C^*-Algebras and K-Theory

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Introduction.

Recently functorial approaches have been introduced to the study of C^*-algebras. These approaches connect operator algebras on one end and algebraic topology on the other end. Among them notables are: Extension theory of Brown-Douglas-Fillmor [2], K-theory of Taylor-Karoubi etc., and KK-theory of Kasparov[4].

In this note we introduce K-theory of C^*-algebras. Readers will immediately notice that we are following the line of Taylor[5]. But we mention that Taylor only considered the case of commutative Banach algebras. However, it is widely known that Taylor's proof goes through for the case of general C^*-algebras without any resistance. But unfortunately it is not written down anywhere. In doing so, we simplify many proofs and moreover we clarify the boundary map. We will see that the boundary map is very natural in the context of C^*-algebras. It is nothing but the index map of Fredholm operators in some cases. And also, we mention that the proof of (3) of Theorem 2.6 is a new approach and is not printed in any place as far as the author knows. The rest of the note is organized as follows.

In Section 1, elementary theory of C^*-algebras of what we need in this lecture are discussed. We give a few examples of C^*-algebras. These examples are directly related to the K-theory of C^*algebras one way or another. In Section 2, K-groups are constructed and basic properties are discussed. In Section 3, the exactness of long exact sequence are proved.

1. C^*-algebras

1.1 Let X be a compact Hausdorff space. Let $C(X)$ be the space of all continuous complex-valued functions on X. Under the point-wise addition and point-wise multiplication $C(X)$ is a commutative (complex) algebra. That is, it is a vector space which is also a commutative ring. In addition to these algebraic structures it has two more, namely norm and involution. A norm $\|\cdot\|: C(X)\to \boldsymbol{R}$ is defined by setting

$$\|f\|=\sup\{|f(x)|: x\in X\},\ f\in C(X).$$

Then $C(X)$ is a Banach space under this supremum norm. An involution $*: C(X)\to C(X)$ is defined by

$$f^*(x)=\overline{f(x)}$$

where the bar "——" denotes the usual complex conjugation. Moreover, the crucial structures (multiplication, norm and involution) are interwoven by the following indentity (it is the so-called C^*-condition):

$$\|f^*f\|=\|f\|^2.$$

1.2 Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{L}(\mathcal{H})$ be the space of all bounded linear operators $T: \mathcal{H}\to\mathcal{H}$. Then $\mathcal{L}(\mathcal{H})$ is an algebra under the pointwise addition and composition as a multiplication. It has a norm defined by

$$\|T\|=\sup\{\|T(x)\|: \|x\|\le 1, x\in\mathcal{H}\}$$

where the norm in the paranthesis denotes the norm on $\mathcal{H}$ induced by the given inner proact $(,)$ in $\mathcal{H}$. Let x be any fixed element in $\mathcal{H}$. Then $y\to(x,Ty)$ is a bounded linear functional on $\mathcal{H}$. Thus by the Riesz Representation Theorem there is a unique $w\in\mathcal{H}$ such that

$$(x, Ty)=(w, y).$$

Call $w=T^*x$. Then $T^*\in\mathcal{L}(\mathcal{H})$. Thus we have an involution on $\mathcal{H}$ defined by $T\to T^*$ and moreover

$$\|T^*T\|=\|T\|^2.$$

1.3 An algebra A which is also a Banach space is called a ***Banach algebra*** if for any $x, y\in A$ we have

$$\|xy\|\le\|x\|\|y\|.$$

Definition A C^*-algebra A is a Banach algebra with the involution* satisfying the following:

(i) $(x^*)^*=x$ (idempotency)

(ii) $(x+y)^*=x^*+y^*$, $(\alpha x)^*=\bar{\alpha}x^*$ (conjugate linearity)

(iii) $(xy)^*=y^*x^*$

(iv) $\|x^*x\|=\|x\|^2$. (C^*-norm condition)

As mentioned before the condition (iv) plays many crucial roles in the the theory of

C^*-algebras. We have already two most important examples of C^*-algebras. In fact, they are the only ones in the following sense. The Gelfand-Naimark theorem says that: (1) Every communtative C^*-algebra with unit is isometrically *-isomorphic to $C(X)$ for some compact Hausdorff space X. (2) Every C^*-algebra is isomerically *-isomorphic to some *-closed, norm-closed subalgebra of $\mathcal{L}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

1.4 A C^*-algebra with unit is called *unital*. Unit preserving homomorphism is called unital homomorphism. Let X and Y be compact Housdorff spaces. Let ϕ be a unital *-homomorphism from $C(Y)$ into $C(X)$. Then there is a continuous map $\pi: X \to Y$ such that for all $f \in C(Y)$

$$(\phi(f))(x) = f(\pi(x)), \quad x \in X.$$

Thus $\|\phi(f)\| \leq \|f\|$. Hence as an operator $\|\phi\| \leq 1$. If unital-homomorphism ϕ is one-to-one and onto, then π must be a homeomorphism. Hence in this case $\|\phi(f)\| = \|f\|$ for all f in $C(Y)$. Thus for commutative unital C^*-algebras, unital *-homomorphism is necessary continuous (i.e., bounded) and *-isomorphism (merely as *-algebra) preserves the norm. Even more we have the following fundamental theorem.

