Private Provision of Price-excludable Public Goods: Welfare Effects of Monopolistic Price Discrimination

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We examine the problem of monopolistic provision of a price-excludable public good. Characterization and comparison of uniform pricing and third-degree price discrimination are made to derive the lower and upper bounds for the welfare change. In contrast to a private-good analog, it is shown that, in most cases, for welfare to increase it is *sufficient but not necessary* that total output increases. (*JEL* Classifications: H41, D60, L12)

I. Introduction

Third-degree price discrimination with a private good has long been an important topic for economists. Since first investigated by Pigou (1920), the main concern has been the effect on social welfare of third-degree price discrimination. Schmalensee(1981) and Varian(1985) have recently reexamined this question and concluded that a *necessary but not sufficient* condition for price discrimination to increase social welfare is that output increases. In other words, when output decreases under price discrimination, social welfare is necessarily lower.

In this paper, third-degree price discrimination problem is examined when the good of interest is public, not private. The major characteristic of a public good is that it can be jointly consumed, whereas the consumption of a private good is exclusive. Many public goods, however, have another feature that their consumption can be excluded even though it can be jointly shared. Examples are cable televisions, high-

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ways, parks, movies, any art performances, development of computer softwares, telephone lines etc. The problem we are interested in is third-degree price discrimination with a price-excludable public good. For example, a cable operator who has franchises over several regional areas that have different demands, may be required to charge a single price or allowed to charge different prices for a channel across the areas. Different entry fees (discounts for some customers) for movies and parks, different rates for day-time and night-time telephone calls and different rates for day-time and night-time uses of electricity (if they are possible to be identified) might be based on price discrimination with a public good.

A strand of literature has analyzed the problem of private provision of price-excludable public goods. This problem has been dealt with under the market structure of perfect competition or monopoly. Oakland (1974) considered a perfectly competitive market. Brennan and Walsh (1981), Brito and Oakland (1981), Burns and Walsh(1981) looked at a monopolistic provision problem to study various pricing schemes such as uniform pricing, two-part tariff and variable prices. No attempt, however, has been made to look into the problem of third-degree price discrimination with a public good.

Conditions characterizing uniform pricing and price discrimination are specified and compared. The effect of price discrimination on social welfare is analyzed by deriving the lower and upper bounds for welfare change. As is well known, with a private good under linear demands, total output is unchanged and total welfare is lower in the presence of price discrimination. Interestingly, with a public good, output can be larger or smaller under price discrimination even with linear demands and it is shown that, in most cases, a *sufficient but not necessary* condition for social welfare to increase is that output increases. For a public good, it is possible and even common that social welfare might be higher under price discrimination even though output is smaller.

On the demand side, we consider n types of demands. Let $t \in T = \{1, 2, \ldots, n\}$. A consumer of type t has a demand function, $D(p, t)$ for $t \in T$, where p denotes price.[1] Let $f(t)$ be the number of consumers of type t and let $F(t) \equiv \sum_{a \leq t} f(a)$. Usual assumptions are made on the demand functions. First, the demand functions are twice continuously differentiable. Second, $D(p, t) < D(p, t')$ for $t < t'$ and for all p, which means that demand schedules don't intersect each other.[2] Third, marginal

[1] $D(p, 0) \equiv 0$.

revenues are decreasing in output.

On the cost side, to keep our analysis simple, it is assumed that the monopolist produces the public good under conditions of constant marginal cost. Therefore, total costs, $C(x)$, is

$$C(x) = cx,$$

where x denotes the amount of output.

The problem facing the monopolist can be set up in a general way as follows. The monopolist tries to solve

$$\max_{(p_1, p_2, \ldots, p_n; x)} \pi(p_1, p_2, \ldots, p_n; x) \equiv \sum_{t \in T} p_t x_t f(t) - cx$$

$$\text{subject to} \quad x_t = \min\{x, D(p_t, t)\} \quad \text{for } t = 1, 2, \ldots, n.$$

Uniform pricing is the case where the firm is constrained to charge a single price, i.e. $p_t = p$ for all t and third-degree price discrimination is the case where the firm can charge different prices to different consumers. In the problem of uniform pricing, however, rather than solving out this optimization problem by kuhn-tucker conditions, it will prove to be useful and intuitive to make use of another tool.

This paper flows as follows. In Section II, conditions implied by uniform pricing are derived, noting the aspects different from those of a private good. In Section III, properties that must hold under price discrimination are found. In Section IV, we compare between uniform pricing and price discrimination in terms of the changes in output and social welfare. In Section V, a simple and illustrative example is given. Section VI concludes.

II. Uniform Pricing

A uniform pricing is analyzed as a benchmark case to be compared with third-degree price discrimination. Suppose that a monopolist who provides a public good has to charge a single price to all customers. Such a single price could result from government restrictions or lack of necessary information to distinguish the customers.

Let $R(p, x)$ be the revenue when the firm produces x at p. Then,

$$R(p, x) \equiv \sum_{t \in T} p \cdot \min\{D(p_t, t), x\} \cdot f(t).^{3}$$

[2]This assumption is introduced only for the convenience of setting up the model and this analysis will do without this assumption.

[3]If the good were a private good, $R(p, x) = p \cdot x$.

Let $ED(p)$ (effective demand at p) be the profit maximizing output at p. Then,

$$ED(p) = \text{argmax}_x \{R(p, x) - cx\}.$$

Note that $ED(p) = D(p, t)$ for some $t \in T$.

