# The Comparative Static Analysis of a Monocentric City: The Indirect Utility Function Approach

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We presented the indirect utility function approach to the comparative static analysis of the open and closed monocentric city. Although our results are essentially identical to those derived in Brueckner (1987), our proofs of the propositions regarding the effect of a change in income and of marginal commuting cost seem more straightforward.

## I. Introduction

The internal structure of long-run equilibrium models of the monocentric city have been extensively studied following the work of Alonso (1964), Muth (1969) and Mills (1967). The Alonso-Muth model has been extended to explicitly consider time (De Salvo 1985), to address the issue of the nonlinear budget constraint (Brown 1985), or to make income of urban residents endogenous (Sasaki 1987; Pines and Sadka 1986).

One important task of research on urban spatial structure is examining how the equilibrium structure of a city changes as a result of a change in income or the commuting cost. The first rigorous comparative static analysis of the Alonso model was carried out by Wheaton (1974), and De Salvo (1977) was the first to present comparative statics results of the Muth model. Each of the authors mentioned in the previous paragraph performed some comparative static analysis.

In a recent paper, Brueckner (1987) presented an excellent exposition of the Muth-Mills model together with a comparative static

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analysis which generalized and improved upon the previous work of Wheaton (1974).

Although some authors used the indirect utility function in studying urban economic models (Kanemoto 1980; Polinski and Shavell 1976; Solow 1973), every paper mentioned above took a direct utility function approach. The purpose of this paper is to demonstrate that comparative statics results can be derived in a more straightforward manner by using the indirect utility function.

## II. The Model

The monocentric model we work with is attributable to Alonso (1964). There are two types of the monocentric model, the closed city model and the open city model. Both types consist of an identical set of equations, but the structure of these equations is different as the different sets of variables are treated as endogenous in each type. In the open city model, the utility level is exogenous and the population size is endogenous, and vice versa in the closed city model. As a result, the open city model is a recursive system whereas the closed city model consists of a system of simultaneous equations.

Residents with identical preference and income $(y)$ consume land $(q)$ and the composite commodity $(c)$. $q$ and $c$ are assumed to be a normal goods. The price of the composite commodity is normalized as one and the rent per unit of land is $p$. Each resident incurs marginal commuting cost $(t)$ for each mile of roundtrip, and $x$ measures the distance from the Central Business District (CBD).

The level of utility which a resident living at $x$ obtains is defined by

$$V(p(x), y - tx) = V(p(x), I(x)) = u \tag{1}$$

where, $V(\cdot)$ is the indirect utility function which is non-decreasing in $I$, income net of commuting cost, non-decreasing and quasi-convex in $p$. We assume that $V_p = \partial V(\cdot)/\partial p < 0$, $V_I = \partial V(\cdot)/\partial (y - tx) > 0$, $V_{pI} = \partial V_p / \partial (y - tx) = \partial V_I / \partial p \doteq V_{Ip} < 0$, $V_{pp} = \partial V_p / \partial p > 0$ and $V_{II} = \partial V_I / \partial (y - tx) < 0$.[1]

Equation (1) is the locational equilibrium condition which says that every consumer enjoys the same utility level regardless of his

[1]Throughout this paper, the variable with a subscript represents its partial derivative with respect to the variable expressed as the subscript, i.e., $A_a = \partial A/\partial a$.

or her location. From equation (1) the land rent $p$ can be written as $p = p(x, y, t, u)$. The demand for land $q(\cdot)$ can be derived from equation (1) using the Roy's identity.

$$-V_p/V_I = q(x, y, t, u) \tag{2}$$

The demand for the composite commodity $c(x, y, t, u)$ can be obtained from budget constraint.

$$y - tx - pq = c \tag{3}$$

At the boundary of the city, $\bar{x}$, the level of urban rent must be equated with the exogenously given agricultural rent, $r$.

$$p(\bar{x}, y, t, u) = r \tag{4}$$

From equation (4), $\bar{x}$ can be derived as $\bar{x} = \bar{x}(y, t, u, r)$. Finally, since the circular monocentric city should accommodate $N$ residents, the following relationship must hold.

