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A Study on the Process of Constructing Mathematics Teachers’ Practical Knowledge

수학교사의 실천적 지식 구성 과정에 대한 연구

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A Study on the Process of Constructing Mathematics Teachers' Practical Knowledge

A dissertation submitted in partial fulfillment
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by

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Abstract

A Study on the Process of Constructing Mathematics Teachers' Practical Knowledge

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The importance of teachers' practical knowledge has gradually been emphasized in the area of mathematics teacher education, but there is not enough discussion about the meaning of practical knowledge and methods for the construction of teachers' practical knowledge. The present study was conducted to address this issue related to teachers' practical knowledge in the area of mathematics teacher education. This study focused on an analysis of tacit knowledge, which is invariably associated with teaching practices, and it closely examined the meaning of tacit knowledge through a review of the relevant literature. Based on this analysis, the study embodied the meaning of mathematics teachers' practical knowledge and also suggested a teacher training method for the construction of mathematics teachers' practical knowledge, reflecting the meaning of tacit knowledge, and applied the teacher training procedure to an in-service teacher.

According to the research results, it is possible for mathematics teachers to construct practical knowledge through teacher training with the procedural process of consciousness (turning subsidiary awareness into focal awareness) and unconsciousness (turning focal awareness into subsidiary awareness), as this study suggested. In other words, teachers conduct their teaching activities consciously with will to change their practices by being conscious of subsidiary awareness in the tacit dimension and reflecting on them, and their conscious activity becomes habitual through repetitive practice. Thus, mathematics teachers' practical knowledge can be constructed through conscious and unconscious processes. In this study, it was found that the teacher attempted to change her teaching practices through conscious methods, and she reflected on her practices, noting that they were changing in the actual class. However, it takes a long time for changed teaching practices to become habitual in the tacit dimension; thus, future monitoring is necessary to confirm the persistence of the teacher's changed teaching practices.

It is difficult to address practical knowledge because this concept is multifaceted and vague. Practical knowledge, however, is an essential part of mathematics teachers' professionalism. Therefore, an analysis of practical knowledge from different points of view and ways to construct practical knowledge needs to be actively discussed.

Keywords: practical knowledge, tacit knowledge, Polanyi, awareness, reflection, consciousness

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CHAPTER Ⅰ

INTRODUCTION

1. Background on the study

What knowledge is needed for teachers to teach subject matter knowledge to students and how teachers' knowledge of teaching is constructed are important issues in the field of teacher education. In fact, many studies of teacher knowledge have been conducted in the past few decades in mathematics teacher education, and the studies conducted by Ball and her colleagues have had a profound impact on pre-service teacher training programs as well as professional development programs. Ball and her colleagues investigated teacher knowledge for teaching mathematics and conceptualized the knowledge as Mathematical Knowledge for Teaching (MKT), by considering the concept of Pedagogical Content Knowledge (PCK) suggested by Shulman (1986) and Subject Matter Knowledge (SMK)[1] (Ball, Thames & Phelps, 2008; Hill, Ball & Schilling, 2008). MKT seems meaningful in that it gives perspectives on mathematics teachers'

[1]SMK and PCK is a subset of MKT. SMK contains Common Content Knowledge (CCK), Specialized Content Knowledge (SCK) and knowledge at the mathematical horizon. PCK contains Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT) and knowledge of curriculum (Hill, Ball & Schilling, 2008, p. 377)

knowledge concerning both theory and practice and helps pave the way to seek for teachers' knowledge for teaching mathematics.

However, mathematics teachers' knowledge such as MKT may not be applied to all different contexts and situations, so there will be a limit if teachers' actions and events that occur during their class are interpreted and understood with only such formal knowledge as MKT. In fact, Tompson (1992) analyzed the relationship between teacher knowledge and teaching practice and reported that teachers' beliefs about mathematics, no less than teachers' knowledge, affected the direction of their mathematics teaching and practice. The recently conducted study by Schoenfeld (2010) also shows similar finding with Tompson's. Schoenfeld found mathematics teachers' orientations, resources, and goals complexly influenced their moment-to-moment decision making, which determined their actions during their classes. Those findings indicate that mathematics teacher knowledge needs to be understood by paying attention to teachers' personal elements such as their beliefs, intentions, and values in teaching and learning mathematics.

Since the mid-1980s, researchers in teacher education have considered teachers not as passive people who simply apply theoretical knowledge to their teaching practice, but as active people who can transform and reconstruct theoretical knowledge to adjust to their teaching situations and contexts according to their beliefs and judgments (Berliner, 1989). Since then, the perspective that teachers have individual idiosyncratic knowledge obtained through their personal experiences, which is different than theoretical knowledge, was supported, and some researchers started investigating teachers' teaching practices with a phenomenological approach. This is what teachers' practical knowledge is about. Elbaz (1983) observed and analyzed one teacher's school work and found that there was idiosyncratic knowledge that guided her teaching practice. Elbaz (1983) conceptualized as practical knowledge that teachers reconstruct and synthesize the knowledge they obtained based on their values and beliefs to fit their teaching situation. Conelly and Clandinin (1988) agreed with Elbaz that teachers hold and use a special kind of knowledge to actively guide their teaching practice. Clandinin (1985) explored teachers' practical knowledge focusing on teachers' images of their classroom and called teachers' special knowledge personal practical knowledge by emphasizing personal aspects of that knowledge. Conelly and Clandinin (1988) defined teachers' personal practical knowledge as experiential knowledge derived from teachers' lives, concreted and reconstructed knowledge and complex beliefs that teachers themselves constructed by influencing their conscious and unconscious personal experiences. They express the knowledge through their actions.

Many studies followed Elbaz (1983) and Conelly and Clandinin (1988) and investigated teacher knowledge grounded in teachers' teaching practices. Researchers used different terms for teachers' special knowledge, such as ‘practical knowledge’ ‘personal knowledge’ ‘craft knowledge’ and so on, and there was no general agreement on a definition of teachers’ practical knowledge. Most researchers, however, seemed to agree that teachers’ practical knowledge is formed through teachers’ various experiences and is actively used in their teaching practices.

As teachers’ roles and teacher-students interactions are considered important in classrooms, teachers’ practical knowledge that guides their teaching action in class has been highlighted (Ponte, 1994; Ponte & Chapman, 2006). Indeed, studies on teachers’ practical knowledge have been conducted overseas since the 1980s, both theoretical studies to conceptualize the term and empirical studies to identify the knowledge using real teaching situations. As mentioned above, it has been considered that practical knowledge is formed through teachers’ personal experiences and actively used in their teaching practices, but in order to understand the meaning and construction of teachers’ practical knowledge deeply and systematically, the nature of practical knowledge needs to be thoroughly examined. However, it is not easy to identify and understand practical knowledge, because teachers’ practices are based on their coherent beliefs and assumptions about teaching and learning (Kwak et al., 2009) and practical knowledge can be identified not explicitly, but implicitly in teachers’ judgments and actions in their classroom. In fact, some researchers studying teacher knowledge have pointed out that it is difficult to examine teachers’ practical knowledge or change teachers' practices due to the tacit aspects of practical knowledge. Meanwhile, other researchers argue that tacit aspects behind teachers' actions, such as teachers' beliefs, intuitions and insights need to be explained explicitly and justified (Fenstermacher, 1994).

Most teachers have some patterns and routines formed through their teaching experiences and use those patterns and routines unconsciously in their teaching practices. If teachers' knowledge and the beliefs that underlie their teaching remain tacit, teachers cannot reflect on their tacit knowledge, whereas if teachers make efforts to explain and justify their teaching actions, tacit knowledge related to their practice can be recognized consciously and teachers can reflect on their teaching practices more meaningfully. Although tacit knowledge needs to be addressed in teacher education research, most previous studies on teachers' practical knowledge seems remain in discussion of the concept of practical knowledge at a superficial level or an investigation of teachers' practical knowledge as it emerges in class. This trend of research on teachers' practical knowledge implies that the nature of practical knowledge needs to be examined thoroughly and the way in which teachers can be consciously aware of tacit knowledge and the influence of tacit knowledge on teachers' practice and their student learning also needs to be discussed.

Recognizing the importance of teachers' practical knowledge, research on practical knowledge in Korea has been conducted since the early 2000s and has been steadily acknowledged in teacher education. Previous studies on practical knowledge in Korea can be classified into two categories according to their aims: studies aimed at theoretically discussing the meaning and characteristics of practical knowledge and studies aimed at analyzing how teachers form and use their practical knowledge in real classrooms.

Most studies included in the first category stayed at the level of understanding the concept of teachers' practical knowledge that was already defined in previous studies and identified practical knowledge that emerged in actual classroom in the Korean context. Hong (2006) and Gim and Choi (2013), for example, tried to understand teachers' classrooms with a conception of practical knowledge and claimed that the influence of not only teachers' explicit knowledge but also teachers' values, beliefs, feelings and intuitions about their classes should be discussed in depth. It is worth noting that they pointed out the importance of teachers' tacit knowledge existing in teaching phenomena, but they did not provide enough evidence of tacit knowledge in their analyses of the classrooms.

Studies conducted by So and Kim (2010) and Kim (2010) are included in the second category. They found that Korean teachers used their practical knowledge actively in the process of planning and conducting a class and major factors affecting their practical knowledge were teachers' teaching actions and reflections on their actions.

Kim (2010) mentioned that practical knowledge acquired only through teachers' teaching experiences can be inappropriately formed due to teachers' intellectual limitations and errors of judgment, and so teachers should be alert to inappropriate practical knowledge that might be used in their practice. Examples of inappropriate practical knowledge can be found in various classroom situations. For example, many studies pointed out that most students could have difficulties in understanding the relational meaning of the equal sign in the transition from arithmetic to algebra. Since most students only have experiences of the operational meaning of the equal sign, they can have trouble with the relational meaning of equal sign in algebra. However, teachers who are not aware of students' difficulties or do not fully understand the different meanings of the equal sign may misunderstand the students' work or teach through simple repeated learning, rather than providing learning opportunities for students to experience different meanings of the equal sign by giving various contexts. Teachers who have those experiences may construct different practical knowledge in terms of quality with teachers who take pedagogically sensitive and proper action in response to student responses.

Similarly, Kwak et al. (2009) also pointed out that there could be a gap between what actually happens in a classroom and what teachers perceive or understand about what happens because teachers' beliefs and biases may be wrong. The comments of Kim (2010) and Kwak et al. (2009) about practical knowledge indicate that teachers' practical knowledge should not be formed based only on their teaching experiences and teachers need to have critical views of their tacit knowledge inherent in their teaching practice.

The Korean studies on practical knowledge discussed so far were mostly conducted in elementary education and general education fields and few studies on secondary mathematics teachers' practical knowledge have been conducted. There is a deficit of meaningful data on what constitute the characteristics of practical knowledge possessed by Korean mathematics teachers and how their practical knowledge is formed and developed. Furthermore, in mathematics teacher education, there is a lack of discussion about how mathematics teachers can be supported to construct appropriate practical knowledge by making them conscious of tacit aspects influencing their practice and having a critical view.

Meanwhile, there were few studies using the term "practical knowledge" in international mathematics education, but studies that have explored and tried to understand mathematics teachers' teaching activities in classrooms have been continuously conducted. Putnam et al. (1992) and Mellon (2011) reported that mathematics teachers' teaching activities were made by intertwining teachers' knowledge about mathematics content, students and teaching, and learning methods with teachers' beliefs in mathematics and mathematics knowledge construction.

Studies that analyzed teachers' actions and decision making in order to understand teachers' teaching practices have also been conducted. Teachers encounter many decision-making situations during class; especially when confronted with an unexpected moment, they should judge the situation quickly and make a decision immediately. It shows that teachers' practical knowledge is actively used in the moments requiring teachers to make decisions.

Bishop (1976; 1982) and Schoenfeld (2010) analyzed teachers' actions and the process of teachers' decision making during class and found that there were tacit factors influencing teachers' observable actions. Schoenfelds' decision-making model shows that teachers' actions are not unconscious and unintended, but intended based on teachers' knowledge, orientation, and goals.

The studies conducted overseas have paid attention to the tacit dimension behind teachers' teaching actions and analyzed teachers' actions and their class with the consideration for tacit dimensions. Studies on teachers' actions and decision making are meaningful in that the studies showed that teachers' practical knowledge was actively used in the process of decision making and tacit knowledge was reflected in teachers' teaching practices.

To sum up, teacher knowledge should not be considered separately from practice and needs to be understood in the context of teachers' personal aspects. The most important thing about practical knowledge is that there must be tacit knowledge significantly affecting teachers' practices and teachers' practical knowledge cannot be constructed with only theoretical knowledge or only teachers’ teaching experiences. The importance of tacit knowledge has been mentioned, but the meaning of tacit knowledge and a way to guide teachers to be conscious of their tacit knowledge have not yet been thoroughly discussed. As can be seen in the previous study, since tacit knowledge is certainly reflected in teachers’ practices, they need to be educated to construct appropriate practical knowledge, yet systematic approaches to deal with tacit knowledge have not been properly discussed in teacher education. To propel the discussion of tacit knowledge in teacher education further, the meaning of tacit knowledge needs to be closely examined.

The focus of this study is mathematics teachers’ practical knowledge, more specifically the tacit knowledge that is entailed in mathematics teachers’ practices. Thus, this study reconsiders the concept of mathematics teachers’ practical knowledge by complementing the meaning of tacit knowledge and investigates the process of constructing mathematics teachers’ practical knowledge.

2. Research questions

This study aims to examine the meaning of tacit knowledge through a review of relevant literature and suggests a teacher training procedure for the construction of mathematics teachers' practical knowledge that incorporates the meaning of tacit knowledge. The study also analyzes the process of mathematics teachers' practical knowledge construction by applying the teacher training procedure to an in-service middle school teacher. The research questions of this study are as follows:

1. What is the meaning of tacit knowledge, and what is teacher training procedure for construction of the mathematics teachers' practical knowledge reflecting tacit knowledge?

2. What is the process of constructing the mathematics teacher's practical knowledge?

2-1 What does the teacher notice in the thought experiment phase?

2-2 How are the noticed things in the thought experiment reflected in the teacher's actual class?

2-3 What changes in teacher's noticing occur in the process of reflection on teaching?

3. Outline of the study

This study includes a theoretical part and an empirical part. In order to answer the first research question, Chapter 2 and 3 conduct a theoretical analysis. In Chapter 2, the meaning of tacit knowledge is investigated through a review of relevant literature and tacit knowledge is also understood in the context of teaching practice. In Chapter 3, the components for the construction of mathematics teachers' practical knowledge are drawn from the discussion about tacit knowledge and the review of previous studies on the concept of teachers' practical knowledge. On the basis of these components, I suggest specific teacher training procedures to construct mathematics teachers' practical knowledge.

After the theoretical chapters, the procedure for constructing mathematics teachers' practical knowledge is applied to one in-service mathematics teacher. Chapter 4 describes the participant and data collection and analysis, and then reports and discusses the results of the analysis. The subsequent research questions are answered throughout Chapter 4, which is empirically grounded. Finally, in Chapter 5, the findings of this study are summarized and discussed according to the research questions, and the conclusions and suggestions for further research are presented.

CHAPTER Ⅱ

TACIT KNOWLEDGE

The concept of tacit knowledge has been actively discussed among philosophers, epistemologists, social scientists and even researchers studying teacher knowledge (Toom, 2012). Actually Polanyi’s concept of tacit knowledge has been used in fields related to professional training, such as business management and medicine. Especially there is active discussion about tacit knowledge in the field of knowledge management, with the discussion focused on the transformation of knowledge. Nonaka and Takeuchi (1995) suggested the knowledge conversion model from tacit knowledge to explicit knowledge, which was based on Polanyi’s tacit knowledge concept. Their model has had great influence over a number of related studies, but it has also caused a dispute because the definition of tacit knowledge that they supposed is not so clear (Gourlay, 2006; Bratianu, 2010). This controversy directly shows that tacit knowledge is a vague and multifaceted concept. This chapter closely examines the meaning of Polanyi’s tacit knowledge, and focuses on understanding tacit knowledge that has been discussed in terms of teacher knowledge.

1. The meaning of tacit knowledge

Tacit knowledge is the most important concept in Polanyi's epistemology, which started from the criticism of objectivism for excluding the function of the subject from the process of the inquiry of knowledge and the nature of knowledge. Polanyi asserts that the subject's beliefs and personal judgments are involved in the process of intellectual inquiry, and judgments are always made by a subject based on their knowledge, which consists of tacit components (Polanyi, 1962; 1967; Polanyi & Prosch, 1975).

According to Polanyi's epistemology, knowledge cannot be constructed by applying the formal rules and ways mechanically. The process of knowing requires skills of knowing and a subject's beliefs, judgments and intuitions are in action when the subject uses the skills (Eom, 1998). The following statement on knowing and knowing skills shows Polanyi's assertion, which claims that all knowledge involve an element beyond the verbal dimension, i.e. the tacit dimension.

> 'I regard knowing as an active comprehension of the things known, and action that requires skill. Skillful knowing and doing is performed by subordinating a set of particulars, as clues or tools, to the shaping of a skillful achievement, whether practical or theoretical, we may then be said to become 'subsidiarily aware' of these particulars within our 'focal awareness' of the coherent entity that we

> achieve. Clues and tools are things used as such and not observed in themselves' (Polanyi, 1962, vii-viii).

To understand the tacit dimension of knowledge, we should firstly look at two kinds of awareness which were labeled by Polanyi as subsidiary awareness and focal awareness. Polanyi describes focal awareness and subsidiary awareness by using the examples of various simple activities, such as using a tool and swimming. Regarding the two different types of awareness, when driving in a nail with a hammer, we attend to two rather different processes (Polanyi, 1962; 1967). We firstly pay our attention to the act of nailing itself. However, we cannot handle a hammer effectively with only the awareness of the act of nailing. To drive the nail in successfully, we should be aware of the feeling in our palm and fingers, although we do not focally attend to them. In other words, we can use a hammer effectively when we are simultaneously aware of both the hammer's strokes and the feeling in the palm and fingers, although the feeling is not actually seen. The act of hammering which the actor attends to focally is called "focal awareness", and the actor's feeling of the palm and fingers which the actor is aware of intensively but does not attend to focally is called "subsidiary awareness". (Polanyi, 1962, p. 55).

Focal awareness is brought about by a person's attention and can be explicitly identified by the person. However, the person knows the particulars of subsidiary awareness in that the particulars are controlled by the person and guide the person to act successfully, but the person cannot explicitly identify the particulars that are tacit. Focal awareness, which involves the subject's focal attention, is always conscious. However, subsidiary awareness can exist anywhere across a spectrum of the conscious and the subconscious (Polanyi & Prosch, 1975).

Polanyi claims that awareness which is a combination of focal awareness and subsidiary awareness can be applied to all human activities, including artistic and intellectual activities as well as simple activities. During the scholar's work of discovering and solving new problems, the scholar can perform skillfully, but the scholar cannot explicitly explain how to do so and can merely express his or her tacit dimension in their academic activities.

In a similar way, professionals, such as doctors or teachers, cannot skillfully perform their duties of teaching students or diagnosing patients with only relevant theoretical knowledge. To realize something that theory has implicitly taught them, professionals need to train their proper sense. They learn skills, test theoretical knowledge using their sense, and gain expertise by applying their skills to relevant experiences (Polanyi, 1962; Polanyi & Prosch, 1975).

Because tacit knowledge significantly influences teachers' teaching practice, tacit knowledge is often mentioned in studies on the analysis of the teaching phenomenon; but the concept of tacit knowledge has been used vaguely in most of the studies (Toom, 2006). Toom (2012) pointed out that tacit knowledge needs to be used, and also clarified its two different meanings: tacit knowledge as a result and tacit knowing as a process. Tacit knowledge as a result is considered as accumulated products of thoughts and actions and is implicit knowledge encompassing underlying beliefs, attitudes and values. Since tacit knowledge as a result can be only partially known, it is difficult to explicate what it is in detail. Tacit knowing as a process is revealed in experts' skillful and competent actions, and especially when experts take an action or make a decision in an instant, tacit knowing significantly influences these activities. It is possible to explicate tacit knowing retrospectively, but all particulars of tacit knowing cannot be put into words. The explicability of tacit knowing is indicated from Polanyi's comment on the recognition of faces. Polanyi states that it is possible to communicate our knowledge of how to recognize someone's face if we are given adequate methods for articulating ourselves (Polanyi, 1967), but the fact that we know more than we can tell does not change even though we use the methods.

The discussion concerning the explication of teachers' tacit knowledge can also be seen in Fenstermacher's study (1994). When one provides his or her rationale for their action, Fenstermacher calls this practical reasoning and claims that practical reasoning is a form of justification showing that one's action is reasonable in a certain situation. Fenstermacher's claim indicates that only if teachers make efforts to reason and articulate their actions, teachers' knowledge that is relevant to their practice, which may be tacit for themselves, can reach a conscious level. It can also be said that teachers can reflect on their actions in depth through their efforts to articulate their actions.

As can be seen so far, tacit knowledge has two different meanings; one is a result and the other is a process, but in reality it may be difficult to distinguish between tacit knowledge as a result and tacit knowing as a process in the process of the construction and development of knowledge. In addition, the tacit knowledge behind teachers' actions and decisions can be assumed to some extent through the teachers' actions and words. However, it is not possible to fully articulate tacit knowledge, but making efforts to reveal knowledge in proper ways and being aware of the existence of the tacit dimension can be essential for teachers to reflect on their practice and to make their experiences more meaningful.

2. The structure of tacit knowing and construction of knowledge

Polanyi's statement that all human activities are formed by the combination of focal awareness and subsidiary awareness was presented earlier. Two kinds of awareness function in a different way to perform a certain activity. One is focal awareness, which is when a person's attention is given focally; and the other is subsidiary awareness, which is required to perform the activity successfully even though the person does not consciously pay attention to particulars of subsidiary awareness.

Focal awareness and subsidiary awareness can be distinguished conceptually, but human activities are constructed by the combination of the two kinds of awareness and so they cannot be separated in reality. All human activities are products of the process of integrating focal awareness and subsidiary awareness by attending to focal awareness grounded in subsidiary awareness. Besides focal awareness and subsidiary awareness, what Polanyi considers as an essential element of tacit knowing is a subject (Polanyi, 1967; Polanyi & Prosch, 1975). The claim that the personal participation of a subject is always essential in the process of knowing is the core of Polanyi's epistemology (Sanders, 1988; Han, 2002). Polanyi rejects the objectivistic viewpoint that a person acquires objective knowledge and applies it directly to his or her action, and he considers that in reality human activities are constructed through the unification of objective knowledge and personal elements.

Tacit knowing has a triad structure, which consists of focal awareness, subsidiary awareness and a subject who combines focal and subsidiary awareness (Polanyi, 1967). The triad structure of tacit knowing means that although a subject cannot exactly identify what the particulars of subsidiary awareness are, the particulars are controlled by the subject and are combined with the focal entity along with the subject's particular purpose and intention. Especially, an inquiry activity undertaken for the acquisition of meaningful knowledge for a subject is not achieved just by the accumulation of subsidiary awareness, but by the subject's efforts to understand in relation to the particulars to form the coherent entity (Eom, 1998). To sum up, Polanyi states that all human activities are conducted by the interaction between the two kinds of awareness; the most important point is that the combination of focal and subsidiary awareness is based on the subject's intuitions, beliefs and participation. In other words, according to Polanyi's epistemology, it is possible to acquire or understand knowledge only if a subject possesses both an explicit dimension and a tacit dimension of knowledge.

To understand what inquiry of new knowledge or human intellectual growth is, from the viewpoint of Polanyi, the triad structure of tacit knowing needs to be closely examined. In the triad structure, knowing occurs through an object to which a perceiver attends focally and the particulars of which the perceiver is subsidiarily aware. However, if the perceiver shifts his or her focal attention to the particulars, the triadic relation disappears and the original meaning is lost (Polanyi & Prosch, 1975). In other words, it means that the perceiver cannot focally attend to both focal awareness and subsidiary awareness simultaneously. According to Polanyi, for example, when a pianist plays the piano, the pianist attends focally to his or her performance and the movements of his or her fingers function subsidiarily. At some point, however, if the pianist attends focally to the movements of his or her fingers, the beautiful tune disappears and the feeling of the movements of the finger muscles and the fingertips touching the keyboard remains. This example shows that if a perceiver's focal attention shifts to the particulars, the original meaning of focal awareness disappears.

However, going back and forth between the whole of focal awareness and the particulars of subsidiary awareness can deepen our understanding of their relationship (Polanyi, 1967). The relationship between subsidiary awareness and focal awareness becomes clearer and it is also possible to conjecture the whole of focal awareness. A knower can understand an object more deeply if the knower is consciously aware of the particulars of subsidiary awareness as focal awareness, and then conscious awareness becomes subsidiary awareness again with a comprehensive and synthetic form. For example, suppose that there is a mechanic who is good at handling a machine but has no theoretical knowledge about the machine. The mechanic handles the machine with subsidiary awareness that cannot be expressed verbally. If the mechanic learns the principles of operating the machine, however, he could have a much deeper understanding of the machine than before by accepting the principles of operating the machine as his or her subsidiary awareness (Polanyi, 1967). In another case, Nam (2006) explains the bidirectional movement between focal awareness and subsidiary awareness using the example of swimming skills.

According to her, most swimming skills are unconscious movements with subsidiary awareness. However, if the swimmer becomes conscious of his or her movements, and the shift is made from subsidiary awareness to focal awareness, and then the conscious movements become habitual and unconscious again, moving from focal awareness to subsidiary awareness, the swimmer's swimming skills can be improved. This example also shows that the bidirectional movement between focal awareness and subsidiary awareness enhances one's knowing.

Polanyi explains knowing activity with the "from-to-relation". From-to-relation, which represents the functional relation between focal awareness and subsidiary awareness, implies that the relationship between focal awareness and subsidiary awareness is not fixed but relative. Focal awareness, which is signified by integrating the particulars of subsidiary awareness, is perceived as a whole with subsidiaries, and it becomes subsidiary awareness again at a higher level (Polanyi, 1967; Polanyi & Prosch, 1975). Subsidiary awareness is close to the perceiver and guides the perceiver to comprehend the distal integrated whole. Focal awareness exists at the end of the subsidiaries and becomes subsidiary awareness again at a higher level. Through the process of this "from-to relation", a knower ends up making new integrated discoveries (Polanyi, 1967). In other words, subsidiary particulars or clues that are related to specific focal awareness can be focal awareness with a relation to other particulars at a lower level, and such interplay of focal and subsidiary awareness makes a knower develop their knowledge and deepen their understanding. In Polanyi's view, it can be said that humans' intellectual growth is a process that internalizes meaning systems which exist in the external world of a knower grounds them in the knower's subsidiary awareness, and then directs them toward a more comprehensive entity.

