Concavity and Differentiability of Value Function with CRS Return Functions

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This paper investigates concavity and differentiability of the value function of a dynamic optimization problem when involved functions and correspondences exhibit CRS property. For the purpose, the relationship between the value function and the solution of the associated Bellman equation is investigated beforehand. As a byproduct of these investigations, the followings are obtained: a strictly quasi-concave CRS function is strictly concave when at least one of the independent variable is fixed in a 2 or higher dimensional case, and quasi-concave CRS function is concave. (*JEL* Classification: C61)

I. Introduction

It is well known that among approaches to dynamic optimization problems, dynamic programming has its strength in discrete time problems, and that the theory for it is best developed when the return functions are bounded. But many economic models deal with functions of homogeneous of degree 1, or Constant-Return-to-Scale (CRS) functions. Consider a firm's infinite horizon investment decision problem, for example. Usually, the production function is CRS and feasibility set is a cone.[1] In this case, the dynamic programming theory for problems with bounded return functions cannot be directly applied.

Song (1986) has shown that in a discrete time case, with a proper selection of a vector space and a norm, dynamic programming techniques for problems with bounded return functions can be extended to solve problems with CRS return functions. In this paper, we pursue the subject further. Especially, the concavity and differentiability of the

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[1] In the next section, a more detailed description will be given.

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value function is investigated, which is important as a practical matter. In most applications of the dynamic programming, first order conditions for the Bellman equation play a significant role, and they cannot be obtained without differentiability of the value function. Also in many cases, concavity is necessary for further economically meaningful analysis.

Yet complete analysis of the properties of the value function of dynamic problems with CRS return functions is not available. Although Stokey and Lucas (1989) present Song's (1986) theory for the CRS cases, it does not fully explain the properties of value function. This paper is trying to fill in the gap.

The rest of the paper is organized as follows. In Section II, we give a specific economic example, to which our analysis can be applied, and in Section III, we present a general framework for a dynamic problem with a CRS return function, and summarize known results through Song (1986) and Stokey and Lucas (1989). In Section IV, we present a theory about the relationship between value function and solution of the associated Bellman equation. In Section V, we develop a theory for the concavity and differentiability of a value function. And we give some concluding remarks in Section VI.

II. Example

Consider the following firm's decision problem in a competitive environment. It's production requires two kind's of inputs; the first class of inputs are available in the market, and the second not. The technology of production has CRS property and the only way to use more of the second kind of input is to build them up in previous periods with costs. An example of this kind is the firm-specific human capital, which can be increased only through internal job training.

The model can be formulated as follows. In each period, the firm maximizes its short-run profit,

$$\pi_t(h) = \max_{k \geq 0} pF(k, h) - wk.$$

Here, $\pi_t(h)$ is the profit function, F is the production function homogeneous of degree 1 in (k, h), k and h are vectors of inputs available and not available in the market, respectively, and w is a vector of market prices of inputs k. $\pi_t(h)$ is also homogeneous of degree 1, since

$$\begin{aligned}
\pi(\lambda h) &= \max_{k\geq 0} F(k, \lambda h) - wk \\
&= \max_{k\geq 0} \lambda \left\{ F\left(\frac{k}{\lambda}, h\right) - w\frac{k}{\lambda} \right\} \\
&= \lambda \max_{(k/\lambda)\geq 0} \left\{ F\left(\frac{k}{\lambda}, h\right) - w\frac{k}{\lambda} \right\} \\
&= \lambda \max_{k'\geq 0} F(k', h) - wk' \\
&= \lambda \pi(h).
\end{aligned}$$

With this short-run profit function, in the long run, the firm solves

$$\max \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t \{\pi(h_t) - C(h_{t+1} - (I - \Delta)h_t)\},$$

where r is an interest rate, $C(\cdot)$ is a cost function involved in increasing in h, which is assumed to have CRS property, and $I - \Delta$ is $n \times n$ diagonal matrix with entry (i, i) being $1 - \delta_i$, i.e.,

$$I - \Delta = \begin{pmatrix} 1-\delta_1 & 0 & \cdots & 0 \\ 0 & 1-\delta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-\delta_n \end{pmatrix}.$$

Here, δ_i represents depreciation rate of input i.

Now suppose that for each h_t, the maximum amount the firm can build for the next period is $I + A$, where

$$I - A = \begin{pmatrix} 1+\alpha_1 & 0 & \cdots & 0 \\ 0 & 1+\alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1+\alpha_n \end{pmatrix}$$

Then the maximization problem is

$$\begin{aligned}
&\max_{\{h_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t \{\pi(h_t) - C(h_{t+1} - (I - \Delta)h_t\} \\
&\text{s.t.} \quad k_t \geq 0, \\
&\qquad (I - \Delta)h_t \leq h_{t+1} \leq (I + A)h_t, \\
&\qquad h_0 \quad \text{given.}
\end{aligned}$$

Through Song(1986) and Stokey and Lucas (1989), we know how to attack this problem. But we need to know the properties of the firm's long-run profit function in order to obtain economic implications. This paper presents a theory for the purpose.

