Continuous Time Optimization, Reservation Wage Path and External Shocks

Bong-Joon Yoon
State University of New York at Binghamton

This paper studies how the nonstationarity in the reservation wage arises within a continuous time optimization framework. Assuming a Poisson distributed offer arrivals and a simple job search model with a finite search horizon, the solution of the optimization as summarized by a differential equation is examined. It is shown that the unique solution, i.e., the unique reservation wage path, exists and exhibits dynamic stability. As expected, the path presents decreasing reservation wage over time for the period of search, i.e., participation, in the labor market, under absence of external shocks.

With introduction of external shocks which are unanticipated, it is shown that the reservation wage paths exhibit disjoint jumps at the times of the shocks but the paths are decreasing over time at the continuity points. With anticipated shocks, however, the reservation wage paths can have regions of increasing reservation wage. The analysis is extended to examine the reservation wage path along a business cycle. The case of the infinite search horizon is also examined.

I. Introduction

This paper studies how the nonstationarity in the reservation wage arises within the framework of a continuous time optimization for a job searcher. In Section II a job search model is introduced in which the search horizon is finite and arrivals of job offers are Poisson distributed. Assuming the time-invariance of both the wage offer distribution and Poisson intensity, the ensuing continuous time optimization is summarized by a differential equation as in Heckman and Singer (1982) and Mortensen (1984).

No explicit assumptions regarding the wage offer distribution are made. Hence, the analytical solution of the differential equation, i.e., the specific form of the reservation wage path, can not be obtained.

[Seoul Journal of Economics 1988, Vol. 1, No. 4]

However, it can be shown that the solution exists and is unique. Examination of the solution, i.e., the reservation wage path, using a phase diagram provides the following findings. Without external shocks, the reservation wage is declining over time for the period of search in the labor market. Moreover, the path shows dynamic stability, converging to the stationary reservation wage solution in reverse time. The analysis is extended to the infinite search horizon case which presents a constant reservation wage over time.

Section III introduces into the model external shocks in the form of shifts in the instantaneous rate of offer arrivals given by the Poisson intensity parameter. It is found that, with unanticipated external shocks, the reservation wage paths show disjoint jumps at the times of the shocks. But the reservation wage decreases at the continuity points. With anticipated external shocks, the paths can contain regions of increasing reservation wage. Also, the episodes of the out of the labor force state are presented under both unanticipated and anticipated shocks. Further, the analysis is extended to examine the reservation wage path and the episodes of the out of the labor force state along a business cycles. Finally, the conclusions are given in Section IV.

II. The Model

It is assumed that job offer arrivals for the searcher are subject to a Poisson process with the time-homogeneous intensity of offer arrivals parameter λ. Recall of job offers is assumed not permissible. The cost of searching for acceptable job offers is incurred at a constant instantaneous rate of c (amount) per unit time. A search model based on the infinitesimal look-ahead stopping (*ILAS*) rule results in the following continuous time optimization scheme for the determination of the reservation wage (see Heckman and Singer 1982; Mortensen 1984; Ross 1970). Denote by $V(t)$ the return from optimal search at t. If the searcher waits an additional infinitesimal time h, his expected return at $t+h$ is equal to i) Pr (no offer in $[t+h)$) times $V(t+h)$ plus ii) Pr (offer in $[t,t+h)$) times the return from optimal search policy given an offer in $[t,t+h)$ minus iii) search cost over the period $[t,t+h)$. To be optimal according to the *ILAS* rule $V(t)$ needs be equal to the present value at t of the above expected return at $t+h$. That is,

$$V(t) = [1 - \lambda h]\, mV(t+h) + \lambda hm\, \mathrm{E}\{\max\, [W(t+h),\ V(t+h)]\}$$
$$- c\int_0^h \exp(-rt')\ dt' + o(h), \tag{1}$$

where $W(t)$ denotes the present value of (the stream of wages from) the job offer arriving at t and $o(h)$ is such that $\lim_{h\to 0} o(h)/h = 0$. m is the discounting factor defined by exp $(-rh)$ with r being the rate of interest.

The search horizon is assumed finite of length T. Its interpretation includes the finite working life (Gronau 1971) and liquidity constraint (Mortensen 1984). The job offer $W(t)$ is described by the wage w measured in the instantaneous rate. w has a time-invariant distribution with finite $\mathrm{E}\{|w|\}$. The job lasts forever once started so that $W(t) = W = w/r$.

