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Three-dimensional Elastic Green's Solution by Formal Time-integration Method

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시간적분을 이용한 3차원 탄성파 임펄스 반응 해의 계산

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Abstract : An ad hoc method of deriving three-dimensional elastic Greens solution for displacements is proposed in this study; Instead of direct evaluation of four-dimensional Fourier integral that is usual in spectral approaches, we first derived the expression for particle acceleration using the calculus of residues, and then, derived the expression for particle displacement by direct formal integration with respect to time. As a result, we can detour the disconcertion related to causality due to directly evaluating the integral with respect to angular frequency in conventional spectral approach.

요 약 : 본 연구에서는 탄성파 임펄스 반응 해를 구하는 하나의 대안으로서 시간적분(formal time-integration)을 이용하여 시간 영역에서의 이론 변위를 계산하는 방법에 대해 고찰하였다. 이는 복소 적분법을 이용하여 푸리에 적분식을 직접 적분하는 대신 우선 가속도에 대한 해를 유도하고, 이로부터 시간적분을 이용하여 변위에 대한 해를 유도하는 방법이다. 이 방법은 시간에 대한 푸리에 적분식을 복소적분을 이용하여 구하는 경우 야기되는 유인성과 관련된 혼란을 피할 수 있어, 변위에 대한 이론 임펄스 반응 해가 직관적인 델타함수와 층계함수에 대한 미적분 관계만을 사용하여 쉽게 구해진다.

Keywords :

Introduction

The problem of providing the Green's solution for two- or three-dimensional elastic wave equations is of considerable importance in both earthquake and exploration seismology. There have been extensive studies and literatures on the analytic solution of the problems of this kind. These are broadly classified into two categories: the potential approach based on the Helmholtz vector decomposition theorem (White, 1965; Achenbach, 1973; Aki and Richards, 1980; Pilant, 1978; Ben-Menahem and Singh, 1981) and the spectral approach based on the direct evaluation of complex Fourier integral (Eason *et al.*, 1956).

White (1965), Achenbach (1973) and Aki and Richards (1980) presented a specific time domain formula for displacement in a homogeneous, isotropic, unbounded medium excited by a directional force. They defined displacements and body forces in terms of Lamé potentials, and then solved the wave equation for these potentials by evaluating the surface integral given by the form of Kirchhoff's integral formula. Pilant (1978) and Ben-Menahem and Singh (1981) presented the frequency-domain Green's solutions for two- and three-dimensional elastic wave equations. Especially, Pilant (1978) suggested an ad hoc step where he newly defined the scalar potential and the vector potential with $\nabla \cdot \mathbf{A}_P$ and $\nabla \times \mathbf{A}_S$, respectively. On the other hand, Eason *et al.* (1956) treated the problem of determining the distribution of stress in an infinite elastic medium based on spectral approach. They presented general solutions of the equation of motions for various body force by evaluating a four-dimensional Fourier integral. Among these, the three-dimensional Green's solution by impulsive point force was given as an integral form of the solution by periodic point force in cylindrical coordinates.

In this paper, we present another ad hoc method to derive the time domain Green's solution for particle displacement based on the spectral approach. In the conventional spectral approach, the Fourier integral was directly evaluated by means of the calculus of residues. However, such direct evaluation may be a nuisance due to the directivity terms in the integrand and the poorly resolved issue of the "arrow of time" that one meets when evaluating the integral with respect

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to temporal frequency by means of the calculus of residues (Snieder, 1997). To avoid such a nuisance and the disconcertion in causality problem, we first derive the time expression for particle acceleration by means of the calculus of residues, and then use the direct formal integration with respect to time in deriving the time expression for particle displacement.

Integral Representation of the Greens Solution

In a three-dimensional homogeneous, isotropic, unbounded medium, the displacements due to a directional force satisfies the elastic wave equation

$$\frac{\partial^2 u}{\partial t^2}=\alpha^2\frac{\partial^2 u}{\partial x^2}+(\alpha^2-\beta^2)\frac{\partial^2 v}{\partial x\partial y}+(\alpha^2-\beta^2)\frac{\partial^2 w}{\partial x\partial z}+\beta^2\frac{\partial^2 u}{\partial y^2}+\beta^2\frac{\partial^2 u}{\partial z^2}+\frac{f_x}{\rho}$$

$$\frac{\partial^2 v}{\partial t^2}=(\alpha^2-\beta^2)\frac{\partial^2 u}{\partial x\partial y}+\alpha^2\frac{\partial^2 v}{\partial y^2}+(\alpha^2-\beta^2)\frac{\partial^2 w}{\partial y\partial z}+\beta^2\frac{\partial^2 v}{\partial x^2}+\beta^2\frac{\partial^2 v}{\partial z^2}+\frac{f_y}{\rho}$$

