An Exact Pricing Error of the APT within the Arbitrage Framework

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We derive an exact deviation for an individual asset from APT pricing in a finite economy within the arbitrage framework. This deviation is the product of a tradeoff between mean and variance of the efficient arbitrage portfolio, the asset's idiosyncratic variance and the proportion of this arbitrage portfolio represented by the asset. We show that the deviation becomes negligible in an infinite economy if the efficient portfolio is well diversified. (*JEL* Classification: G12)

I. Introduction

The arbitrage pricing theory (APT), introduced by Ross (1976a, b) and extended further by Chamberlain and Rothschild (1983) and Ingersoll (1984), has shown the existence of an approximate pricing relationship in an infinite economy, given a factor structure. This approximate pricing relationship is obtained by employing an arbitrage argument. But this relationship has been criticized for its testability as in Shanken (1982, 1985). Shanken (1982) states: "Ross's theory does not (even in the limit as the number of assets approaches infinity) imply exact linear risk-return relation. Thus the testability of the theory could reasonably be questioned on this ground alone." Connor (1984) derived an exact APT model asymptotically in an infinite economy by using an equilibrium argument. Also the testability of the Connor model has been questioned as Shanken (1982) argues, "the 'equilibrium APT' appears to be subject to substantially the same difficulties encountered in testing the

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CAPM."

Dybvig (1983) and Grinblatt and Titman (1983) derived an approximate pricing APT model in a finite economy by using an equilibrium argument. They argue that the magnitude of mis-pricing is explicitly bounded with additional strong assumptions about preferences, asset supplies and idiosyncratic variances of assets. An explicit bound might not be small enough if these assumptions are not satisfied. Furthermore, this APT model has the same testability problems that the Connor model has since it is based on the equilibrium argument.

This paper derives an exact deviation for an individual asset from APT pricing in a finite economy within the arbitrage framework which Ross (1976a, b) employes. We will use "pricing error" to mean deviation from APT pricing. Dybvig (1983) and Grinblatt and Titman (1983) argue that the pricing error for asset is bounded by $R\sigma^2 w_i$, where R is the risk aversion coefficient, σ^2 is the asset's idiosyncratic variance and w_i is the proportion of total wealth represented by the asset. In our model, the exact pricing error for asset i is given by $T\sigma_i^2 w_i$, where T is a tradeoff between expected return and variance of the efficient arbitrage portfolio, σ_i^2 is the asset's idiosyncratic variance and w_i is the proportion of this arbitrage portfolio represented by the asset.

In an infinite economy, this exact pricing error becomes negligible if the efficient arbitrage portfolio is well diversified. We demonstrate that an exact pricing APT model holds asymptotically in an infinite economy even though Ross (1976b) shows that the sum of squared deviations from exact pricing is bounded. In the Ross model, it is unclear how accurately each asset is priced under the APT. Thus Shanken (1982) states: "'most' of the deviation from linearity must be 'small', although any particular deviation may be 'large'." However, our APT model shows that the APT prices every asset accurately no matter how much the idiosyncratic variance of each asset may be as long as the efficient arbitrage portfolio is well diversified. Hence we show that the APT is testable if the efficient arbitrage portfolio is well diversified. Our APT pricing relationship is derived by considering any set of N assets that follows a factor structure. There are no restrictions on the relation between the subsets of assets under consideration and other assets in the economy. Hence, our APT model can be testable for subsets of the universe of assets.[1] Since our model is not an equilibrium APT model, it may avoid the difficulties of testing CAPM. Shanken (1985) states: "As we have seen, however, nothing in D-R's analysis indicates that a refutable empirical hypothesis can be obtained within the arbitrage framework itself." Our APT model provides a testable empirical hypothesis obtained within the arbitrage framework. Section II provides a brief review of the Arbitrage Pricing Theory. Section III shows an exact pricing error for an individual asset in a finite economy within the arbitrage framework. Section IV investigates the APT model in an infinite economy. It demonstrates that an exact pricing APT model holds asymptotically. Section V provides a summary.