$$\int_0^{\bar{x}} 2\pi x/q \; dx = N \tag{5}$$

In the closed city model, in which $p$, $q$, $c$, $\bar{x}$ and $u$ are endogenous variables, $u$ is determined from equation (5), i.e., $u = u(y, t, N, r)$. In the open city model, $p$, $q$, $c$, $\bar{x}$, and $N$ are endogenous variables, and equation (5) determines $N$, i.e., $N = N(y, t, u, r)$. We will perform a comparative static analysis of an open city in section III, and then move on to analyze a closed city in section IV.

## III. The Comparative Static Analysis of an Open City

Since $u$ is exogenous in the open city model, equations (1) through (5) are independent of each other. Therefore, examination of partial derivatives will be sufficient for determining the effects of the change in exogenous variables on the equilibrium structure of the city.

Partial differentiation of equation (1) with respect to $x$ yields $V_p p_x - tV_I = 0$. Using the Roy's identity, equation (2) can be written as $p_x = -t/q(x)$. Therefore, partial derivatives of $p$ with respect to exogenous variables are:

$$\begin{aligned} p_x &= -t/q(x) < 0, & p_y &= 1/q(x) > 0 \\ p_t &= -x/q(x) < 0, & p_u &= 1/V_p(x) < 0 \end{aligned} \tag{6}$$

Since $q(x, y, t, u) = Q(p, y - tx)$, $q_x = [\partial Q(\cdot)/\partial p]p_x - t$

$\cdot[\partial Q(\cdot)/\partial(y-tx)] = Q_p p_x - tQ_y$. Substituting $(Q_p^* - qQ_y)$ for $Q_p$ using the Slutsky equation and $-t/q(x)$ for $p_x$ using equation (6), $q_x$ can be expressed as $q_x = q_p^* p_x$ where "*" represents the substitution effect.[2]

Similarly, partial derivatives of $q$ with respect to exogenous variables can be derived as follows

$$\begin{aligned} q_x &= q_p^* p_x > 0, \qquad q_y = q_p^* p_y < 0 \\ q_t &= q_p^* p_t > 0, \qquad q_u = q_p^* p_u > 0. \end{aligned} \tag{7}$$

Differentiating equation (3) partially with respect to $x$, $y$, $t$ and $u$, we get

$$\begin{aligned} c_x &= -pq_x < 0, \qquad c_y = -pq_y > 0 \\ c_t &= -pq_t < 0, \qquad c_u = -pq_u - qp_u \gtrless 0. \end{aligned} \tag{8}$$

Differentiating equation (4) partially, we get

$$\begin{aligned} \bar{x}_y &= -[p_y(\bar{x})/p_x(\bar{x})] > 0, \qquad \bar{x}_t = -[p_t(\bar{x})/p_x(\bar{x})] < 0 \\ \bar{x}_u &= -[p_u(\bar{x})/p_x(\bar{x})] < 0, \qquad \bar{x}_r = 1/p_x(\bar{x}) < 0. \end{aligned} \tag{9}$$

Using the Leibnitz's rule,[3] the following partial derivatives of $N$ can be derived from equation (5).

$$\begin{aligned} N_y &= n(\bar{x})\bar{x}_y - \int_0^{\bar{x}} [2\pi x/q(x)^2]\, q_y \, dx > 0 \\ N_t &= n(\bar{x})\bar{x}_t - \int_0^{\bar{x}} [2\pi x/q(x)^2]\, q_t \, dx < 0 \\ N_u &= n(\bar{x})\bar{x}_u - \int_0^{\bar{x}} [2\pi x/q(x)^2]\, q_u \, dx < 0 \\ N_r &= n(\bar{x})\bar{x}_r < 0, \end{aligned} \tag{10}$$

where $n(\bar{x})$ is the number of residents living at the boundary of the city, i.e., $n(\bar{x}) = 2\pi\bar{x}/q(\bar{x})$.

