Pricing Risk-Adjusted Guaranty Insurance for Systematic Catastrophes

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Insurance guaranty funds compensate policyholders for losses resulting from insurance company insolvencies. This paper derives a risk-adjusted premium formula for insurance guaranty funds, when the insurance firm faces catastrophic risks which are correlated with the overall economy. In a normal situation, systematic jump risks lower the risk-adjusted premium, on the contrary to unsystematic jump risks. (*JEL* Classification: G22)

I. Introduction

The insurance companies receive premiums paid by policyholders and invest them in marketable securities. In return, the insurance firms promise to pay claims if specified events occur during a coverage period. Sometimes, the assets of an insurance company may be smaller than its debts, when it has to pay claims filed by policyholders. In such a case, the insurance guaranty funds reimburse policyholders and third-party claimants of the insolvent insurance firm in lieu of the firm.

The insurance guaranty funds charge the premium to the insurance firm for the service of guaranteeing reimbursements. If the insurance firm has to pay the flat premiums to the insurance guaranty funds, then the insurance firm has clear incentives to invest premium received into more risky investment objects because the firm is hedged against insolvencies. The number of insolvencies may increase. A well-

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known risk-incentive problem arises. In order to eliminate risk incentives, the premiums for the guaranty funds should be calculated based on the risk characteristics of the insurance firms. The risk-adjusted premiums are the themes of Merton (1977, 1978), Ronn and Verma (1986) and Cummins (1988).

Specifically, Cummins (1988) derives the risk-adjusted premium formula when the assets of the insurance firm follow a diffusion process and its liabilities follow a jump-diffusion process. In order to derive the formula, Cummins (1988) applies the jump-diffusion option pricing model developed by Merton (1976) which rules out systematic jump risk. Thus the Cummins premium formula cannot allow systematic catastrophes which exist in reality, as Cummins (1988) acknowledges. The systematic effect of catastrophes may be too large to be neglected. Furthermore, Cummins (1988) employs overly strong assumptions in order to derive his formula. First, he assumes that the Fisher relation holds. It is well-known that the Fisher relation does not hold under uncertainty (see Benninga and Protopapadakis 1983 and LeRoy 1984). He also assumes that inflation is allowed in liabilities but not in assets. Prohibiting systematic catastrophes and using implausible assumptions motivate us to investigate the risk-adjusted premium of guaranty fund insurance, when catastrophic risks are systematic and inflation is allowed in assets as well as liabilities.

In order to derive a risk-adjusted premium formula for systematic catastrophes, we employ a general equilibrium asset pricing model developed by Ahn and Thompson (1988) which is a jump-diffusion analog of Cox, Ingersoll and Ross (1985a). Recently, Ahn (1992) derives a call option pricing formula by using the asset pricing model of Ahn and Thompson (1988).[1]

When an insurance company becomes insolvent, the insurance guaranty fund takes over the assets of the company and discharges its liabilities at a cost to the fund of the difference between liabilities and assets. The fund works like a put option, whose exercise price is the value of liabilities. To derive the premium formula for the insurance guaranty fund is equivalent to value the put option.

In section II, we develop the risk-adjusted premium formula for the insurance guaranty fund in real terms. Subsequently, we derive the formula in nominal terms. In section III, we find the effect of system-

[1]Ahn (1992) differs from ours in that Ahn (1992) considers only a call option with a constant exercise price. We examine a put option with the stochastic exercise price.

atic catastrophes on the risk-adjusted premium for the insurance guaranty funds and provide numerical illustrations of the premium. Finally, we provide a summary in section IV.

II. The Risk-Adjusted Premiums

In this section, we derive the risk-adjusted premium formula for the insurance guaranty fund by using a general equilibrium model of Ahn and Thompson (1988). This formula is contrasted to the formula prohibiting systematic catastrophes like Cummins (1988).

We assume that the insurance firm makes a contract agreement with the guaranty fund at the beginning of a specified contract period (i.e., 1 year). The premium of the fund is calculated and charged at the beginning of the period. The fund conducts an audit for the firm at the end of the period. If assets exceed liabilities at the audit date, the firm is allowed to continue operating. On the other hand, if liabilities exceed assets, the guaranty fund pays its obligations to the policyholders and claimants at a cost to the fund of the difference between liabilities and assets.

Our setting closely follows the one of Cummins (1988). We assume that the assets of the insurance firm follow the stochastic differential equation given by

$$dA = (\alpha_A A + \delta N - \theta L)dt + A\sigma_A \, dz_A, \tag{1}$$

where α_A is the instantaneous expected return on assets, N is the number of policies, L is liabilities, δ is the instantaneous rate of premium inflow per policy insured, θ is the instantaneous rate of claim payment per dollar of liabilities, σ_A is the diffusion coefficient of assets and z_A is a standard Wiener process.

