Separation or Not: A Critique of "Appearance-Based" Selection Criteria

In-Koo Cho*

We examine the foundations of traditional equilibrium selection rules that choose a separating equilibrium in a signaling model. With an example from the entry deterrence model of Milgrom and Roberts (1982), where the players are forced to reveal their private information in the post-entry game, we show that a separating equilibrium might not survive some forward induction argument (Kohlberg and Mertens 1986), and that the resulting equilibrium has many intuitive properties. If the players are not forced to reveal their private information in the post-entry game, the forward induction criterion selects the Pareto efficient separating equilibrium. We assert that in contrast to forward induction, "appearance-based" equilibrium selection criteria can provide misleading implications on the value of complete information in the post-entry game. (*JEL* classifications: C72, C73, D21)

I. Introduction

Separating equilibria have been the primary analytical tools in the signaling literature. Their tremendous popularity largely stems from the properties that highlight the underlying incentives of the informed party. These convictions about separating equilibria are reinforced by Cho and Sobel (1990), who prove that in most signaling models in economics, separating equilibria are the only ones that survive forward induction, as found Kohlberg and Mertens (1986).[1] This result provides

*Department of Economics, University of Chicago. This is the revised version of "Equilibrium Analysis of Entry Deterrence: Re-examination." The financial support from the Division of Social Sciences at the University of Chicago and the National Science Foundation (SES-8618596) is gratefully acknowledged.

[1]Throughout this paper, we always use forward induction in the sense of Kohlberg and Mertens (1986): an equilibrium survives forward induction if it survives repeated elimination of inferior strategies.

[**Seoul Journal of Economics** 1994, Vol. 7, No. 3]

a comforting theoretical foundation for the traditional selection criteria largely based on the "appearance" of the equilibrium.

This paper explores the potential pitfalls of such appearance-based selection rules. Despite their casual similarities to forward induction, they have very different consequences. The traditional appearance-based selection rules are largely motivated by the intuitive properties of separating equilibria—presuming, of course, their existence. This paper contends that even when a separating equilibrium does exist, this approach is rather suspect if the separating equilibrium does not survive forward induction. In fact, the equilibrium selected by the forward induction process can be more intuitively appealing than a separating equilibrium.

The most important benefit afforded by using forward induction arguments as the selection criterion is their broad applicability to general games, allowing controlled comparisons of different models. For example, suppose that each of the two models has two different Nash equilibria. Assume that the first equilibrium is subgame perfect, but is Pareto dominated by another imperfect Nash equilibrium. Then, contrasting the perfect equilibrium in one model with the Pareto dominating imperfect equilibrium of the other model is hardly a sensible way to compare the two models. By the same token, if the separating equilibrium does not survive forward induction arguments in one model, but does in another, then comparing these two separating equilibria may be equally misleading.

In order to support our claim, we choose an example from the entry deterrence model of Milgrom and Roberts (1982). Because Milgrom and Roberts originally choose a pure strategy separating equilibrium, this model and the selection rule provide a very useful benchmark case.[2] What is interesting in this particular example is that the equilibrium selected by forward induction can be characterized by a fairly standard profit maximization condition subject to an incentive compatibility constraint. Not only can this be considered more intuitive than the similar maximization problem behind any separating equilibrium, but the forward induction equilibrium also provides larger profit to the incumbent

[2]There is another technical reason for choosing the entry deterrence model. This is one of few economic models which does not satisfy single crossing properties. Given the results of Cho and Sobel (1990), it may not be surprising that the only equilibrium surviving forward induction arguments like Criterion $D1$ (Cho and Kreps 1987) is not a separating equilibrium. Thus, the same critique applies to general class of models which do not have single crossing properties.

firm than any separating equilibrium. Hence, Pareto efficiency arguments, one of the most popular equilibrium selection criteria, do not justify the selection of a separating equilibrium in this example.

But the real benefits of forward induction surface when we come to evaluate the value of information in the post-entry game. One important drawback of the original model of Milgrom and Roberts concerns this very issue: once entry occurs, the private information of the entrant and the incumbent must be truthfully revealed, and the post-entry game is played under complete information. One may ask what happens to the equilibrium outcome if the players are not forced to reveal their private information truthfully. Now we are faced with the task of comparing two different models, each having many sequential equilibria. By applying forward induction, we can identify a unique equilibrium for each case, whether or not a separating equilibrium exists. If the players are not forced to reveal their private information in the post-entry game, then the only equilibrium surviving forward induction is the Pareto efficient separating equilibrium. This is the same equilibrium selected by Milgrom and Roberts. Hence, by following their selection rule, one may find oneself comparing fundamentally different equilibria: (in technical jargon) one stable equilibrium and one unstable equilibrium. If so, any implications drawn from the exercise would be seriously suspect.

The paper is organized as follows. In section II, we precisely describe the model and the refinement concepts. Here, the notion of forward induction is tailored to compute an equilibrium in the entry deterrence model. In section III, we re-examine the original version of the model in Milgrom and Roberts (1982) with an example that illuminates the difference between their selection rule and forward induction arguments. Section IV analyzes the case where the players are not forced to reveal their private information. We demonstrate that the unique equilibrium surviving forward induction is the Pareto efficient separating equilibrium. Section V concludes the paper.

II. Model and Selection Criteria

Consider a market with an incumbent monopolist facing a potential entrant. In the first period, the incumbent monopolist chooses production level Q. Observing the first period quantity produced by the monopolist, the potential entrant decides whether or not to come into the market. If the potential entrant stays out, the incumbent firm remains a monopolist and receives the monopoly profit in the second period. If entry occurs, then the entrant must pay a fixed cost K and the post-entry game becomes a Cournot quantity setting duopoly game.

We denote the market demand by $P = a - Q$, where a is a large positive number. The cost functions of the incumbent and the entrant are $\phi(c_I, Q) = c_I Q$ and $\psi(c_E, Q) = c_E Q$. We assume that each firm has private information about its own efficiency, indexed by c_I and c_E. The actual realizations of c_I and c_E are drawn from two independent distributions G_I and G_E whose supports are $[\underline{c}_I, \overline{c}_I]$ and $[\underline{c}_E, \overline{c}_E]$, respectively. For analytic convenience, we assume that G_I and G_E are differentiable. It is assumed that the incumbent's discount factor is $\delta \in (0, 1)$.