Define by s the reverse time index which measures time left till the end of the search horizon, i.e., $s = T - t$. Then (1) is rewritten as

$$V(s) = [1 - \lambda h]\, mV(s-h) + \lambda hm\, \mathrm{E}\{\max\, [W, V(s-h)]\} \tag{2}$$
$$- c\int_0^h \exp(-rt')\ dt' + o(h).$$

Rearranging terms and passing h to the limit, i.e., $h\to 0$, one obtains from (2) the following differential equation for $V(s)$:[1]

$$dV(s)/ds = -c - rV(s) + \lambda\, \mathrm{E}\{\max[W - V(s),\ 0]\}. \tag{3-a}$$

Denote by $w^*(t)$ the reservation wage at t so that $V(t) = w^*(t)/r$.[2]

$$dw^*(s)/ds = -rc - rw^*(s) + \lambda\, \mathrm{E}\{\max[w - w^*(s),\ 0]\}. \tag{3-b}$$

At the end of the search horizon, T, the searcher's only option is to drop out of the labor force. Denote the return from choosing the option by $V(t=T)$. In additon, denote $V(t=T)/r$ by $w^*(t=T)$. $V(t=T)$ and $w^*(t=T)$ provide the initial conditions, i.e., the values of $V(s=0)$ and $w^*(s=0)$, for the differential equations (3-a) and (3-b) stated in backward time s, respectively. They provide the boundary conditions if the equations are restated in forward time t.

[1]Use is made of the following relations. $\lim_{h\to 0} (c/h) \int_0^h \exp(-rt')\ dt' = c$, and $\lim_{h\to 0} \{V(s) - mV(s-h)\}/h = dV(s)ds + rV(s)$.

[2]For the reservation wage property of the optimal stopping strategy, see DeGroot (1970) and Ross (1970).

Without explicit assumptions regarding the distribution of w, one can not determine the analytical solution of the differential equation (3-a) or (3-b). Even with these assumptions it is difficult to derive the analytical solution explicitly (see Heckman and Singer 1982). Nevertheless, one can examine the existence and other important properties of the solution as in the propositions below.

Proposition 1.
The solution of the differential equation (3-b) subject to the initial condition given by $w^*(s=0)$ exists and is unique.

Proof: Denote the right hand side (*RHS*) of (3-b) by $R(w^*(s))$. $R(w^*(s))$ is continuous in $w^*(s)$, because $\mathrm{E}\{\max[w-w^*(s),\ 0]\}$ is continuous in $w^*(s)$ (see DeGroot 1970). Observe that $\max[w,w_1^*(s)] - \max[w,w_2^*(s)] \leq w_1^*(s) - w_2^*(s)$ for any pair $w_1^*(s)$ and $w_2^*(s)$ such that $w_1^*(s) \geq w_2^*(s)$. Thus,

$$\begin{aligned}|R(w_1^*(s)) - R(w_2^*(s))| \leq & \ | -r[w_1^*(s) - w_2^*(s)] \\ & + \lambda\,\mathrm{E}\{\max[w,w_1^*(s)] - \max[w,w_2^*(s)]\}| \\ \leq & (r+\lambda)\ |w_1^*(s) - w_2^*(s)|\ ,\end{aligned}$$

where $\mathrm{E}\{|w|\}$ and $(r+\lambda)$ are finite positive constants by assumption. Hence, the Lipschitz condition is satisfied so that the solution of (3-b) exists and is unique.

Proposition 1 holds when the differential equation (3-b) is replaced by (3-a) with the initial conditon given by $V(s=0)$. The proof follows immediately from the above proof for (3-b).

Now define by w^{**} the stationary solution of the differential equation (3-b) so that

$$dw^*(s)/ds = 0 = -rc - rw^{**} + \lambda\,\mathrm{E}\{\max[w-w^{**},\ 0]\}\ , \tag{4-a}$$

or equivalently

$$rc = -rw^{**} + \lambda\,\mathrm{E}\{\max[w-w^{**},\ 0]\}\ . \tag{4-b}$$

Proposition 2.
w^{**} exists and is unique. Further, $dw^{**}/dc<0$, $dw^{**}/d\lambda>0$, and $dw^{**}/dr<0$.[3]

Proof: Note that $\mathrm{E}\{\max[w-w^*(s),0]\}$ is convex, nonnegative, con-

[3]For $dw^{**}/d\mu>0$ and $dw^*/d\sigma>0$ where μ and σ denote mean and dispersion of w, see Yoon (1981) and Mortensen (1984).

