Performance Analysis of Wide-Band M-ary FSK Systems in Rayleigh Fading Channels

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***Abstract*—Performance analysis schemes for wide-band M-ary frequency-shift keying systems with a limiter-discriminator-integrator receiver are presented along with analysis and simulation results for 4-ary and 8-ary systems. The probability distribution of clicks is required for performance analysis and is determined in this letter for Rayleigh fading channels.**

***Index Terms*—Error analysis, FM receiver (limiter-discriminator-integrator), frequency-shift keying, Rayleigh channels.**

I. Introduction

A limiter-discriminator-integrator (LDI) receiver has been used for many years in the detection of frequency-shift keying (FSK) signals [1]–[7], and its performance has been mostly analyzed for narrow-band systems. The performance of an LDI detector for narrow-band binary FSK systems has been analyzed for static environments in [2]–[4] and for fading environments in [4] and [5]. Narrow-band M-ary systems with an LDI detector have been investigated in [6] and [7]. Unlike narrow-band systems, wide-band FSK systems with an LDI have not been thoroughly analyzed. For wide-band FSK, a binary case was investigated for static environments in [8]. Narrow-band FSK systems are defined as FSK systems with the data phase difference within a symbol duration being less than 2π, which means that there always occurs a decision error whenever clicks occur. In wide-band systems, data phase difference within a symbol duration is greater than 2π, and this means that a decision error may not be made even when clicks occur.

Recently, the FLEX system, based on a wide-band M-ary FSK, has been introduced as a next-generation paging system [9], [10], and is being commercialized in many countries including Korea, USA, and Japan. The purpose of this paper is to present performance analysis schemes for wide-band M-ary FSK systems with integer-valued normalized frequency deviations, and to show analysis and simulation results for $M = 4$ and 8. The normalized frequency deviations for typical wide-band M-ary FSK systems such as FLEX systems are integer-valued [9], [10].

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II. System Model

The FSK system model is shown in Fig. 1. The M-ary FSK system transmits a sequence of symbols from the set, $\{a_1, a_2, \cdots, a_M\}$, $a_m = 2m - 1 - M$, $m = 1, 2, \cdots, M$, and the transmitted signal may be expressed as

$$s(t) = \sqrt{2S} \cos\left(\omega_c t + \frac{\pi h}{T} \int_{-\infty}^{t} \sum_{n=-\infty}^{\infty} d_n p_T(\tau - nT)\, d\tau\right) \tag{1}$$

where S is the transmit power, ω_c is the carrier radian frequency, T is the symbol duration, d_n is the nth transmit symbol, and $p_T(t)$ is a unit amplitude rectangular pulse of duration T. $h = 2f_d T$ is the normalized frequency deviation, where f_d is the frequency deviation for symbol $a_{M/2+1} = 1$. The maximum frequency deviation is $a_M f_d$ for symbol $a_M = M - 1$.

III. Analysis

The LDI output in Fig. 1(b) is the phase rotation of received signal over the symbol interval $(t_0 - T, t_0)$. It can be expressed as [3]

$$\Delta\Phi = \Delta\phi + \psi + 2\pi N \tag{2}$$

where $\Delta\phi$ is the filtered data phase rotation, ψ is the phase rotation due to noise and fading defined in the interval $(-\pi, \pi)$, and N is the number of clicks in the time interval $(t_0 - T, t_0)$.

A. Error Probability Equation

In FSK systems, the IF filter at the receiver typically introduces intersymbol interference, and all three terms in (2) depend on not only the symbol under consideration but also the symbols on either side of it [3], [6]. In wide-band M-ary FSK systems, a decision error may not be made even when a click occurs, and the probability of symbol error is expressed in terms of the probability $P(N = k)$ that the number of clicks N is equal to k, not the average number of negative clicks $\overline{N}_-$. When a positive symbol a_m in the symbol sequence l, which includes a_m in the middle, is transmitted, the conditional probability of symbol error for wide-band M-ary FSK systems is given as

$$\begin{aligned} P(e|m, l) = {} & P(N < -k_-(m, l)) + P(N = -k_-(m, l)) \\ & \cdot \int_{-\pi}^{\theta_-(m,l)} f_\Psi(\psi|m, l)\, d\psi + P(N = k_+(m, l)) \\ & \cdot \int_{\theta_+(m,l)}^{\pi} f_\Psi(\psi|m, l)\, d\psi + P(N > k_+(m, l)) \end{aligned} \tag{3}$$