$$\frac{\partial^2 w}{\partial t^2}=(\alpha^2-\beta^2)\frac{\partial^2 u}{\partial z\partial x}+(\alpha^2-\beta^2)\frac{\partial^2 v}{\partial z\partial y}+\alpha^2\frac{\partial^2 w}{\partial z^2}+\beta^2\frac{\partial^2 w}{\partial x^2}+\beta^2\frac{\partial^2 w}{\partial y^2}+\frac{f_z}{\rho}, \tag{1}$$

where u, v, w and f_i are cartesian components of the displacement vector and body force vector. ρ is the density, α and β are the velocity of P- and S- wave, respectively.

Taking the Fourier transform of equation (1), the elastic wave equation in the frequency-wavenumber domain is given by a simple linear algebraic equation

$$\begin{bmatrix}\alpha^2k_x^2+\beta^2k_y^2+\beta^2k_z^2-\omega^2 & (\alpha^2-\beta^2)k_yk_x & (\alpha^2-\beta^2)k_xk_z\\ (\alpha^2-\beta^2)k_yk_x & \beta^2k_x^2+\alpha^2k_y^2+\beta^2k_z^2-\omega^2 & (\alpha^2-\beta^2)k_yk_z\\ (\alpha^2-\beta^2)k_zk_x & (\alpha^2-\beta^2)k_zk_y & \beta^2k_x^2+\beta^2k_y^2+\alpha^2k_z^2-\omega^2\end{bmatrix}\begin{bmatrix}U\\V\\W\end{bmatrix}=\frac{1}{\rho}\begin{bmatrix}F_x\\F_y\\F_z\end{bmatrix} \tag{2}$$

where U, V, and W are each component of displacements vector and F_i's are those of the force vector in the frequency-wavenumber domain. Here the Fourier transform pair is defined as follows:

$$G(k_x, k_y, k_z, \omega)=\iiint\int_{-\infty}^{\infty} g(x, y, z, t)e^{-i(\omega t-k_xx-k_yy-k_zz)}dxdydzdt$$

$$g(x, y, z, t)=\frac{1}{(2\pi)^4}\iiint\int_{-\infty}^{\infty} G(k_x, k_y, k_z, \omega)e^{i(\omega t-k_xx-k_yy-k_zz)}dk_xdk_ydk_zd\omega. \tag{3}$$

Using the Cramer's rule, the solution of equation (2) is given by

$$U=\frac{D_1}{D},\ V=\frac{D_2}{D},\ \text{and}\ W=\frac{D_3}{D} \tag{4}$$

where D denotes the determinant of the coefficient matrix of equation (2) and D_i is the determinant obtained from D by replacing its i-th column with the force vector. Therefore, the displacements in time-space domain are given by the inverse Fourier transform of the solution obtained from equation (4), which lead to the direct evaluation of four-dimensional integral via the calculus of residues.

Consider, for example, the vertical displacement due to a vertical impulsive force, $\boldsymbol{f}=f_z\hat{\boldsymbol{k}}$ where $f_z=\delta(x)\delta(y)\delta(z)\delta(t)$. Then, by plugging $F_x=F_y=0$, and $F_z=1$ into equation (4), the vertical displacement in the frequency-wavenumber domain is given as

$$W(k_x, k_y, k_z, \omega)=\frac{(\beta^2k^2-\omega^2)(\alpha^2-\beta^2)(k_x^2+k_y^2)}{\rho(\alpha^2k^2-\omega^2)(\beta^2k^2-\omega^2)}=\frac{1}{\rho}\left(\frac{k_z^2}{k^2}\frac{1}{\alpha^2k^2-\omega^2}+\frac{k_x^2+k_y^2}{k^2}\frac{1}{\beta^2k^2-\omega^2}\right). \tag{5}$$

