Government, Distributive Tax Policy and the Dynamic Inefficiency of Capitalism

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This paper considers the dynamic inefficiency of capitalism, that is exhibited in Lancaster's paper as the underinvestment of capital stock compared to the social optimum, in the presence of the government that cares for the welfare of workers as well as that of capitalists. This paper shows that the inefficiency arises because the determining factor in Nash equilibrium is the value of the maximum portion of output that workers can consume at each point while in Command equilibrium it is the relative weights assigned to the welfare of workers and capitalists in the government objective function. We investigate the factors that influence the inefficiency with particular attention being given to the switching points at which capital stock stops accumulating. The analysis also considers the coordinating role of government by showing that the optimal tax policy improves on Nash equilibrium by sacrificing production efficiency to achieve distributional goal. (*JEL* E62)

I. Introduction

In Lancaster (1973), the process of capitalistic economy is modelled as a differential game between two groups of individuals, workers and capitalists. At each point in time, workers make decisions on the division of current output between their consumption and the part that will be handed over to capitalists while capitalists, given this part of the output, decide how much to consume and invest.

Then, as the game is played, there arises a Prisoner's Dilemma situa-

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tion. Even though capitalists want to increase the capital stock for increased future consumption, that is desirable from the viewpoint of society, they fear that workers will take all the fruits of capital accumulation. Similarly, workers stop saving knowing that capitalists will consume workers' saving instead of investing in capital formation.

This coordination failure results in the dynamic inefficiency where the accumulation of the capital stock stops earlier compared to the social optimum. The inefficiency is attributed to the conflict of interests between two groups and this conflict arises because these groups face separate decision problems over time. Since one group decides over consumption and saving and the other's decision is over consumption and investment, this conflict is a dynamic version of Keynesian conflict.[1]

Hoel (1978) extends Lancaster's model with different specifications on production technology and social welfare function. His main concern is to show that, with a particular utility function for capitalists, there is a possibility for the overaccumulation of capital stock in Nash equilibrium compared to the social optimum. Mehrling (1985) considers infinite time horizon version of differential game of workers and capitalists whose choice variables are rate of wage increase and rate of employment ratio respectively. However, these models did not consider explicitly the role of government.

In recent endogenous growth models with government, for example in Barro (1990) and Lee (1990), the main issue concerned is the implication of the size of government for the growth rate. Consequently, possible frictions that may exist in the economy and the potential role of government to attenuate these frictions are ignored. Alesina and Rodrik (1991) extends Barro's model by considering distributive politics

[1]Frech III (1975) criticized the use of the word Capitalism in Lancaster's paper because one distinctive feature of capitalism is a system of well defined property rights and this system will correct the dynamic inefficiency. However, due to transaction costs in any organization, some sort of inefficiency can arise and the principal-agent problem in a firm is a well-known phenomenon of this inefficiency. From the macroeconomic perspective Lancaster found the principal-agent problem in the fact that workers hand over part of their wealth to capitalists to manage it. This captures another distinctive characteristic of capitalistic economy.

Alternatively, our model can be taken to describe a capitalistic economy where distributional politics between capitalists and workers is engaged behind the economic activity and the government is coordinating the conflict of interests.

in growth process and show that unequal wealth distribution between workers and capitalists implies, via voting equilibrium, higher tax rate on capitalists which influences growth rate adversely. In their paper, given the efficient market mechanism in allocating resources, the conflict of interests arises due to transfer payment to workers. However, this conflict is not economic but political one. Also, workers do not perform any active economic role except that they are compensated with the competitive wages and receive the transfer payment that is determined by political process.

The purpose of this paper is to incorporate government as a coordinator in an economy where there is a friction in Lancaster's sense so that workers as well as capitalists perform important economic functions. For that purpose, we assume the followings. First, government cares for the welfare of workers as well as that of capitalists so that distributional consideration affects government behavior. To implement the distributional goal, government is assumed to impose taxes on capitalists.[2] Second, with the tax proceeds, government provides a flow of productive services that is essential for production process.

This paper considers how the dynamic inefficiency can arise in the presence of government and investigate the factors that affect the inefficiency. In doing this, we derive the time path of tax rates that reflect distributional consideration and production efficiency.

Assuming finite time horizon, we show several results.[3] In equilibrium, there is a switching point before which new investment is made at the maximum rate but after which new investment is zero. Tax rates, which in the model indicate the size of the government, are nondecreasing until switching point and then stay constant at the maximum value for the rest of the time. Tax rates depend on parameters that represent the state of technology, relative weights assigned to the welfare of each group in the government objective function, and the

[2]When tax is neutral with respect to both groups, tax rate is chosen to insure only production efficiency, which implies the tax rate that maximizes capital accumulation. To achieve the distributional goal, government is allowed to treat two groups differentially and we assume that taxes are imposed solely on capitalists.

[3]One purpose of this paper is to consider the dynamic inefficiency with government in Lancasters model so that we stick to the finite time horizon framework. As indicated in footnote 18 in Lancaster, the bangbang solution can not be obtained if we analyse the infinite time horizon problem within the original structure of Lancaster's model. Infinite time version of Lancaster's model can be investigated along the lines as in Mehrling (1986).

socio-economic environment that determines the minimum and maximum portions of output that workers can consume at each point. Generally, tax rates are set above the value that maximizes capital accumulation. Only when the welfare of workers is ignored, tax rate maximizes capital accumulation.

We investigate the inefficiency by comparing Command and Nash equilibria. In Command equilibrium, the particular path the economy follows is determined by the relative weights given to each group while, in Nash, it is the value of the maximum portion of output workers can consume that determines the particular path.

This fact gives rises to the inefficiency of capitalism in the presence of government. Near the end of the time horizon, workers can not be prevented from consuming at the maximum even though the weight attached to workers is small in the government objective function. Similarly, when workers' welfare is heavily counted, even though it is desirable for workers to consume at the maximum rate at the early phase of the horizon, workers can not be induced to do that when the maximum portion that workers can take out as consumption is relatively large. That is because the prospective increase in future consumption compensates workers more than the sacrifice in current consumption.

Focusing on the switching point, we show the following. Large values of the weight given to workers and large values of the maximum portion of output that workers can consume tend to increase the difference of the switching points between the two equilibria. However, when this weight is relatively small, this difference widens when the minimum portion of output that workers can consume is lowered.

By restricting Optimal tax policy to a case where it has a constant value after the switching point, it is shown that optimal tax rate is lower than Nash tax rate so that the switching point comes later than Nash. It is indicated that, in optimal tax rate, production efficiency of the economy is somehow sacrificed to improve on the distributional consequences in Nash equilibrium. That the optimal tax policy suffers from time-inconsistency problem is briefly discussed.

The structure of the paper is as follows. In section II, the model is presented. In the following subsections, we analyze Command, Nash, and the Optimal tax policy equilibria and some comparisons are attempted. In the last section, we suggest possible extensions and conclusion.

II. Model

Consider an economy whose time span is finite and runs from 0 to T. In this economy, there are two groups or classes in the population, workers and capitalists, and the government that cares for the welfare of both groups. The production function is assumed to be:

$$Y_t = AK_t^{\alpha}G_t^{1-\alpha}, \quad 0 < \alpha < 1, \tag{1}$$

where A is the technology parameter and K_t and G_t represent respectively capital stock and government expenditures at time t. Government expenditures provide flow of services like national defense and enforcement of law that is essential for the production of output. Time subscripts will be omitted for notational simplicity.