Premiums are collected in advance and held until claims are paid. We assume that liabilities of the insurance firm follow the stochastic differential equation given by

$$dL = (\alpha_L L + \eta N - \theta L - \lambda_\kappa L)dt + L\sigma_L dz_L + L(Y - 1)dq, \tag{2}$$

where α_L is the instantaneous expected growth rate of liabilities, η is the instantaneous rate of occurrence of new claims, σ_L is the diffusion coefficient of liabilities, z_L is a standard Wiener process, q is a Poisson process with an intensity parameter λ, κ is the expected jump amplitude and $Y - 1$ is the random proportionate change in the liabilities if a jump occurs (i.e., the jump amplitude). The expected jump amplitude κ is $E(Y-1)$, where E is the expectation operator over the random variable Y. We assume that the jump amplitude of q is independent of q, z_A and z_L and that q is independent of z_A and z_L. Let ω_{AL} denote the instantaneous correlation coefficient between z_A and z_L.

Following Cummins (1988), in order to facilitate the discussion, we assume that $\delta N = \theta L = \eta N$ in (1) and (2). This assumption implies a steady state, where premium inflow, claims outflow and the incidence of new claims are equal. This assumption is merely an expositional convenience and is not necessary to obtain premium rates.

In order to derive his premium formula, Cummins (1988) employs overly strong assumptions that the Fisher relation holds, that the market portfolio does not include jump risks, and that the static CAPM holds in a jump-diffusion environment.[2] It is well-known that the Fisher relation does not hold under uncertainty (see Benninga and Protopapadakis 1983; LeRoy 1984; Ahn and Thompson 1992 among others). Jarrow and Rosenfeld (1984) provide empirical evidence that the market portfolio contains jump components. Using daily returns on a weighted portfolio of all stocks on the New York Stock Exchange and the American Stock Exchange from July 1962 to December 1978, they reject the null hypothesis of a continuous sample path process at the 1% significance level. Schwert (1990) reports the possibility of jumps in the market portfolio. He lists the largest one day drop in the history of major stock market indexes from February 1885 through the end of 1988. On October 19, 1987, the Standard & Poor's composite portfolio fell from 282.70 to 224.84 or 22.4 percent. Jarrow and Rosenfeld (1984) also show that if jump risk is non-systematic, then the static CAPM holds only with a constant investment opportunity set. Whether the static CAPM holds, when jump risk is systematic or the investment opportunity set is stochastic, is questionable. We dispense these strong assumptions in deriving the risk-adjusted premium.

First, we derive the risk-adjusted premium in real terms and subsequently in nominal terms. In order to derive the premium formula, we consider a simple production economy of Ahn and Thompson (1988) with perfect markets and continuous trading in which there are a large number of consumers with identical endowments and preferences.

[2]In order to derive his formula, Cummins (1988) uses the jump-diffusion option pricing model of Merton (1976). The Merton model assumes that the market portfolio does not contain jump components. Thus the Cummins formula rules out systematic jump risk.

There are a single good and N production processes whose returns follow jump-diffusion processes. To maximize lifetime expected utility, each consumer chooses C^*, the consumption, a^*, a vector of the proportion of wealth W to be invested in each of the N production processes, the number of contingent claims and the amount of riskless borrowing or lending, given his budget constraint. The indirect utility function J is determined by solving the maximization problem. Since consumers are identical, all wealth is invested in the production processes in equilibrium.

The dynamics of wealth are given by

$$dW = \{W(\alpha_w - \lambda\kappa_M) - C^*\}dt + W\sigma_w dz_w + W(Y_M - 1)dq, \tag{3}$$

where α_W is the expected growth rate of wealth, κ_M is the expected jump amplitude of wealth, σ_w is the diffusion coefficient of wealth and z_w is a standard Wiener process. Let ω_{AW} denote the instantaneous correlation coefficient between z_A and z_W, and ω_{LW} denote the instantaneous correlation coefficient between z_L and z_W. The expected jump amplitude of wealth κ_M is $E(Y_M - 1)$, where $(Y_M - 1)$ is the random proportionate jump change in wealth which is the value of the market portfolio if a jump occurs. We assume that the jump amplitude of q is independent of z_W and that q is independent of z_W.

We assume that the direct utility function is time-additive and state-independent. Its functional form is assumed to be logarithmic as in Cox, Ingersoll and Ross (1985b) and Ahn and Thompson (1988).[3] Then it can be shown that the indirect utility function has the form

$$J(W(t), X(t)) = \left(\frac{1}{\beta}\right) \ln W(t) + d(X(t)), \tag{4}$$

and optimal consumption is

$$C^*(t) = \beta W(t), \tag{5}$$

where β is the rate of impatience and d is a real valued function of a set of state variables X (see Ahn and Thompson 1988).

The real value at time t of any contingent claim paying $F(T)$ at time $T \geq t$ is given by (see Theorem 5 of Ahn and Thompson 1988)

$$F(t) = \mathrm{E}_t[\exp\{-\beta(T - t)\} \frac{J_W(T)}{J_W(t)} F(T)], \tag{6}$$

[3]Cox, Ingersoll and Ross (1985b) make this assumption to derive the valuation formula for a call option on a default-free discount bond.

where E_t is an expectation operator based on the information available at time t and $J_W(T)$ denotes marginal utility of wealth at time T.