## IV. The Comparative Static Analysis of a Closed City

Since $u$ is endogenous in the closed city model, equations (1) through (5) constitute a simultaneous equation system. Therefore, the comparative static analysis of the changes in exogenous variables must consider both the direct and the indirect effect through

[2]Since the substitution effect is always negative, $Q_p^* < 0$. If land is a normal good, $Q_y > 0$. Therefore, $Q_p = Q_p^* - qQ_y < 0$ and $Q_p < Q_p^*$, if land is a normal good.

[3]Leibnitz's rule indicates that $d[\int_{a(t)}^{b(t)} f(x, t)dx]/dt = [db(t)/dt]f(b(t), t) - [da(t)/dt] f(a(t), t) + \int_{a(t)}^{b(t)}[df(x, t)/dt]dx$. Notice that $N = N(y, t, u, r)$, $\bar{x} = \bar{x}(y, t, u, r)$ and $q = q(x, y, t, u)$ in applying the rule in our analysis.

the change in $u$. For example, $dp/dt = p_t + p_u u_t$ since $p = p(x, y, t, u(y, t, N, r))$.[4]

Let us first analyze the effect of changes in the exogenous variables on $u$. Applying the Leibnitz's rule to equation (5) and noting that $u_y$, $u_t$, $u_r$ and $u_N$ are all independent of $x$, partial derivatives of $u$ with respect to $y$, $t$, $r$ and $N$ can be derived as follows.

$$\begin{aligned} u_y &= [-n(\bar{x})\bar{x}_y + \textstyle\int_0^{\bar{x}}(2\pi x/q(x)^2)q_y\,dx]/A > 0 \\ u_t &= [-n(\bar{x})\bar{x}_t + \textstyle\int_0^{\bar{x}}(2\pi x/q(x)^2)q_t\,dx]/A < 0 \\ u_r &= [-n(\bar{x})\bar{x}_r]/A < 0 \\ u_N &= 1/A < 0. \end{aligned} \tag{11}$$

Here, $A = n(\bar{x})\bar{x}_u - \int_0^{\bar{x}}[(2\pi x/q(x)^2)q_u]dx < 0$. Partial derivatives of $u$ can also be expressed in terms of partial derivatives of $p$.[5]

$$\begin{aligned} u_y &= [-\textstyle\int_0^{\bar{x}} p_y\,dx]/B > 0, \\ u_t &= [(N/2\pi) - \textstyle\int_0^{\bar{x}} p_t\,dx]/B < 0 \\ u_r &= x/B < 0, \\ u_N &= (t/2\pi)/B < 0. \end{aligned} \tag{11$'$}$$

Here, $B = \int_0^{\bar{x}} p_u\,dx < 0$. Since equations (11) and (11)$'$ are equivalent, either set can be used for the comparative static analysis.

Next, we note that income net-of-commuting cost $(y - tx)$ decreases as $x$ increases. Therefore, the marginal utility of income increases with $x$.

*Lemma* 1

$V_{px} = \partial V_p(\cdot)/\partial x < 0$ and $V_{Ix} = \partial V_{yI}(\cdot)/\partial x > 0$.[6]

*Proof*: Differentiating equation (1) partially twice with respect to $x$ noting that $p_x = -t/q(x)$, we get $V_{px} = [q(x)/t][(V_p(x)tq_x/q(x)^2) + V_{II}t^2] < 0$, since $V_{II} < 0$. Next, $V_{Ix} = V_{pI}p_x - V_{II}t > 0$, since $V_I(p, y - tx)$, $V_{pI} < 0$, and $V_{II} < 0$.

*Q.E.D.*

Now, we are ready to analyze the total effect of changes in the exogenous variables.

[4] The expression of $dp(x)/dt$ is used to distinguish the total effect from the partial effect, $p_t = \partial p/\partial t$, which implies the change in $p$ with respect to $t$ when the change in $u$ is not considered.

[5] See Brueckner (1987, Appendix) for the derivation.

[6] Notice that $V_y$ and $V_I$ are equivalent under given $t$ and $x$.

