The Equivalence of t-wise and Pareto Optimality: A Generalization*

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We study the optimality of allocations obtained in an economy in which agents are not coordinated by a single consistent system like Walrasian auctioneer. In order to exploit all the opportunities for mutually beneficial trade, people must find a way to coordinate themselves beyond the limitation set by the incomplete market structure. We will consider economies whose market structure does not necessarily permit a full coordination, and completely characterize the condition which must be satisfied to guarantee a Pareto optimal outcome. Our result generalizes previous equivalence conditions.

I. Introduction

In this paper, we study the optimality of allocations obtained in an economy in which agents are not coordinated by a single consistent system like Walrasian auctioneer. In the real world, people are separated by location and time, and the market system is not centralized. In order to exploit all the opportunities for mutually beneficial trade, people must find a way to coordinate themselves beyond the limitation set by the incomplete market structure. In this paper, we will consider economies whose market structure does not necessarily permit a full coordination, and completely characterize the condition which must be satisfied to guarantee a Pareto optimal outcome.

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Such a characterization will serve as a useful tool for further research in this area. It is an extension of the previous work done by Feldman (1973), Rader (1976), and Goldman and Starr (1982), to name a few. The most general formulation was given by Goldman and Starr. They considered an economy where coordination is limited to groups of t or fewer individuals. In such an economy, the resulting allocation may not necessarily by Pareto optimal. An allocation is *t-wise optimal* if it is impossible to find a Pareto improving allocation by rearranging only t or fewer individuals' allocations. If the economy works efficiently, t-wise optimal allocation will be obtained. A sufficient condition and a necessary condition for t-wise optimality to imply Pareto optimality were given by Goldman and Starr, but there was shown to exist a gap between them.

Our goal in this paper is to generalize the coordination structure so that coordination is not limited by the "size" of groups but by more fundamental factors like location or time, and at the same time to find a single sufficient and necessary condition which guarantees Pareto optimality.

The rest of the paper is organized as follows. In the next section, we introduce the model. In sections III and IV, we state the main theorem and provide the proof. In section V, we compare our result with the condition given by Goldman and Starr. Section VI concludes this paper.

II. The Model

We will employ the standard representation of a linear economy, by which we are referring to linear consumer preferences, but the reader can also interpret these as linear approximations to more general differentiable utility functions at the specified consumption bundles.

There are H consumers and L commodities indexed by h and l respectively. Consumers have allocations of commodities which are described by an $L \times H$ nonnegative matrix A the h-th column of which is consumption bundle and will be referred to as a^h. Consumer preferences will be presented in another $L \times H$ matrix $P = [P_l^h]$ where h's utility value of a bundle $x \in \mathbf{R}^L$ is $p^h x$. Furthermore without loss of generality we will assume that $\sum_{h=1}^{H} p^h \gg 0$.

We can assume that this allocation has been arrived at through some trading mechanism which could be markets or barter among coalitions, the specific one is not important. What is important are the characteristics which the trading mechanism insures.

The characteristics of allocations which are of current import is their "degree" of optimality. For example the allocation A is Pareto optimal when consumers have preferences P if there is no feasible trade in the set described by the $L \times H$ matrices

$$\mathbf{Z}(A) = \{Z : A + Z > 0 \text{ and } \sum_{h=1}^{H} z^h = 0\},$$

which improves the position of one consumer without harming another. Or more formally,

Definition
Given consumer preferences P, an allocation A is *Pareto optimal* if it is impossible to find $Z \in \mathbf{Z}(A)$ where $p^h z^h \geq 0$ for all h with strict inequality at least once.

