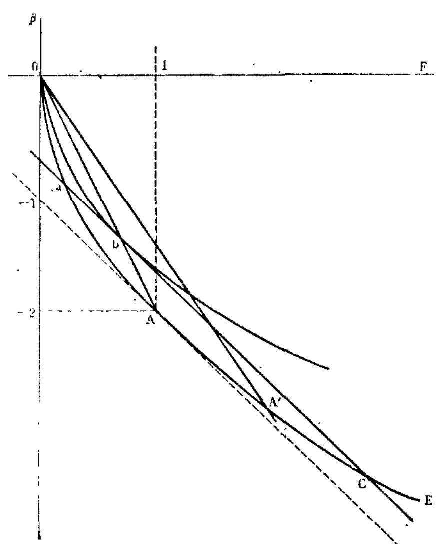

Figure 3: β-γ diagram of Metzler's model

Consider, now, Metzler's inventory cycle model with a coefficient of expectation, v,

$$Y_t-c(1+v)Y_{t-1}+cvY_{t-2}=Y_0. \tag{15}$$

If $v=1$, (15) is the same as (11). If we put

$\beta=-c(1+v)$, and $Y=cv$,

then we get from (6),

$$\cos\theta=\frac{c(1+v)}{2\sqrt{cv}}\,, \tag{16}$$

$$d\cos\theta = \frac{2(1+v)\sqrt{cv} - c(1+v)\sqrt{v/c}}{4cv}\,dc + \frac{2c\sqrt{cv} - c(1+v)\sqrt{c/v}}{4cv}\,dv,$$

$$-\sin\theta\, d\theta = \frac{1+v}{4\sqrt{cv}}\,dc + \frac{c(v-1)}{4\sqrt{v^3}}\,dv,$$

or,

(17) $\dfrac{\partial\theta}{\partial c} = \dfrac{-(1+v)}{4\sqrt{cv}\sin\theta} < 0$ under the condition (2),

(18) $\dfrac{\partial\theta}{\partial v} = \dfrac{\sqrt{c}\,(v-1)}{4\sqrt{v^3}\sin\theta} \lesseqgtr 0$ as $v \lesseqgtr 1$, under the condition (2),

(19) $\dfrac{\partial^2\theta}{\partial v^2} = \dfrac{4\sqrt{cv^3} - 6\sqrt{cv}(v-1)}{-16v^3\sin\theta} - \dfrac{\cos\theta}{\sin\theta}\left(\dfrac{\partial\theta}{\partial v}\right)^2.$

Combining the above we get

(20) $\dfrac{\partial T}{\partial c} = \dfrac{dT}{d\theta}\,\dfrac{\partial\theta}{\partial c} > 0$ under (2)

(21) $\dfrac{\partial T}{\partial v} = \dfrac{dT}{d\theta}\,\dfrac{\partial\theta}{\partial v} \lesseqgtr 0$ as $v \lesseqgtr 1$, under (2)

(22) $\dfrac{\partial^2 T}{\partial v^2}\Big|_{v=1} = \dfrac{d^2T}{d\theta^2}\left(\dfrac{\partial\theta}{\partial v}\right)^2\Big|_{v=1} + \dfrac{dT}{d\theta}\,\dfrac{\partial^2\theta}{\partial v^2}\Big|_{v=1} > 0.$

The inequality (20) shows that under the condition (2), if v is constant and c increases then T increases. Graphically this means that if we move from O to A′ along the line segment OA′ then the period of oscilation of the system (15) increases monotonically from $T=4$ to $T=\infty$. The inequality (21) shows that under the condition (2) if c is costant and v increases from less than 1 to 1, then T decreases monotonically, but if v increases beyond 1 then T increases. The inequality (27) guarantees that $v=1$ minimizes T. Graphically, in Figure 3, as we move along the line segment *ac* which is parallel to AD, the period of oscilation decreases from *a* to *b*, and it increases from *b* to *c*. We also notice that every tangent line to the isoperiod curve at the point on OA is parallel to AD.

References

[1] William J. Baumol, *Economic Dynamics*, 3rd ed., New York, 1970.

[2] L.A. Metzler, "The Nature and Stability of Inventory Cycles," *Review of Economics and Statistics*, 1941.

[3] David J. Smyth, "Quantative Macroeconomic Analysis," mimeo.