*A. The Effect of a Change in the Marginal Commuting Cost*

The total effect of a change in $t$ upon $p$, $q$, $c$, $\bar{x}$ and $u$ can be expressed as $du/dt = p_t + p_u u_t$, $dq/dt = q_t + q_u u_t$, $dc/dt = c_t + c_u u_t$, $d\bar{x}/dt = \bar{x}_t + \bar{x}_u u_t$ and $du/dt = u_t$, respectively.

The signs of the total derivatives other than $du/dt$ cannot be determined directly. Since the sign of $c_u$ is ambiguous, the sign of $dc/dt$ cannot be determined, either.

The decrease in the after commuting cost income following an increase in $t$ will be the greater the larger $x$ is. Thus, the central locations become more attractive and the suburban locations become more unattractive than before relative to each other. This implies that $p(x)$ rotates in the clockwise direction in order to maintain locational equilibrium.

*Proposition* 1
As the marginal commuting cost increases, the rent function rotates clockwise, i.e., there exists $x^*$ such that $dp(x^*)/dt = 0$, $dp(x)/dt > 0$ for $x \in [0, x^*)$ and $dp(x)/dt < 0$ for $x \in (x^*, \bar{x}]$.

*Proof*: From equations (6) and (11), $dp(x)/dt$ can be written as

$$dp(x)/dt = -[x/q(x)] + [1/V_p(x)]\,[-n(\bar{x})\bar{x}_t + \int_0^{\bar{x}}(2\pi x/q^2)q_t dx]/A.$$

The change in $p$ at CBD as a result of an increase in $t$, $dp(0)/dt$, will be as

$$dp(0)/dt = [1/V_p(0)]\,[-n(\bar{x})\bar{x}_t + \int_0^{\bar{x}}(2\pi x/q^2)q_t dx]/A > 0.$$

Using the definition of $A$, the change in $p$ at the urban boundary, $dp(\bar{x})/dt$, can be written as follows.

$$\begin{aligned} dp(\bar{x})/dt = [1/V_p(\bar{x})A]\,[&-(\bar{x}/q(\bar{x}))V_p(\bar{x})n(\bar{x})\bar{x}_u \\ &- n(\bar{x})\bar{x}_t + (x/q(\bar{x}))V_p(\bar{x})\int_0^{\bar{x}}(2\pi x/q^2)q_u dx \\ &+ \int_0^{\bar{x}}(2\pi x/q^2)q_t dx] \end{aligned}$$

From the definitions of $p_x$, $p_t$, $p_u$ in (6) and $\bar{x}_t$ and $\bar{x}_u$ in (9), it follows that

$$-(\bar{x}/q(\bar{x}))V_p(\bar{x})n(\bar{x})\bar{x}_u - n(\bar{x})\bar{x}_t = 0.$$

Thus, $dp(\bar{x})/dt$ can be rewritten as follows using equation (7).

$$dp(\bar{x})/dt = [1/V_p(\bar{x})A]\,\{\int_0^{\bar{x}}(2\pi x^2/q^3)$$

$$\cdot\,[(\bar{x}/x)(V_I(\bar{x})/\,V_I(x))Q_p - Q_p^*]\,dx\}\,.$$

Since $\bar{x} > x$ and $V_{Ix} > 0$ (Lemma 1), $V_I(\bar{x}) > V_I(x)$. Also, since $Q_y > 0$, $Q_p < Q_p^* < 0$. Therefore, $dp(\bar{x})/\,dt < 0$.

Partially differentiating $V(\,p(x, y, t, u), y - tx) = u(y, t, r, N)$ with respect to $t$, we get $V_p(\,p_t + p_u u_t) - xV_I = u_t$. Thus.

$$dp(x)/\,dt = P_t + P_u u_t = [1/\,V_p(x)]\,(u_t + xV_I).$$

Since $V_p(x) < 0$ for all $x$ and $u_t < 0$ is not dependent upon $x$, the sign of $dp(x)/\,dt$ is dependent upon the size of $xV_I(x) > 0$. Since $dp(0)/\,dt > 0$ and $dp(\bar{x})/\,dt < 0$, $u_t + 0V_I(0) = u_t < 0$ and $u_t + \bar{x}V_I(\bar{x}) > 0$. This, together with the fact that $xV_I(x)$ is monotone increasing by Lemma 1, implies that there exists unique $x^*$ such that $u_t + x^*V_I(x^*) = 0$ and thus $dp(x^*)/\,dt = 0$.