The reader can see that the columns of any $Z \in \mathbf{Z}(A)$ are the net trades of consumer h. Within the set $\mathbf{Z}(A)$ are those which can be accomplished without full coordination, i.e., which actively involve less than H consumers. Let $\mathbf{I}^*$ be the set of all possible coalitions, i.e., the set of non-empty subsets of $\{1, 2, \ldots, H\}$. For $I \in \mathbf{I}^*$, we say $Z \in \mathbf{Z}(A)$ is *I-feasible* if Z can be accomplished without consumers outside I participating, i.e., if $z_h = 0$ for $h \notin I$. For any non-empty subset $\mathbf{I}$ of $\mathbf{I}^*$, we say a trade Z is $\mathbf{I}$-feasible, if it is I-feasible for some $I \in \mathbf{I}$. We denote by $\mathbf{Z}(A; \mathbf{I})$ the set of $\mathbf{I}$-feasible trades. We can now define optimality with respect to an incomplete trading system represented by $\mathbf{I} \neq \mathbf{I}^*$.

Definition
Given consumer preferences P, an allocation A is $\mathbf{I}$-*efficient* if it is impossible to find Z in $\mathbf{Z}(A; \mathbf{I})$ such that $p^h z^h \geq 0$ for all h with strict inequality at least once. When $\mathbf{I} = \{I \in \mathbf{I}^* : \# I \leq t\}$, A is also said to be *t-wise optimal.*

One can imagine trading mechanisms in which some groups of individuals (represented by $\mathbf{I}$) can trade amongst themselves, while coordination of other groups are too costly to obtain. Assuming that these groups are in a cooperative mood and are not otherwise constrained, such a mechanism would yield an allocation which is $\mathbf{I}$-efficient. But is there still "waste" in this economy? A standard example shows that this is possible.

(Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$
A is pairwise optimal, but not 3-wise optimal. cf. Feldman 1973). The next question then is under what condition(s) **I**-efficiency is enough to guarantee full Pareto optimality. This question has been studied by Feldman (1973), Ostroy and Starr (1974), Rader (1976), and Goldman and Starr (1982) in the context of pairwise and t-wise optimality. We will investigate this question in a more general context (**I**-efficiency) and give a complete characterization, whereas in the previous literature sufficient conditions and necessary conditions are handled separately.

To answer this question, it would seem that we need more explicit information about consumer preferences. However, it can be shown that for any allocation A there are conditions which guarantee the equivalence of **I**-efficiency and Pareto optimality independent of the preferences. Our next goal is to characterize the necessary and sufficient condition for the equivalence.

III. Equivalence Theorem

We begin with the following simple observation.

Observation
An **I**-efficient allocation A is Pareto optimal (with respect to any preferences) if $\mathbf{Z}(A) = \mathbf{Z}(A\,;\mathbf{I})$.

Two questions naturally arise: 1) Is there a simpler way to check the equivalence? 2) Is the converse true, namely, is it true that the equivalence of **I**-efficiency and Pareto optimality implies the equality of $\mathbf{Z}(A)$ and $\mathbf{Z}(A\,;\mathbf{I})$? The answer to the first question is "Yes". In the following, we will develop a simple method to check the equivalence. This is done by introducing and utilizing a simple concept, which we call circular trade pattern. The answer to the second question is "No". A counter example is given in Section V. However, we will provide a condition in terms of circular trade patterns (to be defined below) which guarantees the equivalence of the two efficiency concepts. This condition can be regarded as a generalization of the elementary principle that an allocation is Pareto optimal if any pair of agents have the same marginal rate of substitution between any two commodities. We provide a condition which applies even when some agents are not consuming some commodities at all.

We start with the observation that in order to check optimality we only need to check circular feasible trades.[1] Let us first define a circular trade.

Definition
A trade $Z \in \mathbf{Z}(A)$ is *circular* if there exist a set of consumers $\{h_1, \ldots, h_t\}$ and a set of commodities $\{l_1, \ldots, l_t\}$, $t \geq 2$, such that

$$\text{(i) } z_{l_i}^{h_i} > 0,\ z_{l_{i+1}}^{h_i} < 0,\ i = 1, \ldots, t\ (t+1 \equiv 1),$$

$$z_l^h = 0, \text{ otherwise, and}$$

$$\text{(ii) } h_i \neq h_{i'} \text{ and } l_i \neq l_{i'} \text{ if } i \neq i'.$$

We need one more term in order to state the forementioned observation.

