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Nonlinear Observer Design via Reduced-Order Dynamic Observer Error Linearization and Extended Nonlinear Observer Canonical Form

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ABSTRACT

Nonlinear Observer Design via Reduced-Order Dynamic Observer Error Linearization and Extended Nonlinear Observer Canonical Form

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This dissertation contributes to the observer design problem for some classes of nonlinear systems. The observer design problem is to construct a dynamic system (called observer) that can estimate the state of a given dynamic system by using available signals which are commonly the input and the output of the given system. While a standard solution (called Luenberger observer) to the problem was solved for linear systems, there has not been a unified solution for general nonlinear systems. However, there have been significant research efforts on the problem of designing observers for special classes of nonlinear systems. Observer error linearization (OEL) is one of the well-known methods, and it is the problem of transforming a nonlinear system into a nonlinear observer canonical form (NOCF) that is an observable linear system modulo output injection. If a nonlinear system can be transformed into the NOCF, then all the nonlinearities

of the system are restricted to the output injection term which is a vector-valued function of the system input and the system output. As a result, we can design a Luenberger-type observer that cancels out the output injection and thus has a linear observer error dynamics in the transformed coordinates. In order to extend the class of systems to which the OEL approach is applicable, a lot of attempts have been made in the past three decades. One of them is to transform a nonlinear system into a higher-dimensional NOCF: system immersion and dynamic observer error linearization (DOEL). In particular, the main idea of DOEL is twofold: the first is to introduce an auxiliary dynamics whose input is system output, and the second is to transform the extended system into a generalized nonlinear observer canonical form (GNOCF) that is an observable linear system modulo generalized output injection depending not only on the system output but also on the state of auxiliary dynamics. By introducing such an auxiliary dynamics, the DOEL problem can be solved for a larger class of systems compared with the (conventional) OEL problem. However, it has a drawback on the dimension of observer. That is, the dimension of observer designed by the DOEL approach is larger than that of the given system, because the dimension of GNOCF equals to the sum of dimensions of the given system and the auxiliary dynamics. Recently, inspired by this fact, a new approach called reduced-order dynamic observer error linearization (RDOEL) was proposed for single output nonlinear systems. In the framework of RDOEL, we also introduce an auxiliary dynamics and transform the extended system into GNOCF in a similar fashion to DOEL, but the coordinate transformation preserves the coordinates corresponding to the state of auxiliary dynamics so that the dimension of GNOCF equals to that of the given system. Although RDOEL is a special case of DOEL (that is, the class of systems to which the RDOEL approach can be applied is a subset of that of DOEL), the RDOEL approach offers a lower-dimensional observer compared to the DOEL approach, and it is also applicable to a larger class of systems compared to the (conventional) OEL approach. In addition, since the framework of RDOEL is coterminous with that of OEL (in fact, the OEL problem is identical to the RDOEL problem with no auxiliary dynamics), most of results for the RDOEL problem can be also used to analyze the OEL problem by slight modification.

In this respect, one of the topics of this dissertation is to deal with the RDOEL problem for multi-output systems. We first formulate the framework of RDOEL for multi-output nonlinear systems and provide three necessary conditions. And then, by means of the necessary conditions, we derive a geometric necessary and sufficient condition in terms of Lie algebras of vector fields. Since the proposed RDOEL problem is a natural extension of the (conventional) OEL problem, the result can be easily translated into a geometric necessary and sufficient condition for the OEL problem, which has not yet been completely established in the case where an output transformation of general form is considered.

The other topic of the dissertation is to introduce an extended nonlinear observer canonical form (ENOCF) whose linear part also depends on the system output and the state of auxiliary dynamics, and to deal with the problem of transforming a single output nonlinear system with an auxiliary dynamics into the ENOCF as an extension of the RDOEL problem. Since the proposed ENOCF admits a kind of high-gain observer, the solvability of the problem allows us to design observers for a class of single output nonlinear systems. We also first present two necessary conditions, and then derive a geometric necessary and sufficient condition for the problem. Furthermore, as a case study, we apply the results to the Rössler system in order to show that the proposed method enlarges the class of applicable systems compared with the RDOEL approach.

Keywords: nonlinear observer design, nonlinear observer canonical form, observer error linearization, system immersion, dynamic observer error linearization, reduced-order dynamic observer error linearization
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Notation and Acronyms

Notation

$\mathbb{R}$	field of real numbers
$\mathbb{R}^n$	real Euclidean space of dimension n
$\mathbb{R}^{m\times n}$	space of $m \times n$ matrices with real entries
I_n	$n \times n$ identity matrix
O	zero matrix of suitable dimension
$A_{m\times n}$	$m \times n$ matrix A
A^T	transpose of A
$\mathrm{diag}(\alpha_1, \ldots, \alpha_n)$	diagonal matrix with the i-th entry $\alpha_i \in \mathbb{R}$
$\mathrm{diag}(A_1, \ldots, A_n)$	block diagonal matrix with the i-th block $A_i \in \mathbb{R}^{m_i\times n_i}$
$:=$	defined as
δ_{ij}	Kronecker delta
$\binom{m}{n}$	binomial coefficient
$\mathrm{rank}(V)$	rank of V
$\mathrm{card}(S)$	cardinality of S
$\mathrm{span}(S)$	span of S
$\square$	end of proof, definition, theorem, remark, and so on

Acronyms

OEL	observer error linearization (or linearizable)
DOEL	dynamic observer error linearization (or linearizable)
RDOEL	reduced-order dynamic observer error linearization (or linearizable)
NOCF	nonlinear observer canonical form
GNOCF	generalized nonlinear observer canonical form
ENOCF	extended nonlinear observer canonical form

Chapter 1

Introduction

1.1 Research Background

In control theory, a state observer (also called a state estimator) is a dynamic system that provides an estimate of the internal state of a given dynamic system by using available signals which are commonly the input and the output of the given system. Knowing the system state is necessary to solve many problems in control theory, for example, state feedback controller design, fault detection and diagnosis, and so on. For this reason, the observer design problem for linear/nonlinear systems has been an important issue in control theory, and has been applied to various fields of application: robot manipulators, aerial and ground vehicles, electric motors, biological systems, chemical systems, image processing, secure communication, and so on. In the case of linear systems, a standard solution called Luenberger observer was developed in [Lue64]. On the other hand, there has been no unified approach to the case of nonlinear systems, although significant research efforts have been devoted to the problem since its advent [Tha73]. However, there have been varied methodologies for special classes of nonlinear systems, such as observer error linearization [BZ83, KI83, KR85], approximate observer error linearization [BL95, BS97, LB97, LB01, Nam97], high-gain observers [BH91, CMG93, DBGR92, GHO92, GK94, HBB10, SSS01], sliding observers [CS91, SHM86, XS01], observers for Lipschitz nonlinear systems [KE03, Raj98, RC98, ZH02], and so on (surveys of various nonlinear observer design approaches can be found in [MH89, NF99]).

In particular, the concept of observer error linearization (OEL) is to transform a nonlinear system into a nonlinear observer canonical form (NOCF) which is an observable linear system modulo output injection. If a nonlinear system is transformed into the NOCF, then all nonlinearities of the system are restricted to the output injection that is a function of the system input and the system output which are available information. As a result, on the transformed coordinates, we can design a Luenberger-type observer which has linear error dynamics because the nonlinearities can be cancelled out by the output injection. Furthermore, we can arbitrarily assign the eigenvalues of the system matrix of the linear observer error dynamics because the linear part of NOCF is observable. This approach was first introduced in [KI83] and [BZ83] for time-invariant and time-varying single output systems respectively, and has been extended to multi-output systems [BBHB09, HP99, KR85, Phe91, XG89] and discrete-time systems [LAM08, LB95, LN91]. Meanwhile, the author of [Kel87] developed a characteristic equation approach which is a different characterization of OEL compared with the original work [KI83]. In addition, since the result of [KI83] is based on coordinate transformation that is a diffeomorphism, in order to relax the condition, the authors of [XZ97] investigated the possibility of taking coordinate transformation as a smooth map with continuous inverse (which is called a semi-diffeomorphism). Besides the above works, many studies have been conducted on the OEL problem, such as introducing generalized output injections depending on time derivatives of system input and/or system output [DGMS94, GMP96, Kel87, LPG99, PG97], employing output transformations [BBHB09, GMP96, KR85] and/or output-dependent time-scale transformations [Gua01, Gua02, Gua05, RPN01, RPN04, WL10], developing constructive algorithms not only to check the possibility of transforming a given system into NOCF but also to design the transformation via a straightforward procedure [BBHB09, BL95, BS97, GMP96, Gua02, Gua05, PG97], designing nonlinear adaptive observers based on NOCF [Mar90, MT92a, MT92b], and so on.

In order to extend the class of systems to which the OEL method can be applied, there have been attempts to immerse a nonlinear system into a higher-dimensional NOCF. Since the first contribution to the system immersion technique

was made in [LM86], the concept has been refined in [BS02, Jou03] and some constructive algorithms to solve the problem have been developed in [BS04, BS06]. Furthermore, inspired by system immersion and dynamic feedback linearization [CLM89, CLM91], the concept of dynamic observer error linearization (DOEL) was first proposed in [NJS04] and generalized in [BYS06]. The main idea of DOEL is twofold:

- The one is to introduce an auxiliary dynamics of which input is the output of a given system.
- The other is to transform the extended system, consisting of the given system and the auxiliary dynamics, into a generalized nonlinear observer canonical form (GNOCF), which is an observable linear system modulo generalized output injection depending not only on the system output but also on the state of auxiliary dynamics, via a coordinate transformation that is a diffeomorphism on the state of the extended system.

In a similar fashion to the (conventional) OEL approach, if there exists an auxiliary dynamics for a given system such that the extended system can be transformed into the GNOCF (i.e. if the given system is dynamic observer error linearizable (DOEL)), then it is also possible to construct a Luenberger-type observer which has linear error dynamics. Moreover, by introducing such an auxiliary dynamics, DOEL is applicable to a class of systems not covered by OEL. Furthermore, in the case of single output systems, one of the results in [BYS06] showed that the concept of DOEL strictly covers that of system immersion. That is to say, if an n-dimensional system is immersible into an $(n + d)$-dimensional NOCF, then it is also DOEL via a d-dimensional auxiliary dynamics, however, the converse is not true. As regards the DOEL problem, the works [BB09, YJS06] made some contributions to multi-output case and there also have been researches on developing constructive algorithms to solve the problem [BB09, YBS07].

As mentioned above, DOEL has an advantage over OEL such that it can be applied to a larger class of systems. However, it also has a drawback such that the dimension of observer is larger than that of a given system because the dimension of GNOCF is equal to that of the extended system composed of the given system

and its auxiliary dynamics. In fact, this implies that the observer estimates not only the state of the given system, which is what we want to estimate, but also the state of auxiliary dynamics, which is already known. Recently, motivated by this fact, the authors of [BB11, YBSS10] proposed a new observer design scheme called reduced-order dynamic observer error linearization (RDOEL) for single output systems, which is a modification of DOEL as well as a natural extension of OEL. Compared with DOEL, RDOEL shares the same idea of introducing such an auxiliary dynamics to a given system and transforming the extended system into GNOCF. In the framework of RDOEL, however, the coordinate transformation preserves a part of coordinates, which corresponds to the state of auxiliary dynamics, so that the extended system is transformed into the system composed of the auxiliary dynamics intact and GNOCF of which dimension is equal to that of the given system. As a result, RDOEL offers a lower-dimensional observer than DOEL, though RDOEL is a special case of DOEL (that is, the class of applicable systems of RDOEL is included in that of DOEL). Moreover, RDOEL also can be applied to a class of systems not covered by OEL due to employing auxiliary dynamics, and most of results for the RDOEL problem can be used to analyze the OEL problem by slight modification because the framework of RDOEL is quite coterminous with that of OEL (they are identical when auxiliary dynamics is not considered; NOCF by OEL and GNOCF by RDOEL have the same dimensions and similar structures, even if auxiliary dynamics is considered). For the RDOEL problem, a complete solution to a special case was derived in [YBS11] and the concept has been extended to discrete-time single output systems [YYS12, YYS13]. However, there has so far been no work dealing with the problem for multi-output systems.

This dissertation deals with two topics in regard to RDOEL. One is to extend the concept of RDOEL to multi-output systems. The other is to propose a new extended NOCF (ENOCF), of which not only output injection but also linear part depend on system output and state of auxiliary dynamics, and then to study the problem of transforming a single output nonlinear system with its auxiliary dynamics into the proposed ENOCF, which is a natural extension of the RDOEL problem for single output systems.

1.2 Organization and Contributions of the Dissertation

The following outlines this dissertation and summarizes the contributions of each individual chapter.

Chapter 2. Mathematical Preliminaries

As a preliminary of the dissertation, we recall some notions in differential geometry and important mathematical tools on them, such as manifolds, vector fields, differential 1-forms, Lie derivatives, Lie brackets, Inverse Function Theorem, Frobenius Theorem, Simultaneous Rectification Theorem, and so on.

Chapter 3. Review of Related Previous Works

In this chapter, we review some established results on observer error linearization [BBHB09, Kel87, KI83, KR85, XG89] and its extensions: system immersion [BS04], dynamic observer error linearization [BYS06, NJS04], and reduced-order dynamic observer error linearization [Yan11] for single output systems, which are closely related to the research in this dissertation.

Chapter 4. Reduced-Order Dynamic Observer Error Linearization (RDOEL) for Multi-output Systems

This chapter defines and deals with the RDOEL problem for multi-output systems. Most of the chapter is based on [CYS12, CYS14b] and the contributions of the chapter are summarized as follows.

- The concept of RDOEL is first extended to multi-output systems. We provide three necessary conditions for the RDOEL problem. Two of them partially identify the class of applicable systems, and the other one presents a condition on output transformation needed to solve the problem. Furthermore, we fully characterize the problem by deriving a geometric necessary and sufficient condition from the necessary conditions.

- From the necessary and sufficient condition for the RDOEL problem, we first establish a geometric necessary and sufficient condition for the OEL problem in the case where output transformation of general form is considered. Moreover, we show by an example that the general output transformation allows us to solve the OEL problem for a class of systems not

covered by previous results considering output transformations with some restrictions.

- Based on the necessary and sufficient conditions for the OEL and RDOEL problems, we develop a procedure to check the solvability and to construct explicit change of coordinates for the problems.

Chapter 5. Extension of RDOEL: System into Extended Nonlinear Observer Canonical Form (ENOCF)

In this chapter, we propose a new NOCF called extended nonlinear observer canonical form (ENOCF) of which not only output injection but also linear part depend on system output and state of auxiliary dynamics. As a natural extension of the RDOEL problem, we address and deal with the problem (called ENOCF problem) of transforming a single output nonlinear system into the ENOCF via an auxiliary dynamics. Since the ENOCF admits a kind of high-gain observer, the solvability of the problem allows us to design observers for a class of nonlinear systems. This chapter is based on [CYS14a] and the contributions of the chapter are listed as follows.

- For the ENOCF problem, we also give two necessary conditions that can partially identify the class of systems for which the problem is solvable, and then establish a geometric necessary and sufficient condition.

- As a case study, we transform the Rössler system into the proposed ENOCF via an auxiliary dynamics, and design an observer for the system by using a high-gain observer design method. This example illustrates that the ENOCF problem can be solved for a larger class of systems compared to the RDOEL problem.

Chapter 6. Conclusions

This chapter concludes this dissertation with some concluding remarks and further issues for future research.

Chapter 2

Mathematical Preliminaries

This chapter provides some brief mathematical background on differential geometry. For a full understanding of the chapter, the reader is referred to the books [Boo75, Mun00, Spi99, War71].

2.1 Manifolds and Differentiable Structures

First, we introduce the notion of manifold and differentiable structures on manifolds. To do this, we need the concepts of topology and topological space.

Definition 2.1.1 (Topology and Topological space)**.** A *topology* on a set M is a collection $\mathcal{T}$ of subsets of M, which are called *open* sets satisfying the following three axioms:

(a) The empty set and M itself are open.

(b) The union of any number of open sets is open.

(c) The intersection of any finite number of open sets is open.

A set M together with a topology $\mathcal{T}$ on M is called a *topological space.* □

A *basis* of a topology $\mathcal{T}$ on M is a subcollection $\mathcal{B} \subset \mathcal{T}$ such that every open subset of M can be represented as a union of elements of $\mathcal{B}$. A topological space is said to be *second countable* if its topology has a countable basis. A *neighborhood* of a point $p \in M$ is an open subset of M containing p. A *Hausdorff space* is a topological space in which any two distinct points have disjoint neighborhoods.

Let M_1 and M_2 be topological spaces. A map $\Phi : M_1 \to M_2$ is said to be *continuous* if the inverse image of any open subset of M_2 under Φ is also an open subset of M_1. The map Φ is called a *homeomorphism*, if it is bijective and both Φ and Φ^{-1} are continuous. If there exists a homeomorphism from M_1 onto M_2, then M_1 is said to be *homeomorphic* to M_2. Furthermore, if M_1 is homeomorphic to M_2, then M_2 is homeomorphic to M_1 also because $\Phi^{-1} : M_2 \to M_1$ is clearly a homeomorphism when $\Phi : M_1 \to M_2$ is a homeomorphism.

Definition 2.1.2 (Topological manifold)**.** A second countable Hausdorff space M is called a (*topological*) *manifold of dimension* n if every point in M has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$. □

Definition 2.1.3 (Coordinate chart (Local coordinate system))**.** For a topological manifold M of dimension n, a *coordinate chart* (also called a *local coordinate system*) of M is a pair (U, x), where U is an open subset of M and x is a homeomorphism from U onto an open subset of $\mathbb{R}^n$. The homeomorphism x is called a *coordinate map* on U. Let $x = (x_1, \ldots, x_n)$ with $x_i : U \to \mathbb{R}$ for $1 \leq i \leq n$. Then, the function x_i is called the *i-th coordinate function* of the coordinate map x, and the n-tuple of real numbers $(x_1(p), \ldots, x_n(p))$ for a point $p \in U$ is called the *local coordinates* of p in the local coordinate system (U, x). □

Let (U, x) and (V, z) be coordinate charts of M with $U \cap V \neq \varnothing$. Then, the homeomorphism $z \circ x^{-1} : x(U \cap V) \to z(U \cap V)$ is called a *coordinate transformation* from x to z on $U \cap V$. Two coordinate charts (U, x) and (V, z) are said to be *C^∞-related* or *C^∞-compatible* if both the maps $z \circ x^{-1} : x(U \cap V) \to z(U \cap V)$ and $x \circ z^{-1} : z(U \cap V) \to x(U \cap V)$ are C^∞ (that is to say, each component function of the maps has continuous partial derivatives of all orders; sometimes we will use the words '*smooth*' or '*differentiable*' to mean 'C^∞'). A collection $\mathcal{A} = \{(U^i, x^i) : i \in I\}$ (I is an index set) of mutually C^∞-related coordinate charts of M with $\bigcup_{i \in I} U^i = M$ is called an *atlas* for M.

Lemma 2.1.1. If $\mathcal{A}$ is an atlas for M, then $\mathcal{A}$ is contained in a unique maximal atlas for M. □

From the above lemma, we can define the notion of smooth manifold.

Definition 2.1.4 (Smooth manifold)**.** A topological manifold M together with a maximal atlas for M is called a *smooth manifold*. □

Now, let us consider the differentiability of a map between smooth manifolds. Let M_1 and M_2 be smooth manifolds. A map $\Phi : M_1 \to M_2$ is said to be *smooth* if, for each $p \in M_1$, there exist two coordinate charts (U, x) and (V, z) on M_1 and M_2, respectively, such that $p \in U$, $\Phi(p) \in V$, and the representation of Φ in the local coordinate systems is smooth.

Definition 2.1.5 (Diffeomorphism)**.** Let M_1 and M_2 be smooth manifolds of the same dimension. A map $\Phi : M_1 \to M_2$ is called a *diffeomorphism*, if it is bijective and both Φ and Φ^{-1} are smooth. If there exits a diffeomorphism from M_1 onto M_2, then M_1 is said to be *diffeomorphic* to M_2. □

Remark 2.1.1. In a similar fashion to the case of homeomorphisms, if M_1 is diffeomorphic to M_2, then M_2 is trivially diffeomorphic to M_1 also. □

Let M_1 and M_2 be smooth manifolds of dimensions n_1 and n_2, respectively. For a smooth map $\Phi : M_1 \to M_2$, the *rank* of Φ at a point $p \in M_1$ is defined as the rank of the Jacobian matrix $\frac{\partial \Phi}{\partial x}(p) \in \mathbb{R}^{n_2 \times n_1}$ and denoted by $\mathrm{rank}(\Phi(p))$, where x is a coordinate map on a neighborhood of p. The rank of Φ does not depend on the choice of coordinate map. By using the notion of rank of a smooth map, the following theorem gives a useful method to check whether a given map is a diffeomorphism or not.

Theorem 2.1.2. Let M_1 and M_2 be smooth manifolds of the same dimension n. A map $\Phi : M_1 \to M_2$ is a diffeomorphism if and only if Φ is a smooth bijective map and $\mathrm{rank}(\Phi(p)) = n$ for all $p \in M_1$. □

The next theorem (known as the 'Inverse Function Theorem') is also a convenient tool to determine whether a map (defined on an open subset of $\mathbb{R}^n$) is a diffeomorphism or not in a local sense.

Theorem 2.1.3 (Inverse Function)**.** Let U be an open subset of $\mathbb{R}^n$ and $\Phi : U \to \mathbb{R}^n$ be a smooth map. If $\frac{\partial \Phi}{\partial x}(p)$ is nonsingular for a point $p \in U$ (i.e. $\mathrm{rank}(\Phi(p)) = n$), then there exists a neighborhood $V \subset U$ of p such that $\Phi|_V : V \to \Phi(V)$ is a diffeomorphism, where $\Phi|_V$ denotes the restriction of Φ to V. □

Lastly, we end this section by introducing the notion of submanifold.

Definition 2.1.6 (Submanifold)**.** Let M be a topological manifold of dimension n and P be a subset of M. For each $p \in P$ and a positive integer $k \leq n$, if there exists a coordinate chart $(U, x) = (U, (x_1, \ldots, x_n))$ of M, where U is a neighborhood of p, such that

$$P \cap U = \{q \in U : x_i(q) = x_i(p),\ i = k+1, \ldots, n\}, \tag{2.1.1}$$

then P is called a *k-dimensional submanifold* of M. □

For the manifold M and its k-dimensional submanifold P, let $\mathcal{T}$ be a topology on M and $\mathcal{T}_P := \{P \cap U : U \in \mathcal{T}\}$. Then, $\mathcal{T}_P$ becomes a topology on P, and thus P together with $\mathcal{T}_P$ is a topological space. Furthermore, if (U, x) is a coordinate chart of M satisfying the condition (2.1.1), then $(P \cap U, x|_{P \cap U})$ is also a coordinate chart of P. Therefore, the submanifold P itself is a manifold of dimension k.

2.2 Vector Fields and Covector Fields

Throughout the rest of this chapter, M is a smooth manifold of dimension n unless otherwise noted and, for a point $p \in M$, $C^\infty(p)$ denotes the set of all smooth real-valued functions that can be defined on a neighborhood of p.

Definition 2.2.1 (Tangent vector and Tangent space)**.** A *tangent vector* v_p to M at a point $p \in M$ is a linear derivation from $C^\infty(p)$ into $\mathbb{R}$. In other words, for all $\phi, \psi \in C^\infty(p)$ and $\alpha, \beta \in \mathbb{R}$, it holds that

(a) $v_p(\alpha\phi + \beta\psi) = \alpha v_p(\phi) + \beta v_p(\psi)$.

(b) $v_p(\phi \cdot \psi) = \phi(p) v_p(\psi) + \psi(p) v_p(\phi)$.

The *tangent space* to M at $p \in M$ is the set of all tangent vectors to M at p and denoted by T_pM. □

Remark 2.2.1. We can observe that the tangent space T_pM is a vector space over the field $\mathbb{R}$ with the vector addition and the scalar multiplication defined by

$$(v_p + w_p)(\phi) := v_p(\phi) + w_p(\phi),$$
$$(\alpha v_p)(\phi) := \alpha v_p(\phi),$$

where $v_p, w_p \in T_pM$, $\phi \in C^\infty(p)$, and $\alpha \in \mathbb{R}$. Moreover, the dimension of T_pM is equal to that of M. □

Based on the concept of tangent space, a vector field on a smooth manifold is defined as follows.

Definition 2.2.2 (Vector field)**.** A *vector field* f on M is a map assigning an element of T_pM to each $p \in M$. The vector field f is said to be *smooth* if, for each $p \in M$, there exist a coordinate chart $(U, (x_1, \ldots, x_n))$ of M and n smooth real-valued functions $f_1, \ldots, f_n$ on U such that

$$f(q) = \sum_{i=1}^{n} f_i(q) \Big(\frac{\partial}{\partial x_i}\Big)_q \qquad \text{for all } q \in U,$$

where U is a neighborhood of p and $(\frac{\partial}{\partial x_i})_q$'s are the tangent vectors to M at q such that $(\frac{\partial}{\partial x_i})_q(x_j) = \delta_{ij}$ for $i, j = 1, \ldots, n$. □

By the above definition, on a fixed coordinate chart $(U, (x_1, \ldots, x_n))$ of M, the representation of a smooth vector field f on M in the local coordinate system is usually written as

$$f(x) = \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i} \tag{2.2.1}$$

or as the column vector

$$f(x) = [f_1(x) \ \cdots \ f_n(x)]^T, \tag{2.2.2}$$

with some smooth real-valued functions f_i's determined by the choice of coordinate chart. It is worth pointing out that the notion of vector field defined above makes it possible to introduce the concept of differential equation on a smooth

manifold. More precisely, we can associate a vector field f with a differential equation $\dot{x} = f(x)$ on a smooth manifold, which is called a *dynamic system* in control theory.

Since a tangent space is a vector space over the field $\mathbb{R}$ as mentioned in Remark 2.2.1, we can define the dual objectives to a tangent space and a vector field, which are called a cotangent space and a covector field, respectively.

Definition 2.2.3 (Cotangent space and Tangent covector (Differential 1-form))**.** The *cotangent space* to M at a point $p \in M$ is the dual space of T_pM and denoted by T_p^*M. An element of the cotangent space T_p^*M is called a *tangent covector* to M at p or a *differential 1-form*. □

Definition 2.2.4 (Covector field (Differential 1-form))**.** A *covector field* (also called a *differential 1-form*) ω on M is a map assigning an element of T_p^*M to each $p \in M$. The covector field ω is said to be *smooth* if, for each $p \in M$, there exist a coordinate chart $(U, (x_1, \ldots, x_n))$ of M and n smooth real-valued functions $\omega_1, \ldots, \omega_n$ on U such that

$$\omega(q) = \sum_{i=1}^{n} \omega_i(q)(\mathrm{d}x_i)_q \qquad \text{for all } q \in U,$$

where U is a neighborhood of p and $(\mathrm{d}x_i)_q$ is the tangent covector to M at q that is dual to $(\frac{\partial}{\partial x_i})_q$ for $1 \leq i \leq n$ and $q \in U$. □

In a similar fashion to the equations (2.2.1) and (2.2.2), the representation of a smooth covector field ω in a local coordinate system can be expressed as

$$\omega(x) = \sum_{i=1}^{n} \omega_i(x)\mathrm{d}x_i$$

or as the row vector

$$\omega(x) = [\omega_1(x) \ \cdots \ \omega_n(x)],$$

with some smooth real-valued functions ω_i's that are also dependent on the choice of local coordinate system. Furthermore, for any smooth function $\phi : M \to \mathbb{R}$,

we can associate ϕ with a covector field $\mathrm{d}\phi$ on M by taking the cotangent vector $(\mathrm{d}\phi)_p$ for each $p \in M$. In fact, the representation of $\mathrm{d}\phi$ in a local coordinate system is given by

$$\mathrm{d}\phi = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} \mathrm{d}x_i, \tag{2.2.3}$$

which is often called the *exterior differentiation* of ϕ. In general, however, the converse does not hold. That is to say, for a covector field ω on M, it is not true that there always exists a smooth function $\phi : M \to \mathbb{R}$ satisfying $\omega = \mathrm{d}\phi$. A covector field ω on M is said to be *exact* if there exists a smooth function $\phi : M \to \mathbb{R}$ such that $\omega = \mathrm{d}\phi$.

2.3 Lie Derivatives and Lie Brackets

In this section, we recall several operators on vector fields and/or covector fields, and then present their basic properties that will be frequently used throughout the dissertation.

Definition 2.3.1. For a smooth covector field ω on M and a smooth vector field f on M, we define a smooth function $\langle \omega, f \rangle : M \to \mathbb{R}$ as

$$\langle \omega, f \rangle(p) := \omega(f(p))$$

for each $p \in M$. □

Let $\omega(x) = [\omega_1(x) \ \cdots \ \omega_n(x)]$ and $f(x) = [f_1(x) \ \cdots \ f_n(x)]^T$ be the representations of ω and f in a local coordinate system, respectively. Then, in the local coordinate system, $\langle \omega, f \rangle$ is written as

$$\langle \omega, f \rangle(x) = \sum_{i=1}^{n} \omega_i(x) f_i(x). \tag{2.3.1}$$

As we can see, the operator $\langle \cdot, \cdot \rangle$ acts like the inner product in linear algebra, when we regard ω and f as a row vector and a column vector, respectively.

Definition 2.3.2 (Lie derivative)**.** For a smooth vector field f on M and a smooth real-valued function ϕ on M, the *Lie derivative* of ϕ along f is a smooth real-valued function on M defined and denoted by

$$(\mathcal{L}_f\phi)(p) := (f(p))(\phi)$$

for each $p \in M$. □

Another equivalent way to define the Lie derivative $\mathcal{L}_f\phi$ is to use the differential 1-form $\mathrm{d}\phi$ as follows:

$$\mathcal{L}_f\phi(p) := \mathrm{d}\phi(f(p)) = \langle \mathrm{d}\phi, f\rangle(p) \qquad \text{for each } p \in M. \tag{2.3.2}$$

Therefore, by the equations (2.2.2), (2.2.3), (2.3.1), and (2.3.2), the representation of $\mathcal{L}_f\phi$ in a local coordinate system can be expressed as

$$\mathcal{L}_f\phi(x) = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} f_i(x).$$

Moreover, since $\mathcal{L}_f\phi$ is a smooth real-valued function on M, the *Lie derivative of order* k (k is a nonnegative integer) can be defined recursively as follows:

$$\mathcal{L}_f^0\phi := \phi, \qquad \mathcal{L}_f^1\phi := \mathcal{L}_f\phi, \qquad \mathcal{L}_f^k\phi := \mathcal{L}_f(\mathcal{L}_f^{k-1}\phi) \quad \text{for } k \geq 2.$$

In a similar way, $\mathcal{L}_g\mathcal{L}_f\phi := \mathcal{L}_g(\mathcal{L}_f\phi)$ when g is another smooth vector field on M.

We can also define the notion of Lie derivative of a smooth covector field ω on M along a smooth vector field f on M. We introduce it briefly as a matrix form in a local coordinate system. Let $\omega(x) = [\omega_1(x) \ \cdots \ \omega_n(x)]$ and $f(x) = [f_1(x) \ \cdots \ f_n(x)]^T$. Then, $\mathcal{L}_f\omega$ is defined by

$$\mathcal{L}_f\omega(x) := f^T(x)\Big(\frac{\partial \omega^T}{\partial x}\Big)^T + w(x)\frac{\partial f}{\partial x}.$$

A more general definition of $\mathcal{L}_f\omega$ can be found in the books [Boo75, Spi99, War71].

Definition 2.3.3 (Lie bracket)**.** For two smooth vector fields f and g on M, the

Lie bracket $[f, g]$ is the vector field on M defined by

$$[f, g]_p(\phi) := (f(p))(\mathcal{L}_g\phi) - (g(p))(\mathcal{L}_f\phi),$$

where $p \in M$, $\phi \in C^\infty(p)$, and $[f, g]_p$ denotes the tangent vector to M at p which is assigned to p by the vector field $[f, g]$. □

If the representations of f and g in a local coordinate system are given by

$$f(x) = \sum_{i=1}^{n} f_i(x)\frac{\partial}{\partial x_i} = [f_1(x) \ \cdots \ f_n(x)]^T,$$
$$g(x) = \sum_{i=1}^{n} g_i(x)\frac{\partial}{\partial x_i} = [g_1(x) \ \cdots \ g_n(x)]^T,$$

then, in the local coordinate system, the Lie bracket $[f, g]$ is written as

$$[f, g]_x = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\frac{\partial g_i}{\partial x_j}\right)f_j(x) - \left(\frac{\partial f_i}{\partial x_j}\right)g_j(x)\right)\frac{\partial}{\partial x_i}$$

or as the vector form

$$[f, g]_x = \frac{\partial g}{\partial x}f(x) - \frac{\partial f}{\partial x}g(x).$$

Since $[f, g]$ is also a vector field on M, we may repeat bracketing of g with f. The following notation is used to simplify this process:

$$ad_f^0 g := g(x), \qquad ad_f g := [f, g], \qquad ad_f^k g := [f, ad_f^{k-1} g] \quad \text{for } k \geq 2.$$

The Lie bracket operator $[\cdot, \cdot]$ has some fundamental properties stated in the next proposition.

Proposition 2.3.1. Let f, g, and h be smooth vector fields on M and $\alpha, \beta \in \mathbb{R}$. Then, we have

(a) Bilinearity over $\mathbb{R}$: $[\alpha f + \beta g, h] = \alpha[f, h] + \beta[g, h]$,
$[f, \alpha g + \beta h] = \alpha[f, g] + \beta[f, h]$.

(b) Anticommutativity: $[f, g] = -[g, f]$.

(c) Jacobi identity: $[[f,g],h]+[[g,h],f]+[[h,f],g]=0$. □

Remark 2.3.1. In fact, a vector space L over the field $\mathbb{R}$ together with a operator $[\cdot,\cdot]$ satisfying the above three properties is called a *Lie algebra* L over $\mathbb{R}$. □

From the definitions of Lie derivative and Lie bracket, we can easily deduce the following facts.

Proposition 2.3.2. Let ϕ and ψ be smooth real-valued functions on M, f and g be smooth vector fields on M, and ω be a smooth covector field on M. Then, it holds that

(a) $\mathcal{L}_f(\phi\psi)=(\mathcal{L}_f\phi)\psi+\phi\mathcal{L}_f\psi$.

(b) $\mathcal{L}_f\langle\omega,g\rangle=\langle\mathcal{L}_f\omega,g\rangle+\langle\omega,[f,g]\rangle$.

(c) $\mathcal{L}_f(\mathrm{d}\phi)=\mathrm{d}\mathcal{L}_f\phi$.

(d) $\mathcal{L}_{(\psi f)}\phi=\psi\mathcal{L}_f\phi$.

(e) $\mathcal{L}_{[f,g]}\phi=\mathcal{L}_f\mathcal{L}_g\phi-\mathcal{L}_g\mathcal{L}_f\phi$.

(f) $\mathcal{L}_{(\psi f)}(\phi\omega)=\psi\mathcal{L}_f(\phi)\omega+\phi(\psi\mathcal{L}_f\omega+\langle\omega,f\rangle\mathrm{d}\psi)$.

(g) $[\phi f,\psi g]=\phi\psi[f,g]+\phi\mathcal{L}_f(\psi)g-\psi\mathcal{L}_g(\phi)f$. □

2.4 Distributions and Codistributions

In this section, we introduce the notions of distribution and codistribution, and review several results to construct a new local coordinate system or a part of it from a set of given vector fields. The results play an important role in solving our problems that will be addressed in Section 4.1 and Section 5.1.

Definition 2.4.1 (Distribution)**.** A *distribution* D on M is a map that assigns a subspace of the tangent space T_pM to each $p\in M$. The distribution D is said to be *smooth* if, for each $p\in M$, there exist a neighborhood U of p and a set $\{X_i : i\in I\}$ (I is an index set) of smooth vector fields on U such that

$$D(q)=\operatorname{span}\{X_i(q) : i\in I\} \qquad \text{for all } q\in U.$$

For a distribution D on M, the *dimension* (or *rank*) of D at a point $p \in M$ is the dimension of $D(p)$. Moreover, the distribution D is said to be *constant dimensional* if the dimension of $D(p)$ is constant on M. □

A vector field f on M is said to *lie in* or *belong to* a distribution D on M if $f(p) \in D(p)$ for all $p \in M$. In this case, we denote it by $f \in D$. For a constant dimensional distribution, the following lemma provides a concept similar to the notion of basis for a vector space in linear algebra.

Lemma 2.4.1. Let D be a constant k-dimensional distribution on M. Then, for each $p \in M$, there exist a neighborhood U of p and k vector fields $X_1, \ldots, X_k$ on U such that $D(q) = \operatorname{span}\{X_1(q), \ldots, X_k(q)\}$ for all $q \in U$. □

Remark 2.4.1. The above k vector fields $X_1, \ldots, X_k$ are called *local generators* on U of the distribution D in the following sense: any vector field $f \in D$ can be expressed on U as a linear combination of X_i's such that $f = \sum_{i=1}^{k} \phi_i X_i$ with some real-valued functions ϕ_i's on U. □

Next, we introduce two kinds of special classes of distributions.

Definition 2.4.2 (Involutive distribution)**.** A distribution D is said to be *involutive* if $[f, g] \in D$ whenever $f, g \in D$. □

Definition 2.4.3 (Integral manifold and Integrable distribution)**.** A submanifold N of M is an *integral manifold* of a distribution D on M if

$$T_q N = D(q) \qquad \text{for all } q \in N.$$

A distribution D on M is said to be *integrable* if, for any $p \in M$, there exists an integral manifold of D containing p. □

If a distribution D is integrable, then it is involutive. In general, however, the converse is not true. The following celebrated theorem of Frobenius gives an additional condition which an involutive distribution should satisfy in order for the distribution to be integrable.

Theorem 2.4.2 (Frobenius)**.** A constant dimensional distribution D on M is integrable if and only if it is involutive. □

The next theorem is another version of the Frobenius Theorem.

Theorem 2.4.3. For a distribution D on M, the following statements are equivalent:

(a) The distribution D is involutive.

(b) For a fixed nonnegative integer $k \leq n$ and each $p \in M$, there exists a coordinate chart $(U, (x_1, \ldots, x_n))$ with a neighborhood U of p such that

$$D(q) = \operatorname{span}\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\} \qquad \text{for all } q \in U.$$

(c) For a fixed nonnegative integer $k \leq n$ and each $p \in M$, there exists a coordinate chart $(U, (x_1, \ldots, x_n))$ with a neighborhood U of p such that

$$\langle \mathrm{d}x_i, f \rangle(q) = 0$$

for all $q \in U$, $f \in \mathcal{D}$, and $k + 1 \leq i \leq n$. □

For a set of given vector fields, the involutivity of the distribution obtained by spanning it is a necessary condition to construct an entire local coordinate system or a part of it by using those vector fields. However, we need some stronger conditions than the involutivity for a sufficient condition. The following theorems and corollary state about them.

Theorem 2.4.4 (Flow-box)**.** Let X be a smooth vector field on M such that $X(p) \neq 0$ for a point $p \in M$. Then, there exists a coordinate chart $(U, (x_1, \ldots, x_n))$ with a neighborhood U of p such that $X = \frac{\partial}{\partial x_1}$ on U. □

Theorem 2.4.5 (Theorem 2.36 in [NvdS90], Simultaneous Rectification)**.** Let $X_1, \ldots, X_n$ be smooth vector fields on $\mathbb{R}^n$, which are linearly independent at a point $p \in \mathbb{R}^n$. Then, there exists a coordinate chart $(U, (x_1, \ldots, x_n))$ with a neighborhood U of p such that

$$X_i = \frac{\partial}{\partial x_i} \quad \text{on } U \qquad \text{for } 1 \leq i \leq n,$$

if and only if $[X_i, X_j] = 0$ on U for $i, j = 1, \ldots, n$. □

Corollary 2.4.6 ([JS02])**.** Let $X_1, \ldots, X_k$ with $k \leq n$ be smooth vector fields on $\mathbb{R}^n$ such that they are linearly independent at a point $p \in \mathbb{R}^n$ and, on a neighborhood U of p,

$$[X_i, X_j] = 0 \qquad \text{for } i, j = 1, \ldots, k.$$

Then, there exist $(n-k)$ smooth vector fields $X_{k+1}, \ldots, X_n$ such that $X_1(p), \ldots, X_n(p)$ are linearly independent and $[X_i, X_j] = 0$ on U for $i, j = 1, \ldots, n$. □

Finally, we introduce the notion of codistribution that is a dual objective to distribution.

Definition 2.4.4 (Codistribution)**.** A *codistribution* Ω on M is a map that assigns a subspace of the cotangent space T_p^*M to each $p \in M$. The codistribution Ω is said to be *smooth* if, for each $p \in M$, there exist a neighborhood U of p and a set $\{\theta_i : i \in I\}$ (I is an index set) of smooth covector fields on U such that

$$\Omega(q) = \operatorname{span}\{\theta_i(q) : i \in I\} \qquad \text{for all } q \in U.$$

The *dimension* (or *rank*) of Ω at a point $p \in M$ is the dimension of $\Omega(p)$, and the codistribution Ω is said to be *constant dimensional* if the dimension of $\Omega(p)$ is constant on M. A covector field ω on M is said to *lie in* or *belong to* Ω if $\omega(p) \in \Omega(p)$ for all $p \in M$. In this case, we denote it by $\omega \in \Omega$. □

Lemma 2.4.7. Let Ω be a constant k-dimensional codistribution on M. Then, for each $p \in M$, there exist a neighborhood U of p and k covector fields $\theta_1, \ldots, \theta_k$ on U such that $\Omega(q) = \operatorname{span}\{\theta_1(q), \ldots, \theta_k(q)\}$ for all $q \in U$. □

Similarly to the local generators of a constant dimensional distribution, the above k covector fields $\theta_1, \ldots, \theta_k$ are called *local generators* of the codistribution Ω on U. That is to say, any covector field $\omega \in \Omega$ can be expressed on U as a linear combination of θ_i's such that $\omega = \sum_{i=1}^k \psi_i \theta_i$ with some real-valued functions ψ_i's on U.

Chapter 3

Review of Related Previous Works

In this chapter, we review some established results on observer error linearization (OEL) and its extensions: system immersion, dynamic observer error linearization (DOEL), and reduced-order dynamic observer error linearization (RDOEL) (particularly for single output systems), which are closely related to the topics that will be studied in this dissertation.

3.1 Observability of Multi-Output Nonlinear Systems

Before we review the previous results, let us recall the notion of observability of multi-output nonlinear systems. Consider a dynamic system given by

$$\begin{aligned} \dot{\xi} &= f(\xi), \qquad \xi \in \mathbb{R}^n, \\ y &= h(\xi), \qquad y \in \mathbb{R}^m, \end{aligned} \tag{3.1.1}$$

where ξ is the state, $y = [y_1 \ \cdots \ y_m]^T$ is the output, $f(\xi)$ is a smooth vector field, and $h(\xi) = [h_1(\xi) \ \cdots \ h_m(\xi)]^T$ is a smooth map. For the multi-output system, observability indices and local observability are defined sequentially as follows.