II. Arbitrage Pricing Theory

The APT assumes returns are generated by a K-factor structure denoted as

$$R = E + Bf + e, \tag{1}$$

where R = an N-dimensional vector of the random asset returns,
E = an N-dimensional vector of the ex-ante expected returns,
B = an $N \times K$ matrix of factor loadings,
f = an K-dimensional vector of mean zero factors, which are assumed to be uncorrelated with each other,
e = an N-dimensional vector of mean zero idiosyncratic disturbances, which are assumed to be uncorrelated with the factors and with each other.

Ross' argument is as follows. Suppose we form an arbitrage portfolio with no systematic risk such that

$$w' 1_N = 0 \text{ and } w'B = 0, \tag{2}$$

where w = an N-dimensional vector of portfolio weights,
1_N = an N-dimensional vector of ones.
Then the ex-post return of the arbitrage portfolio is given by

$$w'R = w'E + w'Bf + w'e = w'E + w'e. \tag{3}$$

The law of large numbers suggests that in an infinite economy,

[1]Dybvig (1983), and Grinblatt and Titman (1983) postulate a factor structure for assets in the economy. As Shanken (1985) points out, this is a fundamental departure from the original APT which requires a factor structure for a given subset of assets.

$$w'e \simeq 0, \tag{4}$$

because the arbitrage portfolio is assumed to be well-diversified. Thus $w'R \cong w'E$. Since this arbitrage portfolio requires zero net investment (i.e. $w' 1_N = 0$), $w'R = 0$ from the absence of arbitrage. It implies that

$$w'E \simeq 0. \tag{5}$$

In sum, any portfolio satisfying (2) must also satisfy (5), given the assumption that the portfolio approximately eliminates the idiosyncratic risk. But in a finite economy where the number of assets is finite, we cannot guarantee that $w'E \simeq 0$, since it may be practically impossible to diversify the idiosyncratic risk completely. However, we assume in this section for pedagogic purposes that $e \simeq 0$. Then (2) implies that $w'E \simeq 0$. Otherwise, there must be an arbitrage opportunity. It is well known that (2) and $w'E \simeq 0$ together implies the existence of the linear return relationship (APT pricing):

$$E = \lambda_0 + B\lambda \quad \text{for} \quad \lambda \in R^K, \tag{6}$$

where λ_0 is an N-dimensional constant vector.

Ross (1976b) has shown that an approximate APT model holds in an infinite economy without assuming $e = 0$. The sum of squared deviation from (6) is bounded as the number of assets approaches infinity, i.e.

$$(E - \lambda_0 - B\lambda)(E - \lambda_0 - B\lambda)' < \infty \quad \text{as} \quad N \to \infty. \tag{7}$$

In Section III, we will derive an exact deviation for each asset from APT pricing in a finite economy within the arbitrage framework under imperfect diversifiability of the idiosyncratic risk.

III. An Exact Pricing Error for Each Asset in a Finite Economy

In a finite economy, undiversifiability of the idiosyncratic risk implies that the absolute value of the idiosyncratic disturbance of the arbitrage portfolio is greater than zero,

$$\| w'e \| > 0. \tag{8}$$

It follows that $w'ee'w > 0$. By taking an expectation operator, we obtain

$$w'E[ee']w = w'\,\mathrm{Var}[e]w = w'Vw > 0, \tag{9}$$

where V is assumed to be an $N \times N$ diagonal covariance matrix of idiosyncratic disturbances defined by Var[e].[2]

The arbitrage portfolios are assumed to require zero net investment and to eliminate factor risks as in Ross (1976a, b),

$$w'1_N = 0 \text{ and } w'B = 0. \tag{10}$$

The arbitrage portfolios are risky since the idiosyncratic disturbance cannot be eliminated completely. Ross (1976b) assumes risk averse investors for whom the coefficient of relative risk aversion is uniformly bounded. He shows that from utility maximization, the variance of the efficient arbitrage portfolio with zero factor risk and zero net investment which provides a positive expected return must be bounded away from zero (i.e. positive). He demonstrates that the minimum variance for the portfolio solving the following problem is strictly positive.