Theorem Let A and B be unital C^*-algebras. Let $\phi: A \to B$ be a unital *-homomorphism. Then for all $x \in A$,

$$\|\phi(x)\| \leqq \|x\|.$$

If ϕ is a *-isomorphism, then ϕ is an isometry.

Proof (Reduction to the commutative case). Notice that for any x in A, $\|\phi(x)\|^2 = \|\phi(x^*x)\|$ and that the C^*-algebra generated by x^*x and the unit 1 is commutative. Thus applying ϕ to this commutative subalgebra, we get

$$\|\phi(x)\|^2 = \|\phi(x^*x)\| \leq \|x^*x\| = \|x\|^2.$$

The proof of the isometry follows easily.

Thus in the category of C^*-algebras and *-homomorphisms, the continuity of the map is automatic. Another consequence of this theorem is that a complete norm satistying the C^*-condition is unique, when it exists.

1.5 Examples

(1) We have already seen that $C(X)$, $\mathcal{L}(\mathcal{H})$ are C^*-algebras.

(2) Let A be a C^*-algebra and B a *-closed and norm closed subalgebra of A. Then B is a C^*-algebra. In this case B is often called a *C^*-subalgebra* of A. Thus for a locally compact Hausdorff space X, the space $C_0(X)$ of all continuous function vanishing at ∞ is a C^*-algebra. In fact, any commutative C^*-algebra without unit is *-isomorphic to

$C_0(X)$ for some locally compact Hausdorff space X.

(3) The full matrix algebra $M_n(\boldsymbol{C})$ of all $n \times n$ matrices over $\boldsymbol{C}$ is a C^*-algebra. The space of all continuous $n \times n$ matrix-valued function on a compact Hausdorff space X is a C^*-algebra with supremum norm.

(4) The Space $M_n(A)$ of all $n \times n$ matrices over a C^*-algebra A is a C^*-algebra.

(5) Let $\mathcal{K}$ denote the space of all compact operators on $\mathcal{H}$, i.e., those operators which transform the unit ball of $\mathcal{H}$ into a compact subset of $\mathcal{H}$. Then $\mathcal{K}$ is a C^*-subalgebra of $\mathcal{L}(\mathcal{H})$. Moreover $\mathcal{K}$ is a norm-closed two sided ideal of $\mathcal{L}(\mathcal{H})$.

(6) Let A_n be a sequence of C^*-algebras. Suppose $A_n \subseteq A_{n+1}$ for all n. Let $A_\infty = \bigcup_{n=1}^{\infty} A_n$. Then A_∞ satisfies all the properties except (possibly) completeness. Let A be the completion of A_∞. Then A is a C^*-algebra. If A_n happens to be finite dimensional C^*-algebras, then A is called approximately finite. More precisely a C^*-algebra A is called AF if there is an increasing sequence of finite dimensional C^*-subalgebras A_n of A such that

$$A = \overline{\bigcup_{n=1}^{\infty} A_n}.$$

(7) Let S_i be an isomety on a Hilbert space $\mathcal{H}$, i.e., $S_i^* S_i = 1$. Suppose that $\sum_{i=1}^{n} S_i S_i^* = 1$. The smallest C^*-algebra containing all S_i is denoted by θ_n and is often called the *Cuntz* algebra. It turns out that θ_n does not depend on the choice of S_i.

(8) Let A and B be two C^*-algebras. Consider the algebraic tensor product $A \odot B$. Then $A \odot B$ is $*$-algebra in a natural way. In general there are many ways to put a C^*-norm on $A \odot B$. Here is one of them. For $x \in A \odot B$, define

$$\|x\|_{\min} = \sup\{\|\pi_1 \otimes \pi_2(x)\| : \pi_1, \pi_2 \text{ representations of } A \text{ and } B, \text{ respectively}\}.$$

The completion of $A \odot B$ under this norm is denoted by $A \underset{\min}{\otimes} B$. In some cases, for instance $B = \mathcal{K}$, it is known that there are only one way to impose a C^*-norm on $A \odot \mathcal{K}$ (or $\mathcal{K} \odot A$). Thus in this case we write $\mathcal{K} \otimes A$ without any danger of confusion. If $B = M_n(\boldsymbol{C})$, then $M_n(\boldsymbol{C}) \otimes A$ can be identified as $M_n(A)$.

(9) Let G be a locally compact abelian group. Let μ denote the unique Haar measure on G (unique up to constant multiple !).