Definition 3.1.1 ([Isi95, MT95], Observability indices)**.** A set of *observability indices* at $\xi_0 \in \mathbb{R}^n$ of the system (3.1.1) is an m-tuple of nonnegative integers $(r_1(\xi_0), \ldots, r_m(\xi_0))$ such that

$$r_i(\xi_0) := \mathrm{card}\{k : 1 \le k \le n, \ s_k(\xi_0) \ge i\} \qquad \text{for } 1 \le i \le m,$$

with

$$s_1(\xi_0) := \mathrm{rank}(\mathcal{D}_1(\xi_0)),$$
$$s_k(\xi_0) := \mathrm{rank}(\mathcal{D}_k(\xi_0)) - \mathrm{rank}(\mathcal{D}_{k-1}(\xi_0)) \qquad \text{for } 2 \le k \le n,$$

where $\mathrm{card}\{\cdot\}$ denotes the cardinality of a set and $\mathcal{D}_k(\xi_0) := \mathrm{span}\{\mathrm{d}\mathcal{L}_f^{j-1}h_i(\xi_0) : 1 \le i \le m,\ 1 \le j \le k\}$ for $1 \le k \le n$. □

Remark 3.1.1. Let $(r_1(\xi_0), \ldots, r_m(\xi_0))$ be the observability indices at ξ_0 of the system (3.1.1). Then, it holds that $n \ge r_1(\xi_0) \ge r_2(\xi_0) \ge \cdots \ge r_m(\xi_0) \ge 0$ for all $\xi_0 \in \mathbb{R}^n$. This property is often called the *lexographic ordering* of observability indices ([KR85]). □

Definition 3.1.2 ([Isi95, MT95], Local observability)**.** The system (3.1.1) is said to be *locally observable* at $\xi_0 \in \mathbb{R}^n$ if there exists a neighborhood $V_0 \subset \mathbb{R}^n$ of ξ_0 such that

$$\mathrm{rank}\big(\mathrm{span}\{\mathrm{d}\mathcal{L}_f^{j-1}h_i(\xi) : 1 \le i \le m,\ 1 \le j \le r_i(\xi)\}\big) = \sum_{i=1}^{m} r_i(\xi) = n \qquad (3.1.2)$$

for all $\xi \in V_0$ after a suitable reordering of h_i's, where $(r_1(\xi), \ldots, r_m(\xi))$ is the observability indices at ξ of the system (3.1.1). The above equation is called the *observability rank condition.* □

Remark 3.1.2. The local observability at ξ_0 of the system (3.1.1) (defined by the observability rank condition (3.1.2)) implies that the distribution, span of 1-forms from each output component $y_i = h_i(\xi)$ and its time derivatives up to order $r_i(\xi) - 1$, has rank n around ξ_0 (after a suitable reordering h_i's). This is a nonlinear version of that a linear system is observable if its observability matrix has full rank [Che99]. Furthermore, if the observability rank condition is satisfied, then it follows from Theorem 2.1.3 (Inverse Function Theorem) that $\Psi(\xi) := [h_1(\xi)\ \cdots\ \mathcal{L}_f^{r_1(\xi_0)-1}h_1(\xi)\ \cdots\ h_m(\xi)\ \cdots\ \mathcal{L}_f^{r_m(\xi_0)-1}h_m(\xi)]^T$ is a diffeomorphism on a neighborhood of ξ_0. □

If the system (3.1.1) is locally observable at $\xi_0 \in \mathbb{R}^n$ and its observability indices at ξ_0 are given by $(r_1(\xi_0), \ldots, r_m(\xi_0)) = (n_1, \ldots, n_m)$ with some positive

integers n_i's such that $n_1 \geq n_2 \geq \cdots \geq n_m$ and $\sum_{i=1}^{m} n_i = n$, then there exists a coordinate transformation Ψ from ξ to x on a neighborhood $V \subset \mathbb{R}^n$ of ξ_0, which is the diffeomorphism given in Remark 3.1.2, such that the system (3.1.1) can be expressed on V as the following form (called *observable form*):

$$\begin{aligned} \dot{x}_{11} &= x_{12}, & \cdots \quad & \dot{x}_{m1} = x_{m2}, \\ &\vdots & & \vdots \\ \dot{x}_{1(n_1-1)} &= x_{1n_1}, & \cdots \quad & \dot{x}_{m(n_m-1)} = x_{mn_m}, \\ \dot{x}_{1n_1} &= f_1(x), & \cdots \quad & \dot{x}_{mn_m} = f_m(x), \\ y_1 &= x_{11}, & \cdots \quad & y_m = x_{m1}, \end{aligned} \tag{3.1.3}$$

where $x_{ij} = \mathcal{L}_f^{j-1} h_i(\xi)$ for $1 \leq i \leq m$ and $1 \leq j \leq n_i$, $f_i : W \to \mathbb{R}$ is a smooth function for $1 \leq i \leq m$, $x = [x_{11} \ \cdots \ x_{1n_1} \ \cdots \ x_{m1} \ \cdots \ x_{mn_m}]^T \in W$, and $W \subset \mathbb{R}^n$ is a neighborhood of x_0 $(= \Psi(\xi_0))$. Therefore, under the assumption, we can regard the system (3.1.1) around ξ_0 as its observable form (3.1.3), without loss of generality.

3.2 Observer Error Linearization (OEL)

As mentioned in Chapter 1, observer error linearization (OEL) is one of the well-known techniques to design observers for a class of nonlinear systems. The OEL problem is a dual concept to feedback linearization [HS81, JR80] and a formal definition of it can be stated as follows.

Definition 3.2.1 (Observer error linearization (OEL))**.** The system (3.1.1) is said to be *observer error linearizable (OEL)*, if there exist two maps $\Phi : V \to \mathbb{R}^n$, $\xi \mapsto z$ as a coordinate transformation and $q : h(V) \to \mathbb{R}^m$, $y \mapsto y_e$ as an output transformation, which are diffeomorphisms onto their images, such that $z = \Phi(\xi)$ and $y_e = q(y)$ transform the system (3.1.1) into a *nonlinear observer canonical form (NOCF)*,

$$\begin{aligned} \dot{z} &= Az + a(y), & z &\in \mathbb{R}^n, \\ y_e &= q(y) = Cz, & y_e &\in \mathbb{R}^m, \end{aligned} \tag{3.2.1}$$

where $V \subset \mathbb{R}^n$ is a neighborhood of an initial state $\xi(0)$, $y_e = [y_{e1} \ \cdots \ y_{em}]^T$ is a new output, $a(y) = [a_1(y) \ \cdots \ a_n(y)]^T$ is a smooth vector-valued function called *output injection*,

$$A = \mathrm{diag}(A_1, \ldots, A_m) \quad \text{with} \quad A_i = \begin{bmatrix} O & I_{n_i-1} \\ O & O \end{bmatrix} \in \mathbb{R}^{n_i \times n_i} \quad \text{for } 1 \le i \le m,$$

$$C = \mathrm{diag}(C_1, \ldots, C_m) \quad \text{with} \quad C_i = [\ 1 \ 0 \ \cdots \ 0\ ] \in \mathbb{R}^{1 \times n_i} \qquad \text{for } 1 \le i \le m,$$

and n_i's are some positive integers such that $n_1 \ge n_2 \ge \cdots \ge n_m$ and $\sum_{i=1}^m n_i = n$. □

If the system (3.1.1) is OEL, then we can construct an observer such that

$$\begin{aligned} \dot{\hat{z}} &= A\hat{z} + a(y) + L(y_e - C\hat{z}), \\ y_e &= q(y), \qquad \hat{\xi} = \Phi^{-1}(\hat{z}), \end{aligned}$$

which has the following linear error dynamics:

$$\dot{e}_z = (A - LC)e_z, \tag{3.2.2}$$

where $e_z := \hat{z} - z$. Since the pair (A, C) in the NOCF (3.2.1) is observable, we can arbitrarily assign the eigenvalues of the matrix $(A - LC)$ so that the observer error dynamics (3.2.2) is exponentially stable [Che99].

The first contribution to the OEL problem was made in [KI83] and [BZ83] for time-invariant and time-varying single output systems, respectively. We review the result of [KI83].

Theorem 3.2.1 ([KI83])**.** When $m = 1$ and $q(y) = y$, the system (3.1.1) is OEL if and only if both the following conditions are satisfied:

(a) $\mathrm{rank}(\mathrm{span}\{\mathrm{d}h(\xi), \mathrm{d}\mathcal{L}_f h(\xi), \ldots, \mathrm{d}\mathcal{L}_f^{n-1} h(\xi)\}) = n$ on V,

(b) $\left[ad_{(-f)}^{k-1} X,\ ad_{(-f)}^{l-1} X\right] = 0$ on V for $k, l = 1, \ldots, n$,

where $V \subset \mathbb{R}^n$ is a neighborhood of $\xi(0)$, $[\,\cdot,\cdot\,]$ denotes the Lie bracket between

vector fields, and X is a vector field that is a solution of the differential equations,

$$\mathcal{L}_X \mathcal{L}_f^{k-1} h(\xi) = \begin{cases} 0 & \text{if } 1 \leq k \leq n-1, \\ 1 & \text{if } k = n, \end{cases}$$

for $1 \leq k \leq n$. □

Remark 3.2.1. The statement (a) in the above theorem just means the local observability at $\xi(0)$ of the given single output system (i.e. the system (3.1.1) when $m = 1$). The statement (b) presents a geometric condition equivalent for the system to be OEL when $q(y) = y$. □

Theorem 3.2.1 gives a geometric characterization of the OEL problem for single output systems in the case where output transformation is not considered (i.e. $y_e = y$). The following theorem provides an algebraic characterization of the same problem.

Theorem 3.2.2 ([Kel87])**.** When $m = 1$ and $q(y) = y$, the system (3.1.1) is OEL if and only if there exist n real-valued functions $a_1(y), \ldots, a_n(y)$ that constitute a set of solutions of the differential equation,

$$0 = \mathcal{L}_f^{n-1} a_1(y) + \mathcal{L}_f^{n-2} a_2(y) + \cdots + a_n(y), \tag{3.2.3}$$

which is called the *characteristic equation.* □

In the rest of this section, we will review the results of [BBHB09, KR85, XG89], which deal with the OEL problem for multi-output systems by using geometric approaches. At first, we introduce a necessary condition for the problem given in [KR85].

Theorem 3.2.3 ([KR85])**.** The system (3.1.1) is OEL, only if it is locally observable at $\xi(0)$ and has a constant observability indices $(n_1, \ldots, n_m)$ on V_0, where V_0 is a neighborhood of $\xi(0)$ and n_i is the dimension of each block A_i in the NOCF (3.2.1) into which the system (3.1.1) can be transformed. □

From the above theorem, without loss of generality, we can impose the following assumption on the system (3.1.1). The assumption will be valid throughout the rest of this section unless otherwise noted.

Assumption 3.2.1. The system (3.1.1) is locally observable at $\xi(0)$ with constant observability indices $(n_1, \ldots, n_m)$ on a neighborhood V_0 of $\xi(0)$, where n_i's are some positive integers such that $n_1 \geq n_2 \geq \cdots \geq n_m$ and $\sum_{i=1}^{m} n_i = n$. □

By the above assumption, the system (3.1.1) can be expressed on a neighborhood of $\xi(0)$ as the observable form (3.1.3). For the observable form (3.1.3), we introduce some notation.

Definition 3.2.2 ([KR85])**.** We denote by $\mathcal{P}(x)$ the ring of polynomials in x_{ij}'s, where $1 \leq i \leq m$ and $2 \leq j \leq n_i$, with coefficients that are smooth real-valued functions of y. The weighted degree of a monomial $c(y)(x_{i_1 j_1})^{l_1} \cdots (x_{i_r j_r})^{l_r}$ is defined as $\sum_{s=1}^{r}(j_s - 1)l_s$ where $l_1, \ldots, l_r$ are nonnegative integers. The weighted degree of a polynomial in $\mathcal{P}(x)$ is the highest weighted degree of any term in the polynomial. $\mathcal{P}^k(x)$ is the set of all the polynomials in $\mathcal{P}(x)$ of which weighted degree is less than or equal to k. □

As regards the notation, the following theorem provides another necessary condition for the OEL problem, which is related to the system dynamics as the observable form (3.1.3).

Theorem 3.2.4 ([KR85])**.** If the system (3.1.3) is OEL, then $f_i(x)$ belongs to $\mathcal{P}^{n_i}(x)$ for $1 \leq i \leq m$. □

Now, we review the main result of [KR85], which is a sufficient condition for the solvability of the OEL problem for multi-output systems.

Theorem 3.2.5 ([KR85])**.** Let $q(y) = [q_1(y) \ \cdots \ q_m(y)]^T$ be an output transformation and $X_1, \ldots, X_m$ be vector fields that are solutions of the equations,

$$\mathcal{L}_{X_i}\mathcal{L}_f^{k-1} q_j(y) = \delta_{ij} \cdot \delta_{kn_j} \qquad \text{for } i, j = 1, \ldots, m \ \text{ and } \ 1 \leq k \leq n_j. \tag{3.2.4}$$

Then, the system (3.1.1) is OEL if it holds that

$$\left[ad_{(-f)}^{k-1} X_i,\ ad_{(-f)}^{l-1} X_j\right] = 0 \quad \text{on} \ \ V \tag{3.2.5}$$

for $i, j = 1, \ldots, m$, $1 \leq k \leq n_i$, and $1 \leq l \leq n_j$, where $V \subset \mathbb{R}^n$ is a neighborhood of $\xi(0)$. □

In the case of single output systems (i.e. when $m = 1$), the sufficient condition given in the above theorem becomes a necessary and sufficient condition. That is to say, the OEL problem is solvable if and only if $[ad_{(-f)}^{k-1}X_1,\ ad_{(-f)}^{l-1}X_1] = 0$ on V for $k, l = 1, \ldots, n$, where X_1 is a solution of $\mathcal{L}_{X_1}\mathcal{L}_f^{k-1}q_1(y) = \delta_{kn}$ for $1 \le k \le n$. However, the authors of [XG89] showed by a counter example that the necessary part does not hold for multi-output systems. The following theorems (given in [XG89] and [BBHB09], respectively) provide geometric necessary and sufficient conditions of the OEL problem for multi-output systems, in the cases when output transformation is not considered or has a structural restriction, respectively.

Theorem 3.2.6 ([XG89])**.** When $q(y) = y$, the system (3.1.1) is OEL if and only if both the following conditions hold:

(a) if we denote (with a possible reordering of h_i's)

$$D(\xi) := \{\mathrm{d}\mathcal{L}_f^{j-1}h_i(\xi) : 1 \le i \le m,\ 1 \le j \le n_i\},$$
$$D_k(\xi) := \{\mathrm{d}\mathcal{L}_f^{j-1}h_i(\xi) : 1 \le i \le m,\ 1 \le j \le n_k\} - \{\mathrm{d}\mathcal{L}_f^{n_k-1}h_k(\xi)\}$$

for $1 \le k \le m$, then it should be satisfied that

$$\mathrm{span}(D_k(\xi)) = \mathrm{span}(D(\xi) \cap D_k(\xi)) \qquad \text{for } 1 \le k \le m \ \text{ and } \ \text{each } \xi \in V,$$

(b) there exist m vector fields $X_1, \ldots, X_m$ which are solutions of the equations,

$$\mathcal{L}_{X_i}\mathcal{L}_f^{k-1}h_j(\xi) = \delta_{ij} \cdot \delta_{kn_j} \qquad \text{for } i, j = 1, \ldots, m \ \text{ and } \ 1 \le k \le n_i, \quad (3.2.6)$$

and satisfy that

$$\left[ad_{(-f)}^{k-1}X_i,\ ad_{(-f)}^{l-1}X_j\right] = 0 \quad \text{on } V$$

for $i, j = 1, \ldots, m$, $1 \le k \le n_i$, and $1 \le l \le n_j$,

where $V \subset \mathbb{R}^n$ is a neighborhood of $\xi(0)$. □

Theorem 3.2.7 ([BBHB09])**.** Suppose that $q(y) = [q_1(y)\ \cdots\ q_m(y)]^T$ is of the form such as $q_i(y) = q_i(y_1, \ldots, y_i)$ for $1 \le i \le m$. Then, the system (3.1.1) is OEL

if and only if there exist m real-valued functions $\phi_1(y), \ldots, \phi_m(y)$ of the form,

$$\begin{aligned}
&\phi_1(y) = \phi_1(y_1), \\
&\phi_2(y) = \begin{cases} \phi_2(y_1, y_2) & \text{if } n_1 > n_2, \\ \phi_2(y_2) & \text{if } n_1 = n_2, \end{cases} \\
&\phi_i(y) = \begin{cases} \phi_i(y_1, \ldots, y_i) & \text{if } n_{i-1} > n_i, \\ \phi_i(y_1, \ldots, y_{i-2}, y_i) & \text{if } n_{i-1} = n_i, \end{cases} \qquad \text{for } 3 \leq i \leq m,
\end{aligned}$$

and m vector fields $X_1, \ldots, X_m$, which are a set of solutions to the equation (3.2.6), such that

$$\left[ad^{k-1}_{(-f)}(\phi_i X_i),\ ad^{l-1}_{(-f)}(\phi_l X_l)\right] = 0 \quad \text{on } V$$

for $i, j = 1, \ldots, m$, $1 \leq k \leq n_i$, and $1 \leq l \leq n_j$, where $V \subset \mathbb{R}^n$ is a neighborhood of $\xi(0)$. □

To our best knowledge, there has so far been no literature providing a geometric necessary and sufficient condition of the OEL problem for multi-output systems, in the case where the general output transformation $y_e = q(y)$ is considered. In Subsection 4.3.2, we will derive it from a direct consequence of one of our results.

3.3 System Immersion

There exists a class of nonlinear systems that cannot be transformed into observable linear systems but can be immersed into higher-dimensional observable linear systems [LM86]. The definition of immersion in differential geometry is a smooth injective map from a smooth manifold into a higher-dimensional smooth manifold. Immersion of a single output nonlinear system into a higher-dimensional observable linear system was defined similarly in [LM86] and it was refined in [BS02, Jou03, BS04] as immersion of a single output nonlinear system into a higher-dimensional NOCF (consisting of an observable linear system and an out-

put injection $a(y)$). For a single output nonlinear system given by

$$\begin{aligned} \dot{\xi} &= f(\xi), \qquad \xi \in \mathbb{R}^n, \\ y &= h(\xi), \qquad y \in \mathbb{R}, \end{aligned} \tag{3.3.1}$$

where $f(\xi)$ is a smooth vector field and $h(\xi)$ is a smooth real-valued function, immersion of the system into a higher-dimensional NOCF can be defined as follows.

Definition 3.3.1 (System immersion)**.** The system (3.3.1) is said to be *immersible* into an $(n+d)$-dimensional NOCF if there exist two maps $\Phi : V \to \mathbb{R}^{n+d}$ as an *immersion* and $q : h(V) \to \mathbb{R}$ as an output transformation such that $z = \Phi(\xi)$ and $y_e = q(y)$ immerse the system (3.3.1) into an $(n+d)$-dimensional NOCF,

$$\begin{aligned} \dot{z} &= Az + a(y), \qquad z \in \mathbb{R}^{n+d} \\ y_e &= q(y) = Cz, \qquad y_e \in \mathbb{R}, \end{aligned} \tag{3.3.2}$$

and $\Phi(\xi) = [\Phi_1(\xi) \ \cdots \ \Phi_{n+d}(\xi)]^T$ satisfies the following condition:

$$\operatorname{rank}(\operatorname{span}\{\mathrm{d}\Phi_1(\xi), \ldots, \mathrm{d}\Phi_n(\xi)\}) = n \qquad \text{for all } \xi \in V,$$

where d is a positive integer, $V \subset \mathbb{R}^n$ is a neighborhood of an initial state $\xi(0)$,

$$A = \begin{bmatrix} O & I_{d+n-1} \\ O & O \end{bmatrix} \in \mathbb{R}^{(n+d)\times(n+d)}, \qquad C = [\ 1\ 0\ \cdots\ 0\ ] \in \mathbb{R}^{1\times(n+d)},$$

and $a(y) = [a_1(y) \ \ldots \ a_{n+d}(y)]^T$ is *output injection.* □

In a similar fashion to the (conventional) OEL approach, if the system (3.3.1) is immersible into the $(n+d)$-dimensional NOCF (3.3.2), then we can also design an observer such that

$$\begin{aligned} &\dot{\hat{z}} = A\hat{z} + a(y) + L(y_e - C\hat{z}) \in \mathbb{R}^{n+d}, \\ &y_e = q(y), \qquad \hat{\xi} = (\Pi \circ \Phi)^{-1}(\Pi(\hat{z})), \end{aligned}$$

with the linear observer error dynamics,

$$\dot{e}_z = (A - LC)e_z,$$

where $\Pi : \mathbb{R}^{n+d} \to \mathbb{R}^n$, $(z_1, \ldots, z_{n+d}) \mapsto (z_1, \ldots, z_n)$ is a projection and $e_z := \hat{z} - z$. Since the pair (A, C) is observable, we can choose $L \in \mathbb{R}^{(n+d)\times 1}$ such that $(A-LC)$ is Hurwitz.

It was shown that the class of systems which are immersible into higher-dimensional NOCF includes the class of systems which are OEL [BS04]. It was also proved in [BS04] that, if the system (3.3.1) is immersible into the $(n+d)$-dimensional NOCF (3.3.2), then it is also immersible into $(n+d+k)$-dimensional NOCF for any nonnegative integer k. The class of systems that are immersible into the NOCF (3.3.2) can be identified in terms of the characteristic equation given by the following theorem.

Theorem 3.3.1 ([BS04])**.** The system (3.3.1) is immersible into the NOCF (3.3.2) if and only if there exists a set of solutions $q(y), a_1(y), \ldots, a_{n+d}(y)$ of the differential equation,

$$\mathcal{L}_f^{n+d} q(y) = \mathcal{L}_f^{n+d-1} a_1(y) + \mathcal{L}_f^{n+d-2} a_2(y) + \cdots + a_{n+d}(y),$$

subject to the condition $\frac{\partial q(h(\xi))}{\partial h} \neq 0$ on a neighborhood of $\xi(0)$. □

The above theorem provides a necessary and sufficient condition for the system immersion problem. Based on the result, some constructive algorithms to design immersion Φ, which immerses the system (3.3.1) into the NOCF (3.3.2), have been developed [Jou03, BS04].

3.4 Dynamic Observer Error Linearization (DOEL)

As mentioned in Section 3.2, OEL is a dual concept of feedback linearization [HS81, JR80]. Similarly, as a dual problem to dynamic feedback linearization [CLM89, CLM91], a new notion of dynamic observer error linearization (DOEL) was first proposed in [NJS04] and, in the case of single output systems, the frame-

work of DOEL was generalized by [BYS06]. The next definition modifies it to fit multi-output systems.

Definition 3.4.1 (Dynamic observer error linearization (DOEL))**.** The system (3.1.1) is said to be *dynamic observer error linearizable (DOEL)* if there exist a dynamic system (called *auxiliary dynamics*),

$$\begin{aligned} \dot{\eta} &= p(\eta, y), \qquad \eta \in \mathbb{R}^d, \\ y_e &= q(\eta, y), \qquad y_e \in \mathbb{R}^m, \end{aligned} \tag{3.4.1}$$

and a coordinate transformation $\Phi : U \times V \to \mathbb{R}^{d+n}$, $(\eta, \xi) \mapsto z$, which is a diffeomorphism onto its image, such that $z = \Phi(\eta, \xi)$ transforms the *extended system* (composed of the given system (3.1.1) and the auxiliary dynamics (3.4.1)),

$$\begin{bmatrix} \dot{\eta} \\ \dot{\xi} \end{bmatrix} = F(\eta, \xi) := \begin{bmatrix} p(\eta, h(\xi)) \\ f(\xi) \end{bmatrix}, \tag{3.4.2}$$

into a $(d+n)$-dimensional *generalized nonlinear observer canonical form (GNOCF)*,

$$\begin{aligned} \dot{z} &= Az + a(\eta, y), \qquad z \in \mathbb{R}^{d+n}, \\ y_e &= q(\eta, y) = Cz, \qquad y_e \in \mathbb{R}^m, \end{aligned} \tag{3.4.3}$$

where $U \times V \subset \mathbb{R}^d \times \mathbb{R}^n$ is a neighborhood of an initial state $(\eta(0), \xi(0))$,

$$\begin{aligned} A &= \operatorname{diag}(A_1, \ldots, A_m) \quad \text{with} \quad A_i = \begin{bmatrix} O & I_{\bar{n}_i - 1} \\ O & O \end{bmatrix} \in \mathbb{R}^{\bar{n}_i \times \bar{n}_i}, \\ C &= \operatorname{diag}(C_1, \ldots, C_m) \quad \text{with} \quad C_i = [\ 1\ 0\ \ldots\ 0\ ] \in \mathbb{R}^{1 \times \bar{n}_i}, \end{aligned}$$

$a(\eta, y) = [a_1(\eta, y) \ \cdots \ a_{d+n}(\eta, y)]^T$ is a smooth vector-valued function called *generalized output injection*, and $\bar{n}_i$'s are some positive integers such that $\bar{n}_1 \geq \bar{n}_2 \geq \cdots \geq \bar{n}_m$ and $\sum_{i=1}^m \bar{n}_i = d + n$. □

Since the generalized output injection $a(\eta, y)$ depends only on available information (η, y), if the system (3.1.1) is DOEL via an auxiliary dynamics such as (3.4.1), then we can design a Luenberger-type observer (including the auxiliary

dynamics) such that

$$\begin{aligned}
&\dot{\eta} = p(\eta, y) \in \mathbb{R}^d, \\
&\dot{\hat{z}} = A\hat{z} + a(\eta, y) + L(y_e - C\hat{z}) \in \mathbb{R}^{d+n}, \\
&y_e = q(\eta, y), \qquad \hat{\xi} = (\Pi_1 \circ \Phi^{-1})(\eta, \hat{z}),
\end{aligned} \tag{3.4.4}$$

with the exponentially stable linear error dynamics,

$$\dot{e}_z = (A - LC)e_z,$$

where $\Pi_1 : \mathbb{R}^{d+n} \to \mathbb{R}^n$, $(z_1, \ldots, z_{d+n}) \mapsto (z_{d+1}, \ldots, z_{d+n})$ is a projection, $e_z := \hat{z} - z$, and $L \in \mathbb{R}^{(d+n)\times m}$ is chosen so that $(A - LC)$ is Hurwitz.

As mentioned at the beginning of this section, the first contribution to the DOEL problem was established by [NJS04]. Since it is not easy to deal with auxiliary dynamics of the general form (3.4.1), the authors of [NJS04] took account of the cases where an auxiliary dynamics is given as a collection of chains of integrators from each system output or a specific linear system, like early researches on dynamic feedback linearization have usually done [AMP95, GMB97, LKJ00]. More precisely, they assumed that the auxiliary dynamics (3.4.1) is of the following form: for $1 \le i \le m$,

$$\begin{aligned}
&\dot{\eta}_{i1} = -\alpha_i \eta_{i1} + y_i && \text{if } d_i \ge 1, \\
&\dot{\eta}_{ij} = -\alpha_i \eta_{ij} + \eta_{i(j-1)} && \text{for } 2 \le j \le d_i \text{ if } d_i \ge 2, \\
&y_{ei} = q_i(\eta, y) = \begin{cases} y_i & \text{if } d_i = 0, \\ \eta_{id_i} & \text{if } d_i \ge 1, \end{cases}
\end{aligned} \tag{3.4.5}$$

where α_i's are some nonnegative real numbers and d_i's are some nonnegative integers such that $\sum_{i=1}^m d_i = d$, and then they derived sufficient conditions for the system (3.1.1) to be DOEL via the auxiliary dynamics (3.4.5) in some special cases. We will review the results. To this end, we need the notion of extended observability indices of the system (3.1.1) corresponding to $(d_1, \ldots, d_m)$, which denotes the observability indices of the extended system composed of the given system (3.1.1) and the auxiliary dynamics (3.4.5).

Definition 3.4.2 ([NJS04])**.** For an m-tuple of nonnegative integers $(d_1, \ldots, d_m)$ such that $\sum_{i=1}^m d_i = d$, a set of *extended observability indices* at $\xi_0 \in \mathbb{R}^n$ of the system (3.1.1) corresponding to $(d_1, \ldots, d_m)$ is an m-tuple of integers $(\bar{r}_1(\xi_0), \ldots, \bar{r}_m(\xi_0))$ that are uniquely associated to the system (3.1.1) as follows:

$$\bar{r}_i(\xi_0) := \mathrm{card}\{k : 1 \le k \le d+n,\ \bar{s}_k(\xi_0) \ge i\} \qquad \text{for } 1 \le i \le m,$$

with

$$\begin{aligned}
\bar{s}_1(\xi_0) &:= \mathrm{card}\{i : 1 \le i \le m,\ d_i \ge 1\} + \mathrm{rank}\{\mathrm{d}h_i(\xi_0) : 1 \le i \le m,\ d_i = 0\}, \\
\bar{s}_k(\xi_0) &:= \mathrm{card}\{i : 1 \le i \le m,\ d_i \ge k\} \\
&\quad + \mathrm{rank}\{\mathrm{d}h_i(\xi_0), \ldots, \mathrm{d}\mathcal{L}_f^{k-d_i-1} h_i(\xi_0) : 1 \le i \le m,\ d_i \le k-1\} \\
&\quad - \mathrm{rank}\{\mathrm{d}h_i(\xi_0), \ldots, \mathrm{d}\mathcal{L}_f^{k-d_i-2} h_i(\xi_0) : 1 \le i \le m,\ d_i \le k-2\}
\end{aligned}$$

for $2 \le k \le d+n$. □

At first, we consider the auxiliary dynamics (3.4.5) when $\alpha_i = 0$ for all i.

Theorem 3.4.1 ([NJS04])**.** The system (3.1.1) is DOEL via an auxiliary dynamics of the form (3.4.5) with $\alpha_i = 0$ for all $1 \le i \le m$, if there exists an m-tuple of nonnegative integers $(d_1, \ldots, d_m)$ such that $\sum_{i=1}^m d_i = d$ and the following statements hold (after suitable reordering of h_i's):

(a) for all $\xi \in V$, it holds that $(\bar{r}_1(\xi), \ldots, \bar{r}_m(\xi)) = (\bar{n}_1, \ldots, \bar{n}_m)$ and

$$\mathrm{rank}\{\mathrm{d}h_i(\xi), \ldots, \mathrm{d}\mathcal{L}_f^{\bar{n}_i - d_i - 1} h_i(\xi) : 1 \le i \le m\} = n,$$

where $(\bar{r}_1(\xi), \ldots, \bar{r}_m(\xi))$ is the extended observability indices at ξ of the system (3.1.1) corresponding to $(d_1, \ldots, d_m)$ and $\bar{n}_i$'s are some positive integers such that $\bar{n}_1 \ge \bar{n}_2 \ge \cdots \ge n_m$ and $\sum_{i=1}^m \bar{n}_i = d+n$.

(b) for all $\xi \in V$ and $1 \le j \le m$, it holds that

$$\begin{aligned}
&\mathrm{rank}(\{\mathrm{d}\mathcal{L}_f^k h_i(\xi) : 1 \le i \le m,\ i \ne j,\ d_i < \bar{n}_j,\ 0 \le k \le \bar{n}_j - d_i - 1\} \\
&\quad \cup \{\mathrm{d}\mathcal{L}_f^k h_j(\xi) : 0 \le k \le \bar{n}_j - d_j - 2\})
\end{aligned}$$

$$= \operatorname{rank}(\{\mathrm{d}\mathcal{L}_f^k h_i(\xi) : 1 \le i \le m,\ i \ne j,\ d_i < \bar{n}_j,\ 0 \le k \le \min(\bar{n}_i, \bar{n}_j) - d_i - 1\}$$
$$\cup \{\mathrm{d}\mathcal{L}_f^k h_j(\xi) : 0 \le k \le \bar{n}_j - d_j - 2\}),$$

(c) there exist m vector fields $X_1, \ldots, X_m$ on $\mathbb{R}^n$, which constitute a set of solutions of the differential equations,

$$\mathcal{L}_{X_i} \mathcal{L}_f^{k-1} h_j(\xi) = \delta_{ij} \cdot \delta_{k(\bar{n}_j - d_j)} \qquad \text{for } i, j = 1, \ldots, m \ \text{ and } \ 1 \le k \le \bar{n}_i - d_j,$$

and satisfy that

$$\left[R_i^k,\ R_j^l\right] = 0 \quad \text{on} \ \ V$$

for $i, j = 1, \ldots, m$, $0 \le k \le \bar{n}_i - 1$, and $0 \le l \le \bar{n}_j - 1$, where R_i^k's are vector fields on $\mathbb{R}^{d+n}$ defined by

$$R_i^k := \begin{bmatrix} (ad_f^k X_i)^T & \zeta_{i1}^k & \ldots & \zeta_{i1d_1}^k & \ldots & \zeta_{im1}^k & \ldots & \zeta_{imd_m}^k \end{bmatrix}_{1 \times (d+n)}^T$$

with

$$\zeta_{ijl}^k = \begin{cases} 0 & \text{if } k = 0, \\ \mathcal{L}_f \zeta_{ij1}^{k-1} - \mathcal{L}_{(ad_f^{k-1} X_i)} h_j & \text{if } k \ne 0 \text{ and } l = 1, \\ \mathcal{L}_f \zeta_{ijl}^{k-1} - \zeta_{ij(l-1)}^{k-1} & \text{if } k \ne 0 \text{ and } 2 \le l \le k, \\ 0 & \text{if } k < l \le d_j, \end{cases}$$

for $i, j = 1, \ldots, m$, $0 \le k \le \bar{n}_i - 1$, and $1 \le l \le d_j$,

where $V \subset \mathbb{R}^n$ is a neighborhood of $\xi(0)$. □

If $\alpha_i = 0$ for all $1 \le i \le m$, then the auxiliary dynamics (3.4.5) consists of chains of integrators from each system output. Hence, the above theorem gives a sufficient condition for the DOEL problem using such an auxiliary dynamics. The authors of [YJS06] discussed about the number of integrators needed for the same problem. Their conclusion is that the number cannot be bounded by a function depending on dimensions of system state and system output, in contrast to the result of dynamic feedback linearization using integrators [LKJ00], although

DOEL is dual to dynamic feedback linearization.

The following is a similar result to Theorem 3.4.1, in the case when α_i's are positive constants so that the auxiliary dynamics (3.4.5) is a stable linear system.

Corollary 3.4.2 ([NJS04])**.** The system (3.1.1) is DOEL via an auxiliary dynamics of the form (3.4.5) if there exist positive constants α_i's for $1 \le i \le m$ and an m-tuple of nonnegative integers $(d_1, \ldots, d_m)$ such that $\sum_{i=1}^m d_i = d$ and the following statements hold (after suitable reordering of h_i's):

(a) both the conditions (a) and (b) in Theorem 3.4.1 are satisfied,

(b) there exist m vector fields $\bar{X}_1, \ldots, \bar{X}_m$ on $\mathbb{R}^n$, which constitute a set of solutions of the differential equations,

$$\mathcal{L}_{\bar{X}_i} \psi_j^k(\xi) = \delta_{ij} \cdot \delta_{k(\bar{n}_j - d_j)} \qquad \text{for } i, j = 1, \ldots, m \ \text{ and } \ 1 \le k \le \bar{n}_i - d_j$$

and satisfy that

$$\left[\bar{R}_i^k,\ \bar{R}_j^l\right] = 0 \quad \text{on } \ V$$

for $i, j = 1, \ldots, m$, $0 \le k \le \bar{n}_i - 1$, and $0 \le l \le \bar{n}_j - 1$, where

$$\psi_j^k(\xi) := \begin{cases} L_f^{k-1} h_j(\xi) & \text{if } d_j = 0, \\ \sum_{l=0}^{k-1} (-\alpha_j)^{k-l-1} \begin{pmatrix} k + d_j - 2 - l \\ d_j - 1 \end{pmatrix} \mathcal{L}_f^l h_j(\xi) & \text{if } d_j \ge 1, \end{cases}$$

$$\bar{R}_i^k := \begin{bmatrix} (ad_f^k \bar{X}_i)^T & \bar{\zeta}_{i11}^k & \cdots & \bar{\zeta}_{i1d_1}^k & \cdots & \bar{\zeta}_{im1}^k & \cdots & \bar{\zeta}_{imd_m}^k \end{bmatrix}_{1 \times (d+n)}^T$$

with

$$\bar{\zeta}_{ijl}^k = \begin{cases} 0 & \text{if } k = 0, \\ \mathcal{L}_f \bar{\zeta}_{ij1}^{k-1} + \alpha_j \bar{\zeta}_{ij1}^{k-1} - \mathcal{L}_{(ad_f^{k-1} X_i)} h_j & \text{if } k \ne 0 \text{ and } l = 1, \\ \mathcal{L}_f \bar{\zeta}_{ijl}^{k-1} + \alpha_j \bar{\zeta}_{ijl}^{k-1} - \bar{\zeta}_{ij(l-1)}^{k-1} & \text{if } k \ne 0 \text{ and } 2 \le l \le k, \\ 0 & \text{if } k < l \le d_j, \end{cases}$$

for $i, j = 1, \ldots, m$, $0 \le k \le \bar{n}_i - 1$, and $1 \le l \le d_j$, where $V \subset \mathbb{R}^n$ is a neighborhood of $\xi(0)$. □

In the case of single output systems, a necessary and sufficient condition for the DOEL problem using auxiliary dynamics of the general form (3.4.1) was derived in terms of the following characteristic equation.

Theorem 3.4.3 ([BYS06])**.** The single output nonlinear system (3.3.1) is DOEL via the auxiliary dynamics (3.4.1) (with $m = 1$) if and only if it holds that

(a) there exist $(d+n)$ real-valued functions $a_1(\eta, y), a_2(\eta, y), \ldots, a_{d+n}(\eta, y)$ that constitute a set of solutions to the differential equation,

$$\mathcal{L}_F^{d+n} q(\eta, y) = \mathcal{L}_F^{d+n-1} a_1(\eta, y) + \mathcal{L}_F^{d+n-2} a_2(\eta, y) + \cdots + a_{d+n}(\eta, y),$$

(b) the map $\Phi(\eta, \xi) = [\Phi_1(\eta, \xi)\ \Phi_2(\eta, \xi)\ \cdots\ \Phi_{d+n}(\eta, \xi)]^T$ defined by

$$\Phi_i(\eta, \xi) := \mathcal{L}_F^{i-1} q(\eta, h(\xi)) - \sum_{j=1}^{i-1} \mathcal{L}_F^{i-1-j} a_j(\eta, h(\xi)) \qquad \text{for } 1 \le i \le n + d$$

is a diffeomorphism on a neighborhood of $(\eta(0), \xi(0))$. □

Moreover, in the same case, it was revealed by the following theorem that the concept of DOEL covers that of system immersion.

Theorem 3.4.4 ([BYS06])**.** If the single output system (3.3.1) is immersible into an $(n+d)$-dimensional NOCF, then it is also DOEL via a d-dimensional auxiliary dynamics such as (3.4.1). □

Lastly, there also have been several attempts to develop a constructive algorithm to solve the DOEL problem using an auxiliary dynamics that is a chain of integrators from system output [Bou07, YBS07] (single output case) or has a lower-triangular structure [BB09] (multi-output case).

3.5 Reduced-Order Dynamic Observer Error Linearization (RDOEL) for Single Output Systems

In the observer (3.4.4) designed by the DOEL approach, the Luenberger-type observer is of dimension $d + n$, while the given system (3.1.1) is of dimension n.

This means that the observer estimates not only the state of the system (3.1.1) but also the state of the auxiliary dynamics (3.4.1), even though the latter is already known. Inspired by the fact, the authors of [BB11, YBSS10] proposed a modified version of DOEL for single output systems, which is often called reduced-order dynamic observer error linearization (RDOEL). The following is a formal definition of it.

Definition 3.5.1 (Reduced-order dynamic observer error linearization (RDOEL) for single output systems)**.** The single output nonlinear system (3.3.1) is said to be *reduced-order dynamic observer error linearizable (RDOEL)* if there exist an *auxiliary dynamics* of the form,

$$\begin{aligned} \dot{\eta} &= p(\eta, y), \qquad \eta \in \mathbb{R}^d, \\ y_e &= q(\eta, y), \qquad y_e \in \mathbb{R}, \end{aligned} \tag{3.5.1}$$

and a coordinate transformation $\Phi : U \times V \to \mathbb{R}^{d+n}$, $(\eta, \xi) \mapsto (w, z)$ with $w = \eta$, which is a diffeomorphism onto its image, such that $z = (\Pi_1 \circ \Phi)(\eta, \xi)$ transforms the *extended system* (consisting of the given system (3.3.1) and the auxiliary dynamics (3.5.1)),

$$\begin{bmatrix} \dot{\eta} \\ \dot{\xi} \end{bmatrix} = F(\eta, \xi) := \begin{bmatrix} p(\eta, h(\xi)) \\ f(\xi) \end{bmatrix}, \tag{3.5.2}$$

into an n-dimensional *generalized nonlinear observer canonical form (GNOCF)*,

$$\begin{aligned} \dot{z} &= Az + a(\eta, y), \qquad z \in \mathbb{R}^n, \\ y_e &= q(\eta, y) = Cz, \qquad y_e \in \mathbb{R}, \end{aligned} \tag{3.5.3}$$

where $U \times V \subset \mathbb{R}^d \times \mathbb{R}^n$ is a neighborhood of an initial state $(\eta(0), \xi(0))$,

$$A = \begin{bmatrix} O & I_{n-1} \\ O & O \end{bmatrix} \in \mathbb{R}^{n \times n}, \qquad C = [\ 1\ 0\ \cdots\ 0\ ] \in \mathbb{R}^{1 \times n},$$

$a(\eta, y) = [a_1(\eta, y)\ \cdots\ a_n(\eta, y)]^T$ is *generalized output injection*, and Π_1 is a projection such that $\Pi_1 : \mathbb{R}^{d+n} \to \mathbb{R}^n$, $(\eta, z) \mapsto z$. □

Remark 3.5.1. Compared with DOEL, RDOEL shares the same idea of adding an auxiliary dynamics such as (3.5.1) on a given system and transforming the augmented system into GNOCF. A significant difference is that, in the framework of RDOEL, the coordinate transformaion $(w, z) = \Phi(\eta, \xi)$ preserves a part of coordinates corresponding to the state of the auxiliary dynamics (i.e. $w = \eta$) so that the extended system (3.5.2) is transformed into the system composed of the auxiliary dynamics (3.5.1) intact and the n-dimensional GNOCF (3.5.3). □

RDOEL has the following advantages over OEL and DOEL. With the aid of auxiliary dynamics, the RDOEL problem can be solved for a class of systems for which the OEL problem is not solvable. Furthermore, RDOEL offers a lower-dimensional observer than DOEL. In actual fact, if the system (3.3.1) is RDOEL via an auxiliary dynamics such as (3.5.2), then we can construct an entire observer including the auxiliary dynamics such that

$$\begin{aligned} &\dot{\eta} = p(\eta, y) \in \mathbb{R}^d, \\ &\dot{\hat{z}} = A\hat{z} + a(\eta, y) + L(y_e - C\hat{z}) \in \mathbb{R}^n, \\ &y_e = q(\eta, y), \qquad \hat{\xi} = (\Pi_1 \circ \Phi^{-1})(\eta, \hat{z}), \end{aligned} \tag{3.5.4}$$

which has the exponentially stable linear error dynamics,

$$\dot{e}_z = (A - LC)e_z,$$

where $e_z := \hat{z} - z$ and $L \in \mathbb{R}^{n\times 1}$ is chosen so that $(A - LC)$ is Hurwitz. As one can see, the dimension of the entire observer (3.5.4) is $d + n$, while that of the observer (3.4.4) designed by the DOEL approach is $2d + n$.

The RDOEL problem for single output systems was fully characterized by the following theorems that provide a geometric necessary and sufficient condition and its algebraic counterpart, respectively.

Theorem 3.5.1 ([Yan11])**.** The system (3.3.1) is RDOEL via the auxiliary dynamics (3.5.1) if and only if both the following conditions are satisfied:

(a) $\frac{\partial q(\eta,y)}{\partial y} \neq 0$ on $U \times h(V)$,

(b) $\left[ad^{k-1}_{(-F)}(\phi X),\ ad^{l-1}_{(-F)}(\phi X)\right] = 0$ on $U \times V$ for $k, l = 1, \ldots, n$,

where $U \times V \in \mathbb{R}^d \times \mathbb{R}^n$ is a neighborhood of $(\eta(0), \xi(0))$, X is a vector field on $\mathbb{R}^{d+n}$ defined by the equations

$$\mathcal{L}_X \mathcal{L}_F^{k-1} h(\xi) = \begin{cases} 0 & \text{if } 1 \leq k \leq n-1, \\ 1 & \text{if } k = n, \end{cases} \qquad \text{for } 1 \leq k \leq n,$$

and ϕ is a real-valued function of η and y such that $\phi(\eta, y) := 1/\frac{\partial q(\eta,y)}{\partial y}$. □

Theorem 3.5.2 ([Yan11])**.** The system (3.3.1) is RDOEL via the auxiliary dynamics (3.5.1) if and only if there exist n real-valued functions $a_1(\eta, y), \ldots, a_n(\eta, y)$ satisfying the differential equation,

$$\mathcal{L}_F^n q(\eta, y) = \mathcal{L}_F^{n-1} a_1(\eta, y) + \mathcal{L}_F^{n-2} a_2(\eta, y) + \cdots + a_n(\eta, y),$$

and it holds that $\frac{\partial q(\eta, h(\xi))}{\partial h} \neq 0$ on a neighborhood of $(\eta(0), \xi(0))$. □

Based on the above results, constructive algorithms to solve the RDOEL problem have been developed for some special cases [BB11, Yan11, YBS11]. Moreover, the concept of RDOEL has been extended to discrete-time systems [YYS12, YYS13]. However, all the results are limited to the case of single output systems. In this respect, we will formulate the RDOEL problem for multi-output nonlinear systems, and derive several necessary conditions and a geometric necessary and sufficient condition in the next chapter.

3.6 Inclusion Relation among OEL, System Immersion, DOEL, and RDOEL

We end this chapter with the verification of the inclusion relation among OEL, system immersion, DOEL, and RDOEL. Since the results of system immersion and RDOEL introduced so far are for the case of single output systems, we only consider the case. Trivially, system immersion, DOEL, and RDOEL are extensions of OEL, and Theorem 3.4.4 shows that system immersion is included in DOEL. In [Yan11], some examples are given to illustrate that system immersion cannot include RDOEL and vice versa. Lastly, in order to show that RDOEL is a special

case of DOEL, we derive a corollary from Theorem 3.4.3 and Theorem 3.5.2.

Corollary 3.6.1. If the system (3.3.1) is RDOEL via the auxiliary dynamics (3.5.1), then it is also DOEL via the same auxiliary dynamics with a new output,

$$\begin{aligned} \dot{\eta} = p(\eta, y) &= \begin{bmatrix} \eta_2 + \bar{a}_1(\eta, y) \\ \vdots \\ \eta_d + \bar{a}_{d-1}(\eta, y) \\ q(\eta, y) + \bar{a}_d(\eta, y) \end{bmatrix}, \\ \bar{y}_e = \bar{q}(\eta, y) &= \eta_1, \end{aligned} \tag{3.6.1}$$

where $\eta = [\eta_1 \ \cdots \ \eta_d]^T$, $p(\eta, y) = [p_1(\eta, y) \ \cdots \ p_d(\eta, y)]^T$, $\bar{a}_i(\eta, y) := p_i(\eta, y) - \eta_{i+1}$ for $1 \leq i \leq d-1$, and $\bar{a}_d(\eta, y) := p_d(\eta, y) - q(\eta, y)$.

