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Finite Element Solutions for the Space Fractional Diffusion Equation with a Nonlinear Source Term

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Finite Element Solutions for the Space Fractional Diffusion Equation with a Nonlinear Source Term

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Abstract

The anomalous diffusion problem has been played a significant role in many areas. In this paper, we consider finite element Galerkin solutions for the space fractional diffusion equation with a nonlinear source term. We derive the variational formula of the semi-discrete scheme by using the Galerkin finite element method in space. Existence of the semi-discrete solution for the equation is shown. The stability and the order of convergence of approximate solutions for the semi-discrete equation have been also discussed.

Furthermore, we derive the fully discrete time-space variational formulation using the backward Euler method. Existence of numerical solutions for the backward Euler fully discrete scheme is shown by using the Brouwer fixed point theorem. The stability and error estimates of solutions for the fully discrete approximate solutions are studied along the lines of the semi-discrete analysis. The order of convergence are obtained as $O(k + h^{\tilde{\gamma}})$, where $\tilde{\gamma}$ is a constant depending on the order of fractional derivative.

Numerical computations are presented, which confirm the theoretical results when the equation has a linear source term. When the equation has a nonlinear source term, numerical results show that the diffusivity depends on the order of fractional derivative as we discuss in theoretical analysis.

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Key words: Anomalous diffusion, Fractional derivative, Galerkin finite element method.
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Chapter 1

Introduction

Fractional calculus is an old mathematical topic but it has not been attracted enough for almost three hundred years. However, it has been recently proven that fractional calculus is a significant tool in the modeling of many phenomena in various fields such as engineering, physics, porous media, economics, biological sciences. One can see related references in [16, 20, 21, 33, 34, 36, 37].

In the classical diffusion model, it is assumed that particles are distributed in a normal bell-shaped pattern based on the Brownian motion. But anomalous diffusion is one of the phenomena in nature and presents in a wide variety of physical situations. This processes of anomalous diffusion are various, including processes with infinite mean-square displacement and processes with distributed orders of fraction leading to the mean-square displacement for power laws with a time dependent exponent. In general, the nature of diffusion is characterized by the mean squared displacement

$$\langle (\Delta r(t))^2 \rangle = \langle (r(t) - \langle r(t) \rangle)^2 \rangle = 2d\kappa_\mu t^\mu,$$

where d is the spatial dimension and κ_μ is the diffusion constant. Here $r(t)$ denotes the position of the particle at time t and $\langle \cdot \rangle$ denotes the mean of the distributed data. For the classical normal diffusion case, the exponent $\mu = 1$. For example, the probability density function of the normal Brownian motion is described as

$$p(x,t) = \frac{1}{\sqrt{4\pi\kappa_1 t}} \exp(-x^2/4\kappa_1 t)$$

which satisfies the classical diffusion equation

$$\frac{\partial p(x,t)}{\partial t} = \kappa_1 \frac{\partial^2 p(x,t)}{\partial x^2}.$$

Otherwise, it is anomalous. The anomalous diffusion is classified as either sub-diffusive(diffusive slowly) when $\mu < 1$ or super-diffusive(diffusive fast) when $\mu > 1$. Lim and Muniandy [25] have considered the probability density function of time fractional Brownian motion is given by, for $0 < \mu < 2$

$$p(x,t) = \frac{1}{\sqrt{4\pi\kappa_\mu t^\mu}} \exp(-x^2/4\kappa_\mu t^\mu)$$

which satisfies the anomalous diffusion equation

$$\frac{\partial p(x,\tau)}{\partial \tau} = \kappa_\mu \frac{\partial^2 p(x,\tau)}{\partial x^2}$$

using the following nonlinear time transformation

$$t \to \tau = t^\mu.$$

As mentioned before, in many real world problems, it is more adequate to use anomalous diffusion described by fractional derivatives than the normal diffusion [3, 13, 15, 26, 27, 33, 34]. This means that fractional diffusion equations are useful for applications in which a cloud of particles spreads faster than the classical diffusion model predicts. In general, the fractional sub-diffusion corresponds to the divergence of microscopic time scales in random walk schemes. One typical fractional super-diffusion equation arises in chaotic dynamics and turbulent processes, where the usual second derivative in space is replaced by a fractional derivative of order $1 < \mu < 2$. In particular, Metzler and Klafter [33, 34] have studied that Lévy flight is one of anomalous diffusions with a diverging mean squared displacement $\langle x^2(t) \rangle \to \infty$ where the closed-form representation of a Lévy stable law $W(x,t) \sim \kappa_\mu t/|x|^{1+\mu}$ with $\mu < 2$. In contrast to the nature of a regular random walk, Lévy flight consists of a self-similar clustering of local sojourns interrupted by long jumps. The associated distribution is heavy tailed with slowly decaying spatial correlations, a signature of non-Gaussian processes with diverging variance. The probability density function of the symmetric Lévy flight

$$\mathcal{F}^{-1}\{\exp(-\kappa_\mu t|x|^\mu)\} \sim \frac{\kappa_\mu t}{|x|^{1+\mu}}$$

corresponds to the space fractional diffusion equation

$$\frac{\partial p(x,t)}{\partial t} = \kappa_\mu \frac{\partial^\mu p(x,t)}{\partial |x|^\mu}.$$

In this paper we discuss approximate solutions for the space fractional diffusion equation with a nonlinear source term. The equation is described as

$$\frac{\partial u(x,t)}{\partial t} = \kappa_\mu \nabla^\mu u(x,t) + f(x,t,u) \tag{1.1}$$

with an initial condition

$$u(x,0) = u_0,\ x \in \Omega \subset \mathbb{R}, \tag{1.2}$$

and boundary conditions

$$u(x,t) = 0,\ x \in \partial\Omega,\ 0 \le t \le T, \tag{1.3}$$

where κ_μ denotes the anomalous diffusion coefficient and $\partial\Omega$ is the boundary of the domain Ω. And the fractional differential operator ∇^μ is

$$\nabla^\mu = \frac{1}{2}{}_aD_x^\mu + \frac{1}{2}{}_xD_b^\mu,$$

where ${}_aD_x^\mu$ and ${}_xD_b^\mu$ are called the left and the right Riemann-Liouville space fractional derivatives of order μ, respectively, defined by

$$\begin{aligned}\mathbf{D}^\mu u := {}_aD_x^\mu u(x) &= D^n {}_aD_x^{\mu-n} u(x) \\ &= \frac{1}{\Gamma(n-\mu)} \frac{d^n}{dx^n} \int_a^x (x-\xi)^{n-\mu-1} u(\xi) d\xi\end{aligned}$$

and

$$\begin{aligned}\mathbf{D}^{\mu *} u := {}_xD_b^\mu u(x) &= (-D)^n {}_xD_b^{\mu-n} u(x) \\ &= \frac{(-1)^n}{\Gamma(n-\mu)} \frac{d^n}{dx^n} \int_x^b (\xi-x)^{n-\mu-1} u(\xi) d\xi.\end{aligned}$$

Here n is the smallest integer such that $n-1 \leq \mu < n$.

Throughout this paper, we will assume that the nonlinear source term $f(x,t,u)$ is locally Lipschitz continuous with constants C_l and C_f such that

$$\|f(u)-f(v)\|_{L^2(\Omega)} \leq C_l \|u-v\|_{L^2(\Omega)} \tag{1.4}$$

and

$$\|f(u)\|_{L^2(\Omega)} \leq C_f \|u\|_{L^2(\Omega)} \tag{1.5}$$

for $u,v \in \{w \in H_0^{\frac{\mu}{2}}(\Omega) | \, \|w\|_{L^2(\Omega)} \leq l\}$.

Baeumer, Kovács and Meerschaert [2, 3] have proved existence and uniqueness of a strong solution for (1.1) using the semigroup theory when $f(x,t,u)$ is globally Lipschitz continuous. Furthermore, when $f(x,t,u)$ is locally Lipschitz continuous, existence of a unique strong solution has also been shown by introducing the cut-off function.

Finite difference methods have been applied in [30, 31, 32] for linear space fractional diffusion problems. It is proven that the standard Grüwald-Letnikov approximates for the fractional derivative gives the unconditional instability even for the implicit method. Using the right-shifted Grüwald approximates which is given by for $1 < \alpha \leq 2$

$$\frac{\partial^\alpha u(x,y,t)}{\partial x^\alpha} = \frac{1}{\Gamma(-\alpha)} \lim_{N_x \to \infty} \frac{1}{h^\alpha} \sum_{k=0}^{N_x} \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} u(x-(k-1)h, y, t),$$

where N_x is a positive integer, they show that the implicit Euler method is unconditionally stable and the explicit Euler method, based on the modified Grüwald-Letnikov approximates, is also stable under the conditions. Using the right-shifted Grüwald-Letnikov approximation, the method of lines has been applied in [27] for numerical approximate solutions.

For the space fractional diffusion problems with a nonlinear source term, Lynch, Carrera, del-Castillo-Negrete, Ferreira-Mejias and Hicks [29] used the so called L2 and L2C methods [36] which are discretization methods of the fractional derivative of order α is replaced by an $\alpha-2$ fractional integral of the second derivative, the second derivative is approximated by the standard three-point and four-point centered finite difference formula, respectively, and the fractional integral is approximated by the modified Grüwald-Letnikov approximates. They compared computational accuracy of them. Baeumer, Kovács and Meerschaert [3] give existence of the solution and computational results using finite difference methods. Choi, Chung and Lee [8] have shown existence and stability of numerical solutions of an implicit finite difference equation obtained by using the right-shifted Grüwald-Letnikov approximation and the backward Euler scheme. For the time fractional diffusion equations, explicit and implicit finite difference methods have been used in [12, 14, 22, 26, 28, 41].

Compared to finite difference methods on the fractional diffusion equation, finite element methods have been rarely discussed. Ervin and Roop [18] have considered finite element analysis for stationary linear advection dispersion equation is given by, for $0 \le \alpha < 1$

$$-D\, a(p\, {}_0D_x^{-\alpha} + q\, {}_xD_1^{-\alpha})Du + b(x)Du + c(x)u = f,$$

where D is a single spatial derivative and ${}_0D_x^{-\alpha}$ and ${}_xD_1^{-\alpha}$ represent the left and the right fractional integral operators, respectively, with $0 \le p,\ q \le 1$ satisfying $p + q = 1$.

Roop [38] has studied finite element analysis for non-stationary linear advection dispersion equations in $\mathbb{R}^2$: for $0 \le \alpha < 1$

$$\frac{\partial u}{\partial t} - \int_{\|\nu\|=1} D_\nu a D_\nu^{-\alpha} D_\nu u M(d\nu) + \mathbf{b} \cdot \nabla u + cu = f,$$

where $\mathbf{b}$ is the velocity of the fluid, cu represents a reaction-absorption term, f is a linear source term, $a > 0$ is the diffusivity constant coefficient, M is a probability measure on the unit circle, D_ν is the directional derivative in the direction of the unit vector ν and $D_\nu^{-\alpha}$ is the α order fractional integral.

