An Equilibrium Index Theory in an Economy with Convex Production Technologies

Seung-Jun Park and Kwan-Koo Yun

The State University of New York at Albany

I. Introduction

Global analysis of economic equilibria for the purpose of counting the number of equilibria has been one of the important tasks of general equilibrium theories. Most of the analysis has relied on the mathematical tools borrowed from differential topology. In applying the tools economists have had to admit strong assumptions, two of which are differentiability and single-valuedness of excess demand fuctions (Dierker 1972; Nishimura 1978; Varian 1975; Yun 1981). If we look at the production sets even with the convexity assumption, we find it natural that there are kinks and flat pieces. Supply fuctions associated with these production sets are non-differentiable and multivalued. Kehoe(1983) and Mas-Colell (1985) concentrated on the problem of multivaluedness. They solved the problem by introducing a single-valued function whose fixed points are equivalent to the equilibria. Here, we solve both of the problems, non-differentiability and multivaluedness, by constructing a Lipschitz continuous function from which we develop an equilibrium index theory. We use generalized Jacobians for Lipschitz continuous functions defined by Clarke(1983) to compute the equilibrium indices.

The content of the paper is as follows. In Section II, we introduce the model and construct a Lipschitz homeomorphism between an open subset of the Euclidean space and the graph of the excess demand correspondence. In Section III, we use the homeomorphism to define regular economies and prove that there exist only a finite number of equilibria in such economies. In Section IV, we define equilibrium indices and prove

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equilibrium index theorem. In Section V, we examine the implications of the equilibrium index theory to the uniqueness of the equilibrium, and compare them with the existing results in the literature.

II. Economies with Convex Production Technologies

We describe the consumption side of the model by individual demand functions d_i: $P \times \boldsymbol{R}^1_{++} \rightarrow \boldsymbol{R}^{\ell+1}$ such that $p \cdot d_i(p, w_i) = w_i$, $i = 1,\ldots,n$. Here, P is a subset of the $\ell + 1$ dimensional Euclidean space, $\boldsymbol{R}^{\ell+1}$, such that $\boldsymbol{R}^{\ell+1}_{++} \subset P \subset \boldsymbol{R}^{\ell+1}_{+} \setminus \{0\}$, where $\boldsymbol{R}^{\ell+1}_{++} = \{x \in \boldsymbol{R}^{\ell+1} \mid x_i > 0, \forall\, i\}$ and $\boldsymbol{R}^{\ell+1}_{+} = \{x \in \boldsymbol{R}^{\ell+1} \mid x_i \geq 0, \forall\, i\}$. Let $e_i \in \boldsymbol{R}^{\ell+1}_{+}$ be the initial endowment of consumer i for all $i = 1,\ldots,n$. Since we wish to concentrate on developing an equlibrium index theory applicable to economies with wider class of technologies, we admit the following rather strong assumption on the demand side.

Assumption 1 (Lipschitz Continuity)

For all $i = 1,\ldots,n$, d_i is Lipschitz continuous, i.e., for all $(p, w_i) \in P \times \boldsymbol{R}^1_{++}$, there exists $\delta > 0$ such that $(p^1, w_i^1) \in P \times \boldsymbol{R}^1_{++}$, $(p^2, w_i^2) \in P \times \boldsymbol{R}^1_{++}$ and $\|(p^1, w_i^1) - (p^2, w_i^2)\| < \delta$ implies

$\| d_i(p^1, w_i^1) - d_i(p^2, w_i^2) \| \leq K \, \|(p^1, w_i^1) - (p^2, w_i^2)\|$, for some $K > 0$.

The following assumption is standard.

Assumption 2 (Homogeneity)

For all $i = 1,\ldots,n$, $d_i(tp, tw_i) = d_i(p, w_i)$ for all $t > 0$.

We describe the production side by individual production sets, $Y_j \subset \boldsymbol{R}^{\ell+1}$, $j = 1,\ldots,m$. Let $Y = \sum_{j=1}^{m} Y_j$.

Assumption 3 (Possibility of No Production)

$0 \in Y_j$ for all $j = 1,\ldots,m$.

Assumption 4 (Irreversibility)
$Y \cap (-Y) = \{0\}$.

Assumption 5 (Free Disposability)
$Y - \mathbf{R}_+^{\ell+1} \subset Y$.

Assumption 6
Y is closed and convex.

Let an economy, $E = \{(d_i, e_i), (Y_j)\}$, be given. For all $j = 1,\ldots,m$, we define the individual supply correspondence s_j: $P \rightarrow \mathbf{R}^{\ell+1}$ as

$$s_j(p) = \{y'_j \in Y_j \mid p \cdot y'_j \geq p \cdot y_j \text{ for all } y_j \in Y_j\}.$$

Let

$$\pi_j(p) = p \cdot s_j(p) \text{ and } w_i(p) = p \cdot e_i + \sum_{j=1}^{m} \theta_{ij} \pi_j(p),$$

where θ_{ij} is consumer i's share of producer j's profit such that $0 \leq \theta_{ij} \leq 1$ and $\sum_{i=1}^{n} \theta_{ij} = 1$. And let

$$s(p) = \sum_{j=1}^{m} s_j(p),\ d(p) = \sum_{i=1}^{n} d_i(p, w_i(p)),\ e = \sum_{i=1}^{n} e_i \text{ and}$$

$$z(p) = d(p) - s(p) - e.$$

The multivalued map, z: $P \rightarrow \mathbf{R}^{\ell+1}$, will be called the excess demand correspondence.

We say that $p^* \in P$ is an *equilibrium* price of the economy, E, if there is $z \leq 0$ such that $z \in z(p^*)$, where $z(\cdot)$ is the excess demand corespondence for the economy, E. An equilibrium price, p^*, is called a *boundary equilibrium,* if p^* is on the boundary of P.