$d[dp(x)/\,dt]\,/\,d^2x = [1/\,V_p(x)^2]\;\{[V_I + xV_I(x)]\,V_p(x) - V_{px}(x)\,[u_t + xV_I(x)]\}$. If $x \in [0, x^*]$, $d[dp(x)/\,dt]\,/\,d^2x < 0$ since $u_t + xV_I(x) \leq 0$. If $x \in (x^*, \bar{x}]$, the sign of $d[dp(x)/\,dt]\,/\,d^2x$ is unclear since $u_t + xV_I(x) > 0$. But, $d[dp(x^*)/\,dt]\,/\,d^2x < 0$, $dp(\bar{x})/\,dt < 0$ and the uniqueness of $x^*$ guarantee that $dp(x)/\,dt < 0$ if $x \in (x^*, \bar{x}]$.

*Q.E.D.*

Since Proposition 1 implies that $p$ decreases at suburban locations as $t$ increases $\bar{x}$ will decrease, i.e., the city shrinks in physical size following an increase in $t$. Therefore, $q$ will decrease in close-in locations because $p$ rises and income net-of commuting cost falls. But, the direction of change in $q$ at suburban locations is ambiguous since both the after commuting cost income and the land rent falls.

*Proposition* 2

As the marginal commuting cost increases, the boundary of the city shrinks and consumption of land decreases in locations near CBD, i.e., $d\bar{x}/dt < 0$; $dq(x)/\,dt < 0$ for $x \in [0, x^*)$ and the sign of $dq(x)/\,dt$ is ambiguous for $x \in (x^*, \bar{x}]$.

*Proof*: From equations (6), (9) and (11) and the fact that $\bar{x}V_p(\bar{x})n(\bar{x})\bar{x}_u + n(\bar{x})q(\bar{x})\bar{x}_t = 0$ (see the Proof of Proposition 1), $d\bar{x}/dt = \bar{x}_t + \bar{x}_u u_t$ can be written as follows.

$$d\bar{x}/dt = [q(\bar{x})/\,t]\,[dp(\bar{x})/\,dt] < 0$$

Using equations (7) and (11), $dq(x)/\,dt = q_t + q_u u_t$ can be expressed as

$$\begin{aligned} dq(x)/dt &= Q_p^* p_t + Q_p p_u u_t \\ &= Q_p^* [(dp(x)/dt) - p_u u_t] \\ &\quad + (Q_p^* - qQ_y) p_u u_t \\ &= Q_p^* [dp(x)/dt] - qQ_y p_u u_t. \end{aligned}$$

For all $x \in [0, x^*)$, $dq(x)/dt < 0$ since $dp(x)/dt > 0$ from Proposition 1, and since $qQ_y p_u u_t > 0$ for all $x \in [0, \bar{x}]$.

*Q.E.D.*

*B. The Effect of a Change in Income*

The total effect of a change in $y$ upon $p$, $q$, $c$, $\bar{x}$ and $u$ is given by $dp/dy = p_y + p_u u_y$, $dq/dy = q_y + q_u u_y$, $dc/dy = c_y + c_u u_y$, $d\bar{x}/dy = \bar{x}_y + \bar{x}_u u_y$ and $du/dy = u_y$ respectively. Signs of these total effects can be determined by following the same procedure as is described in subsection *A* above.

If the marginal utility of income decreases in $y$, the effect of an increase in $y$ of a given amount upon the level of utility will be the greater, the smaller the after commuting cost income is. This implies that the central locations become less attractive and the suburban locations become less unattractive than before relative to each other. Therefore, $p(x)$ will rotate in the counter-clockwise direction to maintain equilibrium.

