공학 수학 I

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Lesson 1: Introduction to matrix

- terminologies
- addition and scalar multiplication
- product of matrices
- transpose of a matrix
Matrix (행렬) & Vector (벡터)
행렬(벡터)의 addition & scalar multiplication

\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}, \begin{bmatrix}
-1 & 0 \\
2 & 0
\end{bmatrix}, \begin{bmatrix}
1 \\
2
\end{bmatrix}
\]

합과 스칼라 곱의 연산법칙

For \(A, B, C \in \mathbb{R}^{m \times n}\) and \(c, k \in \mathbb{R}\),

\[
A + B = B + A \\
(A + B) + C = A + (B + C) \\
A + 0 = A \\
A + (-A) = 0
\]

and

\[
c(A + B) = cA + cB \\
(c + k)A = cA + kA \\
c(kA) = (ck)A \\
1A = A
\]
행렬의 곱

\[
\begin{bmatrix}
1 & 2 & 1 \\
3 & 4 & 1
\end{bmatrix}
\begin{bmatrix}
-1 & 0 & 1 \\
2 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix} =
\]
행렬 곱의 연산법칙

For $A, B, C \in \mathbb{R}^{m \times n}$ and $k \in \mathbb{R}$,

$(kA)B = k(AB) = A(kB)$
$A(BC) = (AB)C$
$(A + B)C = AC + BC$
$C(A + B) = CA + CB$

Transposition

$(A^\top)^\top = A$
$(A + B)^\top = A^\top + B^\top$
$(cA)^\top = cA^\top$
$(AB)^\top = B^\top A^\top$
예: 토지의 용도 변경

예: 회전 변경
Lesson 2: System of linear equations, Gauss elimination

- existence and uniqueness of solution
- elementary row operation
- Gauss elimination, pivoting
- echelon form

선형연립방정식 (system of linear equations) & 해 (solution)

\[a_{11}x_1 + \cdots + a_{1n}x_n = b_1\]
\[a_{21}x_1 + \cdots + a_{2n}x_n = b_2\]
\[\vdots\]
\[a_{m1}x_1 + \cdots + a_{mn}x_n = b_m\]
Existence and uniqueness of solution (해의 존재성과 유일성)
해를 구하는 법

\[ x_1 - x_2 + x_3 = 0 \]
\[ 10x_2 + 25x_3 = 90 \]
\[ -95x_3 = -190 \]
\[ 2x_1 + 5x_2 = 2 \]
\[ -4x_1 + 3x_2 = -30 \]

\[ \begin{bmatrix} 2 & 5 & 2 \\ -4 & 3 & -30 \end{bmatrix} \]

1. 두 식의 위치 교환
2. 한 식을 다른 식에 더하기
3. 한 식에 0 아닌 상수 곱하기
4. 한 식을 상수배하여 다른 식에 더하기

1. 두 행의 위치 교환
2. 한 행을 다른 행에 더하기
3. 한 행에 0 아닌 상수 곱하기
4. 한 행을 상수배하여 다른 행에 더하기
Gauss elimination

\[ a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \]
\[ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \]
\[ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \]

Gauss elimination (partial pivoting)

\[ x_1 - x_2 + x_3 = 0 \]
\[ 2x_1 - 2x_2 + 2x_3 = 0 \]
\[ 10x_2 + 25x_3 = 90 \]
\[ 20x_1 + 10x_2 = 80 \]
Gauss elimination (the case of infinitely many solutions)

\[
\begin{bmatrix}
3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\
0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\
1.2 & -0.3 & -0.3 & 2.4 & 2.1
\end{bmatrix}
\overset{\downarrow}{\rightarrow}
\begin{bmatrix}
3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\
0 & 1.1 & 1.1 & -4.4 & 1.1 \\
0 & -1.1 & -1.1 & 4.4 & -1.1
\end{bmatrix}
\overset{\downarrow}{\rightarrow}
\begin{bmatrix}
3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\
0 & 1.1 & 1.1 & -4.4 & 1.1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Gauss elimination (the case of no solution)

\[
\begin{bmatrix}
3 & 2 & 1 & 3 \\
2 & 1 & 1 & 0 \\
6 & 2 & 4 & 6
\end{bmatrix}
\overset{\downarrow}{\rightarrow}
\begin{bmatrix}
3 & 2 & 1 & 3 \\
0 & -\frac{1}{3} & \frac{1}{3} & -2 \\
0 & -2 & 2 & 0
\end{bmatrix}
\overset{\downarrow}{\rightarrow}
\begin{bmatrix}
3 & 2 & 1 & 3 \\
0 & -\frac{1}{3} & \frac{1}{3} & -2 \\
0 & 0 & 0 & 12
\end{bmatrix}
\]
Echelon form (계단 형태)

Gauss elimination:

\[
\begin{bmatrix} A & b \end{bmatrix} \Rightarrow \begin{bmatrix} R & f \end{bmatrix}
\]

\[
[R, f] = \begin{bmatrix}
  r_{11} & r_{12} & \cdots & \cdots & r_{1n} & f_1 \\
  r_{22} & \cdots & \cdots & \cdots & r_{2n} & f_2 \\
  \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
  r_{rr} & \cdots & \cdots & r_{rn} & f_r \\
  & & & & f_{r+1} \\
  & & & & & f_m
\end{bmatrix}
\]

