Supplementary material of
Back-and-forth Operation of State Observers and
Norm Estimation of Estimation Error

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Abstract—This is a supplementary material for the paper “Back-and-forth Operation of State Observers and Norm Estimation of Estimation Error” that will be presented at the 51st IEEE Conference on Decision and Control, December, 2012.

Consider the linear system

\[ \begin{align*}
    \dot{x} &= Ax + Bu \\
    y &= Cx
\end{align*} \]

(1)

with \( x(0) = x_0 \).

Proposition 1: If \((A, C)\) is an observable pair, then, for any \( d > 0 \) and \( \alpha \) such that \( 0 < \alpha < 1 \), there exist gain matrices \( L_f \) and \( L_b \) such that

\[ \| \exp((A - L_f C) t) \| \leq \alpha, \quad \forall t \in [d/2, d], \] (2)

and

\[ \| \exp(-(A - L_b C) t) \| \leq \alpha, \quad \forall t \in [d/2, d]. \] (3)

Then, the forward observer is given by

\[ \frac{d}{dt} \hat{x}_f = A\hat{x}_f + Bu(t) + L_f(y(t) - C\hat{x}_f) \] (4)

while the backward observer is

\[ \frac{d}{ds} \hat{x}_b = -A\hat{x}_b - Bu(d-s) - L_b(y(d-s) - C\hat{x}_b). \] (5)

In fact, the backward observer is based on the backward-time description of the system (1) written as

\[ \frac{d}{ds} \hat{x} = -A\hat{x} - Bu(d-s), \quad y(d-s) = C\hat{x}, \quad \hat{x}(0) = x(d), \]

with \( \hat{x}(s) = x(d-s) \) for \( s = d - t \in [0, d] \).

I. REAL-TIME APPLICATIONS

A. State Estimation of a Switched System

Consider a switched system given in Fig. 1, which has two modes of operation described by

\[ \Sigma_1 : \begin{cases} 
    \dot{x}_1 = A_1 x_1 + B_1 u, & y = C_1 x_1, \\
    \dot{x}_2 = A_2 x_2 + A_{21} x_1,
\end{cases} \] (6)

\[ \Sigma_2 : \begin{cases} 
    \dot{x}_1 = A_1 x_1 + A_{12} x_2 \\
    \dot{x}_2 = A_2 x_2 + B_2 u, & y = C_2 x_2,
\end{cases} \] (7)

for mode 1, and

for mode 2, where the pair \((A_i, C_i), i = 1, 2,\) is observable. Now suppose that the system configuration switches between modes 1 and 2 (i.e., between (6) and (7)) after every \( T \) seconds, and we want to estimate the states \( x_1 \) and \( x_2 \) completely. Note that, at each mode, the system is not completely observable. For example, at mode 1, the state \( x_2 \) is unobservable.

One can design a separate observer for each mode of operation as follows:

\[ \dot{\hat{x}}_1 = A_1 \hat{x}_1 + B_1 u - L_1 C_1 \hat{x}_1 + L_1 y \] (8a)

\[ \dot{\hat{x}}_2 = A_2 \hat{x}_2 + A_{21} \hat{x}_1 \] (8b)

for \( \Sigma_1 \), and

\[ \dot{\hat{x}}_1 = A_1 \hat{x}_1 + A_{12} \hat{x}_2 \] (9a)

\[ \dot{\hat{x}}_2 = A_2 \hat{x}_2 + B_2 u - L_2 C_2 \hat{x}_2 + L_2 y \] (9b)

for \( \Sigma_2 \), where \( L_1 \) and \( L_2 \) are large enough so that some meaningful estimates \( \hat{x}_1 \) and \( \hat{x}_2 \) are obtained over the interval of length \( T \). In fact, at the end of the first period \([0, T]\), one can obtain the estimate \( \hat{x}_1(T) \) by (8a) for mode 1. For the second interval \([T, 2T]\), this estimate serves as the initial condition of (9a), and the observer (9b) starts to estimate \( x_2(t) \). However, the observer (9b) will exhibit some transients during the initial period of the interval \([T, 2T]\), which may corrupt the estimate \( \hat{x}_1(t) \) being obtained through the observer (9a) because of initially large error between \( \hat{x}_2(t) \) and \( x_2(t) \).

To overcome this problem, the following hybrid-type ob-
server may be utilized instead of (8) and (9)

$$\dot{\xi}_1 : \begin{cases} \dot{x}_1 = A_1 \dot{x}_1 + B_1 u \\ \dot{x}_2 = A_2 \dot{x}_2 + A_{21} \dot{x}_1 \end{cases} \quad (10)$$

$$\dot{\xi}_2 : \begin{cases} \dot{x}_1 = A_1 \dot{x}_1 + A_{12} \dot{x}_2 \\ \dot{x}_2 = A_2 \dot{x}_2 + B_2 u \end{cases} \quad (11)$$

$$\begin{pmatrix} \dot{\xi}_1(kT^-) \\ \dot{\xi}_2(kT^-) \end{pmatrix} = \begin{pmatrix} \dot{\xi}_1(kT) \\ \dot{\xi}_2(kT) \end{pmatrix}, \quad k \geq 1, \quad (12)$$

where the variables $\dot{\xi}_1$ and $\dot{\xi}_2$ will be obtained shortly using the back-and-forth operation such that the inequality,

$$|\dot{x}(k+2T) - x((k+2)T)| \leq \gamma |\dot{x}(kT) - x(kT)|, \quad (13)$$

holds for all $k \geq 1$, $x := (x_1^T, x_2^T)^T$, and a desired parameter $\gamma < 1$. The inequality (13) guarantees the convergence of estimation error to zero due to the fact that $\sup_{t \in [kT, (k+1)T]} |\dot{x}(t) - x(t)| \leq M |\dot{x}(kT) - x(kT)|$ where $M$ is a constant. The latter inequality holds because the dynamics for $\dot{x} - x$ are linear and their growth is bounded over a finite interval.