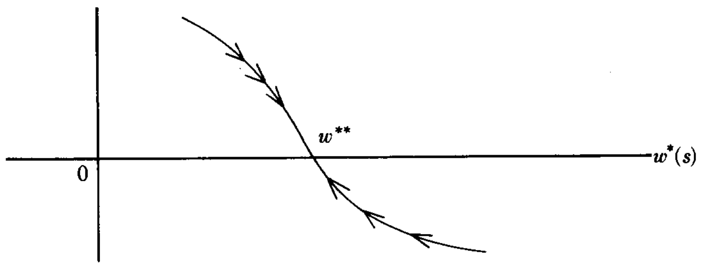


FIGURE 1
PHASE DIAGRAM

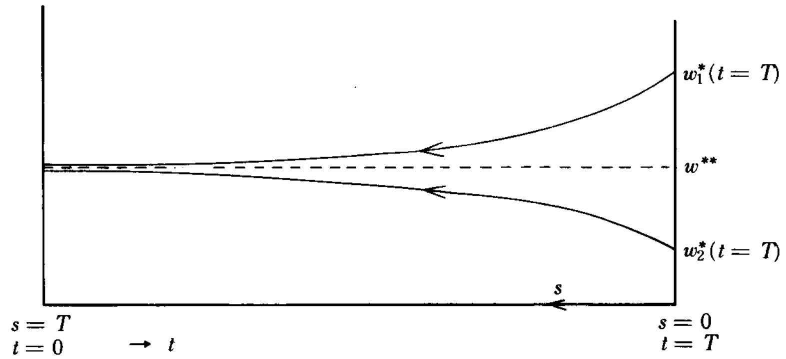


FIGURE 2
DYNAMIC STABILITY OF THE RESERVATION WAGE PATHS

tinuous and monotone decreasing in $w^*(s)$ to 0 over the support of w (see DeGroot 1970). Hence, the right hand side (RHS) of (4–b) is continuous and monotone decreasing in w^{**}. Further the RHS of (4–b) approaches $+\infty$ as $w^{**} \rightarrow -\infty$ and $-rw^{**}$ as $w^{**} \rightarrow +\infty$. Since the left hand side of (4–b) is positive, the existence and uniqueness of w^{**} follow. The signs of the w^{**} derivatives can be shown easily and omitted here.

The stability of the solution of the differential equation (3–b) (and hence of (3–a)) can be analyzed qualitatively using a phase diagram. The phase line in Figure 1 implies a stable equilibrium at w^{**}. If $w^*(s) = w^{**}$, the equilibrium prevails. If $w^*(s) > w^{**}$, then $dw^*(s)/ds < 0$ so that it declines towards w^{**}. If $w^*(s) < w^{**}$, then $dw^*(s)/ds > 0$ so that $w^*(s)$ increases towards w^{**}. In either case, $w^*(s)$ approaches w^{**} over reverse time s.

Depending on the magnitude of the return from dropping out of the labor force (OLF) given by $w^*(t=T)/r$ relative to w^{**}/r, two types of the time path of the reservation wage $w^*(t)$ emerge as shown in Figure 2. Both indicate dynamic stability of the reservation wage. However, the reservation wage is required to be larger than $w^*(t=T)$ for search (participation) in the labor market to be worthwhile. This is because, with $w^*(t) \leq w^*(t=T)$, the searcher is better off by choosing OLF. The path positioned above the equilibrium line implies $w^*(t) < w^*(t=T)$ for $0 \leq t < T$. Hence, the searcher chooses OLF at $t=0$ so that the path is not feasible during the period of search. As a result, a well-known declining reservation wage over (forward) time as established in the discrete time model (see DeGroot 1970; Gronau 1971; Lippman and McCall 1976) among participants with a finite search horizon is reaffirmed. That is, $dw^*(t)/dt < 0$, in the case of the continuous time optimization by a searcher (participant) in the labor market, unless there are external shocks (see Mortensen 1984 for an alternative proof of $dw^*(t)/dt < 0$). These results are stated as Proposition 3.

Proposition 3.
With a finite search horizon, the solution of the differential equation (3-b) is stable, converging to the equilibrium level w^{**} in backward time s. Further, $dw^*(t)/dt < 0$, i.e., the reservation wage declines over time during the period of search in the labor market.

Now the analysis is extended to the infinite search horizon.

