Journal of the Korean
OR/MS Society
Vol. 12, No. 1, APR. 1987

—Research Paper—

The (S-1, S) Inventory Model with Order Size Dependent Delivery Times

Sukho Kang*
Jeonghun Park*

ABSTRACT

This paper considers (S-1, S) inventory models which have wide applications in reparable spare parts inventory systems and multi-echelon systems. We assume a discrete compound Poisson demand and order size dependent delivery times ; when the replenishment order size is n, we assume the delivery time distribution is arbitrary with finite mean b_n

On the basis of the fact the outstanding orders follow a certain queueing process, we introduce the results of Fakinos (1982). We develop the efficient recursive formulae to find the optimal S^* under several performance measures as a function of the decision variable S. The results of this paper can be applied to the multi-echelon systems such as METRIC.

(INVENTORY ; REPARABLE ITEMS ; MULTI-ECHELON ; QUEUES)

1. Introduction

This paper considers (S-1, S) inventory models with a compound Poisson demand and an arbitrary delivery time distribution whose mean is dependent on the replenishment order size. The main goal is to develop the recursive formulae to find the optimal S^* under several performace measures. We assume that a series of customers with a Poisson arrival of rate λ place an order which has an independent and identical discrete distribution $\{f_j\}$ $j=1, 2, \cdots\cdots$. The order sizes and the interarrival times of successive arriving customers are stochastically independent. The delivery time may depend on the replenishment order size on several accounts in many contexts. In this paper, the delivery time distribution is assumed to be arbitrary with a mean b^n when the replenishment order size is n. This is a generalized assumption for the replenishment process, whereas a common one in the literature is to have only one arbitrary distribution independent of the replenishment order size.

(S-1, S) policies are a special case of (s, S) policies where s equals S-1. In (s, S) policies, the order point s and the order up to level S are set and whenever the inventory position (outstanding order plus on hand minus backorder) drops to, or below s, the replenishment order is issued to raise the inventory position to S. Therefore, (S-1, S) policies place orders, namely one-for-one replenishment whenever units are demanded, thus restoring the inventory position to the base stock level S.

*Department of Industrial Engineering,
Seoul National University

The ordering policy described above is appropriate when the item has a low demand and a relatively high unit cost compared with the cost of processing an order. In particular, the reparable spare parts inventroy systems belong to this ordering policy. Sherbrooks (1968) has applied (S-1, S) policies to a multi-echelon model for controlling high value reparable items, so called METRIC using the previous results of Feeney and Sherbrooke (1966). But, he developed computationally intractable formulae with the assumption of a compound Poisson demand, so he assummed in the METRIC a logarithmic-Poisson demand where the order sizes of successive arriving customers follow a logarithmic distribution. In this paper, we will develop easily computable recursive formulae with any compound Poisson demands and a generalized replenishment process which will enjoy direct application to the METRIC without any special modifications. Graves (1985) presents a multi-echelon inventroy model for a reparable item and compares his approximation with that of Sherbrooke (1968). A comprehensive review on reparable inventory systems can be found in Nahmias (1981).

We define $Q(t)$ to be the number of units in the replenishment process, or the outstanding orders at time t. We note if the base stock level S set, then $S-Q(t)$ is the net on hand inventory at time t. As we are interested in the equilibrium state, we omit the time variable t for later use. Q has the range $0 \leq Q < \infty$ in the backorder case and $0 \leq Q < S$ in the lost sales case. We assume that customers behave according to all or nothing policy in the lost sales case. That is, if a customer order is greater than the net on hand inventory, then the customer's whole demand is rejected and vice versa. We will comment on an alternative, partial filling policy in section 4.

In our models, Q follows a queueing process (see Hadley and Whitin 1963, pp. 204-212). For example, Q increases as a demand occurs and decreases as a replenishment order arrives. The queueing models which directly relate to the processes of Q in the lost sales case and the backorder case are the $M/G/S$ group-arrival group-departure loss system and the $M/G/\infty$ group-arrival group-departure system, respectively. Note that two queueing models are essentially the same as the latter is a special case of the former when S approaches infinity. Fakinos (1982) has studied these queueing processes and provides an equilibrium state probability in a rather complex form. One of our goals in this paper is the conversion of the Fakinos' formula into a computationally tractable one.

Once the equilibrium distribution of Q is known, it is straightforward to develop several performance measures in terms of S by using the fact that S-Q is the net on hand inventory.