Deng [13] has discussed finite element numerical approximations for the time and space fractional Fokker-Planck equation: for $0 < \alpha < 1$ and $1 < \beta < 2$

$$\frac{\partial u}{\partial t} = {}_0D_t^{1-\alpha}\Big[\frac{\partial}{\partial x}\frac{E'(x)}{\eta_\alpha} + \kappa_\alpha \frac{1}{2}({}_aD_x^{\mu} + {}_xD_b^{\mu})\Big]u,$$

where $u(x,t)$ is the probability density, $E(x)$ is an external potential, κ_α denotes the anomalous diffusion coeffient, η_α represents the generalized friction coefficient and ${}_aD_x^\mu$, ${}_xD_b^\mu$ are the left and right Riemann-Liouville space fractional derivatives of order μ, respectively.

Ervin, Heuer and Roop [19] has studied finite element analysis for the space general fractional diffusion equations with a nonlocal quadratic nonlinearity but a linear source term. Their equation is given by for $1 < \alpha \leq 2$

$$\frac{\partial u}{\partial t} + D^\alpha u - \nabla \cdot (uB(u)) = f,$$

where u is the density of particles, D^α denotes a general fractional order diffusion operator of order α and $\nabla \cdot (uB(u))$ models particle interactions.

As far as we know, finite element methods have not been considered for the space fractional diffusion equation with nonlinear source terms. In this paper, we will discuss finite element approximate solutions for the problem (1.1) – (1.3) under the assumption of existence of a sufficiently regular solution u of the equation.

We consider the fully discrete Galerkin finite element method. We will use the backward Euler method in time and Galerkin finite element method in space. We will discuss existence, uniqueness and stability of the numerical solutions for the problem (1.1) – (1.3). Also, L^2-error estimate will be considered for the problem (1.1) – (1.3).

The outline of the paper is as follows. In Chapter 2, we introduce the mathematical preliminaries on the space fractional derivatives, which will be used for the analysis. In Chapter 3, the semi-discrete variational formulation for (1.1) based on Galerkin method is introduced. Existence, stability, and L^2-error estimate of the semi-discrete solution are analyzed. In Chapter 4, existence, unconditional stability of approximate solutions for the fully discrete backward Euler method are shown following the idea of the semi-discrete method in Chapter 3. Further, L^2-error estimates are obtained, whose convergence is of $O(k + h^{\tilde{\gamma}})$, where $\tilde{\gamma} = \mu$ if $\mu \neq \frac{3}{2}$ and $\tilde{\gamma} = \mu - \epsilon$, $0 < \epsilon < \frac{1}{2}$ if $\mu = \frac{3}{2}$. Finally, numerical examples are given in Chapter 5 in order to see the theoretical convergence order discussed in previous chapters. We will see that numerical solutions of fractional diffusion equations diffuse more slowly than that of the classical diffusive problem and diffusivity depends on the order of fractional derivatives.

Chapter 2

Theoretical preliminaries

In many interesting partial differential equation problems, existence and uniqueness of classical solution to the problem have not been proved yet, or it is hard to find the explicit formula of the solution even if one can prove the existence and uniqueness of the solution [17]. Thus, the numerical computations to seek a solution of partial differential equation problems is highly demanded in most research areas. In order to obtain numerical approximate solutions for the problem, we have to consider a finite dimensional vector space with a suitable formulation.

In this chapter, we introduce the abstract theories related to the Galerkin finite element method which acts a key role in approximation theory. In general, most of theories to solve partial differential equations are established on the Sobolev space. The Galerkin finite element method is also based on Sobolev space, especially, Hilbert spaces, for obtaining the existence and uniqueness of solution for the problem. Thus, we begin with the definition of Sobolev space in Section 2.1. In Section 2.2 we present basic materials for analysis of finite element method which will be used in later discussion. In Section 2.3 we recall the definitions and theoretical results on Riemann-Liouville fractional calculus. We introduce the left and right fractional derivatives. The equivalence of the fractional derivative spaces with fractional order Hilbert spaces is established.

2.1 Sobolev spaces

The Sobolev spaces are the most important spaces to estimate the solutions for partial differential equations and these spaces are defined based on L^p spaces. We begin with a series of definitions notation on the Sobolev spaces [1]. In the following, Ω denotes an open domain in $\mathbb{R}^n$.

Definition 2.1. *Let $L^p(\Omega)$, $p \geq 1$, be a set of all functions defined by*

$$L^p(\Omega) = \left\{ u \middle| \|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx \right)^{1/p} < \infty \right\}. \tag{2.1}$$

Then it is well known that the function space $\big(L^p(\Omega), \|\cdot\|_{L^p(\Omega)}\big)$ is a complete normed space with respect to the norm $\|\cdot\|_{L^p(\Omega)}$.

We now define the concept of weak derivative of a function.

Definition 2.2. *Let $u, v \in L^1_{loc}(\Omega)$, locally $L^1(\Omega)$, and $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a multi-index notation. Then the $\alpha^{th}-$weak derivative v of u is defined by, for every test function $\phi \in C_c^\infty(\Omega)$*

$$\int_\Omega u(x) D^\alpha \phi(x) dx = (-1)^{|\alpha|} \int_\Omega v(x)\phi(x) dx,$$

where $C_c^\infty(\Omega)$ denotes the space of infinitely differentiable functions with compact support in Ω and

$$D^\alpha = \frac{\partial^{|\alpha|} u}{(\partial x_1)^{\alpha_1} \cdots (\partial x_n)^{\alpha_n}},\ |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n,\ \alpha_i \geq 0.$$

Definition 2.3. *Let $p \geq 1$ and m be a nonnegative integer. Then the Sobolev space $W^{m,p}(\Omega)$ is defined by*

$$W^{m,p}(\Omega) = \{u | D^\alpha u \in L^p(\Omega), \forall |\alpha| \leq m\}. \tag{2.2}$$

Now, we define the semi-norm $|\cdot|_{m,p,\Omega}$ and norm $\|\cdot\|_{m,p,\Omega}$, respectively, in Sobolev space $W^{m,p}(\Omega)$ for $1 \leq p < \infty$,

$$|u|_{m,p,\Omega} = \left(\sum_{|\alpha|=m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p}$$

and

$$\|u\|_{m,p,\Omega} = \left(\sum_{|\alpha| \leq m} |u|^p_{m,p} \right)^{1/p}.$$

Then the normed linear space $\left(W^{m,p}(\Omega), \|\cdot\|_{m,p,\Omega}\right)$ becomes a complete normed linear space. In particular, when $p = 2$, $W^{m,2}$ is denoted by H^m. In this case H^m denotes a Hilbert space, in which the inner product is defined by

$$(u, v)_m = \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v), \tag{2.3}$$

where $(u, v) = \int_\Omega uv dx$.

When the spaces have generalized homogeneous boundary conditions in H^m, denote $H^m_0 = \{u \in H^m | u^{(i)} = 0, i = 0, 1, \cdots, m-1\}$. Hereafter, the seminorm $|\cdot|_m$ and the norm $\|\cdot\|_m$ will stand for $|\cdot|_{m,p,\Omega}$ and $\|\cdot\|_{m,p,\Omega}$, respectively. And $\|\cdot\|_0$ will stand for $\|\cdot\|_{L^2}$. Then we have the Poincaré-Friedrichs inequality [5].

Lemma 2.1. *Suppose Ω is contained in an n-dimensional cube with side length s. Then we have the following inequality*

$$\|v\|_0 \leq s|v|_1, \ \forall v \in H^1_0(\Omega). \tag{2.4}$$

Proof. Since $C^\infty_0(\Omega)$ is dense in $H^1_0(\Omega)$, it suffices to establish the inequality for $v \in C^\infty_0(\Omega)$. We may assume that $\Omega \subset X := \{(x_1, x_2, \cdots, x_n); 0 < x_i < s\}$, and set $v = 0$ for $x \in X \setminus \Omega$. Then

$$v(x_1, x_2, \cdots, x_n) = v(0, x_2, \cdots, x_n) + \int_0^{x_1} \partial_1 v(t, x_2, \cdots, x_n) dt.$$

Since the boundary term vanishes and the Cauchy-Schwarz inequality gives

$$\begin{aligned} |v(x)|^2 &\leq \int_0^{x_1} 1^2 dt \int_0^{x_1} |\partial_1 v(t, x_2, \cdots, x_n)|^2 dt \\ &\leq s \int_0^s |\partial_1 v(t, x_2, \cdots, x_n)| dt. \end{aligned}$$

Since the right-hand side is independent of x_1, it follows that

$$\begin{aligned}\int_0^s |v(x)|^2 dx_1 &\leq \int_0^s s \int_0^s |\partial_1 v(t, x_2, \cdots, x_n)| dt dx_1 \\ &\leq s^2 \int_0^s |\partial_1 v(x)|^2 dx_1.\end{aligned}$$

To complete the proof, we integrate over the other coordinates to obtain

$$\int_X |v|^2 dx \leq s^2 \int_X |\partial_1 v|^2 dx \leq s^2 |v|_1^2.$$

□

Applying Poincaré-Friedrichs' inequality to derivatives, we see that

$$\|\partial^\alpha v\|_0 = |\partial^\alpha v|_0 \leq s|\partial_1 \partial^\alpha v|_0, \ \forall\ |\alpha| \leq m-1, \ \forall v \in H_0^m(\Omega).$$

Taking the mathematical induction on Poincaré-Friedrichs' inequality, we may have the following Lemma 2.2, which tells the seminorm and the norm are equivalent in $H_0^m(\Omega)$.

Lemma 2.2. *If Ω is bounded, the seminorm $|\cdot|_m$ on $H_0^m(\Omega)$ is equivalent to the norm $\|\cdot\|_m$. In fact, if Ω is contained in a cube with side length s, then the following inequality holds;*

$$|v|_m \leq \|v\|_m \leq (1+s)^m |v|_m \tag{2.5}$$

for all $v \in H_0^m(\Omega)$.

Proof. For $v \in H_0^m(\Omega)$, (2.4) implies that

$$|v|_1^2 \leq \|v\|_1^2 = \|v\|_0^2 + |v|_1^2 \leq (1+s^2)|v|_1^2.$$

Similarly we obtain

$$|v|_2^2 \leq \|v\|_2^2 = \|v\|_0^2 + |v|_1^2 + |v|_2^2 \leq s^2|v|_1^2 + s^2|v|_2^2 + |v|_2^2 \leq (1+s)^4|v|_2^2.$$

The successive iterations of this inequality to derivatives $D^\alpha v$, $|\alpha| \leq m-1$ leads to (2.5). This completes the proof. □

2.2 Materials for the analysis

In this section, we present a series of theorems which are basic materials for analysis of finite element approximate solutions.