Definition
A trade $Z \in \mathbf{Z}(A)$ is called a *transfer*, if there exist a commodity j and two consumers k, k' such that $z_j^k < 0$, $z_j^{k'} > 0$, $c_l^h = 0$ for $l \neq j$ and $h \neq k, k'$.

Now we can state the forementioned observation as the following lemma.

Lemma 1
Suppose $p_l^h > 0$ whenever $a_l^h > 0$. Then there exists an I-feasible Pareto improving trade if and only if there exists either a circular I-feasible Pareto improving trade or a I-feasible Pareto improving transfer.

The proof is similar to the one in Ostroy and Starr (1974, p. 1101), and omitted here. (The idea is very simple. When we have a Pareto improving trade, we can obtain another Pareto improving trade with fewer people by subtracting a circular trade in which no one's utility is changed. By continuing this process we can eventually get a Pareto improving circular trade or transfer.) We can further simplify the analysis by focusing on circular trade patterns rather than particular trades. Circular trade patterns are a representation of the trading consumers and the commodities involved in circular trades without reference to quantity.

[1]The first thought in this line is due to Ostroy and Starr (1974). See p. 1101 of their paper.

Definition

An $L \times H$ matrix $C = [c_l^h]$ is a *circular trade pattern* (CTP) if there exist a set of consumers $\{h_1,..., h_t\}$ and a set of commodities $\{l_1,..., l_t\}$ such that

$$\text{(i) } c_{l_i}^{h_i} = 1,\ c_{l_{i+1}}^{h_i} = -1,\ i = 1,..., t\ (t+1 = 1),$$
$$c_l^h = 0, \text{ otherwise, and}$$
$$\text{(ii) } h_i \neq h_{i'} \text{ and } l_i \neq l_{i'} \text{ if } i \neq i'.$$

We say a circular trade pattern is *feasible* if $a_{l_{i+1}}^{h_i} > 0$, $i = 1,..., t$, i.e., if each individual who is supposed to give out a certain commodity actually owns some of that commodity. A circular trade pattern C is said to be *feasible w.r.t. I*, where $I \subset \{1,..., H\}$, if it is feasible and, $c_l^h = 0$ for $h \notin I$. We can define transfer patterns in a similar way.

Definition

An $L \times H$ matrix $C = [c_l^h]$ is called a *transfer pattern* if there exist a commodity j and two consumers k, k' such that $c_j^k = -1$, $c_j^{k'} = 1$, $c_l^h = 0$ for $l \neq j$ and $h \neq k$, k'. It is *feasible* if $a_j^k > 0$. It is said to be *feasible w.r.t. I* if it is feasible and $c^h = 0$ for $h \notin I$.

We can now completely characterize the equivalence of Pareto optimality and **I**-efficiency.

Theorem

Pareto optimality is equivalent to **I**-efficiency, if and only if
(i) every feasible circular trade pattern can be represented as a positive linear combination of circular trade patterns each of which is feasible w.r.t. some $I \in \mathbf{I}$, and
(ii) every feasible transfer pattern can be represented as a sum of transfer patterns each of which is feasible w.r.t. some $I \in \mathbf{I}$.

What do these conditions mean? First of all, condition (ii) requires the market structure to be rich enough to allow all feasible transfers. If this condition is not satisfied, then someone might end up with goods which have no value to him but useful to someone else in the economy. Condition (i) essentially means that separation is all right as long as every circular trade can be accomplished as a sum of several local trades. Some agents might have to serve as intermediaries. The condition implicitly requires the intermediaries to hold enough number of goods. Otherwise, some transaction might not be feasible, and Pareto optimality might not be guaranteed. These goods are used to connect several "local" markets. As earlier literature shows it is enough to have one commodity (interpreted as money) if it is held by everyone. In our generalization, not everyone needs to hold the same commodity. In a sense, we can generate Pareto optimal outcome with several local monetary systems as long as they permit full coordination collectively.