Let α be a continuous homomorphism of G into the group $\mathrm{Aut}(A)$ of all $*$-automorphisms equipped with the topology of pointwise convergence, i.e., for any net $g_j \to g$ in G, $\alpha(g_i)(x) \to \alpha(g)(x)$ in the norm of A for each x. The triple (A, G, α) is called a C^*-dynamical system. We define an involution, multiplication and norm on $K(G, A)$ of continuous functions from G to A with compact supports by

$$y^*(g)=\alpha(g)(y(g^{-1})^*)$$
$$(y\times z)(g)=\int y(h)\alpha(g)(z(h^{-1}g))d\mu(h)$$
$$\|y\|=\int \|y(g)\|d\mu(g).$$

Then $K(G, A)$ becomes a norm *-algebra. Let $L^1(G, A)$ denote the completion of $K(G, A)$. Let $A\underset{\alpha}{\times}G$ be the completion under the greatest C^*-norm on $L^1(G, A)$. This alge bra $A\underset{\alpha}{\times}G$ is called the *crossed product algebra*.

(10) Let A be a C^*-algebra. Let $A^+=A\oplus C$ with the following:

$$(x,\alpha)+(y,\beta)=(x+y,\ \alpha+\beta)$$
$$(x,\alpha)\ (y,\beta)=(xy+\alpha y+\beta x,\ \alpha\beta)$$
$$(x,\alpha)^*=(x^*,\bar{\alpha}).$$

For any $(x,\alpha)\in A^+$, $y\to(x,a)(y,0)$ defines a bounded linear operator from A to A. Taking the operator norm as a norm of $(x,\alpha)\in A^+$, A^+ becomes a C^*-algebra with unit. Notice that $A=(A,0)$ is a maximal ideal of A^+. If A has a unit already, then A^+ is *-isomorphic to $A\oplus C$ (C^*-direct sum). Thus any C^*-algebra can be imbedded in a unital C^*-algebra. This algebra A^+ is said to be the C^*-algebra obtained by adjoining an identity to A.

2. K-groups of C^*-algebras.

2.1 Let A be a unital C^*-algebra. Let $M_n(A)$ be the matrix algebra of entries from A. By imbedding $M_n(A)$ into the left corner of $M_{n+1}(A)$, i.e., for $a\in M_n(A)$

$$a\longrightarrow\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix} \text{ in } M_{n+1}(A)$$

we have an increasing sequence of C^*-algebras $\{M_n(A)\}$. Let

$$M_\infty(A)=\bigcup_{n=1}^{\infty} M_n(A).$$

Two projections (i.e. self-adjoint and idempotent) e, f in $M_\infty(A)$ are *equivalent* if there is an element v in $M_\infty(A)$ such that $v^*v=e$ and $vv^*=f$. Then this relation defines an equivalence relation on the projections of $M_\infty(A)$. It can be seen that if two projections are equivalent, then we can find a unitary u in $M_n(A)$ for some n such that $e=u^*fu$. In fact the converse is also ture.

We denote two equivalent projections by "$e\sim f$".

Proposition Two projections e and f are equiyalent if and only if there is a continuous path, consisting of projections of $M_\infty(A)$, joining e and f.

Proof Choose n so that e, f, and unitary u are in $M_n(A)$ with $e=u^*fu$. Then notice that

$$\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} u^* & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & u^* \end{pmatrix}\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}.$$

Choose a path of unitaries joining $\begin{pmatrix} u^* & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -u^* \end{pmatrix}$ by

$$t \longrightarrow \begin{pmatrix} \cos t & \sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} u^* & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Then

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} u^* & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

joins e and f via projections in $M_n(A)$.

Conversely, if two projections are close ($\|e \sim f\| < 1$), then e and f are unitarily equivalent. Thus a continuous path of projections gives us a bunch of equivalent projections.

Remark We can do the same procedures for $\mathscr{K} \otimes A$(see[1]).

2.2. Let $S(A)$ be the equivalence classes of projections in $M_\infty(A)$. For any two $[e]$, $[f]$ in $S(A)$ we define the addition as follows:

$$[e]+[f]=[e'+f']$$

where $e \sim e'$, $f \sim f'$ and $e'f'=0$. We can always find such e' and f' in $M_\infty(A)$. Then it is easy to see that $[e'+f']$ does not depend on the choice of e' and f'. Now $S(A)$ becomes an abelian monoid under the addition just defined (possible no cancellation law !). There is a standard procedure to obtain a group out of a monoid. Here is how to do it. First, define an equivalence relation "$\sim$" on $S(A) \times S(A)$ by

$$([e],[f]) \sim ([e'],[f'])$$

if there is a $[g]$ in $S(A)$ such that

$$[e]+[f']+[g]=[e']+[f]+[g].$$

Definition The K_0-group, $K_0(A)$, of a unital C^*-algebra A is the Grothendieck group $S(A) \times S(A)/\sim$. We write $([e],[f])$ in $K_0(A)$ as $[e]-[f]$.

Notice that $[e]-[e]$ are all equivalent and this serves as the identity for $K_0(A)$. Even $[e]-[f]=0$ in $K_0(A)$ if and only if there is a g such that $e+g=e+f$. We mention in passing that $K_0(A)$ is commutative.

2.3 **Theorem**

(1) Let A and B be unital C^*-algebras. Let ϕ be a unital $*$-homomorphism of A into

B. Then ϕ induces a group homomorphism.

$$K_0(f) : K_0(A) \longrightarrow K_0(B).$$

That is, $K_0(\cdot)$ is a covariant functor from the category of C^*-algebras and $*$-morphisms to the category of abelian groups and homomorphisms.