In order to show that there exists a weak solution for the problem (1.1) – (1.3), we will use Brouwer's fixed point theorem.

Theorem 2.1. *Let H be a finite-dimensional inner product space. Suppose that $g : H \to H$ is continuous and there exists λ such that $(g(x), x) \geq 0$, for all $x \in H$ with $\|x\| = \lambda$. Then there exists $x^* \in H$ such that $g(x^*) = 0$ and $\|x^*\| \leq \lambda$.*

After showing the existence of a weak solution, we want to find numerical solution of the weak solution. Since the weak solution is obtained in an infinitely dimensional space, it is not easy to calculate the weak solution. Hence, instead of weak solution, it is necessary to find approximate solution of the weak solution. In this case, we may need the following approximation property which tells the error between the solution and the interpolation of the solution. The following lemma will be needed to prove the optimal error estimates [6, 13].

Property 2.1. *For $v \in H^\gamma(\Omega)$, $0 < \gamma \leq n$ and $0 \leq s \leq \gamma$, there exists a constant C depending only on Ω such that*

$$\|v - I^h v\|_{H^s(\Omega)} \leq C h^{\gamma - s} \|v\|_{H^\gamma(\Omega)}, \tag{2.6}$$

where $I^h v$ is the interpolation of v and h is the largest length of intervals.

Proof. Let S_h be a partition of Ω with a grid parameter h such that $\bar{\Omega} = \{\cup K | K \in S_h\}$ and $h = max_{K \in S_h} h_K$, where h_K is the width of the subinterval K. Each K can thus be considered to be a scaled version of a reference $\hat{K}$. Suppose $K_h := h\hat{K} = \{x = hy | y \in \hat{K}\}$ with $h \leq 1$.

Given a function $v \in H^\gamma(K_h)$, we define $w \in H^{\gamma(\hat{K})}$ by $w(y) = v(hy)$. Then $\partial^\alpha w = h^{|\alpha|} \partial^\alpha v$ for $|\alpha| \leq \gamma$. Since the transformation of the length in $\mathbb{R}$ yields an extra factor h^{-1}, we get

$$|w|^2_{l,\hat{K}} = \sum_{|\alpha|=l} \int_{\hat{K}} (\partial^{\alpha w})^2 dy = \sum_{|\alpha|=l} \int_{K_h} h^{2l} (\partial^\alpha v)^2 h^{-1} dx = h^{2l-1} |v|^2_{l,K_h}. \tag{2.7}$$

Assuming $h \leq 1$, after summation the smallest power dominates:

$$\|v\|_{s,K_h}^2 = \sum_{l\leq s} |v|_{l,K_h}^2 = \sum_{l\leq s} h^{-2l+1}|w|_{l,\hat{K}}^2 \leq h^{-2s+1}\|w\|_{s,\hat{K}}^2.$$

Taking $v = v - I^h v$ in the above inequalities and using (2.7), we have

$$\begin{aligned} \|v - I^h v\|_{s,K_h}^2 &\leq h^{-2s+1}\|w - I^h w\|_{s,\hat{K}}^2 \\ &\leq h^{-2s+1}\|w - I^h w\|_{\gamma,\hat{K}}^2 \\ &\leq h^{-2s+1}C|w|_{\gamma,\hat{K}}^2 \\ &\leq h^{2\gamma-2s}C|v|_{\gamma,K_h}^2 \end{aligned}$$

for all $s \leq \gamma$. Here the constant C is independent of h. This completes the proof. □

Since the norm $\|\cdot\|_{H^s(\Omega)}$ is equivalent to the seminorm $|\cdot|_{H^s(\Omega)}$, we may replace the inequality (2.6) by the relation

$$\|v - I^h v\|_{H^s(\Omega)} \leq Ch^{\gamma-s}|v|_{H^\gamma(\Omega)}. \tag{2.8}$$

Hereafter the constant C denotes a generic constant independent of h.

The following Young's inequality will be used judiciously for later analysis [39].

Lemma 2.3. *For $a, b \in \mathbb{R}$ and p, q are positive real numbers connected by the relationship $\frac{1}{p} + \frac{1}{q} = 1$. Then the inequality*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{2.9}$$

holds.

Proof. The result is obvious if $a = 0$ or $b = 0$. So without loss of generality assume that $a > 0$ and $b > 0$. Then

$$ab = e^{\ln ab} = e^{\frac{1}{p}\ln a^p + \frac{1}{q}\ln b^q} \leq \frac{1}{p}e^{\ln a^p} + \frac{1}{q}e^{\ln b^q} = \frac{a^p}{p} + \frac{b^q}{q}.$$

□

It follows from Lemma 2.3 that for $\epsilon > 0$

$$\begin{aligned} ab &= \big((2\epsilon)^{\frac{1}{2}}a\big)\left(\frac{b}{(2\epsilon)^{\frac{1}{2}}}\right) \\ &\le \frac{1}{2}\left(\big((2\epsilon)^{\frac{1}{2}}a\big)^2 + \left(\frac{b}{(2\epsilon)^{\frac{1}{2}}}\right)^2\right) \\ &= \frac{1}{2}\big(2\epsilon a^2 + \frac{b^2}{2\epsilon}\big) \\ &= \epsilon a^2 + \frac{1}{4\epsilon}b^2. \end{aligned}$$

Thus we may use the following form

$$ab \le \epsilon a^2 + \frac{1}{4\epsilon}b^2 \tag{2.10}$$

instead of the form (2.9).

The Gronwall's lemma will play an important role in the semidiscrete analysis when we show the stability of a weak solution. The proof of Gronwall's inequality can be seen in Larsson and Thomée [23].

Lemma 2.4. *Suppose that g is a nonnegative continuous function such that for $t > 0$*

$$g(t) \le a + b\int_0^t g(s)ds,$$

where a and b are nonnegative constants. Then

$$g(t) \le ae^{bt} \text{ for } t > 0.$$

Proof. For $t > 0$, let $h(t) := \int_0^t bg(s)ds$. Then

$$h'(t) = bg(t) \le ab + bh(t).$$

By multiplication with $e^{-bt} > 0$, we have

$$\frac{d}{dt}\big(h(t)e^{-bt}\big) \le abe^{-bt}.$$

Integrating both sides with respect to t, we obtain

$$h(t)e^{-bt} \le \int_0^t ag(s)e^{-bs}ds.$$

Since $g(t) \le a + h(t)$, the proof is completed. □

But when we try to show the stability of fully discrete finite element Galerkin scheme, we may not use the above Gronwall's inequality. Instead of Lemma (2.4), we may need the discrete version of continuous type Gronwall's inequality, which will be called as the discrete Gronwall's inequality [7].

Lemma 2.5. *Assume that $G(n)$, $a(n)$, and $w(n)$ are three sequences of real nonnegative numbers such that*

$$G(n) \leq a(n) + \sum_{i=0}^{n-1} w(i)G(i),\ n = 1, 2, \cdots .$$

Furthermore, assume that $a(n)$ is nondecresing. Then

$$G(n) \leq a(n) \exp\left(\sum_{i=0}^{n-1} w(i)\right).$$

Proof. For $m \leq n$, we can let $H(m) = a(n) + \sum_{i=0}^{m-1} w(i)G(i)$. Then

$$\begin{aligned} H(m) &= a(n) + \sum_{i=0}^{m-2} w(i)G(i) + w(m-1)G(m-1) \\ &= H(m-1) + w(m-1)G(m-1) \\ &\leq (1 + w(m-1))H(m-1) \\ &\leq \exp(w(m-1))H(m-1). \end{aligned}$$

Since $H(0) = a(n)$, the desired result follows. □

2.3 The fractional derivative spaces

In this section, we introduce some basic concepts of fractional derivative and their properties. First, we define the fractional integral operators in terms of the Riemann-Liouville definition given in [13, 18, 37].

Definition 2.4. *Let u be a function defined on $\Omega = (a, b)$ and $\alpha > 0$. Then the left Riemann-Liouville fractional integral of order α is defined as*

$$ {}_aD_x^{-\alpha}u(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1}u(s)ds, \tag{2.11}$$

where $\Gamma(\alpha)$ is the gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\alpha-1}dx$.

Similarly to the left Riemann-Liouville fractional integral, we define the right Riemann-Liouville fractional integral.

Definition 2.5. *Let u be a function defined on $\Omega = (a, b)$ and $\alpha > 0$. Then the right Riemann-Liouville fractional integral of order α is defined as*

$$ {}_xD_b^{-\alpha}u(x) := \frac{1}{\Gamma(\alpha)}\int_x^b (s-x)^{\alpha-1}u(s)ds. \tag{2.12}$$

We now note the semigroup property of the Riemann-Liouville frational integral operators on $\Omega = (a, b)$ as outlined in [13, 18, 37].

Property 2.2. *The semigroup property holds for the Riemann-Liouville fractional integral operators: $\forall\alpha, \beta > 0$, if $u \in L^p(\Omega)$, $p \geq 1$, then*

$$\begin{aligned} {}_aD_x^{-\alpha}\,{}_aD_x^{-\beta}u(x) &= \ {}_aD_x^{-\alpha-\beta}u(x),\ \forall x \in \Omega, \\ {}_xD_b^{-\alpha}\,{}_xD_b^{-\beta}u(x) &= \ {}_xD_b^{-\alpha-\beta}u(x),\ \forall x \in \Omega. \end{aligned} \tag{2.13}$$

Proof. For $u \in L^p(\Omega)$,

$$\begin{aligned} {}_aD_x^{-\alpha}\,{}_aD_x^{-\beta}u(x) &= \frac{1}{\Gamma(\beta)}\int_a^x (x-s)^{\beta-1}\,{}_aD_x^{-\alpha}u(s)ds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x (x-s)^{\beta-1}ds\int_a^s (s-\xi)^{\alpha-1}u(\xi)d\xi \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x u(\xi)d\xi\int_\xi^x (x-s)^{\beta-1}(s-\xi)^{\alpha-1}ds \\ &= \frac{1}{\Gamma(\alpha+\beta)}\int_a^x (x-\xi)^{\alpha+\beta-1}u(\xi)d\xi \\ &= {}_aD_x^{-\alpha-\beta}u(x). \end{aligned}$$

Similarly we obtain

$$ {}_xD_b^{-\alpha}\,{}_xD_b^{-\beta}u(x) = {}_xD_b^{-\alpha-\beta}u(x).$$

□

The fractional integral operators also satisfy the adjoint property [13].

Property 2.3. *The Riemann-Liouville fractional integral operators are adjoints in L^2 sense: $\forall \alpha > 0$,*

$$({}_aD_x^{-\alpha}u, v)_{L^2(\Omega)} = (u, {}_xD_b^{-\alpha}v)_{L^2(\Omega)} \tag{2.14}$$

for all $u, v \in L^2(\Omega)$.