Nam (2006) reinterprets mathematics education theories such as the theory of mathematization by Freudenthal, the historico-genetic principle (see Freudenthal, 1981) and the theory on the levels of mathematical thinking (see van Hiele, 1986) by linking them to Polanyi's view on the relationship between focal and subsidiary awareness. According to Nam, the history of mathematics shows a process in which the essence of mathematical knowledge, which originated from intuition, has been gradually revealed through progressive consciousness, formalization, systematization and organization. In other words, it can be said that the history of mathematics shows the process of becoming conscious of things that were unconsciously used, that is to say the process from the subsidiaries to the focal.

Similarly, the levels of geometrical thinking can also be reinterpreted with the interplay between focal and subsidiary awareness. The levels of geometrical thinking show the process of geometrical thinking that was subsidiarily aware gradually becoming conscious as focal awareness. In other words, it seems that the levels of geometrical thinking are the embodiment of the process of formal and systematic geometrical learning; that is, learners become conscious of their unconscious and habitual actions through reflection and increase their levels of geometrical thinking. Nam utilizes Polanyi's viewpoint in the reinterpretation of mathematical history and the levels of geometrical thinking to indicate that man can discover and explore something depending on subsidiary awareness, which cannot be explicitly articulated, but the knowledge can be developed only if subsidiary awareness is focused upon and consciously explored.

When considering Polanyi's epistemology, which describes all human activities with the relationship between focal awareness and subsidiary awareness, the aforementioned explanation of mathematical knowledge being constructed by consciousness and the reflection on unconscious and habitual things can be applied to teachers' teaching activities. Indeed, many researchers have argued that tacit knowledge like teachers' beliefs and assumptions and teachers' unconscious teaching behaviors need to be conscious and reflected upon (for example, Hager, 2000; Kwak, Nah & Yoo, 2009; Ponte, 1994; Torff, 1999). Schon (1987) mentions that even though teachers continuously perform actions during class, they cannot explicitly explain how to perform such actions, and he calls such knowledge "knowing-in-action". He also argues that knowing-in-action is not appropriate for teaching and such knowledge needs to be developed to "reflection-on-action" and to "reflection-in-action" (cited in Paparistodemou, Potari & Pitta, 2006, p. 1). It implies that teachers need to improve their teaching practice by developing from the level of knowing-in-action to higher levels by becoming more conscious of reflecting on their actions. Kwak et al., (2009) and Torff (1999) similarly points out that teachers need to be consciously aware of their beliefs and tacit assumptions, which influence their teaching activities, and also be conscious of the limitations of their beliefs and assumptions. The researchers also emphasize that teacher educators should provide opportunities for teachers to reflect and reconstruct their practice in a systematic and conscious way. The researchers' aforementioned claims imply that teachers, in order to improve their teaching practice, need to be consciously aware of and reflect on their subsidiaries, which they may remain unconsciously of without any effort.

3. Tacit knowing and teaching

According to Toom (2006) tacit knowing can be understood by linking it to teachers' competence. That is, tacit knowing is related to teachers' competence which 'means the effective use of knowledge and skills in specific complex contexts' (Toom, 2006, p. 57): tacit knowing is related to teachers' ability to choose the relevant knowledge and skills in a certain classroom situation to act in a proper and sensible way, and consequently tacit knowing can play an important role in effective practice. Through understanding the relationship between tacit knowing and teachers' practice, Toom emphasizes that tacit knowledge is the foundation of actions and presents the figure of the structure of tacit knowing as follows on the basis of Rolf's interpretation of Polanyi's structure of tacit knowing (cited in Toom, 2006, p. 67).

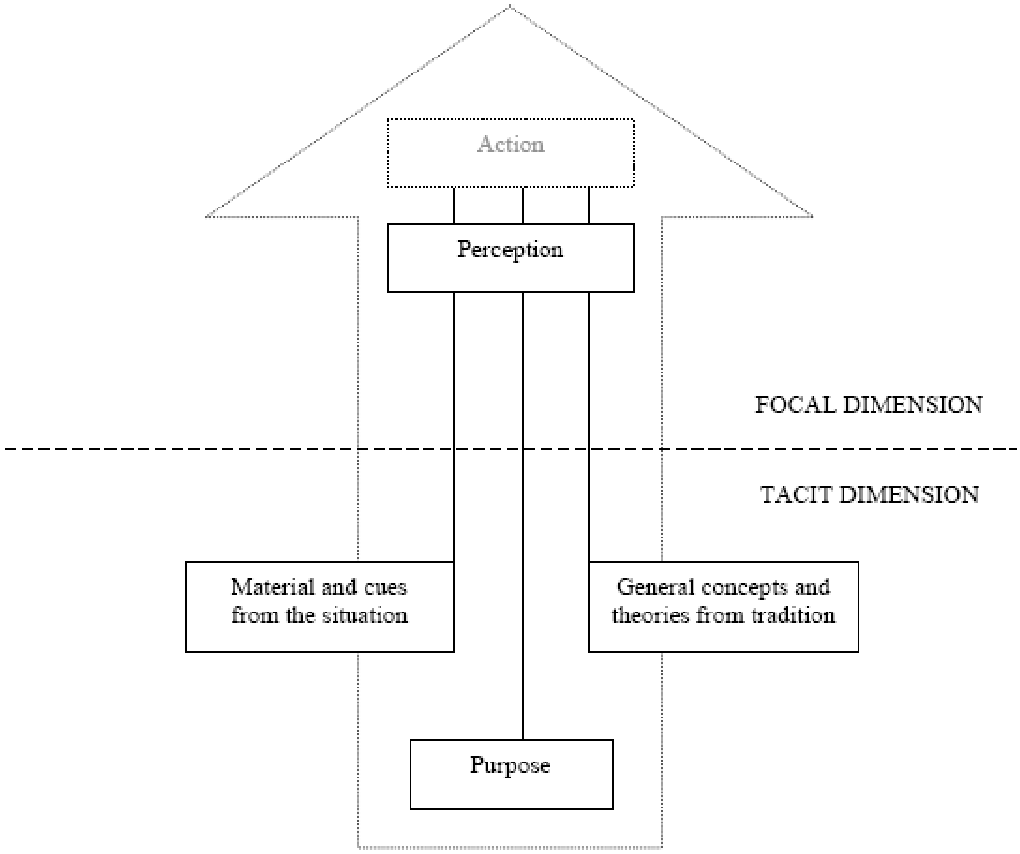

Figure 1. The interpretation of Polanyi's structure of tacit knowing[2] (Toom, 2006, p. 67)

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[2] The original figure and interpretation of Polanyi's structure of tacit knowing was presented by Rolf. The Figure 1 was completed by Toom by adding an arrow boxes with dash line to concretize the direction of the process of tacit knowing.

Figure 1 represents that the perception process involves the combination of general concepts and traditional theories with cues from the situation, and the combination is controlled by the subject's purpose. In other words, tacit knowing leads to perception and action with the integration of theories and situational cues controlled by the subject's purpose. As mentioned earlier, Polanyi's triad structure of tacit knowing shows that the particulars of subsidiary awareness are combined with the focal awareness and the combination is directed by the subject's intention. Figure 1 represents simply the process to action through the combination of the tacit dimension and the focal dimension, on the basis of Polanyi's triad structure. As can be seen in Figure 1, in addition, tacit knowing is formed by the integration of thought and action.

When considering the interpretation of the structure of tacit knowing showing the influence of the tacit dimension behind one's action and practice, a teacher's true understanding of knowledge ought to be seen in the teacher's skillful actions while he or she is teaching. However, only awareness of one's teaching actions cannot warrant effective and efficient practice, and teachers need to be continuously aware of situational cues perceived in their classroom interactions with students and their sensitivities about the cues. Feelings and sensitivities influenced by personal beliefs and values cannot be seen, but teachers need to be aware of those feelings and sensitivities to teach effectively.

Another teaching situation in which tacit knowing functions is the moment a teacher makes a decision during class. Schoenfeld's decision-making model (2010) shows why and how teachers depend on particular resources in their teaching process. Teachers' actions in complex teaching situations are influenced consciously or subconsciously by some factors related to the questions of "why" and "how", rather than being spontaneous and unintended (Schoenfeld, 2010).

By analyzing situations where teachers made decisions, Schoenfeld found that teachers' decisions in both familiar and unfamiliar situations were influenced by their beliefs, resources and goals connected to their actions. The three factors influence teachers' actions and their functions can be understood in a similar way as the aforementioned elements of tacit knowledge and the structure of tacit knowing. Although Schoenfeld did not mention tacit knowledge directly, his finding shows that there is tacit dimension that influences significantly teachers' observable actions and provides empirical evidence of why we should even consider the existence of a tacit dimension behind teachers' actions. The finding implies that there are some aspects of teachers' teaching actions that cannot be verbally expressed, and the acquisition of theoretical knowledge is not sufficient to perform effectively. Teachers need to additionally consider their knowledge, beliefs and goals that influence their decisions.

Studies on the meaning of tacit knowledge and its components have been examined so far. Those studies indicate that the consciousness of and reflection on the tacit dimension behind actions and its function is very important for teachers to construct and improve their practical knowledge. It is thought that tacit knowledge needs to be considered in studies concerning teachers' practical knowledge in a proper way.

The interpretation of the structure of tacit knowing indicates that to construct proper practical knowledge teachers need not only their teaching experiences but also theories by which teachers view their experiences critically and reflect on them. On the basis of their experiences and theories, with a clear intention and purpose, teachers are also able to perceive proper cues from situations. Real class situations are very complex, and in these complex situations, what teachers see and what phenomena they perceive seem to influence their actions significantly. From the point of view of practical knowledge construction, the issue of how to develop teachers' perception and strengthen their situational awareness needs to be addressed.

One needs to be conscious of and reflect upon tacit knowledge because it influences actions. Experts, especially teachers, should be consciously aware that their tacit knowledge influences both their teaching practice and student learning.

3.1. Tacit knowing and awareness

Teachers continuously make perceptions, form judgments, and perform actions in the complex and dynamic class situation. Most teachers, however, are not aware of what they see and how they interpret and understand what they see. In the case of experienced teachers, especially, it is not easy for them to be conscious of their actions because they have formed lots of routines and patterns in familiar situations. However, if teachers are not consciously aware of what they focus on and what they ignore in their class with tacit knowledge, they may ignore, or may not identify, some critical events that occur during class. Teachers' awareness is not explicitly shown but it significantly influences their teaching actions.

The study conducted by Paparistodemou, Potari and Pitta (2006) suggests the importance of teacher awareness that influences teachers' teaching activities and their students' learning. The participating teacher in the study emphasized that providing opportunities for students to experience randomness was essential while she was designing her lesson related to probability. In her actual class, however, she drew students' responses through simple closed questions that required a certain answer and did not provide proper opportunities for her students to experience randomness. Interestingly, however, during the interview conducted after the class, the teacher stated that her students could experience randomness during the class and had followed her without any misconception because they had answered her questions quite well. This result suggests that if teachers cannot be consciously aware of the tacit assumptions behind their actions and any inconsistency between what actually happened in their class and what they perceive as having happened, they cannot truly understand their class and students' learning.

Teachers' awareness of their teaching practice can be compared with the aforementioned interpretation of the structure of tacit knowing. Teachers should be aware of teaching situations by perceiving situational clues from classroom events and adding their understanding and interpretation to them. Teacher awareness can be considered as the ability to perceive and understand a classroom situation. Teachers take actions after interpreting some classroom events, getting a sense of the importance of the events and judging what to do. These phases that occur in the interactions between a teacher and his or her students during class are not separate but rather the whole process of awareness to action, and this process seems similar to the aforementioned process of tacit knowing.

In the next section, the meaning of awareness is presented in detail through an explanation of the relationship between awareness and noticing. It is also discussed how teachers' awareness can be handled in terms of the construction of mathematics teachers' practical knowledge.

3.1.1. Awareness and noticing

To notice is to observe, realize or pay attention to something (Ball, 2011). People always pay attention to something consciously or unconsciously in everyday life and are aware of what they pay attention to. It can be said that attention is necessary for noticing and awareness. However, it is difficult to distinguish through a person's words and actions whether that person is paying attention to something or is noticing and is aware of something. For example, suppose that when a student calculates the subtraction of fractions, the student explains "when subtract one fraction from another fraction, making their denominators same and subtract their numerators" (Mason, 2008). This student might express his thinking with generalization, but might express his thinking based on only the particulars to what he attended to in the process of his calculation. Attention, noticing and awareness are closely interrelated (Mason, 2011); thus it is difficult to distinguish or clearly define them. This study focuses on the role of attention as a requirement for noticing, rather than distinguishing these terms strictly and considering attention and noticing as a developmental stage.

Similar to students learning, noticing and awareness cannot be shown explicitly in teachers' teaching practice. Teachers attend to something and act during class in their idiosyncratic way but what they attend to and how they interpret what they attended to are not explicitly revealed. Teachers determine an action based on what they attend to during class but are not fully aware of the relationships among the things they attend to. If teachers focus merely on their action in particular, they may not be aware of how they develop their classroom activities and how they actually handle them during class. As mentioned earlier, teachers may not teach successfully with only awareness of their actions. Rather, to teach efficiently, they need to be aware of situational clues that they perceive from interactions with their students and the sensitivities of situations; however, teachers do not often pay attention focally to such matters. Noticing something in a teaching situation can be considered as the teacher's perception of the students' responses and situational clues. Teachers may pay attention by gazing at the whole situation to get the overall impression or looking at particulars without forming any relationships among them. They may also attend to by discerning details or recognizing relationships among the discerned details. Noticing at a lower level is revealed in the former way of attention, while noticing at a higher level is revealed in the latter way of attention (van Es, 2011). Being aware of the thing that is noticed requires relevant experiences of attention (Mason, 1998). In other words, one becomes consciously aware of something through efforts to explain the relationships among the noticed details with one's will and intention to pay attention to relevant details.

Previous studies on teacher noticing show that different researchers use different definitions of noticing. Researchers use operational definitions of teacher noticing according to the aim of their study, on the basis of two general processes of noticing: attending to particular events in a teaching situation, and understanding the attended events (Sherin, Jacobs & Philipp, 2011). The former focuses on investigating what teachers attend to or what they ignore, and the latter encompasses teachers' interpretations of the things they pay attention to and the teachers' reactions accordingly. This study follows the latter definition, which considers noticing as an act that operates on both a conscious and a subconscious level (Mason, 2002). When teachers are educated to improve their noticing skills based on the latter definition, they can be given opportunities to be explicitly aware of their action by allowing them to explain what they attend to and how they attend to something.

By analyzing what teachers attend to and how they interpret the things they pay attention to, van Es (2011) found that teachers' noticing skills improved when they changed the way they attend to something. He suggested four levels of noticing of students' mathematical thinking. Teachers attend to the whole class environment, learning and teacher's pedagogy in level 1, but as the level of noticing gets higher, teachers attend to the relationship between students' mathematical thinking and teaching strategies and propose alternative pedagogical solutions. While teachers merely describe what they see in the lower level of noticing, as noticing skills are developed, teachers become sensitized to the situation where they can change their practice by trying to notice an opportunity to act in new and alternative way. Those features in the levels of noticing skills are very similar to the ways of attention that are presented by Mason (2011). Mason describes the ways of attention as follows:

> 1. Holding wholes is attending by gazing at something without particularly discerning details. (…)
>
> 2. Discerning details is picking out bits, discriminating this from that, decomposing or subdividing and so distinguishing and, hence, creating things. (…)
>
> 3. Recognizing relationships is becoming aware of sameness and difference or other relationships among the discerned details in the situation…
>
> 4. Perceiving properties is becoming aware of particular relationships as instances of properties that could hold in other situations.
>
> 5. Reasoning on the basis of agreed properties is going beyond the assembling of things you think you know, intuit, or induce must be true in order to use previously justified properties as the basis for convincing yourself and others, leading to reasoning from definitions and axioms. (Mason, 2011, p. 47)

When considering the relationship between the levels of noticing skills and the ways of attention, it can be possible to improve teachers' noticing skills by guiding them toward noticing significant classroom events and describing and interpreting what they notice. In addition, an improvement in noticing skills can be identified by a change in the way of attention, as Mason (2011) mentioned.

Because noticing and awareness are closely related to teachers' experiences, when teachers notice something that is unfamiliar to them teachers need some external help in not shifting their attention to other things (Mason, 2008). In fact, van Es (2011), in his study, asked participating teachers questions like 'let's take a look at how Lindsey solved that problem' and 'why do you think she chose that method' (p. 137) to make them consistently attend to students' mathematical thinking. It is thought that teacher educators need to consider guiding teachers toward noticing meaningful things to help them to be consciously aware of what they notice by developing their noticing skills.

As aforementioned, people always attend to something and become aware of. Noticing is the lowest level of awareness (Schmidit, 1994), and through consistent experiences of noticing something one can be sensitized to and form a sense impression on the relevant situation. However, people may not be consciously aware when confronted with a similar situation but may be subconsciously aware, which may significantly influence people's actions (Mason, 2011). As can be seen, teachers' awareness has an important influence on their teaching actions, but their awareness is tacit and cannot be explicitly observed. Because awareness and noticing are closely interrelated, however, teachers can be consciously aware by explaining what they noticed and how they interpret the noticed things. In other words, teachers can become sensitive and strengthen their awareness by trying to notice opportunities to act in new way by reflecting on their actions (Mason, 1998; 2011), and consequently teachers may be more likely to change their teaching practice.

According to the discussion about teacher awareness so far, teacher awareness is related to subjective experiences and teacher knowledge, and awareness is reflected in the teachers' actions by perceiving the particulars with the combination of a sense formed by relevant experiences, situational clues and teacher knowledge, according to the teacher's intention and purpose. Even though awareness is tacit, teachers' awareness plays an important role in teachers' noticing and their decision-making in their classroom practice.

It seems that teachers need to have a meta-perspective on their knowledge and actions by developing their awareness of mathematics, pedagogy and action. The formation of a meta-perspective through the development of awareness may help teachers improve the quality of their thinking and practice. Thus, it is thought that teacher awareness is necessary to construct teacher practical knowledge.

3.1.2. Importance of mathematics teacher awareness

The importance of awareness in students' mathematics learning has been emphasized by Gattegno since the 1970s (Powell, 1998), while it is only a short time since teacher awareness was discussed in mathematics teacher education.

Mathematics teachers' awareness can be divided into two aspects, mathematical awareness and pedagogical awareness, but these two aspects cannot be separated and should be closely linked because they are interwoven in the actual classroom setting.

Nam (2007) considers mathematics teachers a manifestor of tacit knowledge. Recorded knowledge in mathematics textbooks are explicit knowledge lacking any explanation of tacit knowledge or the process of discovery. Thus, teachers should help students learn about tacit knowledge which exists alongside explicit knowledge by reproducing the context of knowledge generation. When considering the area formula in a task to calculate the area of a given figure by dividing the figure properly, there is an invariance axiom for movement and an addition axiom as well as the formula for the area of rectangle. These three axioms are the basis for finding the area of a figure (Nam, 2007, p. 150). The essential idea of the area concept of the area formula is hidden, and a teacher who is aware of the hidden idea can help students learn the area formula by providing tasks requiring division or the transformation of given figures. A teacher who is not aware of the idea, however, cannot grasp the hidden idea included in the task, and does not provide appropriate learning opportunities to students or makes students to memorize the area formula by assigning repetitive solving problems requiring complicated calculations.

The most important requisite of teaching mathematics is the awareness of the essence of mathematical knowledge and how the knowledge is connected to other mathematical knowledge. Furthermore, the mathematics teacher ought to be able to transform the essential idea of a mathematical concept to an instructional task and activity. To do so, teachers need to be aware of the way in which they guide their students toward paying attention to the essential idea. This indicates that teachers need to be consciously aware of their awareness in order to integrate the awareness of a mathematical concept with an instructional task or activity.

Teachers need to attend to students' mathematical thinking to transform their mathematical awareness to a learnable form. For example, when a teacher uses the term "angle", the teacher may be subconsciously aware that the size of an angle is unrelated to the lengths of its arms but related to the turning of its arms (Mason, 1998), and the teacher expects that the students would use the term in the same way. Most students, however, 'naturally begin by associating angle with an awareness of a point, of arms, and of space' (Mason, 1998, p. 252) and judge the size of angles by attending to the lengths of their sides. A teacher who is aware of their students' thinking may continuously attend to their interactions with their students and perceive some relevant clues from students' responses, interpret and understand those clues, and take action to provide proper learning opportunities to the students.

Likewise, teachers' mathematical and pedagogical awareness is not explicitly observed but significantly influences teachers' classroom practice. As discussed so far, teacher's awareness cannot be separated from the teacher's self and the teacher's experiences and should be considered within the context of actual teaching practice. Thus, it is important for teachers to realize the influence of the tacit dimension behind their actions on their teaching and students' learning by being more conscious of their teaching actions, and such awareness of tacit dimension plays an important role in constructing mathematics teacher practical knowledge.

3.2. Tacit knowledge and socio-cultural environment

The previous section examined teachers' tacit knowledge in terms of personal aspects. As many researchers have argued, however, teachers' socio-cultural environment influencing their practice cannot be ignored because teachers are beings who adjust to the broader socio-cultural environment around them. However, studies on the relationships between teachers' practice and socio-cultural contexts have been broadly conducted. Therefore, it is difficult to review those studies in detail in this section, but it seems meaningful to examine how teachers interpret, understand and adjust to the socio-cultural environment around them for constructing their idiosyncratic practical knowledge. Thus, in this section, even though it is limited, previous studies on the culture of the teaching profession formed within the socio-cultural context of Korea and some particular teaching styles drawn from the culture of the teaching profession are reviewed. It is expected that the review will help understand the construction of mathematics teachers' practical knowledge in both personal and socio-cultural aspects.

There are several studies on the relationship between teaching styles and the socio-cultural environment in Korea. Especially Lee (1988) and Lee (1990) describe Korean teachers' teaching style as "indoctrination and memory oriented teaching" and "digestive teaching".

Lee (1988) tried to find out why indoctrination and memory-oriented teaching still occurred in Korean classrooms, although new curriculum theories or teaching methods had been introduced in schools. Indoctrination and memory-oriented teaching, termed by Lee, is the teaching style in which a teacher selects the contents of knowledge that students should know, injects the knowledge and forces students to memorize, and engages in unilateral expository instruction. Lee claims that the reason that inquiry instruction, which is generally considered as a desirable teaching and learning method, cannot be applied to Korean classrooms is that teachers are not free from Korea's unique culture that has a hierarchical and competitive structure. Therefore, teachers continue to engage in indoctrination and memory-oriented teaching to survive in the existing cultural system. Lee's claim seems to consider teachers as passive beings who cannot help but accept the pressure of the cultural system, but further discussion is required about whether teachers are continuing to engage in indoctrination and memory-oriented teaching by accepting the Korean socio-cultural system without any doubt and conflict or whether they are choosing the teaching style using rationality to adjust to the system.

Lee's study (1990) that attempted to identify some features of the culture of the teaching profession in a Korean high school shows that teachers are active persons who adjust to the socio-cultural context. According to Lee, teachers are not passive beings who unilaterally follow educational theories and policies; rather they adapt their professional knowledge and skills to the social conditions around them and perform their teaching activities in accordance with the principles formed by their interpretation of the conditions. Korean high school teachers teach their students by interpreting and adapting their subject matter knowledge and textbooks containing the contents to be learned based on parental demand for their children's improvement and students' learning ability. Lee termed the teachers' teaching style metaphorically as "digestive teaching" that was formed by teachers through the acceptance and adaptation of the peculiar situations with the college entrance examination system and parents' enthusiasm for their children's education. According to Lee, the digestive teaching style follows some principles as follows: teachers emphasize some content knowledge within the textbooks, which are expected to be on the college entrance examination, and give information about the format of the examination. They present the emphasized knowledge in condensed form and use familiar examples which students can clearly understand and memorize. Digestive teaching is different from teaching by rote memorization in which teachers simply inject selected content knowledge into students' brain and force them to memorize. Digestive teaching is a teaching style unique to Korean teachers. It involves the acceptance and adaptation of the teachers' perspective, a cultural situation with a competitive structure, and classroom conditions characterized by time and space limitation.

Similar to aforementioned studies, McNeil (1982) analyzed classroom instruction as a social phenomenon and showed how teachers reacted to social and institutional constraints. McNeil identified that American teachers taught subject matter knowledge in a peculiar way and described it using the term, "defensive teaching". Defensive teaching is a method in which teachers restrict students' access to knowledge to control them, and use the simplification and mystification of knowledge to prevent discussion about complex topics.

Defensive teaching by American teachers was influenced by sociocultural factors. Because teachers felt there was a lack of administrative support and had a weak sense of authority, they used the defensive technique to do their work easily and efficiently (McNeil, 1982).

Defensive teaching seems different from digestive teaching that has been revealed from the features of the culture of the teaching profession in Korea. In defensive teaching, teachers control all students regardless of their learning ability in the same way by simplifying the content; however, in digestive teaching teachers teach connected knowledge logically, not simple information, and place less limitations on students who have high learning ability to access knowledge. In the case of Korea, the digestive teaching style is the result of teachers' adjustment to the peculiar situation in Korea. Certain factors such as parents' desires for their children to attend a prestigious university, the college entrance examination and competitive structure have led Korean teachers to adopt the digestive teaching style. The difference between defensive teaching and digestive teaching indicates that the different sociocultural conditions of teachers of two countries may influence their teaching practice to some extent. In other words, different sociocultural environments may cause various teaching styles, and teachers' teaching behavior can be different in accordance with their sociocultural conditions. Teaching occurs in a complex and complicated situation and so to clearly explain teaching seems difficult, but teachers' teaching behavior based on the existing conditions around them has certain features.

Kim (1996), who observed a Korean elementary school, looked closely at both the internal conditions of the school and the social conditions to investigate the patterns of classroom teaching that had resulted from those conditions. The findings of the study showed that there were two main teaching patterns, teaching by explanation and teaching by assignment, and the two patterns seemed to be influenced by external factors such as the progress of classwork and examinations. Teachers seemed to have relative autonomy in their classrooms, but in the broader contexts of their school organization and society, they were restricted by both the power of the principal and other teachers and the progress of classwork, students' performance and parents' demands. It seemed that those social and institutional conditions affect the two dominant teaching patterns of Korean teachers.

Meanwhile, it is also identified that sociocultural factors may affect Korean mathematics teachers' teaching practice. Lee (2010) argues that Korean mathematics teachers are conscious of external factors such as competition and success in the college entrance examination within a social environment that is of great interest in education and develop a strong responsibility to deal with those social conditions. The interesting point in this study is that mathematics teachers emphasized structure and the essence of mathematical knowledge in their class.