2. The Equilibrium Distribution

The main result is that the equilibrium distribution of the outstanding orders Q in the lost sales case is expressed recursively by the following formula.

$$P(Q=0) = P_o = [1 + \sum_i P_i/P_o]^{-1},$$

$$P(Q=n)=\begin{cases} P_n=\dfrac{\lambda}{n}[f_1\cdot b_1\cdot P_{n-1}+2f_2\cdot b_2\cdot P_{n-2}+\cdots+nf_n\cdot b_n\cdot P_o] \\ \qquad\qquad for\ n=1,2,\cdots,S \\ 0 \qquad\qquad otherwise. \end{cases}$$

Alternatively, this formula can be expressed as follows for computational purposes.

$$P_o=[\sum_{i=0}^{S}a_i]^{-1},$$

$$P_n=\begin{cases} a_n\cdot P_o & for\ n=1,2,\cdots,S \\ 0 & otherwise, \end{cases}$$

where

$$a_o=1,$$

$$a_n=\frac{\lambda}{n}[f_1\cdot b_1\cdot a_{n-1}+2f_2\cdot b_2\cdot a_{n-2}+\cdots+nf_n\cdot b_n\cdot a_o]$$

$$for\ n=1,2,\cdots,S.$$

Clearly, P_0 is determined by the normalization condition $\sum_{i=0}^{s}P_i=1$. Note that the equilibrium distribution is independent of the form of the delivery time distributions and depends only on their means $b_i, i=1,2,\cdots\cdots,S$. The distribution in the backorder case can be found by setting S to infinity and P_0 equals $exp[-\lambda\sum_{i=1}^{\infty}f_i\cdot b_i]$ in this case.

The recusive formula can be derived by applying the methods of Adelson (1966) who has analyzed the arriving compound poisson processes to the queueing process of Fakinos (1982). we provide a brief explanation. Let Q_i be the number of groups of outstanding orders of size i in the backorder case, then the outstanding orders in the backorder case is

$$Q=\sum_{i=1}^{\infty}i\cdot Q_i\ . \tag{3}$$

The random variables Q_i, $Q_2,\cdots$ are stochastically independent by the independence assumptions made in the earlier section. It is easy to know that Q_i follows the $M/G/\infty$ queueing process with a Poisson arrival of rate λf_i and an arbitrary service time having a mean b_i. Therefore, the equilibrium distribution of Q_i is a Poisson with mean $\lambda f_i\cdot b_i$. Since Q is a weighted sum of a number of random variables of independent Poisson processes, it has a compound Poisson distribution. The recursive formula is derived from the probability generation functing (PGF) of Q by using the Maclaulin power series and differentiations.

The distribution of Q in the lost sales case is obtained by truncating at S the corresponding distribution in the backorder case. As an another approach, we derive (1) directly from the Fakinos' formula in the Appendix.

3. Performance measures

Performance measures can be defined in various ways. In this section, we develop salient ones in inventory systems as a function of the decision variable S.

For the backorder case, we consider the expected number of backorders in a unit of time and the expected backorder level.

In order to find the expected number of backorders, we consider the following. Customers arrive with a Poisson process of rate λ and the order sizes which a series of customers place have a discrete distribution f_j $j=1, 2, \cdots$. Let's suppose that a customer places an order of size i. There will be one backorder if the net on hand inventory is i-1 (or, the outstanding orders are S-(i-1)), two backorders if the net on hand inventory is i-2, and i backordrs if the net on hand inventory is 0. Therefore, the expected number of backorder is

$$\lambda \cdot \sum_{i=1}^{\infty} f_i \, [P_{S-i+1} + 2P_{S-i+2} + \cdots + (i-1)P_{S-1} + i\,(P_S + P_{S+1} + \cdots)]$$
$$= \lambda \cdot \sum_{i=1}^{\infty} f_i \cdot [\sum_{j=1}^{i} j \cdot P_{S-i+j} + i(1 - \sum_{k=1}^{S} P_k)]. \tag{4}$$

The expected backorder level is

$$\sum_{j=S+1}^{\infty} (j-S) \cdot P_j \cdot \tag{5}$$

We recall that the equilibrium distribution of the outstanding orders Q used in (4) and (5) is obtained from (1) by setting S to infinity, as we explained in the previous section.

Now, we consider another interesting criterion, time in the backorder case. Possibly, we can imagine inventory systems to which additional costs (e. g. the penalty cost of backorder) are not accruing if a customer's order is filled within a given time T. We can obtain generalized performance measures by developing the equilibrium distribution of Q', the outstanding orders which have been in the on order condition at least T.