The monopolist's problem is

$$\max_p \pi(p) \equiv R\{p, ED(p)\} - c \cdot ED(p).$$

Let $p^u \equiv \text{argmax}\ \pi(p)$ and t^u be such that $ED(p^u) = D(p^u, t^u)$. Then, it should hold that

$$\sum_{t \le t^u} \left\{D(p^u, t) + p^u \frac{\partial D(p^u, t)}{\partial p}\right\} f(t) + \left\{D(p^u, t^u) + p^u \frac{\partial D(p^u, t^u)}{\partial p}\right\} \{1 - F(t^u)\}$$
$$= c \frac{\partial D(p^u, t^u)}{\partial p},$$

which is

$$\sum_{t \le t^u} p^u \left\{1 - \frac{1}{\varepsilon(p^u, t)}\right\} \frac{\dfrac{\partial D(p^u, t)}{\partial p}}{\dfrac{D(p^u, t^u)}{\partial p}} f(t) + p^u \left\{1 - \frac{1}{\varepsilon(p^u, t^u)}\right\} \{1 - F(t^u)\} = c, \quad (1)$$

where $\varepsilon(p, t) = -\{p/D(p, t)\}\{\partial D(p, t)/\partial P\}$ i.e. elasticity of consumer t's demand at the price p.

From now on, let us focus on two-type case. In many cases in the real world, price discrimination is performed based upon two groups of consumers along some characteristic such as time(peak demand or off-peak demand), age(senior discount), status(student discount) etc. Furthermore, analyzing a two-type case enables us to grasp better the story that leads to the above conditions.

With two types of demands, $x = D_t(p)$ $(t = 1, 2)$, what $ED(p)$ stands for can be seen vividly. At a given price p, $ED(p)$ is a collection of parts of two demand curves as illustrated in Figure 1. In this case, rationing occurs at prices less than marginal costs c. Two cases which appear depending on the level of the profit maximizing price are examined one by one by characterizing the properties which should hold for each case.

A. No-Rationing

Let us look at the case where nobody is rationed at p^u. This occurs

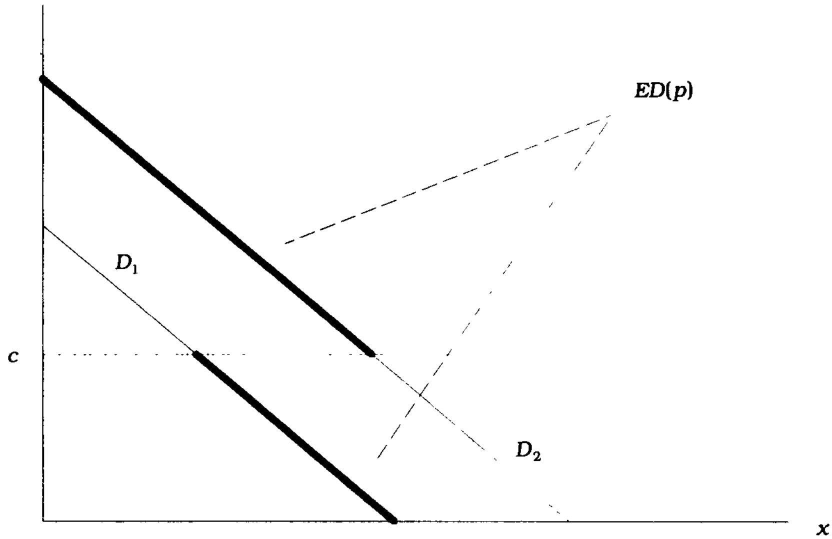

FIGURE 1

when $p^u > c$. Without loss of generality, suppose $D_1(p^u) \leq D_2(p^u)$. As shown above,

$$ED(p^u) = D_2(p^u) \text{ and } TD(p^u) = D_1(p^u) + D_2(p^u).$$

The monopolist's profit is

$$\pi(p^u) = \max_p \left[p\{D_1(p) + D_2(p)\} - c\, D_2(p)\right].$$

It should hold that

$$p^u\left\{1 - \frac{1}{\varepsilon_1(p^u)}\right\}\frac{\frac{\partial D_1}{\partial p}}{\frac{\partial D_2}{\partial p}} + p^u\left\{1 - \frac{1}{\varepsilon_2(p^u)}\right\} = c. \qquad (2)^4$$

Equation (1) has a nice interpretation. It is a usual marginal revenue = marginal cost condition for the monopolist, but modified to allow for the fact that we are dealing with a public good. $p^u\{1 - 1/\varepsilon_1(p^u)\}$ is the marginal revenue from customer 1. Under no-rationing, however, output is determined along customer 2's demand curve (i.e larger demand). $(\partial D_1/\partial p)/(\partial D_2/\partial p)$ measures the change in the amount of output

[4]Another fact implied by equation (2) is that, at p^u, one of the elasticities, $\varepsilon_1(p^u)$ and $\varepsilon_2(p^u)$, can be less than 1, i.e. one of the marginal revenues can be negative.

demanded by consumer 1 (small demand) when output is in-creased by 1 unit along the customer 2's demand curve. That is, $p^u\{1 - 1/\varepsilon_1(p^u)\}(\partial D_1/\partial p)/(\partial D_2/\partial p) = p^u\{1 - 1/\varepsilon_1(p^u)\}(\partial D_1/\partial p)(\partial p/\partial D_2)$ is the marginal revenue from customer 1 with a public good when he is inframarginal in the sense that the monopolist's output is not determined on his demand curve. To put it differently, the marginal revenue from each customer is treated differently with different weights. In the special case of two linear demands with a same slope, this weight is 1.

B. Rationing

Suppose that some customer is rationed at p^u. Rationing occurs when $p^u < c$. Without loss of generality, suppose $D_1(p^u) \leq D_2(p^u)$. Then,

$$ED(p^u) = D_1(p^u) \text{ and } TD(p^u) = 2D_1(p^u).$$

The monopolist's profit is

$$\pi(p^u) = \max_p\ [p(2D_1(p)\} - c\, D_1(p)].$$

It should hold that

$$2p^u\left\{1 - \frac{1}{\varepsilon_1(p^u)}\right\} = c. \tag{3}$$

Rationing is a peculiar feature due to the basic nature of a public good that is absent from a private good. Rationing takes place when, at some price, meeting all demands is unprofitable. The monopolist would raise the price above c to meet consumer 2's demand if profit could be increased by doing so. When consumer 2 is rationed, his demand curve does not play an active role in determining output. Consumer 2's demand is replaced by consumer 1's and the monopolist's output is determined as if there are 2 people with $D_1(p)$.