Lee identified that teachers seemed to lead their class unilaterally and engage in expository teaching to deal with social demands and interests efficiently and effectively, but when looking closely at their interactions with their students, it was found that teachers guided students to access the essence of a mathematical concept. This finding implies that teachers conduct their teaching practice by accommodating and adapting to personal orientations toward mathematics teaching and sociocultural conditions.

The studies reviewed so far seem meaningful in that the studies tried to understand classroom teaching by linking classroom features and the broader sociocultural context. However, the studies do not show how teachers are influenced by sociocultural factors and what actual process of change teachers go through.

Yu (2006) investigated teachers' change of their professionalism by analyzing the characteristic of their reflection and the process of reflection while they were participating in a professional development school setting. The findings of the study can be summarized as follows: there were two types of reflection, interpretive reflection and technical reflection. Teachers who showed interpretive reflection doubted and resisted the new way of teaching when applying it to their class, while teachers who showed technical reflection tended to think within the existing system and practice and return to their old way of teaching. Technical reflection especially was continuously shown throughout teachers' participation in the program, as they adhered to their personal teaching principles and the existing system or returned to the old way of teaching. These findings imply that teachers' practice is more likely to develop through continuous attempts and returns, not monotonic improvement.

It is thought that closely examining how teachers accommodate and adapt to personal and sociocultural factors to change their practice is essential to understand the process of constructing mathematics teachers' practical knowledge.

CHAPTER Ⅲ

CONSTRUCTION OF MATHEMATICS TEACHERS' PRACTICAL KNOWLEDGE

1. The concept of mathematics teachers' practical knowledge and its components

In this section, components of mathematics teachers' practical knowledge are drawn from previous studies on practical knowledge and the discussion on tacit knowledge that is presented in Chapter 2.

Ways of conceptualizing practical knowledge can be divided into two strands: the first via direct observation of classrooms, and the second is based on an epistemological investigation of teacher knowledge. Definitions of teachers' practical knowledge from both perspectives emphasize different aspects of practical knowledge. In this section, different aspects are discussed and components of practical knowledge are drawn from the discussion and considerations of tacit knowledge.

Elbaz, a well-known researcher, tried to conceptualize teachers' practical knowledge based on empirical data. Through the observation of one teacher's daily school work, Elbaz (1983) identified the extensive and idiosyncratic knowledge that guided the teacher's overall teaching behavior but was different with theoretical knowledge, and conceptualized the knowledge as practical knowledge. Elbaz defined practical knowledge as knowledge that teachers synthesize and reconstruct their knowledge based on their values and beliefs to reflect their actual situations. Elbaz' definition highlights personal and experiential aspects of practical knowledge.

Clandinin (1985) and Connelly and Clandinin (1988) describes that teachers' special knowledge used by teachers for teaching practice has both theoretical and practical aspects and combined with their personal background and individuality. Connelly and Clandinin (1988) have a similar point of view to Elbaz on practical knowledge and investigate practical knowledge, focusing on the teachers' image that is presented by Elbaz as one of the terms which explains the structure of practical knowledge. The fact that Clandinin (1985) focuses on the image of teachers indicates that she emphasizes personal aspect of practical knowledge; in fact she termed teachers' special knowledge as 'personal practical knowledge'. Clandinin (1985) described personal practical knowledge as follows:

> '…By personal as defining knowledge, is meant that knowledge which can be discovered in both the actions of the person and, under some circumstances, by discourse or conversation. By 'knowledge' in the phrase 'Personal practical knowledge' is meant that body of convictions, conscious or unconscious, which have arisen from experience, intimate, social, and traditional, and which are expressed in a person's actions. (…) 'Personal practical knowledge' is knowledge

which is imbued with all the experiences that make up a person's being. Its meaning is derived from, and understood in terms of, a person's experiential history, both professional and personal' (p. 362).

As can be seen in Clandinin's explanation stated above, she argues that personal practical knowledge should be understood as knowledge that is formed by professional and personal experiences. In other words, Clandinin's personal practical knowledge emphasizes personal and experiential aspects of practical knowledge.

van Driel, Beijaard and Verloop (2001) states that practical knowledge is the product of teachers' teaching experiences consisting of their knowledge and beliefs toward teaching practices and conceptualizes practical knowledge as action-oriented and person-bound knowledge. van Driel et al.'s definition of practical knowledge emphasizes teachers' beliefs along with personal and experiential aspects.

Summing up, studies on conceptualization of practical knowledge on the basis of empirical data emphasize teachers' personal and experiential aspects commonly and indicate that practical knowledge is strongly influenced by tacit elements such as teachers' beliefs.

Many scholars, however, argue that teachers' practical knowledge cannot be appropriately constructed with only teachers' teaching experiences. Dewey (1904) is one of the scholars who highlighted the limit of teachers' experiences for construction of appropriate teaching practices. According to Dewey, teachers who depend only on their teaching experiences to teach are more likely to shape inappropriate teaching habits, thus teachers need theoretical knowledge that can play an important role in observing and reflecting their teaching actions with critical perspectives. In fact, most novice teachers have a tendency to attend to classroom management and student control most of the time in class (Dewey, 1904; Schoenfeld, 2010). However, as Dewey (1904) pointed out, repetition of those experiences makes teachers adhere to classroom control and students' external behavior and teachers simply repeat their teaching practice, depending only on their experience, without an awareness of students' thinking and psychology that teachers should be focusing on in class. Because such teachers' orientations cannot be changed through accumulation of teaching experience alone, theoretical knowledge that helps teachers understand and reflect on their experience with new perspectives and teachers' conscious efforts to change their teaching practice are required. In other words, to construct proper practical knowledge through organization of teachers' experience and reflection on that, theoretical knowledge by which teachers can critically view their experiences is essential.

However, theory may not be applied directly to teachers' practice because teachers transform and reconstruct theoretical knowledge in accordance with their beliefs, judgments and values to adapt to their surrounding situations.

Teachers' practice should invariably be accompanied by tacit knowledge and thus tacit knowledge should be appropriately dealt with for construction of practical knowledge in teacher education.

Meanwhile, Fenstermacher (1994) examined teacher knowledge epistemologically and discussed the nature of practical knowledge. According to Fenstermacher, practical knowledge is bound by time and situation and characterized as tacit, thus to know something practically means to understand the thing within situation, action and event. Fenstermacher (1994) observes that some epistemologists and educational researchers tend to deny practical knowledge as valuable because practical knowledge, not like formal knowledge, depends on the situation and cannot be explicated in verbal form. Fensternacher thus argues that practical knowledge requires justification of teachers' performance to be accepted as knowledge. This means that reasonableness of teachers' action and evidence need to be presented by verbally explaining some tacit elements which underpin teachers' actions. In other words, teachers' beliefs, images and intuitions and their mental activity such as reflection should be inferred and explicated. Fenstermacher's argumentation emphasizes the necessity of consciousness of tacit dimensions influencing teachers' teaching practices, but a further discussion is not offered in terms of practical aspects (i. e., about how tacit dimension can be addressed in teacher education). In Chapter 2 of the present study, the way in which tacit knowledge can be dealt with in teacher education was discussed to construct practical knowledge, and teacher awareness was derived as a component for the construction of practical knowledge.

To sum up, teachers' practical knowledge has a different characteristic with teacher knowledge that has been considered in only explicit dimensions: that is to say practical knowledge encompasses tacit dimension behind explicit dimension. In addition, from the review of previous studies on practical knowledge and the discussion of tacit knowledge, the three components, teachers' teaching experience, theoretical knowledge and awareness, are derived for the construction of mathematics teachers' practical knowledge. Theoretical knowledge in the present study is considered as explicit knowledge, such as pedagogical content knowledge and specialized content knowledge that have been known as mathematics teachers' professional knowledge. As aforementioned, teachers' practical knowledge is constructed by integration of explicit dimension and tacit dimension and thus teachers need systematic opportunities to have the three components be harmonized in order to construct proper practical knowledge through teacher education.

In the next section, strategies to strengthen mathematics teacher awareness are suggested.

2. Strategies for the reinforcement of mathematics teacher awareness

Teacher education for the construction of mathematics teacher practical knowledge as suggested by this study focuses on making organic integration of the three components: teachers' teaching experience; theoretical knowledge; and awareness. As aforementioned, teachers' practical knowledge consists of both explicit and tacit dimensions. These two dimensions can be distinguished conceptually, but in reality, they cannot be separated because the explicit dimension is rooted in the tacit dimension. The tacit dimension is not exposed explicitly but significantly influences teachers' teaching practice and thus teachers can construct proper practical knowledge by being conscious of their teaching behavior that was unconscious before, reflecting and reapplying it to their teaching practice. This process of construction of practical knowledge can be seen as interplay between focal awareness and subsidiary awareness, as presented by Polanyi. In this section, reflecting and noticing are suggested as strategies to embody the process of constructing practical knowledge through the consciousness of tacit dimension and habituation of the conscious teachers' action.

Teacher awareness can play a critical role in connecting theory and practice, between thinking and actions, and theoretical knowledge can help teachers systematically perceive their teaching experience. Like this, theoretical knowledge, teaching experiences and awareness are interrelated in a complex manner with teaching practice, thus the integration of the three components are required for construction of practical knowledge.

McCutcheon (1995) emphasizes that to improve teachers' practical knowledge, an opportunity is needed to discuss and reflect on practices, exposing and being consciously aware of their practical knowledge. Teachers' reflection and sensitivity towards a situation can lead to the development of teachers' awareness of mathematics and pedagogy as well as their actions. As a result, training teachers' noticing skills can be used as a strategy to raise teachers' sensitivity. Because training of teacher noticing skills is based on systematic reflection (Mason, 2002), reflection and noticing can be effective to strengthen teacher awareness with the connection of theory and practice.

2.1. Reflection

Since teacher professionalism has been emphasized in the late 20th, many researchers have mentioned reflective thinking, grounded in Dewey's concept of reflective thinking, as an important element for improvement of teacher professionalism (Korthagen et al., 2001), and various attempts have been made to put reflective thinking to practical use in teacher education. Reflection does not mean simply 'thinking' (Korthagen et al., 2001), but means the interplay between knowing and practice, reflecting the newly-obtained knowing in practice and verifying it, by externalizing the knowing underlying practice and reconstructing (Seo, 2005). Because reflective thinking in teacher education has been extensively researched, it is possible to classify previous studies in various ways according to the purpose and content of study. However, in this section, studies on process and content of reflection are examined.

2.1.1. Process of reflection

According to Cho (2006), studies on the process of reflection in reflective teacher education have mostly focused on explaining teachers' reflection as a process to handle educational problems. The present study considers teachers' reflection as a strategy for construction of practical knowledge through which teachers look back on their practices and understand it with a new perspective, not merely considering it as a tool for solving educational problems. Studies are therefore discussed in terms of construction of practical knowledge.

Korthagen et al. (2001) suggested the ALACT model (see Figure 2), which is a process of reflection based on the cognitive psychology perspective, which integrates of theory and practice in prospective teacher education. This model describes a cyclical process with five phases, which include teacher's action, looking back on the action, awareness of essential aspects, creating alternative methods of action, and trial with a new action. What Korthagen (2001) emphasizes in this model is to have prospective teachers start from their reality to trigger their autonomous experiential learning.

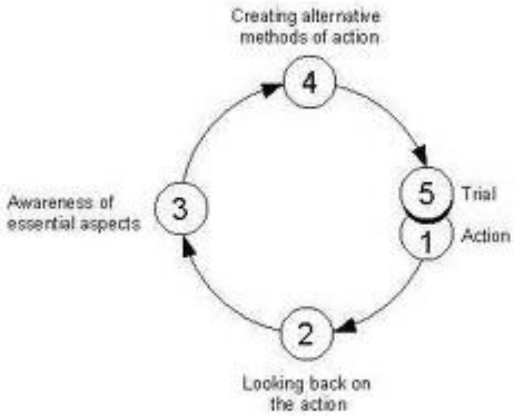

Figure 2. The ALACT model describing the ideal process of reflection (Korthagen, 2001, p. 7)

This model seems meaningful in that it tries to integrate theory and practice within teachers themselves through the process from teachers' real experience to inquiry of perceived phenomenon and trial of new action. However, this model cannot show specifically what teachers actually learn from the process and what qualitative improvement in their reflections actually occurred. Considering that the majority of prospective teachers have a tendency to attend to classroom management in a dynamic classroom situation (Schoenfeld, 2010), this model cannot say how content of reflection to which prospective teachers pay attention ia changing: from classroom management to other essential aspects such as students' thinking. Korthagen et al. (2001) seemed to be aware of such aspects too, and noted that prospective teachers perceived classroom phenomenon with a limited view at the early stages of their teaching careers. He thus proposed that teacher educators should provide prospective teachers with proper situations where a conflict occurs between gestalts exposed naturally by prospective teachers and gestalts to be changed. However, Korthagen's suggestion cannot explain how and in what way teacher educators can change prospective teachers' gestalts to what teacher educators intend. That is the limit of this model and need to revise the model in terms of the limit.

2.1.2. Content of reflection

There are a number of studies on the processes and levels of reflection, while studies on the content of reflection in regard to what teachers reflect on in real classrooms are not relatively abundant (Kwak et al., 2007). Teachers start describing and inquiring about what they perceive consciously or unconsciously (Kwak et al., 2007), thus teacher educators should be concerned with what teachers perceive and reflect in their classroom and how content of their reflection transforms for improvement of their professionalism.

As aforementioned, the majority of novice teachers focus on classroom control, rather than attending to students' thinking, so they are more likely to shape inappropriate teaching habits depending only on their teaching experience, without considering pedagogical principles and theories of teaching and learning (Dewey, 1904). However, such improper teaching habits cannot change naturally as teachers become more experienced or time passes. In order to transform teaching habits, what teachers perceive in classroom situations has to be changed. That is, teachers' way of experience should be changed. It is similar to Marton and Booth's (1997) perspective on learning, which means that learning is to be aware of other aspects of a phenomenon that a learner cannot perceive beforehand and the relationship between the learner and phenomenon become transformed through altered perceptions.

However, it is difficult for teachers to attend by themselves to phenomenon which they ignore or is unfamiliar. As a result, they require external help to change their attention. There are various ways to facilitate teachers' reflection in teacher education, but teacher noticing that can strengthen teacher awareness through intended and systematic reflection, as used in the present study. According to Mason (1998), noticing may be considered as psychological-reflection, which starts with a technical level of reflection and then 'moves inwards towards sensitizing oneself (with the aid of colleagues) to notice situations in which alternative actions are possible, and them changing practices by choosing to act differently' (p17). It is assumed that noticing based on systematic reflection may help teachers perceive other aspects of their teaching practice which they are not aware of beforehand and make an effort to change their actions, and practical knowledge with newly-perceived ways can be constructed. In order to apply noticing to the present study, the next section identifies some implications from the studies on mathematics teachers' noticing.

2.2. Noticing

To develop teachers' professionalism, some researchers in mathematics teacher education have recently paid attention to teachers' noticing skills. Studies on teacher noticing in mathematics teacher education have not yet been actively conducted, but previous studies can be classified into two strands: one is about conceptualization of teacher noticing; the other is about development of noticing skills.

In the first strand of conceptualization of teacher noticing, there are slightly different definitions of noticing, but researchers define the term based on two main processes of noticing: attending to a particular event in a teaching situation and an understanding of the attended event. The operational definition of noticing is applied according to the aims of each study. The former focuses on investigating what teachers attend or ignore in complex and dynamic classroom situations (for example, Alsawaie & Alghazo, 2010; Star & Strickland, 2008), and the latter encompasses teachers' interpretation of details attended by them and their reactions (for example, Jacobs, Lamb, Philipp & Schappelle, 2011; Schifter, 2011). The present study follows the latter definition, which allows considering noticing as something that can be exposed throughout conscious and subconscious levels (Mason, 2002) and providing teachers with opportunities to be aware consciously of their actions to some extent through explication of what they notice and how they notice.

The studies in the second strand are reviewed and some limits of the studies are identified, and in order to use noticing as a strategy for construction of practical knowledge, some suggestions to deal with those limits identified from the review are presented as follows.

First, most of previous studies attempt to develop teachers' noticing skills using a video-based analysis method (for example, Alsawaie & Alghazo, 2010; Sherin, Russ et al., 2011; Star & Strickland, 2008; Star et al., 2011; Stockero & Zoest, 2013; van Es, 2011). This method may benefit the observation of natural developmental processes in terms of teachers' noticing skills; however, if asking teachers to notice without any help, teachers may be more likely to attend to general aspects of the classroom, such as the whole class environment or teachers' movements. With this method, not only does it take time to notice significant classroom events from a professional perspective, but also teachers' noticing may depend on the researcher's scaffolding. Thus, it is difficult to improve teachers' noticing skills steadily. Because noticing can occur when teachers reflect on their actions and notice subtle differences in classroom phenomena. In order to trigger teacher noticing, other approaches, for example, relevant theoretical knowledge such as teaching and learning theories or specialized content knowledge or discussion with their colleagues should be considered.

Second, Jacob et al. (2010) investigated pre-service and in-service teachers' noticing ability of students' thinking and found that professional noticing ability did not develop with only teaching experiences. This finding implies that coherent and systematic opportunities are needed to develop teacher noticing ability. It is not easy for teachers to see things that were not perceived by them before without any guidance, thus in order to trigger teachers' professional noticing, systematic dimensions about what to notice in a dynamic classroom situation need to be given to teachers.

Third, since things that are not noticed by people cannot affect their actions, in order to change actions or behavior, senses to notice various aspects of professional practice need to be broadened and developed (Mason, 2002). It is thought that to develop teachers' orientation to notice critical events which affect student learning, various perspectives on mathematics, pedagogy and psychology may play an important role in changing teachers' practice.

Sherin et al. (2011) found the fact that teachers are inclined to notice things by continuously comparing the classroom situation to what they expected to occur before class. This finding suggests that if teachers make a detailed and organized plan for a lesson and have a chance for thought experiment before class, they are more likely to notice things, in actual class, intentionally and deliberately based on what they expected before. It is assumed that teachers' orientation to notice critical incidents while in class can be encouraged by a thought experiment before class. It is therefore important to guide teachers to experience noticing in a systematic and structured way.

Fourth, as aforementioned, noticing enables teachers to become sensitive to notice opportunities to act in an alternative way and perceive things that teachers are not aware of beforehand, thus noticing can be used as a strategy to change teachers' practice. It means that noticing is triggered as a result of interaction between thinking and actions and is considered within teachers' practice. However, there is a tendency to train teachers to learn noticing skills in mathematics teacher education, separating noticing from teachers' actual classroom practice. To develop teachers' noticing skills, teacher education programs should include a process that teachers experience noticing while teaching their own actual class.

To sum up, in Section 1 of this chapter, it is identified, based on review of previous studies on practical knowledge and the discussion on tacit knowledge in Chapter 2, that teachers' practical knowledge has both explicit and tacit dimensions and the three components are necessary to construct mathematics teachers' practical knowledge. It is also emphasized that in order to improve teachers' practice and construct appropriate practical knowledge, teachers need to be consciously aware of tacit dimensions, because tacit dimension of practical knowledge is not explicitly expressed but significantly influences teachers' teaching practice.

According to the discussion on teacher awareness in Chapter 2, awareness is closely related to teacher knowledge and personal experience and teachers' awareness is reflected in their actions while teaching. As Mason (2008) states, teacher awareness plays an important role in attending to aspects of practice, interpreting and making on the spot decisions but teacher awareness remains tacit without a conscious effort to explicate teachers' own actions. Thus, in Section 2, reflection and noticing are suggested as strategies to achieve conscious, unconscious teacher awareness. It is assumed that teachers can be consciously aware of their unconscious teaching practice through reflection and noticing skills and have experience in acting in a new way by reflecting on their consciousness in actual class. Teachers can construct their practical knowledge through the cyclic process of consciousness and practices. That is, teachers act consciously in a new way and their actions become habitual as relevant experiences are accumulated.

Considering the three components for construction of mathematics teacher practical knowledge and the four suggestions to develop teacher noticing skills, the next section presents specific teacher training procedures for the construction of mathematics teacher practical knowledge.

3. Teacher training procedures for the construction of mathematics teachers' practical knowledge

Specific procedures for the construction of mathematics teachers' practical knowledge are suggested in this section. The procedure consists of five phases, starting with learning theory and then designing tasks, undertaking a thought experiment, conducting a class and analyzing the class with colleagues. A rationale for each phase is explained below.

3.1. Learning theory

Many teachers tend to attribute students' low performance to a lack of students' learning ability and motivation. Such tendencies by teachers makes them focus more on learning activities and strategies that could intrigue their students, rather than focusing on students' thinking and learning processes. Such teachers' beliefs, in which students' interest and involvement is a necessary and sufficient condition for student learning, exist tacitly behind teachers' teaching actions and may cause teachers' limited view of student learning (Nuthall, 2004).

In order to change such teachers' beliefs, teachers need to recognize that their understanding of students' thinking processes and their appropriate responses can significantly influence their students' learning.

Many researchers argue that teachers need to see the necessity to reflect in their own teaching and to change it, and theory is considered as a useful tool to observe and interpret teachers' teaching and leads to a stable and persistent change of their practices (Mellon, 2011; Mason, 2002; Tsamir, 2008). Rhine (1998), for example, reported that teachers who participated in Cognitively Guided Instruction (CGI) changed their orientation by getting continuous opportunities to investigate their students' thinking based on theory.

As discussed in the previous section, teachers' professional noticing ability can be developed not only by their teaching experiences but also theoretical knowledge. Because teachers' noticing is not something which happens all of sudden (Mason, 2011), but something which happens through trigger or stimulus, theoretical knowledge can play an important role in noticing some critical events during class and improving teachers' noticing ability.

Utilizing theory in teacher education can be useful in that theory can help teachers interpret and understand students' thinking and function as the lens by which teachers can observe and reflect on their teaching practices. It is presumed that in order to lead teachers to reflect on their previous teaching practices and notice subtle changes in classroom phenomena, theory should be a fundamental condition for the construction of teachers' practical knowledge. Therefore, in the present study, learning mathematical content knowledge and teaching and learning theory is set as the first phrase of the teacher training procedure.

3.2. Task design

Significant attention has been paid to the design of mathematical tasks in mathematics teacher education recently. Indeed, the importance of activities related to mathematical task design has been gradually emphasized in teacher education programs for the development of teachers' professionalism. Biza, Nandi and Zachariades (2007) and Potari (2013), for example, claim that teachers can develop their sensitivity of students' thinking, their awareness of mathematical structure and tasks' influence by participating in task design and its implementation activities.

In this section, the discussion that task design activity is appropriate as the second phase of the teacher training for the construction of mathematics teacher practical knowledge is presented in terms of two aspects. One aspect is related to how teachers use their mathematical and pedagogical knowledge during task design. The other aspect is related to how task design activity can lead teachers to notice students' learning opportunities.

First, teachers actively use their mathematical and pedagogical knowledge consciously or unconsciously in the process of mathematical task design. Many studies show that mathematics teachers use their knowledge to design a task (Crespo & Sinclair, 2008; Lee, Lee & Park, 2013; Liljedahl et al., 2007; Prestage & Perks, 2007). Utilizing teachers' knowledge to task design does not mean simple application of the knowledge because utilization of their knowledge should be considered within their teaching practice, so personal elements such as their beliefs and values may be involved in designing a task.

Chapman (2013) especially defines mathematics teacher knowledge related to a task as teacher's mathematical-task knowledge and emphasizes teachers' task design ability and the development of teacher's mathematical-task knowledge. According to Chapman (2013), mathematical-task knowledge for teaching is related to the following knowledge: understanding the nature of worthwhile tasks; identifying, selecting and creating tasks that are rich mathematically, pedagogically and personally for students to afford the learning of mathematics meaningfully; knowledge of levels of cognitive demands for tasks and goals for the task; knowledge of students' understanding, interests and experiences; an understanding of how the tasks teachers select and how they use them to influence how students come to make sense of mathematics; knowledge of what aspects of a task to highlight, how to organize and orchestrate the work of the students (p. 1-2). As can be seen in the list of relevant knowledge, teacher's mathematical-task knowledge is the knowledge that teachers apply from meta-perspectives.

Crespo (2003) showed that there was a relation between teachers' task design activity and their beliefs toward mathematics teaching and learning. When asking pre-service teachers to pose mathematical tasks in the beginning of the problem posing practices, they mostly made computational problems, single steps problems that led students to correct answers and problems that avoided students' errors and ambiguity. However, in subsequent problem posing practices, they tried to pose cognitively more complex and exploratory problems. Through the problem posing practices, pre-service teachers especially changed their views and beliefs toward mathematics teaching and learning, and they especially tended to pose more challenging problems that led students to think mathematically by using students' errors and incorrect work. The pre-service teachers' changes through the problem posing practices indicate that the teachers changed their focus from their own view of problems to students' views. The change of teachers' focus seemed to play an important role in shifting teachers' beliefs. According to Mason (2011), the shift of teachers' focus is the change that should accompany an improvement of teacher noticing ability.

Summing up, teacher knowledge is actively used in the process of task analysis and task design and the knowledge used by teachers is not merely theoretical and static knowledge, but related to real teaching practice. It is interwoven with teachers' beliefs toward mathematics and its teaching and learning, teachers' goals and their knowledge. It is thought that task design activities can provide teachers with opportunities to see their knowledge from a meta-perspective and help teachers to develop their awareness.

Second, mathematical tasks are not for merely solving and checking correct answers, but for providing various learning opportunities and promoting mathematics learning. Indeed, many researchers stress the important role of a mathematical task as a mediator that links the teaching to learning (Henningsen & Stein, 1997; Kullberg, 2010; Stenin & Lane, 1996; Sullivan et al., 2011; Watson & Mason, 2005).

The important thing for a task to positively influence teaching and learning processes is that teachers who implement the task should be aware of the nature of the task and potential learning opportunity inherent in the task. Stein and his colleagues found that task features and cognitive demands might change in the class implementation due to various factors (Henningsen & Stein, 1997; Stein & Lane, 1996; Stein et al., 1996). Teachers' goals and their knowledge of mathematics and students affected how teachers implemented mathematical tasks and teachers' orientation and students' leaning propensity affected students' performance (see Figure 3). In fact, many studies have shown that teachers did not provide proper learning opportunities to students due to lack of teachers' awareness of tasks' potential or teachers' orientation (for example, Kullberg, 2013; Sullivan et al., 2010; Sullivan & Mousley, 2001).