Let H_I and H_E be two probability distribution functions over $[\underline{c}_I, \overline{c}_I]$ and $[\underline{c}_E, \overline{c}_E]$, respectively. Here, H_I represents the potential entrant's conjecture about the incumbent firm's type conditioned on its observation of the first period quantity level; similarly, H_E is the incumbent firm's conjecture about the entrant's type, conditioned on the fact that the entry has occurred.

By forced revelation of private information, we mean that the private information of the players must be truthfully revealed. By the post-entry game (H_I, H_E), we mean the simultaneous move quantity setting game where the conjecture about the entrant is H_E and about the incumbent is H_I. In particular, if the players are forced to reveal their private information, then H_I and H_E must be concentrated on the true types of each player. Under the specifications given above, the post-entry game (H_I, H_E) has a unique Bayesian Nash equilibrium for each (H_I, H_E). We write $\Pi_I^c(c_I: H_I, H_E)$ and $\Pi_E^c(c_E: H_I, H_E)$ for the expected Cournot duopoly profits of the c_I incumbent and the c_E entrant in the post-entry duopoly game. We write $Q_I^c(c_I: H_I, H_E)$ and $Q_E^c(c_E: H_I, H_E)$ as the pair of equilibrium quantities produced by each type of the incumbent and the entrant. Similarly, we write the monopolist profit for the c_I incumbent as $\Pi_I^M(c_I)$ and the monopolist profit maximizing quantity as $Q_I^M(c_I)$.

The first period quantity setting strategy of the incumbent firm is a (measurable) function

$$\sigma : [\underline{c}_I, \overline{c}_I] \rightarrow R_+.$$

The entry decision of the potential entrant is a (measurable) function

$$\tau : [\underline{c}_E, \overline{c}_E] \times R_+ \rightarrow \{0, 1\},$$

where $\tau(\cdot) = 1$ represents the case where entry is made and $\tau(\cdot) = 0$ represents the case where the potential entrant decides to stay out of the market. We allow the players to use randomized strategies whenever necessary. We define a sequential equilibrium in the usual way[3], that is a pair of strategy profile and system of beliefs which satisfy sequential rationality and consistency.

Our limit pricing model generally has a continuum of sequential equilibrium outcomes because the consistency of sequential equilibrium imposes no restrictions on beliefs off the equilibrium path. In order to reduce the set of equilibrium outcomes, we use forward induction arguments. Since the structure of signaling games is simpler than that of the limit pricing model, we must extend the original notion of refinements from signaling games. In order to make this paper self-contained, we discuss a minimal amount of material on the refinement of sequential equilibrium. For a comprehensive analysis, readers are urged to consult Kohlberg and Mertens (1986), Cho and Kreps (1987), Banks and Sobel (1987), and Cho and Sobel (1990).

By a signaling game we mean an extensive form game with the following structure. One player, called the Sender, has private information about his own type $t \in T$ where the probability distribution over T is common knowledge. Based on the private information, the Sender sends a message $m \in M$ to the other player, called the Receiver, possibly in a randomized fashion. Then, conditioned on the message received from the Sender, the Receiver chooses his own action $a \in A$, also possibly in a randomized fashion. Given a true type t, a message m and an action a, we write the payoff functions of the Sender and the Receiver as $u(t, m, a)$ and $v(t, m, a)$, respectively. We write the best response correspondence of the Receiver induced by his belief $\mu(t \mid m)$ over the type of the Sender as

$$MBR(\mu, m) = \arg\max_{r \in \Delta(A)} \sum_{a \in A} \sum_{t \in T} v(t, m, a) r(a \mid m) \mu(t \mid m),$$

where $\Delta(A)$ represents the space of probability distributions over A. The collection of (mixed) best responses induced by some belief is denoted by

$$MBR(T, m) = \bigcup_{\mu \in \Delta(T)} MBR(\mu, m).$$

Fixing a sequential equilibrium and a message m which is used with

[3]We do not state the definition of sequential equilibrium with mixed strategies, since the extension of the definition is straightforward.

probability zero in the equilibrium, we write U_t^* for the equilibrium payoff of the type t Sender. Define two subsets of $MBR\,(T,\,m)$,

$$P(t|m) = \left\{ r \in MBR(T,m) : U_t^* < \sum_{a\in A} u(t,m,a) r(a|m) \right\}$$

and

$$P^\circ(t|m) = \left\{ r \in MBR(T,m) : U_t^* = \sum_{a\in A} u(t,m,a) r(a|m) \right\}.$$

Criterion $D1$ requires that strategy m of the type t Sender should be eliminated if $\exists\, t' \in T$ such that $P(t|m) \cup P^\circ(t|m) \subset P(t'|m) \neq \emptyset$.

Our limit pricing model can be decomposed into three parts: the first period, when the incumbent chooses the quantity; the second period, when the potential entrant makes the entry decision; and the post-entry game. The linear structure of the model guarantees the existence of a unique (Bayesian) Nash equilibrium in the post-entry game whether or not the post-entry game is played under complete information. Once we replace the post-entry game by the equilibrium payoff vector, the second period of the game becomes a signaling game. In the second period, we compute a sequential equilibrium surviving Criterion $D1$. As demonstrated in the next section, Criterion $D1$ chooses a unique outcome from the second period of the game. We replace the second period of the game by the payoff vector of the unique $D1$ equilibrium outcome. Then the whole game is reduced to a signaling game where the incumbent monopolist corresponds to the Sender. We again compute a $D1$ equilibrium of the reduced game. Since each $D1$ equilibrium of the reduced game is followed by a unique $D1$ equilibrium for the rest of the game, we regard the $D1$ equilibrium of the reduced game as the solution of the whole game. Let us call this procedure STABAC(STAbility and BACkward induction).

