A Computational Test for the Existence of a Monotonic Core Payoff Configuration

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The objects of study in this paper are *transferable utility games* and prescriptions for such games in the form of an array of outcomes for all coalitions or *payoff configurations*. We aim to identify a necessary and sufficient condition on games for the existence of a payoff configuration which is a *core* allocation and also satisfies an equity requirement named *population monotonicity*. Unlike the conditions found in the previous literature, we identify an 'operational' condition: *given a game, its zero-normalization is non-negative and the value of the objective function in the linear programming problem associated with the game is zero at its solution.* (*JEL* Classifications: C71, D63)

I. Introduction

The objects of study in this paper are *transferable utility games* in coalitional form, and prescriptions for such games in the form of an array of outcomes for all coalitions or *payoff configurations*.[1] We study the problem of the existence of a payoff configuration which is a *core* allocation and also satisfies an equity requirement named *population monotonicity*.

Most solutions for games in coalitional form are concerned with outcomes only for the grand coalition. Typical examples for such solutions are the *core*, the *Shapley value* and the *nucleolus*, etc. However, since games in coalitional form contain the full information about what each

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[1]The term *payoff configuration* was firstly used by Hart (1985).

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coalition can obtain, it is quite natural that a solution prescribes an outcome for every coalition. Indeed, the *Shapley value for non-transferable utility games* implicitly deals with outcomes for all coalitions. The *Harsanyi value* and the *consistent Shapley value* explicitly prescribe an outcome for each coalition.[2]

Harsanyi interpreted an outcome for a coalition as an optimal threat of the coalition against its complementary coalition. There is another interpretation by Hart (1985): an outcome for a coalition is viewed as a payoff vector that the members of the coalition agree upon if the coalition forms.

Given a game in coalitional form, it is uncertain what coalitions can actually be formed. So, it may be desirable that solutions prescribe outcomes for all coalitions. From such a viewpoint of coalition formation, Dutta and Ray (1989), Sprumont (1990) and other authors stressed the need of prescribing outcomes for all coalitions.

Since a payoff configuration is an array of outcomes or payoff vectors for all coalitions, it has a structure of a variable population. In many economic models with a variable population, *population monotonicity* has been adopted as an important equity requirement.[3] In the context of games in coalitional form, population monotonicity means that *if a new agent joins a coalition, then all existing members of the coalition get better off.* For example, imagine cost sharing of a public good. As a new agent joins a coalition, the tax basis also increases but it does not reduce anyone's consumption of the public good.[4] Thus, it is unfair that joining of a new agent hurts some of the existing members.

In this paper, we introduce the core as our basic solution concept for transferable utility games. We focus on the problem of the existence of a (population) monotonic payoff configuration among the core allocations. Sprumont (1990) identified a necessary and sufficient condition on games for the existence of such a payoff configuration. However, since this condition is in the form of an existence condition, it is very

[2]For definitions of the *Shapley value* and the *Harsanyi value* for non-transferable utility games, and for an analytic comparison between the two solutions, refer to Hart (1985). For a definition of the *consistent Shapley value*, refer to Maschler and Owen (1992).

[3]*Population monotonicity* is a 'powerful' axiom in characterizing many well-known solutions in the field of the social choice theory. Examples are the *Kalai-Smorodinsky* solution and the *egalitarian* solution in the bargaining theory. For a comprehensive survey on this literature, see Thomson (1995).

[4]See Moulin (1990) for an example of cost sharing of a public good.

difficult to see whether or not a game satisfies this condition. We aim to identify an alternative necessary and sufficient condition which offers an operational criterion for checking whether a game has a monotonic core payoff configuration.

We associate with each game a linear programming problem. Then, our necessary and sufficient condition is: *given a game,* (i) *its zero-normalization is non-negative*[5] and (ii) *the value of the objective function in the associated linear programming problem is zero at its solution.* Consequently, in order to check the existence of a monotonic core payoff configuration for a game, we need only to obtain its zero-normalization and to solve the linear programming problem associated with the game.

The paper is organized as follows. Section II formalizes the model and some definitions including population monotonicity, and presents a preliminary result of Sprumont (1990). Section III formulates our necessary and sufficient condition and gives a proof. Section IV contains some concluding comments.

II. Preliminaries

Let $N = \{1, 2, \ldots, n\}$ be the set of players. For each $S \subseteq N$, $\mathbb{R}^S$ is the cartesian product of $|S|$ copies of $\mathbb{R}$ indexed by the elements of S.