As shown in many studies, learning opportunities inherent in tasks are not consistent in the design and implementation, rather they are changed by many factors and thus teachers can notice the change of learning opportunity sensitively and provide proper opportunities to students only if teachers perceive learning opportunities. However, it is not easy for teachers to notice subtle differences or changes in the moment and properly deal with that because a real class situation is too complex and dynamic. Because of this, teachers' decision making in the moment and their actions affect students' learning significantly. As a results, teachers can take proper action if they have sufficient situational sensitivity to notice learning opportunities.

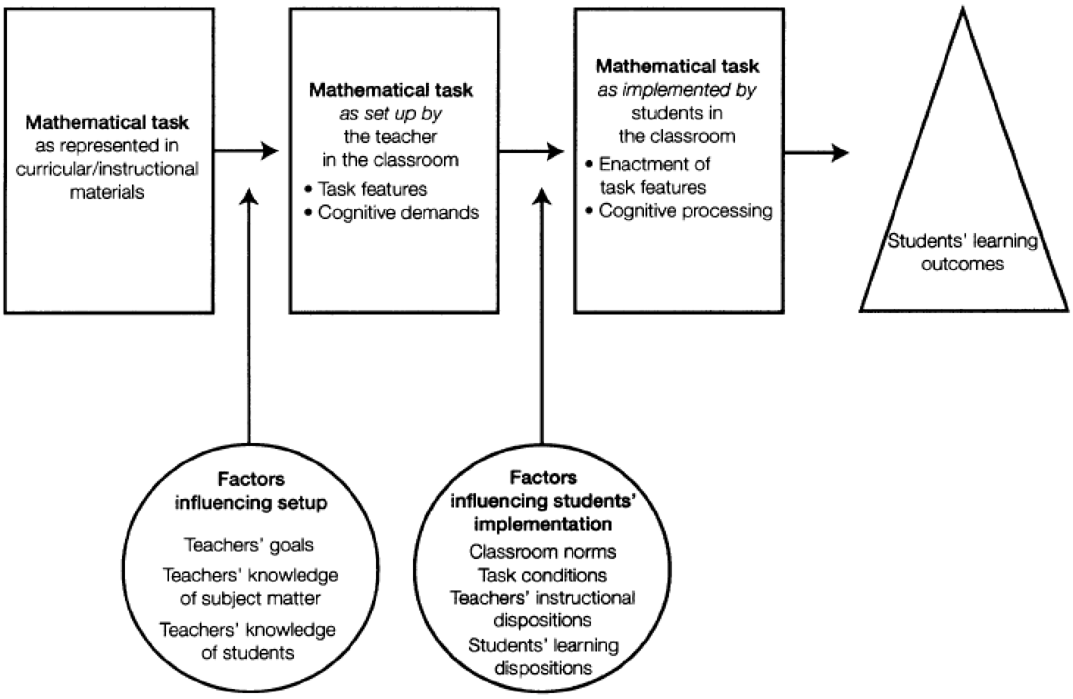

Figure 3. Relationship among various instructional task-related variables and student learning (Stein & Lane, 1996, p. 56)

Student learning may not occur in natural and unintended settings, and teachers' clear intention and goals can increase students' learning potential. In this way, task design activity seems meaningful in that it can help teachers make their intention and plan clearer to notice learning opportunities for students.

Teachers need to be aware that in a real class situation, various factors, such as their knowledge, beliefs and interaction with students, may cause a change in potential learning opportunities which are inherent in a task. Teachers also need to continuously observe and attend to students' responses and notice a certain situation where teachers can give proper learning opportunity to students.

In conclusion, task design, as the second phase for construction of mathematics teacher practical knowledge, focuses on teachers' systematic task design with their clear intentions and goals.

3.3. Thought experiment

The importance of a thought experiment has been emphasized by many researchers in mathematics teacher education (Hiebert, Morris & Glass, 2003; Davis & Sumara, 1997). Thought experiment helps teachers conduct their class systematically and deliberately by anticipating students' learning trajectory and imaging the lesson process in more detail.

Mathematics teachers have coherent beliefs and attitudes toward mathematics and its teaching and learning, as they are experienced in teaching. These personal elements significantly influence teachers' practice tacitly. It is difficult to change teachers' beliefs because it plays a role in filtering external information entering teachers, and thus to change teachers' practice is very difficult too. It is not enough to ask teachers to merely apply their theoretical knowledge, and they should be guided to consciousness and reflection on their teaching action.

The necessity of thought experiment was previously proposed in the discussion about the ways to develop teacher noticing ability. Noticing occurs based on one's experience and thus it is not unintentional and unplanned (Mason, 2011). Rather, it is an intended action which occurs subconsciously. Systematic thought experiment seems necessary for teachers to notice things that are valuable and meaningful for student learning in actual class.

Teachers need alternative actions and sensitivity to situations where those actions can be taken in order to act - not habitually- but in a new way in a real class. As aforementioned, noticing is not an unintended and unplanned action; it occurs based on experience and is sensitized to a certain situation. Because noticing is related to teacher knowledge and teaching experience, if teachers act in a new way other than older way, they need to be sensitized to some situations where they can act in an alternative way by imaging, reconstructing and reflecting in the situation through thought experiment.

The most important thing for teachers to notice things professionally and to act in a new way is systematic thought experiment. When undertaking thought experiment, teachers need a structured guide about what to attend and thus this present study provides participating teachers with three dimensions to be attended to when conducting thought experiment.

The three dimensions for thought experiment are derived from the teaching triad as proposed by Potari and Jaworski (2002) and other studies concerning aspects which mathematics teachers should be aware. Potari and Jaworski defined the concept of the teaching triad in the process of identification of general characteristics of investigative mathematics teaching (See Figure 4). They identified generalized characteristics of complex mathematics teaching, defined three activity domains in which teachers engaged and suggested the teaching triad as a tool for planning and reflection on a class. First, management of learning describes the teacher's role in the constitution of the classroom learning environment, such as planning of tasks, activities and establishing norms. Next, sensitivity to students describes the teacher's knowledge of students and attention to their needs and the ways in which the teacher interacts with individuals and guides group interactions. Finally, mathematical challenge describes the challenges offered to students to engender mathematical thinking and activity, such as tasks set, questions posed and emphasis on metacognitive processing.

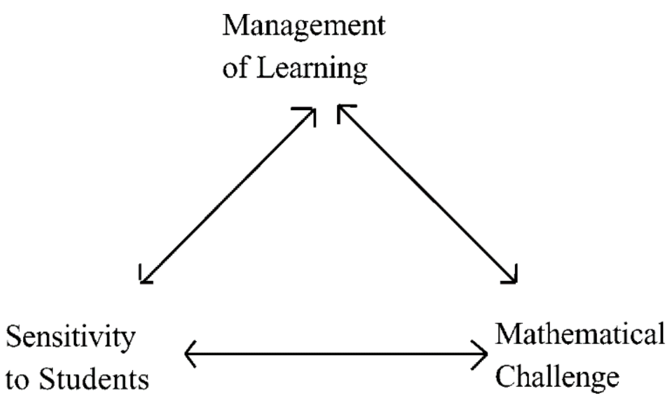

Figure 4. The teaching triad. (Potari & Jaworski, 2002, p. 353)

Cobb and Steffe (1983) provide new perspectives about teachers' pedagogical content knowledge. He mentions that teacher knowledge for teaching mathematics should be based on how students construct mathematical ideas and that knowledge can be generated from the psychological domain rather than the mathematical domain. It emphasizes that in order to teach effectively, teachers should not merely convey mathematical content knowledge to students, but should take into consideration students' misconceptions and their developmental level. The psychological domain outlined by Cobb and Steffe and the sensitivity to students of the teaching triad seems to have something in common. In order to deal with the sensitivity to students more broadly by considering students' misconception and their developmental level, the sensitivity to students is considered as the psychological domain in the thought experiment proposed by this present study.

As many researchers have emphasized, it is beyond doubt that teachers should have profound mathematical content knowledge. Considering the relationship between mathematical content and tasks, the nature of tasks should vary, depending on mathematical content. Thus, when teachers design a mathematical task they should understand whether the mathematical content to be taught requires procedural knowledge or conceptual understanding. According to Mason (2008), all mathematical topics are based on essential actions that students perform during class. For example, the action of distinguishing a repetitive unit is the essence of counting and measurement and the action of combining and separating is in essence an activity of addition and subtraction (Mason, 2008). A teacher who has a lack of awareness and profound understanding of mathematical topics to teach may have a difficulty in selecting proper tasks and teaching strategies. Thus, teachers need to consider, in the thought experiment phase, mathematical dimensions, such as the structure, representation and characteristics of mathematical concept. Mathematical challenge in the teaching triad is related to tasks and teachers' questions to promote students' mathematical thinking and activity. Teachers' profound understanding about mathematical topics must take precedence in order to develop students' mathematical thinking through proper tasks and questions. Thus, the mathematical challenge described in the teaching triad is considered as a mathematical dimension in this present study to encompass understanding of mathematical topics and task design and learning activities to be taught.

Lastly, management of learning in the teaching triad presents the role of teachers in construction of the classroom learning environment, such as task planning, activities and establishing norms. It is thought that management of learning should be considered by teachers in a dynamic classroom situation and additionally, establishing a classroom environment which encourages students' explanation, justification and desirable communication is also important.

Management of learning is suggested as one of the three dimensions that should be considered in the thought experiment phase.

To sum up, teachers will be guided to undertake systematic thought experiments by considering the mathematical dimension, the psychological dimension and the management of learning dimension. It is intended that, through the thought experiment with the three dimensions, teachers develop their awareness by attending to and interpreting complex and dynamic teaching situations structurally and purposefully.

3.4. Conduct of class

In this phase, a teacher conducts a class that is planned through task design and thought experiment phases. The teacher carries out the class by continuously observing the classroom situation to notice learning opportunities for students and identifying situations to act alternatively based on their own experience and through a thought experiment. This results in the teacher imagining a certain situation where he or she can act in a new way rather than habitually. The teacher is encouraged to ask herself what she attends and whether there are any situations to act in a new way during class.

3.5. Analysis of class with colleagues

After each class, the teacher watches a videotaped class with her or his colleagues, and reflects and discusses whether the teacher sensitively noticed issues during the actual class, which the teacher noticed during the task design and thought experiment phases, whether there are any moments to act alternatively and whether any unexpected situations occurred. It is expected that, by reflecting on class, the teacher can have a chance to be conscious of his or her teaching practice by herself or himself or by colleagues. It is thought that sharing ideas and discussing these with colleagues is helpful for the teacher to undertake more meaningful reflection and consciousness because each teacher has a different experience and deals with a certain situation in different ways.

Teacher training procedures for the construction of mathematics teachers' practical knowledge has been described so far and the diagram of the procedures is shown in Figure 5. Because teachers' practice is difficult to change through only one execution, the phases of teacher training are repeated. What is emphasized in the process of constructing mathematics teachers' practical knowledge as suggested in the present study is that the three components for the construction of practical knowledge should be combined through the process of consciousness (turning subsidiary awareness into focal awareness) and unconscious (turning focal awareness into subsidiary awareness). It is expected that the five phases for teacher training, which are learning theory, task design, thought experiment, conducting a class and class analysis, can provide teachers with opportunities for systematic and coherent noticing and reflecting to help teachers construct their practical knowledge with an organic combination of the three components.

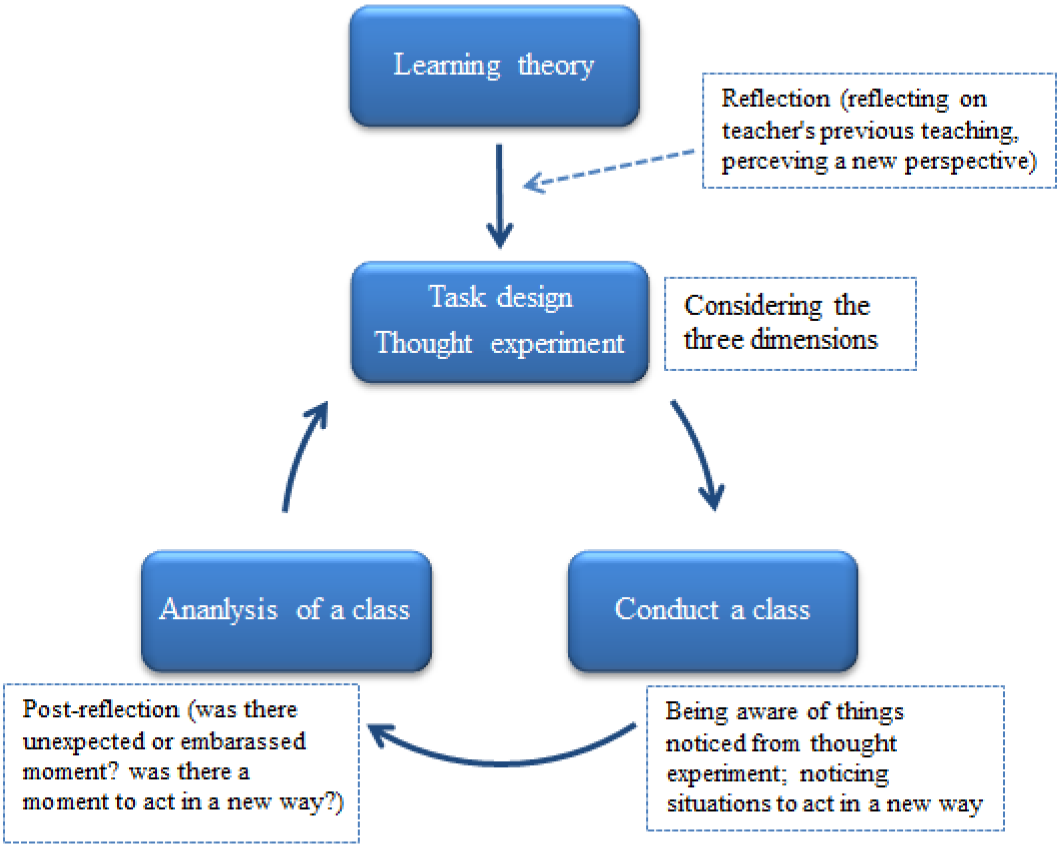

Figure 5. The procedures of the construction of mathematics teachers' practical knowledge

CHAPTER Ⅳ

THE CONDUCT OF THE RESEARCH

In this chapter, the procedure of teacher training for the construction of mathematics teachers' practical knowledge is applied to an in-service teacher and the process of constructing the teacher's practical knowledge is analyzed along with the changes in the teacher's teaching practice. It was identified from Chapters 2 and 3 that teachers' practical knowledge has both explicit dimension and tacit dimension and teachers need to be conscious of their unconscious teaching behaviors in order to construct appropriate practical knowledge. It was also found that teachers' noticing and reflection could be used as a strategy that allows teachers to be conscious of their unconscious awareness. To construct practical knowledge, teachers also need to go through the process of their conscious actions becoming habitual stable again. As teachers conduct their conscious activity and strengthen their awareness of it, their conscious activity becomes habitual. This chapter focuses on teachers' consciousness and the changes in the teachers' teaching practices. This chapter consists of the following five sections:

First, the participating teachers and the procedures of this study are described. The second section provides the analysis of what the teacher notices in the task design and thought experiment phases, after acquiring knowledge related to a didactical analysis of similarity and variation theory, and how the explicit knowledge influences the teacher's noticing. Next, how teacher's noticing in the thought experiment is reflected in the actual class is investigated by classifying into two types, the cases in which the teacher's noticing is reflected and the cases in which it is not. In the fourth section, what the teacher notices in the phase of reflection on class is analyzed through the teacher's consciousness by herself or by her colleagues.

1. Method

1.1. Research participants

Three middle school teachers participated in the present study. They worked at the same school, which was located in a metropolitan city in Korea. They are all females. One teacher who was a subject of this study (called A in this study hereafter) was in charge of grade 8, and the teacher's two colleagues (called B and C hereafter) were in charge of grade 7. Teacher A had 12 years' experience, teacher B had 18 years' experience and teacher C had 7 years' experience. All three teachers were interested in improving their teaching practices and had a regular meeting with colleagues who worked at other middle schools. Teachers B and C were in charge of administrative work related to education information services and spent most school time in the education information services room, which enabled the three teachers to discuss their classes in an atmosphere of freedom.

The three teachers had experiences in different national and regional teacher training programs. Teachers B and C, who were teaching in same grade, were trying to apply a new teaching method learned from teacher training programs or mathematical tasks obtained from other colleagues by modifying them for their classes. Teacher A, however, had participated in various programs with B and C, but unlike them was teaching 8^{th} grade, and thus had a difficulty applying new methods to her class by herself. Teacher A wanted to improve her class but had no colleagues teaching in the same grade with whom to cooperate in designing a task or class, so she did not actively try to apply new methods or design tasks.

Teacher A was the subject of this study who conducted a class, but the other two teachers participated actively in task design and reflection on A’s class. The participating class was a mixed class and classified by level. The participating students were in the middle-low level. Teacher C had been charge of high-level students and was teaching the middle-low level students during the first semester of 2013. This present study was conducted in the second semester of 2013.

1.2. Data gathering process

This study was based on data of Teacher A's teaching practices of five mathematics classes that were conducted in November 2013. Teachers A, B, and C had meetings 2 times for about 2 hours each for learning theory and 5 times for about 2 hours for task design in October. The researcher on this present study participated in all phases of the teacher training, sometimes as an observer and sometimes as a participant, and recorded all classes.

The experiment was conducted according to the procedures for teacher training suggested by this study. The task design and thought experiment phases were separated in the procedure, but were actually conducted simultaneously by modifying a task with the thought experiment.

In the first phase, the participating teachers learned the knowledge related to a didactical analysis of similarity and variation theory. The researcher provided teachers with articles on a didactical analysis of similarity and variation theory and discussed relevant knowledge with the teachers. The researcher triggered the teachers to think about the differences and similarities between their teaching methods and learned knowledge. A brief explanation of variation theory and the reason why variation theory was chosen for this study is provided in the next paragraph.

Variation theory was developed based on phenomenography by Marton and his colleagues and was concerned with how we experience and learn about our surrounding world (Marton & Booth, 1997; Marton & Tsui, 2004). Variation theory had a premise that we are more likely to attend to changing aspects when some aspects are changed and others remain invariable. From the viewpoint of variation theory, learners can differentiate a certain aspect of the object of learning. The main ideas of variation theory are critical features and dimensions of variation. Critical features are things that a learner should differentiate in order to learn a certain concept or ability and can be founded by pre- and post-tests about the concept to be learned or previous studies on the concept. For example, if addition and subtraction of negative numbers is an object of learning, one critical feature of the object of learning can be the different meanings of the operational sign for subtraction and the sign for a negative number. The dimension of variation leads learners to differentiate the critical feature is subtraction or a negative number (Kullberg, 2010). In this case, teachers should guide learners to attend to the critical aspect and differentiate between the subtraction sign and the negative number sign by designing tasks with subtle changes using variation and invariance.

Variation theory allows teachers to create learning conditions by designing tasks systematically according to their assumptions about learning. In other words, variation theory can be a useful tool for the design of tasks as well as reflection in the class because teachers can guide students to attend to critical aspects of a learning object with systematic task design using variation and invariance. Variation theory was used as a tool for task design in this study. More specifically, variation theory was used for the design of optimized tasks for students by modifying tasks and providing suitable examples with proper sequences, considering the anticipation of students' responses before class and reflection in their responses after class. To design a mathematical task and class using variation theory can help teachers express their clear intentions and purpose in the task and design with consideration of students' misconceptions and errors and proper reactions to them. Variation theory can also be a useful tool for the analysis of a class because what students actually learned can be analyzed by comparing the purpose and intention of a task and the students' actual responses (Kullberg, 2010). The present study focuses on guiding teachers to express their intentions for and assumptions about student learning consciously in the design of tasks and classes, especially to understand students' learning. Thus, it is thought that variation theory that enables teachers to design tasks systematically with their clear intention corresponds to the focus of the present study.

Participating teachers seemed to learn variation theory relatively quick. It is presumed that variation theory is suitable for Korean teachers who tend to pay more attention to mathematical content itself than to the context of a problem.

Teacher C seemed to have difficulty designing a task in the beginning of the experiment because she did not have experience of designing a task. However, the teacher adapted to task design through the discussion of her ideas for task design with her colleagues.

The second and third phases were task design and thought experiment. Teachers were asked to share students' possible misconceptions and errors related to similarity based on their experiences or relevant previous studies and take those things into consideration when designing tasks. The teacher was guided to conduct a thought experiment based on mathematical, psychological and management of learning dimensions in these phases. The teacher was asked to take into consideration of her intention for a task, the expected students' responses and her reactions to them in a thought experiment. Data about the teacher's thought experiment was collected from the written papers and audio recording of the teachers' meetings. In the task design phase, the researcher of this study was a supporter who suggested helpful research papers for the teachers and shared ideas. However, the main ideas about the task construction were decided by the participating teachers, and the researcher played a role in mediating and sharing the teachers' opinions in the process of developing the ideas. Tasks and activity papers for the five classes were prepared in advance and revised according to students' responses after each class.

The next phase was the conducting of a class. Teacher A conducted the classes with the designed tasks and activity papers. Five classes were conducted for this study. A camera was set up in the back of the classroom to observe the whole classroom situation and the other camera were handled by the researcher to observe students' group and individual activities.

The final phase was the analysis of a class together. Teachers A, B, and C had a meeting to analyze a class by watching the videotaped class. Teacher A's short interview about a class was conducted informally whenever she had time. The analysis of a class was conducted retrospectively after school or on Saturdays.

1.3. Analysis of data

The present study aims at suggesting the procedure of teacher training for the construction of mathematics teachers' practical knowledge and understanding the process of constructing practical knowledge by applying the procedure to an in-service teacher. First of all, a literature review was conducted to understand the meaning of tacit knowledge and derive the procedure of teacher training for teachers' practical knowledge construction. In addition, a qualitative case study was used for this experiment to investigate the process of constructing practical knowledge in depth. A qualitative case study is to examine, analyze and interpret one case and to understand the case collectively by grasping contexts and critical factors related to the case (Kim, 2012). A qualitative case study seemed suitable for the purpose of this experiment, which was to understand the process of constructing the participating teacher's practical knowledge by examining and analyzing the whole procedure of teacher training.

Data for analysis was collected from the written documents of the teacher's thought experiments and audio recordings, videotaped classes, audio recordings for the analysis of classes and short interviews with the teacher. Based on the collected data, this study analyzed how teacher's noticing was reflected in class by comparing the teacher's noticing in the thought experiment and in the actual class. Aspects of the teacher's noticing were identified from the audio recordings of the teachers' discussions of task design and Teacher A's written documents about the thought experiments. Then the aspects of the teacher's noticing were classified into three dimensions, which were mathematical, psychological and management of learning dimensions.

This study was also focused on how the teacher's noticing in the thought experiment could facilitate the teacher's understanding of students' thinking and her noticing of learning opportunities by identifying whether the teacher took conscious actions in the actual class that were considered in the thought experiment. Thus, the teacher's noticing in the thought experiment, the conduct of class and the analysis of class were classified and compared in terms of the three dimensions. The results of this research are presented in the following sections.

2. Task design on similarity of figures and teacher's noticing in thought experiment

In this section, teacher's task design about the chapter of similarity in grade 8 and her intention of the task design is described and then teacher's noticing in the task design and thought experiment is analyzed based on mathematical, psychological and management of learning dimensions to answer the second research question of the present study.

2.1. Purpose of task design

In the beginning of the chapter on similar figure, a multiple pictures, which have been digitized from the photographs of characters, are provided for the students to identify the most similar figure to the original photograph. In this task, students observe the changes between the original and modified pictures and draw a conclusion that a modified picture, which has been extended both horizontally and vertically in equal ratio from the original, is the most similar to the original picture. After this introductory task, a definition for similar figures is provided, which explains that two figures are similar when one figure is enlarged or reduced in fixed ratio to be congruent to another figure, and the two figures are named similar figures. Following the definition, a symbol for similarity is provided as well as properties of similarity in plane figures and solid figures, and relevant problems are given. Teachers pointed out that the introduction in the textbook only provides a minimum opportunity for observation on similar figures by providing definition and properties, which rids the students of the time to think, and encourages them simply learn the facts to solve various kinds of problems.

Teachers considered that the word “similarity” is commonly used in daily life, so that the students intuitively know the concept of similarity even though they did not learn it formally. Therefore, the teachers reckoned that students should begin from their intuitive knowledge on similarity and move to recognize the difference between the ordinary meaning of the word and that of mathematics. After that, the students can ponder upon how to utilize the word to indicate a mathematical similarity. Teachers considered that the task should be designed so that the students can be provided with an opportunity to conjecture and confirm the mathematical meaning of the word in order for them to refine their intuitive perception of the word.

In teacher A’s previous way of teaching, the teacher explained the contents of the textbook in the order that was provided in the textbook. However, after the teacher learned knowledge related to a didactic analysis of similarity and variation theory, she began to be aware of the difference from her old way of teaching. The teacher recognized that students need to be given a sufficient amount of time to explore the concept of similar figures before learning the definition of similarity, and in the actual task design and thought experiment phases, the task was materialized in a way that the teacher intended using variation theory. Variation theory provided a substantial help for the teacher to design the task in which the examples used in the task were systematically changed so that the students can observe and explore the figures and focus on the essence of the concept of similarity. Additionally, it was utilized to view and understand the task from the students' perspective.

2.2. Noticing during the task design and thought experiment

Task design and thought experiment for the first class

The teacher planned to discuss the everyday meaning of the word "similarity" with the students in the first class. The teacher will mention that the objective of this task is to discuss the everyday meaning of similarity and learn to distinguish its difference from mathematical meaning of similarity. Different pairs of figures are shown to the students who are asked to make intuitive decision on their similarity and provide reasoning. The order of presentation of pairs of figures will begin with triangles and rectangles with different shapes of which sides and angles will be changed gradually. It was intended for students to recognize the properties of similar figures –invariance of angles and consistent ratio of sides- by changing the figures little by little. Teacher A and her colleagues expected that, although the students cannot precisely define similar figures, they will be able to sense the properties of similar figures by observing and conjecturing the similarity of presented figures. Lastly, the students are asked whether or not the two triangles of the same size are similar figures, which will introduce them to the misconception that congruence and similarity are not equal. In the first class, only the question of whether or not congruence is similarity will be posed without the exact definition. This will be elaborated further in the following classes as the concept of similarity is discussed in more details.

Task design and thought experiment for the second class

In the second class, the students are provided with more opportunities to draw similar figures. This activity encourages the students to draw enlarged or reduced figures with the concept of similarity from the previous class and it is designed to solidify the notion that the angles are unchanged and the ratio of the sides is consistent. Through the drawing activity, the students are more familiarized with the properties of similar figures and this will be followed by the definition and introduction of the symbol for similar figures.

