A Power Structure Version of Sen's Paretian Liberal Theorem

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Sen's theorem on the impossibility of a Paretian liberal is established in a general setting. Rights are protected by means of restrictions on the game form, giving individuals special powers in appropriate circumstances. (*JEL* Classifications: D63, D71)

I. Introduction

The discovery by A.K. Sen (1970) that the Pareto criterion and minimal individual rights are in conflict has inspired a wealth of comment, much of it critical of Sen's representation of rights as a restriction on the social choice rule rather than as a limitation on the actions of other agents, including the state, vis-a-vis an individual's private concerns. Nozick (1974), Gardenfors (1981), Sugden (1981), Sugden (1985), and Gaertner *et al* (1992) are in this vein.[1] One of the few critical papers to acknowledge that Sen's theorem would still go through in a properly formulated model is Gaertner *et al* (1992). The purpose of this paper is to prove this conjecture in a game form setting. Campbell (1989) pro-

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[1]See Sen (1992) for a rejoinder. Important recent contributions to the debate include: Deb, Pattanaik, and Razzolini (1993), Pattanaik and Suzumura (1994a), Pattanaik and Suzumura (1994b), Pattanaik (1994), and Pattanaik and Xu (1994).

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vides a proof for the special case of property rights. Our model is based on Pattanaik (1989) in which rights are captured by a *power structure.* A power structure simply assigns to each individual i a set X_i of alternatives *and*, via the game form, the power to ensure that the social outcome belongs to X_i *if* person i chooses to exercise that power. This agrees with Sen's *minimal* rights structure: Rights are asymmetric and limited: Individual i does not necessarily have the power to ensure that the outcome does *not* belong to X_i. If Jack is a conformist he is entitled to discard the *Rolex* watch that he received as a birthday present if he observes Jill wearing a *Timex*—in this case X_i is the set of all outcomes in which i does not wear a *Rolex.* Of course, Jack is not necessarily entitled to *have* a *Rolex* if Jill wears one. So, our model is compatible with a very limited guarantee of rights (and, a fortiori, with any stricter guarantee). We confirm Sen's impossibility theorem not only with minimal rights but under almost any assumption about coalition formation.

II. The Formal Model

X is the set of outcomes. The society, N, is either the finite set $\{1,2,\ldots, n\}$ or the infinite set or positive integers. In the latter case, N represents the set of possible generations. An assignment of rights is characterized by means of a *power structure* which specifies a subset X_i of X for each individual i. Rights are protected by a constitution which requires that each game form (or mechanism) be compatible with the power structure. A game form (M, g) is a pair consisting of a message system M and an outcome function g: $M \rightarrow X$. The message system assigns to each $i \in N$ some set M_i of available messages (or actions, or strategies). Compatibility with the power structure simply means that for each $i \in N$ there is some message $m_i^\circ \in M_i$ such that $g(m) \in X_i$ for all $m \in M$ with $m_i = m_i^\circ$. The game form is *admissible* if it is compatible with the power structure. Note that this representation of rights can handle *conditional* rights as discussed in Gaertner *et al* (1992) and Pattanaik (1989). For example, X_i could be the set of outcomes in which individual h does not smoke in i's presence, either because h has no desire to smoke or because h asks i's permission to smoke and i refuses. In some contexts, X_i is not unique. Dictatorship provides a simple illustration. Individual i is a dictator for g if for each $x \in X$ there is some $m_i \in M_i$ such that $g(s) = x$ whenever $s_i = m_i$. Then, for any outcome x the dictator has a strategy that will ensure that $g(s)$ belongs to $\{x\}$. By assuming only one set X_i for each individual i we provide a stronger inconsistency theorem. Of course, this is in the spirit of Sen's assumption of minimal rights.

Individuals have preferences over the members of X and each individual preference scheme is assumed to be a preorder. (A preorder is a complete and transitive binary relation). A *profile* R is a function from N into the family of preorders on X, and we let $\geq_i$ denote the preorder assigned to i by R. As usual $>_i$ denotes the asymmetric (or strict preference) factor of $\geq_i$ and $\sim_i$ denotes the symmetric (or indifference) factor of $\geq_i$. A priori restrictions on individual preferences are reflected in the *domain* $\mathcal{P}$, which is a given family of profiles.