III. Third-degree Price Discrimination

Suppose that the monopolist can distinguish customers' demands and charge a price for each demand. Senior discount rate for movies, different rates for day-time and night-time telephone calls might be based on price discrimination with a public good. A customer of type t's demand is given as $x_t = D(p_t, t)$ since different prices can be charged now.

Let us observe that there can not occur rationing in third-degree price discrimination. That is, $D(p_t, t) \leq x$ for all t in firm's decision. When the firm produces x, it is clear that the price with $D(p_t, t) > x$ would be unprofitable. The firms can raise the price, still maintaining the same demand.

The monopolist's problem becomes

$$\max_{(p_1, p_2, \ldots, p_n; x)} \pi(p_1, p_2, \ldots, p_n; x) \equiv \sum_{t \in T} p_t D(p_t, t) f(t) - cx$$

$$\text{subject to } D(p_t, t) \leq x \text{ for } t = 1, 2, \ldots, n.$$

Let $(p_1^d, p_2^d, \ldots, p_n^d; x^d) \equiv \text{argmax } \pi(p_1, p_2, \ldots, p_n; x)$.

Form a lagrangean.

$$L = \sum_{t \in T} p_t D(p_t, t) f(t) - cx - \sum_{t \in T} \lambda_t \{D(p_t, t) - x\}.$$

First order conditions are

$$\left\{ D(p_t^d, t) + p_t^d \frac{\partial D(p_t^d, t)}{\partial P_t} \right\} f(t) - \lambda_t \frac{\partial D(p_t^d, t)}{\partial P_t} = 0$$

$$-c + \sum_{t \in T} \lambda_t = 0$$

$$\lambda_t \geq 0$$

$$\lambda_t \{ D(p_t^d, t) - x \} = 0, \quad D(p_t^d, t) \leq x, \quad t = 1, 2, \ldots, n.$$

Combining these first order conditions yields

$$\sum_{t \in T} P_t^d \left\{ 1 - \frac{1}{\varepsilon_1(p_t^d, t)} \right\} f(t) = c \quad \text{and} \quad P_t^d \left\{ 1 - \frac{1}{\varepsilon_1(p_t^d, t)} \right\} \geq 0 \quad \text{for all } t, \tag{4}$$

where $\varepsilon(p_t^d, t)$ is the elasticity of demand.

$p_t^d\{1 - 1/\varepsilon(p_t^d, t)\}$ is the marginal revenue from consumer t at the price p_t^d, i.e. at the output $x_t^d = D(p_t^d, t)$. Since the same unit of output can be supplied to all consumers, marginal revenue for the firm is the sum of the marginal revenues from all consumers. The monopolist's profit is maximized when this sum comes to be equal to marginal costs. There is, however, one constraint that the marginal revenue from every consumer be non negative. At the output x^d, it is suboptimal for the firm to provide all x^d to the consumer t whose marginal revenue is negative. The firm can increase profits by raising the price p_t until revenue from the consumer t gets maximized. Therefore, under third-degree price discrimination, all customers might not be able to consume the same unit of output.

As we did with uniform pricing, let us focus on two-type case to understand better the problem. Two cases that appear depending on the binding conditions will be analyzed by specifying the properties that must prevail in each case.

A. Full-Binding

Full-binding refers to the case where both constraints are binding at the same time. That is, $D_t(p_t^d) = x^d$ (i.e. where $\lambda_t > 0$) $t = 1, 2$.
The profit maximizing prices and output, p_1^d, p_2^d, x^d, can be obtained by :

$$P_1^d\left\{1 - \frac{1}{\varepsilon_1(p_1^d)}\right\} + P_2^d\left\{1 - \frac{1}{\varepsilon_2(p_2^d)}\right\} = c \ \text{ and } \ D_1(p_1^d) = x^d = D_2(p_2^d), \quad (5)$$

where $\varepsilon_t(p_t^d) \geq 1$, $t = 1, 2$.
In determining output, marginal revenues from both consumers play active roles since they are served at the elastic part of their demand curves. In a price discrimination with a private good, the famous inverse elasticity rule applies between price and elasticity. With a public good, however, there is no definite relationship between price and elasticity. It is common with public good that a higher price comes with a higher elasticity.

B. Partial-Binding

Partial-binding refers to the case where one of the constraints is not binding. That is, $D_i(p_i^d) = x^d$, $D_j(p_j^d) < x^d$. $i \neq j$. (i.e. where $\lambda_i > 0$, $\lambda_j = 0$)
The optimal prices and output, p_1^d, p_2^d, x^d, are obtained by :

$$P_i^d\left\{1 - \frac{1}{\varepsilon_i(p_i^d)}\right\} = c, P_j^d\left\{1 - \frac{1}{\varepsilon_j(p_j^d)}\right\} = 0 \ \text{ and } \ D_i(p_i^d) = x^d, \quad (6)$$

where $\varepsilon_i(p_i^d) > 1$, $\varepsilon_j(p_j^d) = 1$ and $D_j(p_j^d) < x^d$.
This case occurs when the conditions for full-binding (5), can't be met. Even though we find out prices that satisfy (5), if consumer j's demand is inelastic at the price, i.e. $\varepsilon_j(p_j^d) < 1$, there is a room for increase in profit. p_j^d can be changed (usually increased) to increase ε_j. Partial-binding corresponds to the situation where whatever output can't induce elastic demands for both consumers that satisfy (5). Then, one of the consumers becomes inframarginal in determining the monopolist's output, which means $\varepsilon_j(p_j^d) = 1$ and $D_j(p_j^d) < x^d$. Cost consideration is taken into account only for the marginal consumer and the inframarginal