An (*n-person*) *transferable utility game in coalitional form* or a *game* is a function v: $\{S | S \subseteq N,\ S \neq \emptyset\} \mapsto \mathbb{R}$, which assigns to each non-empty coalition of N, a real number. A vector $x \in \mathbb{R}^N$ is a *payoff vector* for the game v if $\sum_{i=1}^{n} x_i = v(N)$.

Given a game v, given a non-empty coalition $T \subseteq N$, the *subgame of v with respect to T*, denoted v_T is defined by $v_T \equiv v|_{\{S | S \subseteq T,\ S \neq \emptyset\}}$. The *subgame configuration* of the game v is defined as $(v_T)_{T \subseteq N,\ T \neq \emptyset}$ (for short, (v_T)). This is an array of all subgames of v. An array of vectors $(x^S)_{S \subseteq N,\ S \neq \emptyset}$ in the product $\Pi_{S \subseteq N,\ S \neq \emptyset}\ \mathbb{R}^S$ is a *payoff configuration* for the game v if for all $S \subseteq N$, $\sum_{i \in S} x_i^S = v(S)$.

We naturally extend the usual notion of the core for games to subgames and to subgame configurations as follows.

Definition

Given a game v, given a non-empty coalition $S \subseteq N$, the *core of the sub-*

[5]Formal definitions of the *zero-normalization* and *non-negativity* are given in Section II.

game v_S, denoted $C(v_S)$ is defined by

$$C(v_S) \equiv \{x^S \in \mathbb{R}^S \mid \sum_{i\in S} x_i^S = v_S(S),\ \sum_{i\in T} x_i^S \geq v_S(T)\ \ \forall\, T \subseteq S\},$$

and the *core of the subgame configuration* (v_S), denoted $C((v_S))$ is naturally defined as the product of the core payoff vectors for each subgame, $\Pi_{S\subseteq N}\, C(v_S)$.

Population monotonicity is our equity requirement for payoff configurations. This axiom requires that for any coalition, if a new agent joins the coalition, then it makes every member in the coalition better off. It is formulated as follows.

Definition

A payoff configuration (x^S) is *(population) monotonic* if for all S, $T \subseteq N$ with $S \subseteq T$, for all $i \in S$, $x_i^S \leq x_i^T$.

A payoff configuration (x^S) is a *monotonic core payoff configuration* for v if (x^S) is monotonic and $(x^S) \in \Pi_{S\subseteq N}\, C(v_S)$.

We need the following definitions for technical and expository purposes. A game v is *zero-normalized* if $v(i) = 0$ for all $i \in N$. Given a game v, the *zero-normalization* of v, denoted v^0 is defined by $v^0(S) \equiv v(S) - \Sigma_{i\in S}\, v(i)$ for all $S \subseteq N$. A game v is *non-negative* if for all $S \subseteq N$, $v(S) \geq 0$. A game v is *monotonic* if for all S, $T \subseteq N$ with $S \subseteq T$, $v(S) \leq v(T)$. A game v is a *null* game if $v(S) = 0$ for all S. A game v is a *simple* game if for all S, $v(S) \in \{0, 1\}$. A game v is a *unanimity* game if there exists $T \subseteq N$ with $T \neq \emptyset$ such that

$$v(S) = \begin{cases} 1 & \text{if } \ T \subseteq S \\ 0 & \text{otherwise.} \end{cases}$$

Player i is a *veto player* in the game v if for all $S \subseteq N\backslash i$, $v(S) = 0$. A game with at least one veto player is a *veto-controlled* game.

The next theorem shows a necessary and sufficient condition on zero-normalized games for the existence of a monotonic core payoff configuration.

Theorem 1 (Sprumont, 1990)

Given a zero-normalized game v^0, there exists a monotonic core payoff configuration for v^0 if and only if v^0 is a positive linear combination of monotonic veto-controlled simple games.

Remark: Note that a game has a monotonic core payoff configuration if and only if its zero-normalization has one.[6] Then, Theorem 1 implies that a game has a monotonic core payoff configuration if and only if its zero-normalization is a positive linear combination of monotonic veto-controlled simple games.