By using (5) and applying Ito's lemma to (3), wealth at time T can be written as

$$W(T) = W(t)\exp\{(\alpha_W - \left(\frac{1}{2}\right)\sigma_W^2 - \lambda\kappa_M - \beta)(T-t) \\ +\sigma_W(z_W(T) - z_W(t) + \sum_i^n \ln Y_{M_i}\}, \tag{7}$$

where n is $q(T)$ – $q(t)$. It follows from (4) and (7) that the value of the claim (6) can be rewritten as

$$F(t) = E_t[\exp\{-(\alpha_W - \left(\frac{1}{2}\right)\sigma_W^2 - \lambda\kappa_M)(T-t) \\ -\sigma_W(z_W(T) - z_W(t)) - \sum_i^n \ln Y_{M_i}\}F(T)]. \tag{8}$$

The value of guaranty fund insurance is similar to a put option since its value at the audit date T is Max[0, $L(T)$ – $A(T)$]. Thus, the real value at time t of guaranty fund insurance with $F(T)$ = Max[0, $L(T)$ – $A(T)$] can be written as

$$F(t) = E_t[\exp\{-(\alpha_W - \left(\frac{1}{2}\right)\sigma_W^2 - \lambda\kappa_M)(T-t) - \sigma_W(z_W(T) - z_W(t)) - \sum_i^n \ln Y_{M_i}\} \\ \mathrm{Max}[L(t)\exp\{(\alpha_L - \left(\frac{1}{2}\right)\sigma_L^2 - \lambda\kappa)(T-t) + \sigma_L(z_L(T) - z_L(t)) + \sum_i^n \ln Y_i\} \\ -A(t)\exp\{(\alpha_A - \left(\frac{1}{2}\right)\sigma_A^2)(T-t) + \sigma_A(z_A(T) - z_A(t))\},\ 0]]. \tag{9}$$

In order to find a closed form solution for the guaranty insurance, we assume that the random variables Y and Y_M are jointly bivariate lognormally distributed. Let μ – $(1/2)\delta^2$ and δ^2 denote the mean and the variance of the logarithm of Y respectively. Similarly, define μ_M – $(1/2)$ δ_M^2 and δ_M^2 to be the mean and the variance of the logarithm of Y_M. Let ρ denote the correlation coefficient of $\ln Y$ and $\ln Y_M$, which are bivariate normal. Let τ denote time to expiration ($\tau \equiv T - t$).

Theorem 1

With the assumptions made above, the real risk-adjusted premium of guaranty fund insurance is

$$F(L,A,\tau;\sigma_L,\sigma_A,\mu,\mu_M,\delta,\delta_M,\rho,\lambda) = \sum_{n=0}^{\infty} \frac{e^{-\lambda^*\tau}(\lambda^*\tau)^n}{n!}[L(t)N(d_1(n))$$
$$-A(t)\exp[\lambda\tau\{\exp(\mu-\mu_M+\delta_M^2-\rho\delta\delta_M)-\exp(-\mu_M+\delta_M^2)\} \tag{10}$$
$$+n\mu-n\rho\delta\delta_M]N(d_2(n))],$$

where $\lambda^* = \lambda\exp(\mu-\mu_M+\delta_M^2-\rho\delta\delta_M)$, $\upsilon_n^2 = \sigma_A^2 - 2\omega_{AL}\sigma_A\sigma_L + \delta_L^2 + n\delta^2/\tau$,

$$d_1(n) = [-\ln\frac{A(t)}{L(t)} + \lambda\tau\{\exp(\mu-\mu_M+\delta_M^2-\rho\delta\delta_M)-\exp(-\mu_M+\delta_M^2)\}$$
$$+n\mu-n\rho\delta\delta_M+\left(\frac{1}{2}\right)\upsilon_n^2\tau]/\sqrt{\upsilon_n^2\tau},$$

$d_2(n) = d_1(n) - \sqrt{\upsilon_n^2\tau}$, and $N(\cdot)$ is the standard normal cumulative distribution function.

Proof: see the appendix.

If the market portfolio does not contain jump components, then μ_M and δ_M are zero such that (10) reduces to

$$F(L,A,\tau;\sigma_L,\sigma_A,\mu,\delta,\lambda)$$
$$= \sum_{n=0}^{\infty} \frac{e^{-\lambda^*\tau}(\lambda^*\tau)^n}{n!}\{L(t)N(d_1(n))-A(t)e^{\lambda_\kappa\tau+n\mu}N(d_2(N))\} \tag{11}$$

or

$$= \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\{e^{-\lambda_\kappa\tau}L(t)N(d_1(n))-A(t)e^{n\mu}N(d_2(N))\}, \tag{12}$$

where $\lambda^* = \lambda\exp(\mu)$, $d_1(n) = [-\ln\{A(t)/L(t)\} + \lambda_\kappa\tau + n\mu + (1/2)\upsilon_n^2\tau]/\sqrt{\upsilon_n^2\tau}$, $d_2(n) = d_1(n) - \sqrt{\upsilon_n^2\tau}$, $\upsilon_n^2 = \sigma_A^2 - 2\omega_{AL}\sigma_A\sigma_L + \delta_L^2 + n\delta^2/\tau$ and $N(\cdot)$ is the standard normal cumulative distribution function.