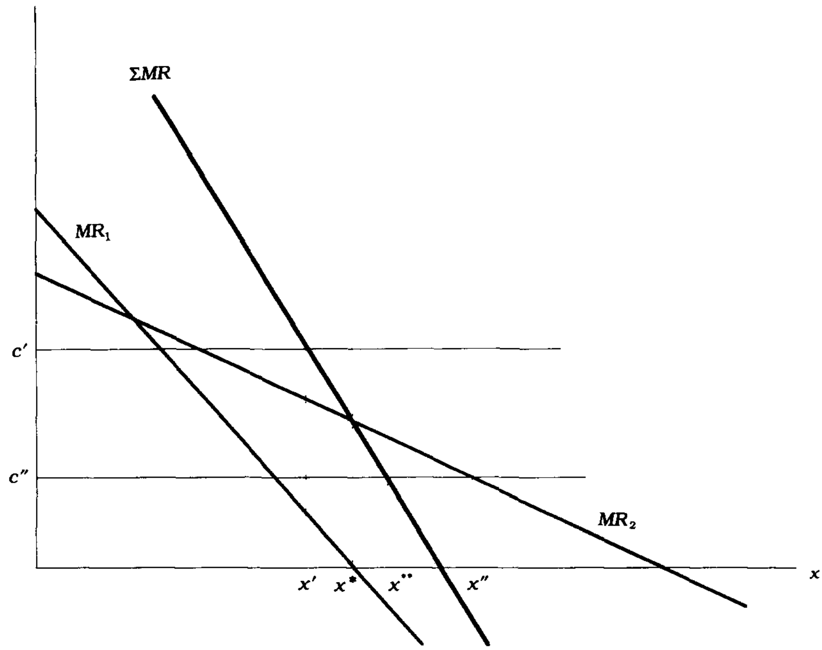

Figure 2

consumer is served as if there was no cost to supply him.

A graph can help us to understand what happens in full or partial binding. In Figure 2, $MR_i(x)$ shows a usual marginal revenue schedule and $\Sigma MR(x)$ is the vertical sum of two marginal revenue schedules, i.e.

$$\Sigma MR(x) = MR_1(x) + MR_2(x).$$

For $x \leq x^*$, $MR_i \geq 0$, $i = 1, 2$. When $c = c'$, full-binding occurs at x'. When $c = c''$, however, condition (5) is satisfied at x^{**}, but $MR_1(x^{**}) < 0$, i.e. $\varepsilon_1(x^{**}) < 1$. Therefore, full-binding can't occur and we have partial-binding that

$$x^d = x'' (= x_2^d) \text{ with } MR_2(x'') = c \text{ and } x_1^d = x^* \text{ with } MR_1(x^*) = 0.$$

IV. Comparison

In this section, comparison is made between uniform pricing and third-degree price discrimination by using the properties derived in the last two sections. We focus particularly on the differences in output and welfare.

A. From No-Rationing to Full-Binding

Suppose that nobody is rationed under uniform pricing and allowing price discrimination induces full-binding.

No-rationing is characterized by

$$p^u\left\{1-\frac{1}{\varepsilon_1(p^u)}\right\}w(p^u)+p^u\left\{1-\frac{1}{\varepsilon_2(p^u)}\right\}=c \quad \text{and} \quad p^u>c,$$

$$\text{where} \quad w(p^u)=\frac{\dfrac{\partial D_1}{\partial p}}{\dfrac{\partial D_2}{\partial p}}.$$

Full-binding by

$$p_1^d\left\{1-\frac{1}{\varepsilon_1(p_1^d)}\right\}+p_2^d\left\{1-\frac{1}{\varepsilon_2(p_2^d)}\right\}=c \quad \text{and} \quad \varepsilon_t(p_t^d)\geq 1,\ \ t=1,2.$$

Let p_1^u be such that $D_1(p_1^u)=D_2(p^u)$. Then, $p_1^u<p^u$. The condition for the change in output can be derived.

Lemma 1

Under the transition from no-rationing to full-binding,

$$x^d \leq (\geq) x^u \quad (\text{and} \quad p^u \leq (\geq) p_2^d)$$

$$\text{if and only if} \quad p_1^u\left\{1-\frac{1}{\varepsilon_1(p_1^u)}\right\}\leq(\geq)p^u\left\{1-\frac{1}{\varepsilon_1(p^u)}\right\}w(p^u).$$

Proof: Consider $T=p_1^u\{1-1/\varepsilon_1(p_1^u)\}+p^u\{1-1/\varepsilon_2(p^u)\}$. T is the marginal revenue for the discriminating monopolist at x^u.

$[p_1^u\{1-1/\varepsilon_1(p_1^u)\}\leq(\geq)\, p^u\{1-1/\varepsilon_1(p^u)\}\,\omega(p^u)] \Rightarrow [T\geq(\leq)\ c] \Leftrightarrow [x^d\leq(\geq)\ x^u]$.

Since full-binding occurs under discrimination,

$$[x^d=x_2^d\leq(\geq)\ x^u] \Leftrightarrow [p^d\leq(\geq)\ p_2^d].$$

Q.E.D.

Since marginal revenue decreases with output, $p_1^u\{1-1/\varepsilon_1(p_1^u)\}<p^u\{1-1/\varepsilon_1(p^u)\}$. The direction of change in output depends on the magnitude of decrease in marginal revenue and the weight on marginal rev-

enue under uniform pricing, $\omega(p^u)$. For example, with linear demands of a same slope, $\omega(p^u) = 1$, which means $T < c$ and $x^d < x^u$.

Let us turn to welfare aspect. Let consumer t's net surplus at p be $S_t(p)$. The change in welfare, $\Delta W \equiv W^d - W^u$, is

$$\Delta W = \sum_t \{S_t(p_t^d) - S_t(p^u)\} + \left(\sum_t p_t^d x_t^d - \sum_t p^u x_t^u\right) - c(x^d - x^u) \qquad (7)$$

To look at the relationship between change in output and welfare, it would be useful if ΔW could be described in terms of Δx ($\equiv x^d - x^u$). The following lemma derives the lower and upper bounds for ΔW in terms of Δx.

Lemma 2

Under the transition from no rationing (where $x_1^u < x_2^u$) to full-binding,

$$\left(\sum_t p_t^d - c\right)(x^d - x^u) \le \Delta W \le (p^u - c)(x^d - x^u) + p^u(x^d - x_1^u),$$

where $x_1^u = D_1(p^u)$.