Proof. If the system (3.3.1) is RDOEL via the auxiliary dynamics (3.5.1), then, by Theorem 3.5.2, there exist n functions $a_1(\eta, y), \ldots, a_n(\eta, y)$ such that

$$\mathcal{L}_F^n q(\eta, y) = \sum_{j=1}^{n} \mathcal{L}_F^{n-j} a_j(\eta, y). \tag{3.6.2}$$

Let $\bar{a}_i(\eta, y) := a_{i-d}(\eta, y)$ for $d+1 \leq i \leq d+n$. Meanwhile, by (3.6.1), we have

$$\mathcal{L}_F^d \bar{q}(\eta, y) = q(\eta, y) + \sum_{i=1}^{d} \mathcal{L}_F^{d-i} \bar{a}_i(\eta, y). \tag{3.6.3}$$

Hence, it follows from (3.6.2) and (3.6.3) that

$$\mathcal{L}_F^{d+n} \bar{q}(\eta, y) = \mathcal{L}_F^n q(\eta, y) + \sum_{i=1}^{d} \mathcal{L}_F^{d+n-i} \bar{a}_i(\eta, y) = \sum_{i=1}^{d+n} \mathcal{L}_F^{d+n-i} \bar{a}_i(\eta, y).$$

Therefore, the condition (a) in Theorem 3.4.3 is satisfied.

In addition, for the map $\Phi(\eta, \xi) = [\Phi_1(\eta, \xi) \ \cdots \ \Phi_{d+n}(\eta, \xi)]^T$ defined by the condition (b) in Theorem 3.4.3, we obtain from (3.6.1) and (3.6.3) that

$$\Phi_i(\eta, \xi) := \mathcal{L}_F^{i-1} \bar{q}(\eta, h(\xi)) - \sum_{j=1}^{i-1} \mathcal{L}_F^{i-1-j} \bar{a}_j(\eta, h(\xi))$$

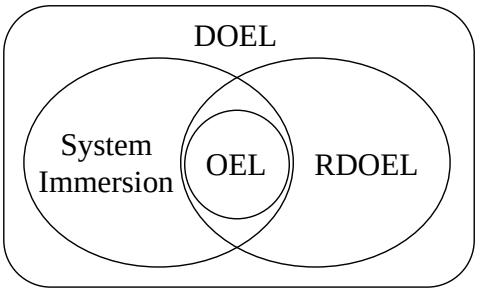

Figure 3.1: Inclusion relation among OEL, system immersion, DOEL, and RDOEL

$$= \begin{cases} \eta_i & \text{for } 1 \le i \le d, \\ \mathcal{L}_F^{i-d-1} q(\eta, h(\xi)) - \sum_{j=1}^{i-d-1} \mathcal{L}_F^{i-d-1-j} a_j(\eta, h(\xi)) & \text{for } d+1 \le i \le d+n. \end{cases}$$

By Definition 3.5.1, it is not difficult to see that $\Phi(\eta, \xi)$ is identical to the coordinate transformation for RDOEL via the auxiliary dynamics (3.5.1). Thus, $\Phi(\eta, \xi)$ is a diffeomorphism on a neighborhood of $(\eta(0), \xi(0))$.

Since both the conditions in Theorem 3.4.3 hold, the system (3.3.1) is DOEL via the same auxiliary dynamics with a new output (3.6.1). □

In summary, Figure 3.1 illustrates the inclusion relation among OEL, system immersion, DOEL, and RDOEL in the case of single output systems.

Chapter 4

Reduced-Order Dynamic Observer Error Linearization (RDOEL) for Multi-Output Systems

In this chapter, we formulate and study the RDOEL problem for multi-output systems. We present three necessary conditions and then provide a geometric equivalent condition for the solvability of the RDOEL problem. Furthermore, from the equivalent condition, we derive a geometric necessary and sufficient condition of the (conventional) OEL problem for multi-output systems in the case under consideration of output transformation of the general form $y_e = q(y)$, which has not been established yet despite several attempts in the past. In addition, by means of the results, we develop a procedure to check the solvability and to construct explicit change of coordinates for OEL and RDOEL. Lastly, some examples are given to illustrate the theoretical results. Most of the chapter is based on [CYS12, CYS14b].

4.1 Problem Statement

In this section, we define the RDOEL problem for the system (3.1.1) by the following definition that is a generalization of Definition 3.5.1 (RDOEL for single output systems) to fit multi-output systems.

Definition 4.1.1 (Reduced-order dynamic observer error linearization (RDOEL))**.** The system (3.1.1) is said to be *reduced-order dynamic observer error linearizable*

(RDOEL) if there exist a dynamic system (called *auxiliary dynamics*),

$$\dot{\eta} = p(\eta, y), \qquad \eta \in \mathbb{R}^d, \tag{4.1.1}$$

and two maps $\Phi : U \times V \to \mathbb{R}^{d+n}$, $(\eta, \xi) \mapsto (w, z) = (\eta, z)$ as a coordinate transformation and $Q : U \times h(V) \to \mathbb{R}^{d+m}$, $(\eta, y) \mapsto (w, y_e) = (\eta, q(\eta, y))$ as an output transformation, which are diffeomorphisms onto their images, such that $\Pi_1 \circ \Phi$ and $\Pi_2 \circ Q$ transform the *extended system* composed of the given system (3.1.1) and the auxiliary dynamics (4.1.1),

$$\begin{aligned} \begin{bmatrix} \dot{\eta} \\ \dot{\xi} \end{bmatrix} &= F(\eta, \xi) := \begin{bmatrix} p(\eta, y) \\ f(\xi) \end{bmatrix}, \\ y &= h(\xi), \end{aligned} \tag{4.1.2}$$

into an n-dimensional *generalized nonlinear observer canonical form (GNOCF)*,

$$\begin{aligned} \dot{z} &= Az + a(\eta, y), \qquad z \in \mathbb{R}^n, \\ y_e &= q(\eta, y) = Cz, \qquad y_e \in \mathbb{R}^m, \end{aligned} \tag{4.1.3}$$

where $U \times V \subset \mathbb{R}^d \times \mathbb{R}^n$ is a neighborhood of an initial state $(\eta(0), \xi(0))$,

$$\begin{aligned} &\eta = [\eta_1 \ \cdots \ \eta_d]^T, && p(\eta, y) = [p_1(\eta, y) \ \cdots \ p_d(\eta, y)]^T, \\ &y_e = [y_{e1} \ \cdots \ y_{em}]^T, && q(\eta, y) = [q_1(\eta, y) \ \cdots \ q_m(\eta, y)]^T, \\ &z = [z_{11} \ \cdots \ z_{1n_1} \ \cdots \ z_{m1} \ \cdots \ z_{mn_m}]^T, \\ &\Pi_1 : \mathbb{R}^{d+n} \to \mathbb{R}^n, \quad (\eta, z) \mapsto z, \\ &\Pi_2 : \mathbb{R}^{d+m} \to \mathbb{R}^m, \quad (\eta, y_e) \mapsto y_e, \\ &A = \mathrm{diag}(A_1, \ldots, A_m) \quad \text{with} \quad A_i = \begin{bmatrix} O & I_{n_i - 1} \\ O & O \end{bmatrix} \in \mathbb{R}^{n_i \times n_i} \quad \text{for } 1 \le i \le m, \\ &C = \mathrm{diag}(C_1, \ldots, C_m) \quad \text{with} \quad C_i = [\ 1 \ 0 \ \cdots \ 0\ ] \in \mathbb{R}^{1 \times n_i} \quad \text{for } 1 \le i \le m, \end{aligned}$$

$a(\eta, y) = [a_{11}(\eta, y) \ \cdots \ a_{1n_1}(\eta, y) \ \cdots \ a_{m1}(\eta, y) \ \cdots \ a_{mn_m}(\eta, y)]^T$ is *generalized output injection*, and n_i's (for $1 \le i \le m$) are some positive integers such that $n_1 \ge n_2 \ge \cdots \ge n_m$ and $\sum_{i=1}^m n_i = n$. □

Remark 4.1.1. In actual fact, the output transformation Q is a part of the coordinate transformation Φ in the following sense: $Q(\eta, y) = (\Pi_3 \circ \Phi)(\eta, \xi)$ where Π_3 is a projection such that $\Pi_3 : \mathbb{R}^{d+n} \to \mathbb{R}^{d+m}$, $(\eta, z) \mapsto (\eta, z_{11}, z_{21}, \ldots, z_{m1})$. In other words, it holds that $y_{ei} = q_i(\eta, y) = q_i(\eta, h(\xi)) = z_{i1}$ for $1 \leq i \leq m$. This is due to the fact that $y_e = q(\eta, y) = Cz$ in the GNOCF (4.1.3). □

Remark 4.1.2. In the OEL problem, it is usually required that the output transformation $q(y)$ is a diffeomorphism onto its image (see Definition 3.2.1). In Definition 4.1.1, if the auxiliary dynamics (4.1.1) is not employed, then $Q(\eta, y)$ becomes $q(y)$. In this sense, the condition that $Q(\eta, y)$ is a diffeomorphism onto its image is a natural extension of the above condition on $q(y)$ in the OEL problem. Moreover, the RDOEL problem is identical to the OEL problem if we do not consider the auxiliary dynamics (cf. Definition 3.2.1 and Definition 4.1.1), and the framework of RDOEL is quite similar to that of OEL even if the auxiliary dynamics is considered (e.g. the same dimension and structure of the pair (A, C) in the NOCF (3.2.1) and the GNOCF (4.1.3)). Therefore, we can say that the RDOEL problem is not only a modified version of the DOEL problem but also a natural extension of the OEL problem. □

In the case of single output systems, we proved by Corollary 3.6.1 that RDOEL is a special case of DOEL. The following theorem shows that it also holds for the case of multi-output systems.

Theorem 4.1.1. If the system (3.1.1) is RDOEL via the auxiliary dynamics (4.1.1), then it is also DOEL via the same auxiliary dynamics with a new output.

Proof. If the system (3.1.1) is RDOEL via the auxiliary dynamics (4.1.1), then, by Definition 4.1.1, there exists a diffeomorphism $[\eta^T \ z^T]^T = \Phi(\eta, \xi)$ on a neighborhood of $(\eta(0), \xi(0))$ such that the extended system (4.1.2) is transformed into the system composed of the auxiliary dynamics (4.1.1) and the n-dimensional GNOCF (4.1.3) on the (η, z)-coordinates. In a similar fashion to the proof of Corollary 3.6.1, we show that $\Phi(\eta, \xi)$ can also transform the extended system into a $(d+n)$-dimensional GNOCF with a new output.

We set $\bar{z} = [\bar{z}_{11} \ \cdots \ \bar{z}_{1(d+n_1)} \ \cdots \ \bar{z}_{m1} \ \cdots \ \bar{z}_{mn_m}]^T \in \mathbb{R}^{d+n}$ as a new coordinate

and $\bar{y}_e = [\bar{y}_{e1} \ \cdots \ \bar{y}_{em}]^T \in \mathbb{R}^m$ as a new output as follows:

$$\bar{z}_{1k} = \begin{cases} \eta_k & \text{for } 1 \le k \le d, \\ z_{i(k-d)} & \text{for } d+1 \le k \le d+n_1, \end{cases} \qquad \bar{y}_{e1} = \eta_1,$$

$$\bar{z}_{ij} = z_{ij}, \qquad \bar{y}_{ei} = y_{ei} = q_i(\eta, y), \qquad \text{for } 2 \le i \le m \ \text{ and } \ 1 \le j \le n_i.$$

Then, we have $\bar{z} = \Phi(\eta, \xi)$ and it follows from (4.1.1), (4.1.3), and the above equation that the extended system (4.1.2) is represented as a $(d+n)$-dimensional GNOCF on the $\bar{z}$-coordinates such that

$$\begin{aligned} \dot{\bar{z}}_{1k} &= \bar{z}_{1(k+1)} + \bar{a}_{1k}(\eta, y) && \text{for } 1 \le k \le d+n_1-1, \\ \dot{\bar{z}}_{1(d+n_1)} &= \bar{a}_{1(d+n_1)}(\eta, y), && \bar{y}_{e1} = \bar{z}_{11} \\ \dot{\bar{z}}_{ij} &= \bar{z}_{i(j+1)} + \bar{a}_{ij}(\eta, y) && \text{for } 2 \le i \le m \ \text{ and } \ 1 \le j \le n_i - 1, \\ \dot{\bar{z}}_{in_i} &= \bar{a}_{in_i}(\eta, y) && \bar{y}_{ei} = \bar{z}_{i1}, \end{aligned}$$

where $\bar{a}_{1k}(\eta, y) := p_k(\eta, y) - \eta_{k+1}$ for $1 \le k \le d-1$, $\bar{a}_{1d}(\eta, y) := p_d(\eta, y) - q_1(\eta, y)$, $\bar{a}_{1k}(\eta, y) := a_{1(k-d)}(\eta, y)$ for $d+1 \le k \le d+n_1$, and $\bar{a}_{ij}(\eta, y) := a_{ij}(\eta, y)$ for $2 \le i \le m$ and $1 \le j \le n_i$. Consequently, the system (3.1.1) is also DOEL via the auxiliary dynamics (4.1.1) with the new output $\bar{y}_e$. □

Although RDOEL is a special class of DOEL, it has an advantage over DOEL such that RDOEL offers a lower-dimensional observer than DOEL as mentioned in Section 3.5. Furthermore, since RDOEL is a natural extension of OEL (for more details, see Remark 4.1.2), research for the RDOEL problem can also contribute to the study of the conventional OEL problem. That is, most of the results for the RDOEL problem can be naturally converted into the ones for the OEL problem by slight modification of eliminating effects from auxiliary dynamics (i.e. changing F, (η, ξ), and (η, y) into f, ξ, and y, respectively). Indeed, we will provide a geometric necessary and sufficient condition for the RDOEL problem under considertation of the general auxiliary dynamics (4.1.1) and the general output transformation $y_e = q(\eta, y)$, and then derive from the result the first geometric necessary and sufficient condition for the OEL problem in the case when the general output transformation $y_e = q(y)$ is considered.

4.2 Necessary Conditions

In this section, we provide three necessary conditions for the RDOEL problem. The first one is the observability of the system (3.1.1), the second one is about the inverse output transformation $Q^{-1}(\eta, y_e)$, and the last one is concerned with the observable form (3.1.3) of the system (3.1.1).

4.2.1 Observability

First, we show that the observability of the original system (3.1.1) is a necessary condition for the RDOEL problem. Furthermore, its observability indices are constant on V and equal to $(n_1, \ldots, n_m)$, where $V \subset \mathbb{R}^n$ is a neighborhood of $\xi(0)$ and n_i (for $1 \leq i \leq m$) is the dimension of the i-th block of the matrix A in the GNOCF (4.1.3) into which the system (3.1.1) can be transformed with the aid of an auxiliary dynamics of the form (4.1.1).

Suppose that the system (3.1.1) is RDOEL with the auxiliary dynamics (4.1.1). Then, there exist a coordinate transformation $\Phi : U \times V \to \mathbb{R}^{d+n}$, $(\eta, \xi) \mapsto (\eta, z)$ and an output transformation $Q : U \times h(V) \to \mathbb{R}^{d+m}$, $(\eta, y) \mapsto (\eta, y_e) = (\eta, q(\eta, y))$, where $U \times V \subset \mathbb{R}^d \times \mathbb{R}^n$ is a neighborhood of $(\eta(0), \xi(0))$. Since Q is a diffeomorphism onto its image, its inverse map $Q^{-1}(\eta, y_e) = [\eta^T \ \tilde{q}(\eta, y_e)^T]^T = [\eta^T \ y^T]^T$ also exists, where $\tilde{q}(\eta, y_e) = [\tilde{q}_1(\eta, y_e) \ \cdots \ \tilde{q}_m(\eta, y_e)]^T$. As a result, the extended system (4.1.2) is transformed into the following system:

$$\begin{aligned} \begin{bmatrix} \dot{\eta} \\ \dot{z} \end{bmatrix} &= F_z(\eta, z) := \begin{bmatrix} \tilde{p}(\eta, y_e) \\ Az + \tilde{a}(\eta, y_e) \end{bmatrix}, \\ y_e &= q(\eta, y) = Cz, \end{aligned} \tag{4.2.1}$$

where $\tilde{p}(\eta, y_e) := p(\eta, \tilde{q}(\eta, y_e)) = p(\eta, y)$ and $\tilde{a}(\eta, y_e) := a(\eta, \tilde{q}(\eta, y_e)) = a(\eta, y)$.

Theorem 4.2.1. The system (3.1.1) is RDOEL with the auxiliary dynamics (4.1.1), only if it is locally observable at $\xi(0)$ and has the constant observability indices $(n_1, \ldots, n_m)$ on V, where $V \subset \mathbb{R}^n$ is a neighborhood of $\xi(0)$ and n_i (for $1 \leq i \leq m$) is the dimension of the i-th block of the matrix A in the GNOCF (4.1.3) into which the extended system (4.1.2) can be transformed.

Proof. Henceforth, when $\alpha = [\alpha_1 \ \cdots \ \alpha_n]^T$ and $\Gamma = [\gamma_{ij}]_{m\times n}$, we use the following notation: $\mathrm{d}\alpha := [\mathrm{d}\alpha_1 \ \cdots \ \mathrm{d}\alpha_n]^T$, $\alpha \bmod \beta := [\alpha_1 \bmod \beta \ \cdots \ \alpha_n \bmod \beta]^T$, and $\mathcal{L}_F\Gamma := [\mathcal{L}_F\gamma_{ij}]_{m\times n}$, where 'mod' denotes the modulo operation.

If the system (3.1.1) is RDOEL with the auxiliary dynamics (4.1.1), then the extended system (4.1.2) can be transformed into the system (4.2.1) by Φ and Q on $U \times V$. For the system (4.2.1), let

$$\tilde{r}_i(\eta, z) := \mathrm{card}\{k : 1 \le k \le n,\ \tilde{s}_k(\eta, z) \ge i\} \qquad \text{for } 1 \le i \le m,$$

with

$$\tilde{s}_k(\eta, z) := \mathrm{rank}(\mathcal{E}_k(\eta, z)) - \mathrm{rank}(\mathcal{E}_{k-1}(\eta, z)) \qquad \text{for } 1 \le k \le n,$$

where

$$\begin{aligned}
\mathcal{E}_0(\eta, z) &:= \mathrm{span}\{\mathrm{d}\eta_1, \ldots, \mathrm{d}\eta_d\} = \mathcal{E}_0, \\
\mathcal{E}_k(\eta, z) &:= \mathrm{span}\big(\{\mathrm{d}\eta_1, \ldots, \mathrm{d}\eta_d\} \cup \{\mathrm{d}\mathcal{L}_{F_z}^{j-1} y_{ei} \text{ at } (\eta, z) : 1 \le i \le m,\ 1 \le j \le k\}\big) \\
&= \mathrm{span}\big(\mathcal{E}_{k-1}(\eta, z) \cup \{\mathrm{d}\mathcal{L}_{F_z}^{k-1} y_{ei} \text{ at } (\eta, z) : 1 \le i \le m\}\big) \quad \text{for } 1 \le k \le n.
\end{aligned}$$

We will complete the proof of Theorem 4.2.1 after showing that the following three claims are true.

Claim 1: For $1 \le i \le d$ and $0 \le j \le k \le n$, it holds that

$$\mathrm{d}\mathcal{L}_{F_z}^{j} \eta_i \text{ at } (\eta, z) \equiv 0 \mod \mathcal{E}_k(\eta, z).$$

Proof of Claim 1. By the definition of $\mathcal{E}_k(\eta, z)$ for $0 \le k \le n$, it is true that $\mathcal{E}_0(\eta, z) \subset \mathcal{E}_1(\eta, z) \subset \cdots \subset \mathcal{E}_n(\eta, z)$. Therefore, we only need to prove that $\mathrm{d}\mathcal{L}_{F_z}^{k} \eta_i$ at $(\eta, z) \equiv 0 \mod \mathcal{E}_k(\eta, z)$ for $1 \le i \le d$ and $0 \le k \le n$. The proof is by induction on k starting from $k = 0$. The claim is trivial when $k = 0$. If $k = 1$, then it follows from the equation (4.2.1) that

$$\begin{aligned}
\mathrm{d}\mathcal{L}_{F_z} \eta_i \text{ at } (\eta, z) &= \frac{\partial \tilde{p}_i}{\partial \eta}(\eta, y_e)\mathrm{d}\eta + \frac{\partial \tilde{p}_i}{\partial y_e}(\eta, y_e)\mathrm{d}y_e \\
&\equiv 0 \mod \mathcal{E}_1(\eta, z) \qquad \text{for } 1 \le i \le d.
\end{aligned}$$

Hence, the claim is true when $k = 1$. Suppose that $2 \leq k \leq n$ and the claim is true for $k-1$, i.e., $\mathrm{d}\mathcal{L}_{F_z}^{j}\eta_i$ at $(\eta, z) \equiv 0 \mod \mathcal{E}_{k-1}(\eta, z)$ for $1 \leq i \leq d$ and $0 \leq j \leq k-1$. Then, from the induction hypothesis, we obtain that

$$\begin{aligned}
\mathrm{d}\mathcal{L}_{F_z}^{k}\eta_i \text{ at } (\eta, z) &= \mathcal{L}_{F_z}^{k-1}(\mathrm{d}\mathcal{L}_{F_z}\eta_i) \text{ at } (\eta, z) = \mathcal{L}_{F_z}^{k-1}\left(\frac{\partial \tilde{p}_i}{\partial \eta}\mathrm{d}\eta + \frac{\partial \tilde{p}_i}{\partial y_e}\mathrm{d}y_e\right) \text{ at } (\eta, z) \\
&= \sum_{l=0}^{k-1}\left(\mathcal{L}_{F_z}^{k-1-l}\Big(\frac{\partial \tilde{p}_i}{\partial \eta}\Big)\mathrm{d}\mathcal{L}_{F_z}^{l}\eta + \mathcal{L}_{F_z}^{k-1-l}\Big(\frac{\partial \tilde{p}_i}{\partial y_e}\Big)\mathrm{d}\mathcal{L}_{F_z}^{l}y_e\right) \text{ at } (\eta, z) \\
&\equiv \frac{\partial \tilde{p}_i}{\partial y_e}\mathrm{d}\mathcal{L}_{F_z}^{k-1}y_e \text{ at } (\eta, z) \quad \mod \mathcal{E}_{k-1}(\eta, z) \\
&\equiv 0 \quad \mod \mathcal{E}_k(\eta, z) \qquad \text{for } 1 \leq i \leq d.
\end{aligned}$$

Consequently, the claim is also true for k and thus the proof of Claim 1 is done.

Claim 2: For $1 \leq k \leq n$ and each $(\eta, \xi) = \Phi^{-1}(\eta, z) \in U \times V$, it holds that

$$\mathcal{E}_k(\eta, z) = \mathrm{span}(\mathcal{E}_0 \cup \mathcal{D}_k(\xi)),$$

where $\mathcal{D}_k(\xi) := \mathrm{span}\{\mathrm{d}\mathcal{L}_f^{j-1}h_i(\xi) : 1 \leq i \leq m,\ 1 \leq j \leq k\}$ which is already defined in Definition 3.1.1.

Proof of Claim 2. The proof is by induction on k starting from $k = 1$. By the existence of Q^{-1} such that $Q^{-1}(\eta, y_e) = [\eta^T\ q(\eta, y_e)^T]^T = [\eta^T\ y^T]^T$, it holds that

$$\begin{aligned}
\begin{bmatrix} \mathrm{d}\eta \\ \mathrm{d}h(\xi) \end{bmatrix} &= \begin{bmatrix} \mathrm{d}\eta \\ \mathrm{d}y \end{bmatrix} = J_{Q^{-1}}(\eta, y_e)\begin{bmatrix} \mathrm{d}\eta \\ \mathrm{d}y_e \end{bmatrix} \\
&= \begin{bmatrix} I_d & O_{d\times m} \\ \frac{\partial \tilde{q}}{\partial \eta}(\eta, y_e) & \frac{\partial \tilde{q}}{\partial y_e}(\eta, y_e) \end{bmatrix}\begin{bmatrix} \mathrm{d}\eta \\ \mathrm{d}y_e \end{bmatrix},
\end{aligned} \tag{4.2.2}$$

where $J_{Q^{-1}}$ represents the Jacobian of Q^{-1}. Since Q^{-1} is also a diffeomorphism on $Q(U \times h(V))$, $J_{Q^{-1}}$ is nonsingular on $Q(U \times h(V))$. Therefore, we have that

$$\begin{aligned}
\mathcal{E}_1(\eta, z) &= \mathrm{span}\{\mathrm{d}\eta_1, \ldots, \mathrm{d}\eta_d, \mathrm{d}y_{e1}, \ldots, \mathrm{d}y_{em}\} \\
&= \mathrm{span}\{\mathrm{d}\eta_1, \ldots, \mathrm{d}\eta_d, \mathrm{d}h_1(\xi), \ldots, \mathrm{d}h_m(\xi)\} \\
&= \mathrm{span}(\mathcal{E}_0 \cup \mathcal{D}_1(\xi)),
\end{aligned}$$

and thus the claim is true when $k = 1$.

Since f and h do not depend on η, it holds that $\mathcal{L}_f^{k-1}h(\xi) = \mathcal{L}_F^{k-1}h(\xi)$ for any $k \geq 1$, where F is the vector field of the extended system (4.1.2) and thus it is the representation of F_z in the (η, ξ)-coordinates. Hence, it follows from the equation (4.2.2) and Claim 1 that

$$\begin{aligned}
\mathrm{d}\mathcal{L}_f^{k-1}h(\xi) &= \mathcal{L}_F^{k-1}\big(\mathrm{d}h(\xi)\big) = \mathcal{L}_{F_z}^{k-1}\left(\frac{\partial \tilde{q}}{\partial \eta}\mathrm{d}\eta + \frac{\partial \tilde{q}}{\partial y_e}\mathrm{d}y_e\right) \text{ at } (\eta, z) \\
&= \sum_{j=0}^{k-1}\left(\mathcal{L}_{F_z}^{k-1-j}\Big(\frac{\partial \tilde{q}}{\partial \eta}\Big)\mathrm{d}\mathcal{L}_{F_z}^{j}\eta + \mathcal{L}_{F_z}^{k-1-j}\Big(\frac{\partial \tilde{q}}{\partial y_e}\Big)\mathrm{d}\mathcal{L}_{F_z}^{j}y_e\right) \text{ at } (\eta, z) \\
&\equiv \frac{\partial \tilde{q}}{\partial y_e}\mathrm{d}\mathcal{L}_{F_z}^{k-1}y_e \text{ at } (\eta, z) \mod \mathcal{E}_{k-1}(\eta, z) \qquad \text{for } 1 \leq k \leq n.
\end{aligned}$$

In addition, $\frac{\partial \tilde{q}}{\partial y_e}$ is nonsingular on $Q(U \times h(V))$ because $J_{Q^{-1}}$ is nonsingular on $Q(U \times h(V))$ in the equation (4.2.2). Suppose that $2 \leq k \leq n$ and the claim is true for $k-1$, i.e., $\mathcal{E}_{k-1}(\eta, z) = \mathrm{span}(\mathcal{E}_0 \cup \mathcal{D}_{k-1}(\xi))$. Then, by the induction hypothesis and the above equation, it holds that

$$\begin{aligned}
\mathcal{E}_k(\eta, z) &= \mathrm{span}\big(\mathcal{E}_{k-1}(\eta, z) \cup \{\mathrm{d}\mathcal{L}_{F_z}^{k-1}y_{ei} \text{ at } (\eta, z) : 1 \leq i \leq m\}\big) \\
&= \mathrm{span}\big(\mathcal{E}_0 \cup \mathcal{D}_{k-1}(\xi) \cup \{\mathrm{d}\mathcal{L}_f^{k-1}h_i(\xi) : 1 \leq i \leq m\}\big) \\
&= \mathrm{span}(\mathcal{E}_0 \cup \mathcal{D}_k(\xi)).
\end{aligned}$$

Therefore, Claim 2 is true.

Claim 3: It holds that $(\tilde{r}_1(\eta, z), \ldots, \tilde{r}_m(\eta, z)) = (n_1, \ldots, n_m)$ on $\Phi(U \times V)$.

Proof of Claim 3. By the equation (4.2.1) and Claim 1, it is easy to see that

$$\begin{aligned}
&\mathrm{d}\mathcal{L}_{F_z}^{k-1}y_{ei} \text{ at } (\eta, z) \\
&= \begin{cases}
\mathrm{d}z_{i1} & \text{if } k = 1 \\
\mathrm{d}\Big(z_{ik} + \sum\limits_{j=1}^{k-1}\mathcal{L}_{F_z}^{k-1-j}\tilde{a}_{ij}(\eta, y_e)\Big) \text{ at } (\eta, z) & \text{if } k \geq 2 \text{ and } n_i \geq k \\
\mathrm{d}\Big(\sum\limits_{j=1}^{n_i}\mathcal{L}_{F_z}^{k-1-j}\tilde{a}_{ij}(\eta, y_e)\Big) \text{ at } (\eta, z) & \text{if } k \geq 2 \text{ and } n_i < k
\end{cases} \\
&\equiv \begin{cases}
\mathrm{d}z_{ik} & \text{if } n_i \geq k \\
0 & \text{if } n_i < k
\end{cases} \mod \mathcal{E}_{k-1}(\eta, z)
\end{aligned}$$

for $1 \le i \le m$, $1 \le k \le n$, and all $(\eta, z) \in \Phi(U \times V)$. Therefore, we have

$$\begin{aligned}\tilde{s}_k(\eta, z) &= \operatorname{rank}\big(\mathcal{E}_k(\eta, z)\big) - \operatorname{rank}\big(\mathcal{E}_{k-1}(\eta, z)\big) \\ &= \operatorname{rank}\big(\operatorname{span}\{\mathrm{d}z_{ik} : 1 \le i \le m,\ n_i \ge k\}\big) \\ &= \operatorname{card}\{1 \le i \le m : n_i \ge k\} \quad \text{on } \Phi(U \times V) \qquad \text{for } 1 \le k \le n.\end{aligned}$$

It implies that each $\tilde{s}_k(\eta, z)$ is constant on $\Phi(U \times V)$ and indicates the number of n_i's greater than or equal to k. Therefore, it is not difficult to see that $\tilde{r}_i(\eta, z) := \operatorname{card}\{\ k : 1 \le k \le n,\ \tilde{s}_k(\eta, z) \ge i\}$ is equal to the i-th observability index of the system (4.2.1) without the generalized output injection $\tilde{a}(\eta, y_e)$. That is, $\tilde{r}_i(\eta, z) = n_i$ on $\Phi(U \times V)$ for $1 \le i \le m$, and thus Claim 3 is true.

Now, let us go back to the proof of Theorem 4.2.1. Since f and h do not depend on η, $\mathrm{d}\mathcal{L}_f^{k-1} h_i(\xi)$ does not depend on $\mathrm{d}\eta_1, \ldots, \mathrm{d}\eta_d$ in the (η, ξ)-coordinates for any $k \ge 1$ and $1 \le i \le m$. Hence, it follows from Claim 2 that

$$\begin{aligned}\tilde{s}_1(\eta, z) &= \operatorname{rank}\big(\operatorname{span}(\mathcal{E}_0 \cup \mathcal{D}_1(\xi)\big) - \operatorname{rank}(\mathcal{E}_0) \\ &= \operatorname{rank}\big(\mathcal{D}_1(\xi)\big) = s_1(\xi), \\ \tilde{s}_k(\eta, z) &= \operatorname{rank}\big(\operatorname{span}(\mathcal{E}_0 \cup \mathcal{D}_k(\xi)\big) - \operatorname{rank}\big(\operatorname{span}(\mathcal{E}_0 \cup \mathcal{D}_{k-1}(\xi)\big) \\ &= \operatorname{rank}\big(\mathcal{D}_k(\xi)\big) - \operatorname{rank}\big(\mathcal{D}_{k-1}(\xi)\big) = s_k(\xi) \qquad \text{for } 2 \le k \le n.\end{aligned}$$

The above equations mean that $r_i(\xi) = \tilde{r}_i(\eta, z)$ for $1 \le i \le m$. Thus, by Claim 3, we have $(r_1(\xi), \ldots, r_m(\xi)) = (n_1, \ldots, n_m)$ on V. As a result, the system (3.1.1) satisfies the observability rank condition as follows:

$$\operatorname{rank}\big(\operatorname{span}\{\mathrm{d}\mathcal{L}_f^{j-1} h_i(\xi) : 1 \le i \le m,\ 1 \le j \le n_i\}\big) = \sum_{i=1}^{m} n_i = n \quad \text{for all } \xi \in V,$$

after a suitable reordering of h_i's. That is to say, the system (3.1.1) is locally observable at $\xi(0)$. □

Remark 4.2.1. Actually, observability of a given system in the sense defined by Definition 3.1.2 is a necessary condition not only for RDOEL but also for many other nonlinear observer design schemes including OEL, system immersion, DOEL, high-gain observers, and so on. However, it is worth pointing out that

dimensions of each block in both the NOCF (3.2.1) by OEL and the GNOCF (4.1.3) by RDOEL are determined by the observability indices of the given system (see Theorem 3.2.3 and Theorem 4.2.1), while it cannot hold in the cases of system immersion and DOEL. In fact, this property is one of the factors that make it possible to convert most of results on the RDOEL problem into the ones for the OEL problem naturally. □

From now on, by Theorem 4.2.1, we assume that the system (3.1.1) is locally observable at $\xi(0)$ with the constant observability indices $(n_1, \ldots, n_m)$ on V. Then, the system (3.1.1) can be expressed as the observable form (3.1.3) and, without loss of generality, we can regard the system (3.1.1) as its observable form (3.1.3). For convenience, we write $\dot{x} = f(x)$ and $y = h(x)$. Thereby, we can also regard the extended system (4.1.2) as the system

$$\begin{aligned} \begin{bmatrix} \dot{\eta} \\ \dot{x} \end{bmatrix} &= F(\eta, x) := \begin{bmatrix} p(\eta, y) \\ f(x) \end{bmatrix}, \\ y &= h(x) = [x_{11} \ \cdots \ x_{m1}]^T. \end{aligned} \tag{4.2.3}$$

Remark 4.2.2. The reason why we regard the system (3.1.1) as the observable form (3.1.3) is to provide a more explicit analysis. Although most of the results and the analysis on them, which will be presented throughout the rest of this chapter, are described in the x-coordinates on which the system (3.1.1) is represented as its observable form (3.1.3), they can be converted into the ones described in the general ξ-coordinates. □

4.2.2 Inverse Output Transformation

This subsection is devoted to present the second necessary condition for the RDOEL problem, which is related to the inverse output transformation $Q^{-1}(\eta, y_e)$. Before providing it, we introduce some notation that will be used frequently in the rest of the chapter.

Definition 4.2.1. For the observability indices $(n_1, \ldots, n_m)$ of the system (3.1.3), let $\chi(j) := \mathrm{card}\{1 \le i \le m : n_i \ge j\}$ for $1 \le j \le n_1$, which indicates the number of n_i's greater than or equal to j. □

Remark 4.2.3. Since it holds that $n_1 \geq n_2 \geq \cdots \geq n_m$ by the lexographic ordering of observability indices mentioned in Remark 3.1.1, we have $\chi(j) := \mathrm{card}\{1 \leq i \leq m : n_i \geq j\} = \max\{1 \leq i \leq m : n_i \geq j\}$. Therefore, the following two inequalities are equivalent: $n_i \geq j$ and $1 \leq i \leq \chi(j)$. In addition, it is easy to see that $m = \chi(1) \geq \chi(2) \geq \cdots \geq \chi(n_1) \geq 1$. □

Definition 4.2.2. For the system (4.2.3), we denote by $\mathcal{P}_e(x)$ (respectively, $\mathcal{P}(x)$) the ring of polynomials in x_{ij}'s, where $1 \leq i \leq \chi(2)$ and $2 \leq j \leq n_i$, with coefficients that are smooth real-valued functions of η and y (respectively, y only). The weighted degree of a monomial $c(\eta, y)(x_{i_1 j_1})^{k_1} \cdots (x_{i_r j_r})^{k_r}$ is defined as $\sum_{s=1}^{r}(j_s - 1)k_s$ where $k_1, \ldots, k_r$ are nonnegative integers. The weighted degree of a polynomial in $\mathcal{P}_e(x)$ or $\mathcal{P}(x)$ is the highest weighted degree of any term in the polynomial. We denote by $\mathcal{P}_e^k(x)$ (respectively, $\mathcal{P}^k(x)$) the set of all the polynomials in $\mathcal{P}_e(x)$ (respectively, $\mathcal{P}(x)$) whose weighted degree is less than or equal to k. $\mathcal{P}_e^0(x)$ (respectively, $\mathcal{P}^0(x)$) represents the set of all smooth real-valued functions of η and y (respectively, y only). When $k \geq 1$, $\mathcal{P}_{eo}^k(x)$ (respectively, $\mathcal{P}_o^k(x)$) denotes the set of polynomials in $\mathcal{P}_e^k(x)$ (respectively, $\mathcal{P}^k(x)$), which do not depend on any x_{ij} such that $j \geq k+1$. For the system (4.2.1), $\mathcal{P}_e^k(z)$, $\mathcal{P}^k(z)$, $\mathcal{P}_{eo}^k(z)$, and $\mathcal{P}_o^k(z)$ are defined in a similar fashion by replacing x and y with z and y_e, respectively. □

Remark 4.2.4. It is easy to see that $\mathcal{P}_{eo}^1(x) = \mathcal{P}_e^0(x)$ and $\mathcal{P}_o^1(x) = \mathcal{P}^0(x)$. When $k \geq 2$, $\phi(\eta, x) \in \mathcal{P}_{eo}^k(x)$ (respectively, $\phi(x) \in \mathcal{P}_o^k(x)$) implies that not only its weighted degree is less than or equal to k, but also it is a polynomial of x_{ij}'s, where $1 \leq i \leq \chi(2)$ and $2 \leq j \leq \min\{k, n_i\}$, with coefficients that are elements of $\mathcal{P}_e^0(x)$ (respectively, $\mathcal{P}^0(x)$). Additionally, since there does not exist any x_{ij} such that $j \geq n_1 + 1$, it holds that $\mathcal{P}_e^k(x) = \mathcal{P}_{eo}^k(x)$ for all $k \geq n_1$. The same interpretations are also valid when x is replaced by z. □

Remark 4.2.5. The concept of the weighted degree was introduced in [KR85]. Definition 4.2.2 is a natural extension of Definition 3.2 in [KR85] (Definition 3.2.2 in Section 3.2) to fit the case when the auxiliary dynamics (4.1.1) is employed. □

As regards $\mathcal{P}(z)$, we present a proposition, a lemma, and its corollary, which will be used frequently in the rest of the chapter.

Proposition 4.2.2. If $\psi(\eta, z) \in \mathcal{P}_e^k(z)$ for any $k \geq 0$, then it holds that

$$\frac{\partial \psi}{\partial \eta_l} \in \mathcal{P}_e^k(z),$$
$$\frac{\partial \psi}{\partial z_{ij}} = \begin{cases} 0 & \text{if } j > k+1, \\ * \in \mathcal{P}_e^{k-j+1}(z) & \text{if } j \leq k+1, \end{cases}$$

for $1 \leq l \leq d$, $1 \leq i \leq m$, and $1 \leq j \leq n_i$. The same analysis is also valid when z is replaced by x.

Proof. Since the proof is apparent from Definition 4.2.2, we omit it. □

Lemma 4.2.3. If $\psi(\eta, z) \in \mathcal{P}_e^k(z)$ for any $k \geq 0$, then $\mathcal{L}_{F_z}\psi \in \mathcal{P}_e^{k+1}(z)$.

Proof. For $1 \leq i \leq \chi(2)$ and $2 \leq j \leq n_i$, it follows from the equation (4.2.1) that

$$\mathcal{L}_{F_z} z_{ij} = \dot{z}_{ij} = \begin{cases} z_{i(j+1)} + \tilde{a}_{ij}(\eta, y_e) & \text{if } j < n_i, \\ \tilde{a}_{in_i}(\eta, y_e) & \text{if } j = n_i. \end{cases} \tag{4.2.4a}$$

One can observe that $\mathcal{L}_{F_z} z_{ij} \in \mathcal{P}_e^j(z)$ while $z_{ij} \in \mathcal{P}_e^{j-1}(z)$. In addition, for any $c(\eta, y_e) \in \mathcal{P}_e^0(z)$, it holds that

$$\begin{aligned} \mathcal{L}_{F_z} c(\eta, y_e) &= \sum_{k=1}^{d} \frac{\partial c}{\partial \eta_k}\dot{\eta}_k + \sum_{i=1}^{m} \frac{\partial c}{\partial y_{ei}}\dot{z}_{i1} \\ &= \sum_{k=1}^{d} \frac{\partial c}{\partial \eta_k}\tilde{p}_k + \sum_{i=1}^{m} \frac{\partial c}{\partial y_{ei}}\tilde{a}_{i1} + \sum_{n_i \geq 2} \frac{\partial c}{\partial y_{ei}} z_{i2} \\ &= c_0(\eta, y_e) + \sum_{i=1}^{\chi(2)} c_i(\eta, y_e) z_{i2} \in \mathcal{P}_e^1(z), \end{aligned} \tag{4.2.4b}$$

where

$$c_0(\eta, y_e) := \sum_{k=1}^{d} \frac{\partial c}{\partial \eta_k}(\eta, y_e)\tilde{p}_k(\eta, y_e) + \sum_{i=1}^{m} \frac{\partial c}{\partial y_{ei}}(\eta, y_e)\tilde{a}_{i1}(\eta, y_e) \in \mathcal{P}_e^0(z),$$
$$c_i(\eta, y_e) := \frac{\partial c}{\partial y_{ei}}(\eta, y_e) \in \mathcal{P}_e^0(z) \qquad \text{for } 1 \leq i \leq \chi(2).$$

Therefore, it is not difficult to see that $\mathcal{L}_{F_z}\psi \in \mathcal{P}_e^{k+1}(z)$ by Definition 4.2.2, the equations (4.2.4), and the Leibniz rule. □

Corollary 4.2.4. If $\psi(\eta, z) \in \mathcal{P}_{eo}^k(z)$ for any $k \geq 1$, then $\mathcal{L}_{F_z}\psi \in \mathcal{P}_{eo}^{k+1}(z)$.

Proof. Since $\mathcal{P}_{eo}^k(z) \subset \mathcal{P}_e^k(z)$, we have $\mathcal{L}_{F_z}\psi \in \mathcal{P}_e^{k+1}(z)$ by Lemma 4.2.3. Moreover, $\psi(\eta, z) \in \mathcal{P}_{eo}^k(z)$ implies that ψ does not depend on any z_{ij} such that $j \geq k+1$. Hence, it follows from (4.2.4) that $\mathcal{L}_{F_z}\psi$ does not contain any z_{ij} such that $j \geq k+2$. Therefore, it is concluded that $\mathcal{L}_{F_z}\psi \in \mathcal{P}_{eo}^{k+1}(z)$. □

Suppose that there exist a neighborhood $U \times W \subset \mathbb{R}^d \times \mathbb{R}^n$ of $(\eta(0), x(0))$ and two maps $\Phi : U \times W \to \mathbb{R}^{d+n}$, $(\eta, x) \mapsto (\eta, z)$ and $Q : U \times h(W) \to \mathbb{R}^{d+m}$, $(\eta, y) \mapsto (\eta, y_e)$, which are diffeomorphisms onto their images, such that the system (4.2.3) is transformed into the system (4.2.1) by Φ and Q. Then, the inverse output transformation $Q^{-1}(\eta, y_e) = [\eta^T \ \tilde{q}(\eta, y_e)^T]^T = [\eta^T \ y^T]^T$ also exists and we have

$$x_{ij} = \mathcal{L}_f^{j-1} y_i = \mathcal{L}_{F_z}^{j-1} \tilde{q}_i(\eta, y_e) \qquad \text{for } 1 \leq i \leq m \text{ and } 1 \leq j \leq n_i. \tag{4.2.5}$$

Trivially, $x_{i1} = y_i = \tilde{q}_i(\eta, y_e) \in \mathcal{P}_e^0(z)$ for $1 \leq i \leq m$. The next lemma shows that the representation of x_{ij} in the (η, z)-coordinates also belongs to $\mathcal{P}_e^{j-1}(z)$ for $1 \leq i \leq \chi(2)$ and $2 \leq j \leq n_i$.

Lemma 4.2.5. If the system (4.2.3) can be transformed into the system (4.2.1) by Φ and Q, then it holds that

$$\begin{bmatrix} x_{1j} \\ \vdots \\ x_{\chi(j)j} \end{bmatrix} = \begin{bmatrix} \psi_{1j} + \sum_{k=1}^{\chi(j)} \frac{\partial \tilde{q}_1}{\partial y_{ek}} z_{kj} \\ \vdots \\ \psi_{\chi(j)j} + \sum_{k=1}^{\chi(j)} \frac{\partial \tilde{q}_{\chi(j)}}{\partial y_{ek}} z_{kj} \end{bmatrix} \qquad \text{for } 2 \leq j \leq n_1, \tag{4.2.6}$$

where $\psi_{ij} \in \mathcal{P}_{eo}^{j-1}(z)$.