Clearly this theorem does provide an algorithm by which the equivalence can be tested. Remember that there are only a finite number of CTP's. Notice also that if we put $\mathbf{I} = \{I \in \mathbf{I}^{*}: \# I \leq t\}$, then the theorem gives a necessary and sufficient condition for equivalence of Pareto and t-wise optimality. The criterion stated in the theorem then has a close relationship with seemingly quite different condition put forward by Goldman and Starr (1982). We will further investigate this point in Section V. In the next section we will first prove the theorem.

IV. Proof of the Theorem

To facilitate the proof, we first introduce a function which transforms an optimality property into numbers.

Definition
When consumer preferences are described by P, the *value* of a CTP C, denoted by $v(C\,; P)$, is defined by

$$v(C\,;P) = \prod_{i=1}^{t} \frac{p_{l_i}^{h_i}}{p_{l_{i+1}}^{h_i}}, \text{ if } p_{l_{i+1}}^{h_i} > 0 \text{ for all } i = 1,\ldots, t$$

$$\infty, \text{ otherwise.}$$

The following lemma illustrates how the transformation works.

Lemma 2
(i) If a circular trade is feasible and Pareto improving, then the corresponding circular trade pattern is feasible and has value larger than 1.
(ii) If a circular trade pattern is I-feasible and has a finite value larger than 1, then there corresponds a circular trade which is I-feasible and Pareto improving.

Proof: (i) Feasibility part is clear. Next, if $p_{l_{i+1}}^{h_i} = 0$ for some i then $v(C\,;P) = \infty$ and the condition is satisfied. Suppose $p_{l_{i+1}}^{h_i} > 0$ for all i. Denote the circular trade by $[z_l^h]$. Then,

$$z_{l_{i+1}}^{h_{i+1}} = -z_{l_{i+1}}^{h_i} \leq \frac{p_{l_i}^{h_i}}{p_{l_{i+1}}^{h_i}} z_{l_i}^{h_i} \quad \text{for all } i$$

with strict inequality for some i. Without loss of generality we can assume $z_{l_1}^{h_1} \neq 0$. Then

$$z_{l_1}^{h_1} < \left[\prod_{i=1}^{t} \frac{p_{l_i}^{h_i}}{p_{l_{i+1}}^{h_i}} \right] z_{l_1}^{h_1}$$

and the condition follows.

(ii) Since $1 < v(C\,;P) < \infty$, $p_{l_i}^{h_i}$ and $p_{l_{i+1}}^{h_i}$ are both positive for all i. Define a circular trade $Z = [z_l^h]$ as follows,

$$\begin{aligned}
z_{l_1}^{h_1} &= 1 \\
z_{l_{i+1}}^{h_i} &= -\frac{p_{l_i}^{h_i}}{p_{l_{i+1}}^{h_i}} z_{l_i}^{h_i}, \quad i = 1,\dots, t-1 \\
z_{l_{i+1}}^{h_{i+1}} &= -z_{l_{i+1}}^{h_i}, \quad i = 1,\dots, t-1 \\
z_{l_1}^{h_t} &= -1 \\
z_l^h &= 0, \text{ otherwise.}
\end{aligned}$$

Then,

$$\begin{aligned}
p^h z^h &= 0, \text{ for } h \neq h_t, \text{ and} \\
p^{h_t} z^{h_t} &= p_{l_t}^{h_t} z_{l_t}^{h_t} + p_{l_1}^{h_t} z_{l_1}^{h_t} \\
&= p_{l_1}^{h_t} \left[\prod_{i=1}^{t} \frac{p_{l_i}^{h_i}}{p_{l_{i+1}}^{h_i}} - 1 \right] > 0.
\end{aligned}$$

By choosing ε sufficiently small and positive, we can get a feasible trade εZ which is Pareto improving.

Q.E.D.

Given an allocation A, preference profile P' is *value-preserving* transformation of P if $v(C\,;P') \gtreqless 1 \Leftrightarrow v(C\,;P) \gtreqless 1$, for each feasible CTP C. Efficiency properties of an allocation are the same as long as two preferences are value-preserving transformations of each other. The following lemma simplifies the proof of the Theorem later.

Lemma 3

Let consumer preferences be described by P. If $v(C\,;P) < \infty$ for every feasible circular trade pattern C, then there is a value-preserving transformation P' with $P' \gg 0$.