The resulting distribution of Q' is

$$P'_0 = exp(-\lambda \sum_{i=1}^{\infty} f_i \cdot b'_i)$$
$$P'_n = \begin{cases} \frac{\lambda}{n}[f_1 \cdot b'_1 \cdot P'_{n-1} + 2f_2 \cdot b'_2 \cdot P'_{n-2} + \cdots + nf_n \cdot b'_n \cdot P'_0] \\ \qquad\qquad for\ n=1, 2, \cdots \\ 0 \qquad\qquad otherwise, \end{cases} \tag{6}$$

where

$for\ i=1, 2, \cdots$

$$b'_i = \alpha_i \cdot b_i$$

$$\alpha_i = \frac{1}{b_i} \int_T^{\infty} (1 - B_i(t))\,dt$$

B_i : the distribution function of the delivery time when the order size is i.

The equilibrium distribution in this case depends on the form of the delivery time distributions.

To prove (6), we define Q'_i to be the number of groups of outstanding orders of size i in the backorder case which have been in the on order condition at least T. Takācs (1969) shows that α_i is the probability that the groups of the outstanding orders of size i in the backorder case will still be in the on order condition after a given time. T. In order for Q'_i to be n ($n=0, 1, 2,\cdots\cdots$), exactly $k(k=0, 1, 2,\cdots\cdots)$ groups should be replenished in the time interval T among $n+k$ groups of the outstanding orders of size i. Since Q_i has a Poisson with mean $\lambda f_i \cdot b_i$, the distribution of Q'_i is

$$P(Q'_i=n)=\sum_{k=0}^{\infty}\frac{e^{-\lambda f_i \cdot b_i}(\lambda f_i b_i)^{n+k}}{(n+k)!}\binom{n+k}{k}\cdot\alpha_i^n(1-\alpha_i)^k$$

$$=\frac{e^{-\lambda f_i \cdot b_i}(\lambda f_i \cdot b'_i)^n}{n!}\qquad for\ n=0, 1, 2,\cdots \tag{7}$$

or, Q'_i has a Poisson with mean $\lambda f_i \cdot b'_i$. Now, the proof of (6) is straightforward if we relate Q'_i to Q_i and follow the same procedure in section 2. A backorder in this case is defined as the unit which has been backordered at least T.

In order to develop the expected number of backorders in a unit of time, we follow the same logic used to develop (4) and add one additional consideration that the order of size i itself should remain in the on order condition after time T with probability α_i . Therefore, the expected number of backorder is

$$\lambda\sum_{i=1}^{\infty}\alpha_i\cdot f_i\cdot[\sum_{j=1}^{i}j\cdot P'_{S-i+j}+i(1-\sum_{k=1}^{S}P'_k)]. \tag{8}$$

Indeed, (8) is a generalization of (4) which is a special case when T equals zero.

The expected backorder level is

$$\sum_{j=s+1}^{\infty}(j-s)\cdot P'_j. \tag{9}$$

Now, we have the last case, the lost sales case. We develop the expected number of lost sales in a unit of time using (1). We assume that customers adopt the all or nothing policy explained in secion 1. If a customer places the order of size i and the net on hand inventory is less than i (or, the outstanding order is greater than S-i), then the customer will reject the order. Therefore, the expected number of lost sales is

$$\lambda[\sum_{i=1}^{S}i\cdot f_i\cdot(P_{S-i+1}+P_{S-i+2}+\cdots+P_S)+\sum_{i=S+1}^{\infty}i\cdot f_i]. \tag{10}$$

From (10), we can easily obtain the expected number of lost customers in a unit of time by omitting the multiplication factor i in both terms.

We defined the performance measures as decreasing functions of S. But, they are essentially the same with those of Feeney and Sherbrook (1966) who developed the measures in increasing functions of S.

4. Discussion

We have presented (S-1, S) inventory models which can be widely applicable to practical inventory systems. To facilitate the use of our models, we have developed formulae in a recursive form. The essential feature of our paper is the dependence on results from a queueing process. This approach is common in the literature (e.g. Chandrasekhar Das 1977, Barrer 1957).

To complete our models, we comment on the partial filling policy in the lost sales case in which an order is partially filled until the net on hand inventory reaches zero stock. Here, we change the assumption of the delivery time distribution and assume it to be arbitrary with a mean b, regardless of the order size. In this case, Feeney and Sherbrooke (1966) has found the equilibrium distribution of Q and that is

$$
\begin{aligned}
P_o &= [1+\sum_{i=1}^{s} P_i/P_o]^{-1} \\
P_n &= \sum_{i=0}^{n} \frac{(\lambda b)^i}{i\,!} \cdot f^{i*}(n) \cdot P_o \qquad for\ n=1, 2, \cdots, S-1 \\
P_s &= \sum_{i=0}^{s} \frac{(\lambda b)^i}{i\,!} \cdot \sum_{j=s}^{\infty} f^{i*}(j) \cdot P_o.
\end{aligned}
\tag{11}
$$

where f^{i*} is the i-fold convolution of $\{f_j\}$.