Proof: $S_t(p)$ is convex in p, since $S_t''(p) = -D_t'(p) > 0$. Under no-rationing and full-binding, therefore,

$$S_t(p_t^d) - S_t(p^u) \ge S_t'(p^u)(p_t^d - p^u), \text{ where } S_t'(p^u) = -D_t(p^u) = -x_t^u, \text{ and}$$
$$S_t(p_t^d) - S_t(p^u) \le S_t'(p_t^d)(p_t^d - p^u), \text{ where } S_t'(p_t^d) = -D_t(p_t^d) = -x_t^d = -x^d.$$

$$\Delta W \ge -\sum_t x_t^u(p_t^d - p^u) + \left(\sum_t p_t^d x^d - \sum_t p^u x_t^u\right) - c(x^d - x^u)$$
$$= (x^d - x^u)\left(\sum_t p_t^d - c\right) + p_1^d(x^u - x_1^u) \ge \left(\sum_t p_t^d - c\right)(x^d - x^u)$$

$$\Delta W \le \sum_t x^d(p^u - p_t^d) + \left(\sum_t p_t^d x^d - \sum_t p^u x_t^u\right) - c(x^d - x^u)$$
$$= \sum_t p^u(x^d - x_t^u) - c(x^d - x^u) = (p^u - c)(x^d - x^u) + p^u(x^d - x_t^u)$$

Q.E.D.

We can easily get the sufficient condition for the increase in welfare.

***Proposition* 1**

Under the transition from no-rationing to full-binding, the sufficient condition for welfare increase is $p_1^d(x_1^u - x^u) \le (p_2^d - c)(x^d - x^u)$. Therefore, if output is larger under price discrimination, welfare is higher.

Firstly, a nice thing about this proposition is that the sufficient condition for welfare increase is composed of observable variables so that we can actually calculate out this condition at least theoretically. Secondly, since $(x_1^u - x^u)$ is negative, the sufficient condition could be satisfied even when $(x^d - x^u) < 0$. In other worlds, the proposition shows that higher welfare can be obtained even though output is smaller under price discrimination. When price discrimination is performed with a *private* good, compared with uniform pricing, a decrease in output by a discriminating monopolist always lowers welfare. (That is, a decrease in output is a sufficient condition for a decrease in welfare.) The feature that, with a *public* good, a decrease in output by a discriminating monopolist is a necessary (not sufficient) condition for a decrease in welfare results from the basic property of a public good that a given unit of output can be jointly consumed by everybody, whereas a private good consumption is exclusive.

Under price discrimination with a private good, different prices charged to customers have a negative effect on welfare since they lead to different marginal utilities across markets. With a public good, however, given some unit of output, different prices can exert positive effects on welfare. Lindahl mechanism shows that pricing by marginal valuations yields maximum welfare. Uniform pricing induces same marginal utilities across consumers. Price discrimination might decrease output, but differential treatment of consumers could improve welfare. The negative effect from lower level of output can be outweighed by the advantageous effect from differential treatment, which improves welfare.

Lindahl mechanism and price discrimination are similar in that consumers are treated differently, but different from each other in that the sum of marginal valuations are equated to marginal costs in Lindahl mechanism, while the sum of marginal revenues to marginal costs in price discrimination.

B. From No-Rationing to Partial-Binding

Suppose that nobody is rationed under uniform pricing and allowing price discrimination induces partial-binding.

No-rationing is characterized by

$$p^u\left\{1-\frac{1}{\varepsilon_1(p^u)}\right\}\omega(p^u)+p^u\left\{1-\frac{1}{\varepsilon_2(p^u)}\right\}=c \quad \text{and} \quad p^u>c.$$

Partial-binding by

$$p_i^d\left\{1-\frac{1}{\varepsilon_i(p_i^d)}\right\}=c,\quad p_j^d\left\{1-\frac{1}{\varepsilon_j(p_j^d)}\right\}=0,\quad D_i(p_i^d)=x^d.$$

In this case, the relationship between changes in output and welfare is ambiguous, which stems from the nature of partial-binding that cost consideration is taken only for the marginal consumer while revenues are maximized for the inframarginal consumer. Even though output is larger under price discrimination, the high price charged to the inframarginal consumer for revenue maximization can outweigh the positive effect on welfare from larger outputs.

An example would help us to clarify our understanding. Consider two demands; $p_1 = 10 - 2x_1$, $x_2 = 10 - 2p_2$ with marginal cost of c. It can be shown that when $c = 1$, Δx ($\equiv x^d - x^u$) > 0 but ΔW ($\equiv \Delta W^d - \Delta W^u$) < 0, which is shown in Figure 3. In this transition from no-rationing to partial-binding, however, the notable aspect of public-good price discrimination still remains to appear, i.e. lower outputs under price discrimination could come along with higher welfare. In the example raised above, when $2 < c < 2.5$, $\Delta x < 0$ but $\Delta W > 0$ as shown in Figure 4.[5]

In sum, the main feature of price discrimination with a public good that lower output could improve welfare, still appears in this case, even though there is no definite condition relating Δx and ΔW.

C. From Rationing to Full-Binding

Suppose that some customer is rationed under uniform pricing and allowing discrimination induces binding.

Rationing is characterized by

$$2p^u\left\{1-\frac{1}{\varepsilon_1(p^u)}\right\}=c \quad\text{and}\quad x^u=D_1(p^u).$$

Full-Binding by

$$p_1^d\left\{1-\frac{1}{\varepsilon_1(p_1^d)}\right\}+p_2^d\left\{1-\frac{1}{\varepsilon_2(p_2^d)}\right\}=c \quad\text{and}\quad x^d=D_1(p_1^d)=D_2(p_2^d).$$

Let p_2^u be such that $D_2(p_2^u) = x^u$. Conditions for the change in output and welfare can be derived.

[5]One thing to be noticed from the example is that a larger demander under uniform pricing could be a smaller one under price discrimination.