Lesson 3: Rank of a matrix, Linear independence of vectors

- linear combination (of vectors)
- linear independence (of vectors)
- rank (of a matrix)
- practice using MATLAB
Linear combination (of vectors) & linear independence (of a set of vectors)

Example

\[ \mathbf{a}_1 = \begin{bmatrix} 3 & 0 & 2 & 2 \end{bmatrix} \]
\[ \mathbf{a}_2 = \begin{bmatrix} -6 & 42 & 24 & 54 \end{bmatrix} \]
\[ \mathbf{a}_3 = \begin{bmatrix} 21 & -21 & 0 & -15 \end{bmatrix} \]
Rank of a matrix

**DEF:** rank $A = $ 행렬 $A$에서 선형독립인 row vector의 최대 수

$$
\begin{bmatrix}
3 & 0 & 2 & 2 \\
-6 & 42 & 24 & 54 \\
21 & -21 & 0 & -15
\end{bmatrix}
$$

Properties of ‘rank’

**THM:** elementary row operation을 해서 얻는 모든 행렬들은 같은 rank를 가진다. (Rank는 elementary row operation에 대하여 invariant 하다.)

$$
\begin{bmatrix}
3 & 0 & 2 & 2 \\
-6 & 42 & 24 & 54 \\
21 & -21 & 0 & -15
\end{bmatrix}
$$
Properties of ‘rank’

**THM:** rank $A$ is a linearly independent column vector of the maximum number.
(Therefore $\text{rank } A = \text{rank } A^T$.)
Properties of 'rank'

- For $A \in \mathbb{R}^{m \times n}$, $\text{rank } A \leq \min\{m, n\}$.
- For $v_1, \ldots, v_p \in \mathbb{R}^n$, if $n < p$, then they are linearly dependent.
- Let $A = [v_1, v_2, \ldots, v_p]$ where $v_i \in \mathbb{R}^n$.
  - If $\text{rank } A = p$, then they are linearly independent.
  - If $\text{rank } A < p$, then they are linearly dependent.

Ex:

$$
\begin{bmatrix}
3 & 0 & 2 & 2 \\
-6 & 42 & 24 & 54 \\
21 & -21 & 0 & -15
\end{bmatrix}
$$

MATLAB을 사용한 실습

http://www.mathworks.com
### Lesson 4: Vector space

- vector space (in $\mathbb{R}^n$), subspace
- basis, dimension
- column space, null space of a matrix
- existence and uniqueness of solutions
- vector space (in general)

**Vector space**
선형연립방정식의 해: 존재성과 유일성

\[ Ax = b \quad \text{with } A \in \mathbb{R}^{m \times n} \text{ and } b \in \mathbb{R}^m \]

1. existence: a solution \( x \) exists iff
   - \( b \in \) column space of \( A \)
   - \( \text{rank } A = \text{rank } [A \ b] \)
2. uniqueness: when a solution \( x \) exists, it is the unique solution iff
   - \( \dim(\text{null space of } A) = 0 \)
   - \( \text{rank } A = n \)
3. existence & uniqueness: the solution \( x \) uniquely exists iff
   - \( \text{rank } A = \text{rank } [A \ b] = n \)
4. existence for any \( b \in \mathbb{R}^m \): a solution \( x \) exists for any \( b \in \mathbb{R}^m \) iff
   - \( \text{rank } A = m \)
5. unique existence for any \( b \in \mathbb{R}^m \): the unique solution \( x \) exists for any \( b \in \mathbb{R}^m \) iff
   - \( \text{rank } A = m \) and \( \text{rank } A = n \) (i.e., \( A \in \mathbb{R}^{n \times n} \) has ‘full rank’)

Ex: \( \text{rank } A = r < n \quad \Rightarrow \)

**Homogeneous case**

\[ Ax = 0 \quad A \in \mathbb{R}^{m \times n} \]
   - non-trivial solution exists iff \( \text{rank } A = r < n \)
   - 방정식의 수가 미지수의 수보다 적은 경우 항상 non-trivial solution을 가진다.

Q: Dimension of the ‘solution space’ =
Nonhomogenous case

\[ Ax = b \neq 0 \quad A \in \mathbb{R}^{m \times n} \]

- Any solution \( x \) can be written as

\[ x = x_0 + x_h \]

where \( x_0 \) is a solution to \( Ax = b \) and \( x_h \) is a solution to \( Ax = 0 \).

