# Differentiability of the Value Function: A New Characterization

**Taesung Kim***

This paper characterizes the differentiability of the value function. We provide a characterization of the necessary and sufficient conditions for the differentiability of the value function. This generalizes the well-known differentiability result of Benveniste and Scheinkman (1979) which shows that the concavity restriction on the return function and the convex graph restriction on the constraint correspondence are sufficient to prove the differentiability. In addition to generalization, our proof is quite simple and different from that of Benveniste and Scheinkman in not using the concavity assumptions.

We also show the differentiability of the indirect function in the envelope theorem under quite weak assumptions. This generalizes the established results regarding the differentiability of the support function and that of the cost function. (*JEL* Classification: C60)

## I. Introduction

In working with dynamic models of economic problems the method of dynamic programming, introduced by R. Bellman, has been widely used. By defining a value function or indirect function properly, this method reduces an optimization problem of an arbitrary (possibly infinite) number of periods to a simple problem of two periods without affecting the optimal solution of the problem. In understanding certain basic properties of the optimal solution, establishing the differentiability of the value function turned out to be very useful.

It has been shown by Benveniste and Scheinkman (1979) that under fairly general assumptions the value function is *once* differentiable.

*Associate Professor, Department of International Economics, Seoul National University, Seoul 151-742, Korea. I am grateful to Kim Border and Wan-Jin Kim who gave much useful feedback and many suggestions.

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Since then several authors have established *twice* differentiability of the value function with additional assumptions (see Araujo and Scheinkman 1977; Boldrin and Montrucchio 1989; Santos 1991). Specifically, Benveniste and Scheinkman (1979) showed that the concave return function and the constraint correspondence with convex graph are sufficient conditions for the differentiability of the value function and their proof uses Rockafellar's lemma (see Rockafellar 1970, Theorem 25.1, p. 242) about the properties of the subgradient of concave function. However, the complete characterization of *once* differentiability of the value function is still unknown. So, the main purpose of this note is to present the necessary and sufficient condition of *once* differentiability of the value function, therefore generalizing the result of Benveniste and Scheinkman.

In section II, we completely characterize the differentiability of the value function (Theorem 1). As one corollary to our characterization, we show that if the optimal solution is unique, then the value function is differentiable (Corollary 1). Benveniste and Scheinkman's differentiability result turns out to be another corollary to our characterization theorem (Theorem 2).

Since the differentiability of value function in dynamic programming is basically of the same nature as that of indirect function in the envelope theorem, in section III we generalize the differentiability result of indirect function in the envelope theorem. This result also implies the established results regrading the differentiability of the support function. In getting the derivative of indirect function with respect to the parameter, the standard envelope theorem requires *twice* differentiability and appropriate rank condition on the objective function since proofs usually rely on the implicit function theorem. Instead, we give necessary and sufficient conditions for the differentiability of the indirect function under *once* differentiability of the objective function (Theorem 3). We also give an example of nondifferentiable indirect function not satisfying our condition.

## II. Main Results

We begin by stating a standard deterministic model of dynamic programming. A detailed exposition of this model can be found in Harris (1987) and Stokey, Lucas and Prescott (1989). A social planner is interested in choosing $\{x_t\}_{t=1}^{\infty}$ which maximizes

$$\sum_{t=0}^{\infty}\beta^t F(x_t, x_{t+1}) \tag{1}$$

$$\text{subject to } x_{t+1} \in \Gamma(x_t),\ t = 0, 1, 2, \ldots,$$
$$x_o \in X \text{ given, } \beta \in (0, 1),$$

where $X \subset R_n$, the real-valued function $F : X \times X \to R$ is called the *return function*, $\Gamma: X \to \to X$, the *constraint correspondence*, and $\beta$, the *discount factor*. Corresponding to the problem (1), we have a functional equation of the form

$$v(x) = \sup_{y \in \Gamma(x)} [F(x, y) + \beta v(y)] \text{ for all } x \in X. \tag{2}$$

This functional equation is called *Bellman equation* or the *Principle of Optimality*. It is well known that the solution $v$ to the problem (2), evaluated at $x_o$, gives the maximum value in (1) when the initial state is $x_o$ and that a sequence $\{x_{t+1}\}_{t=1}^{\infty}$ attains the maximum value in (1) if and only if it satisfies

$$v(x_t) = F(x_t, x_{t+1}) + \beta v(x_{t+1}),\ t = 0, 1, 2, \ldots$$

See, for instance, Stokey, Lucas and Prescott (1989). Let

$$g(x) = \{y \in \Gamma(x) : v(x) = F(x, y) + \beta v(y)\}. \tag{3}$$

and let

$$D = \{(x, y) \in X \times X : y \in \Gamma(x)\},$$

where $g$ is called the *optimal policy correspondence*. We first introduce the conditions on $\Gamma$ and $F$ which are often assumed in the dynamic programming literature.