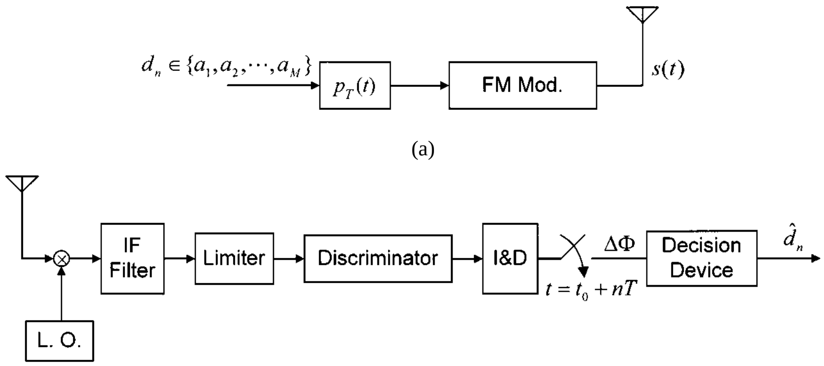

Fig. 1. System model. (a) Transmitter. (b) Receiver.

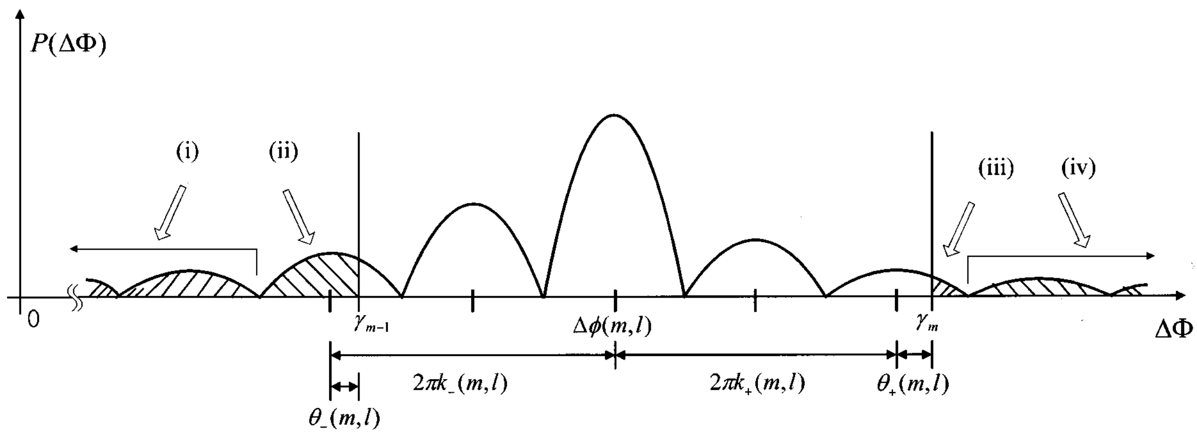

Fig. 2. PDF of LDI output with the parameters in (3).

where

$$\begin{aligned} \theta_-(m, l) &= \gamma_{m-1} - \Delta\phi(m, l) + 2\pi k_-(m, l) \\ \theta_+(m, l) &= \gamma_m - \Delta\phi(m, l) - 2\pi k_+(m, l) \\ k_-(m, l) &= \lceil (\Delta\phi(m, l) - \pi - \gamma_m)/2\pi \rceil \\ k_+(m, l) &= \lceil (\gamma_m - \Delta\phi(m, l) - \pi)/2\pi \rceil \end{aligned}$$

and $\lceil x \rceil$ is the smallest integer greater than x. $\Delta\phi(m, l)$, $P(N = k(m, l))$, and $f_\Psi(\psi|m, l)$ denote, respectively, the filtered data phase rotation, the probability that N is equal to k, and the probability density of ψ for symbol a_m and sequence l. The decision threshold γ_m is set to the middle point between distortion free data phase rotations associated with symbols a_m and a_{m+1}; $\Delta\phi(m, l) = a_m \pi h$, $\Delta\phi(m+1, l) = a_{m+1}\pi h$. γ_m is $(2m - M)\pi h$, $m = 1, 2, \cdots, M-1$ and $\gamma_M = -\gamma_0 \infty$.