Vector space

: set of vectors with “addition” and “scalar multiplication”

For \( A, B, C \in V \) and \( c, k \in \mathbb{R} \),

\[
\begin{align*}
A + B &= B + A \\
(A + B) + C &= A + (B + C) \\
A + 0 &= A \\
A + (-A) &= 0
\end{align*}
\]

and

\[
\begin{align*}
c(A + B) &= cA + cB \\
(c + k)A &= cA + kA \\
c(kA) &= (ck)A \\
1A &= A
\end{align*}
\]
Examples of vector space

Normed space

: vector space with "norm"

ex: for $v \in \mathbb{R}^n$, the norm is $\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$
Inner product space

: vector space with “inner product”

1. \((c_1 A + c_2 B, C) = c_1 (A, C) + c_2 (B, C)\)
2. \((A, B) = (B, A)\)
3. \((A, A) \geq 0\) and \((A, A) = 0\) iff \(A = 0\)
Determinant (of a matrix)

For $A \in \mathbb{R}^{n \times n}$,

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} =$$
Elementary row operation & determinant

1. 두 행을 바꾸면 determinant의 부호가 반대가 됨
2. 똑같은 행이 존재하는 행렬의 determinant는 0
3. 한 행의 상수 배를 다른 행에 더해도 determinant 불변
4. 한 행에 0 아닌 c를 곱하면 determinant는 c배가 됨
   (c = 0인 경우도 성립하지만 쓸모는 없음)
Properties of ‘determinant’

- The determinant of a matrix $A$ is equal to the determinant of its transpose $A^T$.
- If any row or column is zero, then the determinant of the matrix is zero.
- If two rows or two columns of the matrix are identical, then the determinant is zero.

**THM**: A matrix $A \in \mathbb{R}^{m \times n}$ has rank $r (\geq 1)$ iff
- $A$ has a $r \times r$ submatrix whose determinant is non-zero, and
- determinants of submatrices of $A$, whose size is larger than $r \times r$, are zero (if exists).
Cramer’s rule

\[ Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} x = b, \quad A \in \mathbb{R}^{n \times n}, \quad \det A =: D \neq 0 \]

Cramer’s rule:

\[ x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad \cdots \quad x_n = \frac{D_n}{D} \]

where

\[ D_k = \begin{bmatrix} a_1 & \cdots & a_{k-1} & b & a_{k+1} & \cdots & a_n \end{bmatrix} \]

Ex:

\[ \begin{align*}
2x - y &= 1 \\
3x + y &= 2
\end{align*} \]

Lesson 6: Inverse of a matrix

- inverse (of a matrix)
- Gauss-Jordan elimination (computing inverse)
- formula for the inverse
- properties of inverse and nonsingular matrices
Inverse of a matrix

For $A \in \mathbb{R}^{n \times n}$, the inverse of $A$ is a matrix $B$ such that

$$AB = I \quad \text{and} \quad BA = I$$

and we denote $B$ by $A^{-1}$.

$A^{-1}$ exists iff $\text{rank } A = n$ iff $\det A \neq 0$ iff $A$ is ‘non-singular’

Computing the inverse: Gauss-Jordan elimination
A formula for the inverse

For $A = [a_{ij}] \in \mathbb{R}^{n \times n}$,
Properties about nonsingular matrix, inverse, and determinant

- Inverse of ‘diagonal matrix’ is easy.
- \((AB)^{-1} = B^{-1}A^{-1}\)
- \((A^{-1})^{-1} = A\)
- For \(A, B, C \in \mathbb{R}^{n \times n}\), if \(A\) is nonsingular (i.e., \(\text{rank } A = n\)),
  - \(AB = AC\) implies \(B = C\).
  - \(AB = 0\) implies \(B = 0\).
- For \(A, B \in \mathbb{R}^{n \times n}\), if \(A\) is singular, then \(AB\) and \(BA\) are singular.
- \(\det(AB) = \det(BA) = \det A \det B\)

Lesson 7: Eigenvalues and eigenvectors

- eigenvalues and eigenvectors
- symmetric, skew-symmetric, and orthogonal matrices
Eigenvalue and eigenvector of a matrix
Find eigenvalues and eigenvectors of

\[
A = \begin{bmatrix}
-2 & 2 & -3 \\
2 & 1 & -6 \\
-1 & -2 & 0
\end{bmatrix}.
\]

\[-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0\]

\[\lambda_1 = 5, \quad \lambda_2 = \lambda_3 = -3\]

\[
A - 5I = \begin{bmatrix}
-7 & 2 & -3 \\
2 & -4 & -6 \\
-1 & -2 & -5
\end{bmatrix} \Rightarrow \begin{bmatrix}
-7 & 2 & -3 \\
0 & -\frac{24}{7} & -\frac{45}{7} \\
0 & 0 & 0
\end{bmatrix}
\]

\[
A + 3I = \begin{bmatrix}
1 & 2 & -3 \\
2 & 4 & -6 \\
-1 & -2 & 3
\end{bmatrix} \Rightarrow \begin{bmatrix}
1 & 2 & -3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
Symmetric, skew-symmetric, and orthogonal matrices

Lesson 8: Similarity transformation, diagonalization, and quadratic form

- similarity transformation
- diagonalization
- quadratic form
Similarity transformation

행렬 $A \in \mathbb{R}^{n \times n}$가 $n$개의 선형독립인 e.vectors를 가질 때...
언제 행렬 $A$가 $n$개의 선형독립인 e.vectors를 갖나? (1)

언제 행렬 $A$가 $n$개의 선형독립인 e.vectors를 갖나? (2)

$$A_1 = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \quad \lambda_1 = -1, \quad \lambda_2 = -3$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}, \quad \lambda_1 = \lambda_2 = -2$$

$$A_3 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad \lambda_1 = \lambda_2 = -2$$
Diagonalization