The same procedure has been used in many economic models including Milgrom and Roberts (1986) and van Damme and Noldeke (1990). In general, the procedure described above is ad hoc. Since a stable equilibrium outcome need not include an equilibrium that induces a stable equilibrium in every subgame, there is no reason to believe that an equilibrium computed by STABAC survives forward induction, or vice versa. However, the entry deterrence model described in this paper satisfies the sufficient conditions that render STABAC consistent with strategic stability (Cho 1993), in the sense that any equilibrium outcome eliminated by STABAC is not a stable outcome of the whole game.[4] Roughly speaking, if all types of the informed party have the same preference over the response of the other party, and that response changes monotonically with respect to the belief about the informed party, then any equilibrium outcome eliminated by STABAC is not stable. We later identify a *unique* equilibrium outcome for the entry deterrence model. Since STABAC is generally implied by strategic stability, those two equilibrium outcomes are indeed stable outcomes of the model, and we can compare the two versions in a sensible way. Accordingly, we use STABAC and forward induction interchangeably throughout this paper.

III. With Forced Revelation: Example

The selection rule of Milgrom and Roberts is simply to choose the pure strategy separating equilibrium which provides the largest profit to the incumbent firm. This section presents an example to show Milgrom and Roberts' selection can be very different from the forward induction process. To simplify the analysis, consider a special informational structure, where the type space of the incumbent is $\{\overline{c}_I, \underline{c}_I\}$ and that of the potential entrant is the closed interval $[\underline{c}_E, \overline{c}_E]$. The assumption of such a two point type space for the incumbent firm permits us to parameterize the conjecture about incumbent's type by a real number $h \in [0, 1]$. By assuming that the potential entrant's type space is a closed interval, we have only to consider pure strategy entry decisions for the potential entrant. We write $U(c_I, h, Q)$ as the expected utility of the type c_I incumbent when he produces Q in the first period and is believed to be the high cost producer with probability h.

Assume that the post-entry game is played under complete information, and suppose that the parameters of the model are as in Table 1.

First, we compute the separating equilibrium which is Pareto efficient among pure strategy separating equilibrium outcomes in terms of the payoff of the incumbent; this will be called the Milgrom and Roberts outcome. One can verify that in this outcome, the inefficient producer chooses its monopoly output $Q_I^M(\overline{c}_I) = 196$ in the first period, and the efficient producer operates at $Q_I(\underline{c}_I) = 202.288$. Notice that $Q_I(\underline{c}_I)$ yields the maximum expected utility for the efficient producer among all separating equilibria. That is,

[4]We need the strategy space of each player to be discrete and finite. Otherwise, strategic stability is not well-defined.

TABEL 1

Inverse Demand Function	$P = a - Q$ where $a = 400$
Entry Fee	$K = 4$
Cost Function of the Incumbent Firm	$\phi(c_I, Q) = c_I Q$
Cost Function of the Potential Entrant	$\psi(c_E, Q) = c_E Q$
Type of the Incumbent Firm	$c_I \in \{\overline{c}_I = 8, \underline{c}_I = 2\}$
Prior over the Incumbent's Type	$h^\circ = Pr(c_I = \overline{c}_I) = 0.6$
Type of the Entrant	$c_E \in [\underline{c}_E, \overline{c}_E] = [163,203]$
Prior over the Entrant's Type	Uniform Distribution
Discount Factor	$\delta = 0.9$

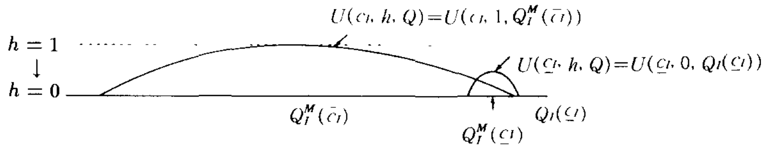

FIGURE 1

The Slope dh/dQ of the Indifference Curve of the $\underline{c}_I$ Incumbent is steeper than that of the $\overline{c}_I$ incumbent at $(Q, h) = (Q_I(c_I), 0)$. The Expected Utility Increases as h Decreases.

$$Q_I(\underline{c}_I) = \arg\max_{Q} U(\underline{c}_I, h' = 0, Q) \tag{1}$$

$$\text{subject to } U(\overline{c}_I, h = 1, Q_I^M(\overline{c}_I)) \geq U(\overline{c}_I, 0, Q).$$

Since the self-selection constraint is binding in the separating equilibrium, we have

$$U(\overline{c}_I, h = 1, Q_I^M(\overline{c}_I)) = U(\overline{c}_I, h = 0, Q_I(\underline{c}_I)).$$

Notice that the monopolist profit maximizing level for the $\underline{c}_I$ incumbent $Q_I^M(\underline{c}_I) = 199 < Q_I(\underline{c}_I) = 202.288$. Therefore,

$$\frac{\partial U(\underline{c}_I, h = 0, Q_I(\underline{c}_I))}{\partial Q} < 0. \tag{2}$$

In order to complete the specification of sequential equilibrium, we must assign a posterior probability conditioned on $Q \neq Q_I^M(\overline{c}_I)$ and $Q \neq Q_I(\underline{c}_I)$. For any equilibrium belief off the equilibrium path, the posterior h should satisfy

$$U(c_I, h, Q) \leq U(c_I, h = 0, Q_I(\underline{c}_I)), \quad c_I \in \{\underline{c}_I, \overline{c}_I\},$$

which implies that h must not be close to 0 if $Q < Q_I(\underline{c}_I)$ is close to $Q_I(\underline{c}_I)$.