In (3) $k_-(m, l)$ and $k_+(m, l)$ are the smallest numbers of clicks in the negative and positive directions, respectively, which may cause errors. $\theta_-(m, l)$ and $\theta_+(m, l)$ are in the interval $(-\pi, \pi)$. Fig. 2 is an illustration of (3), in which $P(\Delta\Phi)$ is a probability density function (pdf) of the LDI output $\Delta\Phi$. In this figure, region (i) represents the probability that the number of negative clicks exceeds $k_-(m, l)$ and corresponds to the first term in (3). Region (ii) represents the probability that the number of negative clicks is $k_-(m, l)$ and ψ is less than $\theta_-(m, l)$. This probability corresponds to the second term in (3). When a symbol with small positive frequency deviation, such as $a_m = 1$ in an M-ary system, is transmitted, there exists a considerable amount of filtered noise whose frequency is larger than that of the signal. This means that clicks may occur in the positive direction. Thus, for symbols with small frequency deviation in wide-band M-ary systems, positive clicks cannot be ignored. The third and fourth terms in (3) are associated with positive click probabilities and represented by regions (iii) and (iv) in Fig. 2.

In wide-band M-ary FSK systems, the receiver IF filter bandwidth is typically set such that just one previous and one following symbol affect phase rotation in the form of intersymbol interference. In the calculation of the error probability $P(e|m)$ for symbol a_m, M^2 sequences should be considered, since there are M possible previous and M possible following symbols. M^2 possible sequences for each symbol may be divided into two groups: M sequences with the same previous and following symbols (SPF sequences) and $M(M-1)$ sequences with different previous and following symbols (DPF sequences). Since a sequence and its reversed sequence, such as $+1+3-3$ and $-3+3+1$, give the same performance by symmetry [3], the number of DPF sequences to be considered may be reduced to $M(M-1)/2$. When the sequences are equally probable, the error probability for symbol a_m may be written as

$$P(e|m) = \frac{1}{M^2}\left[\sum_{l_{\mathrm{SPF}}=1}^{M} P(e|m, l_{\mathrm{SPF}}) + 2\sum_{l_{\mathrm{DPF}}=1}^{M(M-1)/2} P(e|m, l_{\mathrm{DPF}})\right]. \quad (4)$$

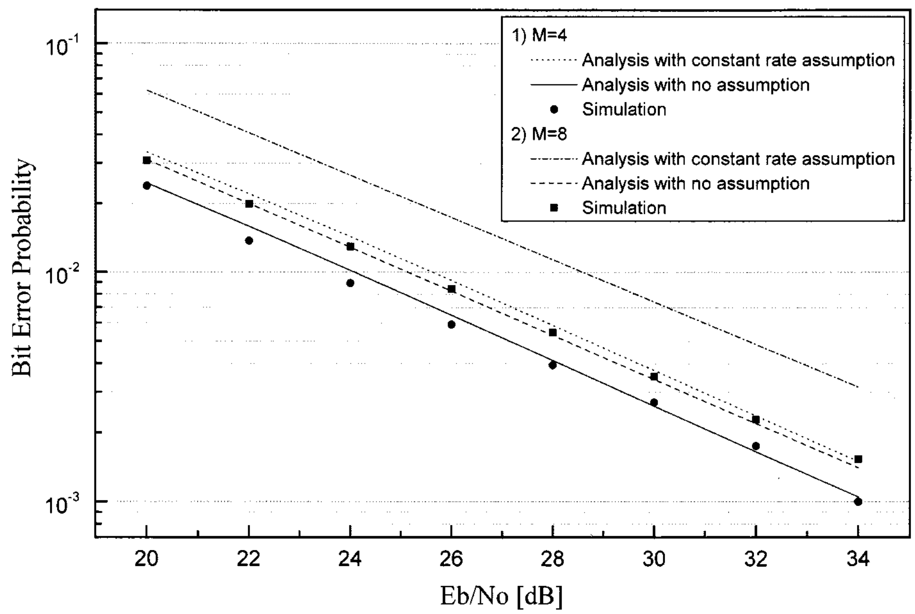

Fig. 3. Bit-error probability for $M = 4, 8$, $h = 2$, and $F_{\mathrm{DM}}T = 0.025$.