Remark: Note that the condition in Theorem 1 is in the form of an existence condition. So, it is hard to check whether a game satisfies this condition. Especially, when the number of players is large, it is almost impossible to check games with this condition. Sprumont (1990) offered another necessary and sufficient condition which is of the same type as the *balancedness* condition for the non-emptiness of the core.[7] Unfortunately, the second condition is not so operational because it is complicated by an infinite number of vectors of 'subbalanced' weights. As pointed out by Sprumont (1990), the two conditions do not offer an operational criterion for checking whether a game has a monotonic core payoff configuration.

III. A Necessary and Sufficient Condition for a Computational Test

In this section, we identify an alternative necessary and sufficient condition which can perform a computational test for the existence of a monotonic core payoff configuration. Firstly, we associate with each zero-normalized game a linear programming problem in the following way.

We arbitrarily fix the order of non-empty coalitions such as $(S_1, \ldots, S_{2^n-1})$. Let $\hat{v}^1, \ldots, \hat{v}^l$ be all the distinct non-null, monotonic veto-controlled simple games. For each $i = 1, \ldots, l$, let $w^i \in \mathbb{R}^{2^n-1}$ be such that

$$w^i_j = \hat{v}^i(S_j) \qquad \forall j = 1, \ldots, 2^n - 1,$$

where S_j is the j-th coalition in the fixed order of coalitions. Let the matrix W be defined by $W = (w^1 \ldots w^l)$. Let $y \in \mathbb{R}^{2^n-1}$ and $a \in \mathbb{R}^l$. Then, given a game v, *the linear programming problem associated with* v is:

$$\min_{a,y} \sum_{i=1}^{2^n-1} y_i \quad \text{s.t.} \quad Wa + y = b \quad \text{and} \quad a, y \geqq 0,^8$$

[6]See Sprumont (1990, p. 383).

[7]See Theorem 2 in Sprumont (1990).

[8]Vector inequalities: $x \geqq y \Leftrightarrow x_i \geq y_i \ \forall i$; $x \geq y \Leftrightarrow x \geqq y$ and $x \neq y$; $x > y \Leftrightarrow x_i > y_i \ \forall i$.

where $b \equiv (v^0(S_1), \ldots, v^0(S_{2^n-1}))$ and v^0 is the zero-normalization of v.

Remark: To identify the matrix W which is composed of all the distinct non-null monotonic veto-controlled simple games, we follow the next steps.

Step 1. We identify $2^{(2^n-1)} - 1$ non-null simple games.
Let V^s be the class of non-null simple games.

Step 2. For each $v \in V^s$, for each coalition S with $v(S) = 1$, if there exists a coalition S' such that $S' \supseteq S$ and $v(S') = 0$, then we delete v from V^s.

Then, only non-null monotonic simple games remain. Let V^m be the class of such games.

Step 3. For each $v \in V^m$, if for all $i \in N$, there exists a coalition S such that $i \notin S$ and $v(S) = 1$, then delete v from V^m.

Then, we end up with all the non-null monotonic veto-controlled simple games.[9]

The following lemma shows that non-negativity of v^0 is a necessary condition for the existence of a monotonic core payoff configuration for v.

***Lemma* 1**

If there exists a monotonic core payoff configuration for v, then its zero-normalization v^0 is non-negative.

Proof: Suppose to the contrary that there exists $S \subseteq N$ such that $|S| \geq 2$ and $v^0(S) < 0$. Since there exists a monotonic core payoff configuration for v, it follows from Theorem 1 that v^0 is a positive linear combination of monotonic games. Since a positive linear combination of monotonic games is also monotonic, it follows that v^0 is a monotonic game. Since $v^0(i) = 0$ for all i, it follows from monotonicity of v^0 that for all $S \subseteq N$ with $|S| \geq 2$, $v^0(S) \geq 0$, which contradicts the hypothesis.

Q.E.D.

[9]Step 1-3 are computer-programmable.

The next lemma by Gale (1960) presents a necessary and sufficient condition on linear equation systems for the existence of a non-negative solution in terms of linear programming. This lemma plays a key role in the proof of Theorem 2.

***Lemma* 2.** (Gale, 1960)
Let A be an $m \times n$ matrix with $n \geq m$, and $x \in \mathbb{R}^n$ and $c, y \in \mathbb{R}^m$ with $c \geqq 0$. Then, the linear equation system $Ax = c$ has a non-negative solution if and only if the value of the objective function in the following linear programming problem

$$\min_{x,y} \sum_{i=1}^{m} y_i \quad \text{s.t.} \quad Ax + y = c \quad \text{and} \quad x, y \geqq 0$$

is zero at its solution.