Cummins (1988) derives a formula similar to (12) by assuming no jumps in the market portfolio and allowing inflation in only liabilities but not in asset.[4] Thus our formula (10) includes Cummins (1988) as a special case.

[4]Equation (15) of Cummins (1988) is not correct even with his assumptions. Equation (15) of Cummins (1988) is

$$E_N[W(\cdot)] = e^{-r^{**}\tau}\phi\left[\frac{-\ln x-\mu^*\tau}{\delta^*\sqrt{\tau}}\right] - xe^{N\gamma}\phi\left[\frac{\ln x-\mu^*\tau-\sigma^{*2}\tau}{\delta^*\sqrt{\tau}}\right],$$

where $\mu^* = r - r_L + \lambda\kappa - \sigma^2/2 - N\alpha/\tau$ and $\tau = -\alpha + N\xi^2/2$. Instead, the corrected

Now consider the risk-adjusted premium of guaranty fund insurance in nominal terms. We assume that the price level follows the stochastic differential equation given by

$$dP = P\alpha_P dt + P\sigma_P\, dz_P, \tag{13}$$

where α_P is the expected rate of the percentage change in inflation and σ_P is the diffusion coefficient of the percentage change in inflation. We utilize a nominal version of the Ahn and Thompson Model (1992) to derive the nominal premium.

The following theorem gives the nominal risk-adjusted premium.

Theorem 2

With the assumptions made above, the nominal risk-adjusted premium of guaranty fund insurance is given by

$$\begin{aligned}&\bar{F}(L, A, \tau, P\,;\, \sigma_L, \sigma_A, \mu, \mu_M, \delta, \delta_M, \rho, \lambda)\\ &= P(t)F(L, A, \tau;\, \sigma_L, \sigma_A, \mu, \mu_M, \delta, \delta_M, \rho, \lambda),\end{aligned} \tag{14}$$

where F is given as in Theorem 1.[5]

We can easily calculate the risk-adjusted premium for guaranty fund insurance per dollar of liabilities by using Theorems 1 and 2. It is given by

$$\begin{aligned}\frac{F}{L} = \frac{\bar{F}}{\bar{L}} = \sum_{n=0}^{\infty}\frac{e^{-\lambda^*\tau}(\lambda^*\tau)^n}{n!}&[N(d_1(n))\\ &-\frac{A(t)}{L(t)}\exp[\lambda\tau\{\exp(\mu-\mu_M+\delta_M^2-\rho\delta\delta_M)\\ &-\exp(-\mu_M+\delta_M^2)\}+n\mu-n\rho\delta\delta_M]N(d_2(n))],\end{aligned} \tag{15}$$

where λ^*, $N(d_1(n))$ and $N(d_2(n))$ are the same as defined in Theorem 1. It is interesting to note that the premium for insurance per dollar of liabilities is independent of being measured in nominal or real terms.

formula should be, in terms of his notations,

$$E_N[W(\cdot)] = e^{-r^{**}\tau}\phi\left[\frac{-\ln x-\mu^*\tau}{\delta^*\sqrt{\tau}}\right]-xe^{N\alpha}\phi\left[\frac{\ln x-\mu^*\tau-\sigma^{*2}\tau}{\delta^*\sqrt{\tau}}\right],$$

where $\mu^* = r - r_L + \lambda\kappa - \sigma^2/2 - N\alpha/\tau + N\xi^2/2$.

He erroneously states that his formula cannot be simplified to a weighted sum of Black-Scholes option values with a Poisson parameter λe^{μ}. In fact, our formula (11) shows that the formula can be simplified to a weighted sum.

[5]Proof of Theorem 2 can be available upon request from the author.

Thus F/L is homogeneous of degree zero in P in this economy. Furthermore, this premium is not affected by the real or nominal interest rate. This is a contrast to Cummins (1988) that includes the interest rate in his formula measured in nominal terms. His formula includes the interest rate since he makes a strange assumption that inflation is allowed in liabilities but not in assets. Our formula is a generalization of Cummins (1988) in that it allows catastrophes to be correlated with the overall economy while his formula does not. In the next section, we examine the effect of this correlation on the premium in detail.