III. Model, Assumptions, and Known Results

In general, the dynamic programming problem is

$$\max_{\{h_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(k_t, k_{t+1}) \tag{1}$$

$$\begin{aligned} \text{s.t.}\quad & k_{t+1} \in \Gamma(k_t), \\ & \Gamma \text{ is a correspondence;} \\ & \Gamma: D \rightarrow D, \quad D \subset R^n, \\ & k_0 \in D \text{ given.} \end{aligned}$$

Now assume the following.

Assumption 1

$D = R^n$ or R_+^n. $A = \{(k, y) \in D \times D \mid y \in \Gamma(k)\}$, and $F: A \rightarrow R$ is a continuous CRS function. $0 < \beta < 1$.

Assumption 2

For all $k \in D$, $\Gamma(k)$ is non-empty, compact, continuous, and $y \in \Gamma(k)$ if and only if $\lambda y \in \Gamma(\lambda k)$ for all $\lambda > 0$. And if $y \in \Gamma(k)$, then $\|y\| \leq \alpha \|k\|$ for some $\alpha \in (0, \beta^{-1})$.

The corresponding functional equation (Bellman equation) to the problem (1) is

$$v(k) = \max_{y \in \Gamma(k)} \{F(k, y) + \beta v(y)\}. \tag{2}$$

In order to solve (1) through the Bellman equation (2), we deal with the following space. Let S be the space of continuous and real valued functions with CRS property on the domain D. Define a norm of $f \in S$ by

$$\|f\| = \sup_{\|x\|=1} |y(x)|.$$

Theorem 1

The normed vector space S is complete.

Proof: See Song (1986)or Stokey and Lucas (1989).

Q.E.D.

Theorem 2. (Modification of Blackwell's sufficient conditions for a contraction)
Let $T: S \to S$ be an operator satisfying
(i) (monotonicity) If $f, g \in S$, and $f(x) \leq g(x)$ for all $x \in D$, then $Tf(x) \leq Tg(x)$ for all $X \in D$.
(ii) (discounting) There exists some $\gamma \in (0, 1)$ such that for all $f \in S$, all $a \geq 0$, and all $x \in D$,

$$T(f + a)(x) \leq Tf(x) + \gamma a$$

where $(f + a)(x) = f(x) + a \| x \|$.
Then T is a contraction with modulus γ.

Proof: See Song (1986) or Stokey and Lucas (1989).

Q.E.D.

Now define an operator $T: S \to S$ for the dynamic problem (1) as follows.

$$Tv(x) = \max_{y \in \Gamma(k)} \{F(k, y) + \beta v(y)\}.$$

Maximum is attained since $F(k, y) + \beta v(y)$ is continuous in y and the set Γ is compact. By the theorem of maximum, $Tv(k)$ is continuous, The new function Tv is CRS since

$$\begin{aligned} Tv(\lambda k) &= \max_{y \in \Gamma(\lambda k)} \{F(\lambda k, y) + \beta v(y)\} \\ &= \lambda \max_{(y/\lambda) \in \Gamma(k)} \left\{ F\left(k, \frac{y}{\lambda}\right) - \beta v \frac{y}{\lambda} \right\} \\ &= \lambda \max_{(y') \in \Gamma(k)} \{F(k, y') + \beta v(y')\} \\ &= \lambda \{Tv(k)\} \quad \text{for all} \quad \lambda > 0. \end{aligned}$$

Hence, T is well defined.

Theorem 3
T has a unique fixed point $v \in S$, and for all $v_0 \in S$, $\| T^n v_0 - v \| \leq (\alpha\beta)^n \| v_0 - v \|$ for $n = 0, 1, 2, \ldots$. And given v, the policy correspondence $g: D \to D$, defined by

$$g(k) = \{y \in \Gamma(k) \mid v(k) = F(k, y) + \beta v(y)\},$$

is compact valued and upper-hemi-continuous.

Proof: See Song (1986) or Stokey and Lucas (1989).

Q.E.D.