Diagonalization이 안되는 경우
Quadratic form

\[ Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128 \]
못 다룬 것들

교재의 연습 문제:
- trace,
- positive definite matrix, positive
  semi-definite matrix

out of the scope:
- (induced) norm of a matrix,
- (generalized eigenvectors,)
  Jordan form

further study:

http://snuon.snu.ac.kr [최신제어기법]
http://snu.ac.kr [최신제어기법]
http://lecture.cdsl.kr [선형대수 및 선형시스템 기초]

Lesson 9: Introduction to differential equation

- function, limit, and differentiation
- differential equation, general and particular solutions
- direction field, solving DE by computer
Function, limit, and differentiation

Basic concepts and ideas
\[ y'(x) + 2y(x) - 3 = 0 \]
\[ y'(x) = -27x + x^2 \]
\[ y'(t) = 2t \]
\[ y''(x) + y'(x) + y(x) = 0 \]
\[ y''(x)y'(x) + \sin(y(x)) + 2 = 0 \]
\[ \left\{ \begin{array}{l}
y_1'(x) + 2y_2(x) + 3 = 0 \\
y_2'(x) + 2y_1(x) + y_2(x) = 2 \\
2 \frac{\partial y}{\partial x}(x, z) + 3 \frac{\partial y}{\partial z}(x, z) - 2x = 0
\end{array} \right. \]

* ODE (ordinary differential equation) / PDE (partial differential equation)

* Solving DE:

* Explicit/implicit solution

Why do we have to study DE?
General solution and particular solution

Direction fields (a geometric interpretation of $y' = f(x, y)$)

An idea of solving DE by computer
Lesson 10: Solving first order differential equations

- separable differential equations
- exact differential equations
Separable DE

\[ f, g: \text{continuous functions} \]

\[ g(y)y' = f(x) \quad \Rightarrow \quad g(y)dy = f(x)dx \]
\[ y' = g \left( \frac{u}{x} \right) \]

replacing \( ay + bx + k \) with \( v \)

\[ (2x - 4y + 5)y' + (x - 2y + 3) = 0 \]
Exact differential equation: introduction

(observation:) For $u(x, y)$,

$$du = \frac{\partial u}{\partial x}(x, y)dx + \frac{\partial u}{\partial y}(x, y)dy : \text{differential of } u.$$ 

So, if $u(x, y) = c$ (constant), then $du = \ldots$.

---

Exact differential equation

Given DE: $M(x, y) + N(x, y)\frac{du}{dx} = 0$

If $\exists$ a function $u(x, y)$ s.t.

$$\frac{\partial u}{\partial x}(x, y) = M(x, y) \quad & \quad \frac{\partial u}{\partial y}(x, y) = N(x, y)$$

then

$$u(x, y) = c$$

is a general sol. to the DE.

The DE is called "exact DE".
How to check if the given DE is exact?

How to solve the exact DE?
Lesson 11: More on first order differential equations

- integrating factor
- linear differential equation
- Bernoulli equation
- obtaining orthogonal trajectories of curves
- existence and uniqueness of solutions to initial value problem
Integrating factor

\[ P(x, y)dx + Q(x, y)dy = 0 \]

\[ (e^{x+y} + ye^y)dx + (xe^y - 1)dy = 0 \]
Linear DE

\[ y' + p(x)y = r(x) \]
Bernoulli DE

\[ y' + p(x)y = g(x)y^a, \quad a \neq 0 \text{ or } 1 \]

Verhulst logistic model (population model):

\[ y' = Ay - By^2, \quad A, B > 0 \]
Orthogonal trajectories of curves

Existence of solutions to initial value problem

\[ y' = f(x, y), \quad y(x_0) = y_0 \]

**THM 1**: IF \( f(x, y) \) is continuous, and bounded such that \( |f(x, y)| \leq K \), in the region

\[ R = \{ (x, y) : |x - x_0| < a, |y - y_0| < b \} \]

THEN the IVP has at least one sol. \( y(x) \) on the interval \( |x - x_0| < \alpha \) where \( \alpha = \min(a, b/K) \).
Uniqueness of solutions to initial value problem

\[ y' = f(x, y), \quad y(x_0) = y_0 \]

**THM 2:** IF \( f(x, y) \) and \( \frac{\partial f}{\partial y}(x, y) \) are continuous, and
bounded such that \( |f(x, y)| \leq K \) and \( \left| \frac{\partial f}{\partial y}(x, y) \right| \leq M \) in \( R \),
THEN the IVP has a unique sol. \( y(x) \) on the interval 
\( |x - x_0| < \alpha \) where \( \alpha = \min(a, b/K) \).

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**Lesson 12: Solving the second order linear DE**

- overview
- homogeneous linear DE
- reduction of order
- homogeneous linear DE with constant coefficients
Overview: Linear ODEs of second order

\[ y'' + p(x)y' + g(x)y = r(x), \quad y(x_0) = K_0, \quad y'(x_0) = K_1 \]

1. The homogeneous linear ODE:

\[ y'' + p(x)y' + g(x)y = 0 \]  \hspace{1cm} (1)

has two "linearly independent" solutions \( y_1(x) \) and \( y_2(x) \).