Proof: Suppose that P has a zero entry $p_n^m = 0$. Define the set

$$\mathbf{D} = \{C : C \text{ is feasible circular trade pattern and } c_n^m = \pm 1\}.$$

Since $\nu(C; P) < \infty$, $C \in \mathbf{D}$ implies $c_n^m = 1$ and $\nu(C; P) = 0$. Hence, we can replace p_n^m by $p'^m_n > 0$ small enough so that $\nu(C; P') < 1$ for all $C \in \mathbf{D}$, where P' denotes the matrix obtained when we replace p_n^m by p'^m_n. By repeating this process if necessary, we can eventually obtain a matrix which preserves the value and has no zero elements.

Q.E.D.

The following notation will simplify the proof. Given $L \times H$ matrices A and B, denote the inner product by AB, i.e., $AB = \sum_{h,l} a_l^h b_l^h$. Given an $L \times H$ matrix P, define $\ln P = [\ln p_l^h]$. Then, it is clear that

$$\ln \nu(C; P) = C \ln P, \tag{1}$$

Where C is a CTP. Now we can present the proof.

Proof of the Theorem

(Sufficiency) Suppose, to the contrary, that **I**-efficiency does not imply Pareto optimality i.e., there exists a preference matrix P for which the allocation is **I**-efficient but not Pareto optimal. **I**-efficiency together with (ii) implies $p_l^h > 0$ if $a_l^h > 0$. Hence, $\nu(C; P) < \infty$ for every feasible CTP C. Then, by Lemmas 1 and 2, there exists a circular trade pattern C with value > 1 (transfer cannot be Pareto improving if $P_l^h > 0$ whenever $a_l^h > 0$), whereas all CTP's feasible w.r.t. **I** have value ≤ 1. By Lemma 3, we may assume that $P \gg 0$. Suppose $C = \sum_k \alpha_k C_k$, $\alpha_k \geq 0$ for all k, where each C_k is feasible w.r.t. **I**, then,

$$\begin{aligned} 0 < \ln \nu(C; P) &= C \ln P \quad \text{by (1)} \\ &= (\sum_k \alpha_k C_k) \ln P \\ &= \sum_k (\alpha_k C_k \ln P) \\ &= \sum_k \alpha_k \ln \nu(C_k; P) \leq 0, \end{aligned}$$

a contradiction.

(Necessity) Let $\{C_k\}$ be the set of all CTP's feasible w.r.t. **I**. Suppose, to the contrary to (i), that there exists a feasible circular trade pattern C outside of the convex cone generated by $\{C_k\}$ in the set of $L \times H$ matrices. Then, we can find an $L \times H$ matrix μ satisfying $C_k \mu \leq 0$, for all k and $C\mu > 0$. (by the separating hyperplane theorem). From these values of μ, define consumer preferences P by $p_l^h = \exp(\mu_l^h)$. When these are the consumer preferences, $v(C_k\,; P) \leq 1$ for all k, while $v(C\,; P) > 1$. So this allocation is **I**-efficient (by Lemma 2 (i)) but not Pareto optimal (by Lemma 2 (ii)), a contradiction.

Next, Suppose that (ii) does not hold; i.e., there is a transfer pattern $C = [c_l^h]$ and k, k', j such that $c_j^k = -1$, $c_j^{k'} = 1$, $c_l^h = 0$ if $h \neq k$, k' or $l \neq j$, with $a_j^k > 0$, which cannot be represented as a sum of transfer patterns feasible for **I**. Define

$$H_j = \{h \in H : h = k \text{ or there are } h_2,\ldots,h_t \text{ such that } h_1 = k, \\ h_{t+1} = h, \text{ and } \{h_i, h_{i+1}\} \in \mathbf{I},\ a_j^{h_i} > 0 \text{ for } i = 1, 2,\ldots, t.\}$$