IV. Relationship between Value Function and Solution of Bellman Equation

Now given $k \in D$, define $B(k)$ as

$B(k) = \{(k_t)_{t=0}^{\infty} \mid k_t \in D, \text{ and } k_{t+1} \in \Gamma(k_t) \text{ for all } t = 0, 1, 2, \ldots, \text{ and } k_0 = k\}$

Define U_n: $B(k) \rightarrow R$ and U_n: $B(k) \rightarrow \overline{R}$ (extended real line) by

$$U_n(k^s) = \sum_{t=0}^{n} \beta^t F(k_t, k_{t+1}), \text{ and}$$
$$U(k^s) = \lim_{n \rightarrow \infty} U_n(k^s),$$

where $k^s = \{k_t\}_{t=0}^{\infty} \in B(k)$. Also Define υ^*: $D \rightarrow \overline{R}$ (extended real line) by

$$\upsilon^*(k) = \sup_{k^s \in B(k)} U(k^s).$$

A sufficient and necessary condition for a function ξ: $D \rightarrow \overline{R}$ to be υ^* is

Condition (a)
for each $k \in D$, $\xi(k) \geq U(k^s)$ for all $k^s \in B(k)$, and

Condition (b)
for any $\varepsilon > 0$, there exists $k^s \in B(k)$ such that $\xi(k) \leq U(k^s) + \varepsilon$ for all $k \in D$.

Note that υ^* is homogeneous of degree 1 with the usual convention that $\infty \cdot \lambda = \infty$ and $-\infty \cdot \lambda = -\infty$ for all $\lambda > 0$. To see this, first observe that $k^s \in B(k)$ if and only if $\lambda k^s \in B(\lambda k)$ for all $\lambda > 0$, which is a result of assumption 2, where $\lambda k^s = \{\lambda k_t\}_{t=0}^{\infty}$. Then

$$\begin{aligned} \upsilon^*(\lambda k) &= \sup_{k^s \in B(\lambda k)} U(k^s) \\ &= \lambda \sup_{(1/\lambda)k^s \in B(k)} U\left(\frac{k^s}{\lambda}\right) \\ &= \lambda \upsilon^*(k). \end{aligned}$$

Let g be a correspondence from D into the collection of subsets of $\Gamma(k)$. We say that it attains v, a solution to (2), if $v(k) = F(k, y) + \beta v(y)$ for all $y \in g(k)$.

Theorem 4

v^* satisfies (2) in the sense that

$$v^* = \sup_{y \in \Gamma(k)} \{F(k, y) + \beta v^*(y)\}.$$

Proof: Fix $k \in D$. For any $y \in \Gamma(k)$ and for any sequence $\{k_t\}_{t=0}^{\infty} \in B(y)$, $(k, y, k_1, k_2, \ldots) \in B(k)$. Hence from condition (a) above,

$$v^*(k) \geq F(k, y) + \beta F(y, k_1) + \beta \sum_{t=1}^{\infty} \beta^t F(k_t, k_{t+1}).$$

From condition (b), given $\varepsilon > 0$, we can choose $\{k_t\}_{t=0}^{\infty} \in B(y)$ such that

$$v^*(k) \leq F(y, k_1) + \sum_{t=1}^{\infty} \beta^t F(k_t, k_{t+1}) + \varepsilon \text{ for all } y.$$

Hence

$$\begin{aligned} v^*(k) &\geq F(k, y) + \beta F(y, k_1) + \beta \sum_{t=1}^{\infty} \beta^t F(k_t, k_{t+1}) \\ &\geq F(k, y) + \beta\{v^*(y) - \varepsilon\} \\ &= F(k, y) + \beta v^*(y) - \beta\varepsilon \text{ for all } y \in \Gamma(k) \text{ and all } \varepsilon < 0. \end{aligned}$$

This implies $v^*(k) \geq F(k, y) + \beta v^*(y)$ for all $y \in \Gamma(k)$. Again from condition (b), given $\varepsilon > 0$, we can choose $\{k_t^*\}_{t=0}^{\infty} \in B(k)$ such that

$$v^*(k) \leq F(y, k_1^*) + \sum_{t=1}^{\infty} \beta^t F(k_t^*, k_{t+1}^*) + \varepsilon.$$

Hence $v^*(k) \leq F(k, k_1^*) + \beta v^*(k_1^*) + \varepsilon$ since

$$\sum_{t=1}^{\infty} \beta^t F(k_t^*, k_{t+1}^*) \leq \beta v^*(k_1^*).$$

k was fixed arbitrarily. Therefore, $v^* = \sup_{y \in \Gamma(k)} \{F(k, y) + \beta v_*(y)\}$ for each $k \in D$ follows immediately.

Q.E.D.

Theorem 5

If v satisfies (2) with $\lim_{n \to \infty} \beta^n v(k_n) = 0$ for all $k \in D$, and all $k^s \in B(k)$, then $v(k) = v^*(k)$ for all $k \in D$.