2. Let \( y_h(x) = c_1 y_1(x) + c_2 y_2(x) \) with two constant coefficients \( c_1 \) and \( c_2 \), which is again a solution to (1).

3. Solve

\[ y'' + p(x)y' + g(x)y = r(x) \]  \hspace{1cm} (2)

without considering the initial condition. Let the solution be \( y_p(x) \).

4. The general solution is

\[ y(x) = y_h(x) + y_p(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x). \]

Determine \( c_1 \) and \( c_2 \) with the initial condition.

Homogeneous linear ODEs of second order

\[ y'' + p(x)y' + g(x)y = 0 \]

Claim: Linear homogeneous ODE of the second order has two linearly independent solutions.
How to obtain a basis if one sol. is known? (Reduction of order)
Obtaining another $y_2(x)$ with a known $y_1(x)$
Homogeneous linear ODEs with constant coefficients

\[ y'' + ay' + by = 0 \]
Lesson 13: The second order linear DE

- case study: free oscillation
- Euler-Cauchy equation
- existence and uniqueness of a solution to IVP
- Wronskian and linear independence of solutions
Modeling: Free oscillation
Euler-Cauchy equation

\[ x^2 y'' + a x y' + b y = 0 \]
Existence and uniqueness of a solution to IVP

\[ y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = K_0, \quad y'(x_0) = K_1 \]

**THM:** IF \( p(x) \) and \( q(x) \) are continuous (on an open interval \( I \ni x_0 \)), THEN \( \exists \) a unique sol. \( y(x) \) (on the interval \( I \)).

Wronskian and linear independence of solutions

With \( y_1(x) \) and \( y_2(x) \) being the solutions of

\[ y'' + p(x)y' + q(x)y = 0, \]

Wronski determinant (Wronskian) of \( y_1 \) and \( y_2 \) is defined by

\[ W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1y'_2 - y_2y'_1 \]

**THM:**
1. two sol. \( y_1, y_2 \) are linearly dep. on \( I \) \( \iff \) \( W(y_1(x), y_2(x)) = 0 \) at some \( x^* \in I \)
2. If \( W(y_1(x), y_2(x)) = 0 \) at some \( x^* \in I \), then \( W(y_1(x), y_2(x)) \equiv 0 \) on \( I \).
3. If \( W(y_1(x), y_2(x)) \neq 0 \) at some \( x^* \in I \), then \( y_1 \) and \( y_2 \) are linearly indep. on \( I \).
\[ y'' + p(x)y' + q(x)y = 0 \] has two indep. sol. \( y_1 \) and \( y_2 \)

so, it has a general sol. \( y(x) = c_1 y_1(x) + c_2 y_2(x) \)
Any sol. to \(y'' + p(x)y' + q(x)y = 0\) has the form of \(c_1y_1(x) + c_2y_2(x)\)
Nonhomogeneous linear DE

\[ y'' + p(x)y' + q(x)y = r(x) \]
Candidate for $y_p(x)$ in $y'' + p(x)y' + q(x)y = r(x)$

<table>
<thead>
<tr>
<th>Term in $r(x)$</th>
<th>Candidate for $y_p(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k e^{rx}$</td>
<td>$C e^{rx}$</td>
</tr>
<tr>
<td>$k x^n$, $n \geq 0$ integer</td>
<td>$K_n x^n + K_{n-1} x^{n-1} + \cdots + K_0 x + K_0$</td>
</tr>
<tr>
<td>$k \cos \alpha x$</td>
<td>$K \cos \alpha x + M \sin \alpha x$</td>
</tr>
<tr>
<td>$k \sin \alpha x$</td>
<td></td>
</tr>
<tr>
<td>$k e^{\gamma x} \cos \alpha x$</td>
<td>$e^{\gamma x}(K \cos \alpha x + M \sin \alpha x)$</td>
</tr>
<tr>
<td>$k e^{\gamma x} \sin \alpha x$</td>
<td></td>
</tr>
</tbody>
</table>

The above rules are applied for each term $r(x)$.
If the candidate for $y_p(x)$ happens to be a sol. of the homogeneous equation, then multiply $y_p(x)$ by $x$ (or by $x^2$ if this sol. corresponds to a double root of the characteristic eq. of the homogeneous equation).

\[ y'' + 4y = 8x^2 \]
\[ y'' - 3y' + 2y = e^x \]

\[ y'' + 2y' + y = e^{-x} \]

\[ y'' + 2y' + 5y = 1.25e^{0.5x} + 40\cos 4x - 55\sin 4x \]

\[ y'' + 2y' + 5y = 1.25e^{0.5x} + 40\cos 2x \]

\[ y'' + 2y' + 5y = 1.25e^{0.5x} + 40e^{-x}\cos 2x \]
Solution by variation of parameters

\[ y'' + p(x)y' + q(x)y = r(x) \]
Higher order homogeneous linear DE

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \]  \hspace{1cm} (H)