Thus, the vertical displacement in time domain $w(x, y, z, t)$ will be given by the inverse Fourier transform of equation (5) as follow:

$$w(x, y, z, t)=\frac{1}{(2\pi)^4\rho}\iiint\int_{-\infty}^{\infty}\frac{k_z^2}{k^2}\frac{e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}}{(\alpha^2k^2-\omega^2)}dk_xdk_ydk_zd\omega+\frac{1}{(2\pi)^4\rho}\iiint\int_{-\infty}^{\infty}\frac{(k_x^2+k_y^2)}{k^2}\frac{e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}}{(\beta^2k^2-\omega^2)}dk_xdk_ydk_zd\omega, \tag{6a}$$

or by substituting the directivity terms with spatial derivatives operators,

$$w(x, y, z, t)=-\frac{1}{(2\pi)^4\rho}\frac{\partial^2}{\partial z^2}\iiint\int_{-\infty}^{\infty}\frac{e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}}{k^2(\alpha^2k^2-\omega^2)}dk_xdk_ydk_zd\omega-\frac{1}{(2\pi)^4\rho}\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right\}\iiint\int_{-\infty}^{\infty}\frac{e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}}{k^2(\beta^2k^2-\omega^2)}dk_xdk_ydk_zd\omega \tag{6b}$$

where $\mathbf{k}=[k_x, k_y, k_z]$ is the wavenumber vector and $\mathbf{r}=[x, y, z]$ is the position vector of any observation point within the domain. In the following derivation, for convenience, we will use equation (6b) in describing our approach to derive explicit time expression for particle displacement.

Green's Solution by Formal Time-Integration

Green's Solution for Particle Acceleration

We begin our approach with the evaluation of integral with respect to wavenumber by means of complex integral, i.e., the displacement in space-frequency domain. Let us take the triplet integral over the wavenumber space in equation (6b) and call it $\tilde{W}(x, y, z, \omega)$:

$$\tilde{W}(x, y, z, \omega)=\frac{1}{(2\pi)^3\rho}\frac{\partial^2}{\partial z^2}\iint\int_{-\infty}^{\infty}\frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{k^2(\alpha^2k^2-\omega^2)}dk_xdk_ydk_z$$
$$+\frac{1}{(2\pi)^3\rho}\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right\}\iint\int_{-\infty}^{\infty}\frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{k^2(\beta^2k^2-\omega^2)}dk_xdk_ydk_z. \quad (7)$$

These triplet integrals can be easily evaluated by means of the calculus of residue in the spherical coordinates proposed by Griffel (1981) (Appendix A), which precisely gives

$$\tilde{W}=+\frac{1}{4\pi\rho}\frac{\partial^2}{\partial z^2}\frac{e^{-i\omega r/\alpha}}{r\omega^2}-\frac{1}{4\pi\rho\omega^2}\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right)$$
$$+\frac{1}{4\pi\rho}\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right\}\frac{e^{-i\omega r/\beta}}{r\omega^2}-\frac{1}{4\pi\rho\omega^2}\left[\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\frac{\partial^2}{\partial y^2}\right]. \quad (8)$$

Note that

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right)=-\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right). \quad (9)$$

Then, the second term and the forth term are eventually canceled out each other, and thus, the time domain displacement $w(x, y, z, t)$ is given by

$$w(x, y, z, t)=\frac{1}{8\pi^2\rho}\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\int_{-\infty}^{\infty}\frac{e^{i\omega\left(t-\frac{r}{\alpha}\right)}}{-\omega^2}d\omega\right)$$
$$+\frac{1}{8\pi^2\rho}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{1}{r}\int_{-\infty}^{\infty}\frac{e^{i\omega\left(t-\frac{r}{\beta}\right)}}{-\omega^2}d\omega\right). \quad (10)$$

To obtain the explicit form of particle displacement, we have to use the method of calculus of residues once more. However, it is not clear how the pole on $\omega=0$ affects the response, and this results in the poorly resolved issue of the "arrow of the time" (Snieder, 1997).

At this point, we note that the integrand has the form of twice-integration of exponential terms with respect to time. Then, the expression for particle acceleration is readily derived using the basic Fourier transform relation, i.e., $\ddot{w}=F^{-1}\{-\omega^2\tilde{W}\}$. Hence, we have

$$\ddot{w}(x, y, z, t)=\frac{1}{4\pi\rho}\frac{\partial^2}{\partial z^2}\frac{\delta(t-r/\alpha)}{r}+\frac{1}{4\pi\rho}\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right\}\frac{\delta(t-r/\beta)}{r}. \quad (11)$$