Let $\mathcal{L}$ be a family of nonempty subsets of N such that $\{i\} \in \mathcal{L}$ for each $i \in N$. An $\mathcal{L}$-equilibrium with respect to profile R is a message n-tuple $m \in M$ such that for all $C \in \mathcal{L}$ and all $s \in M$, if $g(s) >_i g(m)$ for all $i \in C$ then $s_h \neq m_h$ for some $h \in N - C$. Thus, an $\mathcal{L}$-equilibrium is an extended Nash equilibrium: *given* the actions of individuals outside of coalition C there is nothing that the members of C can do to change the outcome to the advantage of each. If $\mathcal{L} = \{\{i\}: i \in N\}$ then an $\mathcal{L}$-equilibrium is simply a Nash equilibrium. (A strong equilibrium is an $\mathcal{L}$-equilibrium for $\mathcal{L} = \{C \subset N: C \neq \phi\}$).

The game form is *consistent* if a Nash equilibrium exists for each $R \in \mathcal{P}$, and it is Pareto satisfactory if it is consistent and for each $R \in \mathcal{P}$ every Nash equilibrium is Pareto optimal. As usual $x \in X$ is Pareto optimal with respect to profile R is there is no $y \in X$ such that $y >_i x$ for all $i \in N$.

III. Sen's Theorem

Sen (1970) proved that a mapping from profiles into social choices could not satisfy both Pareto optimality and minimal liberalism. We will prove the impossibility result using the power structure form explicitly; it will hold for almost any definition $\mathcal{L}$ of extended Nash equilibrium. Set $W = \bigcap_{i \in N} X_i$. Assume that W is not empty. (Although the Case $W = \phi$ is highly unrealistic, it is discussed briefly in the next section).

***Theorem* 1**

Suppose that there exists a profile $R \in \mathcal{P}$ with respect to which no member of W is Pareto optimal and such that for all $i \in N$ we have $x \sim_i y$ for all $x,y \in X_{i+1}$, with X_{n+1} representing X_1 if N is finite. If $N \notin \mathcal{L}$ there is no admissible game form for which every $\mathcal{L}$-equilibrium is Pareto optimal.

Proof: Let m_i° be the action that guarantees $g(m) \in X_i$ if $m_i = m_i^\circ$. Then $g(m^\circ) \in W$ so $g(m^\circ)$ is not Pareto optimal with respect to the hypothesized profile R. We conclude by showing that m° is an $\mathcal{L}$-equilibrium for R if $N \notin \mathcal{L}$. If $C \in \mathcal{L}$ then $i + 1 \notin C$ for some $i \in C$. Consider $x = g(m)$ for $m \in M$ such that $m_{i+1} = m_{i+1}^\circ$. We have $x \in X_{i+1}$ for all such m and therefore $g(m) \sim_i g(m^\circ)$ for all such m. Therefore, C cannot improve upon $g(m^\circ)$ to the advantage of each of its members. Thus, m° is an $\mathcal{L}$-equilibrium.

Q.E.D.

Of course the profile R used in the theorem is very special. For one thing, externalities abound. But it has always been recognized that Sen's dilemma disappears in the absence of externalites. And the theorem merely asserts that there is *some* logically possible configuration of preferenes at which things go wrong.

Campbell (1989) treats a special case of Theorem 1: X is the set of allocations of commodities obtainable by redistribution and X_i is the set of allocations at which i consumes his initial endowment. The rights are property rights; i has an available strategy that can be used to protect his endowment. In this case W is a singleton, $\{w\}$, the initial endowment allocation, and the required profile can be constructed by means of utility functions such that i's utility depends solely on his own consumption and that of his neighbor, $i + 1$. In other words, Sen's theorem is established for a restricted domain that allows only local "consumption externalities" and for almost any notion of equilibrium. However, the hypothesis of Theorem 1 cannot be satisfied if $X_i = X$ for even one individual. On the other hand, Sen's argument is valid even if only two individuals have rights. What can be said in our framework when if $X_i = X$ for some i? Suppose that $X_i = X$ for all $i > 2$. Individuals other than 1 and 2 play no role, so we might as well assume that $N = \{1,2\}$.