General sol.: \[ y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) \]
where \( y_i(x) \)'s are linearly indep. sol. to (H).
\( y^{(n)}(x) + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0, \quad y^{(i)}(x_0) = K_i \)

**THM:** If all \( p_i \)'s are conti. (on \( I \)), then IVP has a unique sol. (on \( I \)).

**THM:** With all \( p_i \)'s being conti.,

sol. \( \{y_1, \cdots, y_n\} \) are lin. dep. on \( I \)

\[
\Leftrightarrow \quad W(y_1, \cdots, y_n) = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = 0 \text{ at some } x_0 \in I
\]

\[
\Leftrightarrow \quad W(y_1, \cdots, y_n) \equiv 0 \text{ on } I
\]

\[
y''' - 5y'' + 4y = 0
\]
**THM:** With all $p_i$'s being conti., the (H) has $n$ lin. indep. sol. (i.e., there is a general solution).

**THM:** With all $p_i$'s being conti., the general sol. includes all solutions.

Higher order homogeneous linear DE with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0$$

* distinct roots

* multiple roots
Higher order nonhomogeneous linear DE

\[ y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x) \]

* undetermined coefficient method:

* variation-of-parameter formula:

\[ y_p(x) = y_1 \int \frac{W_1 r}{W} dx + y_2 \int \frac{W_2 r}{W} dx + \cdots + y_n \int \frac{W_n r}{W} dx \]

where \( W = W(y_1, \cdots, y_n) \) and \( W_j \): \( j \)-th column in \( W \) replaced by \[
\begin{bmatrix}
0 \\
\vdots \\
0 \\
1
\end{bmatrix}.
\]

Lesson 16: Case studies

- mass-spring-damper system: forced oscillation
- RLC circuit
- elastic beam
Case study: forced oscillation \((m\ddot{y} + cy' + ky = r)\)

\[
y_p(t) = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2} \cos \omega t + F_0 \frac{c\omega}{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2} \sin \omega t, \quad y(t) = y_h(t) + y_p(t)
\]
\[
y(t) = y_h(t) + F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + c^2 \omega^2} \cos \omega t + F_0 \frac{c \omega}{m^2(\omega_0^2 - \omega^2)^2 + c^2 \omega^2} \sin \omega t
\]

Modeling: RLC circuit
RLC circuit: forced response

Elastic beam
<table>
<thead>
<tr>
<th>Lesson 17: Systems of ODEs</th>
</tr>
</thead>
<tbody>
<tr>
<td>▶ introduction</td>
</tr>
<tr>
<td>▶ existence and uniqueness of solutions to IVP</td>
</tr>
<tr>
<td>▶ linear homogeneous case</td>
</tr>
<tr>
<td>▶ linear homogeneous constant coefficient case</td>
</tr>
</tbody>
</table>
Existence and uniqueness of solutions to IVP

\[ y' = f(t, y), \quad y(t_0) = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix} \]

**THM:** If all \( f_i(t, y) \) and \( \frac{\partial f_i}{\partial y_j}(t, y) \) are conti. on some region of \((t, y_1, y_2, \cdots, y_n)\)-space containing \((t_0, k_1, \cdots, k_n)\), then a sol. \( y(t) \) exists and is unique in some local interval of \( t \) around \( t_0 \).

\[ y' = A(t)y + g(t), \quad y(t_0) = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix} \]

**THM:** If \( A(t) \) and \( g(t) \) are conti. on an interval \( I \), then a sol. \( y(t) \) exists and is unique on the interval \( I \).

Linear homogeneous case

\[ y' = A(t)y \]

General sol.: \( y(t) = c_1 y^{(1)}(t) + c_2 y^{(2)}(t) + \cdots + c_n y^{(n)}(t) \)

where \( y^{(i)}(t) \)'s are lin. indep. sol.
Linear homogeneous constant coefficient case

\[ y' = Ay \]
Handling complex e.v/e.vectors
Lesson 18: Qualitative properties of systems of ODE

- phase plane and phase portrait
- critical points
- types and stability of critical points

Phase plane and phase portrait
Critical point (= equilibrium)

Example: undamped pendulum
Types of critical points: node

Types of critical points: saddle / center
Types of critical points: spiral / degenerate node

Stability

**DEF:** stability of a critical point $P_0(=y^*)$:
- all trajectories of $y' = f(y)$ whose initial condition $y(t_0)$ is sufficiently close to $P_0$ remain close to $P_0$ for all future time
- for each $\epsilon > 0$, there is $\delta > 0$ such that,

$$|y(t_0) - y^*| < \delta \quad \Rightarrow \quad |y(t) - y^*| < \epsilon, \quad \forall t \geq t_0$$

**DEF:** asymptotic stability of $P_0 =$ stability + attractivity ($\lim_{t \to \infty} y(t) = y^*$)
Lesson 19: Linearization and nonhomogeneous linear systems of ODE

- linearization
- nonhomogeneous case

Example: second order system
Linearization

\[ y' = f(y) \]

Let \( y = 0 \) be a critical point (without loss of generality; WLOG), and be isolated.