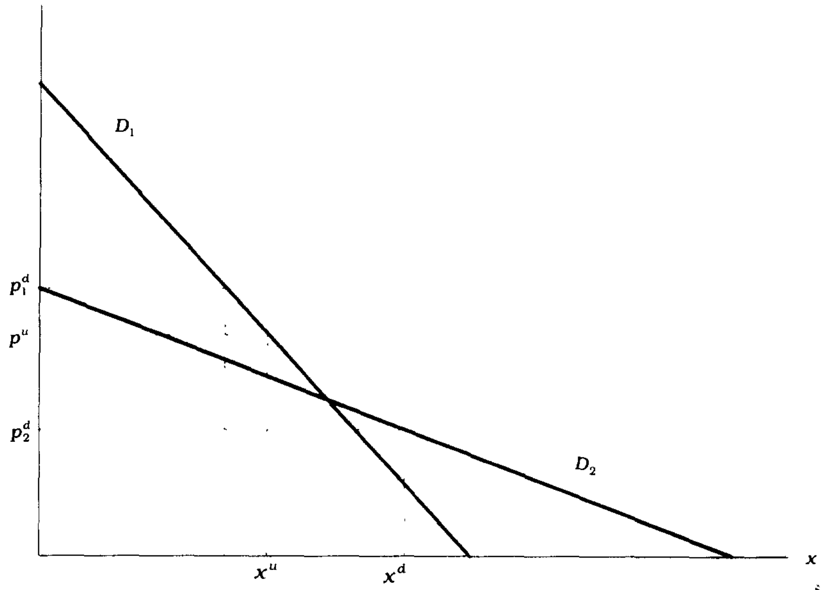

FIGURE 3
[$\Delta x > 0$, $\Delta W < 0$]

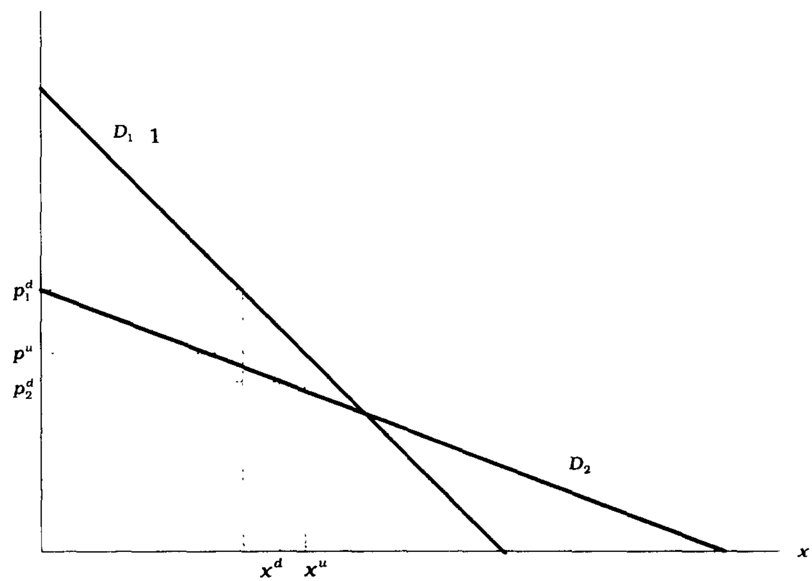

FIGURE 4
[$\Delta x < 0$, $\Delta W > 0$]

Lemma 3

Under the transition from rationing to full-binding,

$$x^d \geq (\leq)\ x^u \ \text{ if and only if } \ p_2^u\left\{1 - \frac{1}{\varepsilon_2(p_2^u)}\right\} \geq (\leq) p^u\left\{1 - \frac{1}{\varepsilon_1(p^u)}\right\}.$$

Proof: Consider $T' = p^u\{1 - 1/\varepsilon_1(p^u)\} + p_2^u\{1 - 1/\varepsilon_2(p_2^u)\}$.

$$[p_2^u\{1 - 1/\varepsilon_2(p_2^u)\} \geq (\leq)\ p^u\{1 - 1/\varepsilon_1(p^u)\}] \Leftrightarrow [T' \geq (\leq)\ c] \Leftrightarrow [x^d \leq (\geq)\ x^u].$$

Q.E.D.

Lemma 4

Under the transition from rationing to full-binding,

$$\left(\sum_t p_t^d - c\right)(x^d - x^u) + (p_2^d - p^u)x^u \leq \Delta W$$
$$\leq (2p^u - c)(x^d - x^u) + (p_2^u - p^u)x^d.$$

Proof: $S_1(p_1^d) - S_1(p^u) \geq -x^u(p_1^d - p^u)$, $S_2(p_2^d) - S_2(p_2^u) \geq -x^u(p_2^d - p_2^u)$,

$$S_1(p^u) - S_1(p_1^d) \geq -x^d(p^u - p_1^d),\ S_2(p_2^u) - S_2(p_2^d) \geq -x^d(p_2^u - p_2^d),$$

where $x_2^u = D_2(p_2^u) = D_1(p^u)$. Substitution yields the bounds.

Q.E.D.

Proposition 2

Under the transition from rationing to full-binding, an increase in output always yields higher welfare and a sufficient condition for welfare increase is

$$(p^u - p_2^u)x^u \leq \left(\sum_t p_t^d - c\right)(x^d - x^u).$$

Here again, firstly, the sufficient condition above is composed of observable variables except p_2^u. p_2^u was defined to satisfy $D_2(p_2^u) = x^u = D_1(p^u)$. When p^u and x^u are observed, therefore, p_2^u could be measured. Secondly, since $(p^u - p_2^u)$ is negative, the sufficient condition can still hold even when $(x^d - x^u) < 0$. In this transition also, welfare could be higher under price discrimination when total output is smaller.

D. From Rationing to Partial-Binding

Suppose that some consumer is rationed under uniform pricing and allowing price discrimination induces partial-binding.