The next theorem shows a necessary and sufficient condition which can perform a computational test for the existence of a monotonic core payoff configuration. Checking up this condition only requires solving a linear programming problem.

Theorem 2
Given a game v, there exists a monotonic core payoff configuration for v if and only if (i) its zero-normalization v^0 is non-negative and (ii) the value of the objective function in the linear programming problem associated with v is zero at its solution.

***Proof*:** By Lemma 1, part (i) of the above condition is a necessary condition for the existence of a monotonic core payoff configuration for v. Let V^+ be the class of games whose zero-normalizations are non-negative. Then, it remains to show that on V^+, part (ii) is a necessary and sufficient condition for the existence of a monotonic core payoff configuration for v.

According to Shapley (1953), every game is a linear combination of unanimity games. Note that every unanimity game is a non-null monotonic veto-controlled simple game. Then, every game is a linear combination of non-null monotonic veto-controlled simple games. Therefore, there exists $a \in \mathbb{R}^l$ such that $v^0 = \Sigma_{i=1}^{l} a_i \hat{v}^i$. That is,

$$b = \sum_{i=1}^{l} a_i w^i = \begin{pmatrix} w_1^1 & \cdots & w_1^i & \cdots & w_1^l \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ w_{2^n-1}^1 & \cdots & w_{2^n-1}^i & \cdots & w_{2^n-1}^l \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_l \end{pmatrix} = Wa.$$

Theorem 1 implies that there exists a monotonic core payoff configuration for v if and only if the linear equation system $Wa = b$ has a non-negative solution.[10]

Since $v \in V^{+}$, it follows that $v^{0}(S) \geq 0$ for all S. Therefore, $b \geqq 0$. Since there are $2^n - 1$ (n-person) unanimity games, it follows that $l \geq 2^n - 1$. Since W is a $(2^n - 1) \times l$ matrix, and $a \in \mathbb{R}^l$ and $b, y \in \mathbb{R}^{2^n-1}$, Lemma 2 completes the proof.

Q.E.D.

IV. Concluding Comments

In this paper, we discovered a necessary and sufficient condition for the existence of a monotonic core payoff configuration in the case of transferable utility games. Unlike Sprumont (1990)'s conditions, our condition offers an operational criterion for checking whether a game has a monotonic core payoff configuration. Since linear programming is involved in the condition, we can routinize the process of checking up the condition by a computer program.

A natural research agenda is to extend the existence problem to the case of non-transferable utility games. Since our result depends on the linearity of transferable utility games (that is, a game is a linear combination of monotonic veto-controlled simple games), it may be impossible to extend our result to the non-transferable utility case. Moulin (1990) also pointed out the difficulty in tackling the existence problem in the case of non-transferable utility games. We leave the agenda to future research.

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References

Dutta, B., and Ray, D. "A Concept of Egalitarianism under Participation Constraint." *Econometrica* 57 (1989): 615-35.

Gale, D. *The Theory of Linear Economic Models*. McGraw-Hill, 1960

Hart, S. "An Axiomatization of Harsanyi's Nontransferable Utility Solution." *Econometrica* 53 (1985): 1295-313.

Maschler M., and Owen, G. "The Consistent Shapley Value for Games with-

[10]Unlike Theorem 1 we need allow the case of zero solution for $Wa = b$ because W does not include a null game.

out Side Payments." in R. Selten, ed., *Rational Interaction.* Springer-Verlag, 1992, 5-12.

Moulin, H. "Cores and Large Cores When Population Varies." *International Journal of Game Theory* 19 (1990): 219-52.

Shapley, L. S. "A Value for n-Person Games." in H.W. Kuhn and A.W. Tucker, eds., *Contribution to the Theory of Games II, Annals of Mathematics Studies* No. 28. Princeton University Press, 1953, 307-17.

Sprumont, Y. "Population Monotonic Allocation Schemes for Cooperative Games with Transferable Utility." *Games and Economic Behavior* 2 (1990): 378-94.

Thomson, W. "Population-Monotonic Allocation Rules." in W.A. Barnett, H. Moulin, M. Salles, and N. Schofield, eds., *Social Choice, Welfare, and Ethics.* Cambridge University Press, 1995, 79-124.