In other words, H_j is the set of consumers to whom commodity j can be delivered through a sequence of **I**-feasible transfers starting with k giving out j. Then, "$h \in H_j$" implies "$\{h, k'\} \notin \mathbf{I}$" (otherwise, C and be represented as a sum of transfers feasible w.r.t. **I**). Define consumers' preferences as follows:

$$p_l^h = 1 \text{ if } a_l^h > 0 \text{ and } (h, l) \notin H_j \times \{j\} \text{ or if } (h, l) = (k', j)$$
$$p_l^h = 0, \text{ otherwise.}$$

Under this preference configuration, commodity j is useless among consumers H_j, and any consumer who can deliver j to k' cannot receive j through a sequence of deliveries from H_j. Notice that no circular trade can be Pareto improving unless it involves consumer in H_j giving out commodity j, since $p_l^h / p_{l'}^h \leq 1$ if $a_{l'}^h > 0$ for (h, l') $\notin H_j \times \{j\}$ (cf. Lemma 2 (i)). It should also involve someone $h \notin H_j$ receiving commodity j, since by construction $p_j^h = 0$ for $h \in H_j$. This contradicts the definition of H_j unless the circular trade is not feasible w.r.t. **I**. Any non-circular Pareto improving trade must also involve someone in H_j giving out j, because $p_l^h > 0$ (actually $= 1$) if $a_l^h > 0$ and $(h, l) \notin H_j \times \{j\}$. It must also involve someone outside H_j receiving commodity j. This cannot be done through transfers feasible w.r.t. **I**. Thus the allocation is **I**-efficient. However, obviously it is not Pareto optimal.

Q.E.D.

Remark

Condition (ii) is not redundant as the following example shows.

Example:
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \mathbf{I} = \{\{1,2\},\{1,3\}\}$$

In this economy (i) is satisfied, A is **I**-efficient, but not Pareto optimal.

V. Comparison with the Goldman-Starr Condition

At first glance, our theorem looks quite different from the Goldman-Starr characterization. Let us first define a new terminology. An allocation A is said to satisfy *t-wise equivalence property* if t-wise optimality implies Pareto optimality *for any preferences*. Let us now state their sufficiency theorem.

Theorem GS[2]
An allocation matrix which may be constructed by successive "extensions" of the 1×1 matrix $A = [1]$ by rows or columns will satisfy t-wise equivalence property.

This theorem follows immediately from the next lemma, where the specific extension is defined. Before we state the lemma, we need to define some new terminology. Given an L-vector a, a^+ denotes the set of indices i such that $a_i > 0$. Given an $L \times L$ matrix S and a set $J \subset \{1, \dots, L\}$, define $[S \text{ proj } J]$ as the $\#J \times \#J$ matrix $\hat{S} = [\hat{s}_i^j]$ such that, for all $i, j \in J$, $\hat{s}_i^j = s_i^j$. A symmetric matrix is said to be *reducible* if the rows and columns can be identically rearranged so that the resulting matrix is block diagonal. If such a rearrangement is impossible, then the matrix is *irreducible.*

Lemma GS[3]
Let A be an allocation matrix which satisfies t-wise equivalence property and let a be an individual allocation vector such that $[(AA')^{t-1} \text{ proj } a^+ \cup \{j\}]$ for all $j \leq L$ is irreducible. Then A augmented by a (called an "extension" of A) exhibits the t-wise equivalence property.

What does Lemma GS imply in terms of our condition given in section II? As we will show in the rest of this section, it is one way to guarantee that the augmentation of A satisfies condition (i) of our

[2]Theorem 5.1 of Goldman and Starr (1982).
[3]Lemma 5.1 of Goldman and Starr (1982).

theorem. (Notice that condition (ii) is automatically satisfied when all the pairwise trades are allowed.) Before going into the details, let us first state some of the fundamental facts we will use later. One can successively prove the following.

Fact 1
ij-element of AA' is positive if and only if there is some consumer holding both i and j.

Fact 2
ij-element of $(AA')^{t-1}$ is positive if and only if there are t goods and $(t-1)$ consumers such that

(i) $i = l_1$ and $j = l_t$,

(ii) h_s holds both l_s and l_{s+1}, $s = 1,\ldots, t-1$.