As depicted in Figure 1, one can verify

$$\frac{\frac{\partial U(\underline{c}_I, h=0, Q_I(\underline{c}_I))}{\partial h}}{\frac{\partial U(\underline{c}_I, h=0, Q_I(\underline{c}_I))}{\partial Q}} = 6.114 < \frac{\frac{\partial U(\overline{c}_I, h=0, Q_I(\underline{c}_I))}{\partial h}}{\frac{\partial U(\overline{c}_I, h=0, Q_I(\underline{c}_I))}{\partial Q}} = 6.280.$$

For sufficiently small $\varepsilon > 0$, and for any conjecture h about the type of the incumbent, whenever the $\overline{c}_I$ incumbent has a weak incentive to deviate from $Q_I^M(\overline{c}_I)$ to $Q_I(\underline{c}_I) - \varepsilon$, the $\underline{c}_I$ incumbent has a strong incentive to deviate from $Q_I(\underline{c}_I)$ to $Q_I(\underline{c}_I) - \varepsilon$. For all h',

$$U(\overline{c}_I, h', Q_I(\underline{c}_I) - \varepsilon) \geq U(\overline{c}_I, h = 0, Q_I(\underline{c}_I)) = U(\overline{c}_I, h = 1, Q_I^M(\overline{c}_I))$$

implies

$$U(\underline{c}_I, h', Q_I(\underline{c}_I) - \varepsilon) > U(\underline{c}_I, h = 0, Q_I(\underline{c}_I)).$$

Therefore, Criterion $D1$ requires that the belief over the information set $Q_I(\underline{c}_I) - \varepsilon$ should be concentrated at $\underline{c}_I$. Then, by (2), the $\underline{c}_I$ incumbent has an incentive to deviate from $Q_I(\underline{c}_I)$ to $Q_I(\underline{c}_I) - \varepsilon$. This proves that the Milgrom and Roberts outcome is pruned by forward induction.

In this example, there exists a unique $D1$ equilibrium outcome. It is a semi-pooling equilibrium where the high cost producer randomizes between $Q_I^M(\overline{c}_I) = 196$ and $Q_I^e = 201.596$ with probability 0.189 for Q_I^e, and the low cost producer chooses Q_I^e with probability 1. The efficient incumbent's strategy Q_I^e is determined by the following maximization problem, subject to the self-selection constraint of the $\overline{c}_I$ incumbent firm:

$$(Q_I^e, h^e) = \arg\max_{Q,\, h'} \, [U(\underline{c}_I, h', Q)] \tag{3}$$

$$\text{subject to } U(\overline{c}_I, h = 1, Q_I^M(\overline{c}_I)) \geq U(\overline{c}_I, h', Q),$$

where h^e is the posterior conditioned on Q_I^e from which we compute the probability that the $\overline{c}_I$ incumbent assigns to Q_I^e.

The uniqueness of equilibrium surviving forward induction can be proved in three steps. First, one can prove that no sequential equilibrium in which the $\underline{c}_I$ incumbent is separated with positive probability survives Criterion $D1$. Therefore, in any $D1$ equilibrium, the $\underline{c}_I$ incumbent must be pooled with the $\overline{c}_I$ incumbent. Let (Q_I^e, h^e) be the pair of the $D1$ equilibrium quantity, which both types of the incumbent choose with positive probability, and the posterior conjecture about the incumbent's type conditioned on Q_I^e. Second, observe that at (Q_I^e, h^e), it must be the case that

$$\frac{\partial U(\overline{c}_I, h^e, Q_I^e) / \partial h}{\partial U(\overline{c}_I, h^e, Q_I^e) / \partial Q} = \frac{\partial U(\underline{c}_I, h^e, Q_I^e) / \partial h}{\partial U(\underline{c}_I, h^e, Q_I^e) / \partial Q}. \tag{4}$$

Otherwise, the $\underline{c}_I$ incumbent can improve its payoff by increasing or decreasing the production from Q_I^e. For example, if

$$\frac{\partial U(\overline{c}_I, h^e, Q_I^e) / \partial h}{\partial U(\overline{c}_I, h^e, Q_I^e) / \partial Q} < \frac{\partial U(\underline{c}_I, h^e, Q_I^e) / \partial h}{\partial U(\underline{c}_I, h^e, Q_I^e) / \partial Q},$$

then whenever the $\overline{c}_I$ incumbent has a weak incentive to deviate from Q_I^e to $Q_I^e + \varepsilon$, the $\underline{c}_I$ incumbent has a strong incentive to deviate to $Q_I^e + \varepsilon$. That is, $\forall h' \in [0, 1]$,

$$U(\overline{c}_I, h^e, Q_I^e) \leq U(\overline{c}_I, h', Q_I^e + \varepsilon) \Rightarrow U(\underline{c}_I, h^e, Q_I^e) < U(\underline{c}_I, h', Q_I^e + \varepsilon).$$

Therefore, Criterion $D1$ requires the belief conditioned on $Q_I^e + \varepsilon$ to be concentrated at $\underline{c}_I$. If the belief is concentrated at $\underline{c}_I$, then for sufficiently small $\varepsilon > 0$,

$$U(\underline{c}_I, h = 0, Q_I^e + \varepsilon) > U(\underline{c}_I, h^e, Q_I^e).$$

Therefore, the $\underline{c}_I$ incumbent will deviate to $Q_I^e + \varepsilon$ and the candidate equilibrium is eliminated by Criterion $D1$. Third, the sequential rationality requires that in any $D1$ equilibrium

$$U(\overline{c}_I, h^e, Q_I^e) \geq U(\overline{c}_I, h = 1, Q_I^M(\overline{c}_I)) \tag{5}$$

It turns out that the pair (Q_I^e, h^e) which solves (4) and (5), satisfies

$$h^e \leq 0.112 < h^\circ = 0.6, \tag{6}$$

and that if (Q_I^e, h^e) and $(\tilde{Q}_I^e, \tilde{h}^e)$ satisfy both (4) and (5), then $\forall c_I \in \{\underline{c}_I, \overline{c}_I\}$,

$$U(c_I, Q_I^e, h^e) \neq U(c_I, \tilde{Q}_I^e, \tilde{h}^e). \tag{7}$$

From the second inequality in (6), we know that the $\overline{c}_I$ incumbent firm must randomize between at least two quantity levels. From (7), we can deduce that if the $\overline{c}_I$ incumbent randomizes, then the $\overline{c}_I$ incumbent must be separated from the $\underline{c}_I$ incumbent with positive probability. Consequently, the $\overline{c}_I$ incumbent receives $U(\overline{c}_I, h = 1, Q_I^M(\overline{c}_I))$ in any $D1$ equilibrium. In this example, there is a unique pair $(Q_I^e, h^e) = (201.596, 0.112)$ satisfying (4), (5), and

$$U(\overline{c}_I, h = 1, Q_I^M(\overline{c}_I)) = U(\overline{c}_I, h^e, Q_I^e).$$

One can easily verify that (Q_I^e, h^e) chosen above indeed solves the constrained maximization problem (3).