Proof: $v(k_n) \geq F(k_n, k_{n+1}) + \beta v(k_{n+1})$ for all $n \geq 0$.
So $U_n(k^s) + \beta^{n+1}v(k_{n+1}) \geq U_n(k^s) + \beta^{n+1}F(k_{n+1}, k_{n+2}) + \beta^{n+2}v(k_{n+2})$
$= U_{n+1}(k^s) + \beta^{n+2}v(k_{n+2})$ for all $n \geq 0$.
Hence, $v(k) \geq U_n(k^s) + \beta^{n+1}v(k_{n+1})$
$\geq U_{n+1}(k^s) + \beta^{n+2}v(k_{n+2})$ for all $n \geq 0$.

This implies

$$\begin{aligned} v(k) &\geq \lim_{n\to\infty}\{U_n(k^s) + \beta^{n+1}v(k_{n+1})\} \\ &= \lim_{n\to\infty} U_n(k^s) \\ &= U(k^s) \quad \text{for all} \quad k \in D \quad \text{and all} \quad k^s \in B(k). \end{aligned}$$

Now fix $k \in D$. Given $\varepsilon > 0$, it is possible to choose $y_1 \in \Gamma(k)$ such that

$$v(k) \leq F(k, y_1) + \beta v(y_1) + \varepsilon, \text{ and}$$

given y_1, it is also possible to choose $y_2 \in \Gamma(y_1)$ such that

$$v(y_1) \leq F(y_1, y_2) + \beta v(y_2) + \varepsilon.$$

So $v(k) \leq F(k, y_1) + \beta F(y_1, y_2) + \beta^2 v(y_2) + \varepsilon(1 + \beta)$.
Continuing this procedure, we can generate a sequence $y^s = \{y_t\}_{t=0}^{\infty}$ such that $y_0 = k$, $y_{n+1} \in \Gamma(y_n)$, and

$$v(y_n) \leq F(y_n, y_{n+1}) + \beta v(y_{n+1}) + \varepsilon.$$

Hence,

$$\begin{aligned} v(k) &\leq \sum_{t=1}^{n} \beta^t F(y_t, y_{t+1}) + \beta^{n+1}v(y_{n+1} + \varepsilon\left(\frac{1-\beta^{n+1}}{1-\beta}\right) \\ &\leq U(y^s) + \frac{\varepsilon}{1-\beta}. \end{aligned}$$

This means for any $\varepsilon > 0$, condition (b) is satisfied. Therefore, $v = v^*$.

Q.E.D.

Theorem 6
If v satisfies (2) with $\lim_{n\to\infty}\beta^n v(k_n) \leq 0$ for all $k \in D$, and $k^s \in B(k)$, then $v(k) \leq v^*(k)$ for all $k \in D$.

Proof: From the second half of the proof of the Theorem 5, we know that given $\varepsilon > 0$, there exists $y^s \in B(k)$ such that

$$v(k) \le U(y^s) + \lim_{n\to\infty} \beta^{n+1} v(y_{n+1}) + \varepsilon$$
$$\le U(y^s) + \varepsilon \quad \text{for all} \quad k \in D.$$

Hence $v(k) \le v^*(k) + \varepsilon$ for all $k \in D$ and for all $\varepsilon > 0$. Therefore, $v(k) \le v^*(k)$ for all $k \in D$.

Q.E.D.

Theorem 7

Assume the hypothesis of Theorem 6 and assume that for any $k^s \in B(k)$, there is $k^{s*} \in B(k)$ such that $U(k^{s*}) \ge U(k^s)$ and $\lim_{n\to\infty} \beta^n v(k^*_n) = 0$. Then $v(k) = v^*(k)$ for all $k \in D$.

Proof: From theorem 6, $v(k) \le v^*(k)$ for all $k \in D$. From condition (b), it is possible to choose $k^s \in B(k)$ such that $v^*(k) \le U(k^s) + \varepsilon$ for a given $\varepsilon > 0$. Then by hypothesis, it is possible to choose $k^{s*} \in B(k)$ such that $U(k^{s*}) \ge U(k^s)$ and $\lim_{n\to\infty} \beta^n v(k^*_n) = 0$. Then from the first part of the proof of the Theorem 5, we conclude that $v(k) \le U(k^{s*})$. Hence $v^*(k) \le v(k) + \varepsilon$ for all $\varepsilon > 0$. Therefore, $v(k) = v^*(k)$ for all $k \in D$.

Q.E.D.

Theorem 8

If $v(k) = v^*(k)$ for all $k \in D$, g attains v^*, and $\lim_{n\to\infty} \beta^n v(k^*_n) \le 0$ for a sequence $k^s \in B(k)$ such that $k_{n+1} \in g(k_n)$, then $v^*(k) = U(k^s)$.

Proof: Since $k_{n+1} \in g(k_n)$, $v^*(k) = U_n(k^s) + \beta^{n+1} v^*(k_{n+1})$ for all $n \ge 0$. Hence

$$v(k_{n+1}) = \lim_{n\to\infty} \{U_n(k^s) + \beta^{n+1} v^*(k_{n+1})\}$$
$$= U(k^s) + \lim_{n\to\infty} \beta^{n+1} v^*(k_{n+1})$$
$$\le U(k^s).$$

But $v^*(k) \ge U(k^s)$ by the definition of v^*. Therefore, $v^*(k) = U(k^s)$.