III. The Effect of Systematic Catastrophes on the Premium

In order to make the analysis simple, we assume that the expected jump amplitudes in liabilities and wealth are zero ($\mu = \mu M = 0$). With the assumption made above, equation (15) can be written as

$$\begin{aligned}\frac{F}{L} = \frac{\overline{F}}{\overline{L}} = \sum_{n=0}^{\infty} \frac{e^{-\lambda^{*}\tau}(\lambda^{*}\tau)^{n}}{n!}[N(d_1(n)) \\ - \frac{A(t)}{L(t)} \exp[\lambda\tau\{\exp(\delta_M^2 - \rho\delta\delta_M) - \exp(\delta_M^2)\} - n\rho\delta\delta_M]N(d_2(n))]\end{aligned} \tag{16}$$

where $\lambda^{*} = \lambda\exp(\delta_M^2 - \rho\delta\delta_M)$, $v_n^2 = \sigma_A^2 - 2\omega_{AL}\sigma_A\sigma_L + \delta_L^2 + n\delta^2/\tau$, $d_1(n) = [-\ln\{A(t)/L(t)\} + \lambda\tau\{\exp(\delta_M^2 - \rho\delta\delta_M) - \exp(\delta_M^2)\} - n\rho\delta\delta_M + (1/2)v_n^2\tau]/\sqrt{v_n^2\tau}$, $d_2(n) = d_1(n) - \sqrt{v_n^2\tau}$ and $N(\cdot)$ is the standard normal cumulative distribution function.

The effect of systematic catastrophes on the risk-adjusted premium can be analyzed by examining the correlation coefficient ρ of liabilities' logarithmic jump with the market portfolio's jump. Table 1 shows the effect of the correlation numerically by using (16).[6] Note that the insurance guaranty fund is a put option with the exercise price being equal to liabilities. For deep-out-of-the-money options (i.e., $A \gg L$), as the correlaton of liabilities' jump with the market portfolio's jump increases, the option becomes less valuable. For $A \gg L$, the premium with $\rho = 1$ is less than the premium with $\rho = 0$ and the premium with $\rho = 0$ is less than the premium with $\rho = -1$. However, for deep-in-the-money option (i.e., $A \ll L$), as the correlation increases, the option becomes more valuable. For $A \ll L$, the premium with $\rho = 1$ is more than the

[6]Table 1 is generated by using $\delta^2 = 0.02$, $\delta_M^2 = 0.1$, $\sigma_A = 0.0415$, $\sigma_L = 0.0045$, $\omega_{AL} = 0.115$ and $\tau = 1$. The values of σ_A, σ_L and ω_{AL} are from Cummins (1988).

TABLE 1
COMPARISON OF GUARANTY FUND PREMIUMS

(λ = 0.33)

A/L Ratio	No jumps	$\rho = -1$	$\rho = 0$	$\rho = 1$
0.8	0.20000	0.20126	0.20151	0.20179
0.9	0.10007	0.10570	0.10643	0.10724
1.0	0.01641	0.02974	0.02950	0.02957
1.1	0.00015	0.00901	0.00793	0.00698
1.2	0.00000	0.00365	0.00308	0.00259
1.3	0.00000	0.00141	0.00114	0.00092
1.4	0.00000	0.00054	0.00042	0.00032

(λ = 0.2)

A/L Ratio	No jumps	$\rho = -1$	$\rho = 0$	$\rho = 1$
0.8	0.20000	0.20070	0.20085	0.20103
0.9	0.10007	0.10346	0.10394	0.10449
1.0	0.01641	0.02473	0.02457	0.02462
1.1	0.00015	0.00563	0.00492	0.00428
1.2	0.00000	0.00216	0.00180	0.00150
1.3	0.00000	0.00078	0.00062	0.00049
1.4	0.00000	0.00027	0.00021	0.00016

(λ = 0.1)

A/L Ratio	No jumps	$\rho = -1$	$\rho = 0$	$\rho = 1$
0.8	0.20000	0.20033	0.20040	0.20049
0.9	0.10007	0.10175	0.10201	0.10231
1.0	0.01641	0.02068	0.02059	0.02062
1.1	0.00015	0.00294	0.00255	0.00221
1.2	0.00000	0.00106	0.00087	0.00072
1.3	0.00000	0.00036	0.00028	0.00022
1.4	0.00000	0.00011	0.00009	0.00006

Note: $\delta^2 = 0.02$, $\delta^2_M = 0.01$, $\sigma_A = 0.0415$, $\sigma_L = 0.0045$, $\omega_{LA} = 0.115$ and $\tau = 1$.

premium with $\rho = 0$ and the premium with $\rho = 0$ is more than the premium with $\rho = -1$. The underlying intuition for the effect of ρ on the premium is provided in the following paragraph.

The effect of the correlation on the premium (i.e., the put option price) is affected by two factors. The first factor is related to the jump risk premia reflecting the hedging service from the put option. As ρ increases, the 'transformed' probability ($\lambda^* = \lambda \exp(\delta^2_M - \rho\delta\delta_M)$) decreases such that it decreases the option price. The second factor is related to the term - $\lambda\tau \exp\{\delta_M^2 - \rho\delta\delta_M\}$. It tends to increase the put option price as ρ increases. As seen in (A8) of the appendix, the expected growth rate α_L of liabilities increases with an increase in the correlation ρ. It is more likely to have a high value of liabilities (i.e., the exercise price) at the expiration date. The probability of being in-the-money at the expiration date increases. The option becomes more valuable.