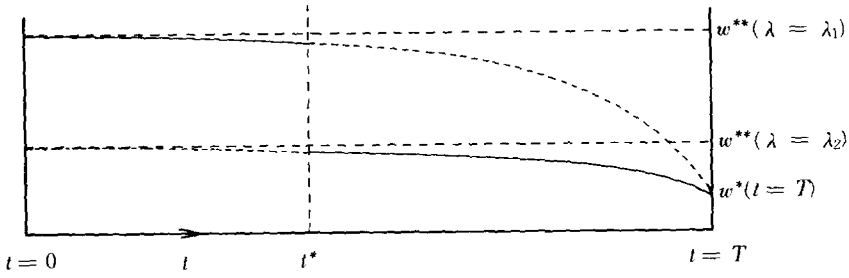
Proposition 4.
With the infinite search horizon, the reservation wage is constant at the equilibrium level w^{**} over time.

Proof: Take the limit on the length of the search horizon, T, i.e., $T \rightarrow +\infty$. Let $t \in (t_1, t_2)$ where t_1 and t_2 are both finite. Then, $T-t = s \in (T-t_2, T-t_1)$ but $T-t_2 = +\infty$ and $T-t_1 = +\infty$. Since $w(s)$ converges to w^{**} as $s \rightarrow +\infty$, $w^*(t)$ equals w^{**} for all finite t.

III. External Shocks, Unanticipated and Anticipated

In above, the reservation wage is shown to decline over time assuming no external shocks. In this section, external shocks are

a) The Case of a Drop in λ: $\lambda_1 > \lambda_2$



1. $w^{**}(\lambda = \lambda_1)$ and $w^{**}(\lambda = \lambda_2)$ denote the levels of w^{**} when $\lambda = \lambda_1$ and $\lambda = \lambda_2$ respectively. $w^{**}(\lambda = \lambda_1) > w^{**}(\lambda = \lambda_2)$ for $\lambda_1 > \lambda_2$ follows from $dw^{**}/d\lambda > 0$ in Proposition 2. Dotted paths here and below indicate the paths under no shift in λ.

b) The Case of a Rise in λ: $\lambda_1 < \lambda_2$.

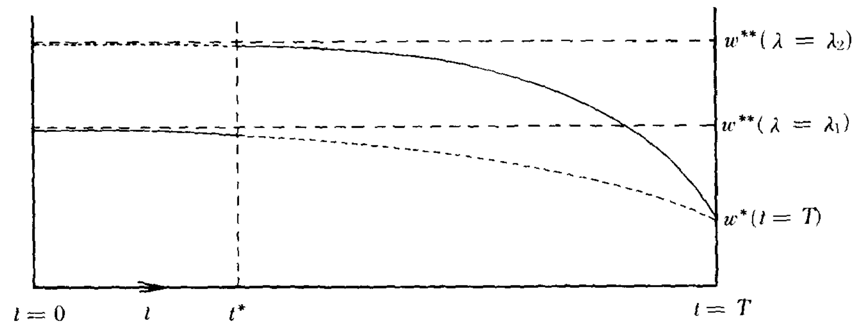


FIGURE 3

THE RESERVATION WAGE PATHS WITH SHIFTS IN THE INTENSITY OF OFFER ARRIVALS λ AT t^{*}

introduced into the model by means of shifts in the intensity of offer arrivals λ. The case of the shifting search cost can be examined analogously and hence omitted here.

If the external shocks are unanticipated then the optimizations prior to and posterior to the shock, both described according to the dynamic programming equation as in (1), are separate problems. This is revealed in Figures 3-a and 3-b showing the reservation wage paths with disjoint jumps at the time of the external shock. The shock is either an upward or a downward shift in the intensity