\[
\begin{align*}
y_1' &= f_1(y_1, y_2) = f_1(0, 0) + \frac{\partial f_1}{\partial y_1}(0, 0)y_1 + \frac{\partial f_1}{\partial y_2}(0, 0)y_2 + h_1(y_1, y_2) \\
y_2' &= f_2(y_1, y_2) = f_2(0, 0) + \frac{\partial f_2}{\partial y_1}(0, 0)y_1 + \frac{\partial f_2}{\partial y_2}(0, 0)y_2 + h_2(y_1, y_2)
\end{align*}
\]

\[ y' = f(y) \quad \Rightarrow \quad y' = Ay = \frac{\partial f}{\partial y} \bigg|_{y=0} y \]

- If no e.v. of \( A \) lies in the imaginary axis, then stability of the critical point of the nonlinear system is determined by \( A \).
  - If \( \text{Re}(\lambda) < 0 \) for all \( \lambda \), it is asymptotically stable.
  - If \( \text{Re}(\lambda) > 0 \) for at least one \( \lambda \), it is unstable.
- If all e.v.‘s are distinct and no e.v. of \( A \) lies in the imaginary axis, then the type of the critical point of the nonlinear system is determined by \( A \).
  - The node, saddle, and spiral are preserved, but center may not be preserved.
Nonhomogeneous linear case

Method of undetermined coefficients (for time-invariant case)
Method of variation of parameters (for time-varying case)

Method of diagonalization (for time-invariant case)
Lesson 20: Series solutions of ODE

- power series method
- Legendre equation

Power series

\[ \sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots \]
\[ \sum_{m=0}^{\infty} a_m(x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + a_{n+1}(x-x_0)^{n+1} + \cdots \]

For a given \( x_1 \),

if \( \lim_{n \to \infty} S_n(x_1) \) exists (or, \( \lim_{n \to \infty} R_n(x_1) = 0 \),

or for any \( \epsilon > 0 \), \( \exists N(\epsilon) \) s.t. \( |R_n(x_1)| < \epsilon \) for all \( n > N(\epsilon) \),

then the series is called “convergent at \( x = x_1 \)” and we write \( S(x_1) = \lim_{n \to \infty} S_n(x_1) \).
Radius of convergence

If

\[ R = \frac{1}{\lim_{m \to \infty} \sqrt[|a_m|]} \text{, or } R = \frac{1}{\lim_{m \to \infty} \left| \frac{a_{m+1}}{a_m} \right|} \]

is well-defined, then the series is convergent for \( x \) s.t. \( |x - x_0| < R \).
Power series method

\[ y''(x) + p(x)y'(x) + q(x)y(x) = r(x) \]

If \( p, q, \) and \( r \) are analytic at \( x = x_0 \),

then there exists a power series solution around \( x_0 \) (i.e., \( R > 0 \)):

\[ y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m. \]
Legendre equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0, \quad n : \text{real number}$$
Legendre polynomial (of degree $n$)
Frobenius method

The DE

\[ y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 \]

where \( b \) and \( c \) are analytic at \( x = 0 \), has at least one sol. around \( x = 0 \) of the form

\[ y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r (a_0 + a_1 x + a_2 x^2 + \cdots). \]
Case 1: distinct roots, not differing by an integer

Case 2: double roots

Case 3: distinct roots differing by an integer

General sol.: $y(x) = c_1 y_1(x) + c_2 y_2(x)$ where

Case 1:

\[
\begin{align*}
y_1(x) &= x^{r_1}(a_0 + a_1 x + \cdots) \\
y_2(x) &= x^{r_2}(A_0 + A_1 x + \cdots)
\end{align*}
\]

Case 2: $r = (1 - b_0)/2$

\[
\begin{align*}
y_1(x) &= x^r(a_0 + a_1 x + \cdots) \\
y_2(x) &= y_1(x) \ln x + x^r(A_1 x + A_2 x^2 + \cdots)
\end{align*}
\]

Case 3: $r_1 > r_2$

\[
\begin{align*}
y_1(x) &= x^{r_1}(a_0 + a_1 x + \cdots) \\
y_2(x) &= ky_1(x) \ln x + x^{r_2}(A_0 + A_1 x + \cdots)
\end{align*}
\]
Example: Euler-Cauchy equation revisited
Example: a simple hypergeometric equation

\[ x(x - 1)y'' + (3x - 1)y' + y = 0 \]
Example: another simple hypergeometric equation

\[ x(x - 1)y'' - xy' + y = 0 \]
Gamma function

\[ \Gamma(\nu) := \int_0^\infty e^{-t} t^{\nu-1} dt \]

has the properties:

1. \( \Gamma(\nu + 1) = \nu \Gamma(\nu) \)
2. \( \Gamma(1) = 1 \)
3. \( \Gamma(n + 1) = n! \)

Bessel’s DE

\[ x^2 y'' + xy' + (x^2 - \nu^2)y = 0, \quad \nu \geq 0 \]
Computing $y_1(x)$
Bessel function of the first kind of order \( n \)

\[
J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}
\]