Proof. The proof is by induction on j starting from $j = 2$. When $j = 2$, it follows from (4.2.1) and (4.2.5) that

$$\begin{aligned} x_{i2} = \mathcal{L}_{F_z}\tilde{q}_i(\eta, y_e) &= \sum_{l=1}^{d} \frac{\partial \tilde{q}_i}{\partial \eta_l}\tilde{p}_l + \sum_{k=1}^{m} \frac{\partial \tilde{q}_i}{\partial y_{ek}}\tilde{a}_{k1} + \sum_{n_k \geq 2} \frac{\partial \tilde{q}_i}{\partial y_{ek}} z_{k2} \\ &= \psi_{i2} + \sum_{k=1}^{\chi(2)} \frac{\partial \tilde{q}_i}{\partial y_{ek}} z_{k2} \qquad \text{for } 1 \leq i \leq \chi(2), \end{aligned}$$

where

$$\psi_{i2} := \sum_{l=1}^{d} \frac{\partial \tilde{q}_i}{\partial \eta_l} \tilde{p}_l + \sum_{k=1}^{m} \frac{\partial \tilde{q}_i}{\partial y_{ek}} \tilde{a}_{k1} \in \mathcal{P}_e^0(z) = \mathcal{P}_{eo}^1(z).$$

Thus, the equation (4.2.6) holds for $j = 2$. Suppose that $3 \le j \le n_1$ and the equation (4.2.6) holds for $j-1$, i.e., it holds that

$$x_{i(j-1)} = \psi_{i(j-1)} + \sum_{k=1}^{\chi(j-1)} \frac{\partial \tilde{q}_i}{\partial y_{ek}} z_{k(j-1)} \qquad \text{for } 1 \le i \le \chi(j-1),$$

where $\psi_{i(j-1)} \in \mathcal{P}_{eo}^{j-2}(z)$. Then, it also follows from (4.2.1), (4.2.5) and the induction hypothesis that

$$\begin{aligned}
x_{ij} &= \mathcal{L}_{F_z} x_{i(j-1)} = \mathcal{L}_{F_z}\Big(\psi_{i(j-1)} + \sum_{k=1}^{\chi(j-1)} \frac{\partial \tilde{q}_i}{\partial y_{ek}} z_{k(j-1)}\Big) \\
&= \mathcal{L}_{F_z}\psi_{i(j-1)} + \sum_{n_k \ge j-1} \left(\Big(\mathcal{L}_{F_z} \frac{\partial \tilde{q}_i}{\partial y_{ek}}\Big) z_{k(j-1)} + \frac{\partial \tilde{q}_i}{\partial y_{ek}} \dot{z}_{k(j-1)}\right) \\
&= \mathcal{L}_{F_z}\psi_{i(j-1)} + \sum_{n_k \ge j-1} \left(\Big(\mathcal{L}_{F_z} \frac{\partial \tilde{q}_i}{\partial y_{ek}}\Big) z_{k(j-1)} + \frac{\partial \tilde{q}_i}{\partial y_{ek}} \tilde{a}_{k(j-1)}\right) + \sum_{n_k \ge j} \frac{\partial \tilde{q}_i}{\partial y_{ek}} z_{kj} \\
&= \psi_{ij} + \sum_{k=1}^{\chi(j)} \frac{\partial \tilde{q}_i}{\partial y_{ek}} z_{kj} \qquad \text{for } 1 \le i \le \chi(j),
\end{aligned}$$

where

$$\psi_{ij} := \mathcal{L}_{F_z}\psi_{i(j-1)} + \sum_{k=1}^{\chi(j-1)} \left(\Big(\mathcal{L}_{F_z} \frac{\partial \tilde{q}_i}{\partial y_{ek}}\Big) z_{k(j-1)} + \frac{\partial \tilde{q}_i}{\partial y_{ek}} \tilde{a}_{k(j-1)}\right).$$

Since $\psi_{i(j-1)} \in \mathcal{P}_{eo}^{j-2}(z)$ and $\frac{\partial \tilde{q}_i}{\partial y_{ek}} \in \mathcal{P}_e^0(z)$, we have $\mathcal{L}_{F_z}\psi_{i(j-1)} \in \mathcal{P}_{eo}^{j-1}(z)$ and $\big(\mathcal{L}_{F_z} \frac{\partial \tilde{q}_i}{\partial y_{ek}}\big) z_{k(j-1)} \in \mathcal{P}_{eo}^{j-1}(z)$ by Lemma 4.2.3 and Corollary 4.2.4. Therefore, one can observe that $\psi_{ij} \in \mathcal{P}_{eo}^{j-1}(z)$ and the equation (4.2.6) also holds for j. □

Now, we give a condition on $\tilde{q}(\eta, y_e)$ as the second necessary condition for the RDOEL problem. The proof of the following theorem shows that the condition on $\tilde{q}(\eta, y_e)$ is equivalent for the Jacobian of Φ^{-1} to be nonsingular on $\Phi(U \times W)$.

Theorem 4.2.6. If the system (4.2.3) can be transformed into the system (4.2.1) by Φ and Q, then $Q^{-1}(\eta, y_e) = [\eta^T \ \tilde{q}(\eta, y_e)^T]^T$ satisfies that

$$\prod_{j=1}^{n_1} \det \tilde{J}_{\chi(j)} \neq 0 \quad \text{on } Q(U \times h(W)), \tag{4.2.7}$$

where $\tilde{J}_i := \begin{bmatrix} \tilde{J}_{11} & \cdots & \tilde{J}_{1i} \\ \vdots & \ddots & \vdots \\ \tilde{J}_{i1} & \cdots & \tilde{J}_{ii} \end{bmatrix}$ for $i = \chi(1), \ldots, \chi(n_1)$ and $\tilde{J}_{\mu\nu} := \frac{\partial \tilde{q}_\mu}{\partial y_{e\nu}}$ for $\mu, \nu = 1, \ldots, m$.

Proof. Consider the inverse coordinate transformation $\Phi^{-1}(\eta, z) = [\eta^T \ x^T]^T$. The exterior differentiation of Φ^{-1} gives

$$\begin{bmatrix} \mathrm{d}\eta \\ \mathrm{d}x \end{bmatrix} = \mathrm{d}\Phi^{-1} = \begin{bmatrix} I_d & O \\ R_{d\times n} & S_{n\times n} \end{bmatrix} \begin{bmatrix} \mathrm{d}\eta \\ \mathrm{d}z \end{bmatrix}, \tag{4.2.8}$$

where

$$S = \frac{\partial x}{\partial z} = \begin{bmatrix} T_{11} & \cdots & T_{1m} \\ \vdots & \ddots & \vdots \\ T_{m1} & \cdots & T_{mm} \end{bmatrix}, \tag{4.2.9}$$

and $T_{\mu\nu}$ is an $n_\mu \times n_\nu$ matrix of which (κ, λ)-th entry is $\frac{\partial x_{\mu\kappa}}{\partial z_{\nu\lambda}}$ for $\mu, \nu = 1, \ldots, m$, $1 \leq \kappa \leq n_\mu$, and $1 \leq \lambda \leq n_\nu$. Since Φ^{-1} is a diffeomorphism on $\Phi(U \times W)$, the matrix S should be nonsingular on $\Phi(U \times W)$. We will show that the equation (4.2.7) is equivalent to the non-singularity of S on $\Phi(U \times W)$.

By Lemma 4.2.5, the representation of x in the (η, z)-coordinates is as follows: for $1 \leq \mu \leq m$ and $1 \leq \kappa \leq n_\mu$,

$$x_{\mu\kappa} = \begin{cases} \tilde{q}_\mu(\eta, y_e) = \tilde{q}_\mu(\eta, z_{11}, \ldots, z_{m1}) \in \mathcal{P}_e^0(z) & \text{if } \kappa = 1, \\ \psi_{\mu\kappa} + \sum_{i=1}^{\chi(\kappa)} \frac{\partial \tilde{q}_\mu}{\partial y_{ei}} z_{i\kappa} \in \mathcal{P}_e^{\kappa-1}(z) & \text{if } 2 \leq \kappa \leq n_\mu, \end{cases}$$

where $\psi_{\mu\kappa} \in \mathcal{P}_{eo}^{\kappa-1}(z)$. Since $\psi_{\mu\kappa} \in \mathcal{P}_{eo}^{\kappa-1}(z)$, it is a function of η and $z_{\nu\lambda}$'s such that $1 \leq \nu \leq m$ and $1 \leq \lambda \leq \min\{\kappa - 1, n_\nu\}$. It means that $x_{\mu\kappa}$ does not

depend on $z_{\nu\lambda}$ such that $\lambda > \kappa$. Furthermore, the coefficient of $z_{\nu\kappa}$ in $x_{\mu\kappa}$ is $\frac{\partial \tilde{q}_\mu}{\partial y_{e\nu}}$. Therefore, $\frac{\partial x_{\mu\kappa}}{\partial z_{\nu\lambda}} = 0$ if $\kappa < \lambda$, and $\frac{\partial x_{\mu\kappa}}{\partial z_{\nu\lambda}} = \frac{\partial \tilde{q}_\mu}{\partial y_{e\nu}}$ if $\kappa = \lambda$. In addition, by Proposition 4.2.2, $\frac{\partial \psi_{\mu\kappa}}{\partial z_{\nu\lambda}} \in \mathcal{P}_e^{\kappa-\lambda}(z)$ when $\kappa > \lambda$. Hence, each block $T_{\mu\nu}$ of the matrix S has the following form (called a *lower triangular-like form*): for $\mu, \nu = 1, \ldots, m$, $1 \le \kappa \le n_\mu$, and $1 \le \lambda \le n_\nu$,

$$(T_{\mu\nu})_{\kappa\lambda} = \begin{cases} 0 & \text{if } \kappa < \lambda, \\ \frac{\partial \tilde{q}_\mu}{\partial y_{e\nu}} = \tilde{J}_{\mu\nu} \in \mathcal{P}_e^0(z) & \text{if } \kappa = \lambda, \\ \frac{\partial \psi_{\mu\kappa}}{\partial z_{\nu\lambda}} \in \mathcal{P}_e^{\kappa-\lambda}(z) & \text{if } \kappa > \lambda \neq 1, \\ \frac{\partial \psi_{\mu\kappa}}{\partial y_{e\nu}} + \frac{\partial}{\partial y_{e\nu}}\left(\sum_{i=1}^{\chi(\kappa)} \frac{\partial \tilde{q}_\mu}{\partial y_{ei}} z_{i\kappa}\right) \in \mathcal{P}_e^{\kappa-1}(z) & \text{if } \kappa > \lambda = 1. \end{cases} \tag{4.2.10}$$

For the matrix S, the Leibniz formula for determinants gives

$$\det S = \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} (S)_{\sigma(k)k}, \tag{4.2.11}$$

where $\mathcal{S}_n$ is the permutation group on $\{1, \ldots, n\}$ and $\operatorname{sgn}(\cdot)$ is the sign function of a permutation. Let $n_0 := 0$ and $r_{ij} := \sum_{s=0}^{i-1} n_s + j$ for $1 \le i \le m$ and $1 \le j \le n_i$. Then, the j-th rows of $T_{i\nu}$'s for $1 \le \nu \le m$ (respectively, columns of $T_{\mu i}$'s for $1 \le \mu \le m$) belong to the r_{ij}-th row (respectively, column) of S. Let $\mathcal{Z}_j := \{r_{ij} : 1 \le i \le \chi(j)\}$ for $1 \le j \le n_1$. Clearly, $\mathcal{Z}_j$'s are disjoint and $\bigcup_{j=1}^{n_1} \mathcal{Z}_j = \{1, \ldots, n\}$. Thus, for any $(\iota, \zeta) \in \{1, \ldots, n\} \times \{1, \ldots, n\}$, there exists a unique 4-tuple $(\mu, \kappa, \nu, \lambda) \in \{1, \ldots, m\} \times \{1, \ldots, n_\mu\} \times \{1, \ldots, m\} \times \{1, \ldots, n_\lambda\}$ such that $\iota = r_{\mu\kappa} \in \mathcal{Z}_\kappa$, $\zeta = r_{\nu\lambda} \in \mathcal{Z}_\lambda$, and

$$(S)_{ij} = (S)_{r_{\mu\kappa} r_{\nu\lambda}} = (T_{\mu\nu})_{\kappa\lambda}. \tag{4.2.12}$$

Let $\mathcal{R}_j := \{\sigma \in \mathcal{S}_n : \sigma(l) = l \text{ if } l \notin \mathcal{Z}_j\}$ for $1 \le j \le n_1$, which is a subgroup of $\mathcal{S}_n$ consisting of all the permutations only on $\mathcal{Z}_j$. In the equation (4.2.11), if $\sigma \neq \sigma_1 \circ \cdots \circ \sigma_{n_1}$ where $\sigma_j \in \mathcal{R}_j$ for $1 \le j \le n_1$, then there exists at least one $l \in \{1, \ldots, n\}$ such that $\sigma(l) \in \mathcal{Z}_\kappa$, $l \in \mathcal{Z}_\lambda$, and $\kappa < \lambda$. For such l, there exist $1 \le \mu \le \chi(\kappa)$ and $1 \le \nu \le \chi(\lambda)$ such that $\sigma(l) = r_{\mu\kappa} \in \mathcal{Z}_\kappa$ and $l = r_{\nu\lambda} \in \mathcal{Z}_\lambda$. As a result, by the lower triangular-like form (4.2.10) of $T_{\mu\nu}$, it holds that $(S)_{\sigma(l)l} = (T_{\mu\nu})_{\kappa\lambda} = 0$ and thus $\operatorname{sgn}(\sigma) \prod_{k=1}^{n} (S)_{\sigma(k)k} = 0$. Consequently, we can observe

from (4.2.10) that the equation (4.2.11) can be rewritten as

$$\begin{aligned}
\det S &= \sum_{\sigma=\sigma_1\circ\cdots\circ\sigma_{n_1}} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} (S)_{\sigma(k)k} \\
&= \sum_{\sigma=\sigma_1\circ\cdots\circ\sigma_{n_1}} \Big(\prod_{k\in\mathcal{Z}_1} \operatorname{sgn}(\sigma_1)(S)_{\sigma_1(k)k} \Big) \cdots \Big(\prod_{k\in\mathcal{Z}_{n_1}} \operatorname{sgn}(\sigma_{n_1})(S)_{\sigma_{n_1}(k)k} \Big) \\
&= \Big(\sum_{\sigma_1\in\mathcal{R}_1} \operatorname{sgn}(\sigma_1) \prod_{k\in\mathcal{Z}_1} (S)_{\sigma_1(k)k} \Big) \cdots \Big(\sum_{\sigma_{n_1}\in\mathcal{R}_{n_1}} \operatorname{sgn}(\sigma_{n_1}) \prod_{k\in\mathcal{Z}_{n_1}} (S)_{\sigma_{n_1}(k)k} \Big) \\
&= \prod_{j=1}^{n_1} \Big(\sum_{\sigma_j\in\mathcal{R}_j} \operatorname{sgn}(\sigma_j) \prod_{k\in\mathcal{Z}_j} (S)_{\sigma_j(k)k} \Big). \qquad (4.2.13)
\end{aligned}$$

In the above equation, $\sigma_j \in \mathcal{R}_j$ means that $\sigma_j(\mathcal{Z}_j) = \mathcal{Z}_j$. Thus, $\{(S)_{\sigma_j(k)k} : k \in \mathcal{Z}_j\} = \{(S)_{r_{\mu k} r_{\nu k}} : 1 \le \mu, \nu \le \chi(j)\}$ and it follows from (4.2.10) and (4.2.12) that

$$\begin{aligned}
\begin{bmatrix} (S)_{r_{1j}r_{1j}} & \cdots & (S)_{r_{1j}r_{\chi(j)j}} \\ \vdots & \ddots & \vdots \\ (S)_{r_{\chi(j)j}r_{1j}} & \cdots & (S)_{r_{\chi(j)j}r_{\chi(j)j}} \end{bmatrix} &= \begin{bmatrix} (T_{11})_{jj} & \cdots & \big(T_{1\chi(j)}\big)_{jj} \\ \vdots & \ddots & \vdots \\ \big(T_{\chi(j)1}\big)_{jj} & \cdots & \big(T_{\chi(j)\chi(j)}\big)_{jj} \end{bmatrix} \\
&= \begin{bmatrix} \tilde{J}_{11} & \cdots & \tilde{J}_{1\chi(j)} \\ \vdots & \ddots & \vdots \\ \tilde{J}_{\chi(j)1} & \cdots & \tilde{J}_{\chi(j)\chi(j)} \end{bmatrix} = \tilde{J}_{\chi(j)}.
\end{aligned}$$

Therefore, for $1 \le j \le n_1$, it holds that

$$\sum_{\sigma_j\in\mathcal{R}_j} \operatorname{sgn}(\sigma_j) \prod_{k\in Z_\zeta} (S)_{\sigma_j(k)k} = \sum_{\sigma\in\mathcal{S}_{\chi(j)}} \operatorname{sgn}(\sigma) \prod_{k=1}^{\chi(j)} J_{\sigma(k)k} = \det \tilde{J}_{\chi(j)}. \qquad (4.2.14)$$

Finally, we can obtain from (4.2.13) and (4.2.14) that

$$\det S = \prod_{j=1}^{n_1} \det \tilde{J}_{\chi(j)}. \qquad (4.2.15)$$

It should be noted that, although S is defined on $\Phi(U \times W)$, $\tilde{J}_{\chi(j)}$ is defined on $Q(U \times h(W))$ for $1 \le j \le n_1$. Hence, S is nonsingular on $\Phi(U \times W)$ if and only if the condition (4.2.7) is satisfied. □

The following example is given for a better understanding of the proof of Theorem 4.2.6. Trough the example, we will verify that the equation (4.2.15) holds for the matrix S defined by (4.2.9) and (4.2.10).

Example 4.2.1. Consider the case where $n = 7$, $m = 3$ and $(n_1, n_2, n_3) = (3, 3, 1)$. The matrix S defined by (4.2.9) and (4.2.10) can be written as

$$S = \left[\begin{array}{ccc|ccc|c} \tilde{J}_{11} & 0 & 0 & \tilde{J}_{12} & 0 & 0 & \tilde{J}_{13} \\ * & \tilde{J}_{11} & 0 & * & \tilde{J}_{12} & 0 & * \\ * & * & \tilde{J}_{11} & * & * & \tilde{J}_{12} & * \\ \hline \tilde{J}_{21} & 0 & 0 & \tilde{J}_{22} & 0 & 0 & \tilde{J}_{23} \\ * & \tilde{J}_{21} & 0 & * & \tilde{J}_{22} & 0 & * \\ * & * & \tilde{J}_{21} & * & * & \tilde{J}_{22} & * \\ \hline \tilde{J}_{31} & 0 & 0 & \tilde{J}_{32} & 0 & 0 & \tilde{J}_{33} \end{array}\right].$$

By the Leibniz formula for determinants, we have

$$\det S = \sum_{\sigma \in \mathcal{S}_7} \operatorname{sgn}(\sigma) \prod_{k=1}^{7} (S)_{\sigma(k)k}.$$

The sets $\mathcal{Z}_j$'s are defined as $\mathcal{Z}_1 := \{1, 4, 7\}$, $\mathcal{Z}_2 := \{2, 5\}$, and $\mathcal{Z}_3 := \{3, 6\}$. If $\sigma(l) \in \mathcal{Z}_1 \cup \mathcal{Z}_2$ for some $l \in \mathcal{Z}_3$, then $(S)_{\sigma(l)l} = 0$ by the lower triangular-like form of each block of the matrix S. Similarly, if $\sigma(l) \in \mathcal{Z}_1$ for some $l \in \mathcal{Z}_2$, then $(S)_{\sigma(l)l} = 0$. This implies that $\prod_{k=1}^{7} (S)_{\sigma(k)k} = 0$ if $\sigma \neq \sigma_1 \circ \sigma_2 \circ \sigma_3$, where $\sigma_j \in \mathcal{R}_j := \{\sigma \in \mathcal{S}_7 : \sigma(l) = l \text{ if } l \notin \mathcal{Z}_j\}$ for $1 \leq j \leq 3$. Hence, it holds that

$$\begin{aligned} \det S &= \sum_{\sigma = \sigma_1 \circ \sigma_2 \circ \sigma_3} \operatorname{sgn}(\sigma_1)\operatorname{sgn}(\sigma_2)\operatorname{sgn}(\sigma_3) \prod_{k \in \mathcal{Z}_1} (S)_{\sigma_1(k)k} \prod_{k \in \mathcal{Z}_2} (S)_{\sigma_2(k)k} \prod_{k \in \mathcal{Z}_3} (S)_{\sigma_3(k)k} \\ &= \prod_{j=1}^{3} \Bigg(\sum_{\sigma_j \in \mathcal{R}_j} \operatorname{sgn}(\sigma_j) \prod_{k \in \mathcal{Z}_j} (S)_{\sigma_j(k)k} \Bigg). \end{aligned}$$

Since the entries of S construct the following structures:

$$\begin{bmatrix} (S)_{22} & (S)_{25} \\ (S)_{52} & (S)_{55} \end{bmatrix} = \begin{bmatrix} (S)_{33} & (S)_{36} \\ (S)_{63} & (S)_{66} \end{bmatrix} = \begin{bmatrix} \tilde{J}_{11} & \tilde{J}_{12} \\ \tilde{J}_{21} & \tilde{J}_{22} \end{bmatrix} = \tilde{J}_2 = \tilde{J}_{\chi(2)} = \tilde{J}_{\chi(3)},$$

$$\begin{bmatrix} (S)_{11} & (S)_{14} & (S)_{17} \\ (S)_{41} & (S)_{44} & (S)_{47} \\ (S)_{71} & (S)_{74} & (S)_{77} \end{bmatrix} = \begin{bmatrix} \tilde{J}_{11} & \tilde{J}_{12} & \tilde{J}_{13} \\ \tilde{J}_{21} & \tilde{J}_{22} & \tilde{J}_{23} \\ \tilde{J}_{31} & \tilde{J}_{32} & \tilde{J}_{33} \end{bmatrix} = \tilde{J}_3 = \tilde{J}_{\chi(1)},$$

it holds that

$$\sum_{\sigma_1 \in \mathcal{R}_1} \operatorname{sgn}(\sigma_1) \prod_{k \in \mathcal{Z}_1} (S)_{\sigma_1(k)k} = \sum_{\sigma \in \mathcal{S}_3} \operatorname{sgn}(\sigma) \prod_{k=1}^{3} \tilde{J}_{\sigma(k)k} = \det \tilde{J}_3 = \det \tilde{J}_{\chi(1)},$$

$$\sum_{\sigma_2 \in \mathcal{R}_2} \operatorname{sgn}(\sigma_2) \prod_{k \in \mathcal{Z}_2} (S)_{\sigma_2(k)k} = \sum_{\sigma \in \mathcal{S}_2} \operatorname{sgn}(\sigma) \prod_{k=1}^{2} \tilde{J}_{\sigma(k)k} = \det \tilde{J}_2 = \det \tilde{J}_{\chi(2)},$$

$$\sum_{\sigma_3 \in \mathcal{R}_3} \operatorname{sgn}(\sigma_3) \prod_{k \in \mathcal{Z}_3} (S)_{\sigma_3(k)k} = \sum_{\sigma \in \mathcal{S}_2} \operatorname{sgn}(\sigma) \prod_{k=1}^{2} \tilde{J}_{\sigma(k)k} = \det \tilde{J}_2 = \det \tilde{J}_{\chi(3)}.$$

Consequently, we obtain

$$\det S = \prod_{j=1}^{3} \det \tilde{J}_{\chi(j)},$$

which satisfies the equation (4.2.15). □

Remark 4.2.6. When we do not consider the auxiliary dynamics (4.1.1), the condition (4.2.7) becomes $\prod_{j=1}^{n_1} \det \tilde{J}_{\chi(j)} \neq 0$ on $Q(h(W))$, and it is also a necessary condition for the OEL problem. However, to our best knowledge, there has so far been no literature providing such a necessary condition. □

4.2.3 System Dynamics

In this subsection, we derive the third necessary condition. It is related to the system dynamics (3.1.3), especially, $f_i(x)$ for $1 \leq i \leq m$. The following theorem states it and plays a key role in deriving a necessary and sufficient condition for the RDOEL problem, which will be given in the next chapter.

Theorem 4.2.7. If the system (3.1.3) is RDOEL, then it holds that $f_i(x) \in \mathcal{P}^{n_i}(x)$ for $1 \leq i \leq m$. □

In order to prove the theorem comfortably, we need the following lemma.

Lemma 4.2.8. Suppose that the system (4.2.3) is transformed into the system (4.2.1) and $\psi(\eta, z) \in \mathcal{P}_e^{k-1}(z)$ (respectively, $\mathcal{P}_{eo}^{k}(z)$) for any $1 \le k \le n_1$. Then, the representation of ψ in the (η, x)-coordinates belongs to $\mathcal{P}_e^{k-1}(x)$ (respectively, $\mathcal{P}_{eo}^{k}(x)$).

Proof. The proof is by induction on k starting from $k = 1$. Let $\psi(\eta, z) \in \mathcal{P}_e^0(z) = \mathcal{P}_{eo}^1(z)$. Then, ψ depends only on η and y_e. Therefore, it holds that $\psi(\eta, y_e) = (\psi \circ Q^{-1})(\eta, y) \in \mathcal{P}_e^0(x) = \mathcal{P}_{eo}^1(x)$, and thus the lemma is true when $k = 1$. Suppose that $2 \le k \le n_1$ and the lemma is true for $1 \le j \le k-1$. By Definition 4.2.2, $\psi(\eta, z) \in \mathcal{P}_e^{k-1}(z)$ (respectively, $\mathcal{P}_{eo}^{k}(z)$) implies that its weighted degree is less than or equal to $k-1$ (respectively, k) and it is a polynomial of z_{ij}'s, where $1 \le i \le \chi(2)$ and $2 \le j \le k$, with coefficients that belong to $\mathcal{P}_e^0(z)$. Since the lemma is true when $k = 1$, all the coefficients also belong to $\mathcal{P}_e^0(x)$ in the (η, x)-coordinates. Hence, if the representation of z_{ij} in the (η, x)-coordinates belongs to $\mathcal{P}_e^{j-1}(x)$ for $1 \le i \le \chi(2)$ and $2 \le j \le k$, then the lemma is also true for k. By the induction hypothesis and the fact that $z_{ij} \in \mathcal{P}_e^{j-1}(z)$, the representation of z_{ij} in the (η, x)-coordinates belongs to $\mathcal{P}_e^{j-1}(x)$ for $1 \le i \le \chi(2)$ and $2 \le j \le k-1$. Thus, in order to complete the proof, we have only to prove that the representation of z_{ik} in the (η, x)-coordinates is an element of $\mathcal{P}_e^{k-1}(x)$ for $1 \le i \le \chi(k)$. By Lemma 4.2.5, it holds that

$$\begin{bmatrix} x_{1k} \\ \vdots \\ x_{\chi(k)k} \end{bmatrix} = \begin{bmatrix} \psi_{1k} + \sum_{i=1}^{\chi(k)} \frac{\partial \tilde{q}_1}{\partial y_{ei}} z_{ik} \\ \vdots \\ \psi_{\chi(k)k} + \sum_{i=1}^{\chi(k)} \frac{\partial \tilde{q}_{\chi(k)}}{\partial y_{ei}} z_{ik} \end{bmatrix} = \begin{bmatrix} \psi_{1k} \\ \vdots \\ \psi_{\chi(k)k} \end{bmatrix} + \tilde{J}_{\chi(k)} \begin{bmatrix} z_{1k} \\ \vdots \\ z_{\chi(k)k} \end{bmatrix},$$

where $\psi_{ik} \in \mathcal{P}_{eo}^{k-1}(z)$ for $1 \le i \le \chi(k)$. Let $\phi_{ik}(\eta, x) := (\psi_{ik} \circ \Phi)(\eta, x) = \psi_{ik}(\eta, z)$, which denotes the representation of ψ_{ik} in the (η, x)-coordinates. Then, $\phi_{ik} \in \mathcal{P}_{eo}^{k-1}(x)$ by the induction hypothesis, and we have

$$\begin{bmatrix} z_{1k} \\ \vdots \\ z_{\chi(k)k} \end{bmatrix} = \left(\tilde{J}_{\chi(k)}\right)^{-1} \begin{bmatrix} x_{1k} - \phi_{1k}(\eta, x) \\ \vdots \\ x_{\chi(k)k} - \phi_{\chi(k)k}(\eta, x) \end{bmatrix},$$

because $\tilde{J}_{\chi(k)}$ is nonsingular for $2 \le k \le n_1$ by Theorem 4.2.6. Since all the entries

of $\tilde{J}_{\chi(k)}$ are elements of $\mathcal{P}_e^0(z)$, their representations in the (η, x)-coordinates also belong to $\mathcal{P}_e^0(x)$. Therefore, one can observe that all the entries in the right-hand side of the above equation belong to $\mathcal{P}_e^{k-1}(x)$, i.e., the representation of z_{ik} in the (η, x)-coordinates is an element of $\mathcal{P}_e^{k-1}(x)$ for $1 \le i \le \chi(k)$. □

Now, let us prove Theorem 4.2.7.

Proof of Theorem 4.2.7. If the system (3.1.3) is RDOEL, then there exist an auxiliary dynamics such as (4.1.1) and two maps Φ and Q transforming the extended system (4.2.3) into the system (4.2.1). Therefore, by Lemma 4.2.5, we have

$$x_{in_i} = \psi_{in_i} + \sum_{k=1}^{\chi(n_i)} \frac{\partial \tilde{q}_i}{\partial y_{ek}} z_{kn_i} \quad \text{for } 1 \le i \le m,$$

where $\psi_{in_i} \in \mathcal{P}_{eo}^{n_i-1}(z)$. Thus, it follows from the above equation and (4.2.1) that

$$\begin{aligned}
f_i(x) = \dot{x}_{in_i} &= \mathcal{L}_{F_z}\left(\psi_{in_i} + \sum_{k=1}^{\chi(n_i)} \frac{\partial \tilde{q}_i}{\partial y_{ek}} z_{kn_i}\right) \\
&= \mathcal{L}_{F_z}\psi_{in_i} + \sum_{k=1}^{\chi(n_i)} \left(\left(\mathcal{L}_{F_z}\frac{\partial \tilde{q}_i}{\partial y_{ek}}\right) z_{kn_i} + \frac{\partial \tilde{q}_i}{\partial y_{ek}} \dot{z}_{kn_i}\right) \\
&= \mathcal{L}_{F_z}\psi_{in_i} + \sum_{k=1}^{\chi(n_i)} \left(\left(\mathcal{L}_{F_z}\frac{\partial \tilde{q}_i}{\partial y_{ek}}\right) z_{kn_i} + \frac{\partial \tilde{q}_i}{\partial y_{ek}} \tilde{a}_{kn_i}\right) + \sum_{n_k \ge n_i+1} \frac{\partial \tilde{q}_i}{\partial y_{ek}} z_{k(n_i+1)} \\
&= \begin{cases} \psi_i + \sum_{k=1}^{\chi(n_i+1)} \frac{\partial \tilde{q}_i}{\partial y_{ek}} z_{k(n_i+1)} & \text{if } n_i < n_1 \\ \psi_i & \text{if } n_i = n_1 \end{cases} \quad \text{for } 1 \le i \le m,
\end{aligned}$$

where

$$\psi_i := \mathcal{L}_{F_z}\psi_{in_i} + \sum_{k=1}^{\chi(n_i)} \left((\mathcal{L}_{F_z}\frac{\partial \tilde{q}_i}{\partial y_{ek}}) z_{kn_i} + \frac{\partial \tilde{q}_i}{\partial y_{ek}} \tilde{a}_{kn_i}\right).$$

Since $\psi_{in_i} \in \mathcal{P}_{eo}^{n_i-1}(z)$ and $\frac{\partial \tilde{q}_i}{\partial y_{ek}} \in \mathcal{P}_e^0(z)$, it holds that $\mathcal{L}_{F_z}\psi_{in_i} \in \mathcal{P}_{eo}^{n_i}(z)$ and $\mathcal{L}_{F_z}\frac{\partial \tilde{q}_i}{\partial y_{ek}} \in \mathcal{P}_e^1(z)$ by Corollary 4.2.4 and Lemma 4.2.3, respectively. As a result, $\psi_i \in \mathcal{P}_{eo}^{n_i}(z)$, and thus the representation of $f_i(x)$ in the (η, z)-coordinates belongs to $\mathcal{P}_e^{n_i}(z)$ (respectively, $\mathcal{P}_{eo}^{n_i}(z)$) if $n_i < n_1$ (respectively, if $n_i = n_1$) for $1 \le i \le m$.

Hence, by Lemma 4.2.8, we have $f_i(x) \in \mathcal{P}_e^{n_i}(x)$ for $1 \le i \le m$. (Note that $\mathcal{P}_{eo}^{n_i}(x) = \mathcal{P}_e^{n_i}(x)$ if $n_i = n_1$.) Since $f_i(x)$ is a function of x only, it is concluded that $f_i(x)$ belongs to $\mathcal{P}^{n_i}(x)$ for $1 \le i \le m$. □

Remark 4.2.7. Actually, Theorem 4.2.7 is motivated by Proposition 3.3 in [KR85] (Theorem 3.2.4 in Section 3.2), which gives a necessary condition for the OEL problem. In this dissertation, we could complete the proof with the aid of Theorem 4.2.6. □

Remark 4.2.8. Although the condition $f_i(x) \in \mathcal{P}^{n_i}(x)$ for $1 \le i \le m$ is a necessary condition of both the OEL and RDOEL problems, it is literally a necessary condition. That is, in the class of systems satisfying the condition, there exists a class of systems that are not OEL but RDOEL (e.g. Example 4.4.2). Meanwhile, the condition is not a necessary condition for the DOEL problem. An example, which does not satisfy the condition but is DOEL, was given in [Noh01]. By the fact and Theorem 4.1.1, we can see that DOEL strictly includes RDOEL. □

This section ends with providing the following lemma, which is dual to Lemma 4.2.3 in some sense.

Lemma 4.2.9. Suppose that $f_i(x) \in \mathcal{P}^{n_i}(x)$ for $1 \le i \le m$ and $\phi(\eta, x) \in \mathcal{P}_e^k(x)$ (respectively, $\phi(x) \in \mathcal{P}^k(x)$) for any $k \ge 0$. Then, $\mathcal{L}_F\phi \in \mathcal{P}_e^{k+1}(x)$ (respectively, $\mathcal{L}_F\phi = \mathcal{L}_f\phi \in \mathcal{P}^{k+1}(x)$).

Proof. It follows from (3.1.3) and (4.2.3) that

$$\begin{aligned}
\mathcal{L}_F c(\eta, y) &= \sum_{i=1}^{d} \frac{\partial c}{\partial \eta_i} p_i + \sum_{n_i=1} \frac{\partial c}{\partial y_i} f_i + \sum_{n_i \ge 2} \frac{\partial c}{\partial y_i} x_{i2} \in \mathcal{P}_e^1(x), \\
\mathcal{L}_F x_{ij} &= \begin{cases} x_{i(j+1)} \in \mathcal{P}^j(x) & \text{if } j < n_i \\ f_i(x) \quad \in \mathcal{P}^j(x) & \text{if } j = n_i \end{cases} \\
&= \mathcal{L}_f x_{ij} \qquad \text{for } 1 \le i \le \chi(2) \text{ and } 1 \le j \le n_i,
\end{aligned}$$

where $c(\eta, y) \in \mathcal{P}_e^0(x)$. If $c(\eta, y)$ does not depend on η, i.e., $c(\eta, y) = c(y) \in \mathcal{P}^0(x)$, then $\mathcal{L}_F c = \mathcal{L}_f c \in \mathcal{P}^1(x)$. Thus, it is easy to see that the lemma is true, by a similarly way to the proof of Lemma 4.2.3 □

4.3 Necessary and Sufficient Conditions

In this section, by means of the necessary conditions given in the previous section, we derive a geometric necessary and sufficient condition for the RDOEL problem, i.e., a geometric equivalent condition for the existence of Φ and Q transforming the extended system (4.2.3) into the system (4.2.1). Since the RDOEL problem is a natural extension of the OEL problem, a geometric necessary and sufficient condition for the OEL problem under consideration of the general output transformation $y_e = q(y)$ also can be deduced from the result. Because the equivalent conditions fully characterize the problems, we can check the solvability for a given system, and it is also possible to construct an explicit change of coordinates for OEL or RDOEL by using the results. We will explain how to do that.

4.3.1 Necessary and Sufficient Condition for RDOEL

In order to derive a geometric necessary and sufficient condition for the RDOEL problem, we need the following consecutive technical lemmas. The first one is a kind of "Leibniz's rule".

Lemma 4.3.1. If X and Y are smooth vector fields and γ is a smooth real-valued function, then, for any nonnegative integer k, it holds that

$$ad^k_{(-Y)}(\gamma X) = \sum_{v=0}^{k}(-1)^v \begin{pmatrix} k \\ v \end{pmatrix} (\mathcal{L}^v_Y \gamma) ad^{k-v}_{(-Y)} X, \tag{4.3.1}$$

where $\binom{k}{v}$ represents the binomial coefficient.

Proof. The proof is by induction on k starting from $k = 0$. The equation (4.3.1) trivially holds when $k = 0$. If $k = 1$, then it follows from Proposition 2.3.2 that

$$\begin{aligned} ad_{(-Y)}(\gamma X) &= [\gamma X,\ Y] = \gamma ad_{(-Y)} X - (L_Y \gamma) X \\ &= \sum_{v=0}^{1}(-1)^v \begin{pmatrix} 1 \\ v \end{pmatrix} (\mathcal{L}^v_Y \gamma) ad^{1-v}_{(-Y)} X. \end{aligned}$$

Suppose that $k \geq 2$ and the the equation (4.3.1) is satisfied for $k - 1$, i.e., the

following equation holds:

$$ad^{k-1}_{(-Y)}(\gamma X) = \sum_{v=0}^{k-1}(-1)^v \begin{pmatrix} k-1 \\ v \end{pmatrix} (\mathcal{L}^v_Y \gamma) ad^{k-1-v}_{(-Y)} X.$$

By the above induction hypothesis and straightforward calculation, we have that

$$\begin{aligned}
ad^k_{(-Y)}(\gamma X) &= \left[ad^{k-1}_{(-Y)}(\gamma X),\ Y\right] \\
&= \sum_{v=0}^{k-1}(-1)^v \begin{pmatrix} k-1 \\ v \end{pmatrix} \left[(\mathcal{L}^v_Y \gamma) ad^{k-1-v}_{(-Y)} X,\ Y\right] \\
&= \sum_{v=0}^{k-1}(-1)^v \begin{pmatrix} k-1 \\ v \end{pmatrix} \left((\mathcal{L}^v_Y \gamma) ad^{k-v}_{(-Y)} X - (\mathcal{L}^{v+1}_Y \gamma) ad^{k-1-v}_{(-Y)} X\right) \\
&= \gamma ad^k_{(-Y)} X + (-1)^k (\mathcal{L}^k_Y \gamma) X \\
&\quad + \sum_{v=1}^{k-1}(-1)^v \left(\begin{pmatrix} k-1 \\ v \end{pmatrix} + \begin{pmatrix} k-1 \\ v-1 \end{pmatrix} \right) (\mathcal{L}^v_Y \gamma) ad^{k-v}_{(-Y)} X \\
&= \sum_{v=0}^{k-1}(-1)^v \begin{pmatrix} k-1 \\ v \end{pmatrix} (\mathcal{L}^v_Y \gamma) ad^{k-1-v}_{(-Y)} X.
\end{aligned}$$

We can observe that the equation (4.3.1) also holds for k, and thus the lemma is true. □

The second lemma is based on the above "Leibniz's rule" and a property of the vector field F of the extended system (4.2.3).

Lemma 4.3.2. Suppose that $f_i(x) \in \mathcal{P}^{n_i}(x)$ for $1 \le i \le m$. Then, for $1 \le i \le m$, $1 \le j \le n_i$, and $0 \le k \le j-1$, it holds that

$$ad^k_{(-F)} \frac{\partial}{\partial x_{ij}} = \frac{\partial}{\partial x_{i(j-k)}} + \sum_{s=0}^{k} \sum_{r=1}^{\chi(k-s+1)} C^{rs}_{kij} \frac{\partial}{\partial x_{r(n_r-k+s)}}, \tag{4.3.2}$$

where F is the vector field of the extended system (4.2.3),

$$C^{rs}_{kij} = \begin{cases} 0 & \text{if } s = 0 \text{ or } n_r < j-s, \\ * \in \mathcal{P}^{n_r-j+s}(x) & \text{if } s \ne 0 \text{ and } n_r \ge j-s, \end{cases} \tag{4.3.3}$$

for $0 \le s \le k$ and $1 \le r \le \chi(k-s+1)$.

Proof. Since the both inequalities $1 \le i \le m$ and $1 \le j \le n_i$ are equivalent to that $1 \le j \le n_1$ and $1 \le i \le \chi(j)$, the proof can be carried out by induction on k for each fixed j. The induction begins with $k = 0$. The equations (4.3.2) and (4.3.3) trivially hold for $1 \le j \le n_1$, $1 \le i \le \chi(j)$, and $k = 0$. Hence, the lemma is true when $j = 1$. If $2 \le j \le n_1$ and $k = 1$, then it holds that

$$\begin{aligned} ad_{(-F)}\frac{\partial}{\partial x_{ij}} &= \left[\frac{\partial}{\partial x_{ij}},\ \sum_{l=1}^{d} p_l \frac{\partial}{\partial \eta_l} + \sum_{n_r \ge 2}\sum_{s=2}^{n_r} x_{rs}\frac{\partial}{\partial x_{r(s-1)}} + \sum_{r=1}^{m} f_r \frac{\partial}{\partial x_{rn_r}}\right] \\ &= \frac{\partial}{\partial x_{i(j-1)}} + \sum_{r=1}^{m} \frac{\partial f_r}{\partial x_{ij}}\frac{\partial}{\partial x_{rn_r}} \\ &= \frac{\partial}{\partial x_{i(j-1)}} + \sum_{s=0}^{1}\sum_{r=1}^{\chi(2-s)} C_{1ij}^{rs}\frac{\partial}{\partial x_{r(n_r-1+s)}} \qquad \text{for } 1 \le i \le \chi(j), \end{aligned}$$

where

$$\begin{aligned} C_{1ij}^{r0} &:= 0 && \text{for } 1 \le r \le \chi(2), \\ C_{1ij}^{r1} &:= \frac{\partial f_r}{\partial x_{ij}} && \text{for } 1 \le r \le \chi(1) = m. \end{aligned}$$

Since we assume that $f_r \in \mathcal{P}^{n_r}(x)$, it follows from Proposition 4.2.2 that $C_{1ij}^{r1} = \frac{\partial f_r}{\partial x_{ij}} = 0$ if $n_r < j-1$. Otherwise, $C_{1ij}^{r1} = \frac{\partial f_r}{\partial x_{ij}} \in \mathcal{P}^{n_r-j+1}(x)$. Therefore, the equations (4.3.2) and (4.3.3) hold for $k = 1$. In addition to this, since we already showed the equations are also valid for $k = 0$, the lemma is true when $j = 2$.