Proof. For $u, v \in L^2(\Omega)$,

$$\begin{aligned} ({}_aD_x^{-\alpha}u, v)_{L^2(\Omega)} &= \frac{1}{\Gamma(\alpha)} \int_a^b \int_a^x (x-s)^{\alpha-1} u(s) v(x) ds dx \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b u(s) \int_s^b (x-s)^{\alpha-1} v(x) dx ds \\ &= (u, {}_xD_b^{-\alpha}v)_{L^2(\Omega)}. \end{aligned}$$

□

Now, we define the Riemann-Liouville fractional derivatives analogous to the fractional integral.

Definition 2.6. *Let u be a function defined on $\mathbb{R}$, $\mu > 0$, n be the smallest integer such that $n - 1 \leq \mu < n$. Then the left Riemann-Liouville fractional derivatives of order μ is defined to be*

$$\begin{aligned} \mathbf{D}^{\mu}u &:= D^n{}_{-\infty}D_x^{\mu-n}u(x) \\ &= \frac{1}{\Gamma(n-\mu)} \frac{d^n}{dx^n} \int_{-\infty}^x (x-\xi)^{n-\mu-1} u(\xi) d\xi. \end{aligned} \tag{2.15}$$

Definition 2.7. *Let u be a function defined on $\mathbb{R}$, $\mu > 0$, n be the smallest integer such that $n - 1 \leq \mu < n$. Then the right Riemann-Liouville fractional derivatives of order μ is defined to be*

$$\begin{aligned} \mathbf{D}^{\mu*}u &:= (-D)^n{}_xD_{\infty}^{\mu-n}u(x) \\ &= \frac{(-1)^n}{\Gamma(n-\mu)} \frac{d^n}{dx^n} \int_x^{\infty} (\xi-x)^{n-\mu-1} u(\xi) d\xi. \end{aligned} \tag{2.16}$$

If $\overline{supp(u)} \subset (a, b)$, then by extending u to be 0 outside a compact set $\overline{supp(u)}$ we may see that

$$\mathbf{D}^{\mu}u = {}_aD_x^{\mu}u, \ \mathbf{D}^{\mu*}u = {}_xD_b^{\mu}u.$$

Thus ${}_aD_x^{\mu}u$ and ${}_xD_b^{\mu}u$ are the left and right Riemann-Liouville fractional derivatives of order μ, respectively, expressed by

$$\mathbf{D}^{\mu}u = {}_aD_x^{\mu}u(x) = \frac{1}{\Gamma(n-\mu)}\frac{d^n}{dx^n}\int_a^x (x-\xi)^{n-\mu-1}u(\xi)d\xi \tag{2.17}$$

and

$$\mathbf{D}^{\mu*}u = {}_xD_b^{\mu}u(x) = \frac{(-1)^n}{\Gamma(n-\mu)}\frac{d^n}{dx^n}\int_x^b (\xi-x)^{n-\mu-1}u(\xi)d\xi. \tag{2.18}$$

We now define the left fractional derivative space [18].

Definition 2.8. *For any given positive number $\mu > 0$, define the semi-norm*

$$|u|_{J_L^{\mu}(\mathbb{R})} = \|\mathbf{D}^{\mu}u\|_{L^2(\mathbb{R})}$$

and the norm

$$\|u\|_{J_L^{\mu}(\mathbb{R})} = \left(\|u\|_{L^2(\mathbb{R})}^2 + |u|_{J_L^{\mu}(\mathbb{R})}^2\right)^{\frac{1}{2}},$$

where the left fractional derivative space $J_L^{\mu}(\mathbb{R})$ denotes the closure of $C_0^{\infty}(\mathbb{R})$ with respect to the norm $\|\cdot\|_{J_L^{\mu}(\mathbb{R})}$.

Similarly, we may define the right fractional derivative space.

Definition 2.9. *For any given positive number $\mu > 0$, define the semi-norm*

$$|u|_{J_R^{\mu}(\mathbb{R})} = \|\mathbf{D}^{\mu*}u\|_{L^2(\mathbb{R})}$$

and the norm

$$\|u\|_{J_R^{\mu}(\mathbb{R})} = \left(\|u\|_{L^2(\mathbb{R})}^2 + |u|_{J_R^{\mu}(\mathbb{R})}^2\right)^{\frac{1}{2}},$$

where the right fractional derivative space $J_R^{\mu}(\mathbb{R})$ as the closure of $C_0^{\infty}(\mathbb{R})$ with respect to the norm $\|\cdot\|_{J_R^{\mu}(\mathbb{R})}$.

Furthermore, with the help of Fourier transform we define the a seminorm for functions in $H^\mu(\mathbb{R})$.

Definition 2.10. *Let $\mu > 0$. Define the semi-norm*

$$|u|_{H^\mu(\mathbb{R})} = \||\omega|^\mu \hat{u}\|_{L^2(\mathbb{R})} \tag{2.19}$$

and the norm

$$\|u\|_{H^\mu(\mathbb{R})} := (\|u\|^2_{L^2(\mathbb{R})} + |u|^2_{H^\mu(\mathbb{R})})^{\frac{1}{2}}. \tag{2.20}$$

Here $H^\mu(\mathbb{R})$ denotes the closure of $C_0^\infty(\mathbb{R})$ with respect to $\| \cdot \|_{H^\mu(\mathbb{R})}$.

Lemma 2.6. *Let $\mu > 0$. A function $u \in L^2(\mathbb{R})$ belongs to $J_L^\mu(\mathbb{R})$ if and only if*

$$|\omega|^\mu \hat{u} \in L^2(\mathbb{R}). \tag{2.21}$$

Specifically,

$$|u|_{J_L^\mu(\mathbb{R})} = \||\omega|^\mu \hat{u}\|_{L^2(\mathbb{R})}. \tag{2.22}$$

Proof. Let $u \in J_L^\mu(\mathbb{R})$ be given. By using the definitions of $\mathbf{D}^\mu u \in L^2(\mathbb{R})$ and the Fourier transform property [18], we obtain

$$\mathcal{F}(\mathbf{D}^\mu u) = (i\omega)^\mu \hat{u}.$$

Using Plancherel's theorem, we have

$$\int_{\mathbb{R}} |\omega|^{2\mu} |\hat{u}|^2 d\omega = \int_{\mathbb{R}} |\mathbf{D}^\mu u|^2 dx.$$

Hence,

$$\||\omega|^\mu \hat{u}\|_{L^2(\mathbb{R})} = |u|_{J_L^\mu(\mathbb{R})}.$$

□

Using Lemma 2.6, we obtain the following theorem.

Theorem 2.2. *Let $\mu > 0$. The spaces $J_L^\mu(\mathbb{R})$ and $H^\mu(\mathbb{R})$ are equal with equivalent seminorms and norms. Also, the spaces $J_L^\mu(\mathbb{R})$ and $J_R^\mu(\mathbb{R})$ are equal with equivalent seminorms and norms.*

Proof. Using Lemma 2.6, we obtain that $J_L^\mu(\mathbb{R})$ and $H^\mu(\mathbb{R})$ are equal with equivalent seminorms and norms. Now we verify that $J_L^\mu(\mathbb{R})$ and $J_R^\mu(\mathbb{R})$ seminorms are equivalent. By using the definitions of $\mathbf{D}^\mu u \in L^2(\mathbb{R})$, the Fourier transform property [18] and Plancherel's theorem, we obtain

$$\begin{aligned} |u|^2_{J_L^\mu(\mathbb{R})} &= \int_{\mathbb{R}} |(i\omega)^\mu \hat{u(\omega)}|^2 d\omega, \\ |u|^2_{J_R^\mu(\mathbb{R})} &= \int_{\mathbb{R}} |(-i\omega)^\mu \hat{u(\omega)}|^2 d\omega. \end{aligned} \tag{2.23}$$

This completes the proof. $\square$

The following lemma on the Riemann-Liouville fractional integral operators will be used in the analysis, which can be proved by using the property of Fourier transform [18].

Lemma 2.7. *For a given $\mu > 0$ and a real valued function u*

$$(\mathbf{D}^\mu u, \mathbf{D}^{\mu *} u) = \cos(\pi\mu) \|\mathbf{D}^\mu u\|^2_{L^2(\mathbb{R})}. \tag{2.24}$$

Proof. By using the Fourier transform property[18], we have

$$\int_{\mathbb{R}} u\bar{v} dx = \int_{\mathbb{R}} \hat{u}\bar{\hat{v}} d\omega \tag{2.25}$$

and

$$\overline{(i\omega)^\mu} = \begin{cases} e^{-i\pi\mu}\overline{(-i\omega)^\mu} & \text{if } \omega \geq 0 \\ e^{i\pi\mu}\overline{(-i\omega)^\mu} & \text{if } \omega < 0. \end{cases} \tag{2.26}$$

Thus

$$\begin{aligned} (\mathbf{D}^\mu u, \mathbf{D}^{\mu *} u) &= \left(D^n{}_{-\infty} D_x^{-\sigma} u, \overline{(-D)^n{}_x D_\infty^{-\sigma} u} \right) \\ &= \left((i\omega)^\mu \hat{u}, \overline{(-i\omega)^\mu \hat{u}} \right) \\ &= \int_{-\infty}^0 (i\omega)^\mu \hat{u} \overline{(-i\omega)^\mu \hat{u}} d\omega + \int_0^\infty (i\omega)^\mu \hat{u} \overline{(-i\omega)^\mu \hat{u}} d\omega. \end{aligned}$$

Using (2.26) in the above equalities, we obtain

$$\begin{aligned}
(\mathbf{D}^\mu u, \mathbf{D}^{\mu*} u) = & \int_{-\infty}^{0} (i\omega)^\mu \hat{u} e^{-r\pi\mu} \overline{(i\omega)^\mu \hat{u}} d\omega \\
& + \int_{0}^{\infty} (i\omega)^\mu \hat{u} e^{i\pi\mu} \overline{(i\omega)^\mu \hat{u}} d\omega \\
= & \cos(\pi\mu) \int_{-\infty}^{\infty} (i\omega)^\mu \hat{u} \overline{(i\omega)^\mu \hat{u}} d\omega \\
& + i \sin(\pi\mu) \int_{0}^{\infty} (i\omega)^\mu \hat{u} \overline{(i\omega)^\mu \hat{u}} d\omega \\
& - i \sin(\pi\mu) \int_{-\infty}^{0} (i\omega)^\mu \hat{u} \overline{(i\omega)^\mu \hat{u}} d\omega.
\end{aligned} \tag{2.27}$$

For $f(x)$ real, we have $\overline{\mathcal{F}(f)(-\omega)} = \mathcal{F}(f)(\omega)$. Thus

$$\int_{0}^{\infty} (i\omega)^\mu \hat{u} \overline{(i\omega)^\mu \hat{u}} d\omega = \int_{-\infty}^{0} (i\omega)^\mu \hat{u} \overline{(i\omega)^\mu \hat{u}} d\omega. \tag{2.28}$$

Combining (2.27) and (2.28), we obtain

$$(\mathbf{D}^\mu u, \mathbf{D}^{\mu*} u) = \cos(\pi\mu)(\mathbf{D}^\mu u, \mathbf{D}^\mu u).$$

$\square$

Remark 2.1. *It follows from (2.24) that we may use the following norm*

$$\|u\|^2_{H_0^{\frac{\mu}{2}}(\mathbb{R})} = \|u\|^2_{L^2(\mathbb{R})} + \kappa_\mu \big|\cos(\pi \cdot \frac{\mu}{2})\big| |u|^2_{H_0^{\frac{\mu}{2}}(\mathbb{R})} \tag{2.29}$$

instead of the norm $\|u\|_{H^\mu(\mathbb{R})}$.