$z(p)$ is nonempty and convex for all $p \in P$. But it may not be compact. However, we may assume without loss of generality that $z(p)$ is compact, when we concentrate on studying the properties of equilibria, as can be seen in the following. Choose a constant C such that, if $y_j \in Y_j$ for all j and $\sum^{m} y_j \geq d(p) - e$, then $\| y_j \| < C$ (see Lemma A.2 in Smale 1982). Let $\hat{Y}_j$

$= Y_j \cap \{v \in \boldsymbol{R}^{\ell+1} \mid \| v \| \leq C\}$ and $\hat{E} = \{(d_i, e_i), (\hat{Y}_j)\}$. We call $\hat{E}$ the false economy of E, and denote the corresponding supply and excess demand correspondences as $\hat{s}_j$'s and $\hat{z}$. Then both E and $\hat{E}$ have the same set of equilibria. For, if p^* is an equilibrium of $\hat{E}$, then there exists $y^*_j \in \hat{s}_j(p^*)$ for all j such that $d(p^*) - e - \sum_{j=1}^{m} y^*_j \leq 0$. And by the convexity of Y_j's and the construction of $\hat{Y}_j$'s, $y^*_j \in s_j(p^*)$ for all j. So, p^* is an equilibrium of E. The converse is obvious by the construction of $\hat{Y}_j$'s. Hence, we assume without loss of generality that $z(p)$ is compact for all $p \in \Delta$.

The excess demand correspondence, z: $P \rightarrow \boldsymbol{R}^{\ell+1}$, has the following properties. It is upper semi-continuous and its values are nonempty, convex and compact. Furthermore, we can show that $p \cdot z(p) = 0$ which is the so-called Walras law. We need the following assumption on the excess demand correspondence.

Assumption 7 (Boundary Condition)

i) $\limsup_{\nu \to \infty} \inf \{\sum_{k=1}^{\ell+1} z_k \mid z \in z(p^\nu)\} = \infty$, for all sequences, (p^ν) in P such that $p^\nu \rightarrow p \in \boldsymbol{R}^{\ell+1}_+ \setminus P$.

ii) There is no boundary equilibrium.

Let q be a ℓ-dimensional vector. We shall use capital letters to denote the *reduced* maps defined in the following. Let $\Pi_j(q) \equiv \pi_j(q, 1)$, $D(q) \equiv \{\bar{x} \in \boldsymbol{R}^\ell \mid (\bar{x}, x_{\ell+1}) \in d(q, 1)$ for some $x_{\ell+1} \in \boldsymbol{R}^1\}$, and $S_j(q) = \{\bar{y} \in \boldsymbol{R}^\ell \mid (\bar{y}, y_{\ell+1}) \in s_j(q, 1)$ for some $y_{\ell+1} \in \boldsymbol{R}^1\}$. Let $S(q) = \sum_{j=1} S_j(q)$ and $Z(q) = D(q) - S(q) - \bar{e}$. Here, $\bar{e}$ is the vector consisting the first ℓ coordinates of e. Note that $S(q) = \{\bar{y} \in \boldsymbol{R}^\ell \mid (\bar{y}, y_{\ell+1}) \in s(q, 1)$ for some $y_{\ell+1} \in \boldsymbol{R}^1\}$ and $Z(q) = \{\bar{z} \in \boldsymbol{R}^\ell \mid (\bar{z}, z_{\ell+1}) \in z(q, 1)$ for some $z_{\ell+1} \in \boldsymbol{R}^1\}$. We also note that $(q^*, 1)$ is an equilibrium if and only if $0 \in Z(q^*)$. This is so by the Walras law and ii) in Assumption 7.

Let Ω be a ℓ-dimensional compact manifold with boundary. And let $C(\Omega)$ be the set of all upper semi-continuous correspondences from Ω to $\boldsymbol{R}^\ell$ whose values are nonempty, convex and compact, and whose zeros are not on the boundary, i.e., $0 \notin f(\partial \Omega)$ for any $f \in C(\Omega)$. We say that f is *homotopic* to g in $C(\Omega)$, denoted by $f \simeq g$, if there exists an upper semi-continuous correspondence $H: \Omega \times [0, 1] \rightarrow \boldsymbol{R}^{\ell}$ such that $H_t \equiv H(\cdot, t) \in C(\Omega)$ for all $t \in [0, 1]$, $H_0 = f$ and $H_1 = g$. For a single-valued function $f \in C(\Omega)$, let $deg(f, \Omega, 0)$ denote the topological degree of f with respect to 0 (for a reference of topological degrees, see Lloyd 1978). For a correspondence $g \in C(\Omega)$, we define the *degree* of g with respect to 0 as $deg(g, \Omega, 0) = deg(f, \Omega, 0)$, where f is a single-valued function homotopic to g. The degree is homotopy invariant, i.e., two maps have the same degree if they are homotopic to each other (see Ma 1972).

Theorem 1

Under Assumptions 1-7, there exists $\Omega \subset \boldsymbol{R}^{\ell}_{++}$, diffeomorphic to an ℓ-dimensional closed disk, such that $q^* \in Int\ \Omega$ for all q^* with $0 \in Z(q^*)$ and that $deg(Z, \Omega, 0) = (-1)^{\ell}$.

Here, $Int\ \Omega$ is the interior of Ω. Let $\Delta = \{p \in P \mid \sum_{k=1}^{\ell+1} p_k = 1\}$, $\Delta^{\circ} = \{p \in \boldsymbol{R}^{\ell+1}_{++} \mid \sum_{k=1}^{\ell+1} p_k = 1\}$ and $\bar{\Delta} = \{p \in \boldsymbol{R}^{\ell+1}_{+} \mid \sum_{k=1}^{\ell+1} p_k = 1\}$. Let c be the center of Δ, i.e., $c = (1/(\ell+1),\ldots,1/(\ell+1))$. Define a fuction $\gamma : \Delta \rightarrow \boldsymbol{R}^{\ell+1}$ and a correspondence $\omega : \Delta \rightarrow \boldsymbol{R}^{\ell+1}$ by $\gamma(p) = p - c$ and $\omega(p) = \{(p_1 z_1, p_2 z_2, \ldots, p_{\ell+1} z_{\ell+1}) \mid z \in z(p)\}$. The values of γ and ω are in fact in the tangent space of Δ°(at any $p \in \Delta^{\circ}$), and ω is an upper semi-continuous correspondences. We use the following lemma for the proof of Theorem 1. Let E^* denote the set of equilibrium prices in Δ.