The similar figure drawing activity from the 1st task allows methodological variation by providing the students with various mathematical tools to draw similar figures in different ways. The teacher distributes the prepared paper triangles to the students and asked the students to draw the triangle enlarged or reduced by two times by using the provided ruler, compass and protractor. The teacher forecasted that the students will draw the figures using a variety of methods and will draw similar figures even though they do not recognize the conditions for similar figures. The teacher projected that, although the students will draw similar figures in many different ways, they will acknowledge that they drew similar figures –enlarged or reduced- by maintaining the angles and ratio of sides.

In addition, the teacher expected that some students will be able to draw a twice-enlarged triangle by connecting four of the provided triangles and this way of drawing will provide a transition to the concept of the ratio of area. The concept of the ratio of area will be mentioned again in Task 2 as the students enlarge and reduce the door of the provided house figure. In the textbook, the ratio of area and volume is covered in utilization of similarity, “Area and volume of Similar Figure”. However, many students mistake the ratio of similarity with ratio of area and volume. Therefore, the teacher reckoned that, rather than separating the lesson on ratio of area and volume from the ratio of similarity, mentioning them together in the lesson on ratio of similarity will help the students to clearly understand the distinction. The teacher, therefore, expected that students will learn the difference between the ratio of area and similarity by putting together paper triangles that were provided, thereby understanding the concept of area ratio.

Task design and thought experiment for the third class

In the beginning of the 3rd class, the teacher mentions similar figure drawing activity on graph paper, which was covered in the 2nd class. This is to remind the students of the previous lesson on properties of similar figures and make them think once again about how to draw similar figures on graph paper. After that, students are taught that when enlarging or reducing similar figures, they need to make modification equally on all sides or they can enlarge or reduce the grid when using graph paper. In other words, when they draw similar figures, they can enlarge or reduce the grid in width and height in the same ratio to draw similar figures. During this activity of enlarging or reducing the grid, the students, who do not fully understand the concept of similarity, will enlarge or reduce the grid in unequal ratio. At this point, the teacher can once again have opportunity to make the students aware of the properties of similar figures. Throughout the second and third classes, the students will participate in similar figure drawing and they will recognize the properties of similar figures that, although there are many ways to draw similar figures, the angle remains the same and the ratio of length should be consistent.

After making the students recognize the properties of similar figure by drawing similar figure, the teacher explains that the previous lessons were on similarity of plane figures, and the lesson proceeds to similarity of solid figures. For this lesson, the teacher prepares cube wooden blocks and distributes them to students. The students are, then, asked to create solid figures that are enlarged by 2 times and 3 times. The intention is to make students understand the difference in numbers of cube blocks that are needed to create twice-enlarged and thrice-enlarged solid figures. Through this activity, the students recognize the difference in numbers of necessary wooden blocks and understand its connection to the ratio of volume. During this activity, the teacher will intentionally give less cube blocks than necessary so that the students will recognize that they need more cube blocks to enlarge the shape by three times and ask for more blocks. The teacher will, then, make the students to focus on how many cube blocks they need in order to enlarge the shape by three times. At this point, the teacher will encourage the students to compare the number of blocks needed to make the figure twice as large and three times as large and introduce the concept of ratio of volume.

When the lesson on similar figures of cube is finished, the students will begin making similar figures by enlarging triangular prism by two times. The teacher selected regular cube and triangular prism because cube is equal on all faces whereas triangular prism is different on its sides and upper side. The teacher reckoned that this will visually reinforce the students' understanding of the property of similarity of solid figures, which is that corresponding sides are similar figures. Through observation on enlarged similar figures of regular hexahedron and triangular prism, the students will understand the difference between twice and thrice-enlarged figures. Additionally, they will observe the difference between similar figures of regular hexahedrons and triangular prism to recognize the properties of similarity for solid figures and ratio of volume.

Task design and thought experiment for the fourth class

The task for the fourth class is on the basis to judge whether or not two figures of the same shape are always similar. This task was designed to reflect on the activities such as drawing and making enlarged or reduced figures and to consolidate the understanding of the properties of similarity for plane and three-dimensional figures. In addition, the teacher incorporated picture examples to help students, who are not familiar with mathematical language when they explain the reasons for similarity of two figures, and to help them to think of counterexamples when two figures of the same shape are not always similar.

The examples from the first task show the teacher's intention (see Figure 6). The teacher expected that the similarity of regular pentagon from 1) will be relatively easy to understand for the students. Additionally, the teacher expected that the students will consider the similarity of general polygon when they think about regular pentagon. That is, it is possible that they will generalize the similarity of regular pentagon to conjecture whether or not regular polygons are always similar. Isosceles right triangle, which was introduced in 4) can be compared to isosceles triangle to explore the reason why two isosceles right triangles are always similar. The two circles, which were introduced in 5), will be compared to two sectors, quarter circles, and half circles in 6) to discover whether they are similar and to consider the supporting basis. The central angle for sector can change; thus, two sectors are not always similar. However, the central angle for quarter circle and half circle is pre-determined; thus, the students will recognize that they are similar by comparing the central angle. The following example is on the cylinder shape from 7). The students would identify in 5) that the bottom side is always similar; therefore, they may answer that the two cylinders are always similar. At this point, the teacher can encourage the students to recognize that the similarity ratio of bottom side can be different from the ratio of height.

1) Two regular pentagons	2) Two isosceles trapezoid	3) Two parallelogram
4) Two isosceles triangles	5) Two circles	6) Two sectors
7) Two cylinders	8) Two rectangular solids	9) Two tetrahedron

Figure 6. Task 1 in the fourth class

Task design and thought experiment for the fifth class

The task for the 5th class is designed by using activity sheet from the 2nd class in which the students drew twice-enlarged similar figure. In the 2nd class, the students were instructed to draw similar figures using diverse tools and some students drew similar figures using the center of similarity. In order to make the task interesting for the students, the task on center of similarity was designed by using the students' activity sheet. Through the presentation of student who drew similar figure by using the center of similarity, the students will explore about the properties of similar figures that were drawn. Task 2 was designed to highlight that students are often unable to find the location of similarity and that not all similar figures have the center of similarity, as well as the fact that congruence is a specific case of similarity. In task 2, the teacher presents different cases of the center of similarity by grouping them by whether the center is inside, outside or in the periphery, and also shows the cases in which similar figures are drawn inversely. By using GeoZebra, the students will observe how the figures are drawn as the center of similarity changes. Additionally, the students will be presented with how the similar figures are enlarged or reduced as the ratio changes, and they will be shown the overlap of two figures by adjusting the value to 1. At this point, the students will naturally say 1:1 and will accept that the two figures can overlap and are similar when the ratio is 1. After covering diverse types of center of similarity, the students will be introduced to two right triangles of which the height ratio increased equally, but do not have the center of similarity. The students will, then, be asked whether or not the statement is true, which asserts that since the two figures are not in the location of similarity, they are not similar. The two right triangles are certain to be similar; therefore, the students will think that the statement is false and naturally recognize that not all similar figures have the center of similarity.

Table 1. Teacher's noticing within mathematical, psychological and management of learning dimensions during task design and thought experiment phases.

Mathematical dimension	Psychological dimension	Management of learning dimension
Discerning (distinguishing between similar figures and non-similar figures, attending to sides and angles) Constructing (drawing similar figures using	Considering students' responses (attempting at anticipating students' response and preparing alternatives) Considering students' prior knowledge (prior	Organizing classroom activity (whole class discussion, group work, individual work) Interacting (interaction between the teacher and students, among students)

various mathematical tools and different methods) Measuring (size of angles, length of sides, area of figures) Recognizing relationships (the relation among the ratio of length, area and volume for similar figures) Generalizing (determining similarity of any two figures with same shape and justifying)	knowledge and prior activities for task design) Encouraging students' participation (designing tasks for observing and conjecturing) Demanding students' reflection on their action (questioning 'why') Dealing with misconceptions (congruence is not similarity, similarity figures should have the center of similarity)	Considering questions to facilitate communication of students and the teacher Asking to presenting a different way of solving a task

So far, we have discussed what teacher A considered in designing task and conducting thought experiment from the 1^{st} to 5^{th} classes, and it was subcategorized according to mathematical, psychological and management of learning dimensions as Table 1 indicates. As stated in the research process, the researcher of this present study proposed the teachers to design task and conduct thought experiment in consideration of mathematical, psychological, and management of learning dimensions, and induced the teacher to systematically and intentionally notice the dynamic and complicated teaching circumstances.

Based on the knowledge related to variation theory and a didactical analysis on similarity concept, the teacher recognizes the difference from their existing habitual and unconscious teaching activities. In this regard, learning theory, which is the first phase in teacher training for construction of practical knowledge, played an important role in raising consciousness of the teacher. In addition, the teacher attempted to more finely understand and reflect the students' perspective in task design and thought experiment based on the knowledge related to variation theory and similarity concept. By doing so, the teacher endeavored to design her classes more intentionally and consciously.

In the next chapter, I will analyze how the teachers' noticing, which they obtained during task design and thought experiment, is reflected in actual classrooms.

3. Noticing and reflection in class

In this section, it will be analyzed how the teacher A's noticing during task design and thought experiment is manifested in her execution. To do so, I will separately look at when the teacher's noticing, which was revealed in task design and thought experiment, is reflected in classroom and when it is not reflected based on teacher's behavior, routine and language.

3.1. When noticing from thought experiment is reflected

3.1.1. Inducing understanding of similarity concept by systematically proposing examples

1) In the beginning of the 1^{st} class, the students and the teacher discussed the everyday meaning of the word, “similarity”. The students said that they say things are similar when they are similar in appearance. After discussing this, the teacher showed the students how dictionaries define “similarity”. The teacher said to the students that they would first learn the everyday meaning of similarity followed by mathematical utilization of the word. After that, the teacher showed the students a pair of figures during task 1 of 1^{st} class and asked them whether or not they are similar figures and the supporting reason <Episode IV-1>. The intention of the teacher in Task 1 is to encourage the students to focus on the length of side and the size of angle.

First, the teacher showed the students equilateral triangle and quadrilateral for which the students could easily judge similarity or not. When the students were asked to explain why the two figures were not similar, the students explained that they are different in appearance. The teacher, then, continuously posed questions on the students’ use of the word “appearance”, so that they could slowly transition to its mathematical term. In line 30 of <Episode IV-1>, the students said that the two figures are different in appearance and demonstrated that they used the term “appearance” in everyday setting to indicate the concept of “similarity”. At this point, the teacher replaced the word “appearance” with “shape”, and the students begin to use the word “shape” after that. The students already knew the mathematical terms for a figure such as angle and side; thus, the teacher asked questions on how they could judge whether some figures are not similar, so that the students could answer by using more mathematical context. The students answered by stating that the number of side is different and they also mentioned the word, angle. When the students mentioned the word, angle, the teacher asks, “what about the angle? The number of the angles?”, to confirm the students’ understanding. The students answered the size of angle and clarified their intention. After the terms, size of the angle and sides, are mentioned, the students began to naturally use these terms to explain the reason for non-similarity.

<Episode IV-1>

21 Teacher: They look very so alike. We say it many times when they look resemble each other and we say this when? (…) Look, What we are going to learn today is similar figure. We use the word “similar” among resembling figures, and today we are going to find out whether the way we use the word similar in our daily life is same to the way it is used in mathematics or not. (…) What figures can you find here?

22 Students: An equilateral triangle and a square.

23 Teacher: Yes, an equilateral triangle and?

24 Students: A rectangle.

25 Teacher: It is not easy to tell what kind of rectangle this one is. It is a little confusing. Do you think they look similar in the 1st drawing?

26 Students: No, they are not similar to each other.

27 Teacher: Anyone who thinks they look similar?

28 Students: (Laughing)

29 Teacher: No one. Why is that? Can anyone give me a reason?

30 Students: They are different in appearance. They are different.

31 Teacher: Their shape is different~

32 Students: Their shapes are different when we look at them.

33 Teacher: Yes, they look different. That is right. Can anyone tell me a good reason why we can judge that the shapes are different?

34 Students: The number of sides is different.

35 Teacher: The number of sides is different, and?

36 Kwon: Because this is a triangle and that is a rectangle.

37 Students: Angle.

38 Teacher: Angle? Yes, what about the angle? The number of the angles?

39 Lyn: No, this is 90°, and that is 60°.

40 Teacher: Oh~~ I missed the part when someone said about the size of the angle. Oh, so one is a triangle and the other is a rectangle, and they are different right? Then, we don’t say they are similar in mathematics.

The teacher showed two seemingly similar isosceles triangles and two isosceles triangles that do not seem similar as the third and fourth image to make the students focus on the changes of side length and angle size. The students who answered that the two isosceles triangles that were shown thirdly are similar began to contemplate when the fourth set of two isosceles triangles was presented. Kwon said that these two triangles are not similar; however, he could not clearly support himself since he only determined similarity based on visual judgment (line 63, 71), and the students who said that they are similar could only base their opinion on the fact that they are two isosceles triangles (line 74, 76, 81). These responses of the students seem as though they determine similarity based on visual judgment, rather than mathematical standards. Kwon raised his hand as if he suddenly remembered something (line 79), and said that the angles are different, as shown in line 83. With Kwon's comment as a cue, the students concentrated on size of the angles, and became able to understand the difference between the third and the forth sets of two isosceles triangles. Following Kwon's comment, Hyun added that the difference in reduction ratio is the reason for non-similarity. The teacher asked another question to clarify Hyun's concept of ratio, and Hyun replied that if the figure is reduced, the two figures should become similar; however, they are not similar because they have different sides. The teacher hoped that the students would focus on the word "same ratio", and added further explanation that "overally smaller, rather than on only one side" to help their understanding. In <Episode IV-1> and <Episode IV-2>, the teacher persistently asked the students to provide reasoning for their judgment. Such demands of the teacher provided opportunity for students to raise their consciousness and make reflection by encouraging them to verbally express their visual judgment.

<Episode IV-2>

58 Teacher: Yes, there are two isosceles triangles here, and how would you describe their relationship? Can you say they are similar?

59 Students: Similarity// It’s not congruence, they look similar.

60 Teacher: Not congruence but similarity? They are similar but not congruence. Why?

61 Jin: That one was minimized into this one.

62 Teacher: Yes, so the bigger one was minimized to the smaller one. It became smaller. So you think they are similar because it is the smaller version of the big one. Good. Then what about this one? How would you describe them this time? (the fourth example)

63 Kwon: They are not similar.

64 Teacher: They are not?

65 Kwon: No, they are not.

66 Students: They are similar because both of them are isosceles triangles.

67 Kwon: Wook, are they similar?

68 Teacher: Jae, we are looking at the drawing number 4. What do you think about the two figures?

69 Students: Similarity. They are similar.

70 Teacher: Are these two similar?

71 Kwon: I guess.

72 Teacher: You think they are similar… Why is that? Jin?

73 Kwon: Why is that?

74 Jin: That is because the length of the down side is…(speaking vaguely)

75 Teacher: Why do you think they are similar, Jin?

76 Jin: Because they are both same isosceles triangles. In the case of isosceles triangle…

77 Teacher: Can you repeat the part about the isosceles triangle again?

78 Jin: Well~

79 Kwon: (Raising the hand highly) They are not similar.

80 Teacher: Okay, Kwon has a different opinion. But let’s listen to what Jin thinks first and then move on to Kwon’s opinion. Please make sure to compare their opinions with your own. I think everyone has their own idea even though you are not saying that out loud. Jin, your opinion first.

81 Jin: (Laughing) Because they are the same isosceles triangles…

82 Teacher: You think they look similar because they are both isosceles triangle. Jin said the reason is because they are two same isosceles triangles. What about Kwon?

83 Kwon: Their angles are all different.

84 Teacher: Different angles~ Their angles are different.

85 Lyn: They are not similar.

86 Teacher: Okay, Lyn, why is that?

87 Lyn: Their angles are different.

88 Teacher: Different angles. That’s Lyn’s opinion as well. Any other opinions?

89 Hyun: They are not similar.

90 Teacher: Why is that, Hyun?

91 Hyun: Because the ratio is different.

92 Teacher: What ratio is different?

93 Hyun: Being similar means the ratio is the same, but that was not minimized…

94 Lyn: Oh~ That’s right. The one on the right.

95 Hyun: It has to become same if it is minimized

96 Teacher: If it is minimized?

97 Kwon: It means the angles are the same.

98 Lyn: (Even if it is minimized) But, that one cannot look like this one.

99 Hyun: That part has to minimized as well

100 Student1: But that has longer bottom(base) now.

101 Teacher: Yes. This one was dented downward~

102 Students: Laughing

103 Teacher: It was dented by squeezing here. It was dented only one side. Jina is saying that it has to be shrank in every way (students laughing), and yes,

she means that they are not similar because only one side was smaller rather than overally smaller. And Hyun said about the ratio. I guess you would not understand about the ratio fully now, but anyway she talked about that (…)

In order to give a lesson on the misconception of the students that congruence is not similarity, the teacher showed two congruent equilateral triangles. As in <Episode IV-2>, the students distinguished congruence from similarity. Kwon and Wook said they are similar; however, the teacher proceeded after confirming their misconception without inquiring their supporting reasons or further developing the discussion. The teacher determined that the students were unable to say that congruence is similarity since they did not have a clear concept of ratio and similarity. Instead, the teacher let the students to ponder on this concept for some time and re-visited the relationship between congruence and similarity in later class.

<Episode IV-3>

114	Teacher:	Now what do you think about these? (showing two equilateral triangles)
115	Students:	Similar! They look similar.
116	Teacher:	You guys are sure there are similar this time?
117	Students:	Yes!
118	Kwon:	Because it was shortened from the sides and in the bottom.
119	Students:	They are all minimized by the same amount.
120	Teacher:	They were minimized to look same.

121 Hyun: All angles are 60°

122 Teacher: Minimized to look similar, the size of the angle is same by 60°, and they are both equilateral triangles. Is there anyone who doesn't think they are not similar?

123 Students: No.

124 Teacher: Lyn, would you look at the picture for the last time?

125 Students: This is congruence. Congruence.

126 Teacher: Congruence. This is congruence (writing down on the blackboard) Then, what about the congruence? Do we say they look similar?

127 Kwon: Yes.

128 Wook: Sort of.

129 Teacher: Do we say they are similar when they look same?

130 Students: Aren't being the same and similar different?

131 Teacher: Being same and similar?

132 Students: Different.

133 Wook: Aren't they similar?

134 Lyn: Being similar is a little bit different~

135 Students: They are not similar (cannot hear clearly due to the mixing of the voices of the students saying they are similar and those saying the opposite)

136 Teacher: Okay.

<Episode IV-1>, <Episode IV-2> and <Episode IV-3> demonstrated the process in which the students develop the concept of the word, similarity, from everyday usage to mathematical usage. It seems that this is attributed to the fact that the teacher carefully placed examples so that the students can recognize the invariant angle size and the ratio of side length, which are the properties of similarity. The students identified the differences among the examples and

understood the invariant properties of similarity. Specifically, the teacher consistently focused on words that the students used during interactions and asked the reason for their answer to lead the students to explain their thought, become conscious of their actions and reflect. Through the interaction with the students, the teacher induced students' understanding of the concept of similarity. Such way of teaching is a stark contrast from the teacher's previous teaching in which the teacher introduced a definition of similarity and immediately began problem-solving. It can be interpreted that the awareness of the teacher on the new way of teaching based on variation theory, which induced the students to concentrate on invariance by changing other conditions when the ratio of angle size and side length remains unchanged, was strengthened through positive interaction with the students.

2) During the fourth lecture, the teacher prepared an activity in which the students distinguish figures that are always similar from the figures that are not, and provide supporting reasons based on the knowledge of properties of plane figures and solid figures that were obtained from previous classes. Through this activity, the teacher intended to provide the students with the opportunity to reflect on the previous activities and encourage them to conjecture whether the provided figures can always be similar and to find a counterexample. By grouping similar figures and non-similar figures, the students provided supporting reasons, and by doing so, the teacher intended to induce the students

to make the judgment based on the properties of similarity, rather than simple assumption. The teacher prepared the examples, through the thought experiment phase, to be judged on their similarity and how the examples will be utilized in real classroom, and <Episode IV-4> will discuss how the teacher's intention was executed.

When the group discussion on Task 1 was near the end, the teacher hung the group board onto the blackboard to commence the classroom discussion.

The teacher started the discussion on the similarity of isosceles right triangle by showing the isosceles triangle that the teacher showed in the first class. In Task 1 during the first class (see Task 1 for the first class in Appendix), the teacher asked whether the two isosceles triangles are similar based on the students' intuition rather than the properties of similarity. The two examples of two isosceles triangles from 3) and 4) of Task 1 were presented for comparison and the difference between the two was mentioned. The pair from 3) appears similar; however, in the pair from 4) the teacher attracted students' attention to the difference of the given pair by mentioning that one is pointed figure and the other is blunted figure. Through this activity, the fact that two isosceles triangles are not always similar was stated, and the teacher asked whether two isosceles right triangles are similar. By showing the difference from previous examples, The teacher intended the students to concentrate on the fact that two isosceles triangles from 4) have different angles, and due to this fact, the teacher mentioned that one is sharp and the other is blunted (line 20). When the students were asked whether the two isosceles right triangles are similar, they focused on the angle and answered that the two isosceles right triangles are always similar because the two angles at the end are equal.

When the teacher designed the task, she intended to discuss the similarity of regular pentagon with the students and develop the discussion into the similarity of two regular polygons. However, the teacher did not mention regular polygon while discussing regular pentagon. Instead, the teacher suddenly remembered and asked the question on the similarity of regular polygons in line 35 while comparing the two isosceles right triangle and isosceles triangles. This demonstrates that the teacher is constantly mindful of the intended goal of the lesson throughout the class. In fact, the teacher presented the examples of equilateral triangles, squares and regular pentagon in order to expand the discussion from the similarity of two regular pentagons to that of two regular polygons. Through this, the teacher led the students to recognize that even though the type of regular polygon may be different, each has the same side length and the angle size, which means that they suffice the conditions of similarity.

What is noteworthy of what the teacher mentioned is line 33 in which the teacher says, “If the angle is the same, it can be enlarged and reduced in all directions.” While designing the task on similarity, the teacher focused on the misconception of the students in regard to AA similarity condition. Based on the teacher's experience, some students answered AA similarity condition as AAA similarity or ASA similarity. The same misconception was confirmed in the thesis that the teacher read in order to design the task. Therefore, the teacher recognized the necessity to instruct the students that sides can be enlarged or reduced in equal ratio if the angles of two figures are equal. The teacher's awareness of the students' misconception is constantly and consciously reflected in what the teacher says.

<Episode IV-4>

18 Teacher: (…) When we talked about the similar figures for the first time in the previous class, I showed you an isosceles triangle (showing the figure on the screen). Can we say that an isosceles triangle is always similar?

19 Students: No

20 Teacher: As I just showed you on the screen, there are isosceles triangles that are similar but what about this? (showing the two different types of isosceles triangles from the first class) It can be sharp like this one and it can also be blunt like this. Guys, Let's put your team board on the blackboard.

21 Teacher: Now, can we say isosceles triangles are similar or not?

22 Students: No

23 Teacher: Can we say they are usually similar?

24 Students: No.

25 Teacher: You placed isosceles right angle in the similar section? (pointing at the team)

26 Lyn: Yes, because it has right angle.

27 Teacher: Okay, this team said they are similar as well (pointing Kwon's team)

Why do you think like that?

28 Kwon: Is that our team? That one?

29 Teacher: Yes, why is that? Kwon?

30 Kwon: Because it is isosceles right triangle. The sizes of the two angles are same.

31 Teacher: The size of the two angles at each side is always same. Are you with me so far? In the case of isosceles right triangle, one angle is always 90 degree and the length of the two sides are same, so each angle has to have same degree. What is that?

32 Students: 45°

33 Teacher: That is right. 45°. Then, as we talked previously, we can say they are similar when the corresponding angles have same degree. If we draw an isosceles right triangle, it must have one angle of 90° and that means the other two angles would be 45° automatically. What about the side? If the size of angles is same, then you can change the length of sides and the direction of the sides as well. Is it true? We need to think about this. Are isosceles right triangles are similar or not?

34 Students: No

35 Teacher: That’s right. I actually missed something about regular pentagon. What about regular polygons? For example, equilateral triangle, square, regular pentagon… Can we say they are always similar?

36 Lyn: No.

37 Teacher: No?

38 Lyn: What? (Lyn was not exactly paying attention to which polygon the teacher was asking about)

39 Teacher: What about this? (pointing equilateral triangle, square, regular pentagon) Can we?

40 Students: Yes

41 Teacher: Why?

42 Kwon: Because they are always just same.

43 Teacher: Without any exception?

44 Students: Because the size of the angles and the length of sides are always same.

45 Teacher: That’s right. Other polygons have the same length sides and angles of the same size like regular pentagon. That is why we can say two equilateral triangles are similar. Two square are also similar. These regular polygons, these different kinds of them, have similarity. (…)

In <Episode IV-5>, the teacher discussed whether the two circles are similar and compared sectors, quadrant and half-circles to lead the students to concentrate on the angle in order to determine the similarity. The teacher presented two quadrants after teaching the similarity of two sectors. This is to induce the students’ understanding by making comparisons that the angles of sectors do not always remain equal, but the property of similarity of angle is maintained in the case of quadrant. Such intention of the teacher is evident in her question, “what is fixed?” (line 84).

<Episode IV-5>

74 Teacher: Yes, we can say they are similar. So we can place number 5 here, the two circles are similar. Let’s compare it with number 6. Sector which is a part of a circle, does it have similarity?

75 Kwon: Can sector change the size of its angle? (using her both arms to express the spreading of the angle)

76 Teacher: Yes. Then what can you tell me about sector?

77 Students: It changes according to the angle.

78 Teacher: That’s right. It can change its shape by angle. So can we say two sectors

are always similar or not?

79 Students: No

80 Teacher: That is right. The answer is no. One more question at this point. Yoon, quadrant, that is one piece of a circle when it is divided by four parts, is always similarity?

81 Kwon: Yes.

82 Teacher: What makes you say that?

83 Kwon: Because it is fixed. One-fourth.

84 Teacher: What is fixed?

85 Kwon: The angle.

86 Teacher: Right. The angle is fixed. Then what about two half circles?

87 Kwon: Similar.

3.1.2. Recognizing the concept of similarity that exists behind diverse methods

In the second class, the teacher prepared the similar figure drawing activity. The students learned two properties of similarity in the first class, so the students can draw similar triangles with the consideration of the properties in this class. The teacher distributed prepared paper triangles and the students were asked to draw twice enlarged or reduced triangles freely using protractor, ruler or compass. The teacher expected that such strategy to facilitate various ways of drawing enables students to approach similar triangle drawing using method that is suitable for individual and it also confirms students' understanding of similarity since they have to draw using the properties of similarity, which is that corresponding angles are equal and the ratio of corresponding sides are consistent. While the students were drawing similar triangles, the teacher asked them what they are drawing and instructed them to draw in consideration of the previous classes, so that the students could reflect on their activity.