Comparing (3) with (1), we find that (3) does not have the separation restriction $h' = 0$. Therefore, the *D*1 equilibrium must give the efficient incumbent at least as much as what the efficient incumbent receives from the Milgrom and Roberts outcome. In this semi-pooling *D*1 equilibrium, the high cost incumbent firm receives the same expected payoff 70589.972 as in the Milgrom and Roberts outcome; however, the low cost incumbent firm receives the expected payoff 73135.122 which is greater than what it receives from the Milgrom and Roberts outcome, 73135.059. The unique semi-pooling *D*1 equilibrium weakly Pareto dominates the Milgrom and Roberts outcome in terms of the interim expected utility of the incumbent firm.

IV. Without Forced Revelation

In order to test the impact of complete information on the post-entry game, we need to compute an equilibrium from the case where the players are not forced to reveal their private information. In order to maintain the consistency in selecting an equilibrium, we again apply STABAC in two steps. First, we show that a single pair (Q, H_I), where Q is the incumbent's first period quantity choice and H_I is the posterior conjecture about the incumbent, may induce two different sequential equilibria. But, we prove that only one equilibrium survives Criterion *D*1 following the pair (Q, H_I). Then, we prove that only one equilibrium survives STABAC.

By the subform (Q, H_I), we mean the part of the whole game that follows the incumbent's first period quantity choice Q and the posterior conjecture H_I about the type of the incumbent. Given a pair of conjectures (H_I, H_E), we can compute a unique Bayesian Nash equilibrium in the post-entry game. We distinguish two kinds of sequential equilibria based on the equilibrium probability of entry. If the probability of entry in a sequential equilibrium of the subform (Q, H_I) is positive, then the equilibrium is called a *somebody* equilibrium. Otherwise, we call the sequential equilibrium a *nobody* equilibrium.

One can easily show that for every c_E, H_I and H_E,

$$\frac{\partial \prod_E^c (c_E : H_E, H_I)}{\partial c_E} < 0, \tag{8}$$

and

$$\frac{\partial Q_E^c(c_E : H_E, H_I)}{\partial c_E} < 0. \tag{9}$$

Suppose that there exists a type γ of the entrant who is indifferent between entering and staying out of the market. By (8), such a type must be unique and $c_E < \gamma$ must enter the market. Therefore, in the post-entry game, the incumbent's conjecture about the entrant's type must be $G_E(c_E \mid c_E \leq \gamma)$ if entry occurs with positive probability. For notational convenience, we write $H_E\big|_{c_E \leq \gamma}$ for the posterior probability of H_E conditioned on the event $\{c_E \mid c_E \leq \gamma\}$. Furthermore, the indifferent type of the potential entrant $\gamma = \gamma(H_I)$ is determined by solving

$$\Pi_E^c(\gamma : G_E\big|_{c_E \leq \gamma}, H_I) = K. \tag{10}$$

Notice that a *somebody* equilibrium exists if and only if

$$\gamma(H_I) > \underline{c}_E,$$

which implies that

$$\Pi_E^c(\underline{c}_E : G_E\big|_{c_E \leq \gamma(H_I)}, H_I) > K.$$

There may exist an equilibrium where *nobody* enters the market. If entry is not made with positive probability, then the post-entry game falls off the equilibrium path and there must exist H_E such that

$$\Pi_E^c(c_E : H_E, H_I) \leq K \quad \forall\, c_E,$$

which is equivalent to

$$\Pi_E^c(\underline{c}_E : H_E, H_I) \leq K. \tag{11}$$

On the other hand, if (11) holds, then we can construct a *nobody* equilibrium in the subform (Q, H_I). It must be emphasized that (10) and (11) can hold simultaneously. If the solution concept is sequential equilibrium, we cannot predict whether or not entry will be made simply by specifying the posterior conjecture about the incumbent firm.

If the subform (Q, H_I) has both *somebody* and *nobody* equilibria, then the *nobody* equilibrium is not a sensible outcome. We know that the most efficient potential entrant has the strongest incentive to enter the market. Such a fact, which is common knowledge among players, must be reflected in any reasonable beliefs. In other words, any reasonable belief conditioned on the event of entry should assign large probability to the efficient type of the entrant. We want to support the *nobody* equilibrium by a reasonable belief off the equilibrium path. As it turns out, whenever the *somebody* equilibrium exists, the *nobody* equilibrium can only be supported by beliefs off the equilibrium path, which assign large probability to the inefficient type of the potential entrant. Therefore, the *nobody* equilibrium should not be considered a sensible outcome of the game, because it is supported only by such unreasonable beliefs. This argument is formalized by Proposition 1.

***Proposition* 1**
If the subform (Q, H_I) has both a *somebody* and a *nobody* equilibria, then the *somebody* equilibrium is the only equilibrium surviving Criterion $D1$ in the subform (Q, H_I).

Proof: See the appendix.

From now on, by an equilibrium in the subform (Q, H_I), we mean the *somebody* equilibrium whenever it exists. Since the conjecture about the entrant in the post-entry game is uniquely determined as $G_E\big|_{c_E\le\gamma}$, we may simplify the notation as $\Pi^c_E(c_E{:}G_E\big|_{c_E\le\gamma}, H_I)=\Pi^c_E(c_E{:}\ \gamma, H_I)$, $\Pi^c_I(c_I{:}G_E\big|_{c_E\le\gamma}, H_I) = \Pi^c_I(c_I{:}\ \gamma, H_I)$, $Q^c_E(c_E{:}\ G_E\big|_{c_E\le\gamma}, H_I) = Q^c_E(c_E{:}\ \gamma, H_I)$, and $Q^c_I(c_I{:}\ G_E\big|_{c_E\le\gamma}, H_I) = Q^c_I(c_I{:}\ \gamma, H_I)$.