Lemma 1

There is a convex subset G of Δ, diffeomorphic to an ℓ-dimensional closed disk, such that

i) $c \in Int\ G$

ii) $E^* \subset Int\ G$

iii) $\gamma(p)/\|\gamma(p)\| \neq \omega/\|\omega\|$ for all $p \in \partial G$ and all $\omega \in \omega(p)$.

Proof of Lemma 1

Construct a sequence of subsets, $\{G^{\nu}\}$, $G^{\nu} \subset \Delta^{\circ}$, such that 1) each

G^{ν} is convex and diffeomorphic to an ℓ-diensional closed disk; 2) $c \in G^{\nu} \subset G^{\nu+1}$ for all ν; and $\bigcup_{\nu=1}^{\infty} G^{\nu} = \Delta^{\circ}$.

Then, we can find $\bar{\nu}$ such that G^{ν} has properties i) and ii) for all $\nu \geq \bar{\nu}$. For, if not, there is a sequence in Δ°, $\{p^{\mu}\}$, such that $p^{\mu} \to p \in \bar{\Delta}$ and $0 \in z(p^{\mu})$ for all μ. If $p \in \bar{\Delta} \setminus \Delta$, then the first part of Assumption 7 is violated. If $p \in \Delta$, then $0 \in z(p)$ by the upper semi-continuity of z. So, the second part of Assumption 7 is violated.

Suppose that for all $\nu \geq \bar{\nu}$ there are $p^{\nu} \in \partial G^{\nu}$ and $z^{\nu} \in z(p^{\nu})$ satisfying

$$\frac{\gamma(p^{\nu})}{\|\gamma(p^{\nu})\|} = \frac{\omega^{\nu}}{\|\omega^{\nu}\|},$$

where $\omega^{\nu} = (p_1^{\nu} z_1^{\nu}, \ldots, p_{\ell+1}^{\nu} z_{\ell+1}^{\nu}) \in \omega(p^{\nu})$. We may assume without loss of generality that $p^{\nu} \to p \in \bar{\Delta} \setminus \Delta^{\circ}$. Taking the dot product of z^{ν} with both sides of the above equation and rearranging, we obtain

$$\sum_{k=1}^{\ell+1} z_k^{\nu} = -(\ell+1) \frac{\|p^{\nu} - c\|}{\|\omega^{\nu}\|} \omega^{\nu} \cdot z^{\nu} < 0.$$

Hence, $p \in \Delta \setminus \Delta^{\circ}$ by the first part of Assumption 7. Thus, $\omega^{\nu} / \|\omega^{\nu}\| \to (p-c)/\|p-c\|$. In particular, for $k \in I = \{k \mid p_k = 0\}$, $p_k^{\nu} z_k^{\nu} / \|\omega^{\nu}\| \to -\frac{1}{\ell+1} / \|p-c\|$. By the upper semi-continuity of z at p, $\{z^{\nu}\}$ is contained in a compact set, and hence has a convergent subsequence denoted $\{z^{\eta}\}$. Then, for all $k \in I$,

$$\frac{p_k^{\eta} \ z_k^{\eta}}{\|\omega^{\eta}\|} \to \frac{-1/(\ell+1)}{\|p-c\|} \text{ and } z^{\eta} \to z \in z(p).$$

This is possible only if $z_k \leq 0$ for all $k \in I$. Since we assumed the absence of a boundary equilibrium in Assumption 7, this is a contradiction. Hence, there is a ν' such that $G^{\nu'}$ satisfies all the properties in the lemma. Let $G = G^{\nu'}$.

Q.E.D.

Let $Proj$: $\boldsymbol{R}^{\ell+1} \to \boldsymbol{R}^{\ell}$ be defined by $Proj(x_1, x_2, \ldots, x_{\ell+1}) = (x_1, x_2, \ldots, x_{\ell})$. $Proj(x)$ is denoted by $\bar{x}$. Then, $\bar{p} = Proj(p)$, $\bar{c} = Proj(c)$, $\bar{E}^* =$

$Proj(E^*)$, and $\bar{G} = Proj(G)$. Also, let $\bar{\omega}(\bar{p}) = Proj(\omega(p))$. This is well-defined because p and $\bar{p}$ have one-to-one correspondence with each other in Δ.

Proof of Theorem 1

Observe that $\bar{G}$ is diffeomorphic to an ℓ-dimensional closed disk and contains $\bar{c}$ and $\bar{E}$ in its interior. For $t \in [0, 1]$ let

$$H(\bar{p}, t) = -t\bar{\gamma}(\bar{p}) + (1-t)\bar{\omega}(\bar{p}).$$

Here $\bar{\gamma}(\bar{p}) = (\bar{p}-\bar{c})$. Observe that $\bar{p} \in \partial\bar{G}$ implies that $(\bar{p}-\bar{c})/\|\bar{p}-\bar{c}\| \neq \bar{\omega}/\|\bar{\omega}\|$ for all $\bar{\omega} \in \bar{\omega}(\bar{p})$. Note also that H is upper semi-continuous and $H(\cdot, t)$ is compact, convex valued for all $t \in [0, 1]$. So, H defines a homotopy between $-\bar{\gamma}$ and $\bar{\omega}$ in $C(\bar{G})$. Thus we have that

$$deg(\bar{\omega}, \bar{G}, 0) = deg(-\bar{\gamma}, \bar{G}, 0) = (-1)^{\ell}.$$