At each point in time, workers, capitalists, and government are playing a game. Workers move first and take a fraction of the output, a, as their consumption, handing over the remaining output to the rest of the economy. Workers' choice variable, a, is bounded from below by $\underline{a}$ and from above by $\bar{a}$. We assume $\underline{a} > 0$ and $\bar{a} < 1$. $\underline{a}$ can be interpreted as the minimum portion of output that is needed for the maintenance of workers and $\bar{a}$ as the portion of output above which our economy can not sustain its capitalistic structure. Generally, these values are determined by the socio-economic structure of the economy and may change over time. In the model, however, we assume that $\underline{a}$ and $\bar{a}$ are exogenously given and do not change over time.[4]

Then, given the choice of a, government imposes taxes with tax rate τ on the remaining output, $(1 - a)Y$. From the tax proceeds, government provides productive services, G, that is essential for the production process. Government satisfies the budget constraint for all t:

$$G = \tau(1 - a)AK^{\alpha}G^{1-\alpha}. \tag{2}$$

[4]Since $\underline{a}$ and $\bar{a}$ can be interpreted to include portions of output that are devoted to make the economy run peacefully. For example, these values are influenced by the presence of labor unions and the efficiency with which government deals with the unions.

In the case of fixed coefficient production function, a positive interpretation on a is suggested by Mehrling (1986). In this case, $\dot{a}/a$ becomes identical to $\dot{w}/w$, where w is wage rate. Assuming wage bargaining where workers can increase wages faster when the threat of unemployment is reduced, the rate of employment where wage increase is 0 determines upper bound on $\bar{a}$.

With $(1 - \tau)(1 - a)Y$ at their disposal, capitalists make decisions between their consumption and investment. Let b, $0 \leq b \leq 1$, denote the capitalists' consumption as a fraction of $(1 - \tau)(1 - a)Y$.

Since we assume that capital stock does not depreciate, the investment or the rate of capital accumulation is given by:

$$\dot{K} = (1 - b)(1 - \tau)(1 - a)AK^{\alpha}G^{1-\alpha}. \tag{3}$$

If we substitute equation (2) into (3), we can express the rate of capital accumulation as:

$$\frac{\dot{K}}{K} = (1 - b)(1 - \tau)(1 - a)A^{\frac{1}{\alpha}}(1 - a)^{\frac{1-\alpha}{\alpha}}\tau^{\frac{1-\alpha}{\alpha}}. \tag{4}$$

Given a and b, tax rate or the size of the government which is $G/(1 - a)Y$ in our model, affects the growth rate of capital stock in two opposing ways. First, with output held constant, an increase in tax rate means smaller addition to the capital stock. Second, an increase in tax rate induces an increase in G and Y, which has a positive effect on capital stock. First effect is dominated when tax rate is low and two effects are balanced when tax rate equals $1 - \alpha$ so that at that point the rate of capital accumulation is maximized. Since $1 - \alpha$ is the elasticity of output net of workers' consumption with respect to G, production efficiency is also achieved when $\tau = 1 - \alpha$. (See Appendix A)

The objectives of workers and capitalists are assumed to maximize the value of consumption stream over the time horizon. And, we assume there is no future discounting.[5] Then, workers choose a to maximize:

$$\int_0^T aAK^{\alpha}G^{1-\alpha}dt. \tag{5}$$

Similarly, capitalists choose b to maximize:

$$\int_0^T b(1 - \tau)(1 - a)AK^{\alpha}G^{1-\alpha}dt. \tag{6}$$

The objective of government is assumed to maximize a weighted sum of welfares of workers and capitalists. Then, without future discounting, government chooses tax policy to maximize:

$$\int_0^T [wa + (1 - w)b(1 - \tau)(1 - a)]AK^{\alpha}G^{1-\alpha}dt, \tag{7}$$

where w is the weight assigned to the welfare of workers and $1 - w$ is the weight on the capitalists' welfare. These weights are given exogenously and do not change over time.

[5]See the conclusion part of the paper.

To consider how the dynamic inefficiency emerges in the presence of the government, we first solve and compare Command and Nash equilibria. Then, to investigate the potential coordinating role of the government, we consider the equilibrium under the optimal tax policy.

A. Command Equilibrium

To make the comparison with Nash equilibrium more easy, we assume that lump-sum taxes are not available to the government in Command equilibrium. Then, government maximizes equation (7) by choosing optimal paths of a, b, and τ subject to the budget constraint and the capital accumulation constraint.

To solve the Command equilibrium, we set up the Hamiltonian, by substituting the government budget constraint into equation (7), as:

$$\begin{aligned} H^{CE} &= [wa + (1-w)b(1-\tau)(1-a) + p(1-b)(1-a)(1-\tau)] \\ &\quad \times A^{\frac{1}{\alpha}}(1-a)^{\frac{1-\alpha}{\alpha}}\tau^{\frac{1-\alpha}{\alpha}}K. \end{aligned} \tag{8}$$

The costate variable, p, represents the value in terms of present consumption to the government of a marginal increase in the capital stock. p must be continuous in time and satisfies the condition:

$$\dot{p} = -[wa + (1-w)b(1-\tau)(1-a) + p(1-b)(1-a)(1-\tau)]A^{\frac{1}{\alpha}}(1-a)^{\frac{1-\alpha}{\alpha}}\tau^{\frac{1-\alpha}{\alpha}}. \tag{9}$$

Note that p is non-increasing over time and whose value is 0 at time T, $p(T) = 0$, the transversality condition.

The conditions for optimality, with some simplifications, are given by the following equations:

$$\frac{\partial H^{CE}}{\partial a} = \frac{\alpha - a}{1-a}w - (1-w)b(1-\tau) - p(1-b)(1-\tau) = 0 \tag{10}$$

$$\frac{\partial H^{CE}}{\partial b} = (1-w)(1-\tau)(1-a) - p(1-a)(1-\tau) = 0 \tag{11}$$

$$\begin{aligned} \frac{\partial H^{CE}}{\partial \tau} &= (1-\alpha)(1-w)b(1-a) - (1-w)b(1-a)\tau \\ &\quad + p(1-\alpha)(1-b)(1-a) - p(1-b)(1-a)\tau + (1-\alpha)wa = 0. \end{aligned} \tag{12}$$

Equation (12) gives us the optimal condition for tax rate. Assuming interior solution, we derive:[6]

$$\tau = (1-\alpha)[1 + \frac{1}{1-a}\frac{wa}{(1-w)b + p(1-b)}]. \tag{13}$$

Given the choices of a and b, an increase in tax rate has three different effects. First, workers' welfare in current period is increased due to the increase in G and Y. Second, the welfare of capitalists in current period is affected negatively from the increase in tax rate and positively from the increase in the output. Third, capital stock is changed so that the welfare of both workers and capitalists in the future is affected.

If we consider only the second and third effects, tax rate should be set to achieve production efficiency, which is obtained at $\tau = 1 - \alpha$. However, when we add the first effect, in order for the three effects to be balanced off, tax rate has to be larger than $1 - \alpha$ and as the weight given to workers becomes larger, tax rate diverges further from $1 - \alpha$. It turns out that tax rate is larger than $1 - \alpha$ by the factor of $[1/(1 - \alpha)]$ $\{wa/[(1 - w)b + p(1 - b)]\}$, which is roughly the ratio of the first effect to the sum of second and third effects.