***Problem* 1**

$$\underset{w}{\text{Minimize}}\ w'Vw$$

subject to $w'1_N = 0$, $w'B = 0$ and $0 < c \le w'E$.

In fact, he used additional technical assumptions such as non-negligibility of type B agents whose relative risk aversion is uniformly bounded, and the existence of at least one asset with limited liability. But we do not need these assumptions. The only assumptions we need in deriving the APT model are the absence of arbitrage and risk averse investors. A much simpler proof is provided in the following lemma.

***Lemma* 1**

The minimum variance of Problem 1 is strictly positive.

Proof: The Kuhn-Tucker conditions for Problem 1 are

$$2Vw - x1_N - yB - zE = 0, \tag{11}$$

$$w'1_N = 0, \tag{12}$$

[2]
$$V = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \sigma_{N-1}^2 & 0 \\ 0 & 0 & \cdots & \sigma_N^2 \end{bmatrix}$$

$$w'B=0, \text{ and} \tag{13}$$

$$c-w'E=0, \tag{14}$$

where x, y and z are Lagrange multipliers. Multiplying both sides of (11) with w and using (12) and (13) gives

$$2w'Vw-zw'E=0. \tag{15}$$

Since z is positive, the strict positivity of $w'E$ implies the strict positivity of $w'Vw$. The proof is complete.

The result of Lemma 1 is intuitively appealing since if a zero-investment, zero-loadings arbitrage portfolio with positive expected return has zero variance, it implies an arbitrage opportunity. Ross (1976b) demonstrates, by using Lemma 1, that the sum of squared deviation from APT pricing is bounded as the number of assets under consideration approaches infinity. In order to obtain the pricing structure, we consider a problem which is in a sense dual to Problem 1. Our approach is similar to the one to obtain a single beta representation of the Capital Asset Pricing Model (CAPM). Consider the following problem:

$$\underset{w}{\text{Maximize}} \quad w'E$$

subject to $w'1_N=1$ and $0<w'Vw\leq d$,
where V is the covariance matrix of asset returns.

The first order condition of the above problem provides a single beta representation for expected return on each asset i in terms of the covariance of random return on the efficient portfolio w with random return on the individual asset i. The pricing relationship for the APT can be obtained by the first order condition for the dual problem of Problem 1. Similarly, the return on asset i under APT pricing is determined by the covariation of random return on the efficient arbitrage portfolio with random return on asset i. This efficient arbitrage portfolio is the solution of the dual problem. We formulate the following dual problem to Problem 1.

***Problem* 2**

$$\underset{w}{\text{Maximize}} \; w'E$$

subject to $w^{*\prime}1_N=0$, $w'B=0$ and $0<w'Vw\leq d$.

We use the arbitrage portfolio to mean the portfolio with zero-investment and zero-loadings. Also we use the efficient arbitrage portfolio(w^*) to mean the solution to Problem 2. We will demonstrate that the expected return on the efficient arbitrage portfolio with non-zero idiosyncratic risk must be positive. We will show by employing the same technique used in Lemma 1 that the maximum expected return for Problem 2, $w^{*\prime}E$, is strictly positive.

***Lemma* 2**
The maximum expected return of Problem 2 is strictly positive.

Proof: The Kuhn-Tucker conditions for Problem 2 are

$$E - x1_N - yB - 2zVw^* = 0, \tag{16}$$

$$w^{*\prime}1_N = 0, \tag{17}$$

$$w'B = 0, \text{ and} \tag{18}$$

$$d - w^{*\prime}Vw^* = 0, \tag{19}$$

where x, y and z are Lagrange multipliers. Multiplying both sides of (16) with w^* and using (17) and (18) gives

$$w^{*\prime}E - 2zw^{*\prime}Vw^* = 0. \tag{20}$$

Since z is positive, the strict positivity of $w^{*\prime}Vw^*$ implies the strict positivity of $w^{*\prime}E$. The proof is complete.