We denote $\bar{H}_I$ and $\underline{H}_I$ as the probability distributions concentrated at $\bar{c}_I$ and $\underline{c}_I$, respectively. Clearly, $G(\gamma(\bar{H}_I))$ and $G(\gamma(\underline{H}_I))$ are the largest and the smallest probabilities of entry. In order to simplify the notation, we make the following assumption:

$$\underline{c}_E \le \gamma(\underline{H}_I) < \gamma(\bar{H}_I) \le \bar{c}_E. \tag{12}$$

Otherwise, no separating equilibrium may exist, implying Milgrom and Roberts' selection rule cannot be applied. Nevertheless, STABAC still applies, and it is straightforward to extend the characterization result without (12).

Observe that if $\gamma(H_I) < \gamma(\tilde{H}_I)$, then $\forall\, c_I$,

$$\Pi^c_I(c_I{:}\ \gamma(H_I), H_I) > \Pi^c_I(c_I{:}\ \gamma(\tilde{H}_I), \tilde{H}_I) \tag{13}$$

and

$$Q^c_I(c_I{:}\ \gamma(H_I), H_I) > Q^c_I(c_I{:}\ \gamma(\tilde{H}_I), \tilde{H}_I). \tag{14}$$

Let us denote the second period equilibrium profit of the c_I incumbent as follows:

$$A(c_I, H_I) = (1 - G(\gamma(H_I)))\, \Pi^M_I(c_I) + G(\gamma(H_I))\Pi^c_I(c_I; \gamma(H_I), H_I).$$

We write the overall payoff of the incumbent as

$$U(c_I, H_I, Q) = QP(Q) - \phi(c_I, Q) + \delta A(c_I, H_I).$$

We can prove that if $Q > Q^M_I(c_I) > Q^M_I(\tilde{c}_I)$, then

$$-\frac{A(c_I, H_I) - A(c_I, \tilde{H}_I)}{QP'(Q) + P(Q) - \dfrac{\partial\phi(c_I, Q)}{\partial Q}} > -\frac{A(\tilde{c}_I, H_I) - A(\tilde{c}_I, \tilde{H}_I)}{QP'(Q) + P(Q) - \dfrac{\partial\phi(\tilde{c}_I, Q)}{\partial Q}} > 0. \quad (15)$$

Note that (15) essentially establishes the single crossing property over a subset of the incumbent's signaling space. Since (15) is proved only for the region $Q > Q^M_I(c_I)$, the single crossing property may not hold for the region of small Q.

We call a sequential equilibrium a separating equilibrium if different types of the incumbent choose different first period quantity levels. We call a sequential equilibrium a pooling equilibrium if every type of the incumbent chooses the same first period quantity level. A semi-pooling equilibrium is a sequential equilibrium which is neither a separating nor a pooling equilibrium.

***Theorem* 1**

Only separating equilibrium outcomes survive criterion $D1$ in the "reduced" game.

Proof: See the appendix.

In any separating equilibrium, H_I must be concentrated on the true type of the incumbent c_I. Abusing the notation slightly, we write $U(c_I, c^e_I, Q) = QP(Q) - \phi(c_I, Q) + \delta A(c_I, c^e_I)$ for the payoff for the type c_I incumbent when he produces Q in the first period and is considered the c^e_I incumbent. Remember that

$$\sigma : [\underline{c}_I, \overline{c}_I] \rightarrow R_+$$

is a pure strategy of the incumbent in the first period. Since the self-selection constraint must be satisfied in any separating equilibrium,

$$U(c_I, c_I, \sigma(c_I)) = \max_Q U(c_I, \sigma^{-1}(Q), Q) \quad \forall c_I \in [\underline{c}_I, \overline{c}_I].$$

If σ is a differentiable function, then the first order condition induces the differential equation

$$(DE) \quad \frac{d\sigma(c_I)}{dc_I} = -\frac{\delta \dfrac{\partial A(c_I, c_I)}{\partial c_I^e}}{\sigma(c_I)P'(\sigma(c_I)) + P(\sigma(c_I)) - \dfrac{\partial \phi(c_I, \sigma(c_I))}{\partial Q}}.$$

Sequential rationality for the $\overline{c}_I$ incumbent provides the initial condition for (DE):

$$\sigma(\overline{c}_I) = \arg\max_Q [QP(Q) - \phi(\overline{c}_I, Q)]. \tag{16}$$

We now state Mailath's theorem adapted to our model.

***Theorem* 2** (Mailath 1987)
Suppose that $U(c_I, c_I^e, Q)$ satisfies the following regularity conditions at every c_I, c_I^e, and Q:

$$U(c_I, c_I^e, Q) \in C^2([\underline{c}_I, \overline{c}_I]^2 \times R_+);$$

$$\frac{\partial U(c_I, c_I^e, Q)}{\partial c_I^e} < 0, \quad \frac{\partial^2 U(c_I, c_I^e, Q)}{\partial c_I \partial c_I^e} > 0, \quad \frac{\partial^2 U(c_I, c_I^e, Q)}{\partial Q^2} < 0, \text{ and}$$

$$\frac{\partial^2 U(c_I, c_I^e, Q)}{\partial c_I \partial Q} < 0; \quad \frac{d}{dc_I}\left(\frac{\dfrac{\partial U(c_I, c_I^e, Q)}{\partial c_I^e}}{\dfrac{\partial U(c_I, c_I^e, Q)}{\partial Q}}\right) < 0$$

for $Q = \sigma(c_I)$ solving (DE) with the initial condition (16).
Then the first period quantity strategy characterized by (DE) with the initial condition (16) is the only separating sequential equilibrium of the game.

V. Concluding Remarks

Without forced revelation of information, the selection rule of Milgrom and Roberts identifies the same equilibrium as STABAC, if a separating equilibrium exists. Given many attractive properties of separating equilibria, one may be tempted to apply the separating equilibrium to the case with forced revelation in order to compare the two models. We contend that such an exercise may be misleading. Indeed, the fundamental differences between the two different models may be obscured by the divergent selection criteria. Our analysis suggests that Milgrom and Robert's (admittedly intuitive) outcome in the model with forced revelation may not be strategically stable (in a finite game which is "close" to the model analyzed in this paper). In fact, even comparing two Milgrom and Roberts outcomes may be tantamount to comparing perfect and imperfect equilibrium outcomes.