In order to obtain $\dot{\xi}_1$ and $\dot{\xi}_2$, we prepare the back-and-forth observer (4) and (5) for the $x_1$-subsystem of (6) and for the $x_2$-subsystem of (7), respectively. For each subsystem, the injection gains $L_f$ and $L_b$ are designed such that (2) and (3) hold with $d = T/2$ and

$$\alpha = \frac{\gamma}{\sqrt{2\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}}}$$

where $\alpha_1 = M_1 + L_1 L_2 + L_2$, $\alpha_2 = L_1 M_2 + M_2$, $\alpha_3 = M_2 + L_1 L_2 + L_1$, $\alpha_4 = L_2 M_1 + M_1$ and

$$M_i := \|e^{A_i T}\|, L_i := \left\| \int_0^T e^{A_i s} dA_{ij} \right\|, \quad i, j = 1, 2, i \neq j,$$

and let $R+1$ be the number of round-trips of numerical back-and-forth integrations that are possible within the interval of length $T/2$. Clearly, the number $R$ relies on the computation power.

Let the initial condition $\dot{\xi}_1(0^-)$ and $\dot{\xi}_2(0^-)$ be arbitrary. Fig. 2 illustrates the strategy to obtain $\dot{\xi}_1(kT^-)$ and $\dot{\xi}_2(kT^-)$, for $k = 2$, over the interval $[T, 2T]$ when mode 2 is active. At time $t = T$, the estimate $\dot{x}_1(T)$ and $\dot{x}_2(T)$ are set to $\dot{\xi}_1(T^-)$ and $\dot{\xi}_2(T^-)$ respectively, and they are integrated in the real time by (11). At the same time, the initial condition of the forward observer (for estimating $x_2$) is set by $\dot{\xi}_2(T^-)$, and this forward observer runs in the real time first until $T + T/2$. At $T + T/2$, the backward observer is employed with the terminal state of the forward observer as its initial condition. The round-trip of back-and-forth operation continues $R$ times with the input-output data of the interval $[T, T + T/2]$, after which the forward observer is finally integrated from $T$ to $2T$. Since the time elapsed by the back-and-forth operation and the last forward operation does not exceed $T/2$, the last forward integration will ‘catch up’ with the real time, as indicated in Fig. 2. While these operations are performed, the information about $\dot{\xi}_2(t)$ is collected as illustrated in the figure. At the same time with the start of the last forward observer, we begin the integration of

$$\dot{\xi}_1 = A_1 \dot{\xi}_1 + A_{12} \dot{\xi}_2 \quad (14)$$

with the initial condition $\dot{\xi}_1(T) = \dot{\xi}_1(T^-)$ and with the signal $\dot{\xi}_2(t)$ obtained by back-and-forth operation over the interval $[T, 2T]$. By this procedure, we obtain $\dot{\xi}_1(2T^-)$ and $\dot{\xi}_2(2T^-)$. This procedure repeats in the next interval, with the role for $x_1$ and $x_2$ being switched, and instead of (14), the following equation is used to compute $\dot{\xi}_2$:

$$\dot{\xi}_2 = A_2 \dot{\xi}_2 + A_{21} \dot{\xi}_1. \quad (15)$$

We now proceed with the error analysis. Let $\epsilon := \dot{x} - x$. Then, $\epsilon(kT) = \dot{x}(kT) - x(kT) = \dot{\xi}_2(kT^-) - x(kT)$. We note that, for the $(2k + 1)$-th interval with $k$ being nonnegative integer, $\dot{\epsilon}_1$ is the estimate from the back-and-forth observer while $\dot{\epsilon}_2$ is the state of (15), and for the $2k$-th interval, their roles are reversed. Therefore, in the first interval $[0, T)$, we have that

$$|\epsilon_1(T)| \leq \alpha^R |\epsilon_1(0)|$$

$$|\epsilon_2(T)| \leq \|e^{A_{2s} T}\| |\epsilon_2(0)| + \left\| \int_0^T e^{A_{2s} s} dA_{21} \right\| \|e^{A_{1s} T}\| |\epsilon_1(0)|.$$
and

\[ |e_2(2T)| \leq \alpha^R |e_2(T)| \]
\[ \leq \alpha^R \|e^{A_2T}\| |e_2(0)| + \alpha^{2R} \left\| \int_0^T e^{A_2s} ds A_{21} \right\| |e_1(0)|. \]

The terms within the brackets, \([\cdot]\), are due to the back-and-forth observer, which yields a rich estimation on the entire interval of length \(T\), including the initial period. Finally, it is seen that

\[ |e(2T)| \leq |e_1(2T)| + |e_2(2T)| \]
\[ \leq \alpha^R (M_1 + L_1 L_2 + L_2)|e_1(0)| + \alpha^R (L_1 M_2 + M_2) \times |e_2(0)| \]
\[ \leq \gamma \sqrt{2} (|e_1(0)| + |e_2(0)|) \leq \gamma |e(0)|. \]

This proves the claim (13) for even \(k\). For odd \(k\), the proof is similar and thus omitted.

An underlying reasoning is that the use of the back-and-forth observer has improved the transient response of the state estimation error, thus leading to quality estimates of the state variable over the entire interval.