*Proposition* 3
The rent function rotates counter-clockwise as income increases, i.e., there exists $x^*$ such that $dp(x^*)/dy = 0$, $dp(x)/dy < 0$ for $x \in [0, x^*)$ and $dp(x)/dy > 0$ for $x \in (x^*, \bar{x}]$.

*Proof*: Using equations (6) and (11)', we can derive the following expression for $dp(\tilde{x})/dy$ for any $\tilde{x} \in [0, \bar{x}]$.

$$\begin{aligned} dp(\tilde{x})/dy = &\ \{1/[q(\tilde{x})V_I(\tilde{x})]\}\{1/[\textstyle\int_0^{\bar{x}}(1/V_p(x))dx]\} \\ &\cdot [\textstyle\int_0^{\bar{x}}(1/V_p(x))(V_I(\tilde{x}) - V_I(x))dx]. \end{aligned}$$

The sign of $dp(\tilde{x})/dy$ is determined by the sign of $V_I(\tilde{x}) - V_I(x)$. Since $V_I(x)$ is monotone increasing in $x$, $V_I(0) < V_I(x) < V_I(\bar{x})$. Therefore, $dp(0)/dy < 0$ and $dp(\bar{x})/dy > 0$.

Partial differentiation of $V(p(x, y, t, u), y - tx) = u(y, t, r, N)$ with respect to $y$ yields $V_p(p_y + p_u u_y) + V_I = u_y$ and hence $dp(x)/dy = P_y + P_u u_y = [1/V_p(x)][u_y - V_I(x)]$. Since $dp(0)/dy < 0$, $dp(x)/dy > 0$, $u_y$ is independent of $x$ and $V_I$ is monotone increasing in $x$, there exists $x^*$ such that $dp(x^*)/dy = 0$.

*Q.E.D.*

Following a similar procedure described in the Proof of Proposition 2, we can show that

$$d\bar{x}/dy = [q(\bar{x})/t][dp(\bar{x})/dy] > 0 \tag{12}$$

$$dp(x)/dy = Q_p^*[dp(x)/dy] - qQ_y p_u u_y. \tag{13}$$

Since $qQ_y p_u u_y > 0$, for all $x$, and $dp(x)/dy < 0$ for $x \in [0, x^*)$, $dq(x)/dy > 0$ near the CBD. The sign of $dq(x)/dy$ is ambiguous for $x \in (x^*, \bar{x}]$.

### *C. The Effect of Changes in the Agricultural Rent and Population Size*

The total effect of a change in $r$ upon $p, q, c, \bar{x}$ and $u$ can be summarized as follows.

$$\begin{aligned} &dp/dr = p_u u_r > 0, \quad dq/dr = q_u u_r < 0, \quad du/dr = u_r < 0, \\ &d\bar{x}/dr = (\bar{x}_r/A)[-\textstyle\int_0^{\bar{x}}(2\pi x/q^2)q_u\ dx] < 0. \end{aligned} \tag{14}$$

Equations (9) and (11) are used to derivative the expression for $dx/dr$. The sign of $dc/dr = c_u u_r$ is uncertain since the sign of $c_u$ is ambiguous (see equation (8)).

The total effect of a change in $N$ upon $p, q, c, x$ and $u$ can be summarized as follows.

$$\begin{aligned} dp/dN &= p_u u_N > 0, \qquad dq/dN = q_u u_N < 0, \\ dx/dN &= x_u u_N > 0, \qquad du/dN = u_N < 0. \end{aligned}$$

The sign of $dc/dN = c_u u_N$ cannot be determined as before.

## V. Conclusion

In this paper, we presented an indirect utility function approach to the comparative static analysis of the open and the closed monocentric city. The model we used assumes that residents consume land directly rather than consuming housing service produced out of land and non-land inputs as in the Muth-Mills-Brueckner model. This, however, does not make any real difference.

Although our results are essentially identical to those derived in Brueckner (1987), our proofs of the propositions regarding the effect of a change in income and of the marginal commuting cost seem more straightforward.

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