Lemma 2 argues that risk aversion requires positive expected return for the efficient arbitrage portfolio with non-zero idiosyncratic risk. Intuitively, a risk averse investor would hold an arbitrage portfolio with non-zero idiosyncratic risk only if it provides positive expected return. We will derive the exact pricing error for the APT. Since risk averse investors would hold the efficient arbitrage portfolio (w^*) and since the pricing relationship is determined by w^*, we will focus on the efficient arbitrage portfolio. Lemma 2 implies that if

$$w^{*\prime}1_N = 0,\ w^{*\prime}B = 0 \text{ and } w^{*\prime}Vw^* > 0 \text{ for } w^* \in R^N, \tag{21}$$

then we must have

$$w^{*\prime}E > 0, \tag{22}$$

where w^* is the efficient arbitrage portfolio, the solution to Problem 2. It follows that there is no solution (i.e. the efficient arbitrage portfolio), $w^* \in R^N$, which satisfies the following system,

$$w^{*\prime}1_N = 0, \quad w^{*\prime}B = 0 \quad \text{and} \quad w^{*\prime}Vw^* > 0 \quad \text{and} \quad w^{*\prime}E \le 0. \tag{23}$$

If and only if there is no solution for the system (23), the model holds as shown in the following theorem.[3]

***Theorem* 1**

Exactly one of the following systems has a solution.

System A: $w^{*\prime}1_N=0$, $w^{*\prime}B=0$, $w^{*\prime}Vw^{*}>0$ and $w^{*\prime}E\leq 0$, for $w^{*}\in R^N$, where w^{*} is the efficient arbitrage portfolio.

System B: $E=\lambda_0+B\lambda+Vw^{*}T$ for $\lambda\in R^K$, where λ_0 is an N-dimensional constant vector, T is a positive scalar and w^{*} is the efficient arbitrage portfolio.

Proof: See the Appendix.

Theorem 1 states that if one of these systems has no solution, there is a solution for the other system. From the concavity of preference, there is no solution of $w^{*}\in R^N$ for System A. Thus there is a solution of $\lambda_0\in R^N$, $\lambda\in R^K$, and $T\in R^1$ for System B. System B provides the exact pricing errors from APT pricing, i.e.

$$E-\lambda_0-B\lambda=Vw^{*}T. \tag{24}$$

The exact pricing error on asset i is given by $T\sigma_i^2 w_i$, a positive scalar, where σ_i^2 is the idiosyncratic variance of asset i and w_i is the proportion of the efficient arbitrage portfolio represented by asset i. System B specifies the sign of the exact pricing error for asset i such that it depends on the sign of w_i. If any asset i has been in a short position to form the efficient arbitrage portfolio (w^{*}), the sign of the pricing error for this asset is negative. If it is in a long position, the sign of its pricing error is positive.[4] The positive scalar T is further explained by the following theorem.

***Theorem* 2**

The positive scalar T is

$$T=\frac{w^{*\prime}E}{w^{*\prime}Vw^{*}}=2z, \tag{25}$$

where z is Lagrange multiplier in (16).

[3]Theorem 1 is one of Theorems of the alternative. See Mangasarian (1969) for a detailed discussion of theorems of the alternative.

[4]Uniqueness of the vector w^{*} is obvious. The optimal solution w in Problem 1 is unique from the strictly convex objective function and the compact set of constraints. Uniqueness of the optimal solution w^{*} in Problem 2 follows from the fact that Problem 2 is equivalent to Problem 1.

Proof: Multiplying both sides of System B with w^* gives

$$w^{*\prime}E = w^{*\prime}1_N\lambda_0 + w^{*\prime}B\lambda + w^{*\prime}Vw^*T, \tag{26}$$

where λ_0^* is an element of λ_0. Using (17), (18) and (20) provides (25). The proof is complete.

The positive scalar T is twice the Lagrange multiplier z in Problem 2. That indicates the marginal effect on the expected return on the efficient arbitrage portfolio of increasing the idiosyncratic risk by 1 unit. Also it is a tradeoff between mean and variance of the efficient arbitrage portfolio. This looks very similar to a linear tradeoff between mean and standard deviation, δ, which, Chamberlain and Rothschild (1983) argue, plays an important role in the analysis of the factor structure. One interesting result in our analysis is that System B is exactly the same as the first order condition of Problem 2 in equation (16). System B is equivalent to the mean-variance efficiency of the arbitrage portfolio. A similar observation has been made by Roll (1977) and Ross (1977) for the CAPM. The CAPM is equivalent to the statement that the market portfolio is mean variance efficient.