Q.E.D.

Theorem 9

Let F, β, and Γ satisfy assumptions 1 and 2, v satisfy (2), and g attain v. Then given $k_0 \in D$, $v(k_0)$ is the value of the objective function in problem (1) at the maximum. Moreover, a sequence $\{k_t\}_{t=0}^{\infty}$ is the solution to problem (1) if and only if it satisfies $k_{t+1} \in g(k_t)$.

Proof: We know v exists and unique from Theorem 3. Since the unit ball is compact in R^n, $|\beta^n v(k_n)| = \beta^n \cdot \|k_n\| \cdot |v(k_n/\|k_n\|)| \le \beta^n \|k_n\| \cdot M$

for all $k_n \neq 0$. $\|k_n\| \leq \alpha\|k_{n-1}\|$ by Assumption 2 since $k_n \in \Gamma(k_{n-1})$. So $\|k_n\| \leq \alpha^n\|k_0\|$. Thus

$$0 \leq |\beta^n v(k_n)| = \beta^n\|k_n\| \cdot |v(k_n/\|k_n\|)| \leq \beta^n\alpha^n\|k_0\| \cdot M.$$

Then $\alpha\beta \leq 1$ implies that $\lim_{n\to\infty}\beta^n v(k_n) = 0$. Therefore, by Theorem 5, v solves (1) and $v(k_0)$ is the value of the objective function.

The fact that if $\{k_t\}_{t=0}^{\infty}$ satisfies $k_{t+1} \in g(k_t)$, it is the solution of problem (1) comes from Theorem 8 and that $\lim_{n\to\infty}\beta^n v(k_n) = 0$.

To prove the converse, suppose that $\{k_t^*\}_{t=0}^{\infty}$ solves (1) with $k_0^* = k_0$. Then

$$\begin{aligned} v(k_0) &= \max_{\{k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(k_t, k_{t+1}) \\ &= \sum_{t=0}^{n} \beta^t F(k_t^*, k_{t+1}^*) + \max_{\{k_t\}_{t=n+2}^{\infty}} \left\{\beta^{n+1}F(k_{n+1}^*, k_{n+2}) + \sum_{t=n+2}^{n} \beta^t F(k_t, k_{t+1})\right\} \\ &= \sum_{t=0}^{n} \beta^t F(k_t^*, k_{t+1}^*) + \beta^{n+1}v(k_{n+2}^*), \quad n = 0, 1, 2, \ldots. \end{aligned}$$

Setting $n = N - 1$ and again $n = N$, we have

$$\begin{aligned} v(k_0) &= \sum_{t=1}^{N-1} \beta^t F(k_t^*, k_{t+1}^*) + \beta^N v(k_N^*) \\ &= \sum_{t=0}^{N} \beta^t F(k_t^*, k_{t+1}^*) + \beta^{n+1}v(k_{N+1}^*), \quad N = 1, 2, \ldots, \end{aligned}$$

so that $v(k_N^*) = F(k_N^*, k_{N+1}^*) + \beta v(k_{N+1}^*)$, $N = 1, 2, \ldots$. Also setting $n = 0$, we have $v(k_0^*) = F(k_0^*, k_1^*) + \beta v(k_1^*)$. Hence by the definition of g it follows that $k_{N+1}^* \in g(k_N^*)$.

Q.E.D.

V. Concavity and Differentiability of Value Function

Now we are ready for the differentiability of the value function if we assume the following.

Assumption 3

F is strictly quasi-concave.

Assumption 4

Γ is convex for all $k \in D$. And for all k, $k' \in D$ and all $0 \leq \theta \leq 1$, $y \in \Gamma(k)$ and $y' \in \Gamma(k')$ implies that $\theta y + (1-\theta)y' \in \Gamma(\theta k + (1-\theta)k')$.

Theorem 10

Let F, β, and Γ satisfy assumptions 1 to 4, v solve (2), and g be policy correspondence. Then v is concave and g is a continuous (single valued) function.

In order to prove this theorem, we need the following lemmas.

Lemma 1

Assume h is CRS and strictly quasi-concave. Then for any $\theta \in (0, 1)$, $h(\theta x_1 + (1 - \theta)x_2) > \theta h(x_1) + (1 - \theta)h(x_2)$ if $x_2 \neq 0$ and $x_1 \neq tx_2$ for all $t \in R$.