It is known in Theorem 2.2 that the left and right fractional derivative spaces and Hilbert space are all equal with equivalent semi-norms and norms. Thus we have the following lemma.

Lemma 2.8. *Let $\mu > 0$. Then spaces $J_L^\mu(\mathbb{R})$, $J_R^\mu(\mathbb{R})$ and $H^\mu(\mathbb{R})$ are equivalent. Also, the semi-norms and norms of $J_L^\mu(\mathbb{R})$, $J_R^\mu(\mathbb{R})$ and $H^\mu(\mathbb{R})$ are equivalent when $\mu \neq n - \frac{1}{2}$, $n \in \mathbb{N}$.*

Now, we restrict the fractional derivative spaces to Ω which is a bounded open interval on $\mathbb{R}$. We may define the left and right fractional derivative spaces and Hilbert space on Ω.

Definition 2.11. *For $\mu > 0$, define the left fractional derivative space $J^{\mu}_{L,0}(\Omega)$, the right fractional derivative space $J^{\mu}_{R,0}(\Omega)$ and $H^{\mu}_0(\Omega)$ denote the closure of $C^{\infty}_0(\Omega)$ with respect to the norm $\|\cdot\|_{J^{\mu}_{L,0}(\Omega)}$, $\|\cdot\|_{J^{\mu}_{R,0}(\Omega)}$ and $\|\cdot\|_{H^{\mu}_0(\Omega)}$, respectively.*

Lemma 2.9. *For $u \in J^{\mu}_{L,0}(\Omega)$,*

$$\mathbf{D}^{-\mu}\mathbf{D}^{\mu}u = u$$

and for $u \in J^{\mu}_{R,0}(\Omega)$

$$\mathbf{D}^{-\mu*}\mathbf{D}^{\mu*}u = u.$$

Proof. Since $u \in J^{\mu}_{L,0}(\Omega)$, there exists a sequence $\{v_n\}_{n=1}^{\infty} \subset C^{\infty}_0(\Omega)$ such that $\lim_{n\to\infty} \|u - v_n\|_{J^{\mu}_{L,0}(\Omega)} = 0$. Using the triangle inequality, we obtain

$$\begin{aligned}\|\mathbf{D}^{-\mu}\mathbf{D}^{\mu}u - u\|_{J^{\mu}_{L,0}(\Omega)} \leq\ & \|\mathbf{D}^{-\mu}\mathbf{D}^{\mu}(u - v_n)\|_{J^{\mu}_{L,0}(\Omega)} \\ & + \|\mathbf{D}^{-\mu}\mathbf{D}^{\mu}v_n - v_n\|_{J^{\mu}_{L,0}(\Omega)} + \|v_n - u\|_{J^{\mu}_{L,0}(\Omega)}.\end{aligned}$$

Applying the Fourier transform property [18], we have $\|\mathbf{D}^{-\mu}\mathbf{D}^{\mu}v_n - v_n\|_{J^{\mu}_{L,0}(\Omega)} = 0$. It follows from the mapping properties [18] that

$$\|\mathbf{D}^{-\mu}\mathbf{D}^{\mu}u - u\|_{J^{\mu}_{L,0}(\Omega)} \leq C\|u - v_n\|_{J^{\mu}_{L,0}(\Omega)}.$$

Thus

$$\|\mathbf{D}^{-\mu}\mathbf{D}^{\mu}u - u\|_{J^{\mu}_{L,0}(\Omega)} \leq (C + 1)\|u - v_n\|_{J^{\mu}_{L,0}(\Omega)}.$$

Taking the limit as $n \to \infty$, we obtain $\mathbf{D}^{-\mu}\mathbf{D}^{\mu}u = u$.

Similarly we have $\mathbf{D}^{-\mu*}\mathbf{D}^{\mu*}u = u.$ □

The fractional differential operators also satisfy a semigroup property which are analogous to fractional integral operators.

Lemma 2.10. *For* $u \in J^{\mu}_{L,0}(\Omega)$, $0 < s < \mu$

$$\mathbf{D}^{\mu}u = \mathbf{D}^{s}\mathbf{D}^{\mu-s}u$$

and for $u \in J^{\mu}_{R,0}(\Omega)$, $0 < s < \mu$

$$\mathbf{D}^{\mu*}u = \mathbf{D}^{s*}\mathbf{D}^{(\mu-s)*}u.$$

Proof. It follows from Property 2.2 and Lemma 2.9 that for $u \in J^{\mu}_{L,0}(\Omega)$

$$u = \mathbf{D}^{-\mu}\mathbf{D}^{\mu}u = \mathbf{D}^{s-\mu}\mathbf{D}^{-s}\mathbf{D}^{\mu}u.$$

Applying $\mathbf{D}^{s}\mathbf{D}^{\mu-s}$ to above equation

$$\mathbf{D}^{s}\mathbf{D}^{\mu-s}u = \mathbf{D}^{s}\mathbf{D}^{\mu-s}\mathbf{D}^{s-\mu}\mathbf{D}^{-s}\mathbf{D}^{\mu}u = \mathbf{D}^{\mu}u.$$

We can similarly prove the result for $J^{\mu}_{L,0}(\Omega)$. $\square$

For the semi-norm on $J^{\mu}_{L,0}(\Omega)$ and $J^{\mu}_{R,0}(\Omega)$ with $\Omega = (a, b)$, the following fractional Poincaré-Friedrichs inequality holds.

Lemma 2.11. *For* $u \in J^{\mu}_{L,0}(\Omega)$*, there is a positive constant* C *such that*

$$\|u\|_{L^2(\Omega)} \leq C|u|_{J^{\mu}_{L}(\Omega)}$$

and for $u \in J^{\mu}_{R,0}(\Omega)$,

$$\|u\|_{L^2(\Omega)} \leq C|u|_{J^{\mu}_{R}(\Omega)}.$$

Proof. It follows from the mapping properties [18] and Lemma 2.9 that for $u \in J^{\mu}_{L,0}(\Omega)$

$$\|u\|_{L^2(\Omega)} = \|\mathbf{D}^{-\mu}\mathbf{D}^{\mu}u\|_{L^2(\Omega)} \leq C\|\mathbf{D}^{\mu}u\|_{L^2(\Omega)} = C|u|_{J^{\mu}_{L,0}(\Omega)}.$$

Similarly we have $\|u\|_{L^2(\Omega)} \leq C|u|_{J^{\mu}_{R}(\Omega)}$. $\square$

Lemma 2.12. *For* $u \in J^{\mu}_{L,0}(\Omega)$, $0 < s < \mu$*, there is a positive constant* C *such that*

$$|u|_{J^{s}_{L}(\Omega)} \leq C|u|_{J^{\mu}_{L}(\Omega)}$$

and for $u \in J^{\mu}_{R,0}(\Omega)$, $0 < s < \mu$

$$|u|_{J^{s}_{R}(\Omega)} \leq C|u|_{J^{\mu}_{R}(\Omega)}.$$

Proof. For $u \in J^{\mu}_{L,0}(\Omega)$, using Lemma 2.10

$$\begin{aligned}\|\mathbf{D}^s u\|_{L^2(\Omega)} &= \|\mathbf{D}^s\mathbf{D}^{-\mu}\mathbf{D}^{\mu}u\|_{L^2(\Omega)}\\ &= \|\mathbf{D}^s\mathbf{D}^{-\mu}\mathbf{D}^{s-\mu}\mathbf{D}^{\mu}u\|_{L^2(\Omega)}\\ &= \|\mathbf{D}^{s-\mu}\mathbf{D}^{\mu}u\|_{L^2(\Omega)}\\ &\leq C\|\mathbf{D}^{\mu}u\|_{L^2(\Omega)}.\end{aligned}$$

Similarly we have $|u|_{J^s_R(\Omega)} \leq C|u|_{J^{\mu}_R(\Omega)}$. □

When the domain Ω is a bounded interval, it is analogous to Lemma 2.8 that the left and right fractional derivative spaces and Hilbert space are all equal with equivalent semi-norms and norms [5, 18].

Theorem 2.3. *Let $\mu > 0$. Then spaces $J^{\mu}_{L,0}(\Omega), J^{\mu}_{R,0}(\Omega)$ and $H^{\mu}_0(\Omega)$ are equivalent. For $\mu \neq n-\frac{1}{2}$, $n \in \mathbb{N}$, the semi-norms and norms of $J^{\mu}_{L,0}(\Omega), J^{\mu}_{R,0}(\Omega)$ and $H^{\mu}_0(\Omega)$ are equivalent.*

Proof. Let $u \in C^{\infty}_0(\Omega)$ and $\tilde{u}$ be the extension of u by zero outside of Ω. Applying the definition (2.19) and (2.23) , we have

$$\|\mathbf{D}^{\mu}u\|_{L^2(\Omega)} = |u|_{J^{\mu}_L(\Omega)} \leq |\tilde{u}|_{J^{\mu}_L(\mathbb{R})} = |\tilde{u}|_{H^{\mu}(\mathbb{R})} = |u|_{H^{\mu}(\Omega)}.$$

Thus, $H^{\mu}_0(\Omega) \subset J^{\mu}_{L,0}(\Omega)$.
Also, using Young's inequality, we obtain

$$\begin{aligned}|u|^2_{H^{\mu}_0(\Omega)} &\leq C\left|\int_{\Omega}\mathbf{D}^{\mu}u\mathbf{D}^{\mu*}udx\right|\\ &\leq \frac{C}{4\epsilon}\|\mathbf{D}^{\mu}u\|^2_{L^2(\Omega)} + C\epsilon\|\mathbf{D}^{\mu*}\|^2_{L^2(\Omega)}\\ &= \frac{C}{4\epsilon}|u|^2_{J^{\mu}_{L,0}(\Omega)} + C\epsilon|u|^2_{J^{\mu}_{R,0}(\Omega)}\\ &\leq \frac{C}{4\epsilon}|u|^2_{J^2_{L,0}(\Omega)} + C\epsilon|\tilde{u}|^2_{J^{\mu}_{R,0}(\mathbb{R})}\\ &= \frac{C}{4\epsilon}|u|^2_{J^2_{L,0}(\Omega)} + C\epsilon|\tilde{u}|^2_{H^{\mu}(\mathbb{R})}\\ &= \frac{C}{4\epsilon}|u|^2_{J^2_{L,0}(\Omega)} + C\epsilon|\tilde{u}|^2_{H^{\mu}_0(\Omega)}.\end{aligned}$$

Thus $J^{\mu}_{L,0}(\Omega) \subset H^{\mu}_0(\Omega)$.