Suppose that $3 \le j \le n_1$, $2 \le k \le j-1$, and the equations (4.3.2) and (4.3.3) are satisfied for $k-1$, i.e., it holds that

$$ad_{(-F)}^{k-1}\frac{\partial}{\partial x_{ij}} = \frac{\partial}{\partial x_{i(j-k+1)}} + \sum_{s=0}^{k-1}\sum_{r=1}^{\chi(k-s)} C_{(k-1)ij}^{rs}\frac{\partial}{\partial x_{r(n_r-k+1+s)}} \qquad \text{for } 1 \le i \le \chi(j),$$

where

$$C_{(k-1)ij}^{rs} = \begin{cases} 0 & \text{if } s = 0 \text{ or } n_r < j-s, \\ * \in \mathcal{P}^{n_r-j+s}(x) & \text{if } s \ne 0 \text{ and } n_r \ge j-s, \end{cases} \tag{4.3.4}$$

for $0 \le s \le k-1$ and $1 \le r \le \chi(k-s)$. Then, from the above induction hypothesis and straightforward calculation, we can obtain that

$$
\begin{aligned}
ad^k_{(-F)}\frac{\partial}{\partial x_{ij}} &= \left[ad^{k-1}_{(-F)}\frac{\partial}{\partial x_{ij}},\ F\right] \\
&= \left[\frac{\partial}{\partial x_{i(j-k+1)}} + \sum_{s=0}^{k-1}\sum_{r=1}^{\chi(k-s)} C^{rs}_{(k-1)ij}\frac{\partial}{\partial x_{r(n_r-k+1+s)}},\ F\right] \\
&= \frac{\partial}{\partial x_{i(j-k)}} + \sum_{l=1}^{m}\frac{\partial f_l}{\partial x_{i(j-k+1)}}\frac{\partial}{\partial x_{ln_l}} \\
&\quad + \sum_{s=0}^{k-1}\sum_{n_r=k-s} C^{rs}_{(k-1)ij}\left(\sum_{l=1}^{d}\frac{\partial p_l}{\partial x_{r1}}\frac{\partial}{\partial \eta_l} + \sum_{l=1}^{m}\frac{\partial f_l}{\partial x_{r1}}\frac{\partial}{\partial x_{ln_l}}\right) \\
&\quad + \sum_{s=0}^{k-1}\sum_{n_r\ge k-s+1} C^{rs}_{(k-1)ij}\left(\frac{\partial}{\partial x_{r(n_r-k+s)}} + \sum_{l=1}^{m}\frac{\partial f_l}{\partial x_{r(n_r-k+1+s)}}\frac{\partial}{\partial x_{ln_l}}\right) \\
&\quad - \sum_{s=0}^{k-1}\sum_{n_r\ge k-s} \mathcal{L}_F C^{rs}_{(k-1)ij}\frac{\partial}{\partial x_{r(n_r-k+1+s)}} \qquad \text{for } 1 \le i \le \chi(j).
\end{aligned}
$$

In the third term of the right-hand side of the above equation, the second summation index $n_r = k-s$ implies that $n_r < j-s$ because $k \le j-1$. Therefore, $C^{rs}_{(k-1)ij} = 0$ for $n_r = k-s$ by the induction hypothesis (4.3.4), and thus the third term vanishes. As a result, the above equation can be rewritten as

$$
\begin{aligned}
ad^k_{(-F)}\frac{\partial}{\partial x_{ij}} &= \frac{\partial}{\partial x_{i(j-k)}} + \sum_{l=1}^{m}\frac{\partial f_l}{\partial x_{i(j-k+1)}}\frac{\partial}{\partial x_{ln_l}} \\
&\quad + \sum_{s=0}^{k-1}\sum_{n_r\ge k-s+1} C^{rs}_{(k-1)ij}\left(\frac{\partial}{\partial x_{r(n_r-k+s)}} + \sum_{l=1}^{m}\frac{\partial f_l}{\partial x_{r(n_r-k+1+s)}}\frac{\partial}{\partial x_{ln_l}}\right) \\
&\quad - \sum_{s=0}^{k-1}\sum_{n_r\ge k-s} \mathcal{L}_F C^{rs}_{(k-1)ij}\frac{\partial}{\partial x_{r(n_r-k+1+s)}} \\
&= \frac{\partial}{\partial x_{i(j-k)}} + \sum_{r=1}^{m}\frac{\partial f_r}{\partial x_{i(j-k+1)}}\frac{\partial}{\partial x_{rn_r}} + \sum_{s=0}^{k-1}\sum_{n_r\ge k-s+1} C^{rs}_{(k-1)ij}\frac{\partial}{\partial x_{r(n_r-k+s)}} \\
&\quad + \sum_{v=0}^{k-1}\sum_{n_u\ge k-v+1}\sum_{r=1}^{m} C^{uv}_{(k-1)ij}\frac{\partial f_r}{\partial x_{u(n_u-k+1+v)}}\frac{\partial}{\partial x_{rn_r}} \qquad (v := s,\ u := r,\ r := u)
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{s=1}^{k}\sum_{n_r\geq k-s+1}\mathcal{L}_F C^{r(s-1)}_{(k-1)ij}\frac{\partial}{\partial x_{r(n_r-k+s)}} \qquad (s:=s+1)\\
&=\frac{\partial}{\partial x_{i(j-k)}}+\sum_{n_r\geq k+1}C^{r0}_{(k-1)ij}\frac{\partial}{\partial x_{r(n_r-k)}}\\
&\quad+\sum_{s=1}^{k-1}\sum_{n_r\geq k-s+1}\left(C^{rs}_{(k-1)ij}-\mathcal{L}_F C^{r(s-1)}_{(k-1)ij}\right)\frac{\partial}{\partial x_{r(n_r-k+s)}}\\
&\quad+\sum_{n_r\geq 1}\left(\frac{\partial f_r}{\partial x_{i(j-k+1)}}+\sum_{v=0}^{k-1}\sum_{n_u\geq k-v+1}C^{uv}_{(k-1)ij}\frac{\partial f_r}{\partial x_{u(n_u-k+1+v)}}-\mathcal{L}_F C^{r(k-1)}_{(k-1)ij}\right)\frac{\partial}{\partial x_{rn_r}}\\
&=\frac{\partial}{\partial x_{i(j-k)}}+\sum_{s=0}^{k}\sum_{r=1}^{\chi(k-s+1)}C^{rs}_{kij}\frac{\partial}{\partial x_{r(n_r-k+s)}} \qquad \text{for } 1\leq i\leq\chi(j),
\end{aligned}
$$

where C^{rs}_{kij}'s are defined by

$$
C^{r0}_{kij}:=C^{r0}_{(k-1)ij}, \tag{4.3.5a}
$$

$$
C^{rs}_{kij}:=C^{rs}_{(k-1)ij}-\mathcal{L}_F C^{r(s-1)}_{(k-1)ij} \qquad \text{for } 1\leq s\leq k-1, \tag{4.3.5b}
$$

$$
C^{rk}_{kij}:=\frac{\partial f_r}{\partial x_{i(j-k+1)}}+\sum_{v=0}^{k-1}\sum_{u=1}^{\chi(k-v+1)}C^{uv}_{(k-1)ij}\frac{\partial f_r}{\partial x_{u(n_u-k+1+v)}}-\mathcal{L}_F C^{r(k-1)}_{(k-1)ij}, \tag{4.3.5c}
$$

for $1\leq r\leq\chi(k-s+1)$. Since it holds that $\chi(k-s+1)\leq\chi(k-s)$ as mentioned in Remark 4.2.3, all the above C^{rs}_{kij}'s are well defined from the induction hypothesis (4.3.4). Thus, one can observe that the equation (4.3.2) also holds for k.

Finally, let us check whether C^{rs}_{kij}'s defined by (4.3.5) satisfy the condition (4.3.3) or not. If $s=0$, then it follows from the induction hypothesis (4.3.4) and (4.3.5a) that $C^{r0}_{kij}=0$ for $1\leq r\leq\chi(k+1)$. If $1\leq s\leq k-1$, then, by the induction hypothesis (4.3.4) and Lemma 4.2.9, it holds that

$$
C^{rs}_{(k-1)ij}=\begin{cases}0 & \text{if } n_r<j-s,\\ *\in\mathcal{P}^{n_r-j+s}(x) & \text{if } n_r\geq j-s,\end{cases}
$$

$$
\mathcal{L}_F C^{r(s-1)}_{(k-1)ij}=\begin{cases}0 & \text{if } s=1 \text{ or } n_r<j-s+1,\\ *\in\mathcal{P}^{n_r-j+s}(x) & \text{if } s\neq 1 \text{ and } n_r\geq j-s+1,\end{cases}
$$

for $1 \leq r \leq \chi(k-s+1)$. Therefore, we obtain from (4.3.5b) that

$$C_{kij}^{rs} = \begin{cases} 0 & \text{if } n_r < j-s, \\ * \in \mathcal{P}^{n_r-j+s}(x) & \text{if } n_r \geq j-s, \end{cases}$$

for $1 \leq s \leq k-1$ and $1 \leq r \leq \chi(k-s+1)$. Lastly, in the equation (4.3.5c), it holds that

$$\frac{\partial f_r}{\partial x_{i(j-k+1)}} = \begin{cases} 0 & \text{if } n_r < j-k, \\ * \in \mathcal{P}^{n_r-j+k}(x) & \text{if } n_r \geq j-k, \end{cases} \tag{4.3.6a}$$

$$C_{(k-1)ij}^{uv} = \begin{cases} 0 & \text{if } v = 0 \text{ or } n_u < j-v, \\ * \in \mathcal{P}^{n_u-j+v}(x) & \text{if } v \neq 0 \text{ and } n_u \geq j-v, \end{cases} \tag{4.3.6b}$$

$$\frac{\partial f_r}{\partial x_{u(n_u-k+1+v)}} = \begin{cases} 0 & \text{if } n_r < n_u-k+v, \\ * \in \mathcal{P}^{n_r-(n_u-k+v)}(x) & \text{if } n_r \geq n_u-k+v, \end{cases} \tag{4.3.6c}$$

$$\mathcal{L}_F C_{(k-1)ij}^{r(k-1)} = \begin{cases} 0 & \text{if } n_r < j-k+1, \\ * \in \mathcal{P}^{n_r-j+k}(x) & \text{if } n_r \geq j-k+1. \end{cases} \tag{4.3.6d}$$

Combining the equations (4.3.6b) and (4.3.6c), we have

$$\begin{aligned} & C_{(k-1)ij}^{uv} \frac{\partial f_r}{\partial x_{u(n_u-k+1+v)}} \\ & = \begin{cases} * \in \mathcal{P}^{n_r-j+k}(x) & \text{if } v \neq 0 \text{ , } n_u \geq j-v \text{ ,and } n_r \geq n_u-k+v, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Since $n_u \geq j-v$ and $n_r \geq n_u-k+v$ implies that $n_r \geq j-k$, the above equation can be rewritten as

$$C_{(k-1)ij}^{uv} \frac{\partial f_r}{\partial x_{u(n_u-k+1+v)}} = \begin{cases} * \in \mathcal{P}^{n_r-j+k} & \text{if } v \neq 0 \text{ and } n_r \geq j-k, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3.6e}$$

Therefore, it follows from (4.3.5c), (4.3.6a), (4.3.6d), and (4.3.6e) that

$$C_{kij}^{rk} = \begin{cases} 0 & \text{if } n_r < j-k, \\ * \in \mathcal{P}^{n_r-j+k}(x) & \text{if } n_r \geq j-k, \end{cases}$$

for $1 \le r \le \chi(1)$. Consequently, the condition (4.3.3) is satisfied and it is concluded that the lemma is also true for $3 \le j \le n_1$. □

In fact, the above lemma is needed to derive the following lemma which plays a key role in proving a necessary and sufficient condition for the RDOEL problem.

Lemma 4.3.3. Suppose that $f_i(x) \in \mathcal{P}^{n_i}(x)$ for $1 \le i \le m$ and $\phi(\eta, x) \in \mathcal{P}_e^c(x)$ for any $c \ge 0$. Then, for $1 \le i \le m$, $1 \le j \le n_i$, and $0 \le k \le j-1$, it holds that

$$ad^k_{(-F)}\left(\phi \frac{\partial}{\partial x_{ij}}\right) = \sum_{\sigma=0}^{k} \left((C_\phi)^\sigma_k \frac{\partial}{\partial x_{i(j-k+\sigma)}} + \sum_{\rho=1}^{\chi(k-\sigma+1)} (C_\phi)^{\rho\sigma}_{kij} \frac{\partial}{\partial x_{\rho(n_\rho-k+\sigma)}} \right),$$

where

$$\begin{aligned}
(C_\phi)^\sigma_k &:= (-1)^\sigma \begin{pmatrix} k \\ \sigma \end{pmatrix} \mathcal{L}_F^\sigma \phi \in \mathcal{P}_e^{c+\sigma}(x), \\
(C_\phi)^{\rho\sigma}_{kij} &:= \sum_{v=0}^{\sigma} (-1)^v \begin{pmatrix} k \\ v \end{pmatrix} (\mathcal{L}_F^v \phi) C^{\rho(\sigma-v)}_{(k-v)ij} \\
&= \begin{cases} 0 & \text{if } \sigma = 0 \text{ or } n_\rho < j-\sigma, \\ * \in \mathcal{P}_e^{c+n_\rho-j+\sigma}(x) & \text{if } \sigma \ne 0 \text{ and } n_\rho \ge j-\sigma, \end{cases}
\end{aligned}$$

for $0 \le \sigma \le k$ and $1 \le \rho \le \chi(k-\sigma+1)$.

Proof. Let $\phi(\eta, x) \in \mathcal{P}_e^c(x)$ for some nonnegative integer c. Then, for $1 \le i \le m$, $1 \le j \le n_i$, and $0 \le k \le j-1$, it follows from Lemma 4.3.1 and Lemma 4.3.2 that

$$\begin{aligned}
&ad^k_{(-F)}\left(\phi \frac{\partial}{\partial x_{ij}}\right) = \sum_{v=0}^{k} (-1)^v \begin{pmatrix} k \\ v \end{pmatrix} (\mathcal{L}_F^v \phi) ad^{k-v}_{(-F)} \frac{\partial}{\partial x_{ij}} \\
&= \sum_{v=0}^{k} (-1)^v \begin{pmatrix} k \\ v \end{pmatrix} (\mathcal{L}_F^v \phi) \left(\frac{\partial}{\partial x_{i(j-k+v)}} + \sum_{s=0}^{k-v} \sum_{r=1}^{\chi(k-v-s+1)} C^{rs}_{(k-v)ij} \frac{\partial}{\partial x_{r(n_r-k+v+s)}} \right) \\
&= \sum_{\sigma=0}^{k} (-1)^\sigma \begin{pmatrix} k \\ \sigma \end{pmatrix} (\mathcal{L}_F^\sigma \phi) \frac{\partial}{\partial x_{i(j-k+\sigma)}} \\
&\quad + \sum_{v=0}^{k} \sum_{s=0}^{k-v} \sum_{r=1}^{\chi(k-v-s+1)} (-1)^v \begin{pmatrix} k \\ v \end{pmatrix} (\mathcal{L}_F^v \phi) C^{rs}_{(k-v)ij} \frac{\partial}{\partial x_{r(n_r-k+v+s)}}.
\end{aligned}$$

By changing the summation indices s and r into $\sigma := v+s$ and $\rho := r$ respectively, the above equation can be rewritten as

$$\begin{aligned} ad^k_{(-F)}\left(\phi\frac{\partial}{\partial x_{ij}}\right) &= \sum_{\sigma=0}^{k}(-1)^\sigma \begin{pmatrix} k \\ \sigma \end{pmatrix} (\mathcal{L}_F^\sigma \phi)\frac{\partial}{\partial x_{i(j-k+\sigma)}} \\ &\quad + \sum_{\sigma=0}^{k}\sum_{v=0}^{\sigma}\sum_{\rho=1}^{\chi(k-\sigma+1)}(-1)^v \begin{pmatrix} k \\ v \end{pmatrix} (\mathcal{L}_F^v \phi) C^{\rho(\sigma-v)}_{(k-v)ij}\frac{\partial}{\partial x_{\rho(n_\rho-k+\sigma)}} \\ &= \sum_{\sigma=0}^{k}\left((C_\phi)^\sigma_k \frac{\partial}{\partial x_{i(j-k+\sigma)}} + \sum_{\rho=1}^{\chi(k-\sigma+1)}(C_\phi)^{\rho\sigma}_{kij}\frac{\partial}{\partial x_{\rho(n_\rho-k+\sigma)}}\right), \end{aligned}$$

where

$$\begin{aligned} (C_\phi)^\sigma_k &:= (-1)^\sigma \begin{pmatrix} k \\ \sigma \end{pmatrix} (\mathcal{L}_F^\sigma \phi), \\ (C_\phi)^{\rho\sigma}_{kij} &:= \sum_{v=0}^{\sigma}(-1)^v \begin{pmatrix} k \\ v \end{pmatrix} (\mathcal{L}_F^v \phi) C^{\rho(\sigma-v)}_{(k-v)ij}, \end{aligned}$$

for $0 \le \sigma \le k$ and $1 \le \rho \le \chi(k-s+1)$. Since $\phi \in \mathcal{P}_e^c(x)$, it follows from Lemma 4.2.9 that $\mathcal{L}_F^\sigma \phi \in \mathcal{P}_e^{c+\sigma}(x)$ for any $\sigma \ge 0$. Hence, $(C_\phi)^\sigma_k \in \mathcal{P}_e^{c+\sigma}(x)$ for $0 \le \sigma \le k$. By Lemma 4.3.2, it holds that

$$C^{\rho(\sigma-v)}_{(k-v)ij} = \begin{cases} 0 & \text{if } \sigma = v \text{ or } n_\rho < j-\sigma+v, \\ * \in \mathcal{P}^{n_r-j+s-u}(x) & \text{if } \sigma \ne v \text{ and } n_\rho \ge j-\sigma+v, \end{cases}$$

for $0 \le \sigma \le k$, $0 \le v \le \sigma$ and $1 \le \rho \le \chi(k-\sigma+1)$. Since $0 \le v \le \sigma$, the condition that $\sigma \ne v$ and $n_\rho \ge j-\sigma+v$ is equivalent to that $0 \le v \le \sigma-1$ and $0 \le v \le n_\rho - j + \sigma$. Hence, the above equation can be rewritten as

$$C^{\rho(\sigma-v)}_{(k-v)ij} = \begin{cases} * \in \mathcal{P}^{n_r-j+s-u}(x) & \text{if } 0 \le v \le \min\{\sigma-1,\ n_\rho - j + \sigma\}, \\ 0 & \text{otherwise.} \end{cases}$$

If $\sigma = 0$ or $n_\rho < j-\sigma$, then $\min\{\sigma-1,\ n_\rho - j + \sigma\} < 0$. Therefore, it follows

from the above equation that

$$(C_\phi)^{\rho\sigma}_{kij} = \sum_{v=0}^{\sigma}(-1)^v \begin{pmatrix} k \\ v \end{pmatrix} (\mathcal{L}_F^v \phi) C^{\rho(\sigma-v)}_{(k-v)ij} = 0.$$

If $\sigma \neq 0$ and $n_\rho \geq j - \sigma$, then $\min\{\sigma - 1,\ n_\rho - j + \sigma\} \geq 0$. Thus, we have

$$(C_\phi)^{\rho\sigma}_{kij} = \sum_{v=0}^{\min\{\sigma-1,\ n_\rho-j+\sigma\}} (-1)^v \begin{pmatrix} k \\ v \end{pmatrix} (\mathcal{L}_F^v \phi) C^{\rho(\sigma-v)}_{(k-v)ij} \in \mathcal{P}_e^{c+n_\rho-j+\sigma}(x),$$

because $\mathcal{L}_F^v \phi \in \mathcal{P}_e^{c+v}(x)$ and $C^{\rho(\sigma-v)}_{(k-v)ij} \in \mathcal{P}^{n_\rho-j+\sigma-v}(x)$ for $0 \leq v \leq \min\{\sigma - 1,\ n_\rho - j + \sigma\}$. □

Now, we provide a geometric necessary and sufficient condition for the RDOEL problem, and prove it by means of the necessary conditions presented in the previous section and Lemma 4.3.3.

Theorem 4.3.4. The system (3.1.3) is RDOEL via the auxiliary dynamics (4.1.1) if and only if $f_i(x) \in \mathcal{P}^{n_i}(x)$ for $1 \leq i \leq m$ and there exist m vector fields $X_1, \ldots, X_m$ satisfying the following conditions:

(R1) X_i's should be of the following form:

$$X_i = \sum_{r=1}^{\chi(n_i)} \sum_{s=0}^{n_r-n_i} \phi_i^{rs}(\eta, x) \frac{\partial}{\partial x_{r(n_i+s)}} \qquad \text{for } 1 \leq i \leq m,$$

where $\phi_i^{rs} \in \mathcal{P}_e^s(x)$.

(R2) The n vector fields $ad_{(-F)}^{n_i-j} X_i$'s are linearly independent on $U \times W$, where $1 \leq i \leq m$, $1 \leq j \leq n_i$, and $U \times W$ is a neighborhood of $(\eta(0), x(0))$.

(R3) On $U \times W$, it holds that

$$\left[ad_{(-F)}^{n_\mu-\kappa} X_\mu,\ ad_{(-F)}^{n_\nu-\lambda} X_\nu\right] = 0$$

for $\mu, \nu = 1, \ldots, m$, $1 \leq \kappa \leq n_\mu$, and $1 \leq \lambda \leq n_\nu$.

Proof. Throughout the proof, we use the following notation: when $\alpha = [\alpha_1 \ \cdots \ \alpha_n]^T$, we write $\mathrm{d}\alpha := [\mathrm{d}\alpha_1 \ \cdots \ \mathrm{d}\alpha_n]^T$ and $\frac{\partial}{\partial \alpha} := [\frac{\partial}{\partial \alpha_1} \ \cdots \ \frac{\partial}{\partial \alpha_n}]$.

(Proof of Necessity): If the system (3.1.3) is RDOEL via the auxiliary dynamics (4.1.1), then there exist a neighborhood $U \times W \in \mathbb{R}^d \times \mathbb{R}^n$ of $(\eta(0), x(0))$ and two maps $\Phi : U \times W \to \mathbb{R}^{d+n}$, $(\eta, x) \mapsto (w, z) = (\eta, z)$ and $Q : U \times h(W) \to \mathbb{R}^{d+m}$, $(\eta, y) \mapsto (w, y_e) = (\eta, q(\eta, y))$, which are diffeomorphisms onto their images and transform the extended system (4.2.3) into the system (4.2.1). Since $\mathrm{d}\eta = \mathrm{d}w$, it follows from (4.2.8) that

$$\begin{bmatrix} \mathrm{d}\eta \\ \mathrm{d}x \end{bmatrix} = \mathrm{d}\Phi^{-1} = \begin{bmatrix} I_d & O \\ R & S \end{bmatrix} \begin{bmatrix} \mathrm{d}w \\ \mathrm{d}z \end{bmatrix},$$

where S is the matrix defined by the equations (4.2.9) and (4.2.10). By the duality between 1-forms and vector fields, it holds that

$$\begin{bmatrix} \frac{\partial}{\partial w} & \frac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial \eta} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} I_d & O \\ R & S \end{bmatrix}. \tag{4.3.7}$$

Therefore, it follows from (4.2.9) and (4.3.7) that

$$\frac{\partial}{\partial z_{\nu n_\nu}} = \sum_{\mu=1}^{m} \sum_{\kappa=1}^{n_\mu} (T_{\mu\nu})_{\kappa n_\nu} \frac{\partial}{\partial x_{\mu\kappa}} \qquad \text{for } 1 \le \nu \le m.$$

In the above equation, by the lower triangular-like form (4.2.10) of each $T_{\mu\nu}$, $(T_{\mu\nu})_{\kappa n_\nu} = 0$ if $\kappa < n_\nu$. Hence, the above equation becomes

$$\begin{aligned} \frac{\partial}{\partial z_{\nu n_\nu}} &= \sum_{\mu=1}^{m} \sum_{\kappa \ge n_\nu} (T_{\mu\nu})_{\kappa n_\nu} \frac{\partial}{\partial x_{\mu\kappa}} \\ &= \sum_{n_\mu \ge n_\nu} \sum_{\kappa=n_\nu}^{n_\mu} (T_{\mu\nu})_{\kappa n_\nu} \frac{\partial}{\partial x_{\mu\kappa}} \qquad \text{for } 1 \le \nu \le m, \end{aligned}$$

where $(T_{\mu\nu})_{\kappa n_\nu} \in \mathcal{P}_e^{\kappa - n_\nu}(z)$ and, in particular, $(T_{\mu\nu})_{\kappa n_\nu} = \frac{\partial \tilde{q}_\mu}{\partial y_{e\nu}}$ when $\kappa = n_\nu$. Let the representation of $(T_{\mu\nu})_{\kappa n_\nu}$ in the (η, x)-coordinates be $\phi_\nu^{\mu(\kappa - n_\nu)}$. Then, $\phi_\nu^{\mu(\kappa - n_\nu)} \in \mathcal{P}_e^{\kappa - n_\nu}(x)$ by Lemma 4.2.8. Finally, change the indices ν, μ, and κ to $i = \nu$, $r = \mu$, and $s = \kappa - n_\nu$, respectively. Then, the above equation can be

rewritten as

$$\frac{\partial}{\partial z_{in_i}} = \sum_{r=1}^{\chi(n_i)} \sum_{s=0}^{n_r-n_i} \phi_i^{rs} \frac{\partial}{\partial x_{r(n_i+s)}} \quad \text{for } 1 \le i \le m,$$

where $\phi_i^{rs} \in \mathcal{P}_e^s(x)$ and, in particular, $\phi_i^{r0} = \frac{\partial \tilde{q}_r}{\partial y_{ei}}$ because $s = 0$ implies $k = n_\nu$. We define

$$X_i := \frac{\partial}{\partial z_{in_i}} = \sum_{r=1}^{\chi(n_i)} \sum_{s=0}^{n_r-n_i} \phi_i^{rs} \frac{\partial}{\partial x_{r(n_i+s)}} \qquad \text{for } 1 \le i \le m, \tag{4.3.8}$$

such that the condition (R1) is satisfied.

Since we assume that the system (3.1.3) is RDOEL via the auxiliary dynamics (4.1.1), the vector field F of the extended system (4.2.3) can be expressed in the (w, z)-coordinates as follows:

$$\begin{aligned} F = &\sum_{k=1}^{d} \tilde{p}_k(w, y_e) \frac{\partial}{\partial w_k} \\ &+ \sum_{n_i \ge 2} \sum_{j=2}^{n_i} \Big(z_{ij} + \tilde{a}_{i(j-1)}(w, y_e) \Big) \frac{\partial}{\partial z_{i(j-1)}} + \sum_{i=1}^{m} \tilde{a}_{in_i}(w, y_e) \frac{\partial}{\partial z_{in_i}}, \end{aligned}$$

where $\tilde{p}_k(w, y_e) := p_k(w, \tilde{q}(w, y_e)) = p_k(\eta, y)$ for $1 \le k \le d$ and $\tilde{a}_{ij}(w, y_e) := a_{ij}(w, \tilde{q}(w, y_e)) = a_{ij}(\eta, y)$ for $1 \le i \le m$ and $1 \le j \le n_i$. By straightforward calculation, it holds that

$$ad_{(-F)} \frac{\partial}{\partial z_{ij}} = \left[\frac{\partial}{\partial z_{ij}},\ F \right] = \frac{\partial}{\partial z_{i(j-1)}}, \tag{4.3.9}$$

for $1 \le i \le \chi(2)$ and $2 \le j \le n_i$. Therefore, from (4.3.8) and (4.3.9), we have $ad_{(-F)}^{n_i-j} X_i = \frac{\partial}{\partial z_{i(n_i-(n_i-j))}} = \frac{\partial}{\partial z_{ij}}$ for $1 \le i \le m$ and $1 \le j \le n_i$. This implies that $\left\{ ad_{(-F)}^{n_i-j} X_i : 1 \le i \le m,\ 1 \le j \le n_i \right\} = \left\{ \frac{\partial}{\partial z_{ij}} : 1 \le i \le m,\ 1 \le j \le n_i \right\}$. Hence, it is easy to see that the conditions (R2) and (R3) are satisfied.

(Proof of Sufficiency): Suppose that there exist m vector fields $X_1, \ldots, X_m$ satisfying the conditions (R1), (R2), and (R3). Then, by (R2), (R3), Theorem 2.4.5 (Simultaneous Rectification Theorem), and Corollary 2.4.6, there exists a

coordinate chart $(\bar{U} \times \bar{W}, (\bar{w}, z))$, where $\bar{U} \times \bar{W} \subset \mathbb{R}^d \times \mathbb{R}^n$ is a neighborhood of $(\eta(0), x(0))$ and $(\bar{w}, z) = (\bar{w}_1, \ldots, \bar{w}_d, z_{11}, \ldots, z_{mn_m})$, such that

$$\frac{\partial}{\partial z_{\nu\lambda}} = ad_{(-F)}^{n_\nu - \lambda} X_\nu \qquad \text{for } 1 \le \nu \le m \text{ and } 1 \le \lambda \le n_\nu. \tag{4.3.10}$$

Moreover, by (R1) and Lemma 4.3.3, each vector field $\frac{\partial}{\partial z_{\nu\lambda}}$ can be expressed as

$$\frac{\partial}{\partial z_{\nu\lambda}} = ad_{(-F)}^{n_\nu - \lambda} X_\nu = ad_{(-F)}^{n_\nu - \lambda} \left(\sum_{r=1}^{\chi(n_\nu)} \sum_{s=0}^{n_r - n_\nu} \phi_\nu^{rs} \frac{\partial}{\partial x_{r(n_\nu + s)}} \right) \tag{4.3.11}$$

$$= \sum_{r=1}^{\chi(n_\nu)} \sum_{s=0}^{n_r - n_\nu} \sum_{\sigma=0}^{n_\nu - \lambda} \left((C_{\phi_\nu^{rs}})_{n_\nu - \lambda}^{\sigma} \frac{\partial}{\partial x_{r(s+\lambda+\sigma)}} + \sum_{\rho=1}^{\chi(n_\nu - \lambda - \sigma + 1)} (C_{\phi_\nu^{rs}})_{(n_\nu - \lambda) r (n_\nu + s)}^{\rho\sigma} \frac{\partial}{\partial x_{\rho(n_\rho - n_\nu + \lambda + \sigma)}} \right)$$

for $1 \le \nu \le m$ and $1 \le \lambda \le n_\nu$. Since all the $\frac{\partial}{\partial z_{\nu\lambda}}$'s do not depend on $\frac{\partial}{\partial \eta_1}, \ldots, \frac{\partial}{\partial \eta_d}$ and they are linear combinations of $\frac{\partial}{\partial x_{\mu\kappa}}$'s, we have

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} L, \tag{4.3.12}$$

where

$$L := \begin{bmatrix} H_{11} & \cdots & H_{1m} \\ \vdots & \ddots & \vdots \\ H_{m1} & \cdots & H_{mm} \end{bmatrix}_{n \times n}, \tag{4.3.13}$$

and $H_{\mu\nu}$ is an $n_\mu \times n_\nu$ matrix whose (κ, λ)-th entry is the coefficient of $\frac{\partial}{\partial x_{\mu\kappa}}$ in the representation (4.3.11) of $\frac{\partial}{\partial z_{\nu\lambda}}$ for $\mu, \nu = 1 \ldots, m$, $1 \le \kappa \le n_\mu$, and $1 \le \lambda \le n_\nu$. In addition to this, since (η, x) is also a coordinate map on $\bar{U} \times \bar{W}$, the remained vector fields $\frac{\partial}{\partial \bar{w}_1}, \ldots, \frac{\partial}{\partial \bar{w}_d}$ can be expressed as linear combinations of $\frac{\partial}{\partial \eta_1}, \ldots, \frac{\partial}{\partial \eta_d}, \frac{\partial}{\partial x_{11}}, \ldots, \frac{\partial}{\partial x_{mn_m}}$ as follows:

$$\frac{\partial}{\partial \bar{w}} = \begin{bmatrix} \frac{\partial}{\partial \eta} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} M_{d \times d} \\ N_{n \times d} \end{bmatrix}. \tag{4.3.14}$$

As a consequence of (4.3.12) and (4.3.14), we have

$$\begin{bmatrix} \frac{\partial}{\partial \bar{w}} & \frac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial \eta} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} M & O \\ N & L \end{bmatrix}.$$

Trivially, both M and L are nonsingular on $\bar{U} \times \bar{W}$. Therefore, by the duality between vector fields and 1-forms, it holds that

$$\begin{bmatrix} \mathrm{d}\bar{w} \\ \mathrm{d}z \end{bmatrix} = \begin{bmatrix} M^{-1} & O \\ -L^{-1}NM^{-1} & L^{-1} \end{bmatrix} \begin{bmatrix} \mathrm{d}\eta \\ \mathrm{d}x \end{bmatrix}.$$

Let $w = \eta$. Then, $\mathrm{d}w = \mathrm{d}\eta$ and thus it follows from the above equation that

$$\begin{bmatrix} \mathrm{d}w \\ \mathrm{d}z \end{bmatrix} = \begin{bmatrix} I_d & O \\ -L^{-1}NM^{-1} & L^{-1} \end{bmatrix} \begin{bmatrix} \mathrm{d}\eta \\ \mathrm{d}x \end{bmatrix}. \tag{4.3.15}$$

Since L^{-1} is nonsingular at $(\eta(0), x(0))$, the 1-forms $\mathrm{d}w_1, \ldots, \mathrm{d}w_d, \mathrm{d}z_{11}, \ldots, \mathrm{d}z_{mn_m}$ are linearly independent at $(\eta(0), x(0))$. This implies that (w, z) can be also a coordinate map on a neighborhood $U \times W$ of $(\eta(0), x(0))$, and thus there exists a coordinate transformation Φ such that $\Phi : U \times W \to \mathbb{R}^{d+n}, (\eta, x) \mapsto (w, z) = (\eta, z)$, which is a diffeomorphism onto its image.

Next, we verify the existence of an output transformation $Q(\eta, y) = [w^T \ y_e^T]^T$ such that $w = \eta$ and $y_e = q(\eta, y) = Cz$. To this end, we first show that each block $H_{\mu\nu}$ for $1 \le \mu, \nu \le m$ of the matrix L, defined by the equation (4.3.13), has a lower triangular-like form similar to (4.2.10) in the proof of Theorem 4.2.6. As mentioned above, $(H_{\mu\nu})_{\kappa\lambda}$ is the coefficient of $\frac{\partial}{\partial x_{\mu\kappa}}$ in the representation (4.3.11) of $\frac{\partial}{z_{\nu\lambda}}$ for $\mu, \nu = 1, \ldots, m$, $1 \le \kappa \le n_\mu$, and $1 \le \lambda \le n_\nu$. For the vector field $\frac{\partial}{\partial x_{k(s+\lambda+\sigma)}}$ in the right-hand side of the equation (4.3.11), let $\kappa := s + \lambda + \sigma$. Then, $\kappa \ge \lambda$ because $s \ge 0$ and $\sigma \ge 0$. Moreover, by Lemma 4.3.3, its coefficient $(C_{\phi_\nu^{kl}})^s_{n_\nu - \lambda}$ satisfies the following condition:

$$(C_{\phi_\nu^{rs}})^\sigma_{n_\nu-\lambda} = \begin{cases} \phi_\nu^{r0} \in \mathcal{P}_e^0(x) & \text{if } \kappa = \lambda \ (s = 0 \text{ and } \sigma = 0), \\ * \in \mathcal{P}_e^{s+\sigma}(x) = \mathcal{P}_e^{\kappa-\lambda}(x) & \text{if } \kappa > \lambda. \end{cases} \tag{4.3.16}$$

Similarly, for the vector field $\frac{\partial}{\partial x_{\rho(n_\rho - n_\nu + \lambda + \sigma)}}$ in the right-hand side of (4.3.11), let $\kappa := n_\rho - n_\nu + \lambda + \sigma$. Then, it holds that

$$\begin{cases} \kappa < \lambda & \Leftrightarrow \quad n_\rho < n_\nu - \sigma \le n_\nu + s - \sigma & \text{for all } 0 \le s \le n_r - n_\nu, \\ \kappa = \lambda & \Leftrightarrow \quad n_\rho = n_\nu - \sigma \le n_\nu + s - \sigma & \text{for all } 0 \le s \le n_r - n_\nu, \\ & & \text{(equality holds only for } s = 0.) \\ \kappa > \lambda & \Leftrightarrow \quad n_\rho \ge n_\nu + s - \sigma & \text{for some } 0 \le s \le n_r - n_\nu. \end{cases}$$

Therefore, by Lemma 4.3.3, its coefficient $\sum_{r=1}^{\chi(n_\nu)} \sum_{s=0}^{n_r - n_\nu} (C_{\phi_\nu^{rs}})^{\rho\sigma}_{(n_\nu - \lambda) r (n_\nu + s)}$ satisfies that

$$\begin{aligned} &\sum_{r=1}^{\chi(n_\nu)} \sum_{s=0}^{n_r - n_\nu} (C_{\phi_\nu^{rs}})^{\rho\sigma}_{(n_\nu - \lambda) r (n_\nu + s)} \\ &= \begin{cases} 0 & \text{if } \sigma = 0 \text{ or } \kappa < \lambda, \\ \sum_{r=1}^{\chi(n_\nu)} (C_{\phi_\nu^{r0}})^{\rho\sigma}_{(n_\nu - \lambda) r n_\nu} \in \mathcal{P}_e^0(x) & \text{if } \sigma \ne 0 \text{ and } \kappa = \lambda, \\ * \in \mathcal{P}_e^{n_\rho - n_\nu + \sigma}(x) = \mathcal{P}_e^{\kappa - \lambda}(x) & \text{if } \sigma \ne 0 \text{ and } \kappa > \lambda. \end{cases} \end{aligned} \tag{4.3.17}$$

Consequently, from (4.3.16) and (4.3.17), we can observe that each $H_{\mu\nu}$ has the following lower triangular-like form:

$$(H_{\mu\nu})_{\kappa\lambda} = \begin{cases} 0 & \text{if } \kappa < \lambda, \\ * \in \mathcal{P}_e^0(x) & \text{if } \kappa = \lambda, \\ * \in \mathcal{P}_e^{\kappa - \lambda}(x) & \text{if } \kappa > \lambda. \end{cases} \tag{4.3.18}$$

for $\mu, \nu = 1, \ldots, m$, $1 \le \kappa \le n_\mu$, and $1 \le \lambda \le n_\nu$. Since each block of the matrix L has the above lower triangular-like form, it follows from (4.3.13) and (4.3.15) that

$$\begin{aligned} \begin{bmatrix} \mathrm{d}\eta \\ \mathrm{d}y \end{bmatrix} = \begin{bmatrix} \mathrm{d}\eta \\ \mathrm{d}x_{11} \\ \vdots \\ \mathrm{d}x_{m1} \end{bmatrix} &= \begin{bmatrix} I_d & O & \cdots & O \\ * & (H_{11})_{11} & \cdots & (H_{1m})_{11} \\ \vdots & \vdots & \ddots & \vdots \\ * & (H_{m1})_{11} & \cdots & (H_{mm})_{11} \end{bmatrix} \begin{bmatrix} \mathrm{d}w \\ \mathrm{d}z_{11} \\ \vdots \\ \mathrm{d}z_{m1} \end{bmatrix} \\ &=: \begin{bmatrix} I_d & O \\ D_{m \times d} & E_{m \times m} \end{bmatrix} \begin{bmatrix} \mathrm{d}w \\ \mathrm{d}y_e \end{bmatrix}, \end{aligned}$$

with the matrix $E_{m\times m}$ of which all the entries are functions of $\eta, x_{11}, x_{21}, \ldots, x_{m1}$ only. Moreover, by a similar way to the proof of Theorem 4.2.6, it is not difficult to see that $E_{m\times m}$ is also nonsingular on $U \times h(W)$, because the matrix L is nonsingular on $U \times W$ and each block of the matrix L has the lower triangular-like form (4.3.18). Therefore, there exists an output transformation $Q : U \times h(W) \to \mathbb{R}^{d+m}$, $(\eta, y) \mapsto (w, y_e) = (\eta, q(\eta, y))$, which is a diffeomorphism onto its image, such that Q forms a part of the coordinate transformation Φ as follows:

$$y_{ei} = q_i(\eta, y) = q_i(\eta, x_{11}, \ldots, x_{m1}) = z_{i1} \qquad \text{for } 1 \le i \le m,$$

where $q(\eta, y) = [q_1(\eta, y) \ \cdots \ q_m(\eta, y)]^T$. Conversely, there also exists the inverse output transformation $Q^{-1}(w, y_e) = [w^T \ \tilde{q}(w, y_e)^T]^T = [\eta^T \ y^T]^T$.

Finally, we determine the vector field F of the extended system (4.2.3) in the (w, z)-coordinates. Let $F_z := \sum_{k=1}^{d} F_{0k} \frac{\partial}{\partial w_k} + \sum_{r=1}^{m} \sum_{s=1}^{n_r} F_{rs} \frac{\partial}{\partial z_{rs}}$ denote the representation of F in the (w, z)-coordinates. Since $w = \eta$, $F_{0k} = \dot{w}_k = \dot{\eta}_k = p_k(\eta, y)$ for $1 \le k \le d$. In addition, it follows from the equation (4.3.10) that

$$\begin{aligned}\frac{\partial}{\partial z_{ij}} &= ad_{(-F)}^{n_i - j} X_i = \left[ad_{(-F)}^{n_i-(j+1)} X_i, F \right] = \left[\frac{\partial}{\partial z_{i(j+1)}}, \ F \right] \\ &= \sum_{k=1}^{d} \left(\frac{\partial F_{0k}}{\partial z_{i(j+1)}} \right) \frac{\partial}{\partial w_k} + \sum_{r=1}^{m} \sum_{s=1}^{n_r} \left(\frac{\partial F_{rs}}{\partial z_{i(j+1)}} \right) \frac{\partial}{\partial z_{rs}},\end{aligned}$$

for $1 \le i \le m$ and $1 \le j \le n_i - 1$. The above equation implies that $\frac{\partial F_{rs}}{\partial z_{i(j+1)}} = \delta_{ir} \cdot \delta_{js}$ for $i, r = 1, \ldots, m$, $1 \le j \le n_i - 1$, and $1 \le s \le n_r$. Therefore, $F_{ij} = z_{i(j+1)} + \tilde{a}_{ij}(w, z_{11}, \ldots, z_{m1}) = z_{i(j+1)} + \tilde{a}_{ij}(w, y_e)$ for $1 \le i \le m$ and $1 \le j \le n_i - 1$, and $F_{in_i} = \tilde{a}_{in_i}(w, z_{11}, \ldots, z_{m1}) = \tilde{a}_{in_i}(w, y_e)$ for $1 \le i \le m$. Therefore, we have

$$F_z = \sum_{k=1}^{d} p_k(\eta, y) \frac{\partial}{\partial w_k} + \sum_{i=1}^{m} \left(\sum_{j=1}^{n_i - 1} \Big(z_{i(j+1)} + a_{ij}(\eta, y) \Big) \frac{\partial}{\partial z_{ij}} + a_{in_i}(\eta, y) \frac{\partial}{\partial z_{in_i}} \right),$$

where $a_{ij}(\eta, y) := \tilde{a}_{ij}(\eta, q(\eta, y)) = \tilde{a}_{ij}(w, y_e)$ for $1 \le i \le m$ and $1 \le j \le n_i$. We can see that F_z is equal to the vector field of the system (4.2.1). □

Remark 4.3.1. By the equations (4.3.10) and (4.3.12), the condition (R2) holds if and only if the matrix L is nonsingular on $U \times W$. In the proof of Theorem

4.3.4, it is shown that each block of L has the lower triangular-like form (4.3.18) similar to that of S defined by (4.2.9) and (4.2.10) in the proof of Theorem 4.2.6. Therefore, by a similar way to the equation (4.2.7), $\det L$ can be easily calculated and thus we obtain from $\det L \neq 0$ the condition for (R2) to be satisfied. We will illustrate it by examples in the next section. □

4.3.2 Necessary and Sufficient Condition for OEL

As mentioned before, if the auxiliary dynamics (4.1.1) is not employed, then the RDOEL problem becomes the OEL problem. Therefore, we can derive a geometric necessary and sufficient condition for the OEL problem, from a direct consequence of Theorem 4.3.4. The following corollary is that.

Corollary 4.3.5. The system (3.1.3) is OEL if and only if $f_i(x) \in \mathcal{P}^{n_i}(x)$ for $1 \leq i \leq m$ and there exist m vector fields $X_1, \ldots, X_m$ satisfying the following conditions:

(O1) X_i's should be of the following form:

$$X_i = \sum_{r=1}^{\chi(n_i)} \sum_{s=0}^{n_r - n_i} \phi_i^{rs}(x) \frac{\partial}{\partial x_{r(n_i+s)}} \qquad \text{for } 1 \leq i \leq m,$$

where $\phi_i^{rs} \in \mathcal{P}^s(x)$.

(O2) The n vector fields $ad_{(-f)}^{n_i - j} X_i$'s are linearly independent on W, where $1 \leq i \leq m$, $1 \leq j \leq n_i$, and W is a neighborhood of $x(0)$.

(O3) On W, it holds that

$$\left[ad_{(-f)}^{n_\mu - \kappa} X_\mu,\ ad_{(-F)}^{n_\nu - \lambda} X_\nu\right] = 0$$

for $\mu, \nu = 1, \ldots, m$, $1 \leq \kappa \leq n_\mu$, and $1 \leq \lambda \leq n_\nu$. □

Remark 4.3.2. Actually, (R1) and (O1), which determine the structure of the vector fields X_i's, are inspired by the works [BBHB09, BB11] that deal with the OEL and RDOEL problem, respectively. The advantage of Theorem 4.3.4 over the result in [BB11] is that we derive a necessary and sufficient condition of the

RDOEL problem for multi-output systems while a sufficient condition for the case of single output systems was provided in [BB11]. The advantage of Corollary 4.3.5 over the work of [BBHB09] is that we consider the general output transformation $y_e = q(y)$ while an output transformation with a structural restriction was considered in [BBHB09] (for more details, see Theorem 3.2.7). □

To our best knowledge, Corollary 4.3.5 provides the first geometric equivalent condition to the solvability of the OEL problem for multi-output systems, in the case under consideration of a diffeomorphism on system output of the general form $y_e = g(y)$. The condition (O3) was originated from [KI83, KR85] and has been commonly witnessed in [XG89, BBHB09]. Significant differences are found in (O1). Although the authors of [XG89] and [BBHB09] derived geometric necessary and sufficient conditions for the OEL problem, they did not consider output transformation (i.e. $y_e = y$) or assumed that output transformation has a structural restriction such as $y_{ei} = q_i(y_1, \ldots, y_i)$ for $1 \leq i \leq m$, respectively. As shown in the proof of Theorem 4.3.4, each $\phi_i^{r0}(x)$, which constitutes X_i by (O1), coincides with $\frac{\partial \tilde{q}_\mu}{\partial y_{e\nu}}$ where $\tilde{q}(y_e) = [\tilde{q}_1(y_e) \ \cdots \ q_m(y_e)]^T$ is the inverse output transformation of $y_e = q(y)$. Therefore, if $y_e = y$, then $\phi_i^{r0} = \delta_{ir}$. Similarly, if $y_{ei} = q_i(y_1, \ldots, y_i)$ for $1 \leq i \leq m$, then $\phi_i^{r0}(x) = \begin{cases} 0 & \text{when } r > i, \\ \phi_i^{r0}(y_1, \ldots, y_i) & \text{when } r \leq i. \end{cases}$ This fact means that our result has more freedom on designing $\phi_i^{rs}(x)$'s than theirs, and the property makes it possible that the OEL problem can be solved for a class of systems not covered by the previous results. We illustrate it by the first example in Section 4.4.

4.3.3 Procedure to Solve OEL and RDOEL

In this subsection, we explain how to check the solvability of the OEL and RDOEL problems for a given system by means of Corollary 4.3.5 and Theorem 4.3.4. Furthermore, we also describe a procedure to construct an explicit change of coordinates for OEL or RDOEL from the vector fields given by Corollary 4.3.5 or Theorem 4.3.4, respectively.