As the teacher predicted, the students drew similar triangles using various methods and the teacher selected students who used different methods to give presentations. The teacher asked Hyun who drew similar triangles using one angle and the length of two sides. After drawing the given triangle, Hyun took one angle as the basis and extended the two sides using compass, and completed a similar triangle. While listening to Hyun’s presentation, the teacher asked the students how the angle and the length of sides changed in order to make them pay their attention to the fact that the properties of similarity is maintained <Episode IV-6>.

<Episode IV-6>

82 Teacher: Hyun drew this in this way. Can you explain how you did it?
83 Hyun: Me?
84 Teacher: Yes.
85 Hyun: I drew it (the given triangle) using the ruler.
86 Teacher: You drew with using the ruler. Then?
87 Hyun: I put the compass here at the center, and I drew a circle with the radius of this length. Then, if we think of the diameter of the circle as the base, then it would be the twice of this so I chose this as the base. And

(pointing the other side) I put the center here and drew a circle all the way there.

88 Teacher: Oh, so the length would be changed how much when you draw a circle with this length at the center?

89 Students: Twice.

90 Teacher: Twice. Yes, the length was doubled. So what Hyun did was, she drew a triangle. And she doubled this side. And this side too. But something is not changed. What is that?

91 Yeo: The angle

92 Teacher: Which angle? This angle didn't change when the side became longer. Twice. And she connected this end with that end. Then can we say this figure Hyun drew is similar to this figure?

93 Students: Yes.

Next, the teacher reviewed Young's method <Episode IV-7>. Young pasted together four of given triangles to make twice-enlarged similar triangle. As the teacher listened to Young's presentation, just as the teacher did during Hyun's presentation, the teacher asked the students what remains equal and how the length of the sides changed and confirmed their response in order to make them focus on the properties of similar figures. Young's method is what the teacher predicted from task design process and utilized the similar triangle that Young drew in order to develop the classroom discussion into the ratio of area.

<Episode IV-7>

153 Teacher: Let's look at another way. Can you guess who drew this? Young did. And Yeo did it in the same way. Let's take a look at this for a moment. How is it? How many triangles did she put?

154 Kwon: Four.

155 Teacher: One, two, three, four. And what didn't change? You learned it.

156 Students: Angle.

157 Teacher: Right. The angles didn't change. The angles are same with the original figure. If we overlap these two, angles are same. What about sides? How many times did them become bigger?

158 Students: Twice.

159 Teacher: That's right. Twice. And the relationship of the original and this figure is?

160 Students: Similarity.

161 Teacher: Similarity. We can draw it in this way too (…)

As the teacher predicted, the students drew similar triangles using various methods and the teacher led the students to understand that even though similar figures can be drawn using various methods, the properties of similarity remains intact. The teacher asked the students to explain how they drew similar figures using different tools and measuring the angle and side length, and inquired what stayed the same. By doing so, the teacher provided the students with the opportunity to become conscious of their action and reflect on that, which led the students to recognize the concept of similarity. While designing this task, the teacher expected various responses from the students and focused on inducing students' understanding of the concept of similarity of triangle as well as the difference between the ratio of length and ratio of area for similar triangles based on the students' responses. <Episode IV-6> and <Episode IV-7> demonstrates that the teacher's noticing during task design and thought experiment phases influenced her teaching activity in which she observed the students' responses and selected certain responses to utilize in the general discussion.

3.1.3. Expansion of perspective through utilization of mathematical tool

1) Through the activity to draw twice-enlarged similar figures using the provided triangle from the previous lesson, the students recognized the ratio of area and ratio of similarity of similar figures. To reinforce this understanding, the teacher prepared similar figure drawing using graph paper in Task 2. In order to provide an opportunity to explore the ratio of area, which was briefly mentioned in the second lesson, and to induce recognition that similarity ratio and area ratio is different, the teacher provided a picture of a house <Episode IV-8>. The number of door grid of the house was 1 in order for the students to observe the ratio of area. The teacher also expected that students will easily recognize the difference between the original grid and enlarged or reduced grid and observe the change of area size when the students enlarge or reduce the grid to create similar figures. In fact, the students enlarged or reduced to a varying degree and were able to make an attempt at generalizing the ratio of area through observing the

change of the door size of the house figure. The teacher forecasted that the students will enlarge the figure to a varying degree (3 times, 4 times, etc.) and intended to lead the students to reason the ratio of area inductively through the similar figures drawn by the students. The teacher enlarged the figure by two times and showed the activity sheet of the students who enlarged by three times. The teacher, then, confirmed with the students that when the similarity ratio is 1:3, the ratio of area is 1:9. After showing twice and thrice-enlarged figures, the teacher asked the students, "What do you sense by looking at these two?" and hoped that the students would answer the relationship between the ratio of similarity and the ratio of area based on their instincts. However, the teacher's question seemed to have been unclear to the students and the teacher changed the question immediately to "What if the number was 4 times?" (line 82). Based on the experience from previous classes, the teacher was conscious that the students were unable to answer because of the unclear nature of the question and was willing to rectify it. The revised question, "What if it was 4 times?" demonstrates the consciousness of the teacher as well as the willingness to make correction. In addition, this question also demonstrates the teacher's intention to provide the students with the opportunity to take a guess through inductive reasoning.

<Episode IV-8>

64 Teacher: The ratio is 1:3. Let’s take a closer look at this. Loot at the door. How many parts of the door were colored in the original picture?

65 Students: One.

66 Teacher: What about in this door?

67 Students: Nine parts.

68 Teacher: Nine parts. What do they tell us?

69 Students: The area.

70 Teacher: That’s right. What is the ratio of similarity?

71 Students: 1:3

72 Teacher: It was 1:3, and how about this area?

73 Students: 1:9

74 Teacher: Right. 1:9. When I doubled the length, what was the ratio of length?

75 Students: 1:2

76 Teacher: It was 1:2. You remember the ratio of the colored door?

77 Students: 1:4

78 Teacher: What was it? What about the ratio of the area? 1:4. Let’s write that down here again. When the ratio of similarity was 1:2, the ratio of area was 1:4, and what about the one Hyun drew with the ratio of 1:3? Nine parts of the door were colored, so the ratio of the area was 1:9. What do you sense by looking at these two?, So?

79 So: …

80 Teacher: You are not sure yet?

81 So: (quietly) Aren’t they similar?

82 Teacher: What if the number was 4?

83 Students: 1:8 /// 1:16 (Students answering at the same time)

84 Teacher: What if I lengthen the side by four times, how many grids would be colored?

85 Students: 16

86 Teacher: Then, what can we find out from here?

87 Students: Square/// the ratio of area (at the same time)

88 Teacher: It is the ratio of similarity's what? (pointing the ratio of similarity and of the area with the finger for comparison)

89 Students: Square

2) The teacher utilized cube wooden blocks to study the similarity for three-dimensional figures in the third class <Episode IV-9>. The teacher provided the students with intentionally few wooden blocks and the students were instructed to focus on how many wooden blocks are necessary to create thrice enlarged figure. As <Episode IV-9> demonstrates, the teacher asked how many wooden blocks are necessary when the figure is twice-enlarged as well as with what the number of blocks is relevant. The students have already learned the relationship between the ratio of similarity and the ratio of area; thus, they were able to provide correct answer that it is volume. The teacher asked a question on the number of required wooden blocks to develop the concept of the ratio of similarity and led the students to connect the ratio of similarity and the number of wooden blocks. After that, the teacher used the word, volume, instead of the number of wooden blocks. Through this activity, the teacher intended the students to recognize the difference between the ratio of similarity and the ratio of volume. In addition, the teacher vertically wrote down on the board the ratio of similarity and the ratio of volume when the figure is twice and thrice-enlarged while asking them questions so that they can easily compare them, and guided

the students to infer that the ratio of volume is three times as large as the ratio of similarity by pointing to what is written on the board.

<Episode IV-9>

140 Students: We need more cubes!

141 Teacher: How many cubes are you planning to put into that? How many do you need? (giving cubes to the students while keep asking how many cubes they need)

142 Student2: I need more cubes.

143 Teacher: There are teams which don't have enough cubes. How many cubes need to fill that in total?

144 Students: 27 cubes.

145 Teacher: Yes, a total of 27 cubes.

146 Teacher: What do you think about the ones you made compared to the original? What about the width, length, and height.

147 Students: Constantly three times.

148 Teacher: Yes, by three times each. Then, let's talk about what we learned before. What if I enlarge this by twice? The three-dimensional figure which is next to Seung's V gesture, had what ratio with the original building wood blocks?

149 Students: 1:2

150 Teacher: What ratio?

151 Students: 1:2

152 Teacher: What was the ratio between these two? (Asking again while bringing the original cube to the screen)

153 Students: (Students shouting) 1:2

154 Teacher: Yes, the ratio would be 1:2. What about all these edges? They are enlarged by twice right? The ratio of similarity is 1:2. Then what is the

ratio between the wooden blocks and this?

155 Students: 1:3

156 Teacher: Yes, the ratio is 1:3. This is where we need to think about the number of blocks you needed before. When we enlarged this twice, how many blocks were needed to fill this?

157 Students: Eight.

158 Teacher: A total of?

159 Students: Eight.

160 Teacher: Right. Then, this number is related to what?

161 Young: Volume.

162 Teacher: Young just mentioned it. Did you hear that? It is related with what?

163 Students: Volume.

164 Teacher: Volume has what ratio?

165 Students: 1:8

3) From the past teaching experience, the teacher was aware that the students struggle with finding the center of similarity, so the teacher designed the task using GeoZebra to show a variety of locations of similarity in order for students to find corresponding points. Using GeoZebra, the teacher demonstrated how the center of similarity changes dynamically and the students were able to conjecture and confirm the relationship between the image of similar figures and the relation of position between the center of similarity and corresponding points. Because the students made only a hypothesis on the relationship between congruence and similarity, the intention of the teacher was to smoothly lead the students to understand that congruence is a special case of similarity by dynamically moving and delving into figures with a variety of similarity ratio

using GeoZebra. In fact, the students were amazed when GeoZebra showed the change of the center of similarity and studied the relationship by observing the location of the center of similarity, of corresponding points as well as the image of similar figures <Episode IV-10>. The teacher also asked a question while changing the value of similarity ratio on GeoZebra, “What happens when it is close to 1?”, in order for the students to guess and confirm that it will reach congruence (line 220). The students continually changed the similarity ratio; therefore, when the teacher asked the meaning of this value, they were able to answer 1:1. Thus, the students are smoothly accepting that congruence is a special case of similarity with 1:1 ratio.

<Episode IV-10>

220 Teacher: Look at the rectangle. It is overturned, right? Let’s do it again. This is what it looked like when the number is 4 or 5. (watching the screen) But now I will make to be 1 (focusing on the number 1 to show they are congruence). What happens when it is close to 1?

221 Student3: They stick together.

222 Teacher: Okay, then what does this value mean?

223 Students: 1:1, the distance

224 Teacher: Distance? … Oh, oh… (The teacher managed to make the value 1 after many trials) Now, it became itself, right? Then, what is this?

225 Kwon: (Loudly) 1:1

226 Teacher: That is right. The ratio of 1:1

227 Students: Wow

228 Teacher: Yes, 1:1 Then it is itself right? What do we call this kind of situation?

229 Students: Congruence
230 Teacher: Yes, congruence. Then, can we say congruence is also a similarity?
231 Students: Yes

We have, so far, reviewed the cases in which the teacher's noticing in task design and thought experiment phase is reflected in actual classes. The cases can be categorized into 3 groups: a group in which the teacher induced the understanding of the concept of similarity by systematically presenting examples, a group in which the students were led to recognize the concept of similarity using a variety of methods, and a group in which the teacher developed the perspective of the students by using mathematical tools.

The teacher planned and conducted the class differently from the previous way of teaching and obtained awareness on the following mathematical and pedagogical areas. The concept of similarity should be taught by beginning with visual recognition and move towards more structured concept, and one should not teach the proportional relation by giving lessons on algorithm only without the image of similarity. Additionally, the teacher focused on the variant and invariant elements of similarity based on variation theory to design tasks and conduct thought experiment. Not only that, the teacher constantly monitored and noticed these elements while interacting with the students and attempted to provide them with learning opportunities.

As I mentioned earlier, the teacher's previous way of teaching was to follow the order of textbook and define similarity and conduct a variety of problem-solving by using proportional expression. Rather than interacting with the students, the teacher led the class either by simply providing explanation or questioning whether they understand or agree. The teacher was unaware of how her habitual teaching could influence the students' learning. Through this experiment, the teacher recognized the limitations of the existing teaching way by going through processes such as learning theory, task design, and thought experiment. Following this process, the teacher noticed new perspectives on mathematical and pedagogical aspects with respect to the concept of similarity and attempted to change the way of teaching. As a matter of fact, the teacher endeavored to provide the students with the opportunity to observe and explore a variety of figures and identify the properties of similar figures. Rather than simple questions to confirm the students' understanding, the teacher intentionally asked questions that can lead inference so that the students can conjecture and reason on their own. The intentional and conscious class designing of the teacher through thought experiment is well-connected to the actual classroom situation and the teacher is increasingly witnessing its usability. The teacher prepared a number of activities such as visually judging the similarity and drawing similar figures to help the students to understand the concept of similarity. These activities were prepared with clear intention and purpose, which went beyond the classroom. The teacher continually observed and noticed whether the students understood the mathematical concept behind these activities and the teacher was able to provide the intended learning opportunities. This confirms that the teacher adopted a meta-perspective in teaching activity rather than staying within the level of awareness-in-action mathematically and pedagogically.

Table 2 shows what the teacher noticed during the class and it is grouped into mathematical, psychological and management of learning dimensions. In comparison to Table 1, which grouped the teachers' noticing into the mathematical, psychological and management of learning dimensions during thought experiment, what the teachers noticed was mostly reflected at mathematical dimension. At psychological dimension, however, what the teachers noticed was displayed limitingly in terms of interpretation and response to the students' reaction and providing opportunity to inquiry. At management of learning dimension, the noticing was not well-reflected in terms of interaction and questions to promote communication. This will be further elaborated in the next chapter.

The three cases in which the teacher's noticing from thought experiment is well-reflected in actual classrooms are: a case in which the teacher induced the understanding of the concept of similarity by systematically presenting examples, a case in which the students were led to recognize the concept of similarity using a variety of methods, and a case in which the teacher developed the perspective of the students by using mathematical tools. In these cases, the teacher found a right balance among mathematical, psychological and management of learning dimensions, so the teacher's intention was well-carried out in the activities. The fact that the noticing from thought experiment was well-carried out in teacher's execution demonstrates that the teacher became conscious of things during actual class, which she noticed in thought experiment phase and began to construct practical knowledge on the new way of teaching. By learning new theory, the teacher places focal awareness on the existing way of teaching that was unconsciously carried out and put it into action consciously during class, and by repeating this process, the new way of teaching becomes habitual for the teacher, which will be unconsciously carried out. Whether or not the focal awareness of the teacher becomes a subsidiary awareness and manifests itself in unconscious behavior will require follow-up study.

Table 2. Teacher's noticing within mathematical, psychological and management of learning dimensions during classes.

Mathematical dimension	Psychological dimension	Management of learning dimension
Discerning (distinguishing between similar figures and non-similar figures, attending to sides and angles)	Interpreting and reaction against students' responses (different reactions between expected responses and unexpected ones in the	Organizing classroom activity (whole class discussion, group work, individual work) Interacting (focusing on

Constructing (drawing similar figures using various mathematical tools and different methods) Measuring (size of angles, length of sides, area of figures) Recognizing relationships (the relation among the ratio of length, area and volume for similar figures) Generalizing (determining similarity of any two figures with same shape and justifying)	thought experiment phase) Utilizing students' prior knowledge (reminding mathematical knowledge learned from previous lesson, using students' activity sheet) Providing limited opportunities to observe and conjecture Demanding students' reflection on their action (questioning 'why') Dealing with misconceptions (congruence is not similarity, similarity figures should have the center of similarity)	teacher-oriented interaction with students) Trying questions to facilitate communication (unclear questions during classes) Giving support the students to involve actively in activities

3.2. A case in which the noticing from thought experiment is not well-reflected

3.2.1. A case in which the noticing at mathematical dimension does not lead to psychological and management of learning dimensions.

3.2.1.1. Unable to accurately identify the students' level of understanding

This is relevant to the similar triangle drawing activity from the second class. The teacher provided the students with paper triangles, compass, ruler and protractor and asked the students to draw triangles that are twice as large. As each individual participated in the activity, the teacher observed each student in order to select students who will make presentations for whole-class discussion, which the teacher prepared in order to demonstrate a variety of methods. The teacher thought that Yoon utilized the size of one side and two angles to draw similar triangle and asked her to present <Episode IV-11>. However, as Yoon carried out her presentation with her drawing of similar triangle, the teacher identified an error. Yoon first acquired the length of all three sides that are twice as long as the provided triangle. She used a ruler to draw a line that is twice as long as the base line and measured the angle B (see Yoon's activity sheet). After that, she drew a line segment that was twice as long as the side AB, and repeated the same process to draw an angle and a side on the other side. When the two sides did not meet at Point A, Yoon extended the two segments to make them meet. The teacher recognized that Yoon did not understand triangle-determining condition upon listening to her explanation. In order to draw a similar triangle, Yoon drew all three sides twice as long as the provided triangle and measured all three angles. It appears that Yoon thought that in order to draw a similar triangle, all corresponding angles should be the same and the ratio of corresponding sides should be consistent. By observing this situation, the teacher judged that Yoon lacked preliminary knowledge on triangle-determining condition. The teacher once again explained to the class the method Yoon adopted (line 75) and asked them whether they can draw a triangle using this method. The teacher asked this question to see the reaction on required conditions of a triangle. However, the students did not understand the teacher's intention, and answered that they can draw by using a compass. It seems that the students concentrated on the fact that two segments did not meet in Yoon's triangle and reckoned that a compass can reduce the error of measurement. When the students' answer did not meet the teacher's intention, the teacher instructed the students to draw one segment and two half lines at measured angles and explained that the two lines will meet at Point A. The purpose is for the students to follow these steps to finish a triangle and confirm that the lengths of sides are twice as long. This method utilizes triangle-determining condition, one side and two angles; however, the teacher did not directly mention the triangle-determining condition to the students. The teacher recognized the student's lack of understanding on the triangle-determining condition; however, finished the lesson by instructing them on how to draw a triangle instead of mentioning the triangle-determining condition since the teacher could not decide what to do at that moment.

After the class, the teacher felt remorseful that she did not handle it properly when she knew that Yoon did not understand the triangle-determining condition. The teacher did not anticipate the students' reaction and prepare an appropriate response on the triangle-determining condition, so she was taken aback, and was pre-occupied to decide what to do in the next class. Through this task, the teacher intended to provide the students an opportunity to draw similar figures with different ways by using a variety of mathematical tools, so that they can recognize the property of similar figures, which is the size of angle and the ratio of the lengths of sides are consistent regardless of varying methods. However, the teacher expected the students to have an understanding of triangle-determining condition and triangle construction; thus, the teacher did not appropriately estimate the students' level of understanding. <Episode IV-11> demonstrated that the triangle-determining condition and triangle construction could possibly pose difficulties and that the teacher did not notice the meta-cognitive shift, which could be caused by a difficulty of using tools. This testifies that the learning opportunities that the teacher intended were not fully provided.

In other words, it can be judged that the teacher's mathematical noticing on construction and learning of similarity properties were not properly aligned with psychological dimension such as anticipating the students' level or response, which resulted in poor reflection in the classroom.

<Episode IV-11>

59 Teacher: Why don't we talk about our drawings together? Let's look at here. This is drawn by Yoon. (showing her drawing on the screen) Yoon~ Can you explain how you drew this in front of the class?

60 Yoon: Yes

61 Teacher: Oh~ Please come out. You can explain it from here. Let's pay attention to her explanation. How she drew this, okay?

62 Yoon: I measured this length and doubled it.

63 Teacher: Okay, so Yoon measured the length of each side because you wanted to measure the length first and then double the length. So you drew a line which was enlarged by 4cm? (checking it while looking at her) And then?

64 Yoon: Angle…

65 Teacher: You measured the angle? It was 50 degree. So you used protractor to draw the line?

66 Yoon: Yes

67 Teacher: Then?

68 Yoon: So I connected them here.

69 Teacher: (pointing at the each side of vertex A) Can you tell me how these two segments met?

70 Yoon: …

71 Teacher: Yoon, what I am curious about here is whether you drew a line here with measuring 50 degree and 60 degree each.

72 Yoon: Yes

73 Teacher: But when you see here (pointing each side of the corresponding point to A) this point didn't meet but ended right here. Then did you measure the length?

74 Yoon: (nodding) You said it was 80cm, but they didn't meet (explaining that the two segments didn't meet at one vertex when doubled the length with pointing them) so I made them meet by lengthening them.

75 Teacher: But, that means this is not 8cm? Isn't it 8cm? It's not 80cm when you say 80, it's 80mm~ (telling her the unit she used was wrong at the point where she wrote 80cm, 90cm) What Yoon explained is like this (telling other students). I will repeat that again. First, she enlarged the base. Then, she measured the size of the each angle at the end of the base of the original triangle, and draws them and line. But she told that this length was not 8CM, and what do you think? Then, it's not doubled right? So can we say this is similar to that one? Can we say they are similar? (asking to all of the students)

76 Students: No

77 Teacher: Then, is it impossible to draw a similar figure in this way?

78 Students: We can. By compass.

79 Teacher: How can we do that with compass?

80 Students: We cut the half of the figure.

81 Teacher: I think it would be great if Yoon and other students who didn't draw this draw again in this way. She said she drew this after measuring the length first. But let's draw half-lines after measuring each angle, 50 and 60, and find the point where they meet, and measure this length. Only those who didn't draw. (Yoon) You can go back to your seat.

3.2.1.2. The interaction with the students is not adequate

This is relevant to the task, which distinguishes figures that are constantly similar from figures that are not. <Episode IV-12> shows the interaction between the teacher and the students in regard to whether or not two cylinders are always similar. While discussing why any two cylinders are not similar, the teacher wanted the students to identify that the ratio of the edge length is consistent, which is a property of similarity for solid figures. Below is what the teacher said about the intention.

> "The term twice as big has something to do with length for plane figures while it has to do with volume for solid figures. It is very possible that the kids will say the ratio of increasing volume, so I wanted to make them focus on the length of edge."

Such intention of the teacher manifested itself in abruptly asking about solid figures without sufficiently discussing cylinders (line 95) and this made it unclear whether the similarity discussion is focused on cylinder or solid figures. The teacher reckoned that the discussion could be developed from similarity in cylinders to similarity in solid figures and the properties of similarity for solid figures could be drawn. Here, the teacher was attempting to lead the discussion from the teacher's perspective rather than the student's perspective. This is visible from line 99, which shows the communication between the teacher and the students. The students mentioned the word, solid figures; however, they only focused on cylinders to explain why they are not similar, which demonstrates that the communication between the teacher and the students is not well in place. However, the teacher was not aware of this during <Episode IV-12> and judged that the students are not grasping the concept of similarity for solid figures. Thus, the teacher continuously asked questions to the students in an attempt to have them say that the ratio of corner is consistent. During this process, the students seemed very confused not knowing the teacher's intention. However, the teacher failed to recognize the reasons for the students' confusion and hastily ended the discussion by stating that the ratio of similarity for solid figures is the ratio of edge.

After the class, the teacher pointed to <Episode IV-12> as a disconcerting moment and was able to interpret and recognize the situation through a discussion with her colleagues. The teacher identified that she could only consider the mathematical meaning and failed to consider the interaction with the students. The teacher also mentioned that she should have led the students to focus on the relationship between two rectangular solid from line 117 rather than cylinders in order to clarify the ratio of edge. The teacher only concentrated on eliciting what she intended in the lesson and failed to recognize that the questions could be unclear to the students. <Episode IV-12> demonstrates that the teacher only focuses on mathematical dimension in terms of generalization and not connecting this with the other two dimensions. In other words, the teacher led the interaction only from her perspective and attempted to hastily achieve generalization. Therefore, the intention of the teacher was not well-reflected in the classroom.

<Episode IV-12>

89	Students:	It doesn’t work with cylinder.
90	Teacher:	Why is that?
91	Lyn:	Because cylinder becomes like this.
92	Kwon:	And because we need to see the length as a height.
93	Teacher:	Team 7 drew it here. They drew a cylinder with small circle and long height, and this one has big circle but what about the height? It’s not that long, right? Then, can we say they are similar?
94	Students:	No
95	Teacher:	Then, what can you tell me about three-dimensional figure?
96	Seung:	It doesn’t work with three-dimensional figures.
97	Teacher:	What doesn’t work with them? Why doesn’t it work?
98	Seung:	Because the size…/// (Kwon) Because when the height is different…
99	Teacher:	When the height is different?
100	Seung:	If we lengthen the height
101	Teacher:	Lengthen the height? If we look at the upper side,
102	Students:	Similar
103	Teacher:	Yes, because they are circles and they look alike, right?
104	Students:	Yes
105	Teacher:	But there must be ratio of similarity. Does the ratio apply in height as well? Is it applied in height as well? If so, then how can we say?

106 Students: They are similar.

107 Teacher: We can say they are similar. But this is bigger than the ratio. Is the height getting bigger like this big ratio in this picture?

108 Students: No

109 Teacher: No. So what can we learn from this picture? If we say two three-dimensional figures are similar, they should have?

110 Students: The same height.

111 Teacher: Should the height be same?

112 Students: It needs to be constant.

113 Teacher: With what?

114 Student4: With one around it /// (Young) with the length of the base

115 Teacher: It has to be same with one of them? The height needs to be same with what?

116 Student5: With the ratio

117 Teacher: With the ratio? What ratio? The big one? Let’s take a look at rectangular solid too. We can say same thing about this too. Let’s say the ratio of these two circles is 1:2 like you said. Then what is the ratio for the height? Can you say it’s 1:2? (…) Can you? (…) If the ratio of height is 1:2, then what happens? The ratio of two circles is 1:2, and the ratio of height also became 1:2. Can we say these two cylinders are similar?

118 Students: Yes.

119 Teacher: But what about ordinary two cylinders?

120 Students: No

121 Teacher: That’s right. Then what can we learn from this? If we want to say a three-dimensional figure is similar, then?