Now, define $\varphi : \bar{G} \to \boldsymbol{R}^{\ell}_{++}$ by $\varphi(\bar{p}) = \bar{p}/(1-\sum_{k=1}^{\ell}\bar{p}_k)$. Let $\varphi(\bar{G}) = \Omega$. Then, φ is a diffeomorphism onto Ω. Let $\hat{\omega}(q) = \bar{\omega}(\varphi^{-1}(q))$. Then,

$$deg(\hat{\omega}, \Omega, 0) = deg(\bar{\omega}, \bar{G}, 0) = (-1)^{\ell},$$

since $|D\varphi^{-1}(\bar{q})| > 0$ for all $\bar{q} \in \Omega$ (see Theorem 2.3.1 in Lloyd 1978). Because $Z(q) = \bar{z}(\bar{p})$ and $q = \varphi(\bar{p})$, we may define a homotopy between $\hat{\omega}$ and Z in $C(\Omega)$ by

$$\phi(q, t) = \{(t\bar{p}_1 z_1 + (1-t)z_1, \ldots, t\bar{p}_{\ell} z_{\ell} + (1-t)z_{\ell}) \mid z \in Z(q)\}.$$

Thus,

$$deg(Z, \Omega, 0) = deg(\hat{\omega}, \Omega, 0) = (-1)^{\ell}.$$

Q.E.D.

We now define generalized Jacobians of Lipschitz continuous functions as in Clarke(1983). For a Lipschitz continuous function $f: \boldsymbol{R}^{\ell} \to \boldsymbol{R}^{n}$, the generalized Jacobian of f at x, $\partial f(x)$, is defined by

$$\partial f(x) = co\{lim\ Df(x^{\nu}) \mid x^{\nu} \to x \text{ and } x^{\nu} \notin N_f\}.$$

Here, co is the convex hull, N_f is the set of points at which f is not differentiable, and $Df(x^{\nu})$ is the usual Jacobian matrix at a differentiable point x^{ν}.

Since profit functions, π_j's, are Lipschitz continuous (see Proposition 2.2.6, Clarke 1983), we can use the generalized Jacobian of π_j to extend the Hotelling's lemma to non-differentiable cases.

Proposition 1 (Generalized Hotelling's Lemma)

$s_j(p) = \partial \pi_j(p)$ for all $p \in \boldsymbol{R}^{\ell+1}_{++}$ and all j.

Proof

Let $y \in s_j(p)$ and $\phi(p') = \pi_j(p') - p' \cdot y$. Then, ϕ is Lipschitz continuous, $\phi(p') \geq 0$ for all p', and $\phi(p) = 0$. Hence, $0 \in \partial \phi(p) = \partial \pi_j(p) - y$ (see Proposition 2.3.2, Clarke 1983). Thus, $y \in \partial \pi_j(p)$. If π_j is differentiable at p, then $\partial \pi_j(p) = D\pi_j(p) = s_j(p)$, and s_j is single-valued and equal to $D\pi_j(p)$. Suppose $y = \lim_{\nu} D\pi_j(p^{\nu}) = \lim_{\nu} s_j(p^{\nu})$, as $p^{\nu} \rightarrow p$. By the upper semi-continuity of s_j, $y \in s_j(p)$. Since $\partial \pi_j(p)$ is the convex hull of $\{\lim_{\nu} D\pi_j(p^{\nu}) \mid p^{\nu} \rightarrow p\}$ and $s_j(p)$ is convex, $\partial \pi_j(p) \subset s_j(p)$.

Q.E.D.

Corollary 1

$S_j(q) = \partial \Pi_j(q)$ for all $q \in \boldsymbol{R}^{\ell}_{++}$ and all j.

Proof

If $y \in S_j(q)$, then there exists $y_{\ell+1} \in \boldsymbol{R}^1$ such that $(y, y_{\ell+1}) \in s_j(q)$. Let $\varphi(q') = \Pi_j(q') - q' \cdot y - y_{\ell+1}$. By the same type of argument used in the proof of Proposition 1, we obtain $y \in \partial \Pi_j(q)$. On the other hand, by the proposition 2.3.16 of Clarke (1983), $\partial \Pi_j(q) = \partial_1 \pi_j(q, 1) \subset$ *Proj* $\partial \pi_j(q, 1) = S_j(q)$. Here, $\partial_1 \pi_j$, is the generalized Jacobian of π_j with respect to q.

Q.E.D.

From the definition of s_j, it is easy to derive the following well-known property of s_j:

$$y^1 \in s_j(p^1) \text{ and } y^2 \in s_j(p^2) \text{ imply } (y^1 - y^2) \cdot (p^1 - p^2) \geq 0.$$

We say that a correspondence is *monotone* if it has the above property. We note that S_j and $S \equiv \sum_j S_j$ are also monotone.

Now, consider the following maps, f and g, defined below. $Gr(S)$ and $Gr(Z)$ denote the graphs of correspondences S and Z, respectively.

$$Gr(Z) \overset{f}{\leftarrow} Gr(S) \overset{g}{\rightarrow} U = g(Gr(S)) \subset \boldsymbol{R}^{\ell}$$
$$(q,\ D(q)-y-\bar{e}) \leftarrow\!\shortmid (q,\ y) \mapsto q + y$$

Since $D(q)$ is a Lipschitz continuous function, it is easy to see that f is a *Lipschitz homeomorphism,* i.e., f is a homeomorphism, and f and its inverse function are Lipschitz continuous. We now show that g is also a Lipschitz homeomorphism.

Lemma 2

g is a Lipschitz homeomorphism.