Thus the put option price increases or decreases with the correlation, depending on whether the first factor overrides the second factor. For deep-out-of-the-money options, the second factor is overridden by the first factor. Thus the option price decreases as ρ increases. Since the current value of assets is far above that of liabilities, an increases in the terminal value of liabilities (i.e., the exercise price) caused by an increase in ρ may not significantly increase the probability of being in-the-money at the expiration date. However, for deep-in-the-money options, the second factor overrides the first one so that the option price increases as ρ increases.

Cummins (1988) reports the premiums for the A/L ratios ranging from 1.2 to 1.4, which are similar to the average asset/liability ratio for the insurance industry over the past ten years in the U.S. (see Cummins 1988). The insurance guaranty fund is currently in deep-out-of-the-money in the United States. Thus the impact of systematic catastrophes is that as the correlation between catastrophes and the overall markets increases, the premium decreases. In reality, systematic catastrophes may be positively correlated with the overall markets. Systematic catastrophes may lower the risk-adjusted insurance premium. They may help alleviate the risk-incentive problem. The circumstances under which risk-incentive problem is critical do not seem to arise because of systematic catastrophes. This result is the contrast to unsystematic catastrophes which are more likely to cause the problem.

Cummins (1988) simply states without proof "to the extent that jump risk has a systematic component, the guaranty fund premium (obtained by using his formula) may be understated." Since Cummins (1988) can be considered as a special case of $\rho = 0$, his statement is correct if ρ is less than zero. However, it is not if ρ is greater than zero, which may be the case in reality.

As we can anticipate, Table I shows that the premium with jump risks is more valuable than the corresponding one without jumps since the put option provides additional hedging against jump risks. The premium increases with the probability to jump λ since hedging services of the put option increase with λ.

IV. Summary

This paper examines the risk-adjusted premium of the insurance guaranty fund, when liabilities of the insurance firms contain catastrophic risks which may be correlated with the overall economy. The effect of the correlation on the premium depends on whether they are positively correlated and assets of insurance companies are greater than liabilities. In reality, they may be positively correlated and assets may be greater than liabilities. In these normal circumstances, systematic catastrophes lower the risk-adjusted premium of the insurance guaranty fund. Systematic catastrophes help alleviate the risk-incentive problem whereas unsystematic catastrophes cause the problem along with diffusion risks. The theoretical extension of future study may include incorporation of stochastic interest rates.

Appendix

We use the following lemma to prove the theorem.

Lemma
If random variables x, y and z are trivariate normally distributed, then

$$\mathrm{E}[e^{x+y}|x \geq a+z] = \exp\{\mathrm{E}(x)+\mathrm{E}(y)+\frac{1}{2}\mathrm{var}(x+y)\} \\ N\left(\frac{-a+\mathrm{E}(x)-\mathrm{E}(z)+\mathrm{cov}(x,y)-\mathrm{cov}(y,z)-\mathrm{cov}(x,z)+\sigma_x^2}{\sqrt{\mathrm{var}(x-z)}}\right) \tag{A1}$$

$$\mathrm{E}[e^{y+z}|x \geq a+z] = \exp\{\mathrm{E}(y)+\mathrm{E}(z)+\frac{1}{2}\mathrm{var}(y+z)\} \\ N\left(\frac{-a+\mathrm{E}(x)-\mathrm{E}(z)+\mathrm{cov}(x,y)-\mathrm{cov}(y,z)-\mathrm{cov}(x,z)-\sigma_x^2}{\sqrt{\mathrm{var}(x-z)}}\right), \tag{A2}$$

where a is a constant and N is the standard normal cumulative distribution function.

Proof: It is straightforward but tedious to derive, by employing a technique used to obtain a moment generating function, the following formula

$$\mathrm{E}[e^{\alpha X+\beta Y}|x \geq a] = \exp\{\alpha \mathrm{E}(x) + \beta \mathrm{E}(y) + \frac{1}{2}\mathrm{var}(x+y)\}$$
$$N\left(\frac{-a + \mathrm{E}(x) + \mathrm{cov}(x, \alpha x + \beta y)}{\sigma_x}\right), \tag{A3}$$

where α and β are constants. A detailed proof of (A3) is available from the author upon request. Use a technique of changing variables. Let $w \equiv \beta y + \gamma z$. Applying (A3) to find $\mathrm{E}[e^{\alpha X + W}|x \geq a]$, we have

$$\mathrm{E}[e^{\alpha x+\beta y+\gamma z}|x \geq a] = \exp\{\alpha \mathrm{E}(x) + \beta \mathrm{E}(y) + \gamma \mathrm{E}(z) + \frac{1}{2}\mathrm{cov}(\alpha x + \beta y + \gamma z)\}$$
$$N\left(\frac{-a + \mathrm{E}(x) + \mathrm{cov}(x, \alpha x + \beta y + \gamma z)}{\sigma_x}\right). \tag{A4}$$

Let $w \equiv x - z$. Applying (A4) to find $\mathrm{E}[e^{w+y+z}|x \geq a] \equiv \mathrm{E}[e^{x+y}|x \geq a + z]$ yields (Al). Similarly, applying (A4) to find $\mathrm{E}[e^{x+y-w}|w \geq a] \equiv \mathrm{E}[e^{y+z}|x \geq a + z]$ yields (A2).