(2) $K_0(A \oplus B) \cong K_0(A) \oplus K_0(B)$.

(3) Let A_n be an increasing sequence of a C^*-algebra with the same unit and $A = \overline{\cup A_n}$. Then $K_0(A) \cong$ dir. lim $K_0(A_n)$.

Proof (1) It is routine to check that

$$id \otimes \phi : M_n \otimes A \longrightarrow M_n \otimes B$$

is a unital $*$-homomorphism. Thus id$\otimes\phi$ maps projections into projections and it respects equivalence relations and addition. Thus we have a group homomorphism.

(2) Apply (1) to the following split exact sequence

$$0 \to A \to A \oplus B \to B \to 0.$$

(3) Let $\phi_n : A_n \to A_{n+1}$ and $\theta_n : A_n \to A$ be the inclusion maps. The following commutative diagrams

$$\begin{array}{ccccc} \longrightarrow A_n & \xrightarrow{\phi_n} & A_{n+1} \longrightarrow \\ {}_{\theta_n}\searrow & & \swarrow{}_{\theta_{n+1}} \\ & A & \end{array}$$

induces a commutative diagram of K_0-groups

$$\begin{array}{ccccc} \longrightarrow K_0(A_n) & \xrightarrow{\phi_n} & K_0(A_{n+1}) \longrightarrow \\ {}_{\theta_n}\searrow & & \swarrow{}_{\theta_{n+1}} \\ & K_0(A) & \end{array}$$

Thus we have a map θ: dir. lim $K_0(A_n) \to K_0(A)$:

$$\begin{array}{ccccc} \longrightarrow K_0(A_n) & & \longrightarrow & & K_0(A_{n+1}) \longrightarrow \\ & {}_{\pi_n}\searrow & & \swarrow{}_{\pi_{n+1}} & \\ {}_{\theta_n}\searrow & & \text{dir. lim } K_0(A_n) & & \swarrow{}_{\theta_{n+1}} \\ & & \downarrow \theta & & \\ & & K_0(A) & & \end{array}$$

To show that θ is onto, choose any $[e]-[f]$ in $K_0(A)$. We may assume that e and f belong to the same $M_p(A)$. Since $\cup A_n$ is dense in A, $\bigcup_{n=1}^{\infty} M_p(A_n)$ is dense in $M_p(A)$.

Thus we can find e', f' in $M_p(A_n)$ such that $e'\sim e$, $f'\sim f$. Then $\theta(\pi_n([e']-[f']))=[e]-[f]$. To show that θ is one-to-one, suppose that $\theta(a)=0$ for a in dir.lim $K_0(A_n)$. By the definition of direct limit group, there is a b in $K_0(A_n)$ for some n such that $\pi_n(b)=a$. Thus $\theta_{n+1}(b)=\theta(\pi_{n+1}(b))=\theta(a)=0$. Thus it suffices to show that b is zero in $K_0(A_m)$ for some $m\geq n$.

Let $b=[e]-[f]$ with e, f in $M_p(A_n)$. Since $\theta_{n+1}([e]-[f])=[\theta_{n+1}(e)]-[\theta_{n+1}(f)]=0$, there exists a projection g in $M_q(A)$ such that $[\theta_{n+1}(e)]+[g]=[\theta_{n+1}(f)]+[g]$. Increasing n, we may replace g by an equivalent projection in $M_q(A_m)$. Still we denote it by g. Thus we may assume

$$\theta_{n+1}(e)+g\sim\theta_{n+1}(f)+g$$

in $M_r(A)$ and projections in $M_r(A_m)$. Thus there is a unitary u in $M_r(A)$ such that $\theta_{n+1}(e)+g=u^*(\theta_{n+1}(f)+g)u$. Finally we can find a unitary v in $M_r(A_l)$ with $\|v-u\|<\frac{1}{2}$

Then

$$\|v^*(\theta_{n+1}(f)+g)v-(\theta_{n+1}(e)+g)\|=\|v^*(\theta_{n+1}(f)+g)v-u^*(\theta_{n+1}(f)+g)u\|$$
$$\leqq\|v^*(\theta_{n+1}(f)+g)v-u^*(\theta_{n+1}(f)+g)v\|-\|u^*(\theta_{n+1}(f)+e)v-u^*(\theta_{n+1}(f)+g)u\|<1.$$

Hence $v^*(\theta_{n+1}(f)+g)v$ and $\theta_{n+1}(e)+g$ are equivalent in $M_r(A_l)$. Therefore $[\theta_{n+1}(e)]-[\theta_{n+1}(f)]=0$ in $K_0(A_l)$. This completes the proof.