Proof

Suppose $q^1 + y^1 = q^2 + y^2$. Then, $(q^1 - q^2) = -(y^1 - y^2)$. By the monotone property of S, $(q^1 - q^2) \cdot (y^1 - y^2) = -(y^1-y^2)^2 \geq 0$. Thus, $q^1 = q^2$ and $y^1 = y^2$. This shows that g is one to one. g is a restriction of a differentiable function on $Gr(S)$ and is Lipschitz continuous. Now, suppose $(q^1, y^1) = g^{-1}(u^1)$ and $(q^2, y^2) = g^{-1}(u^2)$. Then, $u^1 = q^1 + y^1$ and $u^2 = q^2+y^2$ by the definition of g. And,

$$\begin{aligned}\|u^1 - u^2\|^2 &= \|q^1-q^2\|^2 + \|y^1-y^2\|^2 + 2(q^1-q^2)\cdot \\ &\quad (y^1-y^2) \geq \|q^1-q^2\|^2+\|y^1-y^2\|^2 \\ &= \|(q^1, y^1) - (q^2, y^2)\|^2\end{aligned}$$

The above inequality is the result of the monotonicity of S. Thus, g^{-1} is also Lipschitz continuous.

Q.E.D.

From Lemma 2 and the comments preceding it, we conclude that $h: U \rightarrow Gr(Z)$ defined by $h \equiv f o g^{-1}$ is a Lipschitz homeomorphism. Let $B_\delta(q^*)$ be the open δ neighborhood of q^* and $\bar{B}_\delta(q^*)$ and $\partial B_\delta(q^*)$ be its closure and boundary, respectively.

Lemma 3

Let $B_\delta(q^*) \subset \boldsymbol{R}^\ell_{++}$ and define $A(q^*, \delta)$ by $A(q^*, \delta) = \{q+y \mid y \in S(q) \text{ and } q \in B_\delta(q^*)\}$. If $y^* \in S(q^*)$, then $B_\delta(q^*) + y^* \subset A(q^*, \delta)$.

Proof

Choose any $\bar{q} \in B_\delta(q^*)$ and define

$\alpha(q) = q + S(q) - \bar{q} - y^*$, and

$\beta(q) = q - q^*$

for $q \in B_\delta(q^*)$. We show that α is homotopic to β in $C(B_\delta(q^*))$. The homotopy is given by

$$H(q, t) = t\beta(q) + (1-t)\alpha(q), \text{ for all } t \in [0, 1].$$

Now consider

$$H(q, t) \cdot \beta(q) = \{t \| q-q^* \|^2 + (1-t)(q-q^*) \cdot (y-y^*) + (1-t)(q-\bar{q}) \cdot (q-q^*) \mid y \in S(q)\}.$$

If $q \in \partial B_\delta(q^*)$, then each term in the above expression is non-negative, and either the first or the third term is positive. Thus $0 \notin H(q, t)$ for all $t \in [0, 1]$ and all $q \in \partial B_\delta(q^*)$. So, $deg(\alpha, B_\delta(q^*), 0) = deg(\beta, B_\delta(q^*), 0) = 1$. We conclude that there is $q' \in B_\delta(q^*)$ such that $0 \in \alpha(q') = q' + S(q') - \bar{q} - y^*$. That is, $\bar{q} + y^* \in q' + S(q')$.

Q.E.D.

We note that Lemma 3, in particular, shows that $U = g(Gr(S))$ is an open subset of $\boldsymbol{R}^\ell$.

III. Equilibrium Index Theory

We introduce a concept of transversal intersection between a Lipschitzian manifold $(Gr(Z))$ and a smooth manifold $(\boldsymbol{R}^\ell_{++} \times 0)$. $Gr(Z)$ is *transversal* to $\boldsymbol{R}^\ell_{++} \times 0$ (denoted $Gr(Z) \pitchfork \boldsymbol{R}^\ell_{++} \times 0$), if $(q, 0) \in Gr(Z) \cap \boldsymbol{R}^\ell_{++} \times 0$ implies $Im\, J + \boldsymbol{R}^\ell \times 0 = \boldsymbol{R}^\ell \times \boldsymbol{R}^\ell$ for all $J \in \partial h(h^{-1}(q, 0))$. Here, $Im\, J$ is the image of the matrix J. There is, as yet, no result available in the literature stating that the above concept of transversality is generic in some appropriate sense. Note, however, that this concept of transversality reduces to the usual one when $Gr(Z)$ happens to be smooth.

Assumption 8 (Transversality)
$Gr(Z) \pitchfork \mathbf{R}^{\ell}_{++} \times 0$.

Let $h_1, h_2: U \to \mathbf{R}^{\ell}$ be such that $h(u) = (h_1(u), h_2(u))$. Given $J \in \partial h(u)$, J may be written as $J = \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}$, where J_1, and J_2 are $\ell \times \ell$ matrices.

Lemma 4
$Gr(Z) \pitchfork \mathbf{R}^{\ell}_{++} \times 0$ if and only if $|J_2| \neq 0$ for all $J \in \partial h(h^{-1}(q, 0))$ whenever $0 \in Z(q)$.

Proof
Let e^i denote the ℓ-dimensional column vector with 1 in the i^{th} coordinate and 0 elsewhere. Then, $Im\, J + \mathbf{R}^{\ell} \times 0 = \mathbf{R}^{\ell} \times \mathbf{R}^{\ell}$ if and only if $\begin{bmatrix} e^i \\ 0 \end{bmatrix}$, $i = 1, \ldots, \ell$ and $J(e^i)$, $i = 1, \ldots, \ell$, together span $\mathbf{R}^{\ell} \times \mathbf{R}^{\ell}$. This is equivalent to

$$\begin{vmatrix} I & J_1 \\ 0 & J_2 \end{vmatrix} = |J_2| \neq 0.$$

Here I is the ℓ-dimensional identity matrix.

Q.E.D.