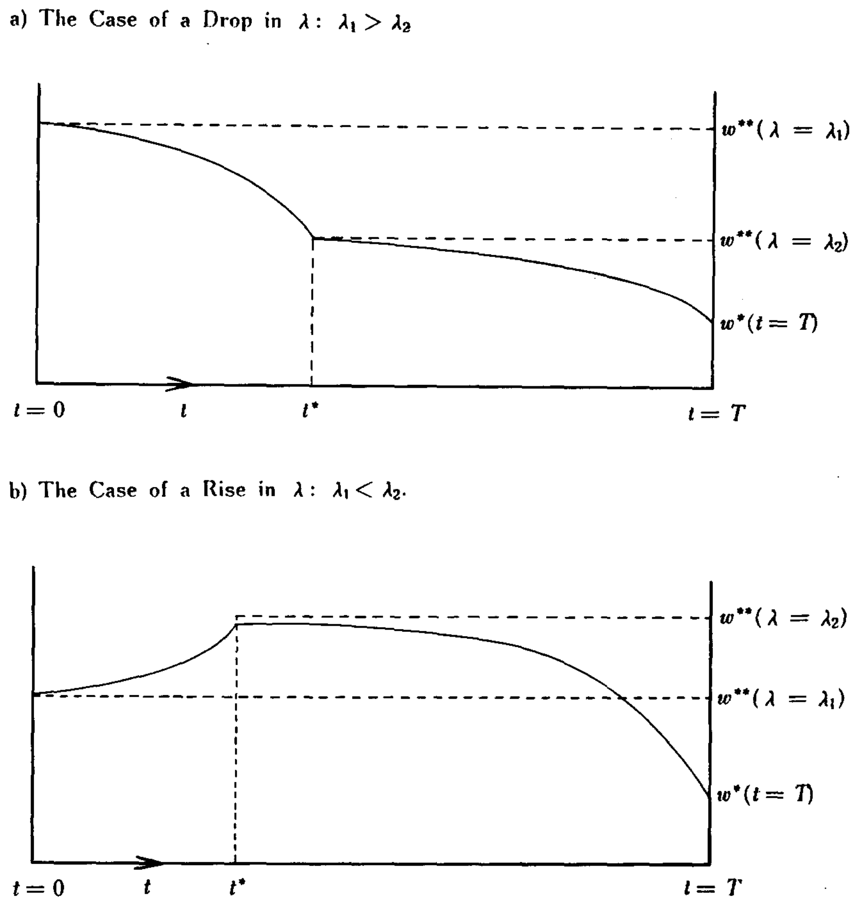
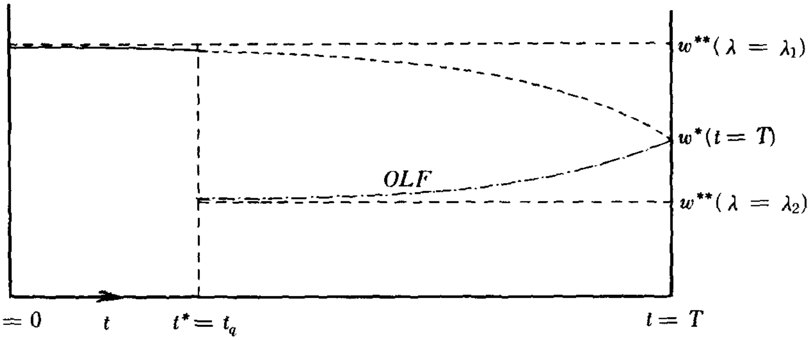


FIGURE 4

THE RESERVATION WAGE PATHS WITH ANTICIPATED SHIFTS IN THE INTENSITY OF OFFER ARRIVALS λ AT t^*.

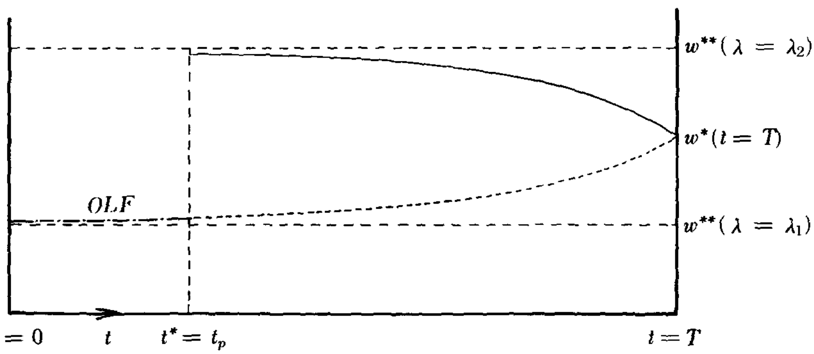
of offer arrivals λ, i.e., $\lambda_1 < \lambda_2$ or $\lambda_1 > \lambda_2$ where 1 and 2 denote the periods before and after the shock.

Often the external shocks are anticipated as with termination of unemployment insurance benefits and the impending recession. Suppose a shock occurs at t^* and is anticipated by the searcher. The post-shock path of the reservation wage, $w^*(t)$, is simply the portion over $[t^*, T)$ of the reservation wage path determined subject to i) the same boundary condition, $w^*(t = T)$, for the differential equation (3-b), but ii) the new value of a parameter or of search cost revised a-1) The Case of an Unanticipated Drop in λ : $\lambda_1 > \lambda_2$ and $w^{**}(\lambda = \lambda_2) < w^*(t = T) < w^{**}(\lambda = \lambda_1)$.[1]



1. t_q denotes the time when the searcher drops out of the labor force.

a-2) The Case of an Unanticipated Rise in λ : $\lambda_1 < \lambda_2$ and $w^{**}(\lambda = \lambda_1) < w^*(t = T) < w^{**}(\lambda = \lambda_2)$.[1]