If $X_1 \cup X_2 \neq X$ there is no admissible and Pareto satisfactory game form. To see why, let m_i° be a strategy such that $g(m) \in X_i$ whenever $m_i = m_i^\circ$. Choose $x \notin X_1 \cup X_2$ and any $R \in \mathcal{P}$ such that $x >_i g(m^\circ) >_i y$ for all $y \in X - \{x, g(m^\circ)\}$ and $i = 1,2$. Then m° is a Nash equilibrium for R, although $g(m^\circ) \in X_1 \cap X_2$ and $g(m^\circ)$ is not Pareto optimal. The assumption that $X_1 \cup X_2$ is a proper subset of X takes us well beyond the minimal rights structure of Sen (1970). What if the power structure gives limited blocking power, and each X_i excludes only a few members of X ? The general impossibility theorem for $N = \{1,2\}$ has already been established in Maskin (1977): Maskin's first theorem proves that every Pareto satisfactory game form is dictatorial if $N = \{1,2\}$. Therefore, if $X_1 \neq X$ then $X_2 = X$ and conversely. If one person has any blocking power at all, however minimal, then the other person has no power or influence whatever. We reproduce Maskin's proof for convenience.

Theorem 2

Suppose that $N = \{1,2\}$ and for any $x \in X$ there is an admissible profile such that $x >_1 y$ and $y >_2 x$ for all $y \neq x$, and for any distinct $x,y,z \in X$ there is an admissible profile such that $x >_i y >_i z$ holds for $i = 1,2$. Then every Pareto satisfactory game form is dictatorial.

Proof: Let $B_1(\alpha) = \{x \in X\colon g(s) \neq x \text{ if } m \in M \text{ and } m_1 = \alpha\}$, and $B_2(\beta) = \{x \in X\colon g(s) \neq x \text{ if } m \in M \text{ and } m_2 = \beta\}$. The proof rests on two lemmas concerning the properties of the B_i. We assume that X has at least two members. (The game form is obviously dictatorial if X is singleton). Suppose that (M,g) is Pareto satisfactory.

Lemma 1

For any $m \in M$, $B_1(m_1) \cap B_2(m_2) = \phi$.

Proof. Suppose $x \in B_1(m_1) \cap B_2(m_2)$. Let $y = g(m_1,m_2)$. Then $x \neq y$ by definition of B_i. Let $R \in \mathcal{P}$ be chosen so that $x >_1 y >_1 z$ and $x >_2 y >_2 z$ for all $z \in X - \{x,y\}$. Then m is a Nash equilibrium for R. But $g(m)$ is not Pareto optimal for R. Therefore, $B_1(m_1) \cap B_2(m_2) = \phi$.

Q.E.D.

Lemma 2

For all $x \in X$, if $x \notin B_2(M_2)$ then there is some $m_1^\circ \in M_1$ such that $g(m) = x$ whenever $m_1 = m_1^\circ$. And if $x \notin B_1(M_1)$ then there is some $m_2^\circ \in M_2$ such that $g(m) = x$ whenever $m_2 = m_2^\circ$.

Proof: Suppose $x \notin B_2(M_2) = \cup \{B_2(m_2)\colon m_2 \in M_2\}$. Choose $R \in \mathcal{P}$ such that both $x >_1 y$ and $y >_2 x$ hold for all $y \in X - \{x\}$. Let e be a Nash equilibrium for R. Because $x \notin B_2(e_2)$ there is some $m_1^\circ \in M_1$ such that $g(m) = x$ for $m_1 = m_1^\circ$ and $m_2 = e_2$. Then $g(e) = x$ because e is a Nash equilibrium for R. And because e is a Nash equilibrium for R, and $g(s) \neq x$ implies $g(m) >_2 x$, we must have $g(m) = x$ for all $m \in M$ such that $m_1 = m_1^\circ$. The case $x \notin B_1(M_1)$ is identical.