Proof of Theorem 1

Let Z_x be $\sigma_L\{z_L(T) - z_L(t)\} + \sum_i^n \ln Y_i$, Z_y be $\sigma_W\{z_W(T) - z_W(t)\} + \sum_i^n \ln Y_{M_i}$ and Z_z be $\sigma_A\{z_A(T) - z_A(t)\}$.

The valuation equation (9) can be rewritten as

$$\begin{aligned} F(t) = {} & \mathrm{E}_t[\exp\{-(\alpha_W - (1/2)\sigma_W^2 - \lambda\kappa_M)\tau - Z_y\} \\ & \mathrm{Max}[0, L(t)\exp\{(\alpha_L - (1/2)\sigma_L^2 - \lambda\kappa)\tau + Z_x\} - A(t)\exp\{(\alpha_A - (1/2)\sigma_A^2)\tau \\ & + Z_z\}]] \end{aligned}$$

or

$$\begin{aligned} = {} & \mathrm{E}_t[\mathrm{E}[\exp\{-(\alpha_W - (1/2)\sigma_W^2 - \lambda\kappa_M)\tau - Z_y(n)\} \\ & \mathrm{Max}\,[0, L(t)\exp\{(\alpha_L - (1/2)\sigma_L^2 - \lambda\kappa)\tau + Z_x(n)\} \\ & - A(t)\exp\{(\alpha_A - (1/2)\sigma_A^2)\tau + Z_z(n)\}]\,|\,q(T) - q(t) = n]]. \end{aligned} \tag{A5}$$

Note that $\mathrm{E}(Z_y(n)) = n(\mu_M - (1/2)\sigma_M^2)$, $\mathrm{var}(Z_y(n)) = \sigma_W^2\tau + n\delta_M^2$, $\mathrm{E}(Z_x(n)) = n(\mu - (1/2)\delta^2)$, $\mathrm{var}(Z_x(n)) = \sigma^2\tau + n\sigma_L^2$, $\mathrm{E}(Z_z) = 0$, $\mathrm{var}(Z_z) = \sigma_A^2\tau$, $\mathrm{cov}(Z_x(n), Z_z)) = \omega_{AL}\sigma_A\sigma_L\tau$, $\mathrm{cov}(Z_y(n), Z_z)) = \omega_{AL}\sigma_A\sigma_L\tau$ and $\mathrm{cov}(Z_x(n), Z_y(n)) = \omega_{LW}\sigma_L\sigma_W\tau + n\rho\delta\delta_M$.

By using (A1) and (A2), the expectation conditional on $g(T) - q(t) = n$ in (A5) can be written as

$$\begin{aligned} & L(t)\exp\{(\alpha_L - \lambda\kappa - \alpha_W + \sigma_W^2 + \lambda\kappa_M - \omega_{LW}\sigma_L\sigma_W)\tau \\ & \quad + n(\mu - \mu_M + \delta_M^2 - \rho\delta\delta_M)\}N(d_1(n)) \\ & \quad - A(t)\exp\{(\alpha_A - \alpha_W + \sigma_W^2 + \lambda\kappa_M - \omega_{AW}\sigma_A\sigma_W)\tau - n(\mu_M - \delta_M^2)\}N(d_2(n)) \end{aligned} \tag{A6}$$

where

$$d_1(n)=\frac{-\ln\left\{\dfrac{L(t)}{A(t)}-(\alpha_A-\omega_{AW}\sigma_A\sigma_W)\tau+(\alpha_L-\lambda\kappa-\omega_{LW}\sigma_L\sigma_W)\tau+n(\mu-\rho\delta\delta_M)+\dfrac{1}{2}v_n^2\tau\right\}}{\sqrt{v_n^2\tau}}$$

$$d_2(n)=d_1-\sqrt{v_n^2\tau}$$

$$v_n^2=\frac{\sigma_L^2-2\omega_{LA}\sigma_L\sigma_A+\sigma_A^2+n\delta^2}{\tau}.$$

Applying (8) for a default-free discount bond yielding \$1 at time T provides the risk-free interest rate given by

$$r=\alpha_W-\sigma_W^2-\lambda\kappa_M-\lambda\{\exp(-\mu_M+\delta_M^2)-1\}. \tag{A7}$$

Applying (8) for liabilities gives the expected growth rate of liabilities given by

$$\alpha_L=\lambda\kappa+\omega_{LW}\sigma_L\sigma_W+\alpha_W-\sigma_W^2-\lambda\kappa_M-\lambda\{\exp(\mu-\mu_M+\delta_M^2-\rho\delta\delta_M)-1\} \tag{A8}$$

and applying (8) for assets gives the expected return on assets given by

$$\alpha_A=\omega_{AW}\sigma_A\sigma_W+\alpha_W-\sigma_W^2-\lambda\kappa_M-\lambda\{\exp(-\mu_M+\delta_M)-1\}, \tag{A9}$$

where $\kappa=\exp(\mu)-1$ and $\kappa_M=\exp(\mu_M)-1$.