2.4 Let A be a unital C^*-algebra. Let $U(n, A)$ $(GL(n, A))$ be the group of all unitary (invertible) elements in $M_n(A)$. Let $U^\circ(n, A)$ $(GL^\circ(n, A))$ be the connected component of the identity in $U(n, A)$ $(GL(n, A))$ be the connected component of the identity in $U(n, A)$ $(GL(n, A))$. Then $U(n, A)/U^\circ(n, A)\cong GL(n, A)/GL^\circ(n, A)$ (Polar decomposition will provide an isomorphism.). Call this group $I_n(A)$. Identify any element u in $U(n, A)$ to the left corner of $U(n+1, A)$ by

$$u\longrightarrow\begin{pmatrix}u & 0\\0 & 1\end{pmatrix}.$$

This identification preserves the equivalence relation, so it induces group homomorphism $I_n(A)\longrightarrow I_{n+1}(A)$.

Definition Let A be a unital C^*-algebra. Then the K_1-group, $K_1(A)$, is defined to be the direct limit group of $I_n(A)$ and of the homomorphism induced by the inclusions, i.e.,

$$K_1(A)=\text{dir. lim } I_n(A).$$

Remarks. (1) Even though $I_n(A)$ is not abelian in general $K_1(A)$ is an abelian group.

(2) Two unitary elements u, v determines the same element in $I_n(A)$ if and only if uv^*

belongs to $U^0(n, A)$ if and only if uv^* and the identity can be continuously joined. Thus $[u]=[v]$ in $I_n(A)$ if and only if there is a continuous path, consiting of elements of $U(n, A)$, connecting u and v, i.e., u and v belong to the same connected component.

2.5 **Theorem** Let A be a unital C^*-algebra. Then

$$K_1(A) \cong U(\mathscr{K}\otimes A)^+)/U^\circ(\mathscr{K}\otimes A)^+)(=I_\infty(A)),\ \text{i.e.,}$$

$K_1(A)$ is isomorphic to the abstract index group of the unital C^*-algebra $(\mathscr{K}\otimes A)^+$. To prove this we need a lemma due to J. Cuntz[1].

Lemma Every $u \in U\ (\mathscr{K}\otimes A)^+$ is equivalent to a unitary of the form $u'+(1-p\otimes 1)$, where p is a projection in $\mathscr{K}$ and u' is unitary in $p\mathscr{K}p\otimes A \subset (\mathscr{K}\otimes A)^+$, 1 denotes the identity in $(\mathscr{K}\otimes A)^+$.

Proof of Lemma Notice that $p\otimes 1$ is an approximate unit for $\mathscr{K}\otimes A$, where p runs through projections in $\mathscr{K}$. Write $u=(x, \alpha)=\lambda+(x, 0)=\lambda+x$ for simplicity. Then $x=u-\lambda$ is an element of $\mathscr{K}\otimes A$ and $|\lambda|=1$. Then there is a projection p such that

$$\|(p\otimes 1)(u-\lambda)(p\otimes 1)-(u-\lambda)\|=\|(p\otimes 1)u(p\otimes 1)+(1-p\otimes 1)-u\|<1.$$

Thus $(p\otimes 1)u(p\otimes 1)+\lambda(1-p\otimes 1)$ is invertible in $(\mathscr{K}\otimes A)^+$. Thus u and $u'+(1-p\otimes 1)$ can be connected continuously in GL $(\mathscr{K}\otimes A)^+)$, where $u'=(p\otimes 1)u(p\otimes 1)\in p\mathscr{K}p\otimes A$. Then the polar decomposition of invertible elements will provide the necessary path consisting of unitaries.

Proof of Theorem Let π_n be the map appeared in the definition of direct limit group. For any n and any $u \in U(n, A)$ imbed u in $(\mathscr{K}\otimes A)^+$ by

$$\theta_n : u \to u+(1-p_n\otimes 1),$$

where p_n denote the identity matrix of size n. These inclusions induce group homomorphisms $\theta_n : I_n(A) \to I_\infty(A)$, thus we have a map

$$\theta : K_1(A) \to I_\infty(A).$$

To prove that this map is onto, take any $u \in U((\mathscr{K}\otimes A)^+)$. We may assume by Lemma that

$$u=u'+1-p_n\otimes 1.$$

Thus $\theta(\pi(u'))=[u]$.

To prove that it is one-to-one, suppose that $\theta(a)=0$. Choose n and v such that $\pi_n(v)=a$, $v\in I_n(A)$. Then since $\theta_n(v)=\theta(\pi_n(v))=0$, $v+(1-p_n\otimes 1)$ is connected to the identity. Thus $v(p_n\otimes 1)=v$ is connected to $p_n\otimes 1$. Hence $[a]=0$ in $I_n(A)$. This completes the proof.

2.6 **Theorem.**

(1) Let A and B be unital C^*-algebras and $\phi: A \to B$ unital *-homomorphism. Then ϕ induces a group homomorphism

$$K_1(\phi): K_1(A) \to K_1(B),$$

i.e., $K_1(\cdot)$ is a covariant functor.

(2) $K_1(A \oplus B) \cong K_1(A) \oplus K_1(B)$.

(3) If $\cup A_n = A$, A_n increasing with the same unit, then

$$K_1(A) \cong \text{dir. lim } K_1(A_n).$$

Proof (1) and (2) are omitted.

(3) It suffices to show that $I_p(A) \cong$ dir. lim $I_p(A_n)$.
As before inclusions $\theta_n: A_n \to A$ induce homomorphisms

$$\theta_n: I_p(A_n) \to I_p(A)$$

and hence θ: dir. lim $I_p(A_n) \to I_p(A)$.