Proof: If $h(x_1) = h(x_2)$, then by the strict quasi-concavity of h, $h(\theta x_1 + (1 - \theta)x_2) > \theta h(x_1) + (1 - \theta)h(x_2)$. If $h(x_1) \neq h(x_2)$, then without loss of generality assume $h(x_1) < h(x_2)$. CRS property and strict quasi-concavity implies that $h(x) \neq 0$ if $x \neq 0$.[2] Let $\mu = h(x_2)/h(x_1)$. then $h(\mu x_1) = \mu h(x_1) = h(x_2)$. Hence

$$\begin{aligned}
&\theta h(x_1) + (1 - \theta)h(x_2) \\
&= \theta h(x_1) + (1 - \theta)h(\mu x_2) \\
&= \{\theta + \mu(1 - \theta)\}h(x_2).
\end{aligned}$$

$$\text{Let } \lambda = \frac{h\{\theta x_1 + (1 - \theta)x_2\}}{h(x_1)}.$$

Then $h\{(\lambda/\mu)x_2\} = h\{\theta x_1 + (1 - \theta)x_2\}$. From strict quasi-concavity,

$$\begin{aligned}
&h\left\{\alpha(\lambda x_1) + (1 - \alpha)\left(\frac{\lambda}{\mu} x_2\right)\right\} \\
&> \alpha h\left\{\alpha(\lambda x_1) + (1 - \alpha)h\left(\frac{\lambda}{\mu} x_2\right)\right\} \\
&= h(\lambda x_1) \\
&= h\{\theta x_1 + (1 - \theta)x_2\} \quad \text{for all} \quad \alpha \in (0, 1).
\end{aligned}$$

Let $\alpha = \theta/\{\theta + (1 - \theta)\mu\}$. Clearly, $0 < \alpha < 1$ with $\theta \in (0, 1)$. Hence

$$h\left\{\frac{\theta}{\theta + (1 - \theta)\mu} \lambda x_1 + \frac{(1 - \theta)\mu}{\theta + (1 - \theta)\mu} \frac{\lambda}{\mu} x_2\right\}$$

[2]Suppose not, i.e., $h(x) = 0$ for some $x \neq 0$. Then $h(\lambda x) = 0$ for all $\lambda > 0$, which contradicts to the strict quasi-concavity of h. So $h(x) \neq 0$ if $x \neq 0$.

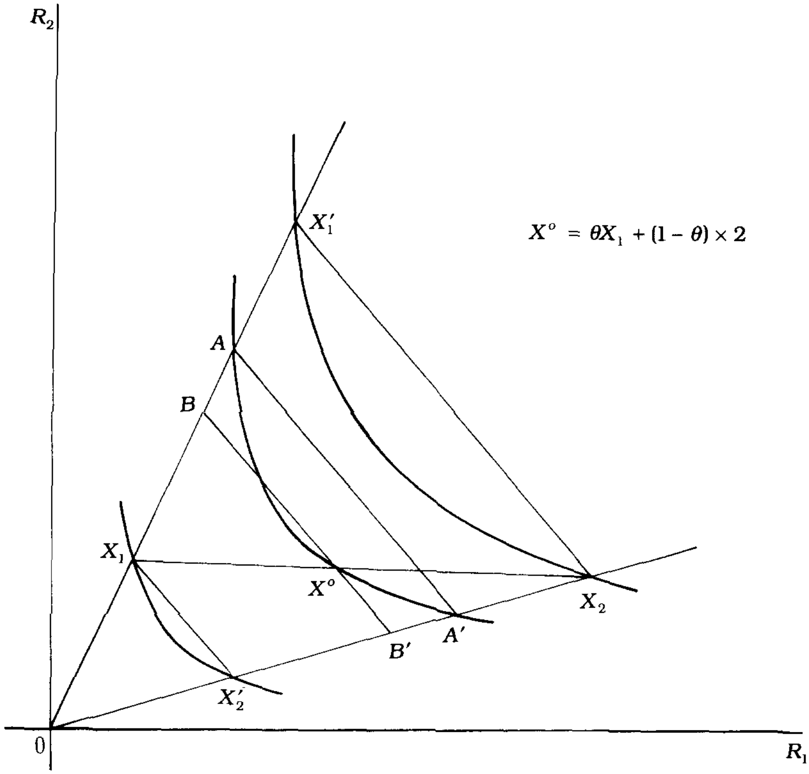

Figure 1

$$= \frac{\lambda}{\theta + (1-\theta)\mu} h\{\theta x_1 + (1-\theta)x_2\}$$
$$> h\{\theta x_1 + (1-\theta)x_2\}.$$

Then $\lambda/\{\theta + (1-\theta)\mu\} > 1$ since $h\{\theta x_1 + (1-\theta)x_2\} \neq 0$ from the assumption that $x_2 \neq 0$ and $x_1 \neq tx_2$ for all $t \in R$. So

$$\lambda h(x_1) > \{\theta + (1-\theta)\mu\}h(x_1), \text{ and}$$
$$\begin{aligned}\lambda h(x_1) &= h\{\theta x_1 + (1-\theta)x_2\}\\ &> \{\theta + (1-\theta)\mu\}h(x_1)\\ &= \theta h(x_1) + (1-\theta)h(x_2).\end{aligned}$$

Therefore, $h\{\theta x_1 + (1-\theta)x_2\} > \theta h(x_1) + (1-\theta)h(x_2)$.