122 Someone: The ratio has to be same.

123 Kwon: The size of the base needs to be same.

124 Teacher: The size of the base needs to be same?

125 Kwon: Yes

126	Young:	The ratio of similarity, of base and height need to be same.
127	Teacher:	Yes. The ratio of similarity of base needs to be same with that of height. Then, what decides the ratio of similarity of base? It has to be the ratio of the sides which compose this base, right? Yes, then ultimately the ratio of similarity of three-dimensional figure is that of what?
128	Young:	Of the base
129	Teacher:	To say the ratio of similarity of base and that of face, what need to say?
130	Young:	The length of the base
131	Teacher:	Yes, the length of the base
132	Jin:	Base.
133	Teacher: :	Yes, we talk about the ratio of similarity with the length of side. Then, ultimately, the ratio of similarity in three-dimensional figure is that of what? (…) We don't call these as sides in three-dimensional figures, How do we call it? Yes, edges. So the ratio of similarity in three-dimensional figure is?
134	Few of students:	The ratio of edge.

3.2.2. The case in which the previous way of teaching appears

3.2.2.1. Unable to provide enough opportunities to inquiry

This is relevant to the activity of drawing twice-enlarged triangle by using paper triangles that the teacher provided. The teacher provided paper triangles because the students can put them together to make enlarged similar triangles, which can teach them the properties of similarity. Not only that, they can smoothly transition into learning the ratio of area and recognize the relation between the ratio of similarity and the ratio of area. Additionally, the teacher was aware from the relevant studies and her teaching experiences that the students are prone to misconception that the ratio of similarity and the ratio of area are equal; therefore, the teacher wanted the students to experience that the ratio of similarity and the ratio of area are different through participating in this task.

In <Episode IV-13>, a student, who created a similar figure using the provided triangles, gives explanation and connects this to the concept of area. As individual activities took place, the teacher saw the twice-enlarged figure that Young made and decided to present his method to the whole class to provide the students with opportunity to explore the triangles' area. First, the teacher explained how the similar figure was made by putting four triangles together and asked the students to confirm that they acknowledged the properties of similarity. Following that, the teacher asked the students how many figures were in the twice-enlarged figure as in line 161 of <Episode IV-13>, so that the students can concentrate on the area. Until then, the students only focused on the ratio of the length of the sides to draw similar figures. However, the teacher thought that they need to understand the number of original figures that fills the similar figure in order to introduce the concept of the ratio of area. By emphasizing that four figures are put together, the teacher expected that the students will easily grasp the concept of area. However, the teacher was disconcerted when the students did not show the anticipated reaction (line 167) and hastily finished the class by mentioning the ratio of area when Young mentioned area (line 169).

<Episode IV-13> shows that the teacher failed to provide sufficient opportunities for students to understand the activities of Young and Yeo, which they carried out implicitly, and although the teacher was aware of the mathematical meaning of the activities, it did not lead to offer opportunity to inquiry for the students. The teacher silently evaluated that Yeo's answer ('height?') was wrong (line 166) and did not show further response. However, the teacher hurriedly explained the relationship between the ratio of similarity and the ratio of area when Young answered by saying "area". On the surface, it may seem that the teacher and the students interact by exchanging questions and answers; however, the teacher unilaterally described the ratio of similarity and area, and simply confirmed the students' understanding on the ratio of area. The task intended to provide the students with the opportunity to explore; however, the teacher's previous way of teaching appeared unconsciously and the task was implemented with one-way communication in Inquiry-Response-Evaluation (IRE) format.

<Episode IV-13>

153 Teacher: Let's take a look at another way. Can you tell who drew this? Young did, and Yeo drew in the same manner. Why don't we look at this? How is it? How many triangles did he put together?

154 Kwon: Four.

155 Teacher: One, two, three, and four. Then, as you already know, what is still same?

156 Students: Angles

157 Teacher: The angles didn't change. The angles are same like the original figure. So they match. Then what about sides? How many times did they become bigger?

158 Students: Twice

159 Teacher: Yes, twice. The original figure and this figure is?

160 Students: Similar.

161 Teacher: We could say they are similar. I think there is one more thing to talk about when I see this figures put together. What would that be? When you drew a similar figure in this way, how many figures needed to fill it when it was doubled compared to the original figure which only needed one?

162 Students: Four

163 Teacher: Four. Then, what can you tell from this? Yeo just said the answer quietly. What is that?

164 Yeo: (Murmuring)

165 Teacher: What? I didn't catch that. What would be the relationship?

166 Yeo: Height?

167 Teacher: Height? How many squares do… What ratio did this figure and the original figure have?

168 Young: Area

169 Teacher: Oh~ Young just said area. That's right. You know the ratio of similarity is 1:2, but the area covers how many squares here? Four. Then, it is the ratio of area, and that would be?

170 Students: 1:4

171 Teacher: Yes, it's 1:4. We can review this again. When the ratio of similarity is 1:2, the ratio of area is?

172 Students: 1:4

173 Teacher: Yes, 1:4. Good job (…)

3.2.2.2. The teacher asks dichotomous or short-answer questions

<Episode IV-14> is relevant to drawing similar figures by enlarging or reducing the grid in the third class. While the students were participating in individual activities, the teacher recognized an error in Minji’s process of enlarging the grid and showed her activity sheet to the entire class so that the students can learn by error. However, the teacher showed Minji’s error on a screen and asked the class, “how many grids create one grid in the case of the width?” The following communication between the teacher and the students reveals that the teacher asks the students short-answer questions in order to help them reach the error. This way of handling the student’s error rids the students of the opportunity to identify the error from meta-perspective. According to Lee (2002), a teacher, who poses questions inducing the meta-processing such as “What is peculiar?”, recognizes the learning ability of the students. The teacher mentioned that the participating students are in a middle to low level group and do not pay attention in the classroom. Considering this, it seems that the perspective of the teacher on the students and her old way of teaching was reflected in her way of handling the error in <Episode IV-14>. <Episode IV-14> shows different way of handling errors from the first class in which the teacher

corrected the error with the students (Refer line 70-103 from <Episode IV-2>). In <Episode IV-2>, the teacher did not instantly judge the students' opinion, but led them to correct their answer by asking them the basis of their opinions and induced different perspectives. <Episode IV-2> shows that the teacher executed as planned and intended. However, it seems that <Episode IV-14> reflects the teacher's habitual teaching behaviors since it handles errors that were revealed during the class.

<Episode IV-14>

30 Teacher: Now Minji tried to change the lattice. How is it? She made how many grids create one grid in the case of the width?

31 Students: Four grids

32 Teacher: Four grids for one. But what about the height?

33 Students: Five grids.

34 Teacher: Five grids for one. Then how does it change? What do you think will happen?

35 Yeo: It's difficult.

36 Teacher: Difficult and what else? How many times did it become bigger in width?

37 Students: Four times.

38 Teacher: Then, how many times is it bigger in width? How many times?

39 Students: Four times.

40 Teacher: We made four grids into one so it means four times. What about the height?

41 Students: Five times.

42 Teacher: Five times, and when we draw similar figures, we need to keep in mind that we need to maintain what?

43 Students: The ratio.

44 Teacher: Yes, the ratio. But in this case, the width is 4 times bigger and the height is 5 times bigger. What happens?

45 Students: The ratio doesn’t fit.

46 Teacher: That’s right. Then, we need to enlarge the original figure…

47 Students: Constantly

We have confirmed that the teacher’s previous way of teaching unconsciously appears while the teacher was conducting the class with the new method. One interesting thing with respect to this is that the teacher coordinated the previous way, which is mainly teacher-oriented, with the new method. The teacher reviewed and emphasized the important lessons from the previous lecture that students should remember either after mathematical activities or before the class begins in Inquisition-Reponses-Evaluation (IRE) format. The teacher also asked the students to confirm and wrote down in their activity sheets what they discussed and studied during individual and group activities. This can be seen as a form of digestive teaching (Lee, 1990). Utilizing digestive teaching, while making an attempt at the new way of teaching, reflects that the teacher felt responsible for students to learn the core of mathematical contents while handling external factors such as the pressure of exam and making progress. Therefore, it seems to have stemmed from the teacher’s effort to find a balance between the new way of teaching and the old way of teaching.

So far, we have analyzed the cases in which the teacher's noticing from thought experiment is not well-reflected in classrooms. In these cases, the teacher could not be conscious or were met with obstacles to do so in actual classroom setting despite the fact that she was consciously aware of some aspects during thought experiment.

First, inadequate noticing in psychological and management of learning dimension in actual classroom setting posed limitation on teacher's practice even though the teacher's noticing at mathematical dimension was successful. This result implies that teachers needs to focus on all three dimensions consciously, which lead them to consider dynamic teaching situations in more structural and systematic way, in a balanced manner in order for mathematical content knowledge and pedagogical knowledge to be linked more meaningfully, not superficially, which could lead to the change in the teacher's practice.

Second, we have confirmed that the teacher unconsciously regresses to the old way of teaching sometimes while practicing a new way. The responsibility of the teacher to manage exams and handle the pressure of making progress as well as the teacher's belief about the students as the reason for the appearance of digestive teaching and the teacher's previous way of teaching. Teacher's beliefs and responsibility are subsidiary awareness that the teacher obtained throughout a long period of teaching. Although the teacher does not explicitly recognize it, it can be confirmed that such subsidiary awareness still function in the process of attempting the new way of teaching. In this experiment, we have confirmed that the teacher's practice changed through her efforts to be consciously aware of their unconscious teaching practice and make improvements; however, the subsidiary awareness that silently linger such as teacher's beliefs about her students and a sense of responsibility can still function strongly. Therefore, it implies that, in order for the changes in practice to sustain and the changed practice to stabilize in the teachers' mind enough to carry out unconsciously, long-term endeavors and social and institutional supports are necessary.

4. Noticing and reflection from class analysis

Teacher A discussed what went on in the classrooms with her colleagues who participated in task design together while watching the recorded video of the class. In this section, we will analyze how the perspective and interpretation of Teacher A changed through a discussion with other teachers and making reflection on the class.

4.1. Recognition of the necessity to understand the students

During the reflection session, Teacher A began to discuss the necessity to predict the response of the students and react to it sensitively. In the interview before the experiment, the teacher mentioned that a big part of class is based on intuition and it is difficult to forecast the students' response before beginning the class. The following is the quote from the Teacher A.

> "… Even though I didn't think about in in advance, once in the classroom, it just happens naturally and intuitively. … I don't really think ahead about what response the students will show to what I prepare and how the discussion will develop from that point on. These are not the things that I anticipate before I begin the class."

In the above confession, the teacher says, "once in the classroom, it just happens naturally and intuitively". This demonstrates that the teacher tacitly has some sort of routine and that not only is it difficult to be aware of the routine, but the teacher does not feel the need to be conscious of it.

However, as the teacher participated in the thought experiment, she anticipated the students' response, planned alternatives accordingly, and executed the class and made reflections afterwards. Throughout this process, the teachers focused more on the students and recognized the importance of thought experiment as well as the necessity to be more sensitive to students. The teachers including Teacher A and her colleagues did not know why and how to participate in the thought experiment in the beginning. However, the following statements testify that they became aware of the necessity as the experiment progressed.

"When I was asked (by the researcher) 'what alternatives do you prepare when you plan for a class?', I was disconcerted by the question. I don't know how the students will react. How can I possible prepare alternatives? But when the researcher asked me whether this and that can be alternatives, I said, 'Yes, possibly.' But when I began the class, the students really showed the responses that the researcher predicted. So I was able to use the alternatives that I prepared.' (Teacher A)

"…As I teach more classes, the more I understand the concept of thought experiment. But it's not easy to predict the students' reactions. And I forget the things I thought about, so maybe I should write them down. I need to write down the things I think of while designing the tasks…" (Teacher C)

The teachers became aware of the necessity and importance to be more conscious with their teaching activities as they thought about the purpose of task design and projected the students' responses during the task design and thought experiment processes of this experiment. The above statement by Teacher C affirms the necessity for teachers to play the class in their heads and become more conscious of possible situations through thought experiment in order for teachers' ideas to be executed in their classrooms.

Teacher A, who conducted the class for the grade eight in this experiment, learned to approach the similarity chapter differently from before. Teacher A also mentioned that more learning opportunities could have been possible for the students if the teacher had more in-depth understanding of the mathematical concepts as well as the students. By stating this, the teacher recognized the importance of teachers' sensitivity.

> "…I feel like I could have done a better job of designing the class so that I could get the students to say more in class. If I had processed the information more than I did, I could have had better interactions with the students, and the students could have been more active in my class… (…)" (Teacher A)

> "I think teachers need to be more sensitive and detail-oriented. I think that's where the students develop misconception. I cannot recall in actual class (about students' responses and teacher's reactions accordingly). So I thought about how I handled the classes so far. …I am envious when I see teachers say things like, 'If I did this, the students will probably react this way'. …" (Teacher A)

4.2. Recognition of the teacher's orientation

In the interview in the beginning of the experiment, Teacher A was most concerned with lack of response from the students, and said that this is due to the fact that they are in the eighth grade and they are middle to low level students. As the Teacher conducted the class for this experiment and discussed the class with other teachers while watching the recorded video of the class, she began to look at the lack of students' response in a new light. The teacher came to understand that the questions asked can be unclear for the students since she does not consider the level of the students, and that her tendency to induce the desired answers from the students hinders the communication with them. The teacher came to be aware of her implicit tendency through her colleagues who discussed together. In the beginning of the experiment, Teacher A and other teachers discussed Teacher A's class while watching the recorded video. Two teachers stopped the video when the students were unable to answer Teacher A's question to confirm Teacher A's intention behind the question and reviewed the situation to point out that Teacher A's question can be unclear to the students. The ambiguity of the question was mentioned multiple times and Teacher A came to understand that the situation stemmed from the ambiguity of the question, rather than the students. Teacher A realized that while the question is clear to herself, it can be ambiguous for the students to whom the concept is new. The below conversation between Teacher A and Teacher B demonstrates that Teacher A adopted a new perspective on the lack of response from the students.

Teacher B: The direction of the question is not clear. The scope is too big for the students, and they are not sure where they were wrong in their answers but they think they are wrong when that look at the face of teacher. So they think they shouldn't speak… I could feel that kind of tendency.

Teacher A: We need to narrow down the scope of the question… That is tough to do. I wish that the students try to think really hard. First they keep thinking about the answer in their minds, and all these thoughts and ideas are mixed in their minds and they don't know how to connect the right ideas, and that's when I want to help them… But I'm not really good at it… I taught the first half class (so far), and what I felt through teaching this second half class is that students are worn out when I try to pull something from them. They become tired so easily. They want fast feedback to move one. If I am so excited about the material and keep going one, they become worn out.

Teacher A recognized that not only the ambiguity of the questions, but also her tendency could be the cause of weak communication with the students. While watching the recorded video of the class, Teacher A said that she was amazed at how talkative she was. Teacher A acknowledged that she was leading the class to induce the answers that she is seeking while ignoring students' responses that are different from her intention. It can be seen that Teacher A only attempted to understand the lack of response from the students by focusing on the special characteristics of the students in the beginning. However, as time progresses, Teacher A concentrates more on the ambiguity of the questions from the lack of understanding on students' level as well as her own tendency. As a result, Teacher A recognized that complex mix of factors influence her teaching practice of which she was previous unaware.

4.3. Teacher's recognition of her purpose

From the initial stage of task design, Teacher A continuously mentioned the lack of time by saying that although she would like to offer a variety of activities to students, there are many things to cover and so she is always under the time pressure to make progress; therefore, she has no other choice but to conduct the class in a teacher-oriented teaching style.

On a constant basis, Teacher A always felt the lack of time during task design and class; therefore, had a strong tendency to point to the lack of time as the reason for non-smooth class during reflection session. However, as the experiment progressed, Teacher A increasingly recognized her own problems, such as the questions, tendency and the understanding of the students, affected her actions during class and began to look at the issue of time pressure from a different perspective.

During the reflection process, the teacher realized her goals acted as a hindrance in judging the situation rather than lack of time, which is what the teacher pointed to as the reason for teacher-oriented teaching.

Teacher A: I focused on how much the students understood without thinking about showing some students' activity sheet (the students who solved incorrectly or in a different way). I was going around the class to check how much they understood. If I showed them the objective first and the students wanted to discuss based on the objective, I could have gone around the class with certain purpose. However, I didn't do that. Rather, my purpose was to check how well they understood. (…) I had to cover this part and move onto the next part. So I had to check the students' understanding to move onto the next lesson… I pick certain students who did a good job and show their works to other students and I move on. (…) Rather than telling the students, 'This is how you do it', I had to choose good students' works, show them to the entire class, and move on. I keep feeling uncomfortable with this.

Researcher: Do you think you could have solved this issue if you had more time?

Teacher A: I do think it was my choice. I made a choice under the circumstances.

On the surface, the teacher believed that a constant lack of time posed limitations on teaching activities. However, the teacher recognized that the teacher can perceive the situation differently based on the prioritized goal in the classroom and the teacher’s action can also change accordingly.

5. Discussion

In this chapter, the teacher training procedures for the construction of mathematics teachers’ practical knowledge were applied to an in-service middle school teacher and the practical knowledge constructing process was analyzed along with the changes in the teaching practice. First, the aspects noticed by the teacher during each phase of the thought experiment and the actual class were analyzed by dividing them into three dimensions, i.e., the mathematical, psychological and management of learning dimensions. Second, the teacher’s actual classes were analyzed and divided into two cases: one in which the aspects noticed by the teacher in the thought experiment were reflected in her teaching practices and; the other where they were not. The analysis of the teacher’s noticing was done to examine which aspects were noticed by the teacher and reflected in the actual class, as well as which aspects were not reflected, and why they were not reflected. This was done, by comparing the aspects noticed by the teacher in the thought experiment and in the actual classes. Finally, what the teacher was consciously aware of and how her noticing changed during her reflection of the classes with her colleagues were analyzed. A discussion of the findings follows.

First, the teacher reflected on her previous teaching experience in terms of her mathematical content knowledge for teaching and variation theory which she learned during the theory learning phase, demonstrating broader perspective on mathematical topics and student learning. Knowledge related to a didactical analysis of similarity and variation theory afforded the teacher the chance to become consciously aware of limits and differences with regard to her old teaching practices and to reflect on her practices consciously. This teacher's consciousness of her old practices influenced what the teacher noticed in the subsequent phases with regard to the task design and thought experiment. It was found that theory provides a means of perception on teaching practices, as Korthagen (2001) noted.

Second, the teacher considered the mathematical, psychological and management of learning dimensions in the task design and thought experiment phases. What the teacher noticed in each dimension was divided into different subcategories. The subcategories of what the teacher noticed in the mathematical dimension were recognizing, constructing, recognizing relationships and generalizing. The subcategories for the psychological dimension were anticipating students' responses and considering reactions, encouraging student participation and reflecting on their actions, and considering students' misconceptions and their previous knowledge. The subcategories for the management of learning dimension were classroom organization, interaction and teacher's questions to promote communication.

The teacher and her colleagues discussed mathematical content first and spent most of their time in discussions when they had meetings on task design, whereas they had difficulty anticipating students' responses and finding their level of learning as well. This difficulty led to trouble in questioning to facilitate interaction with students. This result shows that while teachers tend to have a strong level of awareness of mathematical dimension, they have relative weak awareness of psychological dimension and management of learning dimension. The result which showed the teacher's strong levels of awareness of the mathematical dimension can support Lee's (2010) argument that Korean mathematics teachers place emphasis on teaching the structure or essence of mathematical knowledge.

In addition, the finding of teachers' weak awareness of the psychological dimension is similar to the findings of Kim (2009, 2013). According to Kim, participating teachers tended to consider that anticipating students' misconceptions and responses was not possible: the greatest difficulty in the thought experiment involved devising coping strategies for students' responses and proper questions. However, through the teacher training program which taught the planning and analysis of a virtual class, the teachers paid attention to different ideas held by students and allowed students to express their thoughts during the class. Kim's finding shows that teachers' awareness of psychological dimension can be strengthened by teacher education, as in the present study. Furthermore, the results from both studies imply that providing teachers with an opportunity to be conscious of their strengths and weaknesses can play an important role in their finding will to change their practices.

Third, on the one hand, when the teacher's noticing during the thought experiment was reflected in the actual class, the teacher was observing classroom with consciousness of the aspects that she noticed in the thought experiment, and she noticed some opportunities to act in a different way, differing from her habitual behaviors, during which she provided proper learning opportunities to students. It was also confirmed that the teacher's practices changed as the teacher noticed the mathematical, psychological and management of learning dimensions in a balanced way. It can be said that the teacher's noticing in the thought experiment was reflected in her practices during the class, showing that the teacher's practical knowledge about the new way of teaching was being constructed, as she was acting while being aware of the aspects that were focalized in the thought experiment. This demonstrates the possibility that teachers are consciously aware of their unconscious teaching behaviors and that conscious actions by teachers can be habituated again by repeating the actions consciously in the actual class.

On the other hand, when the teacher's noticing was not reflected in the actual class, it was found that while the teacher was conscious of some aspects during the thought experiment, her consciousness was not linked to the actual class, or there were some factors disturbing her consciousness. The teacher noticed things related to mathematical dimension strongly, whereas she had difficulty anticipating students' responses. This trend impeded her appropriate actions to help students' learning at the mathematically meaningful moment during which she noticed. This occurred because she could not grasp the students' learning levels and interact properly with them. However, as the processes of conducting thought experiment, conducting a class and reflecting on the class were ongoing, the teacher recognized the necessity of developing sensitivity to students, attended to the students consciously, and then attempted to change her practices. This shows that teachers' practices can change through the repeated process of noticing and reflection.

There was one case in which the teacher did not provide a proper opportunity to students to explore while using single responses and dichotomy questions, though she noticed a learning opportunity for students. This case demonstrates the teacher's regression to the old way of teaching due to the disconnection between her noticing and her action during the class. In an interview with the teacher at the beginning of this experiment, she described her teaching style and her students' characteristic as follows: She has taught by explanation with the textbook and her students dislike thinking mathematically. This demonstrates her beliefs that her students are passive and merely listen to her rather than think mathematically by themselves and she should concisely deliver mathematical content knowledge to students without errors. These teacher's beliefs about her students and teaching mathematics seem to function as subsidiary awareness which tacitly influenced her teaching practice.

It appears natural that regression to habitual old ways of teaching may occur during the process of trying a new way of teaching, as there is still subsidiary awareness influencing teaching practices, although the teacher is conscious somewhat of this tacit dimension. It is thought that in order to maintain a change in practices and stably establish a new way of teaching, a steady inquiry of the tacit dimensions, a long-term endeavor and social-institutional support is necessary.

Fourth, the teacher perceived classroom events with new perspectives by reflecting on the class and engaging in discussions with her colleagues. In an interview before the experiment of the present study, the teacher stated that it was difficult to anticipate students' responses and consider reactions to responses due to the dynamic classroom situation and the different characteristics of the students. For this reason, she taught by reacting with feelings. As the experiment went on, however, the teacher mentioned the necessity of understanding students and interpreted situations in which proper interaction with students did not occur with a new point of view, knowing that the situation was not merely due to the students' lack of ability, but it was due to her lack of understanding of the students. In addition, the teacher explained that the situation in which students did not answer her questions was due to the characteristics of the students, who were at an awkward age and who were low performers. However, reflection upon the class and a discussion with her colleagues led the teacher to be aware that her question may be unclear to the students due to her orientation and attempts to draw an intended answer from students and due to her lack of understanding of the students' learning ability levels.

In addition, the teacher had a strong tendency to mention time constraints as one of the reasons for the failure of the class at the beginning of her reflection on the class. However, after the teacher discussed the class with her colleagues, she became consciously aware that many factors, such as her orientation, her questions and a lack of understanding of the students may have caused the problematic situations, not merely the students' abilities or external factors such as time constraints. The consciousness of the teacher's orientation, goals, and the necessity of understanding the students afforded the teacher to gain a strong will to change her practices and try to reflect her consciousness in her actual teaching practices.

To sum up, the teacher developed sensitivity to students and an awareness of mathematical contents and pedagogy as she designed tasks and underwent a thought experiment. She also engaged the teaching act with constant attention to the aspects that she noticed during the thought experiment while in class. During the process of reflection, the teacher reflected on her class, considering the new perspectives which she noticed during the learning theory and task design phases. When interpreting problematic situations and her own teaching actions, the teacher also was conscious of factors behind certain phenomena, such as her orientation, goals, and a lack of understanding the students, especially after discussions with her colleagues. This change in the content of her reflection led her to deepen her reflective thinking and to change her teaching practices.

CHAPTER Ⅴ

SUMMARY AND CONCLUSION

The importance of teachers' practical knowledge has gradually been emphasized in the area of mathematics teacher education, but there is not enough discussion about the meaning of practical knowledge and method for the construction of teachers' practical knowledge. The present study was conducted to address this issue related to teachers' practical knowledge in the area of mathematics teacher education. This study focused on an analysis of tacit knowledge, which is invariably associated with teaching practices, and it closely examined the meaning of tacit knowledge through a review of the relevant literature. Based on this analysis, the study embodied the meaning of mathematics teachers' practical knowledge and also suggested a teacher training method for the construction of mathematics teachers' practical knowledge, reflecting the meaning of tacit knowledge, and applied the teacher training procedure to an in-service teacher. A summary of the results of this study and its conclusions are presented in this chapter.

In Chapter 2, the concept of tacit knowledge as defined by Polanyi was examined and tacit knowledge in the context of teacher education was discussed. It was found that tacit knowing has a triad structure. The structure of tacit knowing shows that particulars of subsidiary awareness, which cannot be identified clearly by the subject, are combined with the focal dimension of which the subject is consciously aware, with the combination controlled by the subject given their intentions and purpose. Considering Polanyi's explanation that all of human knowledge and activities are constructed and developed through the interplay between focal awareness and subsidiary awareness, it appears that teachers can construct practical knowledge through the interplay between focal awareness and subsidiary awareness. The interplay between focal awareness and subsidiary awareness of their teaching practices can be understood as meaning that teachers are consciously aware of their unconscious teaching behavior, reflect on these behaviors, and then act habitually again as they practice their conscious actions for some time.

When considering the structure of tacit knowing, if teachers truly understand formal knowledge, their understanding should be expressed in their skillful action as they teach. However, it is not possible to practice effectively and efficiently with only an awareness of teaching action itself, and teachers should be continually aware of situational clues, which are tacit but which can be perceived by teachers from the interaction with their students, as well as sensitivity to certain situations and students. Teachers can undertake proper teaching practices only if they are aware of feelings and sensitivity levels, as influenced by their personal values and beliefs in the tacit dimension.