***Condition 1***

The constraint correspondence $\Gamma: X \to \to X$ is nonempty, compact-valued and continuous.

***Condition 2***

The return function $F : D \to R$ is continuous on $D$ and continuously differentiable in $x$.

Theorem 1 below shows that under some regularity conditions, the following condition is necessary and sufficient for the differentiability of the value function.

***Condition 3***
$\frac{\partial F}{\partial x_i}(x, y) = \frac{\partial F}{\partial x_i}(x, y')$ for all $y, y' \in g(x)$. So, we write it as $\frac{\partial F}{\partial x_i}(x, g(x))$.

If the optimal policy correspondence $g : X \to\to X$ is a singleton-valued function, then Condition 3 is automatically satisfied. In Theorem 2 below we also show that Condition 3 is satisfied if $F$ is concave in $(x, y)$ and $\Gamma$ has a convex graph. We can now state our main characterization theorem.

***Theorem 1***
Let $\Gamma$ and $F$ satisfy Conditions 1, 2, and let $v$ and $g$ satisfy (2) and (3). Suppose that $x_o \in int\, X$ and $g(x_o) \subset int\, \Gamma(x_o)$. Then if Condition 3 holds, $v$ is continuously differentiable at $x_o$ with

$$\frac{\partial v}{\partial x_i}(x_o) = \frac{\partial F}{\partial x_i}(x_o, g(x_o)) \text{ for } i = 1,2,\ldots,n.$$

On the other hand, if $v$ is differentiable, then Condition 3 is satisfied.

***Proof:*** To show that Condition 3 is sufficient for the continuous differentiability of $v$, let $h_i = (1, \ldots, 0, h, 0, \ldots, 0)$ and $y_o \in g(x_o)$. Since $y_o \in int\, \Gamma(x_o)$ and $\Gamma$ is continuous, it follows that $y_o \in int\, \Gamma(x_o + h_i)$ for $h$ sufficiently small. Therefore, for sufficiently small $h$,

$$\begin{aligned} & v(x_o + h_i) - v(x_o) \\ & \quad \geq \{F(x_o + h_i, y_o) + \beta v(y_o)\} - \{F(x_o, y_o) + \beta v(y_o)\} \\ & \quad \quad \text{(by the definition of } y_o \text{ and the fact the } y_o \in \Gamma(x_o + h_i)) \qquad (4) \\ & \quad = F(x_o + h_i, y_o) - F(x_o, y_o) \\ & \quad = \frac{\partial F}{\partial x_i}(\tilde{x}, y_o) \cdot h, \end{aligned}$$

where $\tilde{x}$ is between $x_o$ and $x_o + h_i$ by the mean value theorem. On the other hand choose $y_h$ such that $y_h \in g(x_o + h_i)$. Since $g$ is an upper hemi-continuous correspondence by Berge's maximum theorem, one can choose $\bar{y} \in g(x_o)$ such that $y_h \to \bar{y}$ as $h \to 0$. Since $\bar{y} \in int\, \Gamma(x_o)$, $y_h \in int\, \Gamma(x_o)$ for sufficiently small $h$. Therefore, for sufficiently small $h$,

$$\begin{aligned} & v(x_o + h_i) - v(x_o) \\ & \quad \leq \{F(x_o + h_i, y_h) + \beta v(y_h)\} - \{F(x_o, y_h) + \beta v(y_h)\} \\ & \quad \quad \text{(by the definition of } y_h \text{ and the fact the } y_h \in int\, \Gamma(x_o)) \qquad (4) \\ & \quad = F(x_o + h_i, y_h) - F(x_o, y_h) \\ & \quad = \frac{\partial F}{\partial x_i}(\hat{x}, y_h) \cdot h, \end{aligned}$$

where $\hat{x}$ is between $x_o$ and $x_o + h_i$ by the mean value theorem. It follows from (4) and (5) that for $h$ positive,

$$\frac{\partial F}{\partial x_i}(\tilde{x},y_o) \le \frac{v(x_o+h_i)-v(x_o)}{h} \le \frac{\partial F}{\partial x_i}(\hat{x},y_h). \tag{6}$$

For $h$ negative, inequality opposite to (6) holds. Note that $\tilde{x}$, $\hat{x} \to x_o$ and $y_h \to \bar{y}$ as $h \to 0$. Since $y_o$, $\bar{y} \in g(x_o)$,

$$\frac{\partial F}{\partial x_i}(x_o,y_o) = \frac{\partial F}{\partial x_i}(x_o,\bar{y}) = \frac{\partial F}{\partial x_i}(x_o,g(x_o)),$$

by Condition 3. Therefore by letting $h \to 0$ it follows from (6) that

$$\frac{\partial v}{\partial x_i}(x_o) = \frac{\partial F}{\partial x_i}(x_o,g(x_o)).$$