Q.E.D.

The above proof can be illustrated in a 2-dimensional diagram.

Three level curves Passing through x_1, x_2, and x^0 are drawn where $x^0 = \theta x_1 + (1-\theta)x_2$, and $h(A) = h(x^0)$. The line segments $\overline{x_1x_2'}$ and $\overline{x_1'x_2}$ are parallel. Then on the line segment $\overline{BB'}$ which passes through x^0 and is parallel to $\overline{x_1x_2'}$, B has the value of $\theta h(x_1) + (1-\theta)h(x_2)$, since

$$\frac{\|x^0 - x_1\|}{\|x_1 - x_2\|} = \theta = \frac{\|B - x_1\|}{\|x_1' - x_1\|} \quad \text{and}$$

$$\frac{\|x_2 - x^0\|}{\|x_2 - x_1\|} = (1-\theta) = \frac{\|x_1' - B\|}{\|x_1' - x_1\|}.$$

Since the level curves are convex to the origin, the line segment BB' is closer to the origin than the line segment $\overline{AA'}$. This implies that

$$h(A) > h(B), \text{ i.e., } h\{\theta x_1 + (1-\theta)x_2\} > \theta h(x_1) + (1-\theta)h(x_2).$$

Notice that the above lemma implies that if at least one of the element in vector x is fixed, the $h(x)$ is strictly concave.

Lemma 2

If $h(x)$ is CRS and quasi-concave, then it is concave.

Proof: The proof is almost identical as that of lemma 1.

Q.E.D.

Lemma 3

Suppose f_n are sequence of concave functions in the space S, and $f_n \to v$ in the topology of S defined in Section II. Then v is concave.

Proof: Let k, k', and θ be given, and let $k^0 = \theta k + (1-\theta)k'$. Assume $k \neq 0$, $k' \neq 0$, and $\theta \in (0, 1)$. In other cases, trivially $\theta v(k) + (1-\theta)v(k') = v(k^0)$. Then given $\varepsilon > 0$, there exist N, N', and N^0 such that

$$\begin{aligned}|v(k) - f_n(k)| &= \|k\| \cdot \left| v\left(\frac{k}{\|k\|}\right) - f_n\left(\frac{k}{\|k\|}\right)\right| \\ &\leq \|k\| \cdot \|f_n - v\| \\ &< \varepsilon\|k\| \quad \text{for all} \quad n \geq N, \\ |v(k') - f_n(k')| &< \varepsilon\|k'\| \quad \text{for all} \quad n \geq N', \quad \text{and} \\ |v(k^0) - f_n(k^0)| &< \varepsilon\|k^0\| \quad \text{for all} \quad n \geq N^0.\end{aligned}$$

Let $\delta = \max\{\varepsilon\|k\|, \varepsilon\|k'\|, \varepsilon\|k^0\|\}$, and $M = \max\{N, N', N^0\}$. Then $v(k) < f_n(k) + \delta$, $v(k') < f_n(k') + \delta$, and $v(k^0) > f_n(k^0) - \delta$ for all $n \geq M$. Hence

$$\begin{aligned}\theta v(k) + (1-\theta)v(k') &< \theta f_n(k) + (1-\theta)f(k') + \delta \\ &\le f_n(k^0) + \delta \\ &< v(k^0) + 2\delta.\end{aligned}$$

Since ε was chosen arbitrarily, $\theta v(k) + (1-\theta)v(k') < v(k^0) + 2\delta$ holds for all $\delta > 0$. Hence $\theta v(k) + (1-\theta)v(k') \le v(k^0)$, i.e., v is concave.

Q.E.D.