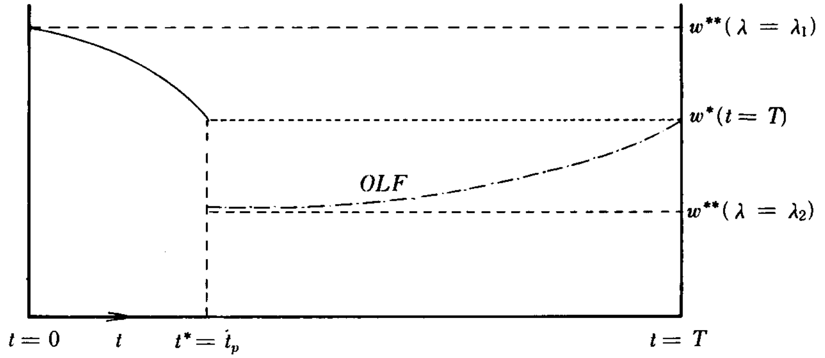
1. t_p denotes the time when the worker ends *OLF* and starts searching for a job, i.e., participates in the labor market.

FIGURE 5

THE RESERVATION WAGE PATHS AND THE LABOR FORCE PARTICIPATION WITH UNANTICIPATED AND ANTICIPATED SHIFTS IN THE INTENSITY OF OFFER ARRIVALS λ AT t^*.

according to the shock.

The optimization prior to the shock, however, is affected differently. Specifically, the boundary condition for the optimization defined over the period $[0, t^*)$ is the reservation wage at t^*, $w^*(t^*)$. But $w^*(t^*)$ is determined via the post-shock optimization, noting the b-1) The Case of an Anticipated Drop in λ : $\lambda_1 > \lambda_2$ and $w^{**}(\lambda = \lambda_2) < w^*(t = T) < w^{**}(\lambda = \lambda_1)$.



b-2) The Case of an Anticipated Rise in λ : $\lambda_1 < \lambda_2$ and $w^{**}(\lambda = \lambda_1) < w^*(t = T) < w^{**}(\lambda = \lambda_2)$.

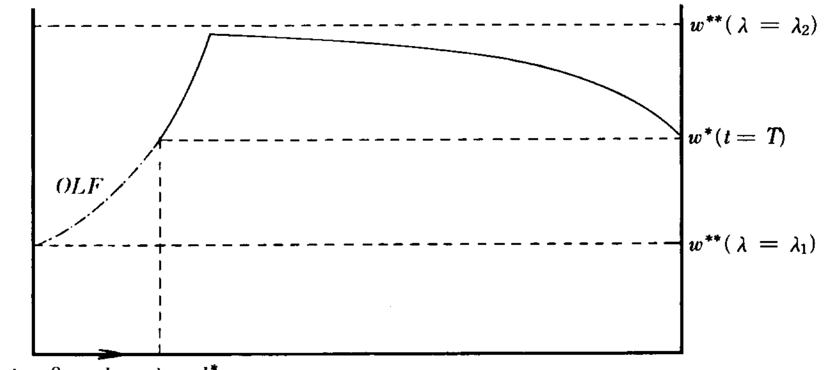
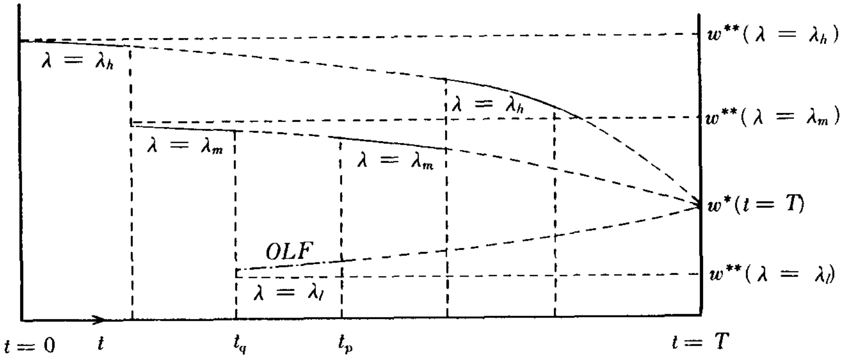


FIGURE 5 (CONTINUED)

backward nature of the Bellman equation (1). As a result, the pre-shock reservation wage path subject to the anticipation of the shock at t^* is different from the segment over $[0, t^*)$ of the original reservation wage path under no shock. Further, the reservation wage can increase over t, for example, if the intensity of offer arrivals shifts upward. Figure 4 illustrates a variety of reservation wage paths under the anticipated shocks.