Similarly, we have $J^{\mu}_{R,0}(\Omega)$ and $H^{\mu}_0(\Omega)$ have equivalent seminorms and norms. □

For the semi-norm on $H_0^\mu(\Omega)$ with $\Omega = (a,b)$, the following fractional Poincaré-Friedrichs inequality holds. The results follows from Lemma 2.11, Lemma 2.12 and Theorem 2.3.

Lemma 2.13. *For $u \in H_0^\mu(\Omega)$, there is a positive constant C such that*

$$\|u\|_{L^2(\Omega)} \leq C|u|_{H_0^\mu(\Omega)}$$

and for $0 < s < \mu$, $s \neq n - 1/2$, $n \in \mathbb{N}$,

$$|u|_{H_0^s(\Omega)} \leq C|u|_{H_0^\mu(\Omega)}.$$

We present an additional estimate for $\mu = n - \frac{1}{2}$, $n \in \mathbb{N}$ [18].

Lemma 2.14. *For $u \in H_0^\mu(\Omega)$, $\mu = n - \frac{1}{2}$, $n \in N$ and $0 < \epsilon < \frac{1}{2}$, there is a positive constant C depending only upon μ, ϵ such that*

$$\begin{aligned} |u|_{H^{\mu-\epsilon}(\Omega)} &\leq C|u|_{J_{L,0}^\mu(\Omega)}, \\ |u|_{H^{\mu-\epsilon}(\Omega)} &\leq C|u|_{J_{R,0}^\mu(\Omega)}. \end{aligned}$$

Proof. It follows from Theorem 2.3 and Lemma 2.12 that for $\mu - \epsilon \neq n - \frac{1}{2}$, $n \in \mathbb{N}$

$$\begin{aligned} |u|_{H^{\mu-\epsilon}(\Omega)} &\leq C|u|_{J_{L,0}^{\mu-\epsilon}(\Omega)} \\ &\leq C|u|_{J_{L,0}^\mu(\Omega)}. \end{aligned}$$

Similarly we have $|u|_{H^{\mu-\epsilon}(\Omega)} \leq C|u|_{J_{R,0}^\mu(\Omega)}$. $\square$

Chapter 3

The semi-discrete approximate solution

In this chapter we will consider existence and stability of the weak solution for the variational form of problem (1.1) – (1.3). Also we will analyze the stability and error estimates of Galerkin finite element solutions for the semi-discrete variational formulation for (1.1) – (1.3).

3.1 The variational form

In this section, we shall show existence and the stability of the weak solution for the equation (1.1) – (1.3) with $1 < \mu < 2$: find a function $u \in H_0^{\frac{\mu}{2}}(\Omega)$ such that

$$(u_t, v) = (\kappa_\mu \nabla^\mu u, v) + (f(u), v), \ \forall v \in H_0^{\frac{\mu}{2}}(\Omega). \tag{3.1}$$

Since it is known in [2, 3] that there is a weak solution of (3.1) when f is locally Lipschitz continuous, we here have only to discuss the stability of the weak solution.

In order to show that we need the following lemma.

Lemma 3.1. *For all $v \in H_0^{\frac{\mu}{2}}(\Omega)$, the following inequality holds:*

$$-(\kappa_\mu \nabla^\mu v, v) \geq \kappa_\mu \Big| \cos(\pi \cdot \frac{\mu}{2}) \Big| |v|^2_{H_0^{\frac{\mu}{2}}(\Omega)}.$$

Proof. Following the ideas in [13, 19], we obtain the following inequality by using the semigroup property (2.13), adjoint property (2.14) and Lemmas 2.7 and 2.13:

$$\begin{aligned}
-(\kappa_\mu \nabla^\mu v, v) &= -\frac{\kappa_\mu}{2}\big\{({}_aD_x^\mu v, v) + ({}_xD_b^\mu v, v)\big\} \\
&= -\frac{\kappa_\mu}{2}\Big\{\int_a^b (D^2{}_aD_x^{-(2-\mu)} v) v dx + \int_a^b ((-D)^2{}_xD_b^{-(2-\mu)} v) v dx\Big\} \\
&= \frac{\kappa_\mu}{2}\Big\{\int_a^b (D_aD_x^{-(2-\mu)} v) Dv dx + \int_a^b (D_xD_b^{-(2-\mu)} v) Dv dx\Big\} \\
&= \frac{\kappa_\mu}{2}\Big\{\int_a^b ({}_aD_x^{-(2-\mu)} Dv) Dv dx + \int_a^b ({}_xD_b^{-(2-\mu)} Dv) Dv dx\Big\} \\
&= \frac{\kappa_\mu}{2}\Big\{\int_a^b ({}_aD_x^{-\frac{(2-\mu)}{2}}{}_aD_x^{-\frac{(2-\mu)}{2}} Dv) Dv dx \\
&\quad + \int_a^b ({}_xD_b^{-\frac{(2-\mu)}{2}}{}_xD_b^{-\frac{(2-\mu)}{2}} Dv) Dv dx\Big\} \\
&= \frac{\kappa_\mu}{2}\Big\{\int_a^b ({}_aD_x^{-\frac{(2-\mu)}{2}} Dv)({}_xD_b^{-\frac{(2-\mu)}{2}} Dv) dx \\
&\quad + \int_a^b ({}_xD_b^{-\frac{(2-\mu)}{2}} Dv)({}_aD_x^{-\frac{(2-\mu)}{2}} Dv) dx\Big\} \\
&= -\kappa_\mu(\mathbf{D}^{\frac{\mu}{2}} v, \mathbf{D}^{\frac{\mu}{2}*} v) \\
&= -\kappa_\mu \cos(\pi\cdot\frac{\mu}{2})\|\mathbf{D}^{\frac{\mu}{2}} v\|^2_{L^2(\Omega)} \\
&\geq \kappa_\mu\big|\cos(\pi\cdot\frac{\mu}{2})\big||v|^2_{H_0^{\frac{\mu}{2}}(\Omega)}.
\end{aligned}$$

This completes the proof. □

We consider the stability of a weak solution u for (3.1).

Theorem 3.1. *Let u be a solution of (3.1). Then there is a constant C such that*

$$\|u(t)\|_{L^2(\Omega)} \leq C\|u(0)\|_{L^2(\Omega)}. \tag{3.2}$$

Proof. Taking $v = u(t)$ in (3.1), we obtain

$$(u_t, u) - (\kappa_\mu \nabla^\mu u, u) = (f(u), u).$$

Since the second term on the left hand side is non-negative from Lemma 3.1, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} &\leq \frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \kappa_\mu \big|\cos(\pi \cdot \frac{\mu}{2})\big| |u|^2_{H_0^{\frac{\mu}{2}}(\Omega)} \\
&\leq (u_t, u) - (\kappa_\mu \nabla^\mu u, u) \\
&= (f(u), u) \\
&\leq \|f(u)\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \\
&\leq C_f \|u\|^2_{L^2(\Omega)}.
\end{aligned}$$

Integrating both sides with respect to t, we obtain

$$\|u(t)\|^2_{L^2(\Omega)} \leq \|u(0)\|^2_{L^2(\Omega)} + C\int_0^t \|u(s)\|^2_{L^2(\Omega)} ds.$$

An application of Gronwall's inequality gives that there is a constant C such that

$$\|u(t)\|^2_{L^2(\Omega)} \leq C\|u(0)\|^2_{L^2(\Omega)}.$$

This completes the proof. □

3.2 The semidiscrete variational form

In this section, we shall show derivation of system in the spatial direction from (3.1) using Galerkin method. Also, we will analyze the stability and error estimates of Galerkin finite element solutions for the semi-discrete variational formulation for (1.1).

Let S_h be a partition of Ω with a grid parameter h such that $\bar{\Omega} = \{\cup K | K \in S_h\}$ and $h = max_{K \in S_h} h_K$, where h_K is the width of the subinterval K. Associated with the partition S_h, we may define a finite-dimensional subspace $V_h \subset H_0^{\frac{\mu}{2}}(\Omega)$ with a basis $\{\varphi_i\}_{i=1}^N$ of the piecewise polynomials of order $n-1$. i.e., $V_h = \text{span}\{\varphi_1, \cdots, \varphi_N\}$.

Then the semidiscrete variational problem is to find $u_h \in V_h$ such that

$$\begin{aligned} (u_{h,t}, v) &= (\kappa_\mu \nabla^\mu u_h, v) + (f(u_h), v),\ \forall v \in V_h, &(3.3)\\ u_h(x,0) &= u_0, &(3.4)\\ u_h(a,t) &= u_h(b,t) = 0. &(3.5)\end{aligned}$$

Also, since we are looking for the solution u_h in the space V_h, u_h and $v \in V_h$ can be represented as

$$u_h(x,t) = \sum_{i=1}^{N} \alpha_i(t)\varphi_i(x),\ v(x,t) = \sum_{j=1}^{N} \beta_j(t)\varphi_j(x).$$

Hence the semidiscrete form (3.3) may be written

$$\begin{aligned} \big(\sum_{i=1}^{N} \alpha_i'(t)\varphi_i(x), \sum_{j=1}^{N} \beta_j(t)\varphi_j(x)\big) &= \big(\kappa_\mu \nabla^\mu \sum_{i=1}^{N} \alpha_i(t)\varphi_i(x), \sum_{j=1}^{N} \beta_j(t)\varphi_j(x)\big)\\ &\quad + \big(f(\sum_{l=1}^{N} \alpha_l(t)\varphi_l(x)), \sum_{j=1}^{N} \beta_j(t)\varphi_j(x)\big).\end{aligned}$$

Factoring the coefficient $\sum_{j=1}^{N} \beta_j$,

$$\sum_{j=1}^{N} \beta_j \Big(\big(\sum_{i=1}^{N} \alpha_i'\varphi_i, \varphi_j\big) - \big(\kappa_\mu \nabla^\mu \sum_{i=1}^{N} \alpha_i\varphi_i, \varphi_j\big) - \big(f(\sum_{l=1}^{N} \alpha_l\varphi_l), \varphi_j\big)\Big) = 0.$$

This equation must holds for arbitrary β_j, $j = 1, 2, \cdots, N$. Thus we obtain form

$$\sum_{i=1}^{N} \alpha_i'(\varphi_i, \varphi_j) - \sum_{i=1}^{N} \alpha_i(\kappa_\mu \nabla^\mu \varphi_i, \varphi_j) = \big(f(\sum_{l=1}^{N} \alpha_l\varphi_l), \varphi_j\big)$$

in which $j = 1, 2, \cdots, N$.