For example, the SPF sequences for symbol $a_m = 1$ in a 4-ary FSK system are $\{-3+1-3, -1+1-1, +1+1+1, +3+1+3\}$, and the DPF sequences are $\{-3+1-1, -3+1+1, -3+1+3, -1+1+1, -1+1+3, +1+1+3\}$. When the positive and negative symbols are equally probable, the overall bit error probability is now expressed as

$$P_b = \frac{1}{(M/2)\log_2 M} \sum_{m=1}^{M/2} P(e|m). \tag{5}$$

B. Error Probability Calculation for the SPF Sequences

$P(e|m, l)$ is defined in (3) in terms of the filtered data phase $\Delta\phi(m, l)$, the pdf $f_\Psi(\psi|m, l)$ of the phase rotation ψ due to noise, and the probability distribution of clicks $P(N = k(m, l))$. In this section, the determination of these parameters for SPF sequences is presented.

The low-pass representation of a band-pass signal in (1) may be expressed in terms of in-phase and quadrature components, $\cos\theta(t)$ and $\sin\theta(t)$, which can be expanded in Fourier series [2]; $\cos\theta(t) = \sum_{n=-\infty}^{\infty} b_n e^{j(n\pi t/T)}$ and $\sin\theta(t) = \sum_{n=-\infty}^{\infty} c_n e^{j(n\pi t/T)}$, in which b_n and c_n are Fourier series coefficients and period $2T$ is assumed. This period assumption is valid for SPF sequences, since the normalized frequency deviation h is assumed to be an integer in this paper.

The in-phase and quadrature signals at the output of the receiver IF filter may be respectively expressed for symbol a_m in sequence l_{SPF} as $i(t|m, l_{\mathrm{SPF}}) = \cos\theta(t) \otimes h(t)$ and $q(t|m, l_{\mathrm{SPF}}) = \sin\theta(t) \otimes h(t)$, where $h(t)$ is the low-pass equivalent impulse response of a receiver IF filter and $\otimes$ denotes convolution. The filtered data phase $\phi(t|m, l_{\mathrm{SPF}})$, the normalized signal amplitude $A(t|m, l_{\mathrm{SPF}})$, and the instantaneous signal-to-noise ratio (SNR) $\rho(t, r|m, l_{\mathrm{SPF}})$ may be respectively calculated as [3]

$$\phi(t|m, l_{\mathrm{SPF}}) = \tan^{-1}\left(q(t|m, l_{\mathrm{SPF}})/i(t|m, l_{\mathrm{SPF}})\right) \tag{6}$$

$$A(t|m, l_{\mathrm{SPF}}) = \sqrt{i^2(t|m, l_{\mathrm{SPF}}) + q^2(t|m, l_{\mathrm{SPF}})} \tag{7}$$

$$\rho(t, r|m, l_{\mathrm{SPF}}) = (E_b/N_0) \cdot \frac{r^2 \cdot A^2(t|m, l_{\mathrm{SPF}}) \log_2 M}{2T \displaystyle\int_{-\infty}^{\infty} |H(f)|^2 \, df} \tag{8}$$

where E_b is average signal energy per bit, N_0 is one-sided power spectral density of white Gaussian noise, and E_b/N_0 is average SNR per bit. $H(f)$ is the frequency response of $h(t)$, and r is a normalized fading envelope whose pdf is $f_R(r) = r\exp(-r^2/2)$, $r \geq 0$.

The filtered data phase rotation $\Delta\phi(m, l_{\mathrm{SPF}})$ is determined by integrating $\dot{\phi}(t|m, l_{\mathrm{SPF}})$ over one symbol interval $(t_0 - T, t_0)$. The pdf $f_\Psi(\psi|m, l_{\mathrm{SPF}})$ is derived in [5] to be a function of the maximum Doppler spread f_{Dm}, E_b/N_0, $A^2(t_0 - T|m, l_{\mathrm{SPF}})$, and $A^2(t_0|m, l_{\mathrm{SPF}})$.