To show that Condition 3 is necessary for the differentiability of $v$, let $y_o \in g(x_o)$. Define $G : R^n \to R$ as

$$G(x) = F(x, y_o) + \beta v(y_o) - v(x).$$

Then for all $x$ near $x_o$, $G(x) \le 0$ and $G(x_o) = 0$ since $y_o \in g(x_o)$. Since $v$ and $F$ is differentiable in $x$, so is $G$. Therefore,

$$\frac{\partial G}{\partial x_i}(x_o) = \frac{\partial F}{\partial x_i}(x_o,y_o) - \frac{\partial v}{\partial x_i}(x_o) = 0.$$

Since this holds for all $y_o \in g(x_o)$, Condition 3 holds

*Q.E.D.*

Notice that if $g$ is singleton-valued, then Condition 3 is automatically satisfied. Therefore, the unique optimal policy for each state guarantees the differentiability of the value function.

***Corollary* 1**

Let $\Gamma$ and $F$ satisfy Conditions 1 and 2, and let $v$ and $g$ satisfy (2) and (3). Assume also that $g$ is a singleton-valued function. If $x_o \in$ *int* $X$ and $g(x_o) \subset$ *int* $\Gamma(x_o)$, then $v$ is continuously differentiable at $x_o$ with

$$\frac{\partial v}{\partial x_i}(x_o) = \frac{\partial F}{\partial x_i}(x_o,g(x_o)) \text{ for } i = 1,2,\ldots,n.$$

Benveniste and Scheinkman (1979) assumed the the graph of $\Gamma$ is convex and $F$ is concave in $(x, y)$ to prove the differentiability of the value function. However, the following theorem shows that convex graph of $\Gamma$ and concavity of $F$ imply Condition 3. Therefore, our result generalizes that of Benveniste and Scheinkman.

***Theorem 2***

Let $\Gamma$ and $F$ satisfy Conditions 1, 2 and assume that

1) $\Gamma: X \to\to X$ has a convex graph, and

2) $F: D \to R$ is concave.

Then Condition 3 is satisfied, therefore $v$ is continuously differentiable at $x_o \in int\, X$ with $g(x_o) \subset int\, \Gamma(x_o)$.

***Proof:*** Let $h_i = (0, \ldots, 0, h, 0, \ldots, 0)$. Since $F$ is concave and the graph of $\Gamma$ is convex, it is easy to see that the value function $v$ is concave. Therefore, $v$ possesses both left- and right-hand derivatives (see e.g., Rockafellar 1970, Theorem 1) and

$$\lim_{h<0, h\to 0} \frac{1}{h}[v(x_o + h_i) - v(x_o)] \geq \lim_{h>0, h\to 0} \frac{1}{h}[v(x_o + h_i) - v(x_o)] \tag{7}$$

On the other hand, let $y_o \in g(x_o)$. Since $y_o \in int\, \Gamma(x_o)$ and $\Gamma$ is continuous, it follows that $y_o \in int\, \Gamma(x_o + h_i)$ for $h$ sufficiently small. So, for sufficiently small $h$,

$$\begin{aligned} & v(x_o + h_i) - v(x_o) \\ & \quad \geq \{F(x_o + h_i, y_o) + \beta v(y_o)\} - \{F(x_o, y_o) + \beta v(y_o)\} \\ & \qquad \text{(by the definition of } y_o \text{ and the fact that } y_o \in \Gamma(x_o + h_i)) \\ & \quad = F(x_o + h_i, y_o) - F(x_o, y_o) \end{aligned}$$

Therefore, it follows that

$$\lim_{h>0, h\to 0} \frac{1}{h}[v(x_o + h_i) - v(x_o)] \geq \sup_{y\in g(x_o)} \frac{\partial F}{\partial x_i}(x_o, y) \tag{8}$$

and

$$\lim_{h<0, h\to 0} \frac{1}{h}[v(x_o + h_i) - v(x_o)] < \inf_{y\in g(x_o)} \frac{\partial F}{\partial x_i}(x_o, y). \tag{9}$$

It follows from (7), (8) and (9) that

$$\sup_{y\in g(x_o)} \frac{\partial F}{\partial x_i}(x_o, y) \leq \inf_{y\in g(x_o)} \frac{\partial F}{\partial x_i}(x_o, y).$$

Therefore, $\partial F / \partial x_i\, (x_o, y_o) = \partial F / \partial x_i\, (x_o, y')$ for all $y, y' \in g(x_o)$.