Using (A7), (A8), and (A9), we can rewrite (A6) as

$$\begin{aligned}&\exp[-\lambda\tau\{\exp(\mu-\mu_M+\delta_M^2-\rho\delta\delta_M)-1\}+n(\mu-\mu_M+\delta_M^2-\rho\delta\delta_M)]L(t)N(d_1(n))]\\&-\exp[-\lambda\tau\{\exp(\mu_M+\delta_M^2)-1\}+n(-\mu_M+\delta_M^2)]\,A(t)N(d_2(n)),\end{aligned} \tag{A10}$$

where

$$d_1(n)=\frac{-\ln\left[\dfrac{L(t)}{A(t)}-\lambda\tau\{\exp(-\mu_M+\delta_M^2)-\exp(\mu-\mu_M+\delta_M^2-\rho\delta\delta_M)\}+n(\mu-\rho\delta\delta_M)+\dfrac{1}{2}v_n^2\tau\right]}{v_n\sqrt{\tau}}$$

$$d_2(n)=d_1(n)-v_n\sqrt{\tau}$$

$$v_n^2=\frac{\sigma_L^2-2\omega_{LA}\sigma_L\sigma_A+\sigma_A^2+n\delta^2}{\gamma}.$$

Substituting (A10) into (A5) gives the put option price given by

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}&\exp\{-\lambda\tau(e^{\xi}-1)\}e^{\xi n}[L(t)N(d_1(n))\\&-A(t)\exp[\lambda\tau\{\exp(\mu-\mu_M+\delta_M^2-\rho\delta\delta_M)\\&-\exp(-\mu_M+\delta_M^2)\}-n\mu+n\rho\delta\delta_M]N(d_2(n))]\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{e^{-\lambda^*\tau}(\lambda^*\tau)^n}{n!} [L(t)N(d_1(n)) \\ - A(t)\exp[\lambda\tau\{\exp(\mu - \mu_M + \delta_M^2 - \rho\delta\delta_M) \\ - \exp(-\mu_M + \delta_M^2)\} - n\mu + n\rho\delta\delta_M]N(d_2(n))], \tag{A11}$$

where $\xi = \mu - \mu_M + \sigma_M^2 - \rho\delta\delta_M$ and $\lambda^* = \lambda\ \exp(\mu - \mu_M + \delta_M^2 - \rho\delta\delta_M)$. This completes the proof.

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References

Ahn, C. "Option Pricing when Jump Risk is Systematic." *Mathematical Finance* 2 (1992): 299-308.

______, and Thompson, H. "Jump-Diffusion Processes and the Term Structure of Interest Rates." *Journal of Finance* 43 (1988): 155-74.

______. "The Impact of Jump Risks on Nominal Interest Rates and Foreign Exchange Rates." *Review of Quantitative Finance and Accounting* 2 (1992): 17-31.

Benninga, S., and Protopapadakis, A. "Real and Nominal Interest Rates under Uncertainty: The Fisher Theorem and Term Structure." *Journal of Political Economy* 91 (1983): 854-67.

Black, F., and Scholes, M. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81 (1973): 736-54.

Cox, J., Ingersoll, J., and Ross, S. "An Intertemporal General Equilibrium Model of Asset Prices." *Econometrica* 53 (1985a): 363-84.

______. "A Theory of the Term Structure of Interest Rates." *Econometrica* 53 (1985b): 385-407.

Cummins, D. "Risk-Based Premiums for Insurance Guaranty Funds." *Journal of Finance* 43 (1988): 823-39.

Jarrow, R., and Rosenfeld, E. "Jump Risks and the Intertemporal Capital Asset Pricing Model." *Journal of Business* 57 (1984): 337-51.

LeRoy, S. "Nominal Prices and Interest Rates in General Equilibrium: Endowment Shocks." *Journal of Business* 57 (1984): 197-214.

Merton, R. "Option Pricing when Underlying Stock Returns are Discontinuous." *Journal of Financial Economics* 3 (1976): 125-44.

______. "Analytic Derivation of the Cost of Deposit Insurance and Loan Guarantees: An Application of Modern Option Pricing Theory." *Journal of Banking and Finance* 1 (1977): 3-11.

______. "On the Cost of Deposit Insurance When There Are Surveillance Costs." *Journal of Business* 51 (1978): 439-52.

Margrabe, W. "The Value of an Option to Exchange One Asset for Another." *Journal of Finance* 33 (1978): 177-98.

Ronn, E., and Verma, A. "Pricing Risk-Adjusted Depositt Insurance: An

Option-Based Model." *Journal of Finance* 41 (1986): 871-95.
Schwert, W. "Stock Volatility and the Crash of '87." *Review of Financial Studies* 3 (1990): 77-102.