To show that θ is onto, choose any unitary u in $M_p(A)$. Since $\cup A_n$ is dense in A, we can find a unitary v in $M_p(A_n)$ for some n with $\|u-v\|$ as small as we wish. Then $[u] = [\theta_n(v)]$. Hence $\theta(\pi_n[v]) = [u]$. To show that θ is one-to-one, suppose that $\theta(a)=0$ for some a $\in$ dir. lim $I_p(A_n)$. Choose n and u such that u in $U(p, A_n)$ and $\pi_n([u])=a$. Thus $\theta_n([u])=0$, this means that the element u in $U(p, A_n)$ is connected to the identity in $U(p, A)$. Let $f: [0,1] \to U(p, A)$ be the continuous connection between 1 and u. By the compactness of the image of f, there is a $\varepsilon > 0$ and $0 < t_1 < \ldots\ldots < t_n = 1$ such that ε-open balls centered at $f(t_i)$ consists of invertibles in $M_p(A)$. We can choose elements g_i of $M_p(A_n)$ which is an element of two adjacent ε-open balls. Then g_i is invertible in $M_p(A)$ hence invertible in $M_p(A_n)$. Now connect g_i and g_{i+1} via line segments of invertibles in $M_p(A_n)$. Hence $[u]=0$ in $I_p(A_n)$. Thus θ is one-to-one.

2.7 Non-unital Cases Let A be a not necessarily unital C^*-algebra. Let A^+ be C^*-algebra obtained by adjoing the identity. Let $\pi: A^+ \to \boldsymbol{C}$ by $\pi(x, \lambda) = \lambda$. Then π is a unital *-homomorphism. The natural map $\pi: A^+ \to \boldsymbol{C}$ induces group homomorphisms.

$$K_0(\pi): K_0(A^+) \to K_0(\boldsymbol{C})$$

$$K_1(\pi): K_1(A^+) \to K_1(\boldsymbol{C}).$$

Definition Let A be a C^*-algebra (not necessarily unital). Then the K_0-group and K_1-group are defined by

$$K_0(A) = \text{Ker}(K_0(\pi))$$

$$K_1(A) = \text{Ker}(K_1(\pi)).$$

Remark It is not hard to that $K_0(\boldsymbol{C}) \cong \boldsymbol{Z}$, the integer group, $K_1(\boldsymbol{C}) = 0$. For unital

C^*-algebra A, we already have seen that $A^+ \cong A \oplus \boldsymbol{C}$ as C^*-direct sum. Thus $K_0(A^+) \cong K_0(A) \oplus K_0(\boldsymbol{C})$ and $K_1(A^+) \cong K_1(A) \oplus K_1(\boldsymbol{C})$. Thus our new definition agrees with the old one when A is unital. By adjoing the identity to A, we have the same theorem for non-unital cases as Theorems 2.3 and 2.5 and 2.6.

3. Exact Sequences of K-groups.

3.1 Let A be a C^*-alggebra and I be a closed two-sided ideal of A. Then A/I is a C^*-algebra in a natural way. Thus we have a short exact sequence in the category of C^*-algebras

$$0 \to I \overset{\mathrm{i}}{\to} A \overset{\pi}{\to} A/I \to 0.$$

Our immediate task is to prove exactness of K-groups out of this short exact sequence. Our ultimate goal of present adventure is to obtain a homology theory of C^*-algebras.

In this chapter the letter "I" will always denote a closed two-sided ideal of A. We will denote the induced morphisms by putting "$*$" in the low left side of the given $*$-homomorphisms. We need the following (this elementary proof is due to M.D. Choi. See [3]).

3.2 **Lemma** Suppose that A is a unital C^*-algebra, I is a closed two sided ideal of A, and $u \in U(A/I)$. Then $u \oplus u^*$ has a unitary lifting in $U(2, \mathrm{A})$.

Proof We have $u^*u = 1 = \pi(1)$, hence there exists an element $v \in A$ such that $v^*v \leq 1$, $\pi(v) = u$. It follows that $\|v\| = 1$ and

$$\begin{pmatrix} v & -\sqrt{1-vv^*} \\ \sqrt{1-v^*v} & v^* \end{pmatrix}$$

is a unitary in $M_2(A)$ which maps onto $u \oplus u^*$.

Remark If v is a partial isometry, then the unitaryf becomes

$$\begin{pmatrix} v & -\mathrm{cok}\ v \\ \ker\ v & v^* \end{pmatrix}$$

3.3 **Theorm** The short exact sequence induces an exact sequence of K^0-groups:

$$K_0(I) \xrightarrow{i_*} K_0(A) \xrightarrow{\pi_*} K_0(A/I).$$

Proof Adjoin the identity and consider

$$I^+ \xrightarrow{i} A^+ \xrightarrow{\pi} (A/I)^+ \cong A^+/I.$$

Then the composition map $\pi \circ i$ is just the complex homomorphism preserving the identity. Thus $\pi_* \circ i_* : K_0(I^+) \to K_0(\boldsymbol{C}) \cong \pi$. Hence $\mathrm{Ker}(\pi_* \circ i_*) = K_0(I)$. Therefore $I_m(i_*) \subseteq \mathrm{Ker}(\pi_*)$.