The above lemma may suggest that we work with the condition that $0 \in Z(q)$ and $K \in \partial h_2(h^{-1}(q, 0))$ imply $|K| \neq 0$. It is likely that this condition is equivalent to the above transversality condition. This equivalence, however, is an open question and we continue to use the characterization in Lemma 4 (see the discussion on page 71 in Clarke 1983). It will be shown later that, when Z is differentiable at an equilibrium price vector q, the above transversality condition reduces to the usual one: $|DZ(q)| \neq 0$.

Proposition 2

Under Assumptions 1-8, there are finite number of equilibrium price vectors.

Proof

By Theorem 1, there exists an equilibrium price vector q and all the equilibrium price vectors are contained in the compact set Ω. We show now that every equilibrium price vector is locally isolated. Let H: $\boldsymbol{R}^\ell \times \boldsymbol{R}^\ell \rightarrow \boldsymbol{R}^\ell \times \boldsymbol{R}^\ell$ be defined by $H(v, u) = (h_1(u) + v, h_2(u))$. If $0 \in Z(q)$ and $h\ (q, 0) = u$, then

$$\partial H(0, u) = \left\{\begin{bmatrix} I & J \\ 0 & \end{bmatrix} \mid J \in \partial h(u)\right\}.$$

By Assumption 8, $L \in \partial H(0, u)$ implies $|L| \neq 0$. And by the inverse function theorem involving Lipschitz continuous functions (see Theorem 7.1.1 in Clarke 1983), H maps a neighborhood of $(0, u)$ homeomorphically onto a neighborhood of $(q, 0)$. If there exists $\{q^\nu\}$ such that $q^\nu \neq q$, $q^\nu \rightarrow q$ and $0 \in Z(q^\nu)$ for each ν, then $h^{-1}(q^\nu, 0) = u^\nu$ converges to $u = h^{-1}(q, 0)$. By choosing $v^\nu = q - h_1(u^\nu)$, we note that $(v^\nu, u^\nu) \rightarrow (0, u)$ and $H(v^\nu, u^\nu) = (q, 0)$, yielding a contradiction. Thus, each equilibrium price vector is locally isolated, and the proposition is proved.

Q.E.D.

For an economy E, we define the *index* of E (denoted ι_E) by $\iota_E = deg(Z, \Omega, 0)$. By Theorem 1, $\iota_E = (-1)^\ell$ for every economy E satisfying Assumptions 1-7. Suppose $q \in E^*$ is an isolated equilibrium price vector for the economy E. Let $B_\delta(q) \subset R^\ell_{++}$ be the open δ neighborhood of q whose closure, $\bar{B}_\delta(q)$, does not contain any equilibrium price vector other than q. We may define the (*local*) *index* of q denoted by ι_q as $\iota_q = deg(Z, \bar{B}_\delta(q), 0)$. This definition is independent of the choice of δ. By the index sum theorem (see theorem 8.1 in Ma 1972), $\iota_E = \sum_{q \in E^*} \iota_q = (-1)$ for the economy E with a finite number of isolated equilibria. Hence, we observe that there is a unique equilibrium price vector, if $\iota_q = (-1)^\ell$ for all $q \in E^*$. For an economy satisfying the transversality condition, the index of an equilibrium price vector can be computed in the following manner.

Theorem 2

If $Gr(Z)$ is transversal to $\mathbf{R}^\ell_{++} \times 0$ at $(q^*, 0)$, then
$\iota_{q*} = sign\ |J_2|$ for any $J \in \partial h(u^*)$, where $u^* = h^{-1}(q^*, 0)$.

comment: Since $\partial h(u^*)$ is a convex set of matrices and $J \in \partial h(u^*)$ implies $|J_2| \neq 0$, the sign of $|J_2|$ is invariant over $J \in \partial h(u^*)$.

Proof

Fix any $J \in \partial h(u^*)$. Given $\delta > 0$ such that $\bar{B}_\delta(q^*) \subset \mathbf{R}^\ell_{++}$, we define:

$\varphi(q) = J_2(q - q^*)$ and $\phi(q) = h_2(q + y^*)$, where $y^* = D(q^*) - \bar{e}$.

We first prove:

Step 1: φ is homotopic to ϕ in $C(\bar{B}_\delta(q^*))$ for some small enough $\delta > 0$.

For each $t \in [0, 1]$, define
$H(q, t) = (1 - t)h(q + y^*) + tJ(q - q^*)$, and
$F(v, q, t) = H(q, t) + (v, 0)$, where $v \in \mathbf{R}^\ell$.
Now, fix t. Then,

$$\partial F(0, q^*, t) = \left\{ \begin{bmatrix} I & K_1 \\ 0 & K_2 \end{bmatrix} \mid K \equiv \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \in \partial_q H(q^*, t) \right\}.$$

Here, $\partial_q H(q^*, t)$ denotes the generalized Jacobian of H with respect to q. We note that

$$\partial_q H(q^*, t) = (1-t)\partial h(q^* + y^*) + tJ \subset \partial h(q^* + y^*).$$

The above inclusion follows since $J \in \partial h(u^*)$ and $\partial h(u^*)$ is a convex set. Thus, all matrices in $\partial F(0, q^*, t)$ are non-singular by the transversality assumption. The inverse function theorem applies and F is a Lipschitz homeomorphism. By shrinking the neighborhood of $(0, q^*)$, if necessary, one can find a local Lipschitz constant $c > 0$ such that

$$\| F(v, q, t) - F(0, q^*, t) \| \geq c \| (v, q) - (0, q^*) \| \geq c \| q - q^* \|,$$

for all (v, q) in the neighborhood of $(0, q^*)$. If we choose a sufficiently small $\delta_t > 0$ such that $\| q - q^* \| < \delta_t$ implies that $(H_1(q^*, t) - H_1(q, t), q)$ is in the neighborhood of $(0, q^*)$, then we have

$$\begin{aligned} \| H_2(q, t) \| &= \| H_2(q^*, t) - H_2(q, t) \| \\ &= \| F(H_1(q^*, t) - H_1(q, t), q, t) - F(0, q^*, t) \| \\ &\geq c \| q - q^* \|. \end{aligned}$$