Q.E.D.

Now, complete the proof of Theorem 2. Choose arbitrary $x \in X$. suppose $x \notin B_2(M_2)$. If $y \notin B_1(M_1)$ for some $y \neq x$ then by Lemma 2 there exist $m_1^\circ \in M_1$ and $m_2^\circ \in M_2$ such that $x = g(m^\circ) = y$, an obvious contradiction. Therefore, $y \in B_1(M_1)$ for all $y \neq x$ and thus $B_2(M_2) = \phi$ by Lemma 1. Therefore, 1 is a dictator by Lemma 2. Similarly, person 2 is a dictator if $B_1(M_1) \neq X$. Therefore, either 1 or 2 is a dictator by Lemma 1.

Q.E.D.

The blocking correspondence B_i is in a sense the inverse of X_i. In particular, if person 1 is a dictator then $B_1 = X$, $X_2 = \phi$, and X_1 is not unique. For each singleton set $\{x\}$ the dictator has a strategy to ensure that the outcome will belong to $\{x\}$. In the two-person case, if one individual has a nontrivial protected personal sphere ($X_i \neq X$) then *only* that individual has a protected personal sphere unless the Pareto criterion is violated or there are no Nash equilibria in some cases. It is easy to get a possibility result if the Pareto criterion is not imposed: Just choose $w \in W$ and set $M_i = \{w\}$ for all $i \in N$ and $g(m) = w$. This is an entirely trivial mechanism, underscoring the point that *minimal* rights are easily guaranteed. They are not at all demanding, but are nevertheless incompatible with the Pareto criterion if existence of Nash equilibria is also required. If existence of equilibrium is not a requirement it is easy to construct admissible game forms for which every Nash equilibrium is Pareto optimal.

What if N has more than two members but $X_i = X$ for all but two of them? In Sen's framework this question does not require separate treatment, but it clearly does with the game form approach. Because our interest is in societies in which every individual is granted some protected personal sphere, however limited, we will not investigate this case. We cannot even appeal to the key Nash implementation theorems in Maskin (1977), Williams (1986), and Saijo (1988). These papers employ a no-veto-power assumption that is in direct conflict with individual rights, which give each individual the power to veto certain outcomes, no matter how highly favored by others.

Finally, consider the case where the grand coalition N is able to coalesce. This is highly unrealistic but for the purpose of mapping the boundary between possibility and impossibility we point out that Sen's problem has a solution in this setting. This is noted formally as Theorem 3 which assumes that for any $R \in \mathcal{P}$ there is a Pareto optimal alternative x such that $x \succeq_i w$ for all $i \in N$. A model without this property would be pathological, with an unbounded or unclosed X, or discontinuous individual preferences.

Theorem 3

Assume that for any $R \in \mathcal{P}$ there is a Pareto optimal alternative x such that $x \succeq_i w$ for all $i \in N$. If $N \in \mathcal{L}$ then there is an admissible game form such that for every domain $\mathcal{P}$ and all $R \in \mathcal{P}$ an $\mathcal{L}$-equilibrium exists, and for all $R \in \mathcal{P}$ every $\mathcal{L}$-equilibrium m gives rise to an outcome $g(m)$ that is Pareto optimal for R.

Proof: The proof is constructive, relying on a game form devised by Eric Maskin in another context (Maskin 1979). Set $M_i = X$ for all $i \in N$ and choose $w \in W$. Define g by setting $g(m) = x$ if $m_i = x$ for all $i \in N$, and $g(m) = w$ if $m_i \neq m_h$ for some i and h. Because $N \in \mathcal{L}$ an $\mathcal{L}$-equilibrium must be Pareto optimal. Because $s_i = w$ for any $i \in N$ implies $g(s) = w \in W \subset X_i$, the game form is admissible. To show that an $\mathcal{L}$-equilibrium exists for any admissible profile we exploit the fact that for any $R \in \mathcal{P}$ there is a Pareto optimal alternative x such that $x \succeq_i w$ for all $i \in N$ (A model without this property would be pathological, with an unbounded or unclosed X, or discontinuous individual preferences). Set $m_i = x$ for all $i \in N$. Then m is an $\mathcal{L}$-equilibrium: The coalition N cannot improve on x because it is Pareto optimal. If $C \neq N$ then $g(s) \in \{w, x\}$ if $s_i = x$ for $i \notin C$. If $g(s) = x$ then there is no challenge to the equilibrium m, and if $g(s) = w$ the coalition C can only lose by deviating from m because $x \succeq_i w$ for all i in N.