Only when $w = 0$, that is when the welfare of workers is ignored, tax rate satisfies the production efficiency condition. But, if some positive weight is attached to the welfare of workers, the Command equilibrium tax rate is greater than the rate that maximizes capital accumulation.

Also, note that, given w, as a increases, tax rate increases. As the portion of output that workers consume increases, more services are provided to make up for the decreased investment of capitalists. Therefore, as w or a increase, the size of the government increases.

Since p is non-increasing and p appears in the denominator of the expression for the tax rate, given the values of other variables, tax rate is non-decreasing over time. As is indicated in Appendix A, since tax rate is non-decreasing over time and has values that are greater than $1 - \alpha$, the speed of the rate of capital accumulation is non-increasing over time.

Equation (11) given the optimal condition for investment decision:

$$\begin{aligned} &b = 1 \text{ if } 1 - w > p, \\ &b = 0 \text{ if } 1 - w < p, \\ &b \in [0,1] \text{ if } 1 - w = p. \end{aligned} \tag{14}$$

[6]When the tax rate as derived in equation (13) in the text is greater than 1, then tax rate is equal to 1. Sufficient condition for the interior solution for tax rate is given by:

$$\frac{\alpha}{1-\alpha} \geq \frac{w\bar{a}}{(1-w)(1-\bar{a})}.$$

Above condition is derived by setting the tax rate in equation (15) less than 1 when evaluated at $a = \bar{a}$.

Whenever the social value of a marginal increase in the capital stock is greater than the weight given to capitalists, capitalists should not consume and invest at the maximum rate. Note that one unit of capitalists consumption is counted only as $1 - w$ units in government objective function.

Utilizing the fact that p is non-increasing and $p(T) = 0$, we can define t'' as the time when $p = 1 - w$. t'' is the switching point for capital accumulation. We shall call the time interval between 0 and t'' as the initial phase and the interval between t'' and T as the last phase. Then, since $b = 1$ when $1 - w > p$, tax rate and $\dot{p}$ during the last phase become:

$$\tau = (1-\alpha)[1 + \frac{aw}{(1-a)(1-w)}].$$
$$\dot{p} = -[wa + (1-w)(1-\tau)(1-a)]A^{\frac{1}{\alpha}}(1-a)^{\frac{1-\alpha}{\alpha}}\tau^{\frac{1-\alpha}{\alpha}}. \tag{15}$$

When a is constant, which is the case in equilibrium (See below), tax rate becomes constant and so is $\dot{p}$. That implies that p declines uniformly for the last phase. If we represent $\dot{p} = - S(a,w)$, then $p(T) = 0$ and $p(t'') = 1 - w$ allow us to solve for t'' as:

$$t'' = T - \frac{1-w}{S(a,w)}. \tag{16}$$

Given that $p = 1 - w$ at the switching point, the switching time is determined by how rapidly p approaches 0 during the last phase.

Some comparative statics are conducted on t''. Given that no further new investment is made in the last phase, an increase in w has an ambiguous effect on $\dot{p}$. That is because workers are positively affected while capitalists are negatively affected. But, given the value of $\dot{p}$, an increase in w forces government to make capitalists invest for longer time period. But, if we combine these two effects, it is shown that $dt''/dw > 0$. That is, the larger is the weight given to workers, the interval for capital accumulation becomes longer. (See Appendix B)

Suppose that the weight attached to workers is greater (less) than that of capitalists. Then, as a increases, the valuation to government of a marginal increase in the capital stock becomes larger (smaller) so that p declines more rapidly (slowly) in the last phase. This implies that the switching point arrives later (sooner) in time. (See Appendix B)

Now, equation (10) gives the optimal decision rules for a. When workers increase consumption level by one unit, that induces changes in the government expenditures which affect workers' welfare, capitalists consumption, and the capital stock. Government chooses a taking these effects into consideration.

Notice that when $\underline{a}$ is greater than α, whole expression becomes negative. In this case, workers should maintain their consumption at the minimum level. α can be interpreted as the elasticity of output with respect to capital stock. So that when the productivity of capital stock is less than the minimum proportion of output that workers can consume, workers should keep their stake at the minimum.

Below, we analyse equation (10) assuming that $\underline{a}$ is less than α. We consider two phases separately.

Last Phase: $1 - w > p$.

During the last phase, $b = 1$, therefore, from equation (10), we derive the following decision rules for a:

$$a = \bar{a} \quad \text{if} \quad \frac{\alpha - a}{1 - a} w > (1 - w)(1 - \tau).$$

$$a = \underline{a} \quad \text{if} \quad \frac{\alpha - a}{1 - a} w < (1 - w)(1 - \tau).$$

$$a \in [\underline{a}, \bar{a}] \quad \text{if} \quad \frac{\alpha - a}{1 - a} w = (1 - w)(1 - \tau).$$

Consider $[(\alpha - a)/(1 - a)]\, w = (1 - w)(1 - \tau)$. Since the tax rate during this interval is given by equation (15), this equation simply becomes $w = 1/2$. Therefore, above conditions become:

$$a = \bar{a}, \quad \text{if} \quad w > \frac{1}{2}, \quad a = \underline{a}, \quad \text{if} \quad w < \frac{1}{2}, \quad a \in [\underline{a}, \bar{a}] \quad \text{if} \quad w = \frac{1}{2}.$$

Initial Phase: $1 - w < p$.

During the initial phase, $b = 0$. And, the tax rate in this interval is given by equation (13) as $\tau = (1 - \alpha)[1 + wa/p(1 - a)]$. To derive the implication for a, we consider $[(\alpha - a)/(1 - a)]\, w = p(1 - \tau)$ in equation (10). With the substitution of tax rate, above equation is simplified as $w = p$.

When $w \leq 1 - w$, that is when $w \leq 1/2$, p is greater than w, implying that $a = \underline{a}$. But when $w > 1/2$, we have the following conditions:

$$\begin{aligned} &a = \underline{a} \ \text{ for } \ 0 \leq t < t', \\ &a \in [\underline{a}, \bar{a}] \ \text{ at } \ t', \\ &a = \bar{a} \ \text{ for } t' < t < t'', \end{aligned}$$

where t' is the time when $p = w$. When $1 - w = p$, it can be easily seen that decision rule for a becomes identical to that in the last phase.

Above considerations indicate that there are two possible paths that the economy can take in Command equilibrium.

PathCE1: either $\alpha \leq \underline{a}$ or $w < 1/2$ and $\alpha > \underline{a}$.

$$a = \underline{a}, \quad b = 0, \quad \tau = (1-\alpha)[1 + \frac{w\underline{a}}{p(1-\underline{a})}] \quad \text{for} \quad 0 \leq t < t''.$$

$$a = \underline{a}, \quad b \in [0, \ 1] \quad \tau = (1-\alpha)[1 + \frac{w\underline{a}}{p(1-\underline{a})}] \quad \text{at} \quad t''.$$

$$a = \underline{a}, \quad b = 1, \quad \tau = (1-\alpha)[1 + \frac{w\underline{a}}{(1-w)(1-\underline{a})}] \quad \text{for} \quad t'' < t \leq T.$$

In cases where the productivity of the capital stock is low or even though the productivity of capital is high enough to satisfy the minimum portion of workers' consumption but the weight given to the welfare of workers is less than half, our economy undergoes two phases. Initially, both workers and capitalists consume at the minimum level. At the moment when the capital stock stops accumulating, capitalists switch their consumption from the minimum to the maximum level while workers still maintain their consumption level at the minimum.