Proof of Theorem 10: By Assumption 3 and Lemma 2, F is concave. Let $f \in S$ be concave, y and y' solve

$$Tf(k) = \max_{y\in\Gamma(k)}\{F(k,y) + \beta f(y)\}, \quad \text{and}$$

$$Tf(k') = \max_{y\in\Gamma(k')}\{F(k',y) + \beta f(y)\}, \quad \text{respectively.}$$

And let $y^0 = \theta y + (1-\theta)y'$ and $k^0 = \theta k + (1-\theta)k'$. By Assumption 4, $y^0 = \in \Gamma(k^0)$. Then

$$\begin{aligned}Tf(k^0) &\ge F(k^0, y^0) + \beta f(y^0) \\ &\ge \theta\{F(k,y) + \beta f(y)\} + (1-\theta)\{F(k', y') + \beta f(y')\} \\ &= \theta Tf(k) + (1-\theta)Tf(k').\end{aligned}$$

Hence choosing f_0 to be concave, we can generate a sequence of concave functions, $\{f_n\}_{n=0}^{\infty}$ in S using T, i.e., $f_n = T^n f_0$. Since $f_n \to v$, by Lemma 3 v is concave. $F(k, y)$ is strictly concave in y by Lemma 1. Then $F(k, y) + \beta v(y)$ is strictly concave. $\Gamma(k)$ is convex for each k. Hence the maximum of (2) is attained at a unique y-value. Therefore, $g(k)$ is a (single-valued) function and the continuity of g follows from the fact that it is upper-hemi-continuous.

Q.E.D.

Theorem 11

Assume the hypotheses of Theorem 10, and let $v_0 \in S$ be concave. Let $\{v_n\}_{n=0}^{\infty}$ be a sequence generated by T, and define $\{g_n\}_{n=0}^{\infty}$ by

$$g_n(k) = \text{argmax}_{y\in\Gamma(k)}\{F(k, y) + \beta v_n(y)\}, \quad n = 0, 1, 2, \ldots.$$

Then $g_n \to g$ pointwise, i.e., for each $k \in D$, $g_n(k) \to g(k)$.

Proof: Let $\phi(y) = \{F(k, y) + \beta v(y)\}$ and $\phi_n(y) = \{F(k, y) + \beta v_n(y)\}$ fixing $k \in D$. Since v and v_n are concave, and $F(k, y)$ is strictly concave by Lemma 1, ϕ and ϕ_n are strictly concave. Since $\Gamma(k)$ is compact for each $k \in D$, there exists y^* such that

$$\| y^* \| = \sup_y \{ \| y \| \mid y \in \Gamma(k) \}.$$

Hence the fact that $\{v_n\} \to v$ in the sense of sup norm on the unit ball in R^n implies $\{v_n\} \to v$ uniformly. Hence $\{\phi_n\} \to \phi$ uniformly. Then by Theorem 3.8 in Stokey and Lucas (1989), the conclusion of this theorem follows immediately.

Q.E.D.

Now assume the following for differentiability.

Assumption 5

For each fixed y, $F(\cdot, y)$ is continuously differentiable on the interior of $\Gamma^{-1}\{y\}$, where $\Gamma^{-1}\{y\} = \{k \in D \mid y \in \Gamma(k)\}$.

Theorem 12

Let F, β and Γ satisfy assumptions 1 to 5, and let v solve (2) and g be the policy correspondence, which is a single valued function under the assumptions. If $k_0 \in \text{Int } D$ and $g(k_0) \in \text{Int } \Gamma(k_0)$, then v is continuously differentiable at k_0 with derivatives given by

$$\frac{\partial v(k_0)}{\partial k_i} = \frac{\partial F\{k_0, g(k_0)\}}{\partial k_i} \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Proof: Since $g(k_0) \in \text{Int } \Gamma(k_0)$, and Γ is continuous, it follows that $g(k_0) \in \text{Int } \Gamma(k)$ for all k in some neighborhood J of k_0. Define $w: J \to R$ by

$$w(k) = F\{k, g(k_0)\} + \beta v\{g(k_0)\}.$$

F is strictly concave by Lemma 1, and differentiable in its first n arguments by Assumption 5. So w is strictly concave and differentiable. Also

$$w(k) \le \max_{y \in \Gamma(k)} \{F(k, y) + \beta v(y)\} = v(k), \quad \text{for all} \quad k \in J,$$

with equality at k_0, because $g(k_0) \in \Gamma(k)$ for all $k \in J$. Thus by Theorem 4.10 in Stokey and Lucas (1989), v is differentiable.

Q.E.D.

VI. Conclusion

The main result of this paper is the establishment of concavity and differentiability of value function when involved functions and correspondences exhibit CRS property. As the example in Section II suggests, some important economic problems fall into this category, which makes our results useful.

In order to establish concavity and differentiability of value function, we had to first clarify the relationship between the value function and the solution of the associated Bellman equation. Since the space we are dealing with is different from that for bounded return functions, we cannot say *a priori* that the relationship in the bounded case carries over to CRS case. But in Section IV we verified that it does.

In conclusion, under assumptions 1 through 5, the solution of Bellman equation exists, is unique, is the solution of the original maximization problem, and is concave and differentiable when involved functions and correspondences exhibit CRS property. As a byproduct on the way to this conclusion, we have that a strictly quasiconcave CRS function is strictly concave when at least one of the independent variable is fixed in a 2 or higher dimensional case, and that a quasiconcave CRS function is concave.

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