There exists a class of systems that can be transformed into NOCF without the aid of any auxiliary dynamics (the case where the OEL problem is solvable).

Since the OEL problem is fully characterized by Corollary 4.3.5, we can check the solvability for a given system. If the problem is solvable, then we need not to use the RDOEL approach, in order not to waste hardware or software resources that are needed to implement an auxiliary dynamics. However, there also exists a class of systems to which OEL is not applicable but the RDOEL problem can be solved. For this reason, the process of applying our results to the given system (3.1.1) is split into the two stages: OEL procedure by Corollary 4.3.5 and RDOEL procedure by Theorem 4.3.4.

As an initial stage, according to Theorem 4.2.1 and Theorem 4.2.7 which state necessary conditions not only for RDOEL but also for OEL, let us first check the observability of the system (3.1.1) and the condition that $f_i(x) \in \mathcal{P}_i^n(x)$ for $1 \le i \le m$ in its observable form (3.1.3). If the system satisfies the conditions, then we move to the first stage - OEL procedure.

The first stage - OEL procedure

Step 1: According to (O1) in Corollary 4.3.5, set

$$X_i = \sum_{r=1}^{\chi(n_i)} \sum_{s=0}^{n_r - n_i} \phi_i^{rs} \frac{\partial}{\partial x_{r(n_i+s)}} \qquad \text{for } 1 \le i \le m,$$

with $\phi_i^{rs} \in \mathcal{P}^s(x)$, and then calculate $ad_{(-f)}^{n_i-j} X_i$ for $1 \le i \le m$ and $1 \le j \le n_i$. Thereby, we can define an $n \times n$ matrix L such that

$$[ad_{(-f)}^{n_1-1} X_1 \cdots X_1 \cdots ad_{(-f)}^{n_m-1} X_m \cdots X_m] = \frac{\partial}{\partial x} L,$$

where $\frac{\partial}{\partial x} = [\frac{\partial}{\partial x_{11}} \cdots \frac{\partial}{\partial x_{1n_1}} \cdots \frac{\partial}{\partial x_{m1}} \cdots \frac{\partial}{\partial x_{mn_m}}]$. Since f is known, all the entries of L are expressed as functions of x and ϕ_i^{rs}'s. The objective is to find ϕ_i^{rs}'s such that both (O2) and (O3) in Corollary 4.3.5 are satisfied.

Step 2: The condition (O2) holds if and only if the matrix L is nonsingular. Therefore, we can obtain some constraint conditions on ϕ_i^{rs}'s, which guarantee (O2), from $\det L \neq 0$. Furthermore, since each block of L has a lower triangular-like form (similar as the equations (4.3.10)-(4.3.13) and (4.3.18)), the method used to calculate $\det S$ in the proof of Theorem 4.2.6 would be helpful in computing $\det L$.

Step 3: Direct calculation of the Lie brackets given in (O3) provides some partial differential equations of ϕ_i^{rs}'s. If there exists a set of solutions of the equations subject to the constraint conditions obtained in Step 2, then the vector fields X_i's with the solutions satisfy (O1)-(O3) in Corollary 4.3.5. That is to say, the OEL problem is solvable. In addition, from the solutions, all the entries of L are determined as functions of x. Since it holds that $\frac{\partial z}{\partial x} = L^{-1}$ by (4.3.15), we can construct an explicit coordinate transformation by solving it. If the partial differential equations subject to the constraint conditions have no solution, then it means that the OEL problem is not solvable. In this case, we move to the second stage - RDOEL procedure.

The second stage - RDOEL procedure

Step 4: Choose an auxiliary dynamics such as $\dot{\eta} = p(\eta, y)$.

Step 5: According to (R1) in Theorem 4.3.4, reset X_i's by replacing $\phi_i^{rs}(x) \in \mathcal{P}^s(x)$ with $\phi_i^{rs}(\eta, x) \in \mathcal{P}_e^s(x)$. After that, compute $ad_{(-F)}^{n_i-j} X_i$'s with $F(x) = [p(\eta, y)^T \; f(x)^T]^T$ and redefine the matrix L such as

$$[ad_{(-F)}^{n_1-1} X_1 \; \cdots \; X_1 \; \cdots \; ad_{(-F)}^{n_m-1} X_m \; \cdots \; X_m] = \frac{\partial}{\partial x} L.$$

Step 6: In a similar way to Step 2, we can obtain some constraint conditions on ϕ_i^{rs}'s from $\det L \neq 0$, which guarantee (R2) in Theorem 4.3.4.

Step 7: The Lie brackets in (R3) of Theorem 4.3.4 also give partial differential equations of ϕ_i^{rs}'s. If there is a set of solutions of the equations subject to the constraints from the preceding step, then the RDOEL problem is solved and we can also design explicit z-coordinates by solving $\frac{\partial z}{\partial x} = L^{-1}$.

Remark 4.3.3. Actually, in the second stage - RDOEL procedure, the auxiliary dynamics plays an important role for the solvability of the RDOEL problem. If there exists an auxiliary dynamics for a given system such that the RDOEL problem can be solved, then it is theoretically possible to design it by the following manner. If the auxiliary dynamics $\dot{\eta} = p(\eta, y)$ is not fixed in Step 4 (equivalently, it is to be designed), then the entries of L defined in Step 5 depend not only on (η, x) and $\phi_i^{rs}(\eta, x)$'s but also on $p(\eta, y)$. Hence, the constraint conditions from (R2) and the partial differential equations by (R3) also depend on $\phi_i^{rs}(\eta, x)$'s and

$p(\eta, y)$. If there exists a set of solutions of the equations subject to the constraints, then, from the solution $p(\eta, y)$, we can also design an auxiliary dynamics that enables the RDOEL problem to be solvable. In general, however, it is very hard to find such a solution because unknown $p(\eta, y)$ makes the constraints and the equations too complicated. This is the reason why we 'choose' a fixed auxiliary dynamics in Step 4 instead of 'designing' it. □

Remark 4.3.4. By Theorem 4.1.1, if a given system is RDOEL via an auxiliary dynamics, then it is also DOEL via the same auxiliary dynamics with a new output. Moreover, the coordinate transformation for RDOEL and the new output $\bar{y}_e = [\eta_1 \ y_{e2} \ \cdots \ y_{em}]^T$ transform the extended system into a $(d+n)$-dimensional GNOCF. In this sense, the second procedure also offers an algorithm to design a coordinate transformation for DOEL. □

In Step 2 and Step 6, since each block of the matrix L has a lower triangular-like form, it is not difficult to obtain constraint conditions on ϕ_i^{rs}'s from $\det L \neq 0$, by using the method in the proof of Theorem 4.2.6. However, in Step 3 and Step 7, it would be a tedious work to calculate Lie brackets among the vector fields. The following lemma and its corollary could help us to reduce the efforts.

Lemma 4.3.6. In the condition (R3) of Theorem 4.3.4, if $\kappa + \lambda > n_1 + 1$, then it always holds that $[ad_{(-F)}^{n_\mu - \kappa} X_\mu, \ ad_{(-F)}^{n_\nu - \lambda} X_\nu] = 0$.

Proof. From the equations (4.3.10)-(4.3.13), one can observe that $ad_{(-F)}^{n_\mu - \kappa} X_\mu$ is the κ-th column of the μ-th column matrices in L. In addition, by the lower triangular-like form (4.3.18) of each block of L, $ad_{(-F)}^{n_\mu - \kappa} X_\mu$ can be written as

$$ad_{(-F)}^{n_\mu - \kappa} X_\mu = \sum_{n_i \geq \kappa} \sum_{j=\kappa}^{n_i} (T_{i\mu})_{j\kappa} \frac{\partial}{\partial x_{ij}},$$

with $(T_{i\mu})_{j\kappa} \in \mathcal{P}_e^{j-\kappa}(x)$. Similarly, we have

$$ad_{(-F)}^{n_\nu - \lambda} X_\nu = \sum_{n_r \geq \lambda} \sum_{s=\lambda}^{n_r} (T_{r\nu})_{s\lambda} \frac{\partial}{\partial x_{rs}},$$

with $(T_{r\nu})_{s\lambda} \in \mathcal{P}_e^{s-\lambda}(x)$. Hence, direct calculation of Lie bracket between the

above two vector fields yields that

$$
\begin{aligned}
&[ad_{(-F)}^{n_\mu-\kappa}X_\mu,\ ad_{(-F)}^{n_\nu-\lambda}X_\nu] \\
&= \sum_{n_i\geq\kappa}\sum_{j=\kappa}^{n_i}\sum_{n_r\geq\lambda}\sum_{s=\lambda}^{n_r}\left((T_{i\mu})_{j\kappa}\frac{\partial(T_{r\nu})_{s\lambda}}{\partial x_{ij}}\frac{\partial}{\partial x_{rs}} - (T_{r\nu})_{s\lambda}\frac{\partial(T_{i\mu})_{j\kappa}}{\partial x_{rs}}\right)\frac{\partial}{\partial x_{ij}}.
\end{aligned}
$$

If $\kappa+\lambda > n_1+1$, then we have $\frac{\partial(T_{r\nu})_{s\lambda}}{\partial x_{ij}} = 0$ because $j \geq \kappa > n_1-\lambda+1 \geq s-\lambda+1$ and $(T_{r\nu})_{s\lambda} \in \mathcal{P}_e^{s-\lambda}(x)$. Similarly, it holds that $\frac{\partial(T_{i\mu})_{j\kappa}}{\partial x_{rs}} = 0$. As a result, $[ad_{(-F)}^{n_\mu-\kappa}X_\mu,\ ad_{(-F)}^{n_\nu-\lambda}X_\nu] = 0$ if $\kappa+\lambda > n_1+1$. □

Corollary 4.3.7. In the condition (O3) of Theorem 4.3.5, if $\kappa+\lambda > n_1+1$, then it always holds that $[ad_{(-f)}^{n_\mu-\kappa}X_\mu,\ ad_{(-f)}^{n_\nu-\lambda}X_\nu] = 0$. □

Lastly, we give a useful tip on the order of computation of the Lie brackets. As mentioned in the proof of Theorem 4.3.4, we take X_i from $\frac{\partial}{\partial z_{in_i}}$ for $1 \leq i \leq m$, which is the last column of the i-th column blocks of the matrix S (see the equations (4.3.7) and (4.3.8)). In the equation (4.2.9), the (i,j)-th block of S is an $n_i \times n_j$ matrix and the observability indices $(n_1,\ldots,n_m)$ satisfy $n_1 \geq n_2 \geq \cdots \geq n_m$. In addition, each block of S has the lower triangular-like form (4.2.10). For these reasons, in general, the following statement holds: the smaller i, the smaller number of ϕ_i^{rs}'s X_i depends on (indeed, if $n_1 > n_2$, then $X_1 = \phi_1^{10}\frac{\partial}{\partial x_{1n_1}}$). Therefore, we calculate Lie brackets among $ad_{(-F)}^{n_1-i}X_1$'s for $1 \leq i \leq n_1$ at first and find ϕ_1^{rs}'s such that the Lie brackets are zero (if there exist). And then, compute Lie brackets among $ad_{(-F)}^{n_1-i}X_1$'s with the solutions ϕ_1^{rs}'s and $ad_{(-F)}^{n_2-j}X_2$'s, in order to get ϕ_2^{rs}'s. After that, we extend the targets of Lie bracket operation to $ad_{(-F)}^{n_1-k}X_3$'s, …, $ad_{(-F)}^{n_m-l}X_m$'s successively. This iterative process could reduce the efforts on calculation of the Lie brackets given in not only (R3) but also (O3).

4.4 Illustrative Examples

In this section, we present two examples in order to demonstrate the results of Theorem 4.3.4 and Corollary 4.3.5. The first example illustrates that the OEL problem can be solved for a larger class of systems when an output transformation of the general form $y_e = q(y)$ is considered.

Example 4.4.1. Consider a multi-output nonlinear system given by

$$\begin{aligned} \dot{x}_{11} &= x_{12}, & \dot{x}_{21} &= e^{-(x_{11}+x_{21})}x_{12}, \\ \dot{x}_{12} &= 2(e^{-(x_{11}+x_{21})}+1)x_{12}^2, & & \\ y_1 &= x_{11}, & y_2 &= x_{21}. \end{aligned} \tag{4.4.1}$$

The above system is already expressed as an observable form and its observability indices are given by $(n_1, n_2) = (2, 1)$. Moreover, the system satisfies the condition,

$$\begin{aligned} f_1(x) &:= \dot{x}_{12} = 2(e^{-(x_{11}+x_{21})}+1)x_{12}^2 \in \mathcal{P}^2(x), \\ f_2(x) &:= \dot{x}_{21} = e^{-(x_{11}+x_{21})}x_{12} \in \mathcal{P}^1(x). \end{aligned}$$

According to the procedure in Subsection 4.3.3, we show that the system is OEL and construct a change of coordinates which transforms the system into NOCF.
Step 1: We set X_1 and X_2 by (O1) in Corollary 4.3.5 as follows:

$$\begin{aligned} X_1 &= \phi_1^{10}\frac{\partial}{\partial x_{12}}, \\ X_2 &= \phi_2^{10}\frac{\partial}{\partial x_{11}} + \phi_2^{11}\frac{\partial}{\partial x_{12}} + \phi_2^{20}\frac{\partial}{\partial x_{21}}, \end{aligned} \tag{4.4.2a}$$

where $\phi_1^{10}, \phi_2^{10}, \phi_2^{20} \in \mathcal{P}^0(x)$ and $\phi_2^{11} \in \mathcal{P}^1(x)$. Since the vector field f of the system is represented as

$$f = x_{12}\frac{\partial}{\partial x_{11}} + 2(E^{-1}+1)x_{12}^2\frac{\partial}{\partial x_{12}} + E^{-1}x_{12}\frac{\partial}{\partial x_{21}},$$

where $E = e^{x_{11}+x_{21}}$, straightforward calculation yields

$$ad_{(-f)}X_1 = \phi_1^{10}\frac{\partial}{\partial x_{11}} + \varphi_1^1\frac{\partial}{\partial x_{12}} + E^{-1}\phi_1^{10}\frac{\partial}{\partial x_{21}}, \tag{4.4.2b}$$

with

$$\varphi_1^1 := \big(4\phi_1^{10}(E^{-1}+1) - \frac{\partial \phi_1^{10}}{\partial x_{11}} - E^{-1}\frac{\partial \phi_1^{10}}{\partial x_{21}}\big)x_{12} \in \mathcal{P}^1(x).$$

From (4.4.2), the matrix L is defined as

$$\begin{bmatrix} ad_{(-f)}X_1 \\ X_1 \\ X_2 \end{bmatrix}^T = \begin{bmatrix} \frac{\partial}{\partial x_{11}} \\ \frac{\partial}{\partial x_{12}} \\ \frac{\partial}{\partial x_{21}} \end{bmatrix}^T \left[\begin{array}{cc|c} \phi_1^{10} & 0 & \phi_2^{10} \\ \varphi_1^1 & \phi_1^{10} & \phi_2^{11} \\ \hline E^{-1}\phi_1^{10} & 0 & \phi_2^{20} \end{array}\right] =: \frac{\partial}{\partial x} L.$$

Step 2: Since each block of L has a lower triangular-like form, by the way used in the proof of Theorem 4.2.6, it holds that

$$\det L = \det\left(\begin{bmatrix} \phi_1^{10} & \phi_2^{10} \\ E^{-1}\phi_1^{10} & \phi_2^{20} \end{bmatrix}\right) \cdot \det([\phi_1^{10}]) = (\phi_1^{10})^2(\phi_2^{20} - E^{-1}\phi_2^{10}).$$

Therefore, (O2) in Corollary 4.3.5 is satisfied if and only if

$$\phi_1^{10} \neq 0 \qquad \text{and} \qquad \phi_2^{20} - E^{-1}\phi_2^{10} \neq 0 \quad \text{on} \quad W, \tag{4.4.3}$$

where W is a neighborhood of $x(0)$.

Step 3: Let us first compute $[X_1,\ ad_{(-f)}X_1]$. Then, we have

$$\begin{aligned} \left[X_1,\ ad_{(-f)}X_1\right] &= \phi_1^{10}\left(\frac{\partial \varphi_1^1}{\partial x_{12}} - \frac{\partial \phi_1^{10}}{\partial x_{11}} - E^{-1}\frac{\partial \phi_1^{10}}{\partial x_{11}}\right)\frac{\partial}{\partial x_{12}} \\ &= 2\phi_1^{10}\left(2(E^{-1}+1)\phi_1^{10} - \frac{\partial \phi_1^{10}}{\partial x_{11}} - E^{-1}\frac{\partial \phi_1^{10}}{\partial x_{21}}\right)\frac{\partial}{\partial x_{12}}. \end{aligned}$$

Since $\phi_1^{10} \neq 0$ by the condition (4.4.3), $[X_1,\ ad_{(-f)}X_1] = 0$ if and only if

$$2(e^{-(x_{11}+x_{21})}+1)\phi_1^{10} - \frac{\partial \phi_1^{10}}{\partial x_{11}} - e^{-(x_{11}+x_{21})}\frac{\partial \phi_1^{10}}{\partial x_{21}} = 0. \tag{4.4.4}$$

Similarly as shown in the proof of Theorem 4.3.4, if the OEL problem is solved, then it holds that $\phi_1^{10} = \frac{\partial \tilde{q}_1}{\partial y_{e1}} = \frac{\partial y_1}{\partial y_{e1}}$ where $\tilde{q}(y_e) = [\tilde{q}_1(y_e)\ \tilde{q}_2(y_e)]^T = y$ is the inverse function of the output transformation $y_e = q(y)$. Hence, if $q(y) = y$ ([XG89]) or $q(y)$ has a structural restriction such that $q_1(y) = q_1(y_1)$ ([BBHB09]), then $\phi_1^{10} = 1$ or $\phi_1^{10} = \phi_1^{10}(y_1) = \phi_1^{10}(x_{11})$, respectively. However, it is easy to see that the equation (4.4.4) does not hold when $\phi_1^{10} = 1$, and it has no solution ϕ_1^{10} depending only on x_{11}. This implies that the OEL problem is not solvable for the

system (4.4.1) when $q(y)$ has such restrictions.

However, if there is no restriction on $q(y)$, then we can find a solution of the equation (4.4.4) such that $\phi_1^{10} = E^2 = e^{2(x_{11}+x_{21})}$ which also satisfies the condition (4.4.3). Let $\phi_1^{10} = E^2$ and $\phi_2^{11} = \psi_1 x_{12} + \psi_0$ where $\psi_1, \psi_0 \in \mathcal{P}^0(x)$ (ϕ_2^{11} can be written as $\phi_2^{11} = \psi_1 x_{12} + \psi_0$ because $\phi_2^{11} \in \mathcal{P}^1(x)$). Then, φ_1^1 is rewritten as $\varphi_1^1 = 2E(E+1)x_{12}$ and we have

$$
\begin{aligned}
&\big[X_1,\ X_2\big] = E^2(\psi_1 - 2\phi_2^{10} - 2\phi_2^{20})\frac{\partial}{\partial x_{12}},\\
&\big[ad_{(-f)}X_1,\ X_2\big] = E^2\left(\frac{\partial \phi_2^{10}}{\partial x_{11}} + E^{-1}\frac{\partial \phi_2^{10}}{\partial x_{21}} - 2\phi_2^{10} - 2\phi_2^{20}\right)\frac{\partial}{\partial x_{11}}\\
&+ E\Bigg\{\left(E\frac{\partial \psi_1}{\partial x_{11}} + \frac{\partial \psi_1}{\partial x_{21}} - (2E-1)\psi_1\right)x_{12} + \left(E\frac{\partial \psi_0}{\partial x_{11}} + \frac{\partial \psi_0}{\partial x_{21}} - 2(E-1)\psi_0\right)\\
&+ 2(E+1)(\psi_1 - 2\phi_2^{10} - 2\phi_2^{20})\Bigg\}\frac{\partial}{\partial x_{12}} + E\left(E\frac{\partial \phi_2^{20}}{\partial x_{11}} + \frac{\partial \phi_2^{20}}{\partial x_{21}} - \phi_2^{10} - \phi_2^{20}\right)\frac{\partial}{\partial x_{11}}.
\end{aligned}
$$

From the above equations, (O3) is satisfied if and only if there exist $\phi_2^{10}, \phi_2^{20}, \psi_1, \psi_0 \in \mathcal{P}^0(x)$ which constitute a set of solutions to the following equations:

$$\psi_1 - 2\phi_2^{10} - 2\phi_2^{20} = 0, \tag{4.4.5a}$$

$$\frac{\partial \phi_2^{10}}{\partial x_{11}} + E^{-1}\frac{\partial \phi_2^{10}}{\partial x_{21}} - 2\phi_2^{10} - 2\phi_2^{20} = 0, \tag{4.4.5b}$$

$$E\frac{\partial \psi_1}{\partial x_{11}} + \frac{\partial \psi_1}{\partial x_{21}} - (2E+1)\psi_1 = 0, \tag{4.4.5c}$$

$$E\frac{\partial \psi_0}{\partial x_{11}} + \frac{\partial \psi_0}{\partial x_{21}} - 2(E+1)\psi_0 = 0, \tag{4.4.5d}$$

$$E\frac{\partial \phi_2^{20}}{\partial x_{11}} + \frac{\partial \phi_2^{20}}{\partial x_{21}} - \phi_2^{10} - \phi_2^{20} = 0. \tag{4.4.5e}$$

Let $\phi_2^{10} = E^2 + 1 = e^{2(x_{11}+x21)} + 1$ and $\phi_2^{20} = E - 1 = e^{x_{11}+x_{21}} - 1$, which are solutions of (4.4.5b) and (4.4.5e) and also satisfy the condition (4.4.3). Then, it follows from (4.4.5a) that $\psi_1 = 2(\phi_2^{10}+\phi_2^{20}) = 2(E^2+E) = 2(e^{2(x_{11}+x_{21})}+e^{x_{11}+x_{21}})$ which is a solution of (4.4.5c). Finally, let $\psi_0 = 0$. Then, the last equation (4.4.5d) holds. Consequently, we find four functions $\phi_1^{10} = e^{2(x_{11}+x_{21})}$, $\phi_2^{10} = e^{2(x_{11}+x21)}+1$, $\phi_2^{20} = e^{x_{11}+x_{21}} - 1$, and $\phi_2^{11} = \psi_1 x_{12} + \psi_0 = 2(e^{2(x_{11}+x_{21})} + e^{x_{11}+x_{21}})x_{12}$ such that the conditions (O2) and (O3) in Corollary 4.3.5 are satisfied. From the four

solutions, we can determine the matrix L as follows:

$$L = \begin{bmatrix} E^2 & 0 & E^2+1 \\ 2(E^2+E)x_{12} & E^2 & 2(E^2+E)x_{12} \\ E & 0 & E-1 \end{bmatrix}.$$

Since it holds that $\frac{\partial z}{\partial x} = L^{-1}$ by (4.3.15), a coordinate transformation $z = \Phi(x)$ for OEL is a solution of the partial differential equation,

$$\frac{\partial \Phi}{\partial x} = L^{-1} = \begin{bmatrix} \frac{E^{-1}-1}{E+1} & 0 & \frac{E+E^{-1}}{E+1} \\ -2E^{-2}x_{12} & E^{-2} & -2E^{-2}x_{12} \\ \frac{1}{E+1} & 0 & \frac{-E}{E+1} \end{bmatrix}.$$

By solving the above equation, we can design a coordinate transformation and an output transformation such that

$$\begin{bmatrix} z_{11} \\ z_{12} \\ z_{21} \end{bmatrix} = \Phi(x) = \begin{bmatrix} 2\ln(e^{x_{11}+x_{21}}+1) - e^{-(x_{11}+x_{21})} - 2x_{11} - x_{21} \\ e^{-2(x_{11}+x_{21})}x_{12} \\ -\ln(e^{x_{11}+x_{21}}+1) + x_{11} \end{bmatrix},$$

$$\begin{bmatrix} y_{e1} \\ y_{e2} \end{bmatrix} = q(y) = \begin{bmatrix} 2\ln(e^{y_1+y_2}+1) - e^{-(y_1+y_2)} - 2y_1 - y_2 \\ -\ln(e^{y_1+y_2}+1) + y_1 \end{bmatrix}.$$

As a result, the system (4.4.1) is transformed into NOCF (in fact, a linear system),

$$\dot{z} = \begin{bmatrix} z_{12} \\ 0 \\ 0 \end{bmatrix} = Az,$$

$$y_e = \begin{bmatrix} z_{11} \\ z_{21} \end{bmatrix} = Cz,$$

by $z = \Phi(x)$ and $y_e = q(y)$. □

The second example is given to illustrate that the RDOEL problem can be solved for a class of systems for which the OEL problem is not solvable and the RDOEL approach offers a lower dimensional GNOCF than the DOEL approach.

Example 4.4.2. Consider the following multi-output nonlinear system:

$$\begin{aligned} \dot{x}_{11} &= x_{12}, & \dot{x}_{21} &= x_{22}, \\ \dot{x}_{12} &= x_{13}, & \dot{x}_{22} &= x_{21}x_{12}, \\ \dot{x}_{13} &= x_{11}x_{13}, \\ y_1 &= x_{11}, & y_2 &= x_{21}. \end{aligned} \tag{4.4.6}$$

The above system is represented as an observable form and satisfies the condition,

$$\begin{aligned} f_1(x) &= x_{11}x_{13} \in \mathcal{P}^2(x) \subset \mathcal{P}^3(x), \\ f_2(x) &= x_{21}x_{12} \in \mathcal{P}^1(x) \subset \mathcal{P}^2(x). \end{aligned}$$

Actually, the system is DOEL via the auxiliary dynamics $\dot{\eta} = y_1$, namely, the extended system can be transformed into a six-dimensional GNOCF ([BB09]). We show that the system cannot be transformed into NOCF without the aid of auxiliary dynamics (i.e. it is not OEL), but it is also RDOEL via a new auxiliary dynamics such as $\dot{\eta} = -\eta + y_1$ and thus can be transformed into a five-dimensional GNOCF.

Steps 1-3: According to (O1) in Corollary 4.3.5, let

$$\begin{aligned} X_1 &= \phi_1^{10}\frac{\partial}{\partial x_{13}}, \\ X_2 &= \phi_2^{10}\frac{\partial}{\partial x_{12}} + \phi_2^{11}\frac{\partial}{\partial x_{13}} + \phi_2^{20}\frac{\partial}{\partial x_{22}}, \end{aligned} \tag{4.4.7}$$

with $\phi_1^{10}, \phi_2^{10}, \phi_2^{20} \in \mathcal{P}^0(x)$ and $\phi_2^{11} \in \mathcal{P}^1(x)$. Then, from Step 2, we obtain the following condition guaranteeing (O2) in Corollary 4.3.5:

$$\phi_1^{10} \neq 0 \qquad \text{and} \qquad \phi_2^{20} \neq 0 \quad \text{on W}, \tag{4.4.8}$$

where W is a neighborhood of $x(0)$. In Step 3, it holds that $[ad_{(-f)}X_1,\ ad^2_{(-f)}X_1] = 0$ if and only if

$$\frac{\partial \phi_1^{10}}{\partial x_{11}} = 0 \qquad \text{and} \qquad \phi_1^{10} - \frac{\partial \phi_1^{10}}{\partial x_{21}}x_{21} = 0. \tag{4.4.9}$$

In the above equation, $\phi_1^{10} = 0$ when $x_{21} = 0$, which violates the condition (4.4.8). This implies that there is no solution satisfying both (O2) and (O3) in Corollary 4.3.5 when $x_{21} = 0$. That is, the OEL problem is not solvable for the system (4.4.6) on the region where $x_{21} = 0$. Therefore, we move to the second stage - RDOEL procedure in Subsection 4.3.3.

Step 4: As mentioned before, we choose an auxiliary dynamics such that

$$\dot{\eta} = -\eta + y_1, \tag{4.4.10}$$

which is an input-to-state stable system in the sense given by [SW95] when we regard the system output y_1 as the input of the auxiliary dynamics.

Step 5: In the equation (4.4.7), we adjust $\phi_1^{10}, \phi_2^{10}, \phi_2^{20} \in \mathcal{P}_e^0(x)$ and $\phi_2^{11} \in \mathcal{P}_e^1(x)$ so that they depend also on η (i.e. X_1 and X_2 satisfy (R1) in Theorem 4.3.4). Since the vector field F of the extended system, which is composed of the system (4.4.6) and the auxiliary dynamics (4.4.10), is given by

$$F = (x_{11} - \eta)\frac{\partial}{\partial \eta} + x_{12}\frac{\partial}{\partial x_{11}} + x_{13}\frac{\partial}{\partial x_{12}} + x_{11}x_{13}\frac{\partial}{\partial x_{13}} + x_{22}\frac{\partial}{\partial x_{21}} + x_{21}x_{12}\frac{\partial}{\partial x_{22}},$$

the other three vector fields are calculated as

$$\begin{aligned}
ad_{(-F)}X_1 &= \phi_1^{10}\frac{\partial}{\partial x_{12}} + \varphi_1^1\frac{\partial}{\partial x_{13}}, \\
ad_{(-F)}^2X_1 &= \phi_1^{10}\frac{\partial}{\partial x_{11}} + (2\varphi_1^1 - \phi_1^{10}x_{11})\frac{\partial}{\partial x_{12}} + \varphi_2^2\frac{\partial}{\partial x_{13}} + \phi_1^{10}x_{21}\frac{\partial}{\partial x_{22}}, \\
ad_{(-F)}X_2 &= \phi_2^{10}\frac{\partial}{\partial x_{11}} + \varphi_3^1\frac{\partial}{\partial x_{12}} + \varphi_4^2\frac{\partial}{\partial x_{13}} + \phi_2^{20}\frac{\partial}{\partial x_{21}} + \varphi_5^1\frac{\partial}{\partial x_{22}},
\end{aligned} \tag{4.4.11}$$

where

$$\begin{aligned}
\varphi_1^1 &:= \phi_1^{10}x_{11} - \frac{\partial \phi_1^{10}}{\partial \eta}(x_{11} - \eta) - \frac{\partial \phi_1^{10}}{\partial x_{11}}x_{12} - \frac{\partial \phi_1^{10}}{\partial x_{21}}x_{22}, \\
\varphi_2^2 &:= \varphi_1^1x_{11} - \frac{\partial \varphi_1^1}{\partial \eta}(x_{11} - \eta) - \frac{\partial \varphi_1^1}{\partial x_{11}}x_{12} - \frac{\partial \varphi_1^1}{\partial x_{12}}x_{13} - \frac{\partial \varphi_1^1}{\partial x_{21}}x_{22} - \frac{\partial \varphi_1^1}{\partial x_{22}}x_{21}x_{12}, \\
\varphi_3^1 &:= \phi_2^{11} - \frac{\partial \phi_2^{10}}{\partial \eta}(x_{11} - \eta) - \frac{\partial \phi_2^{10}}{\partial x_{11}}x_{12} - \frac{\partial \phi_2^{10}}{\partial x_{21}}x_{22}, \\
\varphi_4^2 &:= \phi_2^{11}x_{11} - \frac{\partial \phi_2^{11}}{\partial \eta}(x_{11} - \eta) - \frac{\partial \phi_2^{11}}{\partial x_{11}}x_{12} - \frac{\partial \phi_2^{11}}{\partial x_{12}}x_{13} - \frac{\partial \phi_2^{11}}{\partial x_{21}}x_{22} - \frac{\partial \phi_2^{11}}{\partial x_{22}}x_{21}x_{12},
\end{aligned}$$

$$\varphi_5^1 := \phi_2^{10} x_{21} - \frac{\partial \phi_2^{20}}{\partial \eta}(x_{11} - \eta) - \frac{\partial \phi_2^{20}}{\partial x_{11}} x_{12} - \frac{\partial \phi_2^{20}}{\partial x_{21}} x_{22}.$$

As a consequence of the equations (4.4.7) and (4.4.11), we define the matrix L such that

$$\begin{bmatrix} ad_{(-F)}^2 X_1 \\ ad_{(-F)} X_1 \\ X_1 \\ ad_{(-F)} X_2 \\ X_2 \end{bmatrix}^T = \begin{bmatrix} \frac{\partial}{\partial x_{11}} \\ \frac{\partial}{\partial x_{12}} \\ \frac{\partial}{\partial x_{13}} \\ \frac{\partial}{\partial x_{21}} \\ \frac{\partial}{\partial x_{22}} \end{bmatrix}^T \left[\begin{array}{ccc|cc} \phi_1^{10} & 0 & 0 & \phi_2^{10} & 0 \\ 2\varphi_1^{11} - \phi_1^{10} x_{11} & \phi_1^{10} & 0 & \varphi_3^1 & \phi_2^{10} \\ \varphi_2^2 & \varphi_1^1 & \phi_1^{10} & \varphi_4^2 & \phi_2^{11} \\ \hline 0 & 0 & 0 & \phi_2^{20} & 0 \\ \phi_1^{10} x_{21} & 0 & 0 & \varphi_5^1 & \phi_2^{20} \end{array}\right]^T$$
$$=: \frac{\partial}{\partial x} L. \tag{4.4.12}$$

Step 6: One can observe that each block of the matrix L defined above has a lower triangular-like form. Hence, by the method used in the proof of Theorem 4.2.6, it holds that

$$\det L = \left(\det \begin{bmatrix} \phi_1^{10} & \phi_2^{10} \\ 0 & \phi_2^{20} \end{bmatrix} \right)^2 \cdot \det \left[\phi_1^{10}\right] = \left(\phi_1^{10}\phi_2^{20}\right)^2 \phi_1^{10}.$$

Therefore, (R2) in Theorem 4.3.4 is satisfied if and only if

$$\phi_1^{10} \neq 0 \qquad \text{and} \qquad \phi_2^{20} \neq 0 \quad \text{on} \quad U \times W, \tag{4.4.13}$$

where $U \times W$ is a neighborhood of $(\eta(0), x(0))$.

Step 7: Since the observability indices of the system (4.4.2) are given by $(n_1, n_2) = (3, 2)$, it follows from Lemma 4.3.6 that

$$\begin{aligned} \left[X_1,\ ad_{(-F)} X_1\right] &= 0, \\ \left[X_1,\ X_2\right] &= 0. \end{aligned}$$

Among the other eight Lie brackets, let us first consider $[X_1,\ ad_{(-F)}^2 X_1]$ and $[ad_{(-F)} X_1,\ ad_{(-F)}^2 X_1]$ which will give some partial differential equations of ϕ_1^{10}.

The computation results of them are as follows.

$$
\begin{aligned}
\left[X_1,\ ad^2_{(-F)}X_1\right] &= \left(\phi_1^{10}\frac{\partial\varphi_2^2}{\partial x_{13}} - \phi_1^{10}\frac{\partial\phi_1^{10}}{\partial x_{11}}\right)\frac{\partial}{\partial x_{13}} = 0,\\
\left[ad_{(-F)}X_1,\ ad^2_{(-F)}X_1\right] &= \phi_1^{10}\left(2\frac{\partial\varphi_1^1}{\partial x_{12}} - \frac{\partial\phi_1^{10}}{\partial x_{11}}\right)\frac{\partial}{\partial x_{12}} + \left(\phi_1^{10}\frac{\partial\varphi_2^2}{\partial x_{12}} + \varphi_1^1\frac{\partial\varphi_2^2}{\partial x_{13}}\right.\\
&\qquad \left. - \phi_1^{10}\frac{\partial\varphi_1^1}{\partial x_{11}} - (2\varphi_1^1 - \phi_1^{10}x_{11})\frac{\partial\varphi_1^1}{\partial x_{12}} - \phi_1^{10}x_{21}\frac{\partial\varphi_1^1}{\partial x_{22}}\right)\frac{\partial}{\partial x_{13}}\\
&= -3\phi_1^{10}\frac{\partial\phi_1^{10}}{\partial x_{11}}\frac{\partial}{\partial x_{12}} + \left(\phi_1^{10}\left(\frac{\partial\varphi_2^2}{\partial x_{12}} - \frac{\partial\varphi_1^1}{\partial x_{11}} - \frac{\partial\phi_1^{10}}{\partial x_{11}}x_{11} + \frac{\partial\phi_1^{10}}{\partial x_{21}}x_{21}\right) + 3\varphi_1^1\frac{\partial\phi_1^{10}}{\partial x_{11}}\right)\frac{\partial}{\partial x_{13}}.
\end{aligned}
$$

Since $\phi_1^{10} \neq 0$ by the condition (4.4.13), it follows from the second equation that $[ad_{(-F)}X_1,\ ad^2_{(-F)}X_1] = 0$ if and only if

$$
\frac{\partial\phi_1^{10}}{\partial x_{11}} = 0, \tag{4.4.14}
$$

$$
\frac{\partial\varphi_2^2}{\partial x_{12}} - \frac{\partial\varphi_1^1}{\partial x_{11}} + \frac{\partial\phi_1^{10}}{\partial x_{21}}x_{21} = 0. \tag{4.4.15}
$$

The equation (4.4.14) implies that ϕ_1^{10} does not depend on x_{11}, in other words, $\phi_1^{10} = \phi_1^{10}(\eta, x_{21}) \in \mathcal{P}_e^0(x)$. By this fact, φ_1^1 and φ_2^2 can be rewritten as

$$
\begin{aligned}
\varphi_1^1 &= \phi_1^{10}x_{11} - \frac{\partial\phi_1^{10}}{\partial\eta}(x_{11} - \eta) - \frac{\partial\phi_1^{10}}{\partial x_{21}}x_{22} = \varphi_1^1(\eta, x_{11}, x_{21}, x_{22}),\\
\varphi_2^2 &= \varphi_1^1 x_{11} - \frac{\partial\varphi_1^1}{\partial\eta}(x_{11} - \eta) - \left(\phi_1^{10} - \frac{\partial\phi_1^{10}}{\partial\eta}\right)x_{12} - \frac{\partial\varphi_1^1}{\partial x_{21}}x_{22} + \frac{\partial\phi_1^{10}}{\partial x_{21}}x_{21}x_{12}.
\end{aligned}
$$

By the above equations, the equation (4.4.15) becomes

$$
\frac{\partial\varphi_2^2}{\partial x_{12}} - \frac{\partial\varphi_1^1}{\partial x_{11}} + \frac{\partial\phi_1^{10}}{\partial x_{21}}x_{21} = -2\left(\phi_1^{10} - \frac{\partial\phi_1^{10}}{\partial\eta} - \frac{\partial\phi_1^{10}}{\partial x_{21}}x_{21}\right) = 0.
$$

As a result, we obtain a partial differential equation for $\phi_1^{10}(\eta, x_{21})$ such that

$$
\phi_1^{10} - \frac{\partial\phi_1^{10}}{\partial\eta} - \frac{\partial\phi_1^{10}}{\partial x_{21}}x_{21} = 0. \tag{4.4.16}
$$

Let $\phi_1^{10} = e^\eta$ which is a solution of the equation (4.4.16) and satisfies the condition (4.4.13). Then, $[ad_{(-F)}X_1,\ ad^2_{(-F)}X_1] = 0$. Moreover, since $\varphi_1^1 = \eta e^\eta$ and $\varphi_2^2 =$

$(\eta^2+\eta-x_{11})e^{\eta}$, we have

$$\begin{aligned}
X_1 &= e^{\eta}\frac{\partial}{\partial x_{13}},\\
ad_{(-F)}X_1 &= e^{\eta}\frac{\partial}{\partial x_{12}} + \eta e^{\eta}\frac{\partial}{\partial x_{13}},\\
ad^2_{(-F)}X_1 &= e^{\eta}\frac{\partial}{\partial x_{11}} + (2\eta - x_{11})e^{\eta}\frac{\partial}{\partial x_{12}} + (\eta^2+\eta-x_{11})e^{\eta}\frac{\partial}{\partial x_{13}} + x_{21}e^{\eta}\frac{\partial}{\partial x_{22}}.
\end{aligned}$$

For simple calculations, we temporarily assume that $\phi_2^{10}=0$ and $\phi_2^{11}=0$, which do not violate the condition (4.4.13). Then, X_2 and $ad_{(-F)}X_2$ become

$$\begin{aligned}
X_2 &= \phi_2^{20}\frac{\partial}{\partial x_{22}},\\
ad_{(-F)}X_2 &= \phi_2^{20}\frac{\partial}{\partial x_{21}} + \varphi_5^1\frac{\partial}{\partial x_{22}},
\end{aligned}$$

where

$$\varphi_5^1 = -\frac{\partial\phi_2^{20}}{\partial\eta}(x_{11}-\eta) - \frac{\partial\phi_2^{20}}{\partial x_{11}}x_{12} - \frac{\partial\phi_2^{20}}{\partial x_{21}}x_{22}.$$

With the above new representation of $ad^{n_i-j}_{(-F)}X_i$ for $1\le i\le 2$ and $1\le j\le n_i$, let us continue to check the Lie brackets for (R3) as follows:

$$\begin{aligned}
&\big[X_1,\ ad_{(-F)}X_2\big] = 0,\\
&\big[ad_{(-F)}X_1,\ X_2\big] = 0,\\
&\big[ad_{(-F)}X_1,\ ad_{(-F)}X_2\big] = e^{\eta}\frac{\partial\varphi_5^1}{\partial x_{12}}\frac{\partial}{\partial x_{22}} = -e^{\eta}\frac{\partial\phi_2^{20}}{\partial x_{11}}\frac{\partial}{\partial x_{22}},\\
&\big[ad^2_{(-F)}X_1,\ X_2\big] = e^{\eta}\frac{\partial\phi_2^{20}}{\partial x_{11}}\frac{\partial}{\partial x_{22}},\\
&\big[X_2,\ ad_{(-F)}X_2\big] = \phi_2^{20}\left(\frac{\partial\varphi_5^1}{\partial x_{22}} - \frac{\partial\phi_2^{20}}{\partial x_{21}}\right)\frac{\partial}{\partial x_{22}} = -\phi_2^{20}\frac{\partial\phi_2^{20}}{\partial x_{21}}\frac{\partial}{\partial x_{22}}.
\end{aligned}$$

Since $\phi_2^{20}\neq 0$ by the condition (4.4.13), in order for both (R2) and (R3) to be satisfied, it should hold that

$$\frac{\partial\phi_2^{20}}{\partial x_{11}} = \frac{\partial\phi_2^{20}}{\partial x_{21}} = 0.$$

The equation implies that ϕ_2^{20} depends only on η because $\phi_2^{20} \in \mathcal{P}_e^0(x)$. As a result, we have $\varphi_5^1 = -\frac{\partial \phi_2^{20}}{\partial \eta}(x_{11} - \eta)$ and thus the last Lie bracket between $ad_{(-F)}^2 X_1$ and $ad_{(-F)} X_2$ is calculated as

$$\left[ad_{(-F)}^2 X_1,\ ad_{(-F)} X_2\right] = e^{\eta}\left(\frac{\varphi_5^1}{\partial x_{11}} - \phi_2^{20}\right)\frac{\partial}{\partial x_{22}} = -e^{\eta}\left(\frac{\partial \phi_2^{20}}{\partial \eta} + \phi_2^{20}\right)\frac{\partial}{\partial x_{22}}.$$

From the above equation, $[ad_{(-F)}^2 X_1,\ ad_{(-F)} X_2] = 0$ if and only if

$$\frac{\partial \phi_2^{20}}{\partial \eta} + \phi_2^{20} = 0.$$

Let $\phi_2^{20} = e^{-\eta}$ which is a solution of the above equation and also satisfies the condition (4.4.13). Then, the four functions $\phi_1^{10} = e^{\eta}$, $\phi_2^{10} = 0$, $\phi_2^{11} = 0$, and $\phi_2^{20} = e^{-\eta}$ guarantee that both (R2) and (R3) in Theorem 4.3.4 hold. Consequently, the RDOEL problem is solvable.