During the initial phase, the tax rate, which is greater than $1 - \alpha$, is increasing over time until it reaches its maximum value $(1 - \alpha)[1 + w\underline{a}/(1 - w)(1 - \underline{a})]$ at t''. Correspondingly, the capital stock is increasing at decreasing rate until there is no further new investment at t''.

PathCE2: $w > 1/2$ and $\alpha > \underline{a}$.

$$a = \underline{a}, \quad b = 0, \quad \tau = (1-\alpha)[1 + \frac{w\underline{a}}{p(1-\underline{a})}] \quad \text{for} \quad 0 \leq t < t'.$$

$$a \in [\underline{a}, \ \bar{a}], \quad b = 0, \quad \tau = (1-\alpha)[1 + \frac{a}{1-a}] \quad \text{at} \quad t'.$$

$$a = \bar{a}, \quad b = 0, \quad \tau = (1-\alpha)[1 + \frac{w\bar{a}}{p(1-\bar{a})}] \quad \text{for} \quad t' < t < t''.$$

$$a = \bar{a}, \quad b \in [0, \ 1], \quad \tau = (1-\alpha)[1 + \frac{w\bar{a}}{(1-w)(1-\bar{a})}] \quad \text{at} \quad t''.$$

$$a = \bar{a}, \quad b = 1, \quad \tau = (1-\alpha)[1 + \frac{w\bar{a}}{(1-w)(1-\bar{a})}] \quad \text{for} \quad t'' < t \leq T.$$

This is the case when the productivity of the capital stock is high enough to satisfy the subsistence portion of workers' consumption and at the same time the weight attached to the workers' welfare is relatively high.

Both workers and capitalists start with minimum level of consumption. Then during the time when capitalists restrict their consumption to the minimum, workers switch their consumption level from the minimum to the maximum. Finally, capitalists also switch from the minimum level of consumption to the maximum so that both workers and capitalists consume at the maximum rate.

The tax rate is increasing in initial phase and jumps from $(1-\alpha)/(1-\underline{a})$ to $(1-\alpha)/(1-\bar{a})$ at t', when workers switch their consumption level. Afterwards, tax rate is increasing until t'' and stays constant at $\tau=(1-\alpha)[1+w\bar{a}/(1-w)(1-\bar{a})]$.

In summary, in Command equilibrium, which path is taken by the economy depends on the relative values of $\underline{a}$, w, and α. The value of $\bar{a}$ is important only in the determination of switching point when $w > 1/2$. This contrasts with the Nash equilibrium where $\bar{a}$ plays a crucial role in determining the path of the economy.

Note also that when the weight given to workers is relatively large, there is a time when capitalists' function is just to accumulate whereas workers' function is just to consume at the maximum rate. This phase characterizes simple classical model where it is assumed that workers consume all their income while capitalists are merely investing machines who channel all their profits into capital accumulation.

As the weight attached to workers is increased, the longer will be the time for capital accumulation. However, when $w > 1/2$, $a=\bar{a}$ in the last phase so that the switching point depends on the value of $\bar{a}$. In this case, the interval for capital accumulation becomes longer with the increase in the value of $\bar{a}$. However, when $w < 1/2$, then switching point is determined by the value of $\underline{a}$. In this case, as the value of $\underline{a}$ is increased, the shorter becomes the interval for capital accumulation.[7]

B. Nash Equilibrium

In this subsection, we analyze Nash equilibrium of the game, where each player chooses its optimal response taking the strategies of other

[7]When $w=1/2$ and $\alpha > \underline{a}$, our economy is taking the following path:

$$a=\underline{a},\quad b=0,\quad \tau=(1-\alpha)[1+\frac{w\underline{a}}{p(1-\underline{a})}]\quad \text{for}\quad 0\le t<t''.$$

$$a=\bar{a},\quad b\in[0,\ 1],\quad \tau=(1-\alpha)[1+\frac{w\bar{a}}{(1-w)(1-\bar{a})}]\quad \text{at}\quad t''.$$

$$a\in[\underline{a},\ \bar{a}],\quad b=1,\quad \tau=(1-\alpha)[1+\frac{w\bar{a}}{(1-w)(1-\bar{a})}]\quad \text{for}\quad t''<t\le T.$$

players as given.

Consider workers' problem. Workers maximize equation (5) by choosing the time path of a subject to the capital accumulation constraint and the strategies of other players. To solve the problem, we set up the workers' Hamiltonian as:

$$H^w = [a + m(1 - b)(1 - a)(1 - \tau)]AK^{\alpha}G^{1-\alpha}, \tag{17}$$

where m is the costate variable representing the value to the workers of a marginal increase in the capital stock. m must satisfy the differential equation:

$$\dot{m} = -[a + m(1 - b)(1 - a)(1 - \tau)]\alpha AK^{\alpha - 1}G^{1-\alpha}. \tag{18}$$

Also, we have the transversality condition that $m(T) = 0$.

From the maximum principle, given the time paths of the tax rate and b, the solution to the workers' problem is given by:

$$\begin{aligned} &a = \bar{a} \quad \text{if} \quad 1 - m(1 - b)(1 - \tau) > 0, \\ &a = \underline{a} \quad \text{if} \quad 1 - m(1 - b)(1 - \tau) < 0, \\ &a \in [\underline{a}, \bar{a}\,] \text{ if } \quad 1 - m(1 - b)(1 - \tau) = 0. \end{aligned} \tag{19}$$

When workers give up one unit of current consumption, that contributes to the accumulation of the capital stock only by $(1 - b)(1 - \tau)$. Therefore, condition (19) implies, only when the valuation of $(1 - b)(1 - \tau)$ to the workers, which is given by $m(1 - b)(1 - \tau)$ is greater than 1, workers will abstain from current consumption.

The capitalists' Hamiltonian is:

$$H^c = [b + n(1 - b)](1 - a)(1 - \tau)AK^{\alpha}G^{1-\alpha}, \tag{20}$$

where n is the costate variable that represents the value to the capitalists of a marginal increase in the capital stock. n satisfies the equation:

$$\dot{n} = - [b + n(1 - b)](1 - a)(1 - \tau)\alpha AK^{\alpha - 1}G^{1-\alpha}, \tag{21}$$

and the transversality condition $n(T) = 0$.

Given the time paths of the tax rate and a, values of b that maximize the capitalists' Hamiltonian satisfy the conditions:

$$b = 1 \text{ if } 1 - n > 0, \; b = 0 \text{ if } 1 - n < 0, \; b \in [0, 1] \text{ if } n = 1. \tag{22}$$

When capitalists give up one unit of current consumption, that increases capital stock by one unit whose valuation to capitalists is n, hence above conditions follow.

Note that the decision rules for investment here are different from those in Command equilibrium in two respects. First, whatever weight is assigned to workers in government objective function, this weight is ignored. Second, even though $w = 0$, the valuation of a marginal increase in the capital stock is calculated differently. That is because while capitalists treat G as given in deciding investment, government internalizes the effect of an increase in K on τ and G.

The government Hamiltonian is Nash game is identical to that in Command equilibrium. Consequently, the time path of tax rates is the same as given by equation (13).

In Nash game, the valuation to each player of a marginal increase in the capital stock is different and follows different time path although these valuations are non-increasing and are 0 at time T for all players.