Now, considering that

$$\frac{\partial^2}{\partial x_i^2}\left[\frac{\delta\left(t-\frac{r}{c}\right)}{r}\right]=\frac{x_i^2}{c^2r^3}\delta''\left(t-\frac{r}{c}\right)+\frac{r}{c}\left(-\frac{1}{r^3}+\frac{3x_i^2}{r^5}\right)\delta'\left(t-\frac{r}{c}\right)$$
$$+\left(-\frac{1}{r^3}+\frac{3x_i^2}{r^5}\right)\delta\left(t-\frac{r}{c}\right)$$
$$=\frac{x_i^2}{c^2r^3}\delta''\left(t-\frac{r}{c}\right)+\frac{r}{c}\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{r}\right)\delta'\left(t-\frac{r}{c}\right)+\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{r}\right)\delta\left(t-\frac{r}{c}\right) \quad (12)$$

and using the relation (9) again, the particle acceleration can be explicitly represented with delta functions and their derivatives as follow:

$$\ddot{w}(x, y, z, t)=\frac{1}{4\pi\rho\alpha^2}\frac{z^2}{r^3}\delta''\left(t-\frac{r}{\alpha}\right)+\frac{1}{4\pi\rho\beta^2}\frac{x^2+y^2}{r^3}\delta''\left(t-\frac{r}{\beta}\right)$$
$$+\frac{1}{4\pi\rho}\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right)\left[\frac{r}{\alpha}\delta'\left(t-\frac{r}{\alpha}\right)-\frac{r}{\beta}\delta'\left(t-\frac{r}{\beta}\right)\right]$$
$$+\frac{1}{4\pi\rho}\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right)\left[\delta\left(t-\frac{r}{\alpha}\right)-\delta\left(t-\frac{r}{\beta}\right)\right]. \quad (13)$$

Green's Solution for displacement via Formal Time-integration

The time domain expression for particle displacement can be derived by the formal integration of equation (13) with respect to time using the definitions of the delta function $\delta(t)$ and the Heaviside step function $H(t)$. This formal integration is straightforward, and thus, the vertical particle velocity is give by

$$\dot{w}(x, y, z, t)=\frac{1}{4\pi\rho\alpha^2}\frac{z^2}{r^3}\delta'\left(t-\frac{r}{\alpha}\right)+\frac{1}{4\pi\rho\beta^2}\frac{r^2-z^2}{r^3}\delta'\left(t-\frac{r}{\beta}\right)$$
$$+\frac{1}{4\pi\rho}\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right)\left[\frac{r}{\alpha}\delta\left(t-\frac{r}{\alpha}\right)-\frac{r}{\beta}\delta\left(t-\frac{r}{\beta}\right)\right]$$
$$+\frac{1}{4\pi\rho}\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right)\left[H\left(t-\frac{r}{\alpha}\right)-H\left(t-\frac{r}{\beta}\right)\right], \quad (14)$$

and the vertical displacement is given by

$$w(x, y, z, t)=\frac{1}{4\pi\rho\alpha^2}\frac{z^2}{r^3}\delta\left(t-\frac{r}{\alpha}\right)+\frac{1}{4\pi\rho\beta^2}\frac{r^2-z^2}{r^3}\delta\left(t-\frac{r}{\beta}\right)$$
$$+\frac{1}{4\pi\rho}\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right)\left[\frac{r}{\alpha}H\left(t-\frac{r}{\alpha}\right)-\frac{r}{\beta}H\left(t-\frac{r}{\beta}\right)\right]$$
$$+\frac{1}{4\pi\rho}\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right)\left[\int_{r/\alpha}^{\infty}H\left(t-\frac{r}{\alpha}\right)dt-\int_{r/\beta}^{\infty}H\left(t-\frac{r}{\beta}\right)dt\right]. \quad (15)$$

Substituting the direction cosine $z/r=\cos\theta$ into equation (15) and applying the following integral relation

$$\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}}\tau\delta(t-\tau)d\tau=-H(t-\tau)\tau\Big|_{\frac{r}{\alpha}}^{\frac{r}{\beta}}+\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}}H(t-\tau)d\tau \quad (16)$$

We obtain

$$w(x, y, z, t)=\frac{1}{4\pi\rho\alpha^2}\frac{\cos^2\theta}{r}\delta\left(t-\frac{r}{\alpha}\right)+\frac{1}{4\pi\rho\beta^2}\frac{\sin^2\theta}{r}\delta\left(t-\frac{r}{\beta}\right)$$
$$+\frac{1}{4\pi\rho}\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right)\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}}\tau\delta(t-\tau)d\tau. \qquad (17)$$

Similar form of time domain expressions for arbitrary source function is found in White (1965), Achenbach (1973), and Aki and Richards (1980).

Conclusion

We proposed an ad hoc method of deriving Green's solution for displacement in a homogeneous, isotropic, unbounded medium excited by a directional force. We first derived the expression for particle acceleration by using the Fourier transform relations and calculus of residues, and then, derived the expression for particle displacement by direct formal integration with respect to time. As a result, we can effectively reduced the mathematical difficulty in evaluating the integral and detour the inconsistency between physics and mathematics: the issue of the "arrow of time" which may be encountered when directly evaluating the integral with respect to angular frequency.