Now suppose that $\pi_*([c])=0$ for $c\in K_0(A)\subseteq K_0(A^+)$. We represent $c=[p]-[q]$ for projections p, q in $M_n(A^+)$ for some n. Then by increasing the size of n and by adding an appropriate projection if necessary, we may assume that $c=[p]-[I_k]$. Since $\pi_*([p]-[I_k])=[\pi(p)]-[\pi(I_k)]=0$, $\pi(p)$ and $\pi(I_k)$ are unitary equivalent in some $M_m(A^+/I)$. Let $u\in U(m, A^+/I)$ be a unitary such that $u^*\pi(p)u=\pi(I_k)$. Find $v\in(2m, A^+)$ such that $\pi(v)=u\oplus u^*$. Then

$$\pi(I_k)\oplus 0=(u^*\oplus u)(\pi(p)\oplus 0)(u\oplus u^*)$$

$$[p]-[I_k]=[v^*pv]-[I_k]$$

$$\pi(v^*pv)=(u^*\oplus u)(\pi(p)\oplus 0)(u\oplus u^*)=\pi(I_k)\oplus 0.$$

Hence $v^*pv\in M_\infty(I^+)$, and since $\pi_*\circ i_*([v^*pv]-[I_k])=0$ we see that $[v^*pv]-[I_k]\in K_0(I)$ and thus $c\in I_m(i_*)$.

3.4 **Theorem** The induced sequence of K_1-groups

$$K_1(I)\xrightarrow{i_*}K_1(A)\xrightarrow{\pi_*}K_1(A/I)$$

is exact.

Proof The composition map in the proof of the previous theorem transforms I^+ to the complex numbers. Hence $\pi_*\circ i_*=0$. Thus $I_m(i_*)$ is contained in $\mathrm{Ker}(\pi^*)$. For the reverse containment, suppose a $\in K_1(A^+)$ with $\pi_*(a)=0$. By the Lemma of 2.5 we may choose u in $U(n, A^+/I)$ with $[u]=a$. Then we may assume that $\pi(u)\in U^\circ(m, A^+/I)$ for some m (see the proof of Theorem of 2.6). Hence we can find $c_1\ldots\ldots c_k$ of $M_m(A^+/I)$ such that $\pi(u)=\exp(c_1)\ldots\ldots\exp(c_k)$. Let d_i be a pre-image of c_i in $M_m(A^+)$, i.e., $\pi(d_i)=c_i$. Then $\pi(\exp(d_i))=\exp(c_i)$. Let

$$d=\exp(-d_1)\ldots\ldots\exp(-d_k)u.$$

Then $\pi(d)=1$, hence $d\in I^+$ and u and d belong to the same connected components of $GL(m, A^+)$. Therefore $i_*([d])=[u]=a$.

3.5. **The construction of boundary map** δ. We will construct the connecting map δ: $K_1(A/I)\to K_0(I)$. For certain unitary element in A/I, this map is nothing more than the index map of Fredholm operators. We may assume that A is unital to begin with. Let $a\in K_1(A/I)$. Then we can find $u\in U(n, A/I)$ such that the trivial extension of u determines the element a (2.5). Then by Lemma 3.3, $u\oplus u^*$ has a unitary lifting w in $U(2n, A)$. We set

$$q=I_n\oplus 0,\quad p=w^*qw.$$

Then

$$\pi(p)=\pi(w^*)\pi(q)\pi(w)=\begin{pmatrix}u^* & 0\\ 0 & u\end{pmatrix}\begin{pmatrix}I_n & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}u & 0\\ 0 & u^*\end{pmatrix}=I_n\oplus 0.$$

Hence p is a projection in $M_\infty(I^+)$. Moreover since $\pi_*([p]-[q])=0$, we see that $[p]-[q]\in K_0(I)$. It is not hard to see that the correspondence

$$\delta: a\longrightarrow [p]-[q]$$

is a well-defined map under the search.

3.6 Remarks. (1) Notice that $[p]=[q]$ in $K_0(A)$.

(2) If u has a unitary lifting v in $U(n,A)$, then $p=v^*I_nv=I_n=q$ hence $\delta([u])=0$.

(3) If u has a partial isometry lifting v, i.e., v^*v, vv^* projections in $M_\infty(I)$ and $\pi(v^*v)=\pi(vv^*)=1$, then the unitary in Lemma 3.3 becomes

$$\begin{pmatrix}v & -\text{coker } v\\ \ker v & v^*\end{pmatrix}.$$

Thus

$$p=\begin{pmatrix}v^* & \ker v\\ -\text{coker} & v\end{pmatrix}\begin{pmatrix}I_n & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}v & -\text{coker } v\\ \ker v & v^*\end{pmatrix}=\begin{pmatrix}v^*v & 0\\ 0 & \text{coker } v\end{pmatrix}.$$

Hence $[p]-[q]=\left[\begin{pmatrix}vv^* & 0\\ 0 & \text{coker } v\end{pmatrix}\right]-\left[\begin{pmatrix}I_n & 0\\ 0 & 0\end{pmatrix}\right]=[\text{coker} v]-[\ker v]$.