Given that the time path of the tax rate is given by equation (13), if we put together the decision rules of workers and capitalists, there are four possible combinations of a and b, which are given by:

$$\begin{aligned}
&\text{I: } a = \bar{a},\ b = 1 \text{ if } n < 1 \text{ and } 1 - m(1-b)(1-\tau) > 0.\\
&\text{II: } a = \underline{a},\ b = 1 \text{ if } n < 1 \text{ and } 1 - m(1-b)(1-\tau) < 0.\\
&\text{III: } a = \bar{a},\ b = 0 \text{ if } n > 1 \text{ and } 1 - m(1-b)(1-\tau) > 0.\\
&\text{IV: } a = \underline{a},\ b = 0 \text{ if } n > 1 \text{ and } 1 - m(1-b)(1-\tau) < 0.
\end{aligned}$$

Since the conditions in case II are mutually inconsistent, we can rule it out. As is usual with the finite time horizon problem, we start from the end. Since $n(T) = 0$ and n is non-increasing over time, it must be the case that near the end of the time horizon $n < 1$ so that case I will materialize, the last phase in Nash equilibrium.

If we substitute $b = 1$ and $a = \bar{a}$ into equation (21) and utilize the government budget constraint, the value of n during the last phase changes over time according to:

$$\dot{n} = -(1-\tau)\tau^{\frac{1-\alpha}{\alpha}} Q,$$

where

$$Q = \alpha(1-\bar{a})^{\frac{1}{\alpha}} A^{\frac{1}{\alpha}}.$$

Let us define $\bar{t}$ as the time when $n = 1$, that is $n(\bar{t}) = 1$. $\bar{t}$ is the switching point in Nash equilibrium. Then using the fact that $n(T) = 0$, we can derive the equation that determines the switching point as:

$$\frac{1}{Q} = \int_{\bar{t}}^{T} (1-\tau)\tau^{\frac{1-\alpha}{\alpha}} dt. \tag{23}$$

But, when $a = \bar{a}$ and $b = 1$, equation (13) indicates that tax rate in the last phase is constant at $(1 - \alpha)[1 + w\bar{a}/(1 - w)(1 - \bar{a})]$. Therefore, we can solve equation (23) for $\bar{t}$ as:

$$\bar{t} = T - \frac{1}{\alpha(1-\bar{a})^{\frac{1}{\alpha}} A^{\frac{1}{\alpha}} (1-\tau)\tau^{\frac{1-\alpha}{\alpha}}}. \tag{24}$$

Suppose that either w or $\bar{a}$ increases, then, tax rate is increased so that the valuation to capitalist of a marginal increase in the capital stock becomes smaller. This implies that n declines more slowly in the last phase so that switching point for capital accumulation comes earlier. (See Appendix B)

To determine which case is relevant before $\bar{t}$, we consider the value of m at $\bar{t}$. If we plug $a = \bar{a}$ and $b = 1$ into equation (18), m changes during the last phase according to:

$$\dot{m} = -Q\frac{\bar{a}}{1-\bar{a}}\tau^{\frac{1-\alpha}{\alpha}}.$$

From this equation, using $m(T) = 0$ and Q, we derive:

$$m(\bar{t}) = \frac{\bar{a}}{1-\bar{a}} \frac{\int_{\bar{t}}^{T} \tau^{\frac{1-\alpha}{\alpha}} dt}{\int_{\bar{t}}^{T} (1-\tau)\tau^{\frac{1-\alpha}{\alpha}} dt}.$$

Since tax rate is constant for the last phase, we get:

$$m(\bar{t}) = \frac{\bar{a}}{1-\bar{a}}\frac{1}{1-\tau} \quad \text{or} \quad (1-\tau)m(\bar{t}) = \frac{\bar{a}}{1-\bar{a}}.$$

This implies that case III and IV are potential candidates for the earlier phase in Nash equilibrium. Which case will actually materialize depends on the value of $\bar{a}$. Below, we consider two possible paths in Nash equilibrium.

PathNA1: $\bar{a} > 1/2$.

In this case $(1 - \tau)m > 1$ at $\bar{t}$. Before $\bar{t}$, $n > 1$ and $b = 0$ so that from equation (13), tax rate is given by $(1 - \alpha)[1 + wa/p(1 - a)]$, which is non-decreasing over time. Similarly, m is non-increasing over time. Therefore, before $\bar{t}$, $(1 - \tau)m$ is non-increasing over time, which implies that for the whole time period before $\bar{t}$, $(1 - \tau)m > 1$ so that we will have case IV. At $\bar{t}$, b can take any value between 0 and 1 so that even though $(1 - \tau)m > 1$, $(1 - b)(1 - \tau)m$ can be equal to or greater or less than 1. Therefore, we have:

$$a = \underline{a},\ \ b = 0,\ \ \tau = (1-\alpha)[1+\frac{w\underline{a}}{p(1-\underline{a})}]\ \text{ for }\ 0 \le t < \bar{t}.$$

$$a \in [\underline{a},\ \bar{a}],\ \ b \in [0,\ 1],\ \ \tau = (1-\alpha)[1+\frac{wa}{p(1-a)}]\ \text{ at }\ \bar{t}.$$

$$a = \bar{a},\ \ b = 1,\ \ \tau = (1-\alpha)[1+\frac{w\bar{a}}{(1-w)(1-\bar{a})}]\ \text{ for }\ \bar{t} < t \le T.$$

Initially, both workers and capitalists are satisfied with the minimum level of consumption and the capital stock is increasing at decreasing rate. At the time when the capital stock stops accumulating, tax rate stays at constant level and both workers and capitalist switch their consumption to the maximum.

PathNA 2: $\bar{a} < 1/2$.

When $\bar{a} < 1/2$, $(1-\tau)m(\bar{t})$ becomes less than 1. Therefore, just before $\bar{t}$, case III will be realized. Since $(1-\tau)m$ is non-increasing over time, there is a time $\underline{t}$ that precedes $\bar{t}$ and $(1-\tau)m(\underline{t}) = 1$. This implies that before $\underline{t}$, we will have case IV. At $\bar{t}$, $(1-b)(1-\tau)b$ is less than 1 so that $a = \bar{a}$. Therefore, we have:

$$a = \underline{a},\ \ b = 0,\ \ \tau = (1-\alpha)[1+\frac{w\underline{a}}{p(1-\underline{a})}]\ \text{ for }\ 0 \le t < \underline{t}.$$

$$a \in [\underline{a},\ \bar{a}],\ \ b = 0,\ \ \tau = (1-\alpha)[1+\frac{wa}{p(1-a)}]\ \text{ at }\ \underline{t}.$$

$$a = \bar{a},\ \ b = 0,\ \ \tau = (1-\alpha)[1+\frac{w\bar{a}}{p(1-\bar{a})}]\ \text{ for }\ \underline{t} < t < \bar{t}.$$

$$a = \bar{a},\ \ b \in [0,\ 1],\ \ \tau = (1-\alpha)[1+\frac{w\bar{a}}{(1-w)(1-\bar{a})}]\ \text{ at }\ \bar{t}.$$

$$a = \bar{a},\ \ b = 1,\ \ \tau = (1-\alpha)[1+\frac{w\bar{a}}{(1-w)(1-\bar{a})}]\ \text{ for }\ \bar{t} < t \le T.$$

Our economy undergoes three phases. In the first phase, both workers and capitalists begin with minimum level of consumption. But, during second phase even though workers switch their consumption to the maximum level, capitalists still maintain their consumption level at the minimum. In the final phase, both the workers and capitalists consume at the maximum rate and capital stops accumulating.