The external shocks can affect the labor force status of the sear-

a) The Reservation Wage Path under Unanticipated Shocks along a Business Cycle.[1]



1. t_q and t_p denote the times of the beginning and ending of the out of the labor force state.

b) The Reservation Wage Path under Anticipated Shocks along a Business Cycle.

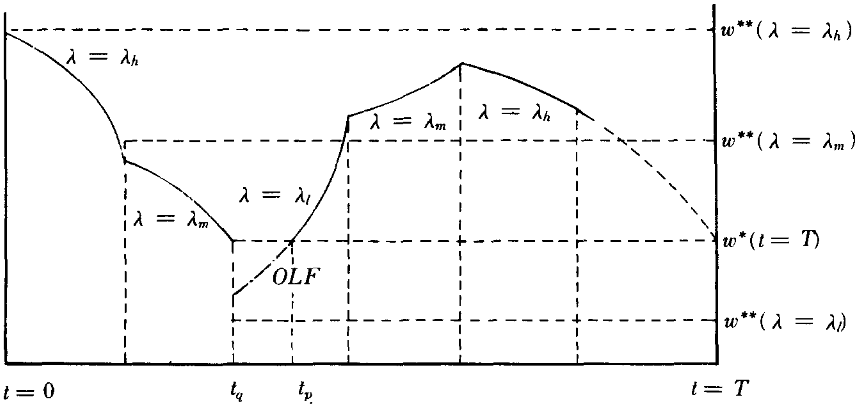
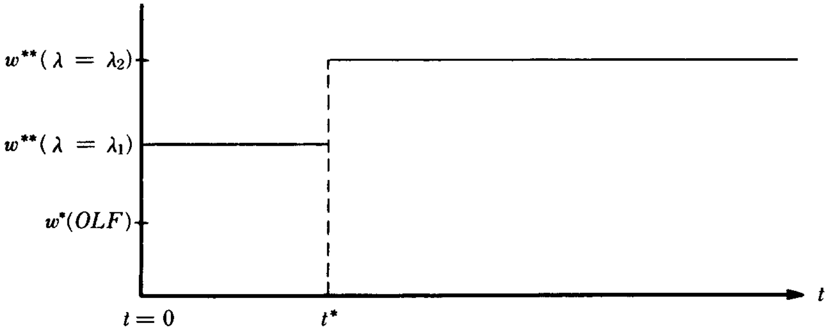


FIGURE 6

ALTERNATING INTENSITY LEVELS OF OFFER ARRIVALS ALONG A BUSINESS CYCLE, THE RESERVATION WAGE PATHS AND LABOR FORCE PARTICIPATION: $\lambda_h > \lambda_m > \lambda_l$.

cher as they alter the reservation wage path. Figure 5 illustrates movements in and out of the labor force when the shocks are both unanticipated and anticipated.

The above analysis can be combined with a business cycle along which the intensity of the offer arrivals alternates with low, medium and high levels, denoted by λ_l, λ_m and λ_h respectively. Figure 6 a-1) The Case of an Unanticipated Rise in λ: $\lambda_1 < \lambda_2$.[1]



1. $w^{*}(OLF)$ is such that $w^{*}(OLF)/r$ equals the gain from dropping out of the labor force.

a-2) The Case of an Unanticipated Drop in λ: $\lambda_1 > \lambda_2$.

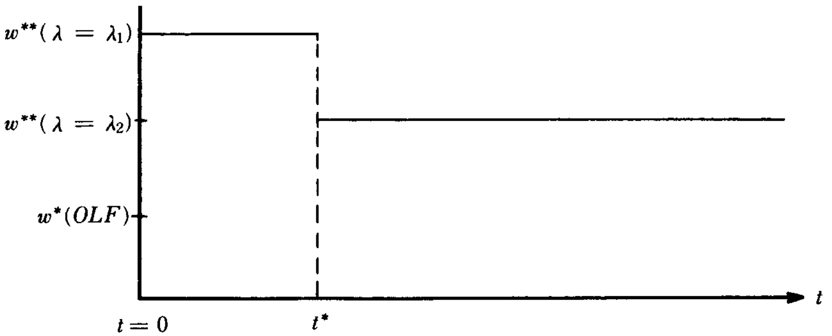
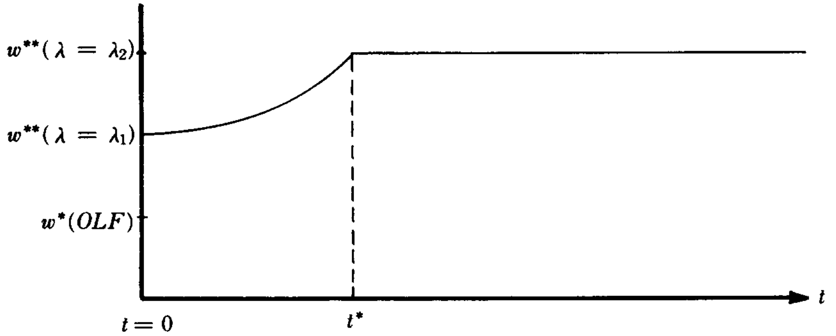