(11) has a rather complex form. For computational purposes, we propose a matrix H which has the properties, $f^{i*}(n)=H^i(0, n)$ $(n=0, 1, \cdots S-1)$ and $\sum_{j=s}^{\infty} f^{i*}(j)=H^i(0, S)$.

The matrix is

$$
H = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \cdot \\ \cdot \\ \cdot \\ S-1 \\ S \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 0 & 1 & 2 & 3 & \cdots & S-1 & S \end{array} \\
\left[\begin{array}{ccccccc}
0 & f_1 & f_2 & f_3 & \cdots & f_{s-1} & \beta_s \\
0 & 0 & f_1 & f_2 & \cdots & f_{s-2} & \beta_{s-1} \\
0 & 0 & 0 & f_1 & \cdots & f_{s-3} & \beta_{s-2} \\
 & & & \cdot & & \cdot & \cdot \\
 & & & & \cdot & \cdot & \cdot \\
 & & & & \cdot & \cdot & \cdot \\
 & 0 & & & & 0 & \beta_1 \\
 & & & & & & 1
\end{array}\right]
\end{array}
\tag{12}
$$

where for $i=1, 2, \cdots, S$

$$\beta_i = \sum_{j=i}^{\infty} f_j.$$

The proof is given in the Appendix.

The expected number of lost sales in this case is

$$\lambda \sum_{i=1}^{\infty} f_i \cdot [\sum_{j=1}^{i} j \cdot P_{s-i+j}]. \tag{13}$$

Using the tradeoffs between the previously developed performance measures and other cost measures such as the expected inventory, we can construct various cost models to determine the optimal S^*.

In this paper, we focused on the mathematical aspect of our results. Detailed explanations of the system, applications and proofs are available in our previous works, but we recommand that readers interested in the application of our results refer to Nahmias (1981).

Appendix

1. Derivation of a Recursive Formula

Fakinos(1982) has developed a generalization of Erlang's B formula which is equivalent to the equilibrium distribution of Q in the lost sales case in our paper. His result is

$$P(Q=j)=P_j=\sum_{A_j}\prod_{i=1}^{s}\frac{(\lambda f_i\cdot b_i)^{n_i}}{n_i\,!}\cdot P_o \qquad for\ j=0,1,2,\cdots,S,$$

where

$$A_j=\{(n_1,\ n_2,\cdots,n_S):n_1+2n_2+\cdots+Sn_S=j\} \qquad for\ j=0,1,2,\cdots,S,$$
$$n_i\in No \qquad for\ i=1,2,\cdots,S$$

The formula can be rewritten as

$$\begin{aligned}P_j&=\frac{1}{j}\sum_{A_j}(n_1+2n_2+\cdots+Sn_S)\prod_{i=1}^{s}\frac{(\lambda f_i\cdot b_i)^{n_i}}{n_i\,!}\cdot P_o\\&=\frac{1}{j}\sum_{A_j}(n_1+2n_2+\cdots+jn_j)\prod_{i=1}^{j}\frac{(\lambda f_i\cdot b_i)^{n_i}}{n_i\,!}\cdot P_o\\&=\frac{1}{j}[\sum_{A_j}n_1\prod_{i=1}^{j}\frac{(\lambda f_i\cdot b_i)^{n_i}}{n_i\,!}\cdot P_o+2\sum_{A_j}n_2\prod_{i=1}^{i}\frac{(\lambda f_i\cdot b_i)^{n_i}}{n_i\,!}\cdot P_o\\&\quad+\cdots+j\sum_{A_j}n_j\prod_{i=1}^{j}\frac{(\lambda f_i\cdot b_i)^{n_i}}{n_i\,!}\cdot P_o].\end{aligned}$$