Now, by using the solutions, we design an explicit change of coordinates for RDOEL. From the solutions, we can determine all the entries of L as functions of x and η as follows:

$$L = \begin{bmatrix} e^{\eta} & 0 & 0 & 0 & 0 \\ (2\eta - x_{11})e^{\eta} & e^{\eta} & 0 & 0 & 0 \\ (\eta^2 + \eta - x_{11})e^{\eta} & \eta e^{\eta} & e^{\eta} & 0 & 0 \\ 0 & 0 & 0 & e^{-\eta} & 0 \\ x_{21}e^{\eta} & 0 & 0 & (x_{11} - \eta)e^{-\eta} & e^{-\eta} \end{bmatrix}.$$

Since it holds that $\frac{\partial z}{\partial x} = L^{-1}$ by (4.3.15), a solution of the partial differential equation,

$$\frac{\partial z}{\partial x} = \begin{bmatrix} e^{-\eta} & 0 & 0 & 0 & 0 \\ -(2\eta - x_{11})e^{-\eta} & e^{-\eta} & 0 & 0 & 0 \\ (\eta - 1)(\eta - x_{11})e^{-\eta} & -\eta e^{-\eta} & e^{-\eta} & 0 & 0 \\ 0 & 0 & 0 & e^{\eta} & 0 \\ -x_{21}e^{\eta} & 0 & 0 & (\eta - x_{11})e^{\eta} & e^{\eta} \end{bmatrix} = L^{-1},$$

can be a new coordinate for RDOEL. By solving the equation, we can design a coordinate transformation and an output transformation such that

$$\begin{bmatrix} z_{11} \\ z_{12} \\ z_{13} \\ z_{21} \\ z_{22} \end{bmatrix} = \begin{bmatrix} x_{11}e^{-\eta} \\ -(2\eta x_{11} - \frac{1}{2}x_{11}^2 - x_{12})e^{-\eta} \\ \big((\eta-1)(\eta x_{11} - \frac{1}{2}x_{11}^2) - \eta x_{12} + x_{13}\big)e^{-\eta} \\ x_{21}e^{\eta} \\ \big((\eta - x_{11})x_{21} + x_{22}\big)e^{\eta} \end{bmatrix},$$

$$\begin{bmatrix} y_{e1} \\ y_{e2} \end{bmatrix} = q(\eta, y) = \begin{bmatrix} y_1 e^{-\eta} \\ y_2 e^{\eta} \end{bmatrix}.$$

As a result, the extended system can be transformed into the following five-dimensional GNOCF:

$$\dot{z} = \begin{bmatrix} z_{12} \\ z_{13} \\ 0 \\ z_{22} \\ 0 \end{bmatrix} + \begin{bmatrix} 3y_1(\eta - \frac{1}{2}y_1)e^{-\eta} \\ -3y_1\big(\eta(\eta - y_1 - 1) + \frac{1}{6}y_1(y_1+5)\big)e^{-\eta} \\ y_1(\eta - y_1)\big(\eta(\eta - \frac{1}{2}y_1 - 3) + y_1 + 1\big)e^{-\eta} \\ -2y_2(\eta - y_1)e^{\eta} \\ -y_2(\eta - y_1)(\eta - y_1 + 1)e^{\eta} \end{bmatrix} = Az + a(\eta, y),$$

$$y_e = \begin{bmatrix} z_{11} \\ z_{21} \end{bmatrix} = Cz,$$

on the z-coordinates. □

Chapter 5

Extension of RDOEL: System into Extended Nonlinear Observer Canonical Form (ENOCF)

As reviewed in Chapter 3, the (conventional) OEL problem is to transform a nonlinear system into NOCF that is an observable linear system modulo output injection depending on the system output. In order to enlarge the class of systems to which we can apply similar approaches, several ideas have been proposed. For instance, system immersion technique is to immerse a given system into a higher-dimensional NOCF, and the concepts of DOEL and RDOEL are first to append an auxiliary dynamics of which input is the output of a given system and then to transform the extended system into GNOCF which is an observable linear system modulo generalized output injection depending on the system output and the state of auxiliary dynamics.

Another idea is to introduce a new NOCF of which not only output injection part but also linear part depends on the system output (i.e. $A = A(y)$ in the NOCF (3.2.1)). For single output nonlinear systems, such an idea was first addressed in [Gua01, RPN01, Gua02, RPN04] by using an output-dependent time-scaling transformation such that

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \alpha(y) \quad \text{with} \quad \alpha(y) > 0. \tag{5.0.1}$$

More precisely, their works have studied the problem of transforming the single

output nonlinear system (3.3.1) into an output-dependent NOCF,

$$\begin{aligned} \dot{z} &= A(y) + a(y), \qquad && z \in \mathbb{R}^n, \\ y &= Cz, && y \in \mathbb{R}, \end{aligned} \tag{5.0.2}$$

where

$$A(y) = \begin{bmatrix} O & \bar{A}(y) \\ O & O \end{bmatrix}_{n\times n}, \qquad \bar{A}(y) = \alpha(y) I_{n-1},$$
$$C = [1\ 0\ \cdots\ 0]_{1\times n}, \qquad a(y) = [a_1(y)\ \cdots\ a_n(y)]^T,$$

via a change of coordinates with the output-dependent time-scaling transformation $\alpha(y)$ defined by (5.0.1). In particular, a complete algorithm to solve the problem together with an output transformation ($y_e = q(y) = Cz$) was developed [Gua05], and the problem was extended to multi-output nonlinear systems by using a multiple output-dependent time-scaling transformation [WL10]. As regards the output-dependent NOCF (5.0.2), another attempt was made in [ZBB07]. The authors of [ZBB07] proposed a modified output-dependent NOCF such that, in the equation (5.0.2),

$$\bar{A}(y) = \mathrm{diag}(\alpha_2(y), \ldots, \alpha_n(y)),$$

with $\alpha_i(y) \neq 0$ for $2 \leq i \leq n$, and addressed the problem of transforming the single output system (3.3.1) into the output-dependent NOCF via just a coordinate transformation without the output-dependent time-scaling transformation. They developed a complete algorithm to design $\alpha_i(y)$'s and a coordinate transformation $z = \Phi(x)$ for the problem. Recently, by combining the concepts of output-dependent NOCF and RDOEL, the authors of [TBZ13] introduced an output-dependent GNOCF (i.e. $A = A(y)$ in the GNOCF (3.5.3)) and provided a sufficient condition for the problem of transforming the single output system (3.3.1) into the proposed output-dependent GNOCF with the aid of an auxiliary dynamics such as (3.5.1).

In this chapter, inspired by the above previous works, we introduce a new

NOCF called extended nonlinear observer canonical form (ENOCF), of which both linear and output injection parts depend on the system output and the state of auxiliary state, and then investigate the problem of transforming a class of single output nonlinear systems into the proposed ENOCF with the aid of auxiliary dynamics. In actual fact, the problem is a natural extension of the RDOEL problem for single output systems and the work [TBZ13]. Most of this chapter is based on [CYS14a].

5.1 Problem Statement

Consider a single output nonlinear system given by

$$\begin{aligned} \dot{\xi} &= f(\xi), \qquad \xi \in \mathbb{R}^n, \\ y &= h(\xi), \qquad y \in \mathbb{R}, \end{aligned} \tag{5.1.1}$$

where ξ is the system state, y is the system output, $f(\xi)$ is a smooth vector field, and $h(\xi)$ is a smooth real-valued function. For the above system, we append an auxiliary dynamics such that

$$\dot{\eta} = p(\eta, y) = \begin{bmatrix} p_1(\eta, y) \\ \vdots \\ p_d(\eta, y) \end{bmatrix}, \qquad \eta \in \mathbb{R}^d, \tag{5.1.2}$$

where $\eta = [\eta_1 \ \cdots \eta_d]^T$ is the auxiliary state and $p(\eta, y)$ is a smooth vector field. After that, on a neighborhood $U \times V \subset \mathbb{R}^d \times \mathbb{R}^n$ of an initial state $(\eta(0), \xi(0))$, we consider a coordinate transformation $\Phi : U \times V \to \mathbb{R}^d \times \mathbb{R}^n$, $(\eta, \xi) \mapsto (w, z)$ with $w = \eta$, which is a diffeomorphism onto its image, and an output transformation $y_e = q(\eta, y)$ such that $z = (\Pi \circ \Phi)(\eta, \xi)$ and $y_e = q(\eta, y)$ transform the extended system consisting of the given system (5.1.1) and the auxiliary dynamics (5.1.2),

$$\begin{aligned} \begin{bmatrix} \dot{\eta} \\ \dot{\xi} \end{bmatrix} &= F(\eta, \xi) := \begin{bmatrix} p(\eta, y) \\ f(\xi) \end{bmatrix}, \\ y &= h(\xi), \end{aligned} \tag{5.1.3}$$

into a system of the following form (called *extended nonlinear observer canonical form (ENOCF)*):

$$\begin{aligned} \dot{z} &= A(\eta, y)z + a(\eta, y), \qquad && z \in \mathbb{R}^n \\ y_e &= q(\eta, y) = Cz, && y_e \in \mathbb{R}, \end{aligned} \tag{5.1.4}$$

where $z = [z_1 \ \cdots \ z_n]^T$, $\Pi : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ is a projection such that $\Pi(w, z) = z$,

$$\begin{aligned} A(\eta, y) &= \begin{bmatrix} O & \bar{A}(\eta, y) \\ O & O \end{bmatrix}_{n\times n}, \\ \bar{A}(\eta, y) &= \operatorname{diag}\big(\alpha_2(\eta, y), \ldots, \alpha_n(\eta, y)\big), \\ C &= [1 \ 0 \ \cdots \ 0]_{1\times n}, \\ a(\eta, y) &= [a_1(\eta, y) \ \cdots \ a_n(\eta, y)]^T, \end{aligned}$$

with $\alpha_i(\eta, y) \neq 0$ for $2 \leq i \leq n$ and all $(\eta, y) \in U \times h(V)$.

If (η, y) is bounded and there exist such transformations Φ and q, then we can design a high-gain observer (including the auxiliary dynamics (5.1.2)) by the method in [BFH98] as follows:

$$\begin{aligned} \dot{\eta} &= p(\eta, y), \\ \dot{\hat{z}} &= A(\eta, y)\hat{z} + a(\eta, y) - \Lambda^{-1}(\eta, y)S_\theta^{-1}C^T(C\hat{z} - y_e), \\ y_e &= q(\eta, y), \\ \hat{\xi} &= \Pi \circ \Phi^{-1}(\eta, \hat{z}), \end{aligned} \tag{5.1.5}$$

where $\Lambda(\eta, y) = \operatorname{diag}(1, \alpha_2(\eta, y), \ldots, \prod_{i=2}^n \alpha_i(\eta, y))$, and S_θ is a solution of the algebraic Lyapunov equation,

$$\theta S_\theta + A_n^T S_\theta + S_\theta A_n - C^T C = 0,$$

with

$$A_n = \begin{bmatrix} O & I_{n-1} \\ O & O \end{bmatrix}_{n\times n}.$$

By the result of [BFH98], a well-chosen θ governs the exponential stability of the observer error dynamics,

$$\dot{e}_z = (A(\eta, y) - \Lambda(\eta, y) S_\theta^{-1} C^T C) e_z,$$

where $e_z := \hat{z} - z$. In this respect, we deal with the problem of transforming the extended system (5.1.3) into the system composed of the auxiliary dynamics (5.1.3) intact and the ENOCF (5.1.4), as a new method to design observers for a class of single output nonlinear systems.

Definition 5.1.1. We say that the ENOCF problem is solved for the system (5.1.1) via the auxiliary dynamics (5.1.2) if there exist a coordinate transformation $\Phi(\eta, \xi)$ and an output transformation $q(\eta, y)$ transforming the extended system (5.1.3) into the system composed of the auxiliary dynamics (5.1.3) and the ENOCF (5.1.4). □

Remark 5.1.1. Since $z_1 = y_e$, the output transformation $q(\eta, y)$ is a part of the coordinate transformation $\Phi(\eta, \xi)$. That is to say, it holds that $y_e = q(\eta, y) = q(\eta, h(\xi)) = z_1$. □

Remark 5.1.2. The ENOCF problem is a natural extension of the RDOEL problem for single output systems in the sense that they are identical when $\alpha_2(\eta, y) = \cdots = \alpha_n(\eta, y) = 1$. The difference is that the $(n-1)$ number of functions α_i's can be designed in the ENOCF problem. In actual fact, the difference makes it possible to solve the ENOCF problem for a class of systems to which the RDOEL problem is not solvable. We will illustrate it by an example in Section 5.4. □

Remark 5.1.3. In the case when $A(\eta, y) = A(y)$ and $d = 1$ (dimension of the auxiliary dynamics (5.1.2)), a sufficient condition for the ENOCF problem to be solved was already given in [TBZ13]. Actually, our research is motivated by the work. However, an equivalent condition has not been found even for the case. Furthermore, a lot of works dealing with dynamic extension of OEL (such as DOEL or RDOEL) have considered high-order auxiliary dynamics even for the case of single output systems (e.g. [BB11, BYS06, YBS07, YBS11, YBSS10,

YYS12]) because it may allow us to solve the problems for a larger class of systems. In these regards, our objective is to derive a necessary and sufficient condition for the ENOCF problem in the case where $A(\eta, y)$ and the general auxiliary dynamics (5.1.2) are considered. □

5.2 Necessary Conditions

In this section, we provide two necessary conditions. One is a condition on the output transformation $q(\eta, y)$ and the observability of the given system (5.1.1). The other is concerned with the observable form of the system (5.1.1), similarly to the RDOEL problem.

5.2.1 Output Transformation and Observability

The following theorem gives the first necessary condition for the ENOCF problem to be solved.

Theorem 5.2.1. If the ENOCF problem is solved for the system (5.1.1) via the auxiliary dynamics (5.1.2), then both the following conditions are satisfied:

(a) the output transformation $y_e = q(\eta, h(\xi))$ satisfies that

$$\frac{\partial q(\eta, h(\xi))}{\partial h} \neq 0 \qquad \text{for all } (\eta, \xi) \in U \times V, \tag{5.2.1}$$

(b) the system (5.1.1) is locally observable at $\xi(0)$, i.e., it satisfies the observability rank condition,

$$\operatorname{rank}\big(\operatorname{span}\{\mathrm{d}\mathcal{L}_f^{k-1} h(\xi) : 1 \leq k \leq n\}\big) = n \qquad \text{for all } \xi \in V, \tag{5.2.2}$$

where $U \times V \subset \mathbb{R}^d \times \mathbb{R}^n$ is a neighborhood of $(\eta(0), \xi(0))$.

Proof. When $x = [x_1 \ \cdots \ x_n]^T$, we denote $\mathrm{d}x := [\mathrm{d}x_1 \ \cdots \ \mathrm{d}x_n]^T$ for convenience. Suppose that the ENOCF problem is solved for the system (5.1.1) via the auxiliary dynamics (5.1.2). Then, there exist a coordinate transformation $[w^T \ z^T]^T =$

$\Phi(\eta,\xi) = [\Phi_1(\eta,\xi) \ \cdots \ \Phi_{d+n}(\eta,\xi)]^T$ and an output transformation $y_e = q(\eta, y)$ such that

$$
\begin{aligned}
w_i &= \Phi_i(\eta,\xi) = \eta_i, \\
z_1 &= \Phi_{d+1}(\eta,\xi) = q(\eta, h(\xi)) = y_e, && (5.2.3a)\\
z_j &= \Phi_{d+j}(\eta,\xi) = \frac{1}{\alpha_j(\eta,h(\xi))}\Big(\mathcal{L}_F\Phi_{d+j-1}(\eta,\xi) - a_{j-1}(\eta,h(\xi))\Big), && (5.2.3b)
\end{aligned}
$$

for $1 \le i \le d$ and $2 \le j \le n$. By the above equations, it holds that

$$
\begin{aligned}
\mathrm{d}w_i &= \mathrm{d}\eta_i, && (5.2.4a)\\
\mathrm{d}z_1 &= \sum_{k=1}^{d} \frac{\partial q}{\partial \eta_k}\mathrm{d}\eta_k + \frac{\partial q}{\partial h}\mathrm{d}h \equiv \frac{\partial q}{\partial h}\mathrm{d}h \mod (\mathrm{d}\eta_1,\ldots,\mathrm{d}\eta_d), && (5.2.4b)\\
\mathrm{d}z_j &= (\mathcal{L}_{F_z} z_{j-1} - a_{j-1})\mathrm{d}\frac{1}{\alpha_j} + \frac{1}{\alpha_j}(\mathrm{d}\mathcal{L}_{F_z} z_{j-1} - \mathrm{d}a_{j-1}), && (5.2.4c)
\end{aligned}
$$

for $1 \le i \le d$ and $2 \le j \le n$. For the $\mathrm{d}z_j$'s ($2 \le j \le n$) in the last equation, we claim that

$$
\mathrm{d}z_j \equiv \Big(\prod_{k=2}^{j}\frac{1}{\alpha_k}\Big)\frac{\partial q}{\partial h}\mathrm{d}\mathcal{L}_F^{j-1}h \mod (\mathrm{d}\eta_1,\ldots,\mathrm{d}\eta_d,\mathrm{d}h,\ldots,\mathrm{d}\mathcal{L}_F^{j-2}h). \qquad (5.2.5)
$$

The proof of the claim is by induction on j starting from $j = 2$. If $j = 2$, then it follows from (5.2.4b) and (5.2.4c) that

$$
\begin{aligned}
\mathrm{d}z_2 &= (\mathcal{L}_{F_z} z_1 - a_1)\mathrm{d}\frac{1}{\alpha_2(\eta,h)} + \frac{1}{\alpha_2}\big(\mathrm{d}\mathcal{L}_{F_z} z_1 - \mathrm{d}a_1(\eta,h)\big)\\
&\equiv \frac{1}{\alpha_2}\frac{\partial q}{\partial h}\mathrm{d}\mathcal{L}_F h \mod (\mathrm{d}\eta_1,\ldots,\mathrm{d}\eta_d,\mathrm{d}h).
\end{aligned}
$$

Therefore, the equation (5.2.5) holds when $j = 2$. Suppose that $3 \le j \le n$ and the equation (5.2.5) holds for $j-1$, i.e.,

$$
\mathrm{d}z_{j-1} \equiv \Big(\prod_{k=2}^{j-1}\frac{1}{\alpha_k}\Big)\frac{\partial q}{\partial h}\mathrm{d}\mathcal{L}_F^{j-2}h \mod (\mathrm{d}\eta_1,\ldots,\mathrm{d}\eta_d,\mathrm{d}h,\ldots,\mathrm{d}\mathcal{L}_F^{j-3}h). \qquad (5.2.6)
$$

Then, by the equation (5.2.4c) and the induction hypothesis (5.2.6), it is easy to

see that

$$\begin{aligned}
\mathrm{d}z_j &= (\mathcal{L}_{F_z} z_{j-1} - a_{j-1})\mathrm{d}\frac{1}{\alpha_j(\eta,h)} + \frac{1}{\alpha_j}\big(\mathrm{d}\mathcal{L}_{F_z} z_{j-1} - \mathrm{d}a_{j-1}(\eta,h)\big) \\
&\equiv \frac{1}{\alpha_j}\Big(\prod_{k=2}^{j-1}\frac{1}{\alpha_k}\Big)\frac{\partial q}{\partial h}\mathrm{d}\mathcal{L}_F^{j-1}h \mod (\mathrm{d}\eta_1,\ldots,\mathrm{d}\eta_d,\mathrm{d}h,\ldots,\mathrm{d}\mathcal{L}_F^{j-2}h).
\end{aligned}$$

Consequently, the equation (5.2.5) also holds for j, and thus it is concluded that the claim is true.

Since h does not depend on η, it holds that $\mathcal{L}_F^k h(\xi) = \mathcal{L}_f^k h(\xi)$ for any non-negative integer k. Moreover, the 1-forms $\mathrm{d}h,\ldots,\mathrm{d}\mathcal{L}_f^{n-1}h$ can be represented as linear combinations of $\mathrm{d}\xi_1,\ldots,\mathrm{d}\xi_n$. Therefore, we obtain from (5.2.4a), (5.2.4b) and (5.2.5) that

$$\begin{aligned}
\begin{bmatrix} \mathrm{d}w \\ \mathrm{d}z \end{bmatrix} &= \begin{bmatrix} I_d & O \\ * & R_{n\times n} \end{bmatrix}\begin{bmatrix} \mathrm{d}\eta \\ \mathrm{d}h \\ \vdots \\ \mathrm{d}\mathcal{L}_f^{n-1}h \end{bmatrix} \\
&= \begin{bmatrix} I_d & O \\ * & R_{n\times n} \end{bmatrix}\begin{bmatrix} I_d & O \\ O & S_{n\times n} \end{bmatrix}\begin{bmatrix} \mathrm{d}\eta \\ \mathrm{d}\xi \end{bmatrix},
\end{aligned} \tag{5.2.7}$$

where

$$R = \begin{bmatrix} \frac{\partial q}{\partial h} & 0 & \cdots & 0 \\ * & \frac{\partial q}{\partial h}\frac{1}{\alpha_2} & \ddots & \vdots \\ * & * & \ddots & 0 \\ * & * & * & \frac{\partial q}{\partial h}\prod_{k=2}^{n}\frac{1}{\alpha_k} \end{bmatrix}, \tag{5.2.8}$$

which is a lower-triangular matrix of which diagonal entries are $\frac{\partial q}{\partial h}$, $\frac{\partial q}{\partial h}\frac{1}{\alpha_2},\ldots$, and $\frac{\partial q}{\partial h}\prod_{k=2}^{n}\frac{1}{\alpha_k}$ in sequence. Since Φ is a diffeomorphism from a neighborhood $U\times V$ of $(\eta(0),\xi(0))$ onto its image, both the matrices R and M are nonsingular on $U\times V$. Therefore, $\frac{\partial q(\eta,h(\xi))}{\partial h} \neq 0$ for all $(\eta,\xi)\in U\times V$ and the 1-forms $\mathrm{d}h,\ldots,\mathrm{d}\mathcal{L}_f^{n-1}h$ are linearly independent on V. Hence, the given system (5.1.1) satisfies the observability rank condition (5.2.2). □

By Theorem 5.2.1, we assume the observability rank condition (5.2.2) of the system (5.1.1). Then, as mentioned before, the system can be expressed on a neighborhood of $\xi(0)$ as the following observable form:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ &\vdots \\ \dot{x}_{n-1} &= x_n, \\ \dot{x}_n &= f_n(x), \\ y &= x_1, \end{aligned} \tag{5.2.9}$$

where $x_i = \mathcal{L}_f^{i-1} h(\xi)$ for $1 \leq i \leq n$, $x = [x_1 \ \cdots \ x_n]^T \in W$, $f_n : W \to \mathbb{R}$ is a smooth function, and $W \subset \mathbb{R}^n$ is a neighborhood of $x(0)$. For convenience, we write $\dot{x} = f(x)$ and $y = h(x) = x_1$. Then, the extended system (5.1.3) is also written as

$$\begin{aligned} \begin{bmatrix} \dot{\eta} \\ \dot{x} \end{bmatrix} &= F(\eta, x) := \begin{bmatrix} p(\eta, x_1) \\ f(x) \end{bmatrix}, \\ y &= h(x) = x_1. \end{aligned} \tag{5.2.10}$$

Henceforth, without loss of generality, we regard the original system (5.1.1) and the extended system (5.1.3) as the observable form (5.2.9) and the system (5.2.10), respectively.

5.2.2 System Dynamics

By Theorem 3.2.4 and Theorem 4.2.7, a necessary condition of both OEL and RDOEL for single output systems is that $f_n(x)$ in the observable form (5.2.9) should be a certain polynomial of weighted degree n, which is defined in Definition 3.2.2. In this subsection, we show that the condition is also a necessary condition for the ENOCF problem. Since we deal with the problem for the single output system (5.1.1), in order to prevent confusion, we modify Definition 4.2.2 to fit it to the case of single output systems.

Definition 5.2.1. For the extended system (5.2.10), we denote by $\mathcal{P}_{se}(x)$ (re-

spectively, $\mathcal{P}_s(x)$) the ring of polynomials in $x_2, \ldots, x_n$ with coefficients that are smooth real-valued functions of η and x_1 (respectively, x_1 only). The weighted degree of a monomial $c(\eta, x_1)x_2^{k_2} \cdots x_n^{k_n}$ is defined as $\sum_{i=2}^{n}(i-1)k_i$. The weighted degree of a polynomial in $\mathcal{P}_{se}(x)$ or $\mathcal{P}_s(x)$ is the highest weighted degree of any term in the polynomial. We denote by $\mathcal{P}_{se}^k(x)$ (respectively, $\mathcal{P}_s^k(x)$) the set of all the polynomials in $\mathcal{P}_{se}(x)$ (respectively, $\mathcal{P}_s(x)$) of which weighted degree is less than or equal to k. $\mathcal{P}_{se}^0(x)$ (respectively, $\mathcal{P}_s^0(x)$) represents the set of all smooth real-valued functions of η and x_1 (respectively, x_1 only). In fact, the subscript 's' means 'single output case'. □

In a similar fashion to Proposition 4.2.2 and Lemma 4.2.3, we give a proposition, a lemma, and its corollary as regards the partial derivatives and the Lie derivatives of elements in $\mathcal{P}_{se}^k(x)$.

Proposition 5.2.2. If $\phi(\eta, x) \in \mathcal{P}_{se}^k(x)$ for any nonnegative integer k, then it holds that

$$\frac{\partial \phi}{\partial \eta_i} \in \mathcal{P}_{se}^k(x),$$
$$\frac{\partial \phi}{\partial x_j} = \begin{cases} 0 & \text{if } j > k+1, \\ * \in \mathcal{P}_{se}^{k-j+1}(x) & \text{if } j \leq k+1, \end{cases}$$

for $1 \leq i \leq d$ and $1 \leq j \leq n$. □

Lemma 5.2.3. For any $0 \leq k \leq n-2$ and $\phi(\eta, x) \in \mathcal{P}_{se}^k(x)$, it holds that

$$\mathcal{L}_F \phi(\eta, x) \in \mathcal{P}_{se}^{k+1}(x),$$

where F is the vector field of the extended system (5.2.10).

Proof. Let $\phi \in \mathcal{P}_{se}^0(x)$. Then, ϕ is a function of η and x_1 only. Thus, it holds that

$$\begin{aligned} \mathcal{L}_F \phi &= \sum_{i=1}^{d} \frac{\partial \phi}{\partial \eta_i} \dot{\eta}_i + \frac{\partial \phi}{\partial x_1} \dot{x}_1 \\ &= \sum_{i=1}^{d} \frac{\partial \phi}{\partial \eta_i} p_i + \frac{\partial \phi}{\partial x_1} x_2 \in \mathcal{P}_{se}^1(x). \end{aligned}$$

Therefore, the lemma is true when $k = 0$. Let $\phi \in \mathcal{P}_{se}^k(x)$ and $1 \leq k \leq n-2$. Then, ϕ is a polynomial of x_i's, where $2 \leq i \leq k+1 \leq n-1$, with coefficients that are elements of $\mathcal{P}_{se}^0(x)$. For any $c(\eta, x_1) \in \mathcal{P}_{se}^0(x)$, we have $\mathcal{L}_F c = \mathcal{P}_{se}^1(x)$ because the lemma is true when $k = 0$. Moreover, while $x_i \in \mathcal{P}_s^{i-1}(x)$, it holds that $\mathcal{L}_F x_i = x_{i+1} \in \mathcal{P}_s^i(x)$ for $2 \leq i \leq n-1$. By these facts and the Leibniz rule, it is easy to see that $\mathcal{L}_F \phi \in \mathcal{P}_{se}^{k+1}(x)$. □

Corollary 5.2.4. For any $0 \leq k \leq n-2$ and $\phi(\eta, x_1) \in \mathcal{P}_{se}^0(x)$, it holds that

$$\mathcal{L}_F^k \phi \in \mathcal{P}_{se}^k(x),$$

where F is the vector field of the extended system (5.2.10).

Proof. This corollary is a direct consequence from Lemma 5.2.3. □

The following theorem shows that the condition $f_n(x) \in \mathcal{P}_s^n(x)$ is also a necessary condition for the ENOCF problem.

Theorem 5.2.5. The ENOCF problem is solved for the system (5.2.9) only if $f_n(x)$ belongs to $\mathcal{P}_s^n(x)$.

Proof. Suppose that the ENOCF problem is solved for the system (5.2.9). Then, there exist an auxiliary dynamics such as (5.1.2) so that the extended system (5.2.10) can be transformed into the system composed of the auxiliary dynamics (5.1.2) intact and the ENOCF (5.1.4). Let $y_e = q(\eta, y)$ be the output transformation. Then, $z_1 = y_e = q(\eta, x_1)$. For $2 \leq i \leq n$, we claim that z_i can be represented as follows:

$$z_i = \sum_{j=0}^{i-2} \Big(C_i^j \mathcal{L}_F^{i-1-j} q + D_i^j \Big), \tag{5.2.11}$$

where $C_i^j, D_i^j \in \mathcal{P}_{se}^j(x)$ for $0 \leq j \leq i-2$ and, in particular, $C_i^0 = \prod_{k=2}^i \frac{1}{\alpha_k}$. The proof of the claim is by induction on i starting from $i = 2$. If $i = 2$, then it follows from the equation (5.2.3b) in the proof of Theorem 5.2.1 that

$$z_2 = \frac{1}{\alpha_2(\eta, x_1)} \Big(\mathcal{L}_F q(\eta, x_1) - a_1(\eta, x_1) \Big) = C_2^0 \mathcal{L}_f q + D_2^0,$$

where

$$C_2^0 := \frac{1}{\alpha_2} \in \mathcal{P}_{se}^0(x), \qquad D_2^0 := -\frac{a_1}{\alpha_2} \in \mathcal{P}_{se}^0(x).$$

Thus, the equation (5.2.11) is satisfied. Suppose that $3 \le i \le n$ and (5.2.11) holds for $i-1$, i.e.,

$$z_{i-1} = \sum_{j=0}^{i-3} \Big(C_{i-1}^j \mathcal{L}_F^{i-2-j} q + D_{i-1}^j \Big),$$

where $C_{i-1}^j, D_{i-1}^j \in \mathcal{P}_{se}^j(x)$ for $0 \le j \le i-3$ and $C_{i-1}^0 = \prod_{k=2}^{i-1} \frac{1}{\alpha_k}$. Then, we also obtain from the equation (5.2.3b) that

$$\begin{aligned} z_i &= \frac{1}{\alpha_i} \Bigg(\mathcal{L}_F \Big(\sum_{j=0}^{i-3} \big(C_{i-1}^j \mathcal{L}_F^{i-2-j} q + D_{i-1}^j \big) \Big) - a_{i-1} \Bigg\} \\ &= \frac{1}{\alpha_i} \Bigg(\sum_{j=0}^{i-3} \Big(\mathcal{L}_F C_{i-1}^j \cdot \mathcal{L}_F^{i-2-j} q + C_{i-1}^j \mathcal{L}_F^{i-1-j} q + \mathcal{L}_F D_{i-1}^j \Big) - a_{i-1} \Bigg) \\ &= \sum_{j=0}^{i-2} \Big(C_i^j \mathcal{L}_F^{i-1-j} q + D_i^j \Big), \end{aligned}$$

where

$$\begin{aligned} C_i^0 &:= \frac{1}{\alpha_i} C_{i-1}^0 = \prod_{k=2}^{i} \frac{1}{\alpha_k}, \\ C_i^j &:= \frac{1}{\alpha_i} (\mathcal{L}_F C_{i-1}^{j-1} + C_{i-1}^j) \qquad \text{for } 1 \le j \le i-3, \\ C_i^{i-2} &:= \frac{1}{\alpha_i} \mathcal{L}_F C_{i-1}^{i-3}, \\ D_i^0 &:= -\frac{a_{i-1}}{\alpha_i}, \\ D_i^j &:= \frac{1}{\alpha_i} \mathcal{L}_F D_{i-1}^{j-1} \qquad \text{for } 1 \le j \le i-2. \end{aligned}$$

By Lemma 5.2.3 and the induction hypothesis, it is easy to see that $C_i^j, D_i^j \in \mathcal{P}_e^j(x)$ for $0 \le j \le i-2$. Therefore, the equation (5.2.11) also holds for i, and thus the claim is true.

Let us go back to the proof of Theorem 5.2.5. By the ENOCF (5.1.4) and the claim (5.2.11), it holds that

$$\begin{aligned} a_n(\eta, x_1) = \dot{z}_n &= \mathcal{L}_F\Big(\sum_{j=0}^{n-2}\big(C_n^j \mathcal{L}_F^{n-1-j} q + D_n^j\big)\Big) \\ &= L_F C_n^0 \cdot \mathcal{L}_F^{n-1} q + C_n^0 \mathcal{L}_F^n q + L_F\Big(\sum_{j=1}^{n-2} C_n^j \mathcal{L}_F^{n-1-j} q + \sum_{j=0}^{n-2} D_n^j\Big) \\ &= C_n^0 \mathcal{L}_F^{n-1}\Big(\frac{\partial q}{\partial x_1} x_2\Big) + E, \end{aligned}$$

where

$$E := C_n^0 \mathcal{L}_F^{n-1}\Big(\sum_{k=1}^{d} \frac{\partial q}{\partial \eta_k} p_k\Big) + L_F C_n^0 \cdot \mathcal{L}_F^{n-1} q + L_F\Big(\sum_{j=1}^{n-2} C_n^j \mathcal{L}_F^{n-1-j} q + \sum_{j=0}^{n-2} D_n^j\Big).$$

By Lemma 5.2.3 and Corollary 5.2.4, we can observer that $E \in \mathcal{P}_{se}^n(x)$. Since $C_n^0 = \prod_{k=2}^n \frac{1}{\alpha_k} \neq 0$ and $\frac{\partial q}{\partial x_1} \neq 0$ by Theorem 5.2.1, it follows from the above equation that

$$\begin{aligned} f_n(x) &= \mathcal{L}_F^{n-1} x_2 \\ &= \frac{1}{C_n^0 \frac{\partial q}{\partial x_1}}\Bigg(a_n - C_n^0 \sum_{k=0}^{n-2} \begin{pmatrix} n-1 \\ k \end{pmatrix} \mathcal{L}_F^{n-1-k}\Big(\frac{\partial q}{\partial x_1}\Big) \cdot \mathcal{L}_F^k x_2 - E\Bigg). \end{aligned}$$

Since it holds that $\mathcal{L}_F^{n-1-k}\Big(\frac{\partial q}{\partial x_1}\Big) \in \mathcal{P}_{se}^{n-1-k}(x)$ and $\mathcal{L}_F^k x_2 = x_{2+k} \in \mathcal{P}_{se}^{1+k}(x)$ for $0 \leq k \leq n-2$, the right-hand side of the above equation belongs to $\mathcal{P}_{se}^n(x)$. Therefore, $f_n(x) \in \mathcal{P}_s^n(x)$ because it does not depend on η. □

NB: Henceforth, by Theorem 5.2.5, we assume that the system (5.1.1) satisfies the condition $f_n(x) \in \mathcal{P}_s^n(x)$ in its observable form (5.2.9).

5.3 Necessary and Sufficient Condition

In the RDOEL problem studied in the previous chapter, the condition, $f_i(x) \in \mathcal{P}^{n_i}(x)$ for $1 \leq i \leq m$, plays an important role in proving Theorem 4.3.4 which states a geometric necessary and sufficient condition. In this section, by means of

the condition $f_n(x) \in \mathcal{P}_s^n(x)$, we also derive a geometric necessary and sufficient condition for the ENOCF problem.

Theorem 5.3.1. The ENOCF problem is solved for the system (5.2.9) via the auxiliary dynamics (5.1.2) if and only if there exist n functions $\phi(\eta, x_1)$, $\alpha_2(\eta, x_1)$, $\ldots$, $\alpha_n(\eta, x_1) \in \mathcal{P}_{se}^0(x)$ such that both the following conditions are satisfied:

(E1) $\phi(\eta, x_1) \neq 0$, $\alpha_2(\eta, x_1) \neq 0$, $\ldots$, $\alpha_n(\eta, x_1) \neq 0$ for all $(\eta, x_1) \in U \times h(W)$.

(E2) $[X_i,\ X_j] = 0$ on $U \times W$ for $i, j = 1, \ldots, n$,

where X_1, $\ldots$, X_n are vector fields defined by

$$\begin{aligned} X_n &:= \phi \frac{\partial}{\partial x_n}, \\ X_i &:= \frac{1}{\alpha_{i+1}}[X_{i+1},\ F] \qquad \text{for } 1 \leq i \leq n-1, \end{aligned}$$

and $U \times W \subset \mathbb{R}^d \times \mathbb{R}^n$ is a neighborhood of $(\eta(0), x(0))$. □

Proof. When $\zeta = [\zeta_1\ \cdots\ \zeta_n]^T$, $\mathrm{d}\zeta := [\mathrm{d}\zeta_1\ \cdots\ \mathrm{d}\zeta_n]^T$ and $\frac{\partial}{\partial \zeta} := [\frac{\partial}{\partial \zeta_1}\ \cdots\ \frac{\partial}{\partial \zeta_n}]$.

(Proof of Necessity): Suppose that the ENOCF problem is solved for the system (5.2.9) via the auxiliary dynamics (5.1.2). Then, there exist the $(n-1)$ functions $\alpha_2(\eta, x_1)$, $\ldots$, $\alpha_n(\eta, x_1)$ satisfying the condition (E1) by the ENOCF (5.1.4). Furthermore, it follows from (5.2.7) in the proof of Theorem 5.2.1 that

$$\begin{bmatrix} \mathrm{d}\eta \\ \mathrm{d}x \end{bmatrix} = \begin{bmatrix} I_d & O \\ * & R^{-1} \end{bmatrix} \begin{bmatrix} \mathrm{d}w \\ \mathrm{d}z \end{bmatrix}, \tag{5.3.1}$$

where the matrix R is given by the equation (5.2.8). Due to the lower-triangularity of R, $\mathrm{d}x_1$ can be represented as a linear combination of $\mathrm{d}w_1$, $\ldots$, $\mathrm{d}w_d$, and $\mathrm{d}z_1$ only. Thus, there exists a function $\tilde{q}(w, z_1)$ such that $y = x_1 = \tilde{q}(w, z_1)$. As a result, the vector field F of the extended system (5.2.10) can be expressed in the (w, z)-coordinates as follows:

$$F = \sum_{k=1}^{d} \tilde{p}_k(w, z_1) \frac{\partial}{\partial w_k} + \sum_{i=1}^{n-1} \Big(\tilde{\alpha}_{i+1}(w, z_1) z_{i+1} + \tilde{a}_i(w, z_1) \Big) \frac{\partial}{\partial z_i} + \tilde{a}_n(w, z_1) \frac{\partial}{\partial z_n},$$

where

$$\begin{aligned}
\tilde{p}_k(w, z_1) &:= p_k(w, \tilde{q}(w, z_1)) = p_k(\eta, y) && \text{for } 1 \le k \le d, \\
\tilde{\alpha}_{i+1}(w, z_1) &:= \alpha_{i+1}(w, \tilde{q}(w, z_1)) = \alpha_{i+1}(\eta, y) && \text{for } 1 \le i \le n-1, \\
\tilde{a}_i(w, z_1) &:= a_i(w, \tilde{q}(w, z_1)) = a_i(\eta, y) && \text{for } 1 \le i \le n.
\end{aligned}$$

Therefore, a straightforward calculation gives

$$\frac{1}{\alpha_{i+1}}\left[\frac{\partial}{\partial z_{i+1}},\ F\right] = \frac{\partial}{\partial z_i} \qquad \text{for } 1 \le i \le n-1. \tag{5.3.2}$$

By the equation (5.3.1) and the duality between 1-forms and vector fields, it holds that

$$\begin{bmatrix} \frac{\partial}{\partial w} & \frac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial \eta} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} I_d & O \\ * & R^{-1} \end{bmatrix}. \tag{5.3.3}$$

Let $\phi := (1/\frac{\partial q}{\partial x_1}) \prod_{k=2}^{n} \alpha_k$ which is the n-th diagonal entry of R^{-1}. Since $\frac{\partial q(\eta, x_1)}{\partial x_1} \neq 0$ for all $(\eta, x_1) \in U \times h(W)$ by Theorem 5.2.1, the function ϕ is well defined on $U \times h(W)$ and satisfies the condition (E1). Moreover, by the lower-triangularity (5.2.8) of R and the equations (5.3.2) and (5.3.3), it is easy to see that

$$\begin{aligned}
X_n &:= \phi \frac{\partial}{\partial x_n} = \frac{\partial}{\partial z_n}, \\
X_i &:= \frac{1}{\alpha_{i+1}}[X_i, F] = \frac{\partial}{\partial z_i} \qquad \text{for } 1 \le i \le n-1.
\end{aligned}$$

Therefore, the condition (E2) is clearly satisfied.

(Proof of Sufficiency): Suppose that there exist n functions $\phi(\eta, x_1)$, $\alpha_2(\eta, x_1)$, $\ldots$, $\alpha_n(\eta, x_1) \in \mathcal{P}^0_{se}(x)$ satisfying the conditions (E1) and (E2). For $1 \le i \le n-1$, we claim that X_i can be represented as

$$X_i = \sum_{j=i}^{n} \phi_i^j \frac{\partial}{\partial x_j}, \tag{5.3.4}$$

where $\phi_i^j \in \mathcal{P}^{j-i}_{se}(x)$ and, in particular, $\phi_i^i = \phi \prod_{k=i+1}^{n} \frac{1}{\alpha_k}$. The proof of the claim is by induction on i starting from $n-1$. The vector field F of the system (5.2.10)

can be written as

$$F = \sum_{k=1}^{d} p_k(\eta, x_1)\frac{\partial}{\partial \eta_k} + \sum_{k=1}^{n-1} x_{k+1}\frac{\partial}{\partial x_k} + f_n(x)\frac{\partial}{\partial x_n}.$$

Thus, it holds that

$$\begin{aligned} X_{n-1} &:= \frac{1}{\alpha_n}[X_n,\ F] = \frac{1}{\alpha_n}\Big[\phi\frac{\partial}{\partial x_n},\ F\Big] \\ &= \frac{1}{\alpha_n}\Bigg(\phi\Big(\frac{\partial}{\partial x_{n-1}} + \frac{\partial f_n}{\partial x_n}\frac{\partial}{\partial x_n}\Big) - \Big(\sum_{k=1}^{d} p_k\frac{\partial \phi}{\partial \eta_k} + x_2\frac{\partial \phi}{\partial x_1}\Big)\frac{\partial}{\partial x_n}\Bigg) \\ &= \phi_{n-1}^{n-1}\frac{\partial}{\partial x_{n-1}} + \phi_{n-1}^{n}\frac{\partial}{\partial x_n}, \end{aligned}$$

where

$$\begin{aligned} \phi_{n-1}^{n-1} &:= \phi\frac{1}{\alpha_n} \in \mathcal{P}_{se}^{0}(x), \\ \phi_{n-1}^{n} &:= \frac{1}{\alpha_n}\Big(\frac{\partial f_n}{\partial x_n} - \sum_{k=1}^{d} p_k\frac{\partial \phi}{\partial \eta_k} + x_2\frac{\partial \phi}{\partial x_1}\Big) \in \mathcal{P}_{se}^{1}(x). \end{aligned}$$

Therefore, the equation (5.3.4) holds when $i = n-1$. Suppose that $1 \le i \le n-2$ and (5.3.4) holds for $i+1$, i.e.,

$$X_{i+1} = \sum_{j=i+1}^{n} \phi_{i+1}^{j}\frac{\partial}{\partial x_j},$$

where $\phi_{i+1}^{j} \in \mathcal{P}_{se}^{j-i-1}(x)$ and, in particular, $\phi_{i+1}^{i+1} = \phi\prod_{k=i+2}^{n}\frac{1}{\alpha_k}$. Then, it follows from the above induction hypothesis that

$$\begin{aligned} X_i := \frac{1}{\alpha_{i+1}}[X_{i+1},\ F] = \frac{1}{\alpha_{i+1}}\sum_{j=i+1}^{n}\Bigg(&\phi_{i+1}^{j}\Big(\frac{\partial}{\partial x_{j-1}} + \frac{\partial f_n}{\partial x_j}\frac{\partial}{\partial x_n}\Big) \\ &- \Big(\sum_{k=1}^{d} p_k\frac{\partial \phi_{i+1}^{j}}{\partial \eta_k} + \sum_{k=1}^{n-1} x_{k+1}\frac{\partial \phi_{i+1}^{j}}{\partial x_k} + f_n\frac{\partial \phi_{i+1}^{j}}{\partial x_n}\Big)\frac{\partial}{\partial x_j}\Bigg). \end{aligned}$$

Since $\phi_{i+1}^{j} \in \mathcal{P}_{se}^{j-i-1}(x)$ by the induction hypothesis, it holds that $\frac{\partial \phi_{i+1}^{j}}{\partial x_n} = 0$ and

$\frac{\partial \phi_{i+1}^j}{\partial x_k} = 0$ for $k > j - i$. Therefore, the above equation can be rewritten as

$$
\begin{aligned}
X_i &= \frac{1}{\alpha_{i+1}} \sum_{j=i+1}^{n} \Bigg(\phi_{i+1}^j \Big(\frac{\partial}{\partial x_{j-1}} + \frac{\partial f_n}{\partial x_j} \frac{\partial}{\partial x_n} \Big) \\
&\quad - \Big(\sum_{k=1}^{d} p_k \frac{\partial \phi_{i+1}^j}{\partial \eta_k} + \sum_{k=1}^{j-i} x_{k+1} \frac{\partial \phi_{i+1}^j}{\partial x_k} \Big) \frac{\partial}{\partial x_j} \Bigg) \\
&= \frac{1}{\alpha_{i+1}} \Bigg(\phi_{i+1}^{i+1} \frac{\partial}{\partial x_i} + \sum_{j=i+2}^{n} \phi_{i+1}^j \frac{\partial}{\partial x_{j-1}} \\
&\quad + \sum_{j=i+1}^{n} \Big(\phi_{i+1}^j \frac{\partial f_n}{\partial x_j} \frac{\partial}{\partial x_n} - \Big(\sum_{k=1}^{d} p_k \frac{\partial \phi_{i+1}^j}{\partial \eta_k} + \sum_{k=1}^{j-i} x_{k+1} \frac{\partial \phi_{i+1}^j}{\partial x_k} \Big) \frac{\partial}{\partial x_j} \Big) \Bigg) \\
&= \frac{1}{\alpha_{i+1}} \Bigg(\phi_{i+1}^{i+1} \frac{\partial}{\partial x_i} + \sum_{j=i+1}^{n-1} \Big(\phi_{i+1}^{j+1} - \sum_{k=1}^{d} p_k \frac{\partial \phi_{i+1}^j}{\partial \eta_k} - \sum_{k=1}^{j-i} x_{k+1} \frac{\partial \phi_{i+1}^j}{\partial x_k} \Big) \frac{\partial}{\partial x_j} \\
&\quad + \Big(\sum_{j=i+1}^{n} \phi_{i+1}^j \frac{\partial f_n}{\partial x_j} - \sum_{k=1}^{d} p_k \frac{\partial \phi_{i+1}^n}{\partial \eta_k} - \sum_{k=1}^{n-i} x_{k+1} \frac{\partial \phi_{i+1}^n}{\partial x_k} \Big) \frac{\partial}{\partial x_n} \Bigg) \\
&= \sum_{j=i}^{n} \phi_i^j \frac{\partial}{\partial x_j},
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_i^i &:= \frac{1}{\alpha_{i+1}} \phi_{i+1}^{i+1} = \phi \prod_{k=i+1}^{n} \frac{1}{\alpha_k} \in \mathcal{P}_{se}^0(x), \\
\phi_i^j &:= \frac{1}{\alpha_{i+1}} \Big(\phi_{i+1}^{j+1} - \sum_{k=1}^{d} p_k \frac{\partial \phi_{i+1}^j}{\partial \eta_k} - \sum_{k=1}^{j-i} x_{k+1} \frac{\partial \phi_{i+1}^j}{\partial x_k} \Big) \in \mathcal{P}_{se}^{j-i}(x) \quad \text{for } 2 \le j \le n-1, \\
\phi_i^n &:= \frac{1}{\alpha_{i+1}} \Big(\sum_{j=i+1}^{n} \phi_{i+1}^j \frac{\partial f_n}{\partial x_j} - \sum_{k=1}^{d} p_k \frac{\partial \phi_{i+1}^n}{\partial \eta_k} - \sum_{k=1}^{n-i} x_{k+1} \frac{\partial \phi_{i+1}^n}{\partial x_k} \Big) \in \mathcal{P}_{se}^{n-i}(x).
\end{aligned}
$$

Hence, the equation (5.3.4) also holds for i, and the claim is true. As a result, it follows from $X_n := \phi \frac{\partial}{\partial x_n}$ and the claim (5.3.4) that

$$
\begin{bmatrix} X_1 & \cdots & X_n \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \end{bmatrix} L, \tag{5.3.5}
$$

where

$$L := \begin{bmatrix} \phi \prod_{k=2}^{n} \frac{1}{\alpha_k} & 0 & \cdots & 0 \\ * & \phi \prod_{k=3}^{n} \frac{1}{\alpha_k} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & \phi \end{bmatrix}.$$

Since the matrix L is a lower-triangular matrix and the condition $(E1)$ is satisfied, L is nonsingular at $(\eta(0), x(0))$. Therefore, the n vector fields X_1, ..., X_n are linearly independent at $(\eta(0), x(0))$. By this fact, (E2), Theorem 2.4.5 (Simultaneous Rectification Theorem), and Corollary 2.4.5, there exists a coordinate chart $(\bar{U} \times \bar{W}, (\bar{w}, z))$, where $\bar{U} \times \bar{W} \subset \mathbb{R}^d \times \mathbb{R}^n$ is a neighborhood of $(\eta(0), x(0))$, such that

$$\frac{\partial}{\partial z_i} = X_i \qquad \text{for } 1 \leq i \leq n. \tag{5.3.6}$$

In addition, since both (η, x) and $(\bar{w}, z)$ are coordinate maps on $\bar{U} \times \bar{W}$, the rest d vector fields $\frac{\partial}{\partial \bar{w}_1}$, ..., $\frac{\partial}{\partial \bar{w}_d}$ can be represented as

$$\frac{\partial}{\partial \bar{w}} = \begin{bmatrix} \frac{\partial}{\partial \eta} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} M_{d\times d} \\ N_{n\times d} \end{bmatrix}. \tag{5.3.7}$$

Therefore, it follows from the equations (5.3.5)-(5.3.7) that

$$\begin{bmatrix} \frac{\partial}{\partial \bar{w}} & \frac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial \eta} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} M & O \\ N & L \end{bmatrix}. \tag{5.3.8}$$

Trivially, both the M and L are nonsingular on $\bar{U} \times \bar{W}$. Thus, by the duality between vector fields and 1-forms, it holds that

$$\begin{bmatrix} \mathrm{d}\bar{w} \\ \mathrm{d}z \end{bmatrix} = \begin{bmatrix} M^{-1} & O \\ -L^{-1}NM^{-1} & L^{-1} \end{bmatrix} \begin{bmatrix} \mathrm{d}\eta \\ \mathrm{d}x \end{bmatrix}.$$

Let $w = \eta$. Then, $\mathrm{d}w = \mathrm{d}\eta$ and thus we obtain from the above equation that

$$\begin{bmatrix} \mathrm{d}w \\ \mathrm{d}z \end{bmatrix} = \begin{bmatrix} I_d & O \\ -L^{-1}NM^{-1} & L^{-1} \end{bmatrix} \begin{bmatrix} \mathrm{d}\eta \\ \mathrm{d}x \end{bmatrix}. \tag{5.3.9}$$

Since L^{-1} is nonsingular at $(\eta(0), x(0))$, the $(d+n)$ differential forms $\mathrm{d}w_1, \ldots, \mathrm{d}w_d, \mathrm{d}z_1, \ldots, \mathrm{d}z_n$ are linearly independent at $(\eta(0), x(0))$. This implies that (w, z) with $w = \eta$ is also a coordinate map on a neighborhood $U_o \times W_o \subset \mathbb{R}^d \times \mathbb{R}^n$ of $(\eta(0), x(0))$, and thus there exists a coordinate transformation $\Phi : U_o \times W_o \to \mathbb{R}^{d+n}$, $(\eta, x) \mapsto (w, z) = (\eta, z)$. In particular, due to the lower-triangularity of L^{-1}, $\mathrm{d}z_1$ is a linear combination of $\mathrm{d}\eta_1, \ldots, \mathrm{d}\eta_d$, and $\mathrm{d}x_1$ only. Therefore, there exists an output transformation such that $y_e = z_1 = q(\eta, x_1) = q(\eta, y)$. Similarly, there also exists a function such that $y = x_1 = \tilde{q}(w, z_1)$.