Similarly, we can argue that the APT is equivalent to the statement that the mean-variance efficient arbitrage portfolio is well-diversified. This will be clearly demonstrated in Theorem 3 and Theorem 4. So far, even though we have worked in the arbitrage framework used by Ross (1976a, b), we have not employed an arbitrage argument. In fact, we will employ an arbitrage argument to explain the limiting behavior of the pricing error in proving Theorem 3 and Theorem 4.

We might argue that the exact pricing error, $T\sigma_i^2 w_i^*$, for the individual asset i, will be negligible, if T, as a function of w_i^*, does not increase too fast, as the number of assets approaches infinity. (As the number of assets approaches infinity, w_i^* will obviously be negligible.) We will show this in the next section.

IV. The APT in an Infinite Economy

In this section, we show that an exact pricing APT model holds asymptotically in an infinite economy if the efficient arbitrage portfolio (w^*) is well diversified. In Section III, we derived the exact pricing errors in a finite economy, i.e.

$$E-\lambda_0-B\lambda=Vw^*T. \tag{24}$$

The exact pricing error for asset i can be written as

$$E_i-\lambda_0^*-B_i\lambda=T\sigma_i^2 w_i^*, \tag{27}$$

where λ_0^* is an element of λ_0, B_i is the i^{th} row of B and σ_i^2 is the i^{th} diagonal element of V (i.e. the idiosyncratic variance of asset i).

If the arbitrage portfolio is well diversified in an infinite economy, the proportion of the efficient arbitrage portfolio represented by asset i is of order $1/N$ in absolute magnitude as in Ross (1976b) i.e.

$$w_i=\pm\frac{a_i}{N}, \tag{28}$$

where a_i is a positive constant. The variance of the efficient arbitrage portfolio is

$$w^{*\prime}Vw=\sum w_i^2\sigma_i=\frac{\sum a_i^2\sigma_i^2}{N^2}. \tag{29}$$

The variance of the efficient arbitrage portfolio is bounded by

$$\underline{a}^2\frac{\sigma^2}{N}\leq w^{*\prime}Vw\leq\bar{a}^2\frac{\sigma^2}{N}, \tag{30}$$

where $\underline{a}$ is the smallest a_i, $\bar{a}$ is the largest a_i and σ^2 is the average variance of the e_i terms (e_i is the i^{th} element of the vector e).

If the number of assets approaches infinity, the variance of the arbitrage portfolio will be negligible. If the arbitrage portfolio is well-diversified, the absolute exact deviation for asset i from APT pricing is

$$|T\sigma_i^2 w_i^*|=\frac{a_i\sigma_i^2 w^{*\prime}E}{Nw^{*\prime}Vw^*}. \tag{31}$$

The following theorem demonstrates that if the number of assets approaches infinity, the pricing error for asset i is negligible.

***Theorem* 3**

The absolute pricing error from exact pricing for asset i,

$$|T\sigma_i^2 w_i^*|\rightarrow 0 \text{ as } N\rightarrow\infty. \tag{32}$$

Proof: If follows from (30) and (31) that the absolute deviation is bounded, i.e.

$$|T\sigma_i^2 w_i^*| = \frac{a_i \sigma_i^2 w^{*\prime}E}{Nw^{*\prime}Vw^*} \leq \frac{a\sigma_i^2 w^{*\prime}E}{N\underline{a}^2 \frac{\sigma^2}{N}} = \frac{\bar{a}\sigma_i^2 w^{*\prime}E}{\underline{a}^2 \sigma^2}. \tag{33}$$

Since the efficient arbitrage portfolio is assumed to be well diversified, $w^{*\prime}Vw^* \rightarrow 0$ as the number of assets approaches infinity as demonstrated in (30). By the absence of arbitrage, if

$$w^{*\prime}1_N = 0,\ w^*B = 0 \text{ and } w^{*\prime}Vw^* \rightarrow 0, \tag{34}$$

we must have

$$w^{*\prime}E \rightarrow 0. \tag{35}$$

(33) and (35) imply that the absolute deviation from exact pricing is negligible. The proof is complete.