Appendix

Proof of Proposition 1

For every $c_E > \underline{c}_E$ whenever $\Pi_E^c(c_E\colon H_E, H_I) \geq K$, $\Pi_E^c(\underline{c}_E\colon H_E, H_I) > K$. Therefore, Criterion $D1$ requires that in the *nobody* equilibrium the belief must be concentrated at $\underline{c}_E$ in the post-entry game. This implies that the *nobody* equilibrium survives the Criterion $D1$ if and only if

$$\Pi_E^c(\underline{c}_E : H_E\big|_{c_E \le \underline{c}_E}, H_I) \le K.$$

We assert that if $\gamma < \gamma'$, then

$$\Pi_E^c(\underline{c}_E : H_E\big|_{c_E \le \gamma}, H_I) > \Pi_E^c(\underline{c}_E : H_E\big|_{c_E \le \gamma'}, H_I).$$

Once the assertion is verified, we have

$$\Pi_E^c(\underline{c}_E : H_E\big|_{c_E \le \underline{c}_E}, H_I) = \Pi_E^c(\underline{c}_E : G_E\big|_{c_E \le \underline{c}_E}, H_I) > \Pi_E^c(\underline{c}_E : G_E\big|_{c_E \le \gamma(H_I)}, H_I) > K,$$

which proves that the *nobody* equilibrium must be eliminated by the Criterion $D1$. Now we prove the assertion.

Let $Q_E(c_E)$ and $Q_I(c_I)$ be the quantities chosen by the type c_E entrant and the type c_I incumbent. Given $Q_I = Q_I(c_I)$, the reaction function $R_E(c_E, Q_I)$ of the type c_E entrant must satisfy the profit maximization first order condition:

$$\int_{\underline{c}_I}^{\bar{c}_I} R_E(c_E, Q_I)P'(R_E(c_E, Q_I) + Q_I(c_I)) + P(R_E(c_E, Q_I) + Q_I(c_I))dH_I(c_I) - \frac{\partial\psi(c_E, R_E(c_E, Q_I))}{\partial Q_E} = 0.$$

Similarly, the c_I incumbent's reaction function $R_I(c_I, Q_E)$ for $Q_E = Q_E(c_E)$ must satisfy

$$\int_{\underline{c}_E}^{\bar{c}_E} R_I(c_I, Q_E)P'(R_I(c_I, Q_E) + Q_E(c_E)) + P(R_I(c_I, Q_E) + Q_E(c_E))dH_E(c_E) - \frac{\partial\phi(c_I, R_I(c_I, Q_E))}{\partial Q_I} = 0.$$

Notice that

$$\frac{\partial \Pi_E^c(c_E:G_E\big|_{c_E \le \gamma}, H_E)}{\partial \gamma} = \left[Q_E^c \int_{\underline{c}_I}^{\bar{c}_I} P'(Q_E^c + Q_I^c) dH_I(c_I) \right] \left[\frac{\partial R_I}{\partial \gamma} \right].$$

Since P is downward sloping, the first term is negative. Therefore, in order to prove our assertion, it suffices show

$$\frac{\partial R_I(c_I, Q_E)}{\partial \gamma} > 0.$$

By differentiating the profit function of the c_I incumbent around (Q_I^c, Q_E^c), we have

$$\left[\frac{H_E'(\gamma)}{H_E(\gamma)}\right]$$

$$\times\Big[(P(R_I(\cdot) + Q_E^c(\gamma:H_E\big|_{c_E \le \gamma}, H_I)) + R_I(\cdot)P'(R_I(\cdot) + Q_E^c(\gamma:H_E\big|_{c_E \le \gamma}, H_I)))H_E(\gamma)$$

$$- \int_{\underline{c}_E}^{\gamma} P(R_I(\cdot) + Q_E(c_E)) + R_I(\cdot)P'(R_I(\cdot) + Q_E^c(\gamma:H_E\big|_{c_E \le \gamma}, H_I))dH_E(c_E\big|c_E \le \gamma)\Big]$$

$$+ \frac{\partial FOC}{\partial Q_I} \frac{\partial R_I(c_I, Q_E)}{\partial \gamma} = 0.$$

The first term is positive since $Q_E^c(c_E:H_E\big|_{c_E \le \gamma}, H_I)$ is a decreasing function of c_E. The second term is negative by the second order condition of profit maximization. Thus,

$$\frac{\partial R_I(c_I, Q_E)}{\partial \gamma} > 0,$$

which proves the assertion. This completes the proof of the proposition.

Q.E.D.

***Proof of Theorem* 1**

Fix a pair of quantity and belief ($\tilde{Q}$, $\tilde{H}_I$), by way of contradiction, where $\tilde{Q}$ is reached with positive probability and $\tilde{H}$ is computed by Bayes' rule. Assume that more than one type of the incumbent uses $\tilde{Q}$ with positive probability. Define $c_I^* = \inf supp\tilde{H}_I$ as the most efficient type of incumbent pooled at $\tilde{Q}$

***Case* 1:** $\tilde{Q} \geq Q_I^M(c_I^*)$

We know that the single crossing property holds in the neighborhood of $\tilde{Q}$. Therefore, for sufficiently small $\varepsilon > 0$, Criterion $D1$ requires the belief of the entrant conditioned on $\tilde{Q} + \varepsilon$ must be concentrated at c_I^*. We also know that

$$\gamma(\tilde{H}_I) > \gamma(\tilde{H}\Big|_{c_I \le c_I^*}).$$

Hence,

$$\Pi_I^c(c_I^*: \gamma(\tilde{H}_I), \tilde{H}_I) < \Pi_I^c(c_I^*: \gamma(\tilde{H}\Big|_{c_I \le c_I^*}), \tilde{H}\Big|_{c_I \le c_I^*}),$$

so that for sufficiently small $\varepsilon > 0$,

$$U(c_I^*: \tilde{H}_I, \tilde{Q}) < U(c_I^*: \tilde{H}\Big|_{c_I \le c_I^*}, \tilde{Q} + \varepsilon).$$

Thus, any sequential equilibrium using $\tilde{Q}$ with positive probability by more than one type of the incumbent must be eliminated by Criterion $D1$.