In static environments, $P(N = k(m, l_{\mathrm{SPF}}))$ may be calculated using the average click rate with the assumption that N is Poisson distributed [1]–[3], whereas in fading environments $P(N = k(m, l_{\mathrm{SPF}}))$ cannot be calculated using the average click rate for a given average SNR. The reason is that the click rate is not constant, as the amplitude of a faded signal varies. However, for a faded signal with the fixed amplitude r, the conditional probability $P(N = k(m, l_{\mathrm{SPF}})|r)$ may be calculated using the Poisson distribution, since the average click rate may be assumed constant in this case. This approximation is valid as long as click rate varies slowly with respect to the amplitude. $P(N = k(m, l_{\mathrm{SPF}}))$ may be numerically determined by averaging $P(N = k(m, l_{\mathrm{SPF}})|r)$ over r

$$P(N = k(m, l_{\mathrm{SPF}})) = \int_0^\infty P(N = k(m, l_{\mathrm{SPF}})|r) f_R(r) \, dr. \tag{9}$$

The calculation of $P(N = k(m, l_{\text{SPF}})|r)$ may be easily derived with $\rho(t, r|m, l_{\text{SPF}})$ and $\dot{\phi}(t|m, l_{\text{SPF}})$ as described in [1] and [2].

C. Error Probability Calculation for DPF Sequences

In this section, the error probability $P(e|m, l_{\text{DPF}})$ for symbol a_m in sequence l_{DPF} is considered. A DPF sequence may be considered as a combination of two SPF sequences. For example, the sequence $+1 + 3 - 3$ may be viewed as the result of the sequence $+1 + 3 + 1$ up to the middle of the center symbol and the sequence $-3 + 3 - 3$ thereafter [3]. Thus, the data phase rotation $\Delta\phi(m, l_{\text{DPF}})$ and the probability of clicks $P(N = k(m, l_{\text{DPF}}))$ for this kind of sequences l_{DPF} may be approximated as arithmetic means of those of two SPF sequences, l_{SPF1} and l_{SPF2}, each of which contributes to a half of the DPF sequence l_{DPF}. Similarly, $f_{\Psi}(\psi|m, l_{\text{DPF}})$ is approximated using the arithmetic means of $A^2(t|m, l_{\text{SPF1}})$ and $A^2(t|m, l_{\text{SPF2}})$ at $t = t_0 - T$ and $t = t_0$.

By means of these approximations, all $\Delta\phi(m, l_{\text{DPF}})$'s, $f_{\Psi}(\psi|m, l_{\text{DPF}})$'s, and $P(N = k(m, l_{\text{DPF}}))$'s of DPF sequences may be calculated from those of M SPF sequences, and the calculation of $P(e|m, l_{\text{DPF}})$ in (3) for DPF sequences may be performed using these approximations. This approach eliminates the need for direct calculations of $\Delta\phi(m, l_{\text{DPF}})$'s, $P(N = k(m, l_{\text{DPF}}))$'s, and $f_{\Psi}(\psi|m, l_{\text{DPF}})$'s of all $M(M-1)/2$ DPF sequences, and reduces computational requirements.

IV. Simulation and Numerical Results

In this section, simulation and numerical analysis results for 4-ary and 8-ary FSK systems with $h = 2$ are presented for Rayleigh fading channels. The symbol rate $1/T$ is set to 1600 Hz. When $M = 4$, this case corresponds to FLEX 3200 b/s operation [9], [10]. The receiver IF filter used is a fourth-order Butterworth with the bandwidth time product $BT = 7.50$ for $M = 4$ and $BT = 17.50$ for $M = 8$. Perfect symbol timing recovery is assumed.

Fig. 3 shows bit-error rates for 4-ary and 8-ary FSK systems in Rayleigh fading environments. The curves in the figure are calculated in three ways as follows: 1) use of Poisson distribution with the constant click rate assumption for fading environments; 2) use of (9) without the constant click rate assumption; and 3) simulation. A close agreement between simulation and numerical analysis without the constant click rate assumption verifies that our numerical analysis method and a scheme to simplify the computations of error probabilities are correct.

V. Conclusion

In this letter, performance analysis schemes for wide-band M-ary FSK systems with an LDI receiver have been presented along with the derivation of error probability formulas. The accuracy of the error probability formulas has been verified by simulation for both 4-ary and 8-ary FSK systems.

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