Fact 3
$[(AA')^{t-1} \text{ proj } J]$ is irreducible if and only if, for any $i, j \in J$, there exist $l_1,\ldots, l_n \in J$ such that

$$\begin{aligned} &il_1\text{-element of } (AA')^{t-1} \text{ is positive,} \\ &l_1l_2\text{-element of } (AA')^{t-1} \text{ is positive,} \\ &\qquad\cdots \\ &l_nj\text{-element of } (AA')^{t-1} \text{ is positive.}^4 \end{aligned} \tag{A}$$

Now suppose that an allocation matrix A satisfies t-wise equivalence property. Suppose also that A together with an L-vector a satisfies the condition stated in Lemma GS. Let us consider a feasible CTP C in the augmented economy. Without loss of generality we may assume that C involves the consumer h^* whose endowment is a. (Otherwise, C could be decomposed into t-wise feasible CTP's.) Denote by i the commodity that h^* is giving out in C. Then, $i \in a^+$. Let j be the good that h^* is receiving in C. Irreducibility of $[(AA')^{t-1} \text{ proj } a^+ \cup \{j\}]$ implies that there are commodities $l_1,\ldots, l_n \in a^+$ such that (A) holds (cf. Fact 3). Fact 2 then implies that, for each l_m, $m = 0, 1,\ldots, n, n+1$ ($l_0 \equiv i$, $l_{n+1} \equiv j$), there are t commodities $l_1^m,\ldots, l_t^m$ and $(t-1)$ consumers such that $h_1^m,\ldots, h_{t-1}^m$ such that $l_m = l_1^m$, $l_{m+1} = l_t^m$ and h_s^m holds both l_s^m and l_{s+1}^m, $s = 1,\ldots, t-1$ (See Figure 1). Thus, C can be decomposed into two CTP's,

[4]To see Fact 3, define an equivalence relation $\sim$ on J; $i \sim j$ iff there exist $l_1,\ldots, l_n \in J$ such that (A) holds. Then, $[(AA')^{t-1} \text{ proj } J]$ is irreducible if and only if J is the single equivalence class.