Case 2: $\tilde{Q} < Q_I^0(c_I^*)$

Since the first period revenue function $QP(Q) - \phi(c_I, Q)$ has a single peak at $Q = Q_I^M(c_I)$ for every c_I, we can find $\tilde{Q}^* > Q_I^c(c_I^*)$ satisfying $\tilde{Q}P(\tilde{Q} - \phi(c_I, \tilde{Q}) = \tilde{Q}^*P(\tilde{Q}^*) - \phi(c_I, \tilde{Q}^*)$. Write $U_{c_I}^*$ for the equilibrium payoff of the type c_I incumbent. If $\tilde{Q}^*$ is used with positive probability in the equilibrium, then we can apply Case 1.

Assume that $\tilde{Q}^*$ is not used with positive probability in the equilibrium. Define H_I^b as a *borderline* equilibrium belief. That is,

$$U(c_I, H_I^b, \tilde{Q}^*) \le U_{c_I}^* \quad \forall c_I$$

and

$$U(c_I, H_I^b, \tilde{Q}^*) = U_{c_I}^* \quad \text{for some } c_I.$$

Since $U_{c_I^*}^* = U(c_I^*, \tilde{H}_I, \tilde{Q})$, it must be the case that

$$\gamma(H_I^b) \ge \gamma(\tilde{H}_I). \tag{A1}$$

Define

$$T^b = \{c_I \,|\, \exists H_I^b \text{ such that } U_{c_I}^* = U(c_I, H_I^b, \tilde{Q}^*)\}.$$

We assert that

$$\inf supp\, \tilde{H}_I \ge \sup T^b.$$

Given this assertion, for any belief H_I concentrated on T^b, $\gamma(\tilde{H}_I) > \gamma(H_I)$. Hence, $U(c_I^*, \tilde{H}_I, \tilde{Q}) = U_{c_I^*}^* < U(c_I^*, \tilde{H}_I, \tilde{Q}^*)$, which implies that the equilibrium should be eliminated by Criterion $D1$.

Now, we prove the assertion. Suppose that $\exists\, c_I' \in T^b$ such that $c_I' > c_I^*$. Since $c_I' \in T^b$, $\exists\, H_I^b$ such that

$$U^*_{c'_I} = U(c'_I, H^b_I, \tilde{Q}^*). \tag{A2}$$

Since $\tilde{Q}$ is used with positive probability in the equilibrium, incentive compatibility requires

$$U^*_{c'_I} \geq U(c'_I, \tilde{H}_I, \tilde{Q}). \tag{A3}$$

Notice that if $c'_I \in supp\ \tilde{H}_I$, then the relation holds as an equality. Since $c'_I > c^*_{I\cdot}$,

$$U(c'_I, \tilde{H}_I, \tilde{Q}) > U(c'_I, \tilde{H}_I, \tilde{Q}^*). \tag{A4}$$

Combining (A2), (A3) and (A4), we have

$$U(c'_I, H^b_I, \tilde{Q}^*) > U(c'_I, \tilde{H}_I, \tilde{Q}^*). \tag{A5}$$

Therefore,

$$\gamma(H^b_I) \leq \gamma(\tilde{H}_I).$$

In order to complete the proof of the assertion, we only have to demonstrate that

$$\gamma(H^b_I) = \gamma(\tilde{H}_I).$$

is impossible, since by (A1) we know that the strict inequality is not possible. By (A5), we know

$$\Pi^q_I(c'_I; \gamma(H^b_I), H^b_I) > \Pi^q_I(c'_I; \gamma(\tilde{H}_I), \tilde{H}_I).$$

Let $\overline{H}_I$ be the probability distribution concentrated at $\overline{c}_I$. Define $\tilde{H}^b_I = (1 - p)H^b_I + p\overline{H}_I$. We know that $\forall p \in (0, 1)$

$$\gamma(\tilde{H}^p_I) > \gamma(\tilde{H}_I) = \gamma(H^b_I),$$

and as $p \to 0$,

$$\gamma(\tilde{H}^p_I) \to \gamma(\tilde{H}_I) = \gamma(H^b_I),$$

Hence, for p close to 0, we have

$$\Pi^c_I(c'_I; \gamma(\tilde{H}^p_I), \tilde{H}^p_I) > \Pi^c_I(c'_I; \gamma(\tilde{H}_I), \tilde{H}_I),$$

which is a contradiction.

Q.E.D.

(Manuscript received February, 1994; final vevision received April, 1994)

Reference

Banks, Jeffrey S., and Sobel, Joel. "Equilibrium Selection in Signaling Games." *Econometrica* 55 (1987): 647-61.

Cho, In-Koo. "Strategic Stability in Repeated Signaling Games." *International Journal of Game Theory* 22 (1993): 107-21.

__________, and Kreps, David M. "Signaling Games and Stable Equilibria." *Quarterly Journal of Economics* 102 (1987): 179-221.

__________, and Sobel, Joel. "Strategic Stability and Uniqueness in Signaling Games." *Journal of Economic Theory* 50 (1990): 381-413.

Eric van Damme, and Noldeke, Georg. "Signaling in a Dynamic Labor Market." *Review of Economic Studies* 57 (1990): 1-23.

Kohlberg, Elon, and Mertens, Jean-Francois. "On the Strategic Stability of Equilibria." *Econometrica* 54 (1986): 1003-38.

Kreps, David M., and Wilson, Robert. "Sequential Equilibria." *Econometrica* 50 (1982): 863-94.

Mailath, George. "Incentive compatibility in Signaling Games with a Continuum of Types." *Econometrica* 55 (1987): 1349-65.

Milgrom, Paul, and Roberts, D. John. "Limit Pricing and Entry under Incomplete Information: An Equilibrium Analysis." *Econometrica* 50 (1982): 443-52.

__________. "Price and Advertising Signals of Product Quality." *Journal of Political Economy* 54 (1986): 796-821.