This approach may be easily extended to two-dimensional problems and problems having a multi-directional driving forces. In addition, the formal expressions for particle accelerations and particle velocities obtained while deriving the expression for the displacement can be additional advantages, which is helpful to have some insight for physical wave phenomena.

References

Achenbach, J. D., 1973, *Wave Propagation in Elastic Solids*: North-Holland Publishing Co., 96-114.

Aki, K., and Richards, P. G., 1980, *Quantitative seismology: theory and methods*: Vol. **1**, Freeman, 70-73.

Arfken, G., 1985, *Mathematical methods for physicists, 3rd Ed.*, Academic Press, 409-411.

Ben-Menahem, A., and Singh, S. J., 1981, *Seismic waves and sources*: Springer, 53-62.

Eason, G., Fulton, J., and Sneddon, I. N., 1956, *The generation of waves in an infinite elastic solid by variable body*: Phil. Trans. Roy. Soc., London, **248**, 575-607.

Griffel, D. H., 1981, *Applied functional analysis*: Ellis Horwood, 67-78.

Pilant, W. L., 1979, *Elastic waves in the earth*: Elservier Scientific, 39-51.

Snieder, R., 1997, *A Guided Tour of Mathematical Physics*: Smizdat Press, 151-155.

White, J. E., 1965, *Seismic waves: Radiation, Transmission, and Attenuation*: McGraw-Hill. 213-216.

Appendix A: Evaluation of *k*-integral by Calculus of Residues

As a typical form of the triplet integral with respect to wavenumber in equation (9) of the main text, consider an integral

$$I(\xi)=\iiint_{-\infty}^{\infty}\frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{k^2(k^2-\xi^2)}dk_xdk_ydk_z \qquad (A.1)$$

where $\mathbf{k}=[k_x, k_y, k_z]$, $\mathbf{r}=[x, y, z]$, and ξ is real and positive. This can be effectively evaluated in spherical coordinates (k, θ, ϕ) with axis along $\mathbf{r}$ (Griffel, 1981):

$$k=\sqrt{k_x^2+k_y^2+k_z^2}=\sqrt{k'^2_x+k'^2_y+k'^2_z}$$
$$\theta=\cos^{-1}\frac{\mathbf{k}\cdot\mathbf{r}}{kr}$$
$$\phi=\tan^{-1}\frac{k_y'}{k_x'} \qquad (A.2)$$

where k_x', k_y' and k_z' are the cartesian components of the vector $\mathbf{k}$ with respect to new axes of which the third is parallel to $\mathbf{r}$. Then, equation (A.2) becomes

$$I(\xi)=\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi}\frac{e^{-ikr\cos\theta}}{k^2-\xi^2}\sin\theta d\phi d\theta dk$$
$$=-\frac{2\pi i}{r}\int_0^{\infty}\frac{e^{ikr}-e^{-ikr}}{k(k^2-\xi^2)}dk=-\frac{\pi i}{r}\int_{-\infty}^{\infty}\frac{e^{ikr}-e^{-ikr}}{k(k^2-\xi^2)}dk. \qquad (A.3)$$

For this, we will use complex integration. The integration variable is now called a complex variable $z=x+iy$ rather than real k. Thus, we write equation (A.1) in the complex plane as

$$I(\xi)=-\frac{\pi i}{r}\int_{-\infty}^{\infty}\frac{e^{izr}-e^{-izr}}{z(z^2-\xi^2)}dz$$
$$=\frac{\pi i}{r\xi^2}\left[\int_{-\infty}^{\infty}\frac{e^{izr}-e^{-izr}}{z}dz-\int_{-\infty}^{\infty}\frac{z(e^{izr}-e^{-izr})}{(z^2-\xi^2)}dz\right]. \qquad (A.4)$$

Note that the first integral precisely gives $2\pi i$, as readily evaluated by complex contour integral. Now the problem is to evaluate the second integral. However, we can fortunately find a solution to this problem provided by Arfken (1985, pp 410-411). Following his derivation except that we move-off the poles in real axis by letting $\xi\rightarrow\xi-i\gamma$ instead of $\xi\rightarrow\xi+i\gamma$ in order to obtain the "outgoing wave", the second integral gives

$$\int_{-\infty}^{\infty}\frac{z(e^{izr}-e^{-izr})}{(z^2-\xi^2)}dz=2\pi ie^{-i\xi r}. \qquad (A.5)$$

As a result, the integral (A.2) becomes

$$I(\xi)=\frac{2\pi^2}{r\xi^2}(e^{-i\xi r}-1). \qquad (A.6)$$