In summarizing Nash equilibrium, we note the following. When $\bar{a} > 1/2$, it is better for workers to keep their consumption to the minimum and hand over resources at the maximum rate while capitalists are determined to make investment. But, when $\bar{a} < 1/2$, even while capitalists are making new investment, the maximum portion of increased future output that can be taken by workers is small so that there is no incentive for workers to abstain from current consumption. In this case, our economy undergoes one additional phase compared to the case when $\bar{a} < 1/2$.

In contrast to the Command equilibrium where relative values of α, w, and $\underline{a}$ are important, in Nash equilibrium, it is now $\bar{a}$ that determines the particular path our economy follows.[8] Also, the larger is the maximum portion of output that can be taken by workers, the shorter will be the interval for capital accumulation. Similarly, the more heavily workers' welfare is counted in government objective function, the shorter will be the time interval for capital accumulation.

C. Dynamic Inefficiency and the Switching Point

Given that we explored the reasons why Nash and Command equilibria diverge from each other, in this subsection, we analyse the implication of the dynamic inefficiency to the switching point of capital accumulation.[9]

t'' and $\bar{t}$ are respectively the switching points in Command and Nash equilibria. For convenience, we reproduce them:

$$t'' = T - \frac{1-w}{[wa+(1-w)(1-\tau)(1-a)]A^{\frac{1}{\alpha}}(1-a)^{\frac{1-\alpha}{\alpha}}\tau^{\frac{1-\alpha}{\alpha}}}.$$

$$\bar{t} = T - \frac{1}{\alpha(1-\bar{a})^{\frac{1}{\alpha}}A^{\frac{1}{\alpha}}(1-\tau)\tau^{\frac{1-\alpha}{\alpha}}}.$$

We show that the accumulation of capital stock in Nash stops earlier than is desirable. Suppose that $w > 1/2$. Then, $a = \bar{a}$ in the last phase

[8]When $\bar{a} = 1/2$, $m(1 - \tau) = 1$ and $1 - m(1 - b)(1 - \tau) \geq 0$ at $\bar{t}$. This implies that our economy follows PathNA1 in this case.

[9]In Lancaster's model, government and distributional aspects are ignored, In this case, when $\bar{a} > 1/2$, both workers and capitalists start with the minimum level of consumption and both groups switch to the maximum level at the same point in time in Nash as well as in Command equilibrium. Therefore, two equilibria undergo identical phases except the timing of the switch. Therefore, the welfare loss in Nash can be calculated rather simply and the inefficiency of capitalism can be represented solely in terms of the switching time.

However, in our model, to make welfare judgement on Nash equilibrium is rather complicated. That is because different combinations of w, $\bar{a}$, $\underline{a}$ induce different phases between two equilibria. Hence, in the text, we focus on the considerations of switching points.

of Command equilibrium so that tax rates are identical during the last phase in both equilibria. In that case, since $(1-\bar{a})(1-\tau) > \alpha(1-\bar{a})(1-\tau)$, it is easily seen that $(1-w)/[w\bar{a}+(1-w)(1-\tau)(1-\bar{a})] < 1/\alpha(1-\bar{a})(1-\tau)$, which implies that $t'' > \bar{t}$ Now, suppose that $w < 1/2$. Then, a becomes $\underline{a}$ in the last phase of Command equilibrium so that Command equilibrium tax rate is smaller than Nash tax rate during the last phase. However, since $(1-\underline{a})^{1/\alpha} > (1-\bar{a})^{1/\alpha}$ and $(1-\tau)\tau^{(1-\alpha)/\alpha}$ has a larger value when evaluated at Command equilibrium tax rate, it is still true that $t'' > \bar{t}$.

To investigate the reasons for the earlier stopping of capital accumulation in Nash equilibrium, we consider two cases. In an economy where welfare of workers is counted relatively low ($w < 1/2$), it is desirable to maintain workers' consumption to the minimum for the entire time horizon. However, in Nash, workers can not be induced to take minimum portion during the last phase so that switching point occurs earlier. In addition, when the maximum portion that workers can consume is relatively small ($\bar{a} < 1/2$), workers consume at the maximum rate even though capitalists are making new investment.

Consider the case where welfare of workers is highly valued ($w > 1/2$). In this case, during the final phase, the behaviors of workers, capitalists, and tax rates are the same for two equilibria. But, since capitalists in Nash game do not internalize the externality associated with G, capital stops accumulating earlier in Nash.

Also, when the stake of workers is small ($\bar{a} < 1/2$), two equilibria exhibit identical phases of capitalistic process. However, two equilibria show divergence in earlier phases when the stake of workers is large ($\bar{a} > 1/2$). That is because even though it is desirable for workers to consume at the maximum rate, the workers in Nash restrict their consumption while capitalists are making new investment. That is to say, there exists a phase where there is two much saving.

We already established the following relationships:

$$\frac{dt''}{dw} > 0, \quad \frac{dt''}{da} > 0 \quad \text{if} \quad w > \frac{1}{2} \quad \text{and} \quad \frac{dt''}{da} < 0 \quad \text{if} \quad w < \frac{1}{2},$$

$$\frac{d\bar{t}}{dw} < 0, \quad \frac{d\bar{t}}{d\bar{a}} < 0.$$

Consider the effect of w on $t'' - \bar{t}$. As the weight attached to workers is increased, the time span for capital accumulation becomes longer in Command equilibrium while in Nash the span becomes shorter. Hence, with the increase in the weight of workers, the switching time diverges further between two equilibria.

When $w > 1/2$, it is $\bar{a}$ that determines the switching point in both equilibria. Furthermore, in this case, as the value of $\bar{a}$ is increased, $t'' - \bar{t}$ becomes larger. However, when $w < 1/2$, it is $\underline{a}$ that determines the switching point in Command equilibrium while $\bar{a}$ determines the switching point in Nash equilibrium. In this case, either when $\bar{a}$ is increased or $\underline{a}$ lowered, $t'' - \bar{t}$ becomes larger.

If it is desirable to reduce the difference between t'' and $\bar{t}$, whatever the value of the weight attached to the welfare of workers, it is better to have an economic or political structure where $\bar{a}$ is smaller. However, when the welfare of capitalists is more heavily counted than that of workers, it is better to have a larger value of $\underline{a}$.

D. Optimal Tax Policy

Now consider optimal tax policy where government chooses optimal time path of tax rates taking the strategies of workers and capitalists into account. The optimal tax policy improves on Nash equilibrium so that even though time paths of tax rates and the switching points are different, our economy under optimal tax policy undergoes the same phases as in Nash.