Thus,

$$H_2(q, t) \neq 0 \text{ for all } q \in B_{\delta_t}(q^*) \text{ with } q \neq q^*.$$

We now show that there is $\delta > 0$ such that

$H_2(q, t) \neq 0$ for all $q \in B_\delta(q^*)$ with $q \neq q^*$ and for all $t \in [0, 1]$. Suppose not. Then, one can find $(q^\nu, t^\nu) \in \boldsymbol{R}^\ell_{++} \times [0, 1]$ converging to (q^*, t^*) such that $q^\nu \neq q^*$, $t^\nu \in [0, 1]$ and $H_2(q^\nu, t^\nu) = 0$. Rewriting the last equation and dividing by $\| q^\nu - q^* \|$, we have

$$\frac{(1 - t^*) h_2(q^\nu + y^*) + t^* J(q^\nu - q^*)}{\| q^\nu - q^* \|} + (t^* - t^\nu) \cdot$$

$$\frac{h_2(q^\nu + y^*) - J(q^\nu - q^*)}{\| q^\nu - q^* \|} = 0.$$

The fraction in the second term is bounded by the Lipschitz property of h_2 and J. Thus, the second term converges to zero vector. So, the first term also converges to the zero vector. But we saw that $\| H_2(q, t^*) \| / \| q - q^* \| \geq c$ for all q close enough to q^*. Thus, a contradiction.

Step 2: ϕ is homotopic to Z in $C(\bar{B}_\delta(q^*))$.

Lemma 2 showed that g: $Gr(S) \rightarrow U$, defined by $g(q, y) = q + y$, was a Lipschitz homeomorphism. Lemma 3 showed that, if $y^* \in S(q^*)$ and $\bar{B}_\delta$

$(q^*) \subset \boldsymbol{R}^{\ell}_{++}$, then $\bar{B}_\delta(q^*) + y^* \subset \{q + y \mid q \in \bar{B}_\delta(q^*)$ and $y \in S(q)\}$. Given $t \in [0, 1]$, tS is a monotone correspondence. Following exactly the same arguments as in the proof of the lemmas, we can show that, for $t > 0$,

a) g_t: $Gr(S) \to U_t$ defined by $g_t(q, y) = q + ty$ is a Lipschitz homeomorphism, and

b) $y^* \in S(q^*)$ and $\bar{B}_\delta(q^*) \subset \boldsymbol{R}^{\ell}_{++}$ imply $\bar{B}_\delta(q^*) + ty^* \subset \{q + ty \mid q \in \bar{B}_\delta(q^*)$ and $y \in S(q)\}$.

Let $y^* = D(q^*) - \bar{e} \in S(q^*)$. We now define H: $\bar{B}_\delta(q^*) \times [0, 1] \to \boldsymbol{R}^{\ell}$. If $t > 0$, $H(q, t)$ is defined as a composition of the following maps:

$$(q, t) \longrightarrow q + ty^* \xrightarrow{g_t^{-1}} (q', y') \longrightarrow D(q') - \bar{e} - y'.$$

This map is well-defined by the observations made in the above. We define $H(q, 0) = Z(q)$. We show that H defines a homotopy between ϕ and Z in $C(\bar{B}_\delta(q^*))$. We first observe that $H(q, 1) = h_2(q + y^*) = \phi(q)$. We now show that $0 \notin H(q, t)$ if $q \neq q^*$. This is obvious when $t = 0$. Suppose $t > 0$. Suppose also that $g_t^{-1}(q + ty^*) = (q', y')$ and $D(q') - \bar{e} - y' = 0$. Then, $q' \in \bar{B}_\delta(q^*)$ by the observation $b)$ in the above. Since q^* is the only equilibrium price vector in $\bar{B}_\delta(q^*)$, $q^* = q'$. Then, $y' = y^*$. By the definition of g_t, $q = q^*$. So, we conclude that $0 \notin H(q, t)$ whenever $q \neq q^*$.

It remains to show that H is upper semi-continuous. Since the image of H is contained in a compact set, it suffices to show that it has a closed graph. But this follows immediately from the fact that the correspondence Γ defined by $\Gamma(q, t) = \{(q', y') \mid q' + ty' = q + ty^*$ and $y' \in S(q')\}$ for $(q, t) \in \bar{B}_\delta(q^*) \times [0, 1]$ has a closed graph.

Since $deg(J_2, \bar{B}_\delta(q), 0) = sign \mid J_2 \mid$ for a non-singular linear map J_2 (see Lloyd 1978), the proof of Theorem 2 is now complete.

IV. Some Examples of Economies and the Uniqueness of an Equilibrium

We consider some economies for which we may compute the indices more explicitly and derive conditions under which these economies have unique equilibria. Let q^* be an equilibrium and $u^* = h^{-1}(q^*, 0)$ throughout the section.

First, we consider a differentiable economy for which the demand and supply functions are single-valued and differentiable. For such an economy, $h_2(u) = Z(q) = D(q) - S(q) - \bar{e}$, where q satisfies $q + S(q) = u$. By the implicit function theorem, we may derive:

$$Dh_2 = DZ\ (I + DS)^{-1}.$$

We know that $I + DS$ is positive definite. Hence, the index of q^* is the sign of $|DZ(q^*)|$. Thus, there is only one equilibrium if $|DZ(q^*)|$ has the same sign for every equilibrium $q^* \in E^*$. Compare this with the result in Yun(1981).