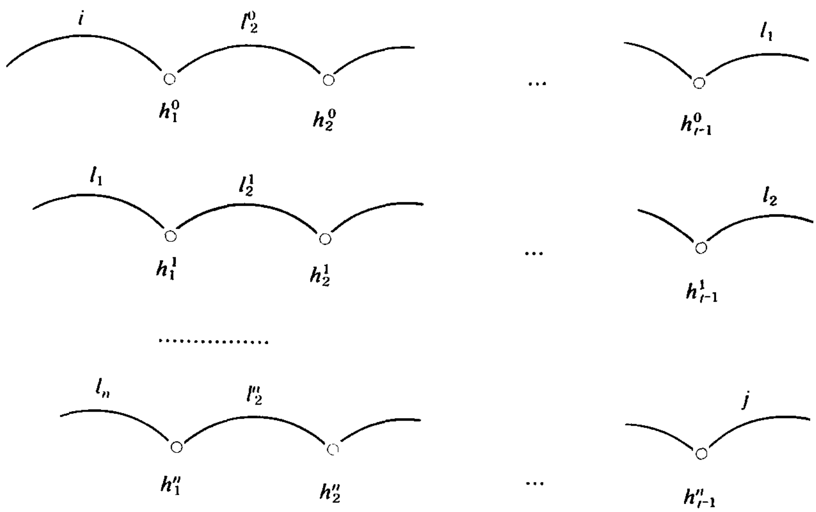

FIGURE 1

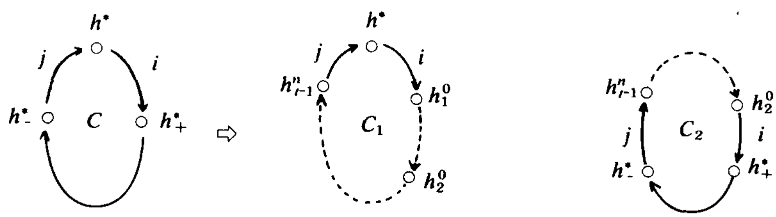

FIGURE 2

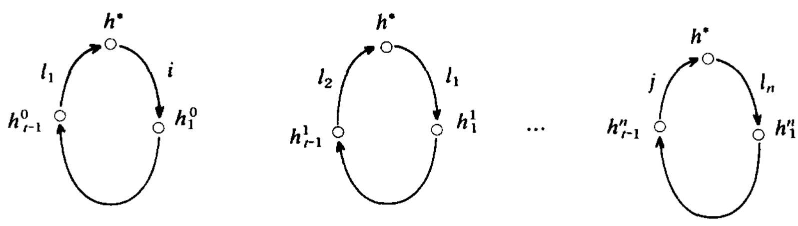

FIGURE 3

C_1 and C_2 as in Figure 2. Since C_2 does not involve h^* as an active trader, C_2 can be represented as a sum of t-wise feasible CTP's. C_1 can also be represented as a sum of t-wise feasible CTP's each of which involves h^*, because $l_m \in a^+$ for all $m = 0, 1, \ldots, n$ (See Fi-

TABLE 1

	c_1^1	c_2^1	c_3^1	c_4^1	c_1^2	c_2^2	c_3^2	c_4^2	c_1^3	c_2^3	c_3^3	c_4^3	c_1^4	c_2^4	c_3^4	c_4^4
C_I	−1	1				−1	1				−1	1	1			−1
C_{II}	−1	1				−1		1	1		−1				1	−1
C_{III}	−1		1		1	−1					−1	1		1		−1
C_{IV}	−1		1			−1		1		1	−1		1			−1
C_V	−1			1	1	−1				1	−1				1	−1
C_{VI}	−1			1		−1	1		1		−1			1		−1
C_1	−1	1			1	−1										
C_2					−1		1				−1	1	1			−1
C_3					−1			1			−1				1	−1
C_4		−1	1								−1	1		1		−1

gure 3). Hence, A augmented by a satisfies the necessary and sufficient condition of our theorem.

Is the G–S condition also necessary for t-wise equivalence? The answer is "No" as the following example shows.

Example : Consider an economy with $L=4$ and $H=3$, where the allocation matrix is given as

$$A=\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix A trivially satisfies 3-wise equivalence property. Now add one more consumer to this economy who has endowment vector a^4 identical to a^3, the endowment vector of the third consumer in the original economy. The matrix $[(AA')^2 \text{ prob } (a^4)^+ \cup \{1\}]$ is reducible, violating the condition in Lemma GS. On the other hand, this example satisfies our condition. In Table 1, six of 24 4-wise feasible circular trade patterns are given in the first 6 rows. Other 4-wise feasible circular trade patterns can be obtained from these by changing the role of $h=1$ and 2 or $h=3$ and 4. Among those 6 patterns given in the top of Table 1, the first 3 are represented as positive sums of 3-wise patterns;

$$C_I = C_1 + C_2,\ C_{II} = C_1 + C_3,\ C_{III} = C_1 + C_4.$$

The others can also be represented by 3-wise patterns. This example serves as a counterexample mentioned in section III, because C_I

is not feasible w.r.t. any coalition of size less than 4.

VI. Conclusion

In this paper, we considered economies whose market structure does not necessarily permit a full coordination, and completely characterized the condition which guarantees a Pareto optimal outcome. The market structure is represented by a set of coalitions among whose members trade can be coordinated. The condition requires that all feasible trades can be accomplished through a series of trades among the coalition members. In fact, we only need to check circular trades, and hence a rather simple algorithm obtains. The condition we obtain generalizes all the previous conditions, in particular the one obtained by Goldman and Starr (1982). Our characterization permits a simpler and more intuitive condition for Pareto optimality in a more general setup. The message is simple: when market structure itself does not guarantee a full coordination, some people might have to serve as intermediaries and they are required to hold enough number of goods. These goods are used to connect structurally separated markets. Our condition is a necessary and sufficient condition, hence provides a complete characterization of the link between optimality in local markets and full Pareto optimality of an economy.

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