We define

$$\begin{aligned}\delta(i)&=0 \quad if\ i\leq 0\\&=1 \quad O.W.\end{aligned}$$

Then, for $k=1,2,\cdots,j$

$$\sum_{A_j}n_k\prod_{i=1}^{j}\frac{(\lambda f_i\cdot b_i)^{n}}{n_i}\cdot p_o$$

$$=\sum_{A_j}\delta(n_k)\frac{(\lambda f_k\cdot b_k)(\lambda f_k\cdot b_k)^{n_k-1}}{(n_k-1)\,!}\cdot\prod_{\substack{i=1\\i\neq k}}^{j}\frac{(\lambda f_i\cdot b_i)^{n_i}}{n_i\,!}\cdot P_o$$

$$= \lambda f_k \cdot b_k \sum_{A_{j,k}} \prod_{i=1}^{j-k} \frac{(\lambda f_i \cdot b_i)^{n_i}}{n_i\,!} \cdot P_0$$

$$= \lambda f_k \cdot b_k \cdot P_{j-k}$$

Thus, we have

$$P_j = \frac{\lambda}{j}\,[f_1 \cdot b_1 \cdot P_{j-1} + 2f_2 \cdot b_2 \cdot P_{j-2} + \cdots + jf_j \cdot b_j \cdot P_0].$$

2. Proof of $f^{i*}(n) = H^i(0, n)$ and $\sum_{j=S}^{\infty} f^{i*}(j) = H^i(0, S)$

We shall prove this by mathematical induction. Clearly, it holds for $i=0$ and $i=1$. Suppose for y > 1

$$f^{y*}(n) = H^y(0, n) \qquad for\ n = 0, 1, \cdots, S-1$$

$$\sum_{j=S}^{\infty} f^{y*}(j) = H^y(0, S).$$

We must show that it holds for $i=y+1$.
For $n=0,\ 1\cdots\cdots,\ S\text{-}1$

$$f^{y+1*}(n) = \sum_{j=0}^{n} f^{y*}(j) \cdot f(n-j)$$

$$= \sum_{j=0}^{S} H^y(0, j) \cdot H(j, n) \quad (\because H(j, n) = 0\ for\ n \leq j \leq S)$$

$$= H^{y+1}(0, n)$$

$$\sum_{j=S}^{\infty} f^{y+1*}(j) = \sum_{j=S}^{\infty} \sum_{m=0}^{j} f^{y*}(m) \cdot f(j-m)$$

$$= \sum_{j=S}^{\infty} [f^{y*}(1) \cdot f(j-1) + f^{y*}(2) \cdot f(j-2) + \cdots$$

$$+ f^{y*}(S-1) \cdot f(j-S+1) + f^{y*}(S) \cdot f(j-S) + \cdots]$$

$$= f^{y*}(1) \cdot \beta_{S-1} + f^{y*}(2) \cdot \beta_{S-2} + \cdots + f^{y*}(S-1) \cdot \beta_1 + [f^{y*}(S) + f^{y*}(S+1) + \cdots] \cdot 1$$

$$= \sum_{j=0}^{S} H^y(0, j) \cdot H(j, S) = H^{y+1}(0, S).$$

Hence, we complete the proof.

Reference

1. Adelson, R. M., "Compound Poisson Distributions," *Opns. Res. Quarterly,* Vol. 17 (1966), pp. 73-75.
2. Barrer, D.Y., "Queueing with Impatient Customers and Ordered Service," *Opns. Res.,* Vol. 5 (1957), pp. 650-656.
3. Chandrasekhar Das, "The (S-1, S) Inventory Model under Time Limit on Backorders," *Opns. Res.,* Vol. 25 (1977), pp. 835-850.
4. Fakinos, D., "The M/G/K Group-Arrival Group-Departure Loss System," *J. Appl. Prob.,* Vol. 19 (1982), pp. 826-834.
5. Feeney, G.J. and Sherbrooke, C.C., "The (S-1, S) Inventory Policy under Compound Poisson Demand," *Management Science,* Vol. 12 (1966), pp. 391-411.
6. Graves, S. C., "A Multi-Echelon Inventory Model for A Reparable Item with One-For-One Replenishment," *Management Science,* Vol. 31 (1985), pp. 1247-1256.
7. Hadley, G. and Whitin, T., *Analysis of Inventory Systems,* Prentice-Hall, Englewood Cliffs, N. J., 1963, pp. 204-212.
8. Nahmias, s., "Managing Reparable Inventory Systems : A Review," *TIMS Studies in the Management Sciences* 16 (1981), pp. 253-277.
9. Sherbrooke, C.C., "METRIC : A Multi-Echelon Technique for Recoverable Item Control," *Opns. Res.,* Vol. 16 (1968), pp. 122-141.
10. Takace, L., "On Erlang's Formula," *Ann. Math. Statistics,* Vol. 40 (1969), pp. 71-78.