*Q.E.D.*

We can apply the similar idea to generalizing the differentiability of the indirect function in the envelop theorem, which will be discussed in the next section.

## III. Applications

Consider a parameterized maximization problem of choosing z:

$$\text{maximize } F(p, z)$$
$$\text{subject to } z \in Z,$$

where $z \subset R^n$ and $p \in P \subset R^m$. For each $p$, let

$$v(p) = \sup_{z \in Z} F(p, z) \tag{10}$$

and let

$$g(p) = \{z \in Z : v(p) = F(p, z)\}.$$

Now we can state the generalized version of the envelope theorem.

***Theorem 3***

Suppose that the continuous function $F : P \times Z \to R$ is continuously differentiable in z. Let $p_o \in int\, P$. Then if $\partial F / \partial p_i\ (p_o, z)$ is same for all $z \in g(p_o)$, the indirect function $v$ is continuously differentiable at $p_o$ with

$$\frac{\partial v}{\partial p_i}(p_o) = \frac{\partial F}{\partial p_i}(p_o, g(p_o)) \text{ for } i = 1, 2, \ldots, m.$$

On the other hand, if $v$ is differentiable at $p_o$, then $\partial F / \partial p_i\ (p_o, z)$ is same for all $z \in g(p_o)$.

***Proof:*** Proof is essentially the same as that in Theorem 1 if we replace $x$ with $p$, $y$ with $z$, $\Gamma(x)$ with $Z$ for all $x$, and equation (2) with equation (10) for the definition of $v$,.

*Q.E.D.*

***Remark 1***

1) Notice that $v$ is continuously differentiable if $g$ is singleton-valued.

2) The standard version of envelope theorem assumes twice differentiability of the objective function and the appropriate rank condition on the first-order condition since proofs usually rely on the implicit function theorem in getting the derivative of indirect function $v$. However, Theorem 3 only assumes once differentiability of the objective function in parameter $p$ since our proof does not rely on the implicit function

theorem.

The continuous differentiability of the support function (see, Rockafellar 1970, Theorem 2, p. 243 or Richter 1987) and that of the cost function in input prices (see, Saijo 1983) can be derived as a corollary of the above envelope theorem. For, these are the cases in which the objective function $F(p, z)$ of Theorem 3 has the form of $p \cdot z$.

***Corollary 2***

Suppose that $f: R^n_+ \to R$ is upper semi-continuous, strictly quasi-concave production function. Then for each $y$, the cost function $c(\cdot, y)$: $R^n_{++} \to R$, defined by

$$c(p,y) = \inf_{z \in Z} pz, \text{ where } Z = \{x \in R^n_+ : f(x) \geq y\},$$

is continuously differentiable at $p_o \in R^n_{++}$ with

$$\frac{\partial c}{\partial p_i}(p_o, y) = g_i(p_o, y),$$

where $g(p, y) = \{z \in Z : c(p, y) = pz\}$.

***Proof:*** Since $f$ is strictly quasi-concave, $g(\cdot, y)$ is single-valued for each $y$. Since $\{x \in R^n_+: f(x) \geq y\}$ is bounded from below and closed by upper semicontinuity of $f$, for each $y$ $g(\cdot, y)$ is a nonempty-valued, continuous function by the maximum theorem. If we let $Z = \{x \in R^n_+: f(x) \geq y\}$ and $F(p, z) = -pz$, then $c(p, y) = -\sup_{z \in Z} F(p, z)$. Therefore, the conclusion of the corollary follows from Theorem 3.

*Q.E.D.*

However, the next example shows that if $\partial F / \partial p_i(p, z)$ is not same for all $z \in g(p)$, indirect function $v$ is not necessarily differentiable.

***Example 1***

Let $F: R \times R \to R$ be

$$F(p,z) = \begin{cases} -(z+2)^2 & \text{if } z \leq -1 \\ -\dfrac{1}{4}pz^3 + z^2 + \dfrac{3}{4}pz + \dfrac{1}{2}p - 2 & \text{if } -1 < z < 1 \\ -(z-2)^2 + p & \text{if } z \geq 1 \end{cases}$$

Then $F$ is continuously differentiable. It is easy to show that

$$g(p) = \begin{cases} -2 & \text{if } p < 0 \\ \{-2, 2\} & \text{if } p = 0 \\ 2 & \text{if } p > 0 \end{cases}$$

and

$$v(p) = \begin{cases} 0 & \text{if } p < 0 \\ p & \text{if } p \geq 0 \end{cases}$$

Note that

$$\frac{\partial F}{\partial p}(0,2) \neq \frac{\partial F}{\partial p}(0,-2)$$

and $v$ is not differentiable at 0.

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