Then we may rewrite (3.3) in a matrix form

$$\mathbf{A}\dot{\mathbf{u}}(t) + \mathbf{B}\mathbf{u} = \mathbf{F}(\mathbf{u}), \tag{3.6}$$

where the $N \times N$ matrices $\mathbf{A}$ and $\mathbf{B}$ and the vectors $\mathbf{u}$ and $\mathbf{F}$ are

$$\begin{aligned}
\mathbf{A} = (a_{ij}), &\qquad a_{ij} = (\varphi_i, \varphi_j), \\
\mathbf{B} = (b_{ij}), &\qquad b_{ij} = -\frac{\kappa_\mu}{2}[(\mathbf{D}^{\frac{\mu}{2}}\varphi_i, \mathbf{D}^{\frac{\mu}{2}*}\varphi_j) + (\mathbf{D}^{\frac{\mu}{2}}\varphi_j, \mathbf{D}^{\frac{\mu}{2}*}\varphi_i)], \\
\mathbf{F}(\mathbf{u}) = (F_j), &\qquad F_j = (f(\sum_{l=1}^{N}\alpha_l\varphi_l), \varphi_j),
\end{aligned}$$

$$\mathbf{u} = (\alpha_1(t), \alpha_2(t), \cdots, \alpha_N(t))^T.$$

It follows from $\sum_{i,j=1}^N \alpha_i\alpha_j(\varphi_i, \varphi_j) = \left(\sum_{i=1}^N \alpha_i\varphi_i, \sum_{j=1}^N \alpha_j\varphi_j\right) \geq 0$ and Lemma 3.1 that the matrices $\mathbf{A}$ and $\mathbf{B}$ are non-negative definite and nonsingular. Thus this system (3.6) of ordinary differential equations has a unique solution since f is locally Lipschitz continuous.

The stability for the semi-discrete variational problem (3.3) can be obtained by following the proof.

Theorem 3.2. *Let u_h be a solution of (3.3). Then there is a constant C such that*

$$\|u_h\|_{L^2(\Omega)} \leq C\|u_0\|_{L^2(\Omega)}. \tag{3.7}$$

Proof. Choosing $v = u_h(t)$ in (3.3), we obtain

$$(u_{h,t}, u_h) - (\kappa_\mu\nabla^\mu u_h, u_h) = (f(u_h), u_h).$$

Since the second term on the left hand side is non-negative from Lemma 3.1, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_h\|^2_{L^2(\Omega)} &\leq \frac{1}{2}\frac{d}{dt}\|u_h\|^2_{L^2(\Omega)} + \kappa_\mu\left|\cos(\pi\cdot\frac{\mu}{2})\right||u_h|^2_{H_0^{\frac{\mu}{2}}(\Omega)} \\
&\leq (u_{h,t}, u_h) - (\kappa_\mu\nabla^\mu u_h, u_h) \\
&\leq \|f(u_h)\|_{L^2(\Omega)}\|u_h\|_{L^2(\Omega)} \\
&\leq C_f\|u_h\|^2_{L^2(\Omega)}.
\end{aligned}$$

Integrating both sides with respect to t, we obtain

$$\|u_h(t)\|^2_{L^2(\Omega)} \leq \|u_0\|^2_{L^2(\Omega)} + C\int_0^t \|u_h(s)\|^2 ds.$$

An application of Gronwall's inequality gives that there is a constant C such that

$$\|u_h(t)\|^2_{L^2(\Omega)} \le C\|u_0\|^2_{L^2(\Omega)}.$$

This completes the proof. $\square$

Now we shall consider estimates of the error between the weak solution of (3.1) and the one of semi-discrete form (3.3). The finite dimensional subspace $V_h \subset H_0^{\frac{\mu}{2}}(\Omega)$ is chosen so that the interpolation $I^h u$ of u satisfies an approximation property (2.6).

Further we need an adjoint problem to find $w \in H^\mu(\Omega) \cap H_0^{\frac{\mu}{2}}(\Omega)$ satisfying

$$\begin{aligned} -\kappa_\mu \nabla^\mu w = &\quad g, \qquad \text{in } \Omega, \\ w = &\quad 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{3.8}$$

Bai and Lü [4] have proved existence of a solution to the problem (3.8). We assume as in Ervin and Roop [18] that the solution w satisfies the regularity

$$\|w\|_{H^\mu(\Omega)} \le C\|g\|_{L^2(\Omega)},\ \mu \ne \frac{3}{2}, \tag{3.9}$$

$$\|w\|_{H^{\mu-\epsilon}(\Omega)} \le C\|g\|_{L^2(\Omega)},\ \mu = \frac{3}{2},\ 0 < \epsilon < \frac{1}{2}. \tag{3.10}$$

For the error estimate we need to define a projection operator. Let $\tilde{u}_h = P_h u$ be the elliptic projection $P_h : H_0^{\frac{\mu}{2}}(\Omega) \to V_h$ of the exact solution u defined by

$$-\kappa_\mu(\nabla^\mu(u - \tilde{u}_h), v) = 0,\ \forall v \in V_h. \tag{3.11}$$

If we let $\theta = u_h - \tilde{u}_h$, $\rho = \tilde{u}_h - u$, then the error is expressed as

$$e_h = u_h - u = (u_h - \tilde{u}_h) + (\tilde{u}_h - u) = \theta + \rho. \tag{3.12}$$

We begin with the following auxiliary result.

Lemma 3.2. *For $\psi, \chi \in V_h \subset H_0^{\frac{\mu}{2}}(\Omega)$,*

$$(\mathbf{D}^{\mu}\psi, \chi) \leq C\|\psi\|_{H_0^{\frac{\mu}{2}}(\Omega)}\|\chi\|_{H_0^{\frac{\mu}{2}}(\Omega)}, \tag{3.13}$$
$$(\mathbf{D}^{\mu*}\psi, \chi) \leq C\|\psi\|_{H_0^{\frac{\mu}{2}}(\Omega)}\|\chi\|_{H_0^{\frac{\mu}{2}}(\Omega)}. \tag{3.14}$$

Proof. It follows from the fractional Poincaré-Friedrichs inequality and the adjoint property (2.14) that for $\psi, \chi \in V_h \subset H_0^{\frac{\mu}{2}}(\Omega)$

$$\begin{aligned}(\mathbf{D}^{\mu}\psi, \chi) &= \int_a^b (\mathbf{D}^{\frac{\mu}{2}}\psi)\mathbf{D}^{\frac{\mu}{2}*}\chi dx \\ &\leq |\psi|_{J_{L,0}^{\frac{\mu}{2}}(\Omega)}|\chi|_{J_{R,0}^{\frac{\mu}{2}}(\Omega)} \\ &\leq C\|\psi\|_{H_0^{\frac{\mu}{2}}(\Omega)}\|\chi\|_{H_0^{\frac{\mu}{2}}(\Omega)}.\end{aligned}$$

Similarly we obtain

$$(\mathbf{D}^{\mu*}\psi, \chi) = \int_a^b (\mathbf{D}^{\frac{\mu}{2}*}\psi)\mathbf{D}^{\frac{\mu}{2}}\chi dx \leq C\|\psi\|_{H_0^{\frac{\mu}{2}}(\Omega)}\|\chi\|_{H_0^{\frac{\mu}{2}}(\Omega)}.$$

□

First, we consider the following estimates on ρ.

Lemma 3.3. *Let $\tilde{u}_h$ be a solution of (3.11) and $u \in H^{\mu}(\Omega) \cap H_0^{\frac{\mu}{2}}(\Omega)$ be the solution of (3.1). Let $\rho(t) = \tilde{u}_h(t) - u(t)$. Then there is a constant C such that*

$$\|\rho(t)\|_{L^2(\Omega)} \leq Ch^{\tilde{\gamma}}\|u(t)\|_{H^{\gamma}(\Omega)}, \tag{3.15}$$
$$\|\rho_t(t)\|_{L^2(\Omega)} \leq Ch^{\tilde{\gamma}}\|u(t)\|_{H^{\gamma}(\Omega)}, \tag{3.16}$$

where $\tilde{\gamma} = \mu$ if $\mu \neq \frac{3}{2}$ and $\tilde{\gamma} = \mu - \epsilon$, $0 < \epsilon < \frac{1}{2}$ if $\mu = \frac{3}{2}$.

Proof. It follows from Lemma 3.1 that for $v \in V_h$

$$\begin{aligned}\kappa_{\mu}|\cos(\pi \cdot \frac{\mu}{2})||u - \tilde{u}_h|^2_{H_0^{\frac{\mu}{2}}(\Omega)} &\leq -\kappa_{\mu}(\nabla^{\mu}(u - \tilde{u}_h), u - \tilde{u}_h) \\ &\leq -\kappa_{\mu}(\nabla^{\mu}(u - \tilde{u}_h), u - v) \\ &\quad -\kappa_{\mu}(\nabla^{\mu}(u - \tilde{u}_h), v - \tilde{u}_h) \\ &\leq C\|u - \tilde{u}_h\|_{H_0^{\frac{\mu}{2}}(\Omega)}\|u - v\|_{H_0^{\frac{\mu}{2}}(\Omega)}.\end{aligned}$$

Using the equivalence of semi-norms and norms, we obtain

$$\|u - \tilde{u}_h\|_{H_0^{\frac{\mu}{2}}(\Omega)} \le C \inf_{v \in V_h} \|u - v\|_{H_0^{\frac{\mu}{2}}(\Omega)} \le C\|u - I^h u\|_{H_0^{\frac{\mu}{2}}(\Omega)}. \quad (3.17)$$

In case of $\mu \neq \frac{3}{2}$ and $v \in V_h$, taking $g = \rho$ in (3.8) and using the property (3.11), Lemma 3.2 and the adjoint property (2.14), we have

$$\begin{aligned} (\rho, \rho) &= -\kappa_\mu(\nabla^\mu w, \rho) \\ &= -\kappa_\mu(\nabla^\mu (w - v), \rho) - \kappa_\mu(\nabla^\mu v, \rho) \\ &= -\kappa_\mu(\nabla^\mu (w - v), \rho) - \kappa_\mu(\nabla^\mu \rho, v) \\ &= -\kappa_\mu(\nabla^\mu (w - v), \rho) \\ &\le C\|w - v\|_{H_0^{\frac{\mu}{2}}(\Omega)} \|\rho\|_{H_0^{\frac{\mu}{2}}(\Omega)}. \end{aligned}$$

Taking $v = I^h w$ in the above inequalities and using (3.17), we have

$$\begin{aligned} \|\rho\|^2_{L^2(\Omega)} &\le C\|w - I^h w\|_{H_0^{\frac{\mu}{2}}(\Omega)} \|\rho\|_{H_0^{\frac{\mu}{2}}(\Omega)} \\ &\le Ch^{\frac{\mu}{2}} \|w\|_{H^\mu(\Omega)} \|u - I^h u\|_{H_0^{\frac{\mu}{2}}(\Omega)} \\ &\le Ch^{\frac{\mu}{2}} \|\rho\|_{L^2(\Omega)} h^{\frac{\mu}{2}} \|u\|_{H^\mu(\Omega)}. \end{aligned}$$

Then we obtain

$$\|\rho\|_{L^2(\Omega)} \le Ch^\mu \|u\|_{H^\mu(\Omega)}.$$

We now differentiate the equation (3.11). Then we obtain $-\kappa_\mu(\nabla^\mu \rho_t, v) = 0$ for all $v \in V_h$. Using again the above duality arguments, we have

$$\|\rho_t\|_{L^2(\Omega)} \le Ch^\mu \|u\|_{H^\mu(\Omega)}.$$

In case of $\mu = \frac{3}{2}$, we can similarly prove (3.15) and (3.16) by applying the assumption (3.10). This completes the proof □

We now consider the estimates on θ.