Second, consider an economy for which the supply correspondence is multi-valued and h is differentiable at u^*. We say that the supply correspondence has a *k-dimensional jump* at q^*, if the dimension of $S(q^*)$ (i.e., the dimension of the linear span of $S(q^*)$) is k. If the supply has a ℓ-dimensional jump at q^*, then there exists an open neighborhood, B, of (q^*, y^*) such that $q = q^*$ for all $(q, y) \in B \cap Gr(S)$. So, in a neighborhood of u^*, we have

$$\begin{aligned} h(u) &= \{(q^*, D(q^*) - y - \bar{e}) \mid q^* + y = u\} \\ &= (q^*, D(q^*) + q^* - u - \bar{e}). \end{aligned}$$

Thus, $\iota_{q^*} = sign\ |J_2| = sign\,(-I) = (-1)^{\ell}$. So, there is a unique equilibrium if the supply correspondence has a ℓ-dimensional jump at every equilibrium. We now consider an equilibrium q^* at which the supply correspondence has a k-dimensional jump with $k < \ell$. In particular, we concentrate on an activity analysis model where k production activities are active. Let $x \in \boldsymbol{R}^k_{++}$ represent the levels of the k production activities, A be an $\ell \times k$ activity matrix with full column rank, and A^t be the transpose of A. Then, generically, we can choose an open neighborhood B of $u^* = q^* + y^*$ in U such that any $(q, y) \in g^{-1}(B)$ can be represented as

$$y = Ax \text{ and } A^t q = 0,$$

for some $x \in \boldsymbol{R}^k_{++}$. Hence,

$$h_2(u) = D(q) - Ax - \bar{e},\ q + Ax = u \text{ and } A^t q = 0.$$

Assume h_2 is differentiable at u^*. Note that a Lipschitz continuous function (in particular, h_2) is differentiable almost everywhere (see Clarke 1983). Differentiating the last two equations and solving for $\frac{dx}{du}$ and $\frac{dq}{du}$, we get:

$$\frac{dx}{du} = A(A^tA)^{-1} \text{ and } \frac{dq}{du} = I - A(A^tA)^{-1}A^t.$$

Then,

$$\frac{dh_2}{du} = \frac{dD}{dq}(I - A(A^tA)^{-1}A^t) - A(A^tA)^{-1}A^t.$$

By Lemma 4 in Kehoe (1980),

$$\iota_{q^*} = sign \left| \frac{dh_2}{du} \right| = sign \begin{vmatrix} \frac{dD}{dq} & A \\ A^t & 0 \end{vmatrix}.$$

Compare this to the result in Kehoe (1980).

Third, we consider an economy for which the supply function is single-valued and Lipschitz continuous. For such an economy,

$$h(u) = (q, Z(q)) \text{ where } q + S(q) = u.$$

Let $G(q, u) = q + S(q) - u$ and N_G be the set of (q, u) at which G is not differentiable. If $(q, u) \notin N_G$, then $DG(q, u) = [I + DS(q), -I]$. So, $\partial G(q^*, u^*)$ is the convex hull of the limits of the form $[I + DS(q), -I]$ and $I + DS(q)$ is positive definite. Thus, for all $(K_1, K_2) \in \partial G(q^*, u^*)$, K_1 is positive definite, and hence non-singular. So, by the implicit function theorem for Lipschitz continuous functions (see Clarke 1983), there exists a Lipschitz continuous function $q(u)$ such that $G(q(u), u) = 0$ for all u sufficiently close to u^*. Hence, we can write $h(u) = (q(u), Z(q(u)))$. If

$u \notin N_h$, then $u \notin N_q$ and $q(u) \notin N_S$. We have:

$$\{J_2 \mid J \in \partial h(u^*)\} = co \ \{lim \ DZ(q(u)) \ [I + S(q(u))]^{-1} \mid u \to u^* \text{ and } u \notin N_h\}$$

So, under the transversality condition,

$$\iota_{q^*} = sign \ \mid DZ(q(u)) \ [I + S(q)]^{-1} \mid \ = sign \ \mid DZ(q) \mid ,$$

for some $q \in N_Z$ sufficiently close to q^*. Hence, under the transversality condition, if $\mid DZ(q) \mid$ has the same sign around every equilibrium, there is only one equilibrium.

Finally, we compare the approach in this paper to the one in Mas-Colell (1985). Let V be an open subset of $\boldsymbol{R}^{\ell+1}$ and $\rho: V \to Y$ be the orthogonal projection to Y. Mas-Colell defines a Lipschitz continuous function $\tilde{h}: V \to \boldsymbol{R}^{\ell+1}$ as $\tilde{h}(v) = \rho(v) - d(v - \rho(v))$, and observes that $p^* = v^* - \rho(v^*)$ is an equilibrium price vector if and only if $\tilde{h}(v^*) = 0$. This function $\tilde{h}$ does a similar role in his case as h does in ours. Mas-Colell only considers the case where $\tilde{h}$ is continuously differentiable at every equilibrium. Aside from the fact that we consider a more general, Lipschitz case, our approach may have some advantages in the study of comparative statics. When the excess demand correspondence is allowed to be multi-valued, it is difficult to do comparative statics with the system of inclusion: $0 \in z(p)$. Thus, it is helpful to construct an equivalent system of simultaneous equations with the aid of such functions as h and $\tilde{h}$. The Jacobian matrix (or the generalized Jacobian) of the system need to be nonsingular, because an implicit function theorem is crucial in doing comparative statics. But, observe that for the above system, $\tilde{h}(v) = 0$, the Jacobian matrix $D\tilde{h}$ is singular, since it is a $(\ell + 1) \times (\ell + 1)$ matrix of rank less than or equal to ℓ. Hence, this system cannot be directly applied in a comparative static analysis. By contrast, the Jacobian matrix Dh is non-singular under the transversality condition for the system considered in this paper. Moreover, the index of an equilibrium is defined as the sign of the Jacobian $\mid Dh \mid$. This sign is important in telling which direction endogenous variables would move as an exogenous variable changes. Hence, the index defined in this paper has a direct relation to the study of comparative statics. For an example illustrating this, see Chapter 2 of Park (1988).

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