In this subsection, we focus on the case when $\bar{a} > 1/2$.[10] Then, government chooses tax policy to maximize:

$$\int_0^T [wa + (1-w)b(1-\tau)(1-a)]AK^{\alpha}G^{1-\alpha}dt \tag{25}$$

subject to the following constraints:

$$G = \tau(1 - a)AK^{\alpha}G^{1-\alpha}. \tag{2}$$

$$\dot{K} = (1 - b)(1 - \tau)(1 - a)AK^{\alpha}G^{1-\alpha}. \tag{3}$$

$$\frac{1}{Q} = \int_t^T (1-\tau)\tau^{\frac{1-\alpha}{\alpha}}dt. \tag{23}$$

$$a = \underline{a},\ b = 0 \ \text{ for } \ 0 \leq t \leq \bar{t} \text{ and } \ a = \bar{a},\ b = 1 \ \text{ for } \ \bar{t} \leq t \leq T. \tag{26}$$

If we substitute equation (2) and condition (26) into (25), the government objective function can be rewritten as:

$$\begin{aligned} & w\underline{a}A^{\frac{1}{\alpha}}(1-\underline{a})^{\frac{1-\alpha}{\alpha}}\int_0^{\bar{t}}\tau^{\frac{1-\alpha}{\alpha}}K_t\ dt \\ & +A^{\frac{1}{\alpha}}(1-\bar{a})^{\frac{1-\alpha}{\alpha}}\int_{\bar{t}}^T[w\bar{a}+(1-w)(1-\tau)(1-\bar{a})]\tau^{\frac{1-\alpha}{\alpha}}K_{\bar{t}}dt, \end{aligned} \tag{27}$$

[10]When $\bar{a} < 1/2$, the analysis can be conducted along the same lines as in the text except that in this case we have one additional phase to consider.

where K_t and $K_{\bar{t}}$ are derived from equation (3) as:

$$K_t = K_0 e^{\int_0^t Z(1-\tau)\tau^{\frac{1-\alpha}{\alpha}} dv} \quad \text{for} \quad 0 \le t \le \bar{t},$$

$$K_t = K_{\bar{t}} = K_0 e^{\int_0^{\bar{t}} Z(1-\tau)\tau^{\frac{1-\alpha}{\alpha}} dv} \quad \text{for} \quad \bar{t} \le t \le T,$$

where Z is a constant which is equal to $A^{1/\alpha}(1-\underline{a})^{1/\alpha}$.[11]

In deriving the optimal tax policy, we restrict the solution to the class where tax rate during the final phase is constant.[12]

In this cast, $\bar{t}$ is solved explicitly as in equation (24). If we substitute equation (24) into the objective function and differentiate with respect to the constant tax rate during the last phase, we derive the condition that determines the optimal constant tax rate for the last phase (See Appendix C):

$$\frac{A^{\frac{1}{\alpha}} K_{\bar{t}}}{Q\alpha^2 \tau(1-\tau)^2} \{(1-\underline{a})^{\frac{1-\alpha}{\alpha}} \alpha w \underline{a}(1-\alpha-\tau)$$
$$+(1-\bar{a})^{\frac{1-\alpha}{\alpha}} \{(1-w)(1-\bar{a})(1-\alpha-\tau)(1-\tau)+w\bar{a}(\alpha^2\tau+1-\alpha-\tau)\}\} = 0. \tag{28}$$

Suppose that $w = 0$, then equation (28) implies that the optimal tax rate is equal to $1-\alpha$. The optimal tax rate is the same as that of Nash and the switching points occur at the same point in time in both equilibria. But, in this case, the switching point in Command equilibrium comes later because the externality associated with G is internalized in Command equilibrium.

Suppose that $w > 0$. If we plug $\tau = 1-\alpha$ into equation (28), the expression becomes positive, implying that the optimal tax rate is greater than $1-\alpha$. Now, if we substitute $\tau = (1-\alpha)[1+w\bar{a}/(1-w)(1-\bar{a})]$, which is Nash tax rate, into equation (28), the expression becomes negative (See Appendix D), indicating that the optimal constant tax rate during the last phase is lower than Nash tax rate.

[11]Specified in this way, optimal tax policy is connected with the problem faced by a government that implements a development program, in which government tries to persuade people to save more to invest further for the benefit of future welfare. In our model, this is equivalent to the optimal choice of the switching time with appropriate choice of tax policy.

[12]The optimal tax policy as specified in the text is well-defined and can be solvable even though the solution may not be obtained in explicit form. However, we can not rule out the possibility that the restricted solution in the text is the unique one.

The fact that the optimal tax rate is greater than $1 - \alpha$, the rate at which production efficiency is achieved, but less than Nash tax rate, which reflects distributional consideration, implies that the optimal tax policy partially corrects the distributional consequences in Nash at the expense of production efficiency.

If we denote the switching point under optimal tax policy as t^*, then, equation (24) implies that $t^* > \bar{t}$. Also, when $w = 0$, we have $t^* < t''$. The optimal tax policy improves on Nash equilibrium by lengthening the interval for capital formation. To the government as a coordinator, the valuation of a marginal increase in the capital stock at the switching point in Nash is greater than $1 - w$ so that it is desirable to make capitalists invest further.

However, when weight attached to the welfare of workers is positive, we can not say unambiguously about the relative values of t^* and t''. That is, we can not rule out the possibility that, under the optimal tax policy, the switching point can come later than that of Command equilibrium.[13]

To investigate what happens to the welfare of workers and capitalists separately is interesting but the analysis becomes too complicated to give any definite results. Consider the case of workers. The situation is different depending on the relative values of w. In Nash, workers consume at the minimum rate before $\bar{t}$. By lowering the tax rate during the last phase, workers are induced to maintain minimum rate of consumption a little longer. When $w < 1/2$, it is desirable to restrict workers' consumption to the minimum to the end and optimal tax policy can do that even though it does it only partially. But suppose that $w > 1/2$. If $\bar{t}$ comes before t', optimal tax policy results in equilibrium that is close to Command equilibrium. However, if $\bar{t}$ comes after t', even though it is desirable for workers to consume at the maximum rate, optimal tax policy induces workers to keep their consumption to the minimum for longer period of time. Furthermore, to obtain the change

[13]We can solve equation (28) using the discriminant of the quadratic forms in τ. But, that does not give an unambiguous answer in comparing the switching points in optimal tax policy and Command equilibrium. This reflects the following fact. When $w > 1/2$, Nash tax rate during the final phase is identical to Command equilibrium tax rate. Therefore, in this case, the optimal tax rate is less than Command equilibrium tax rate. However, when $w < 1/2$, Command equilibrium tax rate is less than Nash tax rate and the comparison between the optimal tax rate and Command equilibrium tax rate does not give unambiguous result.

in the welfare of each group over the whole time horizon, we have to have the optimal tax rates before the switching point. (See Appendix E)

As is well-known, even though government tries to impose lower than Nash tax rate for the last phase, this policy suffers from time inconsistency problem. Here we just indicate the problem and do not consider the solutions to deal with time inconsistency problem that are suggested in the literature.

Suppose that capitalists and workers are faithful to the optimal policy so that they switch at t^*. Suppose that all three players behave according to optimal tax policy and time has elapsed to $\bar{t}$, which is the switching point in Nash. At some time before t^* but after, $\bar{t}$, if government reoptimizes at that point, given the behaviors of workers and capitalist, government has an incentive to set a higher than optimal tax rate and maintain this higher rate to the end. Because, given the behaviors of workers and capitalists, increase in the tax rate implies increased G and Y, which increases the government objective function. If workers and capitalists anticipate this time-inconsistency problem, they do not adhere to the optimal policy and switch their behaviors before t^*. This process converges to $\bar{t}$, which is the switching point under consistent tax policy which is Nash.

III. Extensions and Conclusions

Without repeating the main results of the analysis, we briefly discuss some of the possible extensions of the model.

We assumed that there is no future discounting. If the discount rates are identical among three players, the results in the text will remain valid except that the switching points come earlier. However, when the discount rates are different among three players, the decision rules for workers and capitalists as well as government will be different so that discount rates play crucial role in determining the equilibrium path of the economy.