Finally, let us identify the vector field F of the extended system (5.2.10) in the (w, z)-coordinates. Let $F_z = \sum_{k=1}^{d} F_k \frac{\partial}{\partial w_k} + \sum_{j=1}^{n} F_{d+j} \frac{\partial}{\partial z_j}$ denote the representation of F in the (w, z)-coordinates. Trivially, $F_k = \dot{w}_k = p_k(\eta, y)$ for $1 \le k \le d$ because $w = \eta$. For $1 \le i \le n-1$, by the equation (5.3.6) and the definition $X_i := \frac{1}{\alpha_{i+1}}[X_{i+1}, F]$, it holds that

$$\begin{aligned} \frac{\partial}{\partial z_i} &= \frac{1}{\alpha_{i+1}} \Big[\frac{\partial}{\partial z_{i+1}}, F \Big] \\ &= \frac{1}{\alpha_{i+1}} \Big(\sum_{k=1}^{d} \frac{\partial F_k}{\partial z_{i+1}} \frac{\partial}{\partial w_k} + \sum_{j=1}^{n} \frac{\partial F_{d+j}}{\partial z_{i+1}} \frac{\partial}{\partial z_j} \Big). \end{aligned}$$

Hence, for $1 \le i \le n-1$ and $1 \le j \le n$, we have

$$\frac{\partial F_{d+j}}{\partial z_{i+1}} = \alpha_{i+1} \cdot \delta_{ij}.$$

When $1 \le j \le n-1$, the above equation implies that $\frac{\partial F_{d+j}}{\partial z_{j+1}} = \alpha_{j+1}(\eta, y) = \alpha_{j+1}(w, \tilde{q}(w, z_1))$ and F_{d+j} depends only on w, z_1, and z_{j+1}. Therefore, $F_{d+j} = \alpha_{j+1}(\eta, y) z_{j+1} + \tilde{a}_j(w, z_1)$ for $1 \le j \le n-1$. Similarly, $F_{d+n} = \tilde{a}_n(w, z_1)$. Let $a_j(\eta, y) := \tilde{a}_j(\eta, q(\eta, y)) = \tilde{a}_j(w, z_1)$ for $1 \le j \le n$. Then, one can observe that F_z is equal to the vector field of the system composed of the auxiliary dynamics (5.1.2) and the ENOCF (5.1.4). $\square$

Remark 5.3.1. As mentioned in Remark 5.1.2, the ENOCF problem is a natural extension of the RDOEL problem for single output systems, in the sense that they are identical when $\alpha_2(\eta, x_1) = \cdots = \alpha_n(\eta, x_1) = 1$. Therefore, the existence of $\phi(\eta, x_1)$, which satisfies (E1) and (E2) in Theorem 5.3.1 when $a_2(\eta, x_1) = \cdots = a_n(\eta, x_1) = 1$, is a necessary and sufficient condition of the RDOEL problem for single output systems. As shown in the proof of Theorem 5.3.1, if there exits such a function ϕ, then it holds that $\phi = (1/\frac{\partial q}{\partial x_1}) \prod_{k=2}^{n} \alpha_k = 1/\frac{\partial q}{\partial x_1}$. Therefore, one can observe that the statement of Theorem 5.3.1 with $\alpha_2(\eta, x_1) = \cdots = \alpha_n(\eta, x_1) = 1$ is exactly same to Theorem 3.5.1 which states a necessary and sufficient condition of the RDOEL problem for single output systems. □

Finally, we explain how to check the solvability of the ENOCF problem and to design an explicit coordinate transformation by using the results presented in this chapter. It is quite similar to the procedure described in Subsection 4.3.3. First of all, by Theorem 5.2.1 and Theorem 5.2.5, check the observability rank condition (5.2.2) of the given system (5.1.1) and the condition $f_n(x) \in \mathcal{P}_s^n(x)$ in its observable form (5.2.9). If they are satisfied, then we choose an auxiliary dynamics such as (5.1.2). After that, according to Theorem 5.3.1, set $X_n := \phi \frac{\partial}{\partial x_n}$ with $\phi \in \mathcal{P}_{se}^0(x)$ and calculate $X_i = \frac{1}{\alpha_{i+1}}[X_{i+1}, F]$ with $\alpha_{i+1} \in \mathcal{P}_{se}^0(x)$ successively from $i = n-1$ to $i = 1$. Since F is known, (E2) gives some partial differential equations of ϕ and α_{i+1}'s for $1 \leq i \leq n-1$. If there exists a set of solutions of the equations subject to the conditions given in (E1), then the ENOCF problem is solvable by Theorem (5.3.1). In addition, from the solutions, we can determine all the entries of L (defined by (5.3.5)) as functions of η and x. Since (E1) implies that L is nonsingular, L^{-1} exists. Finally, it follows from (5.3.9) that $\frac{\partial z}{\partial x} = L^{-1}$, and thus we can construct an explicit z-coordinates by solving the equation.

Remark 5.3.2. For a given system, if there exists an auxiliary dynamics with which the ENOCF problem is solvable, then it is theoretically possible to design such an auxiliary dynamics by the same manner explained in Remark 4.3.3, but in practice it is very hard due to the same reason. Although we have not yet developed an algorithm to design it, we present a basic principle of selecting it: it should be input-to-state stable (ISS) in the sense that η is bounded for every bounded y, in order to use the high-gain observer design method [BFH98]. □

5.4 Case Study: Rössler System into ENOCF

As an application of our theoretical results, we transform the Rössler system [Rös76] into the proposed ENOCF via a stable linear auxiliary dynamics, and then design an observer by using the high-gain observer design approach [BFH98].

The Rössler system is a chaotic oscillator whose dynamics is given by

$$\begin{aligned}\dot{\xi}_1 &= -(\xi_2 + \xi_3),\\ \dot{\xi}_2 &= \xi_1 + c_1\xi_2,\\ \dot{\xi}_3 &= c_2 + \xi_3(\xi_1 - c_3),\end{aligned} \tag{5.4.1}$$

where c_1, c_2, and c_3 are positive constant parameters (the original values selected in [Rös76] are 0.2, 0.2, and 5.7, respectively). Figures 5.1-5.3 illustrate varied behaviors over changing the parameters, sensitivity to initial states, and density of periodic orbits of the Rössler system, which are typical properties of chaotic dynamics. Owing to the chaotic properties, the Rössler system has been widely used in secure communication (e.g. see [LH99, NM97] and references therein).

For the Rössler system (5.4.1), we define the system output $y = h(\xi) := \xi_2$ where $\xi = [\xi_1\ \xi_2\ \xi_3]^T$. Then, it holds that

$$\begin{aligned}\mathrm{d}h(\xi) &= \mathrm{d}\xi_2,\\ \mathrm{d}\mathcal{L}_f h(\xi) &= \mathrm{d}\xi_1 + c_1\mathrm{d}\xi_2,\\ \mathrm{d}\mathcal{L}_f^2 h(\xi) &= c_1\mathrm{d}\xi_1 + (c_1^2 - 1)\mathrm{d}\xi_2 - \mathrm{d}\xi_3,\end{aligned}$$

where f denotes the vector field of the system (5.4.1). Therefore, the system (5.4.1) with the system output $y = \xi_2$ satisfies the observability rank condition, and thus it can be expressed as the following observable form:

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= x_3,\\ \dot{x}_3 &= f_3(x) := g_1(x_1) + g_2(x_1)x_2 - c_1x_2^2 + (g_3(x_1) + x_2)x_3,\\ y &= x_1,\end{aligned} \tag{5.4.2}$$

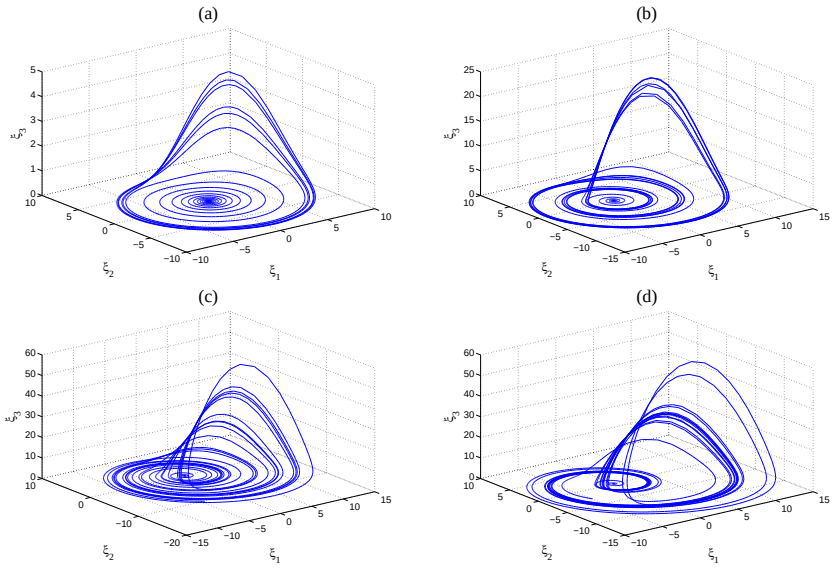

Figure 5.1: Variation in behaviors resulting from change of c_1: state trajectories starting from $(0, 0, 0)$ over $t \in [0, 150]$ of the Rössler system with (a) $c_1 = 0.1$, (b) $c_1 = 0.2$, (c) $c_1 = 0.3$, (d) $c_1 = 0.4$, respectively, $c_2 = 0.2$, and $c_3 = 5.7$

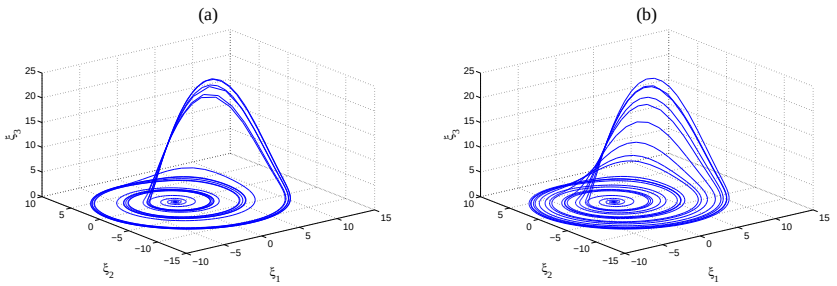

Figure 5.2: Sensitivity to initial states: state trajectories starting from (a) $(0, 0, 0)$ and (b) (-0.001,0,0) over $t \in [0, 150]$ of the Rössler system with $(c_1, c_2, c_3) = (0.2, 0.2, 5.7)$

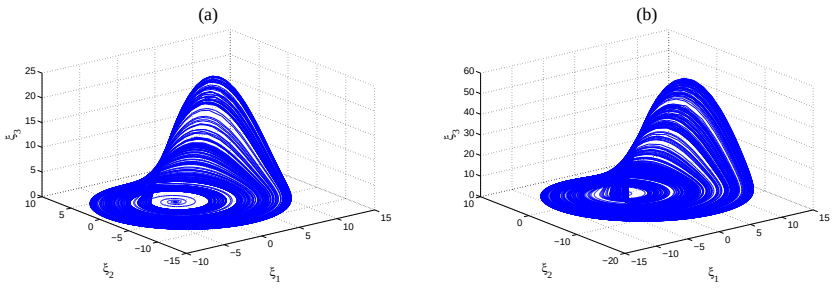

Figure 5.3: Density of periodic orbits: state trajectories starting from $(0, 0, 0)$ over $t \in [0, 1500]$ of the Rössler system with (a) $(c_1, c_2, c_3) = (0.2, 0.2, 5.7)$ and (b) $(c_1, c_2, c_3) = (0.3, 0.2, 5.7)$

where $x_i = \mathcal{L}_f^i h(\xi)$ for $i = 1, 2, 3$, $x = [x_1\ x_2\ x_3]^T$, and $g_i(x_1)$'s are defined by

$$\begin{aligned} g_1(x_1) &:= -c_1 x_1^2 - c_3 x_1 - c_2, \\ g_2(x_1) &:= (c_1^2 + 1)x_1 + (c_1 c_3 - 1), \\ g_3(x_1) &:= -c_1 x_1 + (c_1 - c_3). \end{aligned}$$

One can observe that $f_3(x) \in \mathcal{P}_s^3(x)$. To the system (5.4.2), we append the following auxiliary dynamics:

$$\dot{\eta} = -\eta + y, \tag{5.4.3}$$

which is a stable linear system and also is an input-to-state stable (ISS) system when we regard the system output y as the input of the auxiliary dynamics. Then, the vector field F of the extended system, which consists of the observable form (5.4.2) and the auxiliary dynamics (5.4.3), is represented as

$$F = (-\eta + x_1)\frac{\partial}{\partial \eta} + x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + f_3(x)\frac{\partial}{\partial x_3}.$$

By Theorem 5.3.1, we set

$$X_3 = \phi \frac{\partial}{\partial x_3},$$

where $\phi(\eta, x_1) \in \mathcal{P}_{se}^0(x)$. For simple calculation, we denote $\beta_i(\eta, x_1) := \frac{1}{\alpha_i(\eta, x_1)}$ for $i = 2, 3$. Then, we can obtain X_2 and X_1 from straightforward computation such that

$$\begin{aligned} X_2 &= \frac{1}{\alpha_3}[X_3,\ F] \\ &= \beta_3 \Big(\phi \Big(\frac{\partial}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\frac{\partial}{\partial x_3} \Big) - \Big((-\eta + x_1)\frac{\partial \phi}{\partial \eta} + x_2 \frac{\partial \phi}{\partial x_1} \Big) \frac{\partial}{\partial x_3} \Big) \\ &= \beta_3 \phi \frac{\partial}{\partial x_2} + \beta_3 \Big(g_3 \phi + (\eta - x_1)\frac{\partial \phi}{\partial \eta} + (\phi - \frac{\partial \phi}{\partial x_1})x_2 \Big) \frac{\partial}{\partial x_3} \\ &= \phi_2^2 \frac{\partial}{\partial x_2} + \phi_2^3 \frac{\partial}{\partial x_3}, \\ X_1 &= \frac{1}{\alpha_2}[X_2,\ F] \end{aligned}$$

$$
\begin{aligned}
&= \beta_2\Big(\phi_2^2\Big(\frac{\partial}{\partial x_1} + \frac{\partial f_3}{\partial x_2}\frac{\partial}{\partial x_3}\Big) + \phi_2^3\Big(\frac{\partial}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\frac{\partial}{\partial x_3}\Big) \\
&\quad + (\eta - x_1)\Big(\frac{\partial \phi_2^2}{\partial \eta}\frac{\partial}{\partial x_2} + \frac{\partial \phi_2^3}{\partial \eta}\frac{\partial}{\partial x_3}\Big) - x_2\Big(\frac{\partial \phi_2^2}{\partial x_1}\frac{\partial}{\partial x_2} + \frac{\partial \phi_2^3}{\partial x_1}\frac{\partial}{\partial x_3}\Big) - x_3\frac{\partial \phi_2^3}{\partial x_2}\frac{\partial}{\partial x_3}\Big) \\
&= \beta_2\phi_2^2\frac{\partial}{\partial x_1} + \beta_2\Big(\phi_2^3 + (\eta - x_1)\frac{\partial \phi_2^2}{\partial \eta} - \frac{\partial \phi_2^2}{\partial x_1}x_2\Big)\frac{\partial}{\partial x_2} \\
&\quad + \beta_2\Big(\phi_2^2(g_2 - 2c_1x_2 + x_3) + \phi_2^3(g_3 + x_2) + (\eta - x_1)\frac{\partial \phi_2^3}{\partial \eta} - \frac{\partial \phi_2^3}{\partial x_1}x_2 - \frac{\partial \phi_2^3}{\partial x_2}x_3\Big)\frac{\partial}{\partial x_3} \\
&= \phi_1^1\frac{\partial}{\partial x_1} + \phi_1^2\frac{\partial}{\partial x_2} + \phi_1^3\frac{\partial}{\partial x_3},
\end{aligned}
$$

where

$$
\begin{aligned}
&\phi_2^2 := \beta_3\phi, \\
&\phi_2^3 := \beta_3\Big(g_3\phi + (\eta - x_1)\frac{\partial \phi}{\partial \eta} + (\phi - \frac{\partial \phi}{\partial x_1})x_2\Big), \\
&\phi_1^1 := \beta_2\phi_2^2, \\
&\phi_1^2 := \beta_2\Big(\phi_2^3 + (\eta - x_1)\frac{\partial \phi_2^2}{\partial \eta} - \frac{\partial \phi_2^2}{\partial x_1}x_2\Big), \\
&\phi_1^3 := \beta_2\Big(\phi_2^2(g_2 - 2c_1x_2 + x_3) + \phi_2^3(g_3 + x_2) + (\eta - x_1)\frac{\partial \phi_2^3}{\partial \eta} - \frac{\partial \phi_2^3}{\partial x_1}x_2 - \frac{\partial \phi_2^3}{\partial x_2}x_3\Big).
\end{aligned}
$$

One can observe that $\phi_i^j \in \mathcal{P}_{se}^{j-i}(x)$ for $i = 1, 2$ and $i \le j \le 3$.

The objective is to find $\phi(\eta, x_1)$, $\alpha_2(\eta, x_1)$, and $\alpha_3(\eta, x_1)$ satisfying both (E1) and (E2) in Theorem 5.3.1. Then, we can construct a change of coordinates that transforms the extended system into ENOCF. By straightforward calculation, it holds that

$$
\begin{aligned}
&[X_3,\ X_2] = 0, \\
&[X_3,\ X_1] = \Big(\phi\frac{\partial \phi_1^3}{\partial x_3} - \phi_1^1\frac{\partial \phi}{\partial x_1}\Big)\frac{\partial}{\partial x_3} = \phi\Big(b_2\phi_2^2 - b_2\frac{\partial \phi_2^3}{\partial x_2} - b_3b_2\frac{\partial \phi}{\partial x_1}\Big)\partial x_3 = 0, \\
&[X_2,\ X_1] = \phi_2^2\Big(\frac{\partial \phi_1^2}{\partial x_2}\frac{\partial}{\partial x_2} + \frac{\partial \phi_1^3}{\partial x_2}\frac{\partial}{\partial x_3}\Big) + \phi_2^3\frac{\partial \phi_1^3}{\partial x_3}\frac{\partial}{\partial x_3} \\
&\qquad - \phi_1^1\Big(\frac{\partial \phi_2^2}{\partial x_1}\frac{\partial}{\partial x_2} + \frac{\partial \phi_2^3}{\partial x_1}\frac{\partial}{\partial x_3}\Big) - \phi_1^2\frac{\partial \phi_2^3}{\partial x_2}\frac{\partial}{\partial x_3} \\
&\qquad = \Big(\phi_2^2\frac{\partial \phi_1^2}{\partial x_2} - \phi_1^1\frac{\partial \phi_2^2}{\partial x_1}\Big)\frac{\partial}{\partial x_2} + \Big(\phi_2^2\frac{\partial \phi_1^3}{\partial x_2} + \phi_2^3\frac{\partial \phi_1^3}{\partial x_3} - \phi_1^1\frac{\partial \phi_2^3}{\partial x_1} - \phi_1^2\frac{\partial \phi_2^3}{\partial x_2}\Big)\frac{\partial}{\partial x_3}.
\end{aligned}
$$

From the last equation, $[X_2,\ X_1] = 0$ if and only if both the following partial differential equations hold:

$$\phi_2^2 \frac{\partial \phi_1^2}{\partial x_2} - \phi_1^1 \frac{\partial \phi_2^2}{\partial x_1} = 0, \tag{5.4.4}$$

$$\phi_2^2 \frac{\partial \phi_1^3}{\partial x_2} + \phi_2^3 \frac{\partial \phi_1^3}{\partial x_3} - \phi_1^1 \frac{\partial \phi_2^3}{\partial x_1} - \phi_1^2 \frac{\partial \phi_2^3}{\partial x_2} = 0. \tag{5.4.5}$$

If $\alpha_2(\eta, x_1) = \alpha_3(\eta, x_1) = 1$, then the first equation (5.4.4) is rewritten as

$$\phi_2^2 \frac{\partial \phi_1^2}{\partial x_2} - \phi_1^1 \frac{\partial \phi_2^2}{\partial x_1} = \phi\Big(\phi - 3\frac{\partial \phi}{\partial x_1}\Big) = 0.$$

Since it should be satisfied that $\phi \neq 0$ by (E1), we have

$$\frac{\partial \phi}{\partial x_1} = \frac{1}{3}\phi.$$

Then, the second equation (5.4.5) becomes

$$\begin{aligned}
&\phi_2^2 \frac{\partial \phi_1^3}{\partial x_2} + \phi_2^3 \frac{\partial \phi_1^3}{\partial x_3} - \phi_1^1 \frac{\partial \phi_2^3}{\partial x_1} - \phi_1^2 \frac{\partial \phi_2^3}{\partial x_2} \\
&= \phi\big(\frac{5}{3}\psi - \frac{\partial \psi}{\partial x_1} - 2c_1\phi + \frac{8}{9}\phi x_2\big) + \big(\psi + \frac{2}{3}\phi x_2\big)\frac{1}{3}\phi \\
&\quad - \phi\big(\frac{\partial \psi}{\partial x_1} + \frac{2}{9}\phi x_2\big) - \big(2\psi - g_3\phi + \frac{1}{3}\phi x_2\big)\frac{2}{3}\phi \\
&= \phi\bigg(\Big(\frac{2}{3}\psi - 2\frac{\psi}{\partial x_1} + (\frac{2}{3}g_3 - 2c_1)\phi\Big) + \frac{2}{3}\phi x_2\bigg) = 0,
\end{aligned}$$

where

$$\psi := g_3\phi + (\eta - x_1)\frac{\partial \phi}{\partial \eta} \in \mathcal{P}_{se}^0(x).$$

Hence, the equation (5.4.5) holds if and only if

$$\phi\Big(\frac{2}{3}\psi - 2\frac{\psi}{\partial x_1} + (\frac{2}{3}g_3 - 2c_1)\phi\Big) = 0 \quad \text{and} \quad \frac{2}{3}(\phi)^2 = 0.$$

However, the latter is a contradiction to (E1). Therefore, there does not exist any $\phi \in \mathcal{P}_{se}^0(x)$ such that $\phi \neq 0$ and the partial differential equations (5.4.4) and (5.4.5) are satisfied when $\alpha_2(\eta, x_1) = \alpha_3(\eta, x_1) = 1$. This implies that the RDOEL

problem is not solvable for the system (5.4.2) via the auxiliary dynamics (5.4.3). However, we can find a set of non-vanishing solutions of the partial differential equations (5.4.4) and (5.4.5) such that

$$\phi(\eta, x_1) = 1, \qquad \alpha_2(\eta, x_1) = 1, \qquad \alpha_3(\eta, x_1) = e^{\frac{(\eta - x_1)}{2}}. \tag{5.4.6}$$

From the above solutions, we can determine all the entries of the matrix L (defined by (5.3.5)) as functions of η and x, and it follows from (5.3.9) that

$$\begin{aligned}\frac{\partial z}{\partial x} &= L^{-1} \\ &= \begin{bmatrix} e^{\frac{\eta - x_1}{2}} & 0 & 0 \\ \big(c_3 - c_1 + \frac{\eta}{2} + (c_1 - \frac{1}{2})x_1 - \frac{x_2}{2}\big)e^{\frac{\eta - x_1}{2}} & e^{\frac{\eta - x_1}{2}} & 0 \\ 1 - c_1c_3 - (c_1^2 + 1)x_1 + c_1x_2 & c_3 - c_1 + c_1x_1 - x_2 & 1 \end{bmatrix}.\end{aligned}$$

By solving the above equation, we can construct z-coordinates and an output transformation such that

$$\begin{aligned}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} &= \begin{bmatrix} -2e^{\frac{\eta - x_1}{2}} \\ \big(2(1 - c_1 - c_3) - \eta + (1 - 2c_1)x_1 + x_2\big)e^{\frac{\eta - x_1}{2}} \\ (1 - c_1c_3)x_1 - \frac{1 + c_1^2}{2}x_1^2 + (c_3 - c_1 + c_1x_1)x_2 - \frac{x_2^2}{2} + x_3 \end{bmatrix} =: \Phi_z(\eta, x), \\ y_e &= q(\eta, y) := -2e^{\frac{\eta - y}{2}} = z_1.\end{aligned}$$

As a result, on the (η, z)-coordinates, the extended system is represented as

$$\begin{aligned}\dot{\eta} &= -\eta + y, \\ \dot{z} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & e^{\frac{\eta - y}{2}} \\ 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} a_1(\eta, y) \\ a_2(\eta, y) \\ a_3(\eta, y) \end{bmatrix} = A(\eta, y)z + a(\eta, z), \\ y_e &= q(\eta, y) = [\ 1\ 0\ 0\ ]\ z = Cz,\end{aligned} \tag{5.4.7}$$

where

$$a_1(\eta, y) = 2\Big(c_1 + c_3 - 1 + \eta + (c_3 - 1)y\Big)e^{\frac{\eta - y}{2}},$$

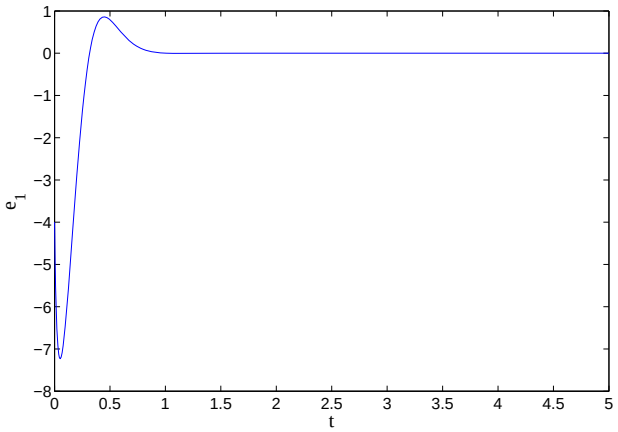

Figure 5.4: Simulation result: observer error $e_1(t) := \hat{\xi}_1(t) - \xi_1(t)$

$$
\begin{aligned}
a_2(\eta, y) = \Big(& (c_1 + c_3 + \frac{\eta}{2})\eta - (1 + c_1 + c_3 - c_1 c_3) y \\
& + (1 - c_1 + \frac{c_1^2}{2}) y^2 - (1 - c_1)\eta y \Big) e^{\frac{\eta - y}{2}}, \\
a_3(\eta, y) = & -c_2 - c_3 y - c_1 y^2.
\end{aligned}
$$

Let $[\eta^T \ z^T]^T = \Phi(\eta, \xi) := [\eta^T \ \Phi_z(\eta, \Phi_x(\xi))^T]^T$, where $\Phi_x(\xi) := [h(\xi) \ \ \mathcal{L}_f h(\xi) \ \ \mathcal{L}_f^2 h(\xi)]^T$ that is the transformation from the Rössler system (5.4.1) into its observable form (5.4.2). Then, $\Phi(\eta, \xi)$ transforms the extended system, composed of the Rössler system (5.4.1) and the auxiliary dynamics (5.4.3), into the system (5.4.7). In addition, since the Rössler system is an oscillator and the auxiliary dynamics is an ISS system, (η, y) is bounded. Therefore, by using the high-gain observer design method [BFH98], we can design an observer such as (5.1.5) in Section 5.1. Actually, in the observer (5.1.5), it is not easy to obtain the inverse coordinate transformation Φ^{-1}. However, by using the Jacobian of Φ (=: J_Φ), we can design a dynamic system, which is equivalent to the observer, such that

$$
\begin{bmatrix} \dot{\eta} \\ \dot{\hat{\xi}} \end{bmatrix} = (J_\Phi)^{-1} \begin{bmatrix} p(\eta, y) \\ A(\eta, y)\hat{z} + a(\eta, y) - \Lambda^{-1}(\eta, y) S_\theta^{-1} C^T (C\hat{z} - y_e) \end{bmatrix},
$$

$$
y_e = q(\eta, y), \qquad \hat{z} = (\Phi_z \circ \Phi_x)(\eta, \hat{\xi}).
$$

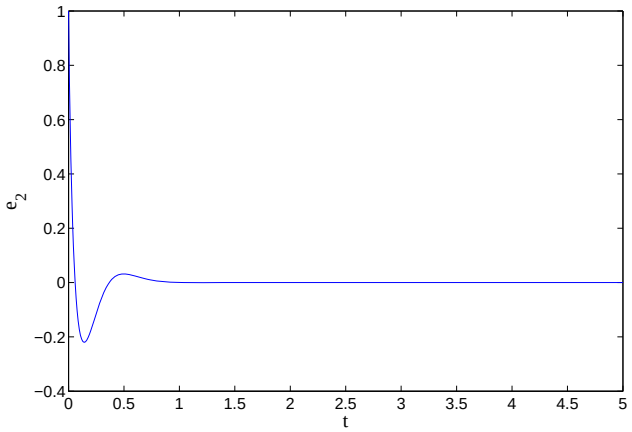

Figure 5.5: Simulation result: observer error $e_2(t) := \hat{\xi}_2(t) - \xi_2(t)$

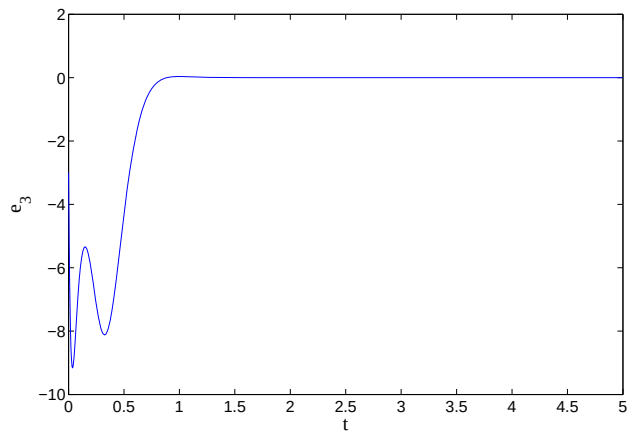

Figure 5.6: Simulation result: observer error $e_3(t) := \hat{\xi}_3(t) - \xi_3(t)$

In order to verify the performance of the observer, we carry out a simulation using MATLAB, in the case when we set $(c_1, c_2, c_3) = (0.2, 0.2, 5.7)$, $\theta = 10$, $\xi(0) = (4, -1, 3)$, $\eta(0) = 0$, and $\hat{\xi}(0) = (0, 0, 0)$. Figures 5.4-5.6 show that the observer errors $e_1 := \hat{\xi}_1 - \xi_1$, $e_2 := \hat{\xi}_2 - \xi_2$, and $e_3 := \hat{\xi}_3 - \xi_3$ converge to zero.

Chapter 6

Conclusions

This chapter summarizes the results of this dissertation that have been addressed so far, and presents some future directions for the research related to this work. In the dissertation, we have dealt with two kinds of problems of designing observers for nonlinear systems as listed below.

- **The RDOEL problem for multi-output nonlinear systems**
  We have introduced the framework of reduced-order dynamic observer error linearization (RDOEL) for multi-output nonlinear systems. The proposed RDOEL problem is a modified version of the dynamic observer error linearization (DOEL) problem, in the sense that it shares the same idea (of introducing an auxiliary dynamics and a generalized output injection term in a generalized nonlinear observer canonical form (GNOCF)) with the DOEL problem. Although RDOEL is a special case of DOEL, RDOEL has an advantage over DOEL such that it offers a lower-dimensional observer compared with DOEL. Furthermore, the RDOEL problem is a natural extension of the (conventional) observer error linearization (OEL) problem, because RDOEL with no auxiliary dynamics is identical to OEL. For the RDOEL problem, we have given three necessary conditions. Two of them can partially identify the class of systems to which the problem is solvable, and the other one presents a condition on output transformation in order for the problem to be solved. Based on the necessary conditions, we have found a geometric necessary and sufficient condition for the RDOEL problem with

the general auxiliary dynamics ($\dot{\eta} = p(\eta, y)$) and the general output transformation ($y_e = q(\eta, y)$). Furthermore, from the result, we also have derived a necessary and sufficient condition for the OEL problem, which is, for our best knowledge, the first geometric necessary and sufficient condition for the OEL problem in the case under consideration of the general output transformation ($y_e = q(y)$). At last, by using the results, we have developed a procedure to check the solvability and to design explicit coordinate and output transformations for OEL and RDOEL.

- **The ENOCF problem for single output nonlinear systems**
  The dissertation has introduced an extended nonlinear observer canonical form (ENOCF) of which linear part also depends on the system output and the state of auxiliary dynamics, and we have dealt with the problem (called the ENOCF problem) of transforming a single output nonlinear system into the ENOCF via an auxiliary dynamics, as an extension of the RDOEL problem. We also provide two necessary conditions, and a geometric necessary and sufficient condition for the ENOCF problem. And the results is applied to the Rössler system to illustrate that the ENOCF problem can be solved for a class of systems which are not covered by the RDOEL framework.

Some further issues for future research related to the topics of this dissertation are listed as follows.

- The ENOCF problem can be extended to multi-output systems, like we have extended the concept of RDOEL to multi-output systems in the dissertation.

- In order to solve the OEL, RDOEL, and ENOCF problems completely, we have to find an explicit coordinate transformation for them. Although the procedure in Subsection 4.3.3 explains how to do that, it is not a complete algorithm yet. So, it may be a good topic of future research to investigate a complete algorithm to design a coordinate transformation for the OEL, RDOEL, and ENOCF problems by a straightforward manner.

- As similar as the above topic, although our results have been made under consideration of auxiliary dynamics of general form, we have no idea yet

how we can design it for a given system. Therefore, it would be also a further topic to construct an auxiliary dynamics in order for the problems to be solvable for the given system.

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국문초록

Nonlinear Observer Design via Reduced-Order Dynamic Observer Error Linearization and Extended Nonlinear Observer Canonical Form

축소 차원 동적 관측기 오차 선형화와 확장된 비선형 관측기 정준형을 통한 비선형 관측기 설계

본 논문은 비선형 시스템에 대한 관측기 설계 문제를 다루고 있다. 관측기 설계 문제란 주어진 시스템의 입력과 출력 정보만을 활용하여 대상 시스템의 상태 변수를 추정할 수 있는 시스템을 설계하는 것이다. 선형 시스템의 경우에는 루엔버거 관측기(Luenberger observer)로 알려진 일반적인 해법이 존재하는 반면, 일반적인 비선형 시스템에 대해 관측기를 설계하는 방법에 대한 연구 결과는 현재까지 보고된 바가 없다. 다만, 특정한 형태의 비선형 시스템에 대해 관측기를 설계하는 문제에 대한 연구는 활발하게 진행되어 오고 있다. 관측기 오차 선형화(observer error linearization) 기법은 이 문제에 대한 가장 잘 알려진 방법론 중의 하나로서, 주어진 비선형 시스템을 좌표 변환을 통해 관측 가능한 선형 시스템과 출력 주입(output injection) 부분들로 구성된 비선형 관측기 정준형(nonlinear observer canonical form)으로 변환시키는 문제이다. 비선형 관측기 정준형으로 변환 가능한 좌표계에서는 시스템의 모든 비선형성이 시스템의 입력과 출력의 함수로 이루어진 출력 주입 부분에 제한되므로, 이를 상쇄시킴으로써 선형 시스템의 경우와 비슷한 형태의 루엔버거형의 관측기(Luenberger-type observer)를 설계하는 것이 가능하고, 이에 따라 선형화된 관측기 오차 동역학(observer error dynamics)을 얻을 수 있다. 관측기 오차 선형화 기법의 출현 이래로, 이를 적용할 수 있는 시스템의 범위를 확장시키기 위한 여러 연구가 진행되어 왔다. 그 중 하나는 주어진 시스템을 보다 높은 차수의 비선형 관측기 정준형으로 변환시키는 방법이다. 이러한 방식에는 시스템 이머젼 기법과 동적 관측기 오차 선형화(dynamic observer error linearization) 기법이 있는데, 그 중에서도 동적 관측기 오차 선형화 기법의 특징은 다음과 같이 크게 두 가지로 요약될 수 있다. 첫째는 대상 시스템의 출력을 입력으로 하는 보조 동역학(auxiliary dynamics)을 설계하는 것이고, 둘째는 보조 동역학을 포함하는 확장된 시스템을 대상 시스템보다 높은 차수의 일반화된 비선형 관측기 정준형(generalized nonlinear observer canonical form)으로 변환하는

것이다. 동적 관측기 오차 선형화 기법에서 제안된 일반화된 비선형 관측기 정준형은 관측 가능한 선형 시스템과 일반화된 출력 주입(generalized output injection)으로 구성되어 있고, 일반화된 출력 주입은 대상 시스템의 출력 뿐만 아니라 보조 동역학의 상태 변수에 대한 함수로 이루어져 있다는 차이점이 있다. 하지만, 이 방법론은 관측기의 차수가 대상 시스템의 차수보다 크다는 단점을 가지고 있다. 이러한 문제를 해결하기 위해, 최근에는 동적 관측기 오차 선형화의 변형된 기법으로서 축소 차원 동적 관측기 오차 선형화(reduced-order dynamic observer error linearization)란 기법이 단일 출력 시스템에 대해 새롭게 제안되었다. 축소 차원 동적 관측기 오차 선형화 기법 역시 보조 동역학을 설계하여 확장된 시스템을 일반화된 비선형 관측기 정준형으로 변환시킨다는 점에서 동적 관측기 오차 선형화 기법과 공통점을 갖지만, 변환된 일반화된 비선형 관측기 정준형의 차수가 대상 시스템의 차수와 같다는 차이점이 있다. 비록 축소 차원 동적 관측기 오차 선형화 기법이 적용 가능한 시스템의 범주는 동적 관측기 선형화 기법이 적용 가능한 시스템 범주를 벗어날 수는 없지만, 축소 차원 동적 관측기 오차 선형화 기법은 동적 관측기 선형화 기법에 비해 더 작은 차수의 관측기를 설계할 수 있다는 이점이 있고, 보조 동역학의 개념을 도입함으로써 관측기 오차 선형화 기법에 비해 더 넓은 범주의 시스템에 적용 가능하다는 장점을 지닌다. 뿐만 아니라, 축소 차원 동적 관측기 오차 선형화 기법의 개념 자체가 관측기 오차 선형화 기법의 개념과 매우 흡사하기 때문에 (보조 동역학을 고려하지 않은 축소 차원 동적 관측기 오차 선형화 문제는 관측기 오차 선형화 문제와 일치한다.) 축소 차원 동적 관측기 오차 선형화 기법에 대한 연구를 통해 기존의 관측기 오차 선형화 기법을 해석할 수도 있다.

이에 따라, 본 논문에서는 축소 차원 동적 관측기 오차 선형화 기법을 다중 출력 시스템에 대해 확장시키고, 이에 대한 연구를 수행하여 궁극적으로는 주어진 다중 출력 시스템이 이 기법에 의해 일반화된 비선형 관측기 정준형으로 변환될 수 있는 필요충분 조건을 제시한다. 이 결과는 현재까지 확립되지 않았던 일반적인 형태의 출력 변환까지 고려하였을 경우의 다중 출력 시스템에 대한 관측기 오차 선형화 문제의 필요충분 조건을 내포하고 있다.

또한, 본 논문에서는 비선형 관측기 정준형의 선형 부분 또한 시스템의 출력과 보조 동역학의 상태 변수에 의해 결정되는 확장된 비선형 관측기 정준형(extended nonlinear observer canonical form)을 제안하고, 축소 차원 동적 관측기 오차 선형화의 확장된 기법으로서 주어진 단일 출력 시스템을 보조 동역학을 설계하여

확장된 비선형 관측기 정준형으로 변환하는 문제를 제안하고 이에 대한 필요충분 조건을 제시한다. 또한 이 결과를 뢰슬러 시스템(Rössler system)에 적용시켜봄으로써 새롭게 제안된 방법론이 축소 차원 동적 관측기 오차 선형화에 비해 더 넓은 범주의 시스템에 적용될 수 있음을 예증한다.

주요어: 비선형 관측기 설계, 비선형 관측기 정준형, 관측기 오차 선형화, 시스템 이머전, 동적 관측기 오차 선형화, 축소 차원 동적 관측기 오차 선형화
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