IV. Consistency

Gibbard (1974) discovered that the configuration of rights itself might be inconsistent, even without the Pareto principle. We address this question briefly by asking which power structures are incompatible with the existence of even one admissible and consistent game form. The case $W = \bigcap_{i \in N} X_i = \phi$ is easily handled. Specify R so that, for each $i \in N$, $x \succ_i y$ for all $x \in X_i$ and all $y \in X - X_i$, unless $X = X_i$ in which case $\succeq_i$ is arbitrary. Choose any $m \in M$. If $g(m) \notin X_i$ then m is not an equilibrium if the game form is admissible: i has a message $m_i^\circ \in M_i$ such that $g(s) \in X_i$ whenever $s_i = m_i^\circ$. Then $g(s) \succ_i g(m)$. Therefore, if m is a Nash equilibrium we must have $g(m) \in X_i$ for all $i \in N$ and this is impossible when $W = \phi$.

If $W \neq \phi$ then one can find consistent and admissible game forms, and in some cases admissible game forms that are not consistent. Consistency alone can always be satisfied because we can choose $w \in W$ and define the game form by setting $M_i = \{w\}$ for all $i \in N$ and $g(w, w, \ldots, w) = w$. The construction in the proof of Theorem 3 shows that there exists a *nontrivial* admissible and consistent game form whenever W is nonempty. (It will not often yield Pareto optimal Nash equilibria, however.) Further, it is clear from Saijo (1988) that for *any* correspondence F from $\mathcal{P}$ into X one can design a game form such that for all $R \in \mathcal{P}$ every member of $F(R)$ is a Nash equilibrium for *some* $m \in M$. The difficulty is in designing game forms for which *all* Nash equilibria belong to $F(R)$, and that is why the no-veto-power assumption is made.

We conclude this section by comparing the power structure representation of rights with the model in Gibbard (1974). In the latter case each individual has a set of variables Y_i that can be used to control features of his life that fall within his protected sphere and hence are to be placed beyond the jurisdiction of any other agency, including the state. The outcome set X is the cartesian product of the Y_i. The rights structure associated with this framework is more complex and demanding than the one we have been using. For each i there is a family D_i of subsets of X and i has the right to ensure that the outcome belongs to the member of D_i of his choice. In the Gibbard model, $D_i = \{(x \in X: x_i = y_i): y_i \in Y_i\}$. If, for example, there are n individuals and each person has a private binary choice, represented by $Y_i = \{0, 1\}$, then D_i is comprised of the two sets $\{x \in X: x_i = 0\}$ and $\{x \in X: x_i = 1\}$. Sen's impossibility theorem requires a much less demanding *asymmetric* assignment of rights. (Of course, if there are k features of i's life over which he has exclusive jurisdiction then D_i will contain 2^k subsets of X, and it will be much larger, with nonbinary choices).