This is a strong result. It shows that an exact pricing APT model holds asymptotically in an infinite economy. Ross (1976b) demonstrates that as the number of assets approaches infinity, the sum of squared deviations is bounded, i.e.

$$\Sigma_i^\infty [E_i - \lambda_0^* - B_i\lambda]^2 < \infty. \tag{36}$$

Shanken (1985) states: "The APT remains silent, however, with respect to the pricing of a given individual security with positive residual variance." The above result shows that the APT provides the negligible deviation for each asset no matter how much the residual variance of the asset may be. Even the sum of squared deviations from exact pricing is negligible in an infinite economy as shown in the following theorem.

***Theorem* 4**

The sum of squared deviations is negligible as the number of assets approaches infinity, i.e.

$$\Sigma_{i=1}^N [E_i - \lambda_0^* - B_i\lambda]^2 \rightarrow 0 \text{ as } N \rightarrow \infty. \tag{37}$$

Proof: See the Appendix.

Shanken (1985) states: "In fact, as emphasized in Shanken (1982), the APT restriction is an approximation, one which prices "most" assets well but permits arbitrarily large deviations from exact pricing on a finite asset. Thus it is difficult to conceive of any (finite) empirical procedure that could be used to refute the actual conclusion of the APT. In this sense, the theory is untestable in principle." He states: "My thesis in Shanken (1982) was i) the arbitrage paradigm has not produced a refutable hypothesis." The Shanken argument can apply to only (36). Theorem 3 and Theorem 4 show that the APT prices every asset very accurately. His argument does not apply to our APT model. Hence the APT is testable in principle if the efficient arbitrage portfolio is well diversified.

V. Summary

The exact pricing error for an individual asset is derived in a finite economy by using an arbitrage framework. The exact pricing error is the product of a trade-off between mean and variance of the efficient arbitrage portfolio, the idiosyncratic variance of the individual asset, and the proportion of this arbitrage portfolio represented by the individual asset. The trade-off between mean and variance is twice the Lagrange multiplier for the idiosyncratic variance of the efficient arbitrage portfolio. That indicates the marginal effect on the expected return on the arbitrage portfolio of increasing the idiosyncratic variance by 1 unit.

In an infinite economy, the pricing error for the individual asset is negligible if the efficient arbitrage portfolio is well diversified. Also the sum of squared deviations from exact pricing is negligible. The APT is equivalent to the statement that the mean-variance efficient arbitrage portfolio is well-diversified. If the portfolio is well diversified, we have an exact APT model asymptotically.

The derivation of an asymptotic exact pricing model in an infinite economy depends critically on how well the efficient arbitrage portfolio is diversified. It would be an interesting research if one demonstrates that the efficient arbitrage portfolio is well diversified in an infinite economy.

Appendix

***Theorem* 1**

Only one of the following systems has a solution.

System A: $w^{*\prime}B=0$, $w^{*\prime}1_N=0$, $w^{*\prime}Vw^*>0$, and $w^{*\prime}E\leq 0$, for $w^*\in R^N$, where w^* is the efficient arbitrage portfolio.

System B: $E = \lambda_0 + B\lambda + Vw^*T$ for $\lambda \in R^K$,
where λ_0 is an N-dimensional constant vector, T is a positive scalar and w^* is the efficient arbitrage portfolio.