Finding \( y_2(x) \)
Bessel function of the second kind of order $\nu$

$$Y_{\nu}(x) = \frac{1}{\sin \nu \pi} \left[ J_{\nu}(x) \cos \nu \pi - J_{-\nu}(x) \right]$$

$$Y_n(x) = \lim_{\nu \to n} Y_{\nu}(x) = \cdots$$
Laplace transform

\[ \mathcal{L}\{f\} = \int_0^\infty f(t)e^{-st} dt = F(s) \]

(Property) Linearity: \( \mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} \)
(Property) $s$-shifting property: $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

Transform table: $f(t) \leftrightarrow F(s)$

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>$t$</td>
<td>$\frac{1}{s^2}$</td>
</tr>
<tr>
<td>$t^2$</td>
<td>$\frac{2!}{s^3}$</td>
</tr>
<tr>
<td>$t^n$</td>
<td>$\frac{n!}{s^{n+1}}$, $n = \text{integer}$</td>
</tr>
<tr>
<td>$t^a$</td>
<td>$\frac{\Gamma(a+1)}{s^{a+1}}$, $a &gt; 0$</td>
</tr>
<tr>
<td>$e^{at}$</td>
<td>$\frac{1}{s-a}$</td>
</tr>
<tr>
<td>$\cos \omega t$</td>
<td>$\frac{s}{s^2 + \omega^2}$</td>
</tr>
<tr>
<td>$\sin \omega t$</td>
<td>$\frac{s}{s^2 - \omega^2}$</td>
</tr>
<tr>
<td>$\cosh at$</td>
<td>$\frac{a}{s^2 - a^2}$</td>
</tr>
<tr>
<td>$\sinh at$</td>
<td>$\frac{a}{s^2 - a^2}$</td>
</tr>
<tr>
<td>$e^{at} \cos \omega t$</td>
<td>$\frac{s-a}{(s-a)^2 + \omega^2}$</td>
</tr>
<tr>
<td>$e^{at} \sin \omega t$</td>
<td>$\frac{\omega}{(s-a)^2 + \omega^2}$</td>
</tr>
</tbody>
</table>
Existence and uniqueness of Laplace transform

IF $f(t)$ is piecewise continuous on every finite interval in $\{t : t \geq 0\}$, and

$$|f(t)| \leq Me^{kt}, \quad t \geq 0$$

with some $M$ and $k$,

THEN $\mathcal{L}\{f(t)\}$ exists for all $\text{Re}(s) > k$. 

Computing inverse Laplace transform

\[ \mathcal{L}^{-1}\{F(s)\} = f(t) = ? \]

* Partial fraction expansion:

Finding coefficients in partial fraction expansion: Heaviside formula

\[ Y(s) = \frac{s+1}{s^3 + s^2 - 6s} = \frac{A_1}{s} + \frac{A_2}{s+3} + \frac{A_3}{s-2} \]

\[ Y(s) = \frac{s^3 - 4s^2 + 4}{s^2(s-2)(s-1)} = \frac{A_2}{s^2} + \frac{A_1}{s} + \frac{B}{s-2} + \frac{C}{s-1} \]
\[ Y(s) = \cdots = \frac{A_3}{(s-1)^3} + \frac{A_2}{(s-1)^2} + \frac{A_1}{s-1} + \frac{B_2}{(s-2)^2} + \frac{B_1}{s-2} \]

\[ Y(s) = \frac{20}{(s^2+4)(s^2+2s+2)} + \frac{s-3}{s^2+2s+2} \]

**Lesson 24: Laplace transform II**

- transform of derivative and integral
- solving linear ODE
- unit step function and t-shifting property
- Dirac’s delta function (impulse)
(Property) Transform of differentiation: $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

(Property) Transform of integration: $\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{1}{s}F(s)$
Solving IVP of linear ODEs with constant coefficients

\[ y'' + ay' + by = r(t), \quad y(0) = K_0, \quad y'(0) = K_1 \]
Unit step function (Heaviside function)

(Property) $t$-shifting property: $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$
(Dirac’s) delta function

\[ \delta(t) \] is a (generalized) function such that

\[ \delta(t) = \begin{cases} 
0, & t \neq 0 \\
\infty, & t = 0 
\end{cases} \quad \text{and} \quad \int_{-a}^{a} \delta(t) \, dt = 1 \quad \text{for any} \ a > 0 
\]

sifting property:

\[ \int_{0}^{\infty} g(t) \delta(t - a) \, dt = g(a), \quad g: \text{conti.}, \ a > 0 \]

---

Lesson 25: Laplace transform III

- convolution
- impulse response
- differentiation and integration of transforms
- solving system of ODEs
(Property) Convolution: \( \mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t) \)

Properties of convolution:

\[
\begin{align*}
  f * g &= g * f \\
  f * (g_1 + g_2) &= f * g_1 + f * g_2 \\
  (f * g) * v &= f * (g * v) \\
  f * 0 &= 0 * f = 0, \quad f * 1 \neq f
\end{align*}
\]
Impulse response
(Property) Differentiation of transform: $\mathcal{L}\{t f(t)\} = -F'(s)$
(Property) Integration of transform: \( \mathcal{L}\{\frac{f(t)}{t}\} = \int_{\hat{s}}^{\infty} F(\hat{s})d\hat{s} \)
Solving system of ODEs