Indeed $[p]-[q]=[\text{coker } v]-[\ker]$ is the Fredholm index of the Fredholm element in $U(n,A/I)$.

3.7 Theorem If I is a closed two-sided ideal of A, then we have an exact sequence

$$K_1(I)\xrightarrow{i_*}K_1(A)\xrightarrow{\pi_*}K_1(A/I)\xrightarrow{\delta}K_0(I)\xrightarrow{i_*}K_0(A)\xrightarrow{\pi_*}K_0(A/I).$$

Proof We have already seen exactness at $K_1(A)$ and $K_0(A)$. We noticed that $i_*\circ\delta=0$ (Remark 3.6). Thus $I_m(\delta)\subseteq\text{Ker}(i_*)$. On the other hand, if $[p]-[I_k]\in K_0(I)$ (any element in K_0-group is of this form) is an element of $\text{Ker}(i_*)$, then we can find a unitary u in $U(n,A)$ such that $p=u^*(I_k\oplus 0)u$. Moreover since $[p]-[I_k]\in K_0(I)$, $\pi(p)$ is a scalar matrix of rank k which may as well assume is $I_k\oplus 0$.

Then

$$I_k\oplus 0=\pi(u^*)(I_k\oplus 0)\ \pi(u).$$

Thus $\pi(u)$ commutes with $I_k\oplus 0$, and hence $\pi(u)$ is of the following form;

$$\pi(u)=a\oplus b,\ \ a\in U(k,A/I),\ \ b\in U(n-k,\ A/I).$$

Then by the construction of δ, $\delta([a])=[p]-[I_k]$. Thus ker $(i_*)\subseteq I_m(\delta)$.

We now prove exactness at $K_1(A/I)$. Take any $a\in K_1(A)$. Represent $a=[u]$, some unitary u in $U(n,A)$ for some n. Then $\pi(u)$ is a unitary in $M_n(A/I)$ which has a unitary lifting, namely u. Thus by Remark 3.6(2), $\delta(\pi_*[u])=0$. Hence $I_m(\pi_*)\subseteq\text{Ker}\delta$. Conversely if $\delta(x)=[p]-[q]=0$, $p=u^*qu$, $u\in(n,A)$ (zero in $K_0(I)$). Then by extending p and q trivially and adding appropriate projections if necessary we can find a unitary v in $M_m(I^+)$ (since $[p]=[q]$ in $K_0(I)$) such that $p=v^*qv$, $q=I_k\oplus 0$. Hence

$$q=vpv^*=vu^*quv^*,\quad uv^*q=quv^*.$$

Thus uv^* commutes with $I_k\oplus 0$, and therefore uv^* is of the form

$$uv^*=g\oplus h,\quad g\in U(k,A), h\in U(m-k,A).$$

And then since $\pi(v)=1$, we have

$$\pi(g)\oplus\pi(h)=\pi(uv^*)=\pi(u)=x\oplus x^*.$$

Hence $\pi(g)=x$, and therefore $[x]\in I_m(\pi^*)$, i.e., Ker $\delta\subset I_m(\pi^*)$.
This completes the proof.

3.8. **Suspension** Let A be a C^*-algebra. Let

$$CA=\{f:[0,1]\to A \text{ continuous}; f(0)=0\}$$
$$SA=\{f\in CA : f(1)=0\}.$$

Then obviously SA is a closed two sieed ideal of C^*-algebra CA. Since CA is contractible, i.e., the identity map of CA onto CA is homotopic to the zero map meaning that there is a homotopy $h:[0,1]\to CA$ such that

$$h(t): CA\to CA,\ \text{*-homomorphism}$$
$$h(0)=0$$
$$h(1)=id.$$

hence $K_0(CA)=K_1(CA)=0$. Now apply Theorem 3.7 to the following short exact sequence

$$0\to SA\to CA\to A\to 0$$

we get an exact sequence

$$K_1(SA)\to 0\to K_1(A)\xrightarrow{\delta}K_0(SA)\to 0\to K_0(A).$$

Therefore δ is an isomorphism. Thus we proved the following.

Theorem There is a natural isomorphism $\delta: K_1(A)\to K_0(SA)$.

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〈국문요약〉

C^*-代數와 K-理論

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C^*-代數에 관한 K-理論을 논술하였다. 이 것은 Taylor의 가환바나하대수에서의 K-理論을 확장한 것으로 많은 새로운 증명을 하고, 새로운 관점을 제시하였다. 특히, K_1-群이 C^*-代數 A의 n次행렬대수 $M_n(A)$의 추상지표군들(Abstract index group)의 직접극한군(direct limit group)과 群동형인 것을 증명하였다. 또한 K_1-群에서 K_0-群으로의 경계함수(Boundary map)를 Fredholm 指標를 이용하여 매우 명확하고도 자연스럽게 정의하고 논술하였다.