Proof:

i) Suppose System A has a solution $w^* \in R^N$. Then we have to show that System B has no solution. On the contrary, suppose that System B has a solution, λ_0, λ, and T. Multiplying w^* on both sides of System B gives

$$w^{*\prime}E = \lambda_0{}^* w^{*\prime} 1_N + w^{*\prime}B\lambda + w^{*\prime}Vw^*T,$$

where is an element of λ_0. From System A,

$$0 > w^{*\prime}E = \lambda_0{}^* w^{*\prime}1_N + w^{*\prime}B\lambda + w^{*\prime}Vw^*T > 0,$$

a contradiction. Hence, System B cannot have a solution.

ii) Suppose System A has no solution. Then we have to show that System B has a solution, λ_0, λ and T. Consider the following sets:

$$C_1 = \{(u, x, y, z): w^{*\prime}B = u,\ w^{*\prime}1_N = x,\ w^{*\prime}Vw^* = y \text{ and } w^{*\prime}E = z\},$$

$$C_2 = \{(u, x, y, z): u = 0,\ x = 0,\ y > 0 \text{ and } z \leq 0\}.$$

Since there is no solution for System A, $C_1 \cap C_2 = \varnothing$. Then there exists a hyperplane that separates C_1 and C_2. That is, there exist non-zero vector p_1 and non-zero scalars p_2, p_3 and p_4 such that

$$w^{*\prime}Bp_1 + w^{*\prime}1_N p_2 + w^{*\prime}Vw^*p_3 + w^{*\prime}Ep_4 \geq up_1 + xp_2 + yp_3 + zp_4,$$

for u, x, y and $z \in clC_2$, $p_1 \in R^K$, $p_2 \in R^1$, $p_3 \in R^1$ and $p_4 \in R^1$. Since y can be an arbitrarily large positive number, it follows that $p_3 < 0$. Since z can be an arbitrarily large negative number, if follows that $p_4 > 0$. Let $u = 0$, $x = 0$, $y = 0$ and $z = 0$. Then

$$w^{*\prime}Bp_1 + w^{*\prime}1_N p_2 + w^{*\prime}Vw^*p_3 + w^{*\prime}Ep_4 \geq 0 \text{ for each } w^* \in R^N.$$

By choosing $w^* = -(Bp_1 + 1_N p_2 + Vw^*p_3 + Ep_4)$, it then follows that

$$-\,|Bp_1 + 1_N p_2 + Vw^*p_3 + Ep_4|^2 \geq 0.$$

Thus $Bp_1 + 1_N p_2 + Vw^*p_3 + Ep_4 = 0$. It implies that

$$E = \lambda_0{}^* 1_N + B\lambda + Vw^*T,$$

where $\lambda_0{}^* = -p_2/p_4$, $\lambda = -p_1/p_4$ and $T = -p_3/p_4 > 0$. The equation can be rewritten as $E = \lambda_0{}^* 1_N + B\lambda + Vw^*T$. Hence, System B has a solution. The proof is complete.

***Theorem* 4**
The sum of squared deviations is negligible as the number of assets approaches infinity, i.e.

$$\Sigma_{i=1}^{N} [E_i - \lambda_0^* - B_i \lambda]^2 \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Proof: If we show $(Vw^*T)' Vw^*T \rightarrow 0$ as the number of assets approaches infinity, the proof is complete. The sum of squared deviations, $T^2 w^{*\prime} V' Vw^*$, can be rewritten as

$$T^2 w^{*\prime} V' Vw^* = \frac{(w^{*\prime} E)^2 \Sigma_{i=1}^{N} a_i^2 \sigma_i^4}{(w^{*\prime} Vw^*)^2 N^2}.$$

Using (30) and (31), we can find a bound of this sum, i.e.

$$T^2 w^{*\prime} V' Vw^* \leq \frac{N(w^{*\prime} E)^2 \bar{a}^2 \bar{\sigma}^4}{\underline{a}^4 (\bar{\sigma}^2)^2} = \frac{w^{*\prime} Ex' E\bar{a}^2 \bar{\sigma}^4}{\underline{a}^4 (\bar{\sigma}^2)^2},$$

where $\bar{\sigma}^2$ is the average variance of the e_i terms, $\bar{\sigma}^4$ is the average fourth moment of the e_i terms and $x = Nw^*$. $x'E$ is bounded in the following way:

$$x' E < \bar{a} 1_N |E| < \infty.$$

As the number of assets approaches infinity, $w^{*\prime} E$ approaches zero from the absence of arbitrage. Thus $T^2 w^{*\prime} V' Vw^* \rightarrow 0$. The proof is complete.

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