Rationing is characterized by

$$2p^u\left\{1-\frac{1}{\varepsilon_1(p^u)}\right\}=c \quad\text{and}\quad x^u=D_1(p^u).$$

Partial-Binding by

$$p_i^d\left\{1-\frac{1}{\varepsilon_i(p_i^d)}\right\}=c,\quad p_j^d\left\{1-\frac{1}{\varepsilon_j(p_j^d)}\right\}=0 \quad\text{and}\quad x^d=D_i(p_i^d). \tag{8}$$

Lemma 5

Under the transition from rationing to partial-binding, the following holds.

$x^d < x^u$, when $i = 1$ in (6), i.e. the consumer 1's demands determines output under both uniform pricing and discrimination.

$x^d > x^u$, when $i = 2$ in (6), i.e. the consumer whose demand determines output under uniform pricing becomes inframarginal under discrimination.

Proof: Suppose $i = 1$ in (6). Then, $p^u\{1-1/\varepsilon_1(p^u)\}=c/2<p_1^d\{1-1/\varepsilon_1(p_1^d)\}=c$, which means $[x^d=x_1^d<x^u$ and $p^u<p_1^d$, $x_2^d\le x_1^d=x^d<x^u]\Rightarrow[p_2^d>p^u]$. Suppose that $i = 2$ in (6). Then, $[p^u\{1-1/\varepsilon_1(p^u)\}=c/2>p_1^d\{1-1/\varepsilon_1(p_1^d)\}=0]\Rightarrow[x_1^d>x^u$ and $p^u>p_1^d]\Rightarrow[x^d=x_2^d\ge x_1^d=x^u]$.

Q.E.D.

Lemma 6

Under the transition from rationing to partial-binding,

$$(p_i^d-c)(x^d-x^u)+p_j^d(x_j^d-x^u)+(p_2^u-p^u)x^u$$
$$\le \Delta W\le(2p^u-c)(x^d-x^u)+(p_2^u-p^u)x^d.$$

Proof: $S_1(p_1^d)-S_1(p^u)\ge -x^u(p_1^d-p^u)$, $S_2(p_2^d)-S_2(p_2^u)\ge -x^u(p_2^d-p_2^u)$,

$$S_1(p^u)-S_1(p_1^d)\ge -x_1^d(p^u-p_1^d),\; S_2(p_2^u)-S_2(p_2^d)\ge -x_2^d(p_2^u-p_2^d).$$

Substitution yields the bounds.

Q.E.D.

***Proposition* 3**

Under the transition from rationing to partial-binding, an increase in output always yields higher welfare and a sufficient condition for welfare increase is, $p_j^d(x^u - x_j^d) + (p^u - p_2^u)x^u \leq (p_i^d - c)(x^d - x^u)$.

***Proof*:** $\Delta x > 0$ is the case where $i = 2$. Then, $p_2^d > c$ since $p_2^d\{1 - 1/\varepsilon_2(p_2^d)\}$ $= c$ with $\varepsilon_2(p_2^d) > 1$ and $x_1^d > x^u$. So, $\Delta W \geq 0$.

Q.E.D.

In this transition again, the sufficient condition is composed of observable variables except p_2^u which could be measured based on the other observable variables. On the other hand, $(x^d - x^u) < 0$ is the case where $i = 1$. and $x^u > x_2^d$. Since $(p^u - p_2^u) < 0$, however, the condition could be satisfied even when $x^d < x^u$ (and $x^u > x_2^d$), i.e. welfare could be higher even if output is smaller under price discrimination.

V. Example

Here, we analyze price discrimination with a specific example to get some intuitive idea. Whenever it is possible, we try to compare the joint good case with a private one.

Example: i) 2 people: $x_1 = 10 - p_1$, $x_2 = 12 - p_2$

ii) monopolist's output: x, monopolist's cost: $C(x) = c{\cdot}x$, where $0 \leq c \leq 20$, i.e. both are served under uniform pricing.

A. Uniform Pricing

The effective demand curve is $ED(p) = \begin{cases} 12 - p & \text{for} \quad p \geq c \\ 10 - p & \text{for} \quad p \leq c \end{cases}$.

Profit function for the monopolist is

$$\pi(p) = p \cdot TD(p) - c \cdot ED(p) \quad \text{where} \quad TD(p) = \begin{cases} (12 - p) + (10 - p) & \text{at } p \geq c \\ 2(10 - p) & \text{at } p \leq c \end{cases}.$$

Solution is summarized as Table 1.

B. Third-degree Price Discrimination

Let's suppose that the monopolist can distinguish the demands and set different prices for each. The monopolist's problem is

TABLE 1

c	rationing or not	p^u	x^u	x_1^u	x_2^u
$7 < c \leq 20$	rationing	$\frac{20+c}{4}$	$\frac{20-c}{4}$	$\frac{20-c}{4}$	$\frac{20-c}{4}$
$c \leq 7$	no-rationing	$\frac{22+c}{4}$	$\frac{26-c}{4}$	$\frac{18-c}{4}$	$\frac{26-c}{4}$

TABLE 2

c	binding or not	p_1^d	p_2^d	x^d	x_1^d	x_2^d
$2 < c \leq 20$	full binding	$\frac{18+c}{4}$	$\frac{26+c}{4}$	$\frac{22-c}{4}$	$\frac{22-c}{4}$	$\frac{22-c}{4}$
$c \leq 2$	partial binding	5	$\frac{12+c}{4}$	$\frac{12-c}{4}$	5	$\frac{12-c}{4}$

$$\max_{p_1, p_2, x} \pi(p_1, p_2; x) = p_1(10 - p_1) + p_2(12 - p_2) - cx$$

$$\text{subject to} \begin{cases} 10 - p_1 \leq x \\ 12 - p_2 \leq x. \end{cases}$$

Solution can be summarized as Table 2.

C. Comparison between Uniform Pricing and Third-degree Price Discrimination.

We can compare Table 1 and Table 2. Furthermore, the change in welfare $\Delta W = W^d - W^u$, can be evaluated by the bounds derived in the last section. Table 3 contains the results.