According to the examination of the meaning of tacit knowledge, there should be a tacit dimension behind teachers' practices, which is not observable but which significantly influences teachers' teaching activities and students' learning. This implies that it is very important to construct appropriate practical knowledge such that teachers can recognize the function of tacit dimension and remain conscious some tacit dimensions and reflect on them. Thus, tacit knowledge should be actively discussed in research related to teachers' practical knowledge.

It was found from the discussion about tacit knowing and teacher awareness that the awareness of mathematics teachers needs to be considered when constructing practical knowledge. Teachers teach students, perceive certain situations, understand the situations, decide what to do, and take the proper action. These processes, which occur during the interaction with students during the class, are not discrete steps but instead represent the overall process from perception to the teacher's action. Most teachers, however, tend to undertake teaching behavior unconsciously. Teacher awareness is not revealed explicitly, but it plays an important role in how they notice, explain, interpret some aspects of their teaching practice and make decisions on the spot (Potari, 2013).

Teacher awareness is closely related to teacher knowledge and subjective experience. Teacher awareness in actual classes is expressed by their action when they perceive certain aspects in light of the combination of their knowledge and situational clues and their sense, which is formed through personal experience according to their purposes and intentions. In order to practice in a proper and sensible manner, teachers need to strengthen their awareness of classroom situations by attending to the essential aspects of the classroom intentionally and consciously, noticing subtle differences in classroom events and reacting appropriately. In other words, it is essential during the construction of meaningful practical knowledge that teachers recognize the significant influence of the tacit dimension behind their actions related to teaching and student learning through a lucid consciousness of their own teaching practices.

In chapter 3, the components necessary for the construction of mathematics teachers' practical knowledge were derived from a review of previous studies related to the concept of practical knowledge and the meaning of tacit knowledge as discussed in chapter2. In addition, the procedure of teacher training for the construction of practical knowledge was suggested on the basis of these components.

First, aspects of the conceptualization of teachers' practical knowledge were placed into two categories, and the aspects stressed in each were identified. The method used to conceptualize practical knowledge was based on direct observation data of teachers' school work, highlighting teachers' personal experiences. As many scholars have argued, however, proper practical knowledge cannot be constructed based on only teachers' experiences, as theoretical knowledge is necessary to organize and reflect on their experiences meaningfully (for example, Dewey, 1904; Mason, 2002; Mellon, 2011). The other means of conceptualizing practical knowledge based on an epistemological investigation of teacher knowledge focuses on the tacit dimension of practical knowledge and emphasizes the necessity of the explication and justification of the tacit dimension, as it affects the teachers' practices. However, there has been no specific discussion of how the tacit dimension can be practically addressed in teacher education. In Chapter 2 in this study, the meaning of tacit knowledge was discussed in the context of teaching practice, and teacher awareness was considered as one of the components necessary for the construction of practical knowledge in mathematics teachers. Consequently, it was found that the construction of meaningful practical knowledge for mathematics teachers requires the three components of their relevant experience, knowledge and awareness.

Teachers' reflections and noticing were suggested as crucial part of a strategy to strengthen teachers' awareness of their teaching practices. It was considered that the development of noticing skills could be an effective means of strengthening teachers' awareness while combining experience and theory in a balanced way, as noticing can help teachers to be sensitized through systematic reflection (Mason, 2002). Previous research on mathematics teacher noticing reviewed and explored how to use noticing as a strategy for the construction of practical knowledge. The results of the review implied that relevant and proper knowledge was necessary to trigger teachers' noticing because noticing occurred while reflecting on actions and perceiving the subtle differences in classroom phenomena. It also implied that teachers need to be guided to attend to the essential aspects of the classroom systematically, interpreting and considering proper reactions through a thought experiment because noticing is closely related to teaching experience and is subconscious but intentional and deliberate. These implications were reflected in establishing the learning theory and thought experiment phases as parts of the teacher training procedure for the construction of mathematics teachers' practical knowledge.

The procedure of teacher training as suggested here had five phases based on the three aforementioned components and teachers' noticing and reflections that were considered an effective means of strengthening teachers' awareness. Learning theory, task design, a thought experiment, conduct of class and analysis of one's class were the five phases. Teachers would learn formal knowledge related to mathematical content and learning theory in the first phase, after which it was expected that learning theory would play an important role in how the teachers reflected on their previous teaching practices and how they perceived subtle changes occurring in their classes. Task design sought to lead teachers to construct a task systematically with clear intention and a clear purpose. The thought experiment phase focused on how teachers could sensitively notice learning opportunities and on situations where they could act in a new way during the actual class by anticipating students' responses and plaining for their reactions through a thought experiment. Teachers would especially be guided during the thought experiment, attending to the three dimensions i.e., the mathematical, psychological and management of learning dimensions. It was expected that they would develop their awareness by attending to and interpreting complex and dynamic classroom situation structurally and deliberately, as they go through the process of implementing and reflecting on their classes based on the thought experiment.

The procedure of teacher training for the construction of practical knowledge was applied to an in-service teacher and the process of constructing practical knowledge was discussed in chapter 4. The results and the discussions were as follows.

First, the participating teacher became conscious of her unconscious and habitual teaching practices and their limits and differences, compared to her old practices with her new knowledge related to similarity concept and variation theory. The teacher was aware that she should provide students with enough opportunities to explore various similar figures before the introduction of a formal definition of similarity. She also recognized how this new way of teaching differed from her old style, in which she explained the definition of similarity at the beginning of the chapter and solved many problems using proportional expressions. The teacher noticed certain aspects related to the mathematical, psychological and management of learning dimensions during the phases of the task design and thought experiment based on explicit knowledge related to a didactical analysis of similarity and variation theory that she learned. The teacher designed tasks by carefully changing examples that were included in the tasks based on variation theory to guide students to pay attention to the essence of similarity concept through an investigation of various figures, and she tried to plan classes consciously and intentionally by perceiving the tasks from students' point of view and reflecting on this.

Second, the teacher provided students with opportunities to grasp the properties of similar figures in the actual class, observing and investigating various figures. She also guided the students to pay attention to the invariance of similarity concept with subtle variances. It was found that the teacher planned teaching activities with a clear purpose and intention, i.e., as to lead students to recognize mathematical concepts included in the activity and not merely to do the activity. She also attempted to provide learning opportunities as intended in the task design and thought experiment, by observing and noticing classroom phenomena in the actual class. It was considered that the teacher came to feel the usefulness of the new way of teaching by connecting her conscious and intentional plan for a class to the actual classroom situations.

Third, the teacher reflected on the aspects that she noticed through teaching practice and considered alternatives ways to act by including her reflections actively in subsequent instruction plans. She then made efforts to notice opportunities to act in a new way during her classes. This shows that the teacher was in the process of constructing practical knowledge about new way of teaching through consciousness, reflection and practice. The procedure of teacher training suggested by the present study provided the teacher with coherent opportunities to notice throughout her intentional and systematic planning, practice and reflection. This helped the teacher to be become more sensitized to situations which she noticed and to develop her awareness of the situations to which explicit knowledge could be applied. It can be said that this teacher training procedure guided the teacher to connect her knowledge to her practices by strengthening her awareness of teaching situations in which explicit knowledge were applied and not merely through the learning of explicit knowledge isolated from the teacher's practice.

Fourth, it was found that the teacher's old ways of teaching were sometimes demonstrated unconsciously or that "Digestive teaching" resulted when the teacher attempted to teach in a new way. This appeared to be influenced by the teacher's beliefs about students as passive and her responsibilities to deal with examinations and progress in class. The teacher's beliefs and responsibilities are particulars of subsidiary awareness formed by her long teaching experience. The teacher could not be aware explicitly of her beliefs and responsibilities but the subsidiary awareness still functioned while her practice was changing to the new method.

In conclusion, it is possible for mathematics teachers to construct practical knowledge through teacher training with the procedural process of consciousness (turning subsidiary awareness into focal awareness) and unconsciousness (turning focal awareness into subsidiary awareness), as this study suggested. In other words, teachers conduct their teaching activities consciously with will to change their practices by being conscious of subsidiary awareness in the tacit dimension and reflecting on them, and their conscious activity becomes habitual through repetitive practice. Thus, mathematics teachers' practical knowledge can be constructed through conscious and unconscious processes. In this study, it was found that the teacher attempted to change her teaching practices through conscious methods, and she reflected on her practices, noting that they were changing in the actual class. However, it takes a long time for changed teaching practices to become habitual in the tacit dimension; thus, future monitoring is necessary to confirm the persistence of the teacher's changed teaching practices.

This study also noted a regression to the teacher's old teaching practices through the function of subsidiary awareness, such as the teacher's beliefs about the students and her responsibilities for examinations and progress in class. Two implications can be derived from this finding. First, teachers are in a society and the institutions around them and therefore construct practical knowledge as they are influenced, adapting to their circumstances. Thus, social and institutional forms of support are necessary to maintain changes in teachers' practices and to construct practical knowledge about the changed practices. Next, it is possible to disturb this change in practices, even when teachers attempt to change their practices by consciousness and reflection, because the tacit dimension is substantially behind the explicit dimension and subsidiary awareness remains tacit and influences the teachers' practices strongly after they become conscious of some part of the tacit dimension. This implies that teachers need to be provided with continued opportunities for investigation, consciousness and reflection on their tacit dimension.

The findings of this study indicate that teachers need to cooperate with their colleagues to become conscious of the tacit dimension during the process of constructing their practical knowledge. The tacit dimension has an aspect of which teachers can become conscious, whereas it has other aspects of which teachers can become conscious through their colleagues, who may have different teaching experiences and perspectives. For example, it may be difficult for an experienced teacher to be conscious of their beliefs about mathematics teaching and learning or routines formed by the teacher's long teaching experience, as they are too obvious for her or him, but the teacher's colleagues, with different perspectives, can trigger teacher's consciousness and reflection.

This study is meaningful in that the meaning of teachers' practical knowledge was embodied with an understanding of tacit knowledge. Furthermore, this study took a theoretical and practical approach to teachers' practical knowledge, suggesting a procedure of teacher training for the construction of practical knowledge and investigating the process of the construction of teachers' practical knowledge based on the meaning of practical knowledge, which consists of both explicit and tacit dimensions.

This study, however, is limited when used to justify the procedure of teacher training and the process of constructing practical knowledge in mathematics teachers and when generalizing the results in that it was a case study. Thus, more empirical studies are needed to verify the results of the study. In addition, it is difficult to address practical knowledge because this concept is multifaceted and vague. Practical knowledge, however, is an essential part of mathematics teachers' professionalism. Therefore, an analysis of practical knowledge from different points of view and ways to construct practical knowledge needs to be actively discussed.

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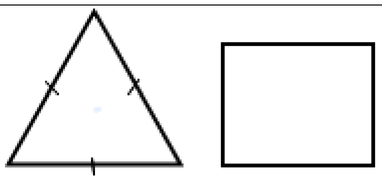
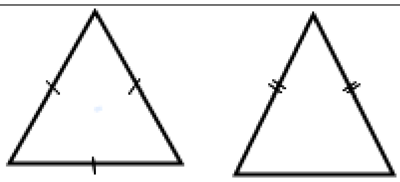
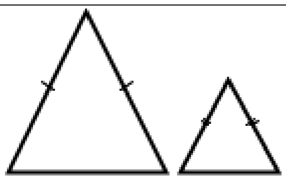
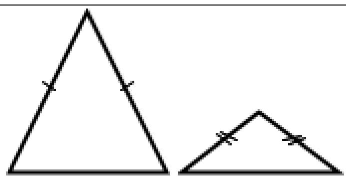
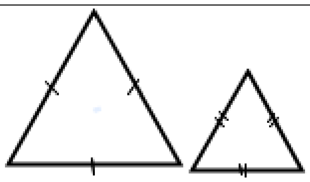
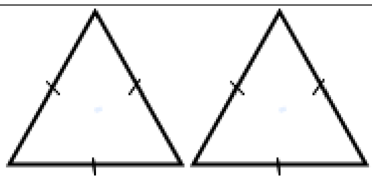
APPENDIX

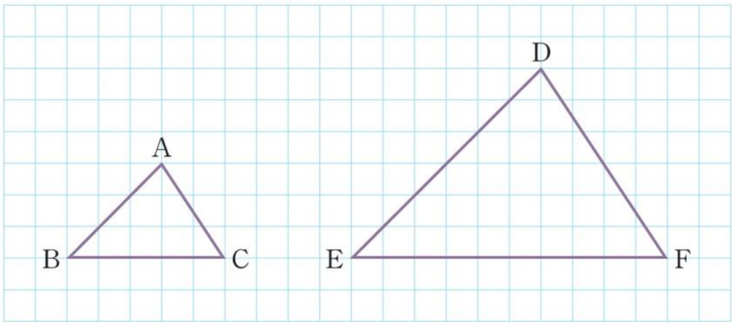
☐ Tasks for the first class

♠ 1. Which one do you think is paired up with the similar figures?

1)	 (○, ×) reason:	2)	 (○, ×) reason:
3)	 (○, ×) reason:	4)	 (○, ×) reason:
5)	 (○, ×) reason:	6)	 (○, ×) reason:

☐ My opinion about the definition of similar figure:

♠ 2. In the following picture, △ABC and △DEF are figures similar to each other. Look at the picture and write down what 'similar figure' is.



※ When you enlarge or reduce by (   ) one figure to become (   ) to other figure, we say they are in similarity relationship, and two figures in similar relationship are called similar figures.

☐ Tasks for the second class

♠ 1. Use ruler, compass or protractor to draw a similar figure of the given figure by (enlarging, reducing) it twice.

♠ 2. Draw a similar figure by using graph paper.

1) Draw a figure similar to the given figure by (enlarging/reducing) it (   ) times.

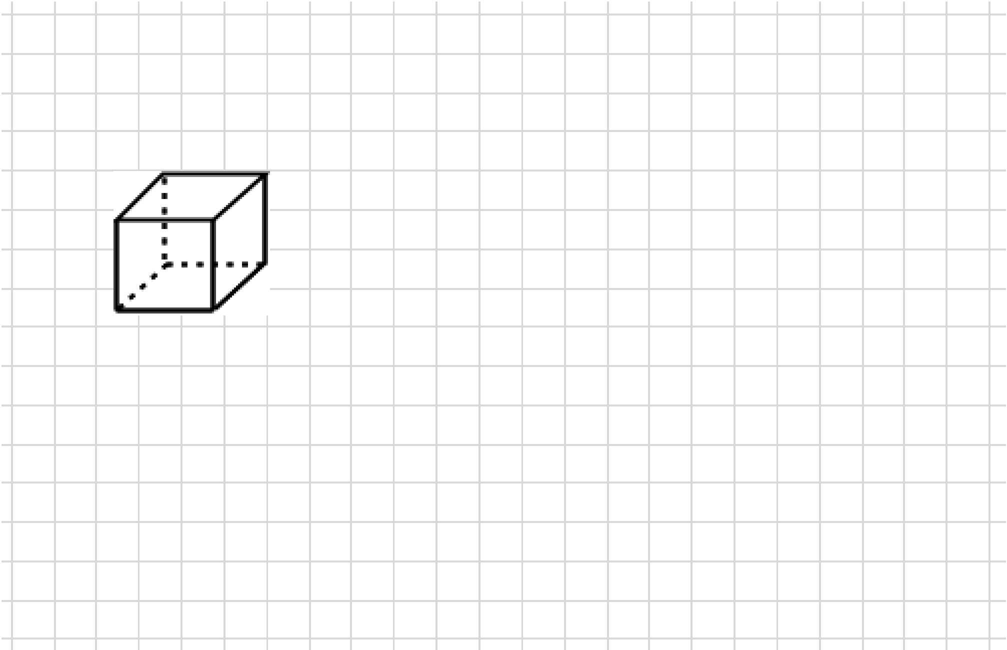
2) Draw a similar figure by (enlarging, reducing) the size of the lattice of the graph paper (   ) times.

☐ Tasks for the third class

♠ 1. Let’s do the following activities with the wooden building blocks.

1) Use the wooden building blocks to make a similar figure which enlarged the cube building block by (   ) times.

2) Draw the sketch map of the similar figure enlarged (   ) times you drew on graph paper in 1).



3) Draw the sketch map of the similar figure which enlarged the triangular prism building block (   ) times.

☐ Tasks for the fourth class

♠ 1. Distinguish the figures which are always similar and those which are not always similar, and explain the reason with drawing.

1) Two regular pentagons	2) Two isosceles trapezoid	3) Two parallelogram
4) Two isosceles triangles	5) Two circles	6) Two sectors
7) Two cylinders	8) Two rectangular solids	9) Two tetrahedron

☐ Figures always similar:

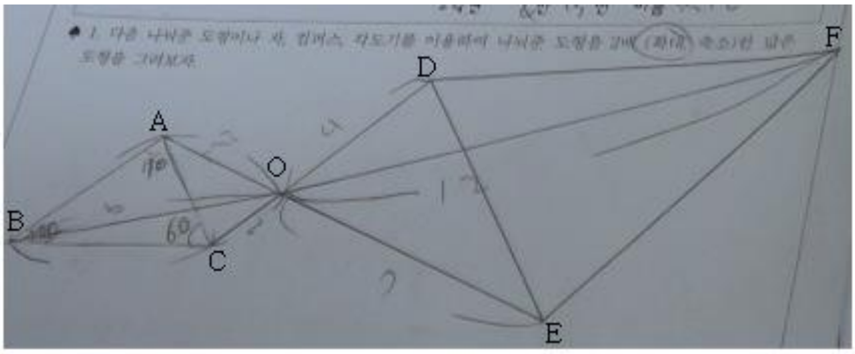
The reason (drawing):

☐ Figures not always similar:

The reason (drawing):

□ Tasks for the fifth class

♠ 1. The following is the picture of the figure which doubled the figure given in the second class.



1) How was $\triangle DEF$ drawn?

2) $\triangle ABC$ $\triangle$

3) What can we find out when we see $\overline{AC}$ and $\overline{DE}$?

4) Can you find any other similar figures other than the two triangles?

<Riview>

(1) The straight line which connects the corresponding points of the two similar figures like the picture above is always ( ), the point O is ( ), and we say the two figures are ( )

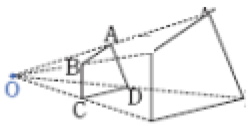
(2) When two figures are in a position of similarity, the ratio of distance from the center of the similarity to the corresponding point is ( ), and it is same as ( )

♠ 2. When two quadrilaterals have similarity, the center of the similarity can be in different places like the following. Let’ s find out the place of vertex according to the center of similarity. (showing GeoZebra)

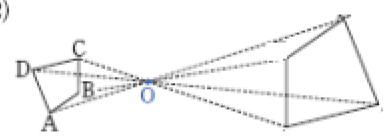
1) The place of vertex when the center of similarity is at ( )

(A→A', B→B', C→C', D→D')

(1)

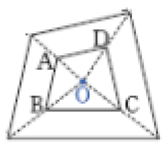


(2)

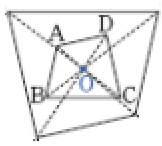


2) The place of vertex when the center of similarity is at ( )

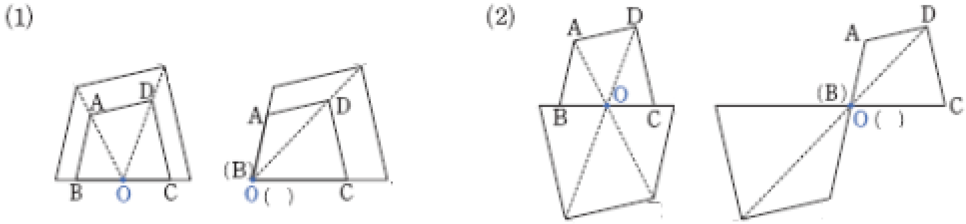
(1)



(2)

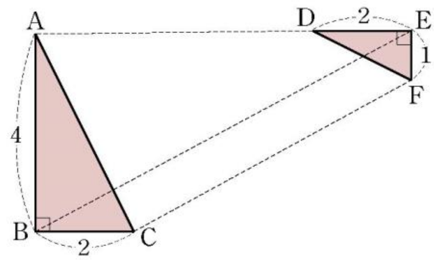


3) The place of vertex when the center of similarity is on the circumference of the figure.



♠ 3. This is an explanation on the following picture.

In the right picture, △ABC and △DEF are not in a position of similarity. Therefore, △ABC and △DEF are not similar.



국문 초록

수학교사의 실천적 지식 구성 과정에 대한 연구

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이 은 정

본 연구는 수학교사 교육에서 교사의 실천적 지식의 중요성이 점차 강조되고 있으나, 실천적 지식의 의미와 구성 방법에 대한 논의가 부족하다는 문제의식에서 출발하였다. 특히, 수학교사의 실천적 지식 구성을 위한 구체적인 방법에 대한 논의가 이루어지기 위해서는 교사의 실천에 반드시 수반되는 암묵적 지식에 대한 분석이 선행되어야 함에 주목하였다. 이에 본 연구에서는 문헌 분석을 통해 교사의 실천에 관여하는 다양한 지식과 더불어 암묵적 지식의 의미를 면밀히 파악하였다. 교수학적 내용지식, 수학을 가르치기 위한 지식 등 분절적으로 수학교사의 전문적 지식에 대해 이루어진 기존의 논의에, 수학교사의 실천을 둘러싼 암묵적 차원에 대한 연구를 보완함으로써 수학교사의 실천적 지식의 의미를 구체화하였으며, 실제 수학교사의 실천적 지식 구성 과정을 확인하기 위하여 과제 설계에서부터 수업 실행에 이르는 일련의 과정을 분석하였다.

폴라니의 인식론에 근거하여 암묵적 지식의 의미를 살펴본 결과는 다음과 같다. 먼저 암묵적 지식은 삼원적 구조로 이루어지며, 인식자, 보조식, 초점식이 그 세 요소이다. 인식자는 보조식의 세목들이 무엇인지 정확하게 식별할 수는 없지만 그것들을 제어할 수 있다. 또, 인식자는 특정한 목적과 의도를 가지고 의식하는 초점 대상에 보조식의 세목들을 관련시킬 수 있다. 이 과정에 의하여 초점식과 보조식이 교대되면서 기술과 지식이 발달한다. 이와 같은 관점에 비추어 보면, 교사는 자신의 신념과 같은 암묵적 지식과 무의식적으로 행하는 교수 활

동을 반성하고 의식화하고, 의식적으로 교수 행위를 하면서 다시 무의식적으로 행하게 되는 과정을 통해 실천적 지식을 구성하고 발달시켜나간다고 볼 수 있다. 요컨대, 교사의 실천적 지식은 명시적 차원과 암묵적 차원이 존재하며, 교사가 암묵적 차원의 작용을 인식하고 이를 성찰하는 것은 교사의 실천적 지식 구성 과정에서 매우 중요하다는 점을 확인하였다.

교사의 식(awareness)은 교사의 지식과 더불어 주관적인 경험과 관련이 있다. 실제 수업 상황에서의 교사의 식(awareness)은 교사의 의도와 목적에 따라 주관적인 경험을 통해 형성된 감각과 상황으로부터의 단서가 교사의 지식과 결합되어 세부적인 사항들을 지각함으로써 교사의 행동으로 드러나게 된다. 적절하고 합리적인 방식으로 수업을 실행하기 위해서, 교사는 수업에서 중요하게 다루어져야 할 측면들에 의도적이고 의식적으로 주의를 기울이고 지속적으로 관찰하면서 수업에서 발생하는 사건들의 미묘한 차이들을 알아차리고 이에 적절한 대응을 경험해나가면서 그 상황에 대한 식(awareness)을 키워나갈 필요가 있다. 교사의 식(awareness)에 대한 논의로부터 도출한 시사점은 교사가 자신의 교수 활동을 의식화함으로써 자신의 행동 이면에 존재하는 암묵적 차원이 수업과 학생의 학습에 중요한 영향을 미치고 있다는 것을 깨닫게 되면서 실천적 지식을 구성할 수 있다는 것이다.

실천적 지식에 대한 선행연구와 암묵적 지식에 대한 논의를 토대로, 수학교사의 실천적 지식 구성을 위해서는 교사의 실천 경험, 이론적 지식 그리고 교사의 식(awareness)의 세 요소가 필요함을 확인하였다. 또한 교사의 식(awareness)을 강화시키기 위한 전략으로 교사의 반성과 알아차림(noticing)을 제안하였다. 알아차림은 체계적인 반성을 기반으로 교사의 민감성을 키울 수 있기 때문에 교사의 식(awareness)을 강화시키고 경험과 명시적인 지식이 조화롭게 균형을 이루는데 효과적인 방법이 될 수 있다. 수학교사의 실천적 지식 구성을 위한 세 요소와 반성과 알아차림 전략을 토대로 교사교육의 절차를 다음과 같은 다섯 단계로 설계하였다. 이론학습, 과제설계, 사고실험, 수업실행, 수업에 대한 분석이 그것이다. 본 연구에서는 중학교 교사가 이 절차에 따라 실천적 지식을 구성하는 과정에 대하여 살펴보았다.

연구 결과, 실천적 지식의 구성 과정은 크게 초점식화와 보조식화로 이루어진

다는 것을 알 수 있었다. 초점식화 과정에서 교사는 암묵적 차원의 보조식들을 의식화하고 반성함으로써 실천을 변화시키려는 의지를 가지고 의식적인 교수 활동을 해가면서 실천을 개선한다. 이어서 보조식화 과정에서 다시 무의식적인 교수 활동을 하게 된다. 본 연구에 참여한 교사는 의식화와 반성에 의한 초점식화 그리고 이를 다시 보조식화 과정을 교대로 거치면서 실천적 지식을 구성하였다. 그러나 변화된 교수 활동이 습관화되어 안정적으로 교사의 암묵적 차원으로 자리 잡기 위해서는 장시간이 필요하기 때문에 이에 대해서는 향후 추적 관찰이 필요할 것으로 보인다.

본 연구에서는 암묵적 지식에 대한 면밀한 분석을 통해 실천적 지식의 의미를 구체적으로 밝히고 이를 토대로 실천적 지식 구성을 지원하는 절차를 제안하였다. 또한, 중학교 수학교사가 이 절차에 따라 실천적 지식을 구성하는 과정에 대하여 살펴봄으로써 실천적 지식에 관한 이론적이고 실제적인 접근을 시도하였다. 실천적 지식은 그 특성이 다면적이고 모호하여 다루는 것이 쉽지 않다. 그러나 실천적 지식은 수학교사의 전문성을 이루는 핵심적인 요소이므로 다각도로 이해하고 분석하여 그 특성을 밝히고 구성 방안을 모색할 필요가 있다.

주요어: 실천적 지식, 암묵적 지식, 폴라니, 식(識, awareness), 반성, 의식화

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