As to the role of government, we assumed that the objective function of the government is the weighted sum of people's welfare. However, as a positive model of capitalistic process where government plays an important role, it is better to assume that government itself is a selfish agent. See, for example, Grossman and Noh (1990).

In the model, w and a, and $\bar{a}$ play important roles. It has been indicated that if we want to reduce the difference of the switching point between Nash and Command equilibria, it is desirable to have a small value of $\bar{a}$ regardless of the relative values of w and a large value of $\underline{a}$ when the welfare of capitalists heavily is counted in government objective function. Incorporating political process for the endogenous determination of these variables in addition to the economic equilibrium in the model would be an interesting extension of the model.

Appendix

A. Suppose that a and b are fixed when we increase τ. Then, ignoring constant terms in equation (4) in the text, we have:

$$\frac{d(\frac{\dot{K}}{K})}{d\tau} = \frac{1}{\alpha}\tau^{\frac{1-2\alpha}{\alpha}}(1-\alpha-\tau), \quad \frac{d^2(\frac{\dot{K}}{K})}{d\tau^2} = \frac{1-\alpha}{\alpha}\tau^{\frac{1-3\alpha}{\alpha}}(1-2\alpha-\tau).$$

These equations imply that the rate of capital accumulation is maximized when tax rate is $1 - \alpha$. The speed of the rate of capital accumulation is decreasing as tax rate is increased from $1 - 2\alpha$.

B. Consider the term $(1 - w)/S(a, w)$ in t'', which is given in equation (16) in the text. If we differentiate this term with respect to w, ignoring denominator which is positive, we derive $- a[\alpha + (1 - \alpha)/\alpha]$ which is negative. This implies that $dt''/dw > 0$. Now, if we differentiate $S(a, w)$ with respect to a, we derive:

$$\frac{dS(a,w)}{da} = A^{\frac{1}{\alpha}}[aw+(1-w)(1-a)](1-a)^{\frac{1-2\alpha}{\alpha}}\tau^{\frac{1-2\alpha}{\alpha}}(2w-1)\frac{1-\alpha}{1-w}.$$

This implies that:

$$\frac{dt''}{da}>0 \text{ if } w>\frac{1}{2}, \quad \frac{dt''}{da}<0 \text{ if } w<\frac{1}{2}.$$

Consider $\bar{t}$, equation (23) in the text. From $d\tau/dw = \bar{a}(1 - \alpha)/[(1 - w)^2(1 - \bar{a})] > 0$ and $\tau > 1 - \alpha$, it follows:

$$\frac{d\bar{t}}{dw} = \frac{\frac{d\tau}{dw}}{\alpha^2(1-\bar{a})^{\frac{1}{\alpha}}A^{\frac{1}{\alpha}}}\frac{1-\alpha-\tau}{(1-\tau)^2\tau^{\frac{1}{\alpha}}} < 0.$$

Now, consider $(1 - \bar{a})^{1/\alpha}(1 - \tau)\tau^{(1-\alpha)/\alpha}$ in $\bar{t}$ equation. If we differentiate this term with respect to $\bar{a}$, we get:

$$-\frac{1}{\alpha}(1-\bar{a})^{\frac{1}{\alpha}}\tau^{\frac{1-2\alpha}{\alpha}}[(1-\tau)\tau-(1-\bar{a})(1-\alpha-\tau)\frac{d\tau}{d\bar{a}}]>0,$$

where $d\tau/d\bar{a}$ = $[w(1-\alpha)/(1-w)]\,[1/(1-\bar{a})^2] > 0$. This implies that $d\bar{t}/d\bar{a} < 0$.

C. If we differentiate equation (27) with respect to the constant tax rate which prevails for the last phase and set it equal to 0, we get:

$$\frac{d\bar{t}}{d\tau} w\underline{a} A^{\frac{1}{\alpha}} (1-\bar{a})^{\frac{1-\alpha}{\alpha}} \tau^{\frac{1-\alpha}{\alpha}} K_{\bar{t}}$$

$$-\frac{d\bar{t}}{d\tau}[w\bar{a}+(1-w)(1-\tau)(1-\bar{a})]A^{\frac{1}{\alpha}}(1-\bar{a})^{\frac{1-\alpha}{\alpha}}\tau^{\frac{1-\alpha}{\alpha}}K_{\bar{t}}$$

$$-\int_{\bar{t}}^{T}(1-w)(1-\bar{a})A^{\frac{1}{\alpha}}(1-\bar{a})^{\frac{1-\alpha}{\alpha}}\tau^{\frac{1-\alpha}{\alpha}}K_{\bar{t}}\ dt \qquad (*)$$

$$+\int_{\bar{t}}^{T}\{w\bar{a}+(1-w)(1-\tau)(1-\bar{a})\}$$

$$\times[A^{\frac{1}{\alpha}}(1-\bar{a})^{\frac{1-\alpha}{\alpha}}\{\frac{1-\alpha}{\alpha}\tau^{\frac{1-2\alpha}{\alpha}}K_{\bar{t}}+\tau^{\frac{1-\alpha}{\alpha}}\frac{dK_{\bar{t}}}{d\bar{t}}\frac{d\bar{t}}{d\tau}\}]\ dt=0,$$

where $$\frac{d\bar{t}}{d\tau}=\frac{1-\alpha-\tau}{\alpha Q(1-\tau)^2\tau^{\frac{1}{\alpha}}},\quad \frac{dK_{\bar{t}}}{d\bar{t}}=K_{\bar{t}}Z(1-\tau)\tau^{\frac{1-\alpha}{\alpha}}.$$

If we plug above relations into ($*$), with some simplifications we derive equation (28) in the text.

D. Consider the following terms in the equation (28) in the text:

$$(1-w)(1-\bar{a})(1-\alpha-\tau)(1-\tau)+w\bar{a}\{\alpha^2\tau+1-\alpha-\tau\}.$$

If we substitute $\tau = (1-\alpha)[1 + w\bar{a}/(1-w)(1-\bar{a})]$ into above equation, the expression becomes negative:

$$-\frac{w\bar{a}\alpha(1-\alpha)^2[w\bar{a}+(1-w)(1-\bar{a})]}{(1-w)(1-\bar{a})}<0.$$

Since Nash tax rate is greater than $1-\alpha$, the expressions in equation (28) evaluated at Nash tax rate become negative.

E. To derive the optimal tax rate for the first phase, we can take tax rate for the last phase as given. So that, in the second term in equation (27), whole expression except $K_{\bar{t}}$ can be treated as constant denoted by R. If we denote the expression in front of the integral sign in the first term as M, equation (27) becomes:

$$M\int_0^{\bar{t}}\tau^{\frac{1-\alpha}{\alpha}}K_t dt+RK_{\bar{t}}.$$

If we differentiate above expression with respect to tax rate and set the resulting equation equal to zero, we get:

$$M\int_0^{\bar{t}} \frac{1}{\alpha}\tau^{\frac{1-2\alpha}{\alpha}} K_t[1-\alpha+\tau Z\int_0^t \tau^{\frac{1-2\alpha}{\alpha}}(1-\alpha-\tau)dv]dt$$
$$+\frac{Z}{\alpha}K_{\bar{t}}\int_0^{\bar{t}}\tau^{\frac{1-2\alpha}{\alpha}}(1-\alpha-\tau)dv=0.$$

If the optimal tax rate is constant before the switching point, above equation implies that that has to be greater than 1 - α. Other than that it is difficult to derive and compare optimal and Nash tax rates during the initial phase.

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