Note that

i) Under the transition from rationing to full-binding, the larger output brings higher welfare.

ii) When $1 < c \leq 7$, output decreases under price discrimination, but welfare improves. As pointed out before, this is a novel aspect absent from a private good case.

iii) Output can be smaller or larger even with linear demands under price discrimination and even if output is smaller, welfare can be higher, which can not happen with a private good.

VI. Conclusion

Little attention has been paid to the problem of third-degree price

TABLE 3

c	Regime		$\Delta x = x^d - x^u$	$\Delta W = W^d - W^u$
$7 < c \leq 20$	rationing	full binding	+	+
$2 < c \leq 7$	no rationing	full binding	-	+
$1 < c \leq 2$	no rationing	partial Binding	-	+
$0 \leq c \leq 1$	no rationing	partial Binding	-	-

*Refer to Figure 5, 6 and 7.

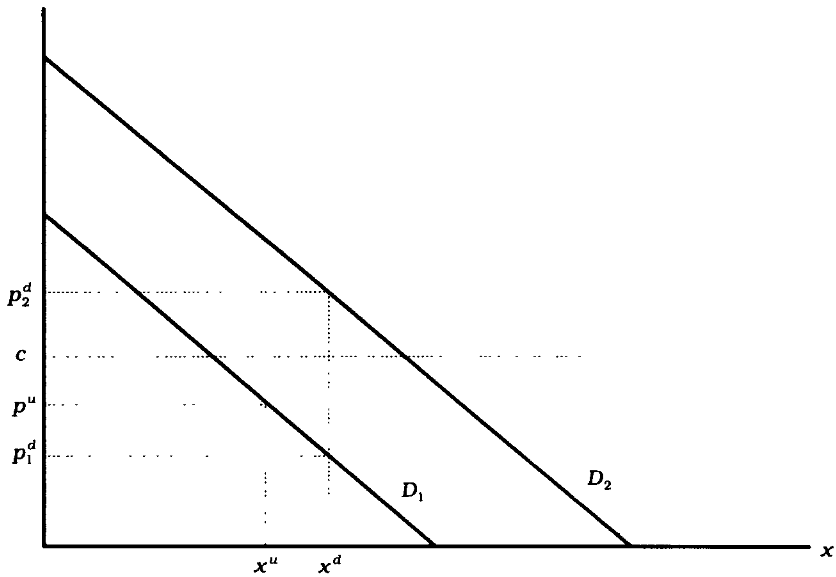

FIGURE 5
RATIONING TO BINDING ($7 < c \leq 20$)

discrimination with a public good. There are many cases where the goods we think of as private are actually public in nature. This paper tries to look at the output and welfare implications of third-degree price discrimination with a public good. We identify interesting and meaningful properties in contrast with those of a private good case, which stems from the basic nature of a public good. If the demand and output are interpreted properly, many cases can be explained by this model even though this model might need to be enriched for a specific industry.

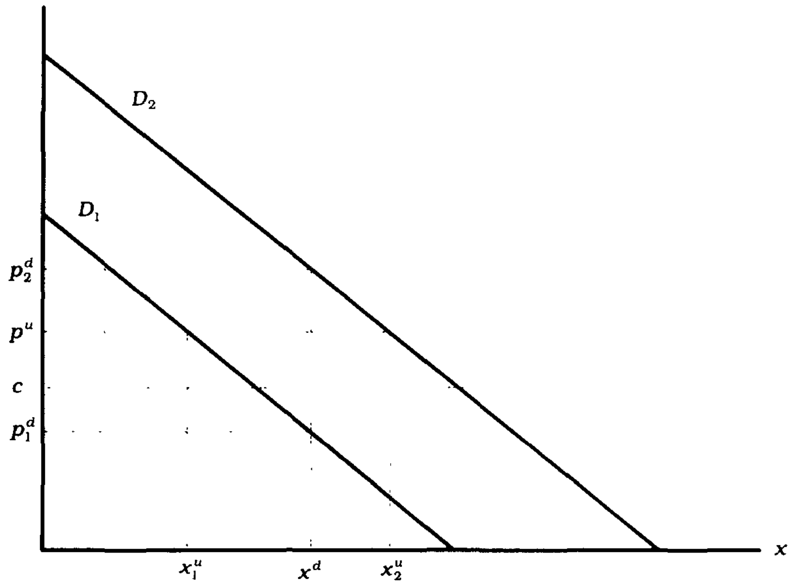

FIGURE 6
NO-RATIONING TO BINDING ($2 \leq c \leq 7$)

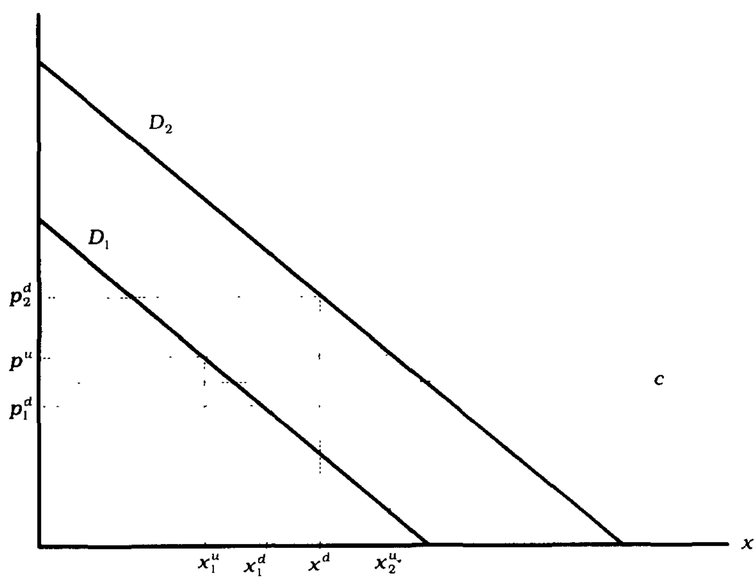

FIGURE 7
NO-RATIONING TO PARTIAL BINDING ($0 \leq c \leq 2$)

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