FIGURE 7

THE RESERVATION WAGE PATHS WITH INFINITE SEARCH HORIZON WHEN THE SHIFT IN THE OFFER ARRIVAL RATE λ IS a) UNANTICIPATED AND b) ANTICIPATED, RESPECTIVELY.

shows the reservation wage paths along the business cycle when the shocks are both unanticipated and anticipated. The paths illustrate how the reservation wage varies along the cycle. They also reveal episodes of *OLF*.

Finally, with the infinite search horizon, if the shock is unanticipated, the reservation wage path shows a disjoint jump at the time of the shock. But the path is horizontal within the continuity region exhibiting constant reservation wage over t. However, with anticib-1) The Case of an Anticipated Rise in λ: $\lambda_1 < \lambda_2$



b-2) The Case of an Anticipated Drop in λ: $\lambda_1 > \lambda_2$.

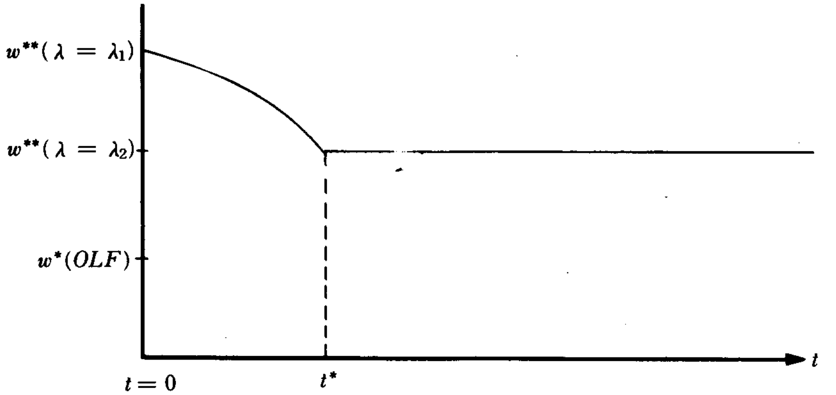


FIGURE 7 (CONTINUED)

pated shocks, the path can contain a segment of increasing reservation wage as shown in Figure 7.

IV. Conclusion

This paper analyzes how the nonstationarity in the reservation wage arises within a continuous time optimization framework. Assuming a Poisson distributed offer arrivals and a simple job search model with a finite search horizon, the solution of the optimization as summarized by a differential equation is examined. It is shown that the solution, i.e., the reservation wage path, exists and is unique. Also, the path preserves dynamic stability. As expected, the path presents decreasing reservation wage over time during the period of participation under absence of external shocks.

With unanticipated shocks, the reservation wage path exhibits disjoint jumps at the times of the shocks but the path is decreasing over time at the continuity points. With anticipated shocks, however, the path can contain regions of increasing reservation wage. The analysis is extended to examine the reservation wage path along a business cyle. The case of the infinite search horizon is also considered.

References

DeGroot, M. *Optimal Statistical Decisions.* New York: McGraw-Hill, 1970.

Gronau, R. "Information and Frictional Unemployment." *American Economic Review* 61 (1971) : 290-301.

Heckman, J., and Singer, B. "The Identification Problem in Econometric Models for Duration Data." In W. Hildenbrand (ed.), *Advances in Econometrics: Proceedings of World Meetings of the Econometric Society.* Cambridge: Cambridge University Press, 1982.

________. "Econometric Duration Analysis." *Journal of Econometrics* 24 (1984) : 63-132.

Lippman, S., and McCall J. "The Economics of Job Search: A Survey." *Economic Inquiry* 14 (1976) : 113-26.

Mortensen, D. "Job Search and Labor Market Analysis." Discussion Paper No. 594. The Center for Mathematical Studies in Economics and Management Science, Northwestern University, 1984.

Ross, S. *Applied Probability Models with Optimization Applications.* San Francisco: Holden-Day, 1970.

Yoon, B. J. "A Model of Unemployment Duration with Variable Search Intensity." *Review of Economics and Statistics* 63 (1981) : 599-609.