V. Some Concluding Remarks

We briefly consider three possible reactions to the version of Sen's impossibility theorem above. First, alternatives to Nash equilibrium can be explored. John Moore and Raphael Repullo (1988) and Dilip Abreu and Arunava Sen (1990), among others, have shown that more sophisticated notions of noncooperative equilibrium lead to a striking enlargement of the family of implementable social choice correspondences. But would the imposition of a rights respecting requirement severely constrain the construction of game forms precipitating another impossibility theorem? Second, Kaushik Basu has proposed a variant of the 'blocking power' definition of rights employed in this paper: Suppose we merely assume that for each i and each $m \in M$ there is some $m_i^\circ \in M_i$ such that $g(s) \in X_i$ if $s_i = m_i^\circ$ and $s_h = m_h$ for all $h \neq i$. For some purposes this may be a more appropriate definition of individual rights; and it may open the door to a positive result, although Sen's theorem appears to be *very* robust. Third, it is vital to identify the (fuzzy) boundary separating the situations in which the Pareto criterion is welcome from cases where it is irrelevant. Pareto optimality is primarily of concern in a model of resource allocation, where the spotlight is on the mechanism for coordinating the production, distribution, and consumption of standard goods and services and the goal is to evaluate the performance of an economic system without getting sidetracked in a dispute over equity. Imposition of the Pareto criterion is an extremely useful way of avoiding that sidetrack while at the same time setting a high standard of performance for the economic system. It is not clear why we should insist upon Pareto optimality, or even existence of equilibrium, in the standard illustration of the Gibbard paradox—in which two neighbors must decide individually what to wear—or even in the more significant example of Sugden (1985 and 1989), in which individuals are given a choice between keeping a political or a nonpolitical diary. Even when that boundary has been determined we will see a region in which the Pareto criterion is of value yet the interrelationships between one person's actions and another's welfare do not reflect the kind of externalities that are the subject of classical welfare economics.

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References

Abreu, D., and Sen, A. "Subgame Perfect Implementation: A Necessary and almost Sufficient Condition." *Journal of Economic Theory* 50 (1990): 286-99.

Campbell, D.E. "Equilibrium and Efficiency with Property Rights and Local Consumption Externalities." *Social Choice and Welfare* 6 (1989): 189-203.

Gaertner, W., Pattanaik, P.K., and Suzumura, K. "Individual Rights Revisited." *Economica* 59 (1992): 161-77.

Gardenfors, P. "Rights, Games, and Social Choice." *Nous* 15 (1981): 341-56.

Gibbard, A. "A Pareto-Consistent Libertarian Claim." *Journal of Economic

Theory 7 (1974): 388-410.

Maskin, E. "Nash Equilibrium and Welfare Optimality." mimeo, M.I.T. (1977).

____________. "Implementation and Strong Nash Equilibrium." In J.J. Laffont, (ed.). *Aggregation and Revelation of Preferences*. North-Holland, New York (1979): 433-40.

Moore, J., and Repullo, R. "Subgame Perfect Implementation." *Econometrica* 46 (1988): 1191-220.

Nozick, R. *Anarchy, State, and Utopia*. Oxford: Basil Blackwell, 1974.

Pattanaik, P.K. "Rights and Liberty." mimeo, University of Birmingham, 1989.

____________. "Rights and Freedom in Welfare Economics." *European Economic Review*, forthcoming. 1994.

Deb, R., Pattanaik, P.K., and Razzolini, L. "Game Forms, Rights, and the Efficiency of Social Outcomes." Southern Methodist University, Department of Economics Working Paper, 1993.

Pattanaik, P.K., and Suzumura, K. "Rights, Welfarism, and Social Choice." *American Economic Review*, forthcoming, 1994a.

____________. "Individual Rights and Social Evaluation: A Conceptual Framework." University of California at Riverside, Department of Economics Working Paper, 1994b.

Pattanaik, P.K., and Xu. "On Preference and Freedom." University of California at Riverside, Department of Economics Working Paper, 1994.

Saijo, T. "Strategy Space Reduction in Maskin's Theorem: Sufficient Conditions for Nash Implementation." *Econometrica* 56 (1988): 693-700.

Sen, A.K. "The Impossibility of a Paretian Liberal." *Journal of Political Economy* 78 (1970): 152-57.

____________. "Minimal Liberty." *Economica* 59 (1992): 161-77.

Sugden, R. *The Political Economy of Public Choice*. Oxford: Martin Robertson, 1981.

____________. "Liberty, Preferences, and Choice." *Economics and Philosophy* 1 (1985): 185-205.

____________. "Rights: Why Do They Matter, and To Whom?" mimeo, University of East Anglia, 1989.

Williams, S.R. "Realization and Nash Implementation: Two Aspects of Mechanism Design." *Econometrica* 54 (1986): 139-152.