Lemma 3.4. *Let u_h and $\tilde{u}_h$ be the solutions of (3.3)–(3.5) and (3.11), respectively. Let $\theta(t) = u_h(t) - \tilde{u}_h(t)$. Then there is a constant C such that*

$$\|\theta(t)\|_{L^2(\Omega)} \le Ch^{\tilde{\gamma}}, \quad (3.18)$$

where $\tilde{\gamma} = \mu$ if $\mu \neq \frac{3}{2}$ and $\tilde{\gamma} = \mu - \epsilon$, $0 < \epsilon < \frac{1}{2}$ if $\mu = \frac{3}{2}$.

Proof. It follows from (3.3) and (3.11) that for $v \in V_h$,

$$\begin{aligned}
(\theta_t, v) - \kappa_\mu(\nabla^\mu \theta, v) &= (u_{h,t}, v) - \kappa_\mu(\nabla^\mu u_h, v) - (\tilde{u}_{h,t}, v) + \kappa_\mu(\nabla^\mu \tilde{u}_h, v) \\
&= (f(u_h), v) - (\tilde{u}_{h,t}, v) + \kappa_\mu(\nabla^\mu \tilde{u}_h, v) \\
&\quad -\kappa_\mu(\nabla^\mu(\tilde{u}_h - u), v) \\
&= (f(u_h), v) + \kappa_\mu(\nabla^\mu u, v) - (\tilde{u}_{h,t}, v) \\
&= (f(u_h), v) + (u_t, v) - (f(u), v) - (\tilde{u}_{h,t}, v) \\
&= (f(u_h) - f(u), v) - (\tilde{u}_{h,t} - u_t, v)
\end{aligned}$$

Then we obtain

$$(\theta_t, v) - \kappa_\mu(\nabla^\mu \theta, v) = (f(u_h) - f(u), v) - (\rho_t, v). \tag{3.19}$$

Replacing $v = \theta$ in (3.19), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2(\Omega)} \leq C_l\|u_h - u\|_{L^2(\Omega)}\|\theta\|_{L^2(\Omega)} + \|\rho_t\|_{L^2(\Omega)}\|\theta\|_{L^2(\Omega)}.$$

Using the Young's inequality

$$\begin{aligned}
\frac{d}{dt}\|\theta\|^2_{L^2(\Omega)} &\leq C\big(\|u_h - \tilde{u}_h\|_{L^2(\Omega)} + \|\tilde{u}_h - u\|_{L^2(\Omega)}\big)\|\theta\|_{L^2(\Omega)} \\
&\quad +\|\rho_t\|_{L^2(\Omega)}\|\theta\|_{L^2(\Omega)} \\
&\leq C\big(\|\theta\|_{L^2(\Omega)} + \|\rho\|_{L^2(\Omega)} + \|\rho_t\|_{L^2(\Omega)}\big)\|\theta\|_{L^2(\Omega)} \\
&\leq C_1\|\theta\|^2_{L^2(\Omega)} + C_2\|\rho\|^2_{L^2(\Omega)} + C_3\|\rho_t\|^2_{L^2(\Omega)}.
\end{aligned}$$

Integration on time t gives

$$\begin{aligned}
\|\theta(t)\|^2_{L^2(\Omega)} &\leq \|\theta(0)\|^2_{L^2(\Omega)} + C\int_0^t \|\theta\|^2_{L^2(\Omega)} ds \\
&\quad +C\int_0^t \big(\|\rho\|^2_{L^2(\Omega)} + \|\rho_t\|^2_{L^2(\Omega)}\big) ds.
\end{aligned}$$

Applying Gronwall's inequality, we obtain

$$\|\theta(t)\|^2_{L^2(\Omega)} \leq C_1\|\theta(0)\|^2_{L^2(\Omega)} + C_2\int_0^t \big(\|\rho\|^2_{L^2(\Omega)} + \|\rho_t\|^2_{L^2(\Omega)}\big) ds.$$

Since

$$\begin{aligned}\|\theta(0)\|_{L^2(\Omega)} &\leq \|u_h(0)-u(0)\|_{L^2(\Omega)} + \|\tilde{u}_h(0)-u(0)\|_{L^2(\Omega)} \\ &\leq Ch^{\tilde{\gamma}}\|u_0\|_{H^\gamma},\end{aligned}$$

we get the desired inequality

$$\|\theta(t)\|_{L^2(\Omega)} \leq C(u)h^{\tilde{\gamma}}.$$

Here $\tilde{\gamma} = \mu$ if $\mu \neq \frac{3}{2}$ and $\tilde{\gamma} = \mu - \epsilon$, $0 < \epsilon < \frac{1}{2}$ if $\mu = \frac{3}{2}$. This completes the proof. □

Combining Lemma 3.3 and Lemma 3.4, we obtain the following error estimates.

Theorem 3.3. *Let u_h and u be the solutions of (3.3) – (3.5) and (1.1) – (1.3), respectively. Then there is a constant $C(u)$ such that*

$$\|u(t)-u_h(t)\|_{L^2(\Omega)} \leq C(u)h^{\mu},\ \mu \neq \frac{3}{2} \tag{3.20}$$

and

$$\|u(t)-u_h(t)\|_{L^2(\Omega)} \leq C(u)h^{\mu-\epsilon},\ \mu = \frac{3}{2},\ 0<\epsilon<\frac{1}{2}. \tag{3.21}$$

Chapter 4

The fully discrete variational form

In this chapter, we consider a fully discrete variational formulation of (1.1) – (1.3). We show existence and uniqueness of numerical solutions for the fully discrete variational formulation. And we analyze the corresponding error estimates.

We consider a discretization in time base on backward Euler method. For the temporal discretization let $k = \frac{T}{M}$ for a positive integer M and $t_m = mk$. Let u^m be the solution of the backward Euler method defined by

$$\frac{u^{m+1} - u^m}{k} = \kappa_\mu \nabla^\mu u^{m+1} + f(u^{m+1}), \tag{4.1}$$

where an initial condition

$$u^0(x) = u_0,\ x \in \Omega = (a, b), \tag{4.2}$$

and boundary conditions

$$u^{m+1}(a) = u^{m+1}(b) = 0,\ m = 0, 1, \cdots, M-1. \tag{4.3}$$

Then we get the fully discrete variational formulation of (1.1) to find $u^{m+1} \in H_0^{\frac{\mu}{2}}(\Omega)$ such that for all $v \in H_0^{\frac{\mu}{2}}(\Omega)$

$$(u^{m+1}, v) - k(\kappa_\mu \nabla^\mu u^{m+1}, v) = (kf(u^{m+1}), v) + (u^m, v). \tag{4.4}$$

Thus a finite Galerkin solution $u_h^{m+1} \in V_h \subset H_0^{\frac{\mu}{2}}(\Omega)$ is a solution of the equation

$$(u_h^{m+1}, v_h) - k\kappa_\mu(\nabla^\mu u_h^{m+1}, v_h) = k(f(u_h^{m+1}), v_h) + (u_h^m, v_h), \forall v_h \in V_h \tag{4.5}$$

with an initial condition

$$u_h^0 = u_0 \tag{4.6}$$

and boundary conditions

$$u_h^{m+1}(a) = u_h^{m+1}(b) = 0,\ m = 0, 1, \cdots, M-1. \tag{4.7}$$

4.1 Existence, uniqueness and stability

In this section, we derive existence and uniqueness of numerical solutions for the fully discrete variational formulation. Also, we show that the backward Euler fully discrete scheme is unconditionally stable.

First, we prove the existence and uniqueness of solutions for (4.5) using the Brouwer fixed-point theorem.

Theorem 4.1. *There exists a unique solution $u_h^{m+1} \in V_h \subset H_0^{\frac{\mu}{2}}(\Omega)$ of (4.5) – (4.7).*

Proof. Let

$$G(u_h^{m+1}) = u_h^{m+1} - k\kappa_\mu \nabla^\mu u_h^{m+1} - kf(u_h^{m+1}) - u_h^m.$$

Then $G(v)$ is obviously a continuous function from V_h to V_h. In order to show the existence of solution for $G(v) = 0$, we adopt the mathematical induction. Assume that $u_h^0, u_h^1, \cdots, u_h^m$ exist for $m < M$. It follows from (1.5), Lemma 3.1 and Young's inequality that

$$\begin{aligned}
(G(v), v) &= (v, v) - (u_h^m, v) - k(\kappa_\mu \nabla^\mu v, v) - k(f(v), v) \\
&\geq \|v\|_{L^2(\Omega)}^2 - \|u_h^m\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + k\kappa_\mu \big|\cos(\pi \cdot \frac{\mu}{2})\big||v|^2_{H_0^{\frac{\mu}{2}}(\Omega)} \\
&\quad - C_f k\|v\|_{L^2(\Omega)}^2 \\
&\geq \|v\|_{L^2(\Omega)}^2 - \|u_h^m\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} - C_f k\|v\|_{L^2(\Omega)}^2 \\
&\geq \|v\|_{L^2(\Omega)}^2 - \frac{1}{2}(\|u_h^m\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2) - C_f k\|v\|_{L^2(\Omega)}^2 \\
&= (\frac{1}{2} - C_f k)\|v\|_{L^2(\Omega)}^2 - \frac{1}{2}\|u_h^m\|_{L^2(\Omega)}^2.
\end{aligned}$$

If we take sufficiently small k so that $k < 1/2C_f$ and
$\|v\|_{L^2(\Omega)} > \|u_h^m\|_{L^2(\Omega)}/(1-2C_f k)$ then the Brouwer's fixed point theorem implies the existence of a solution.

For the proof of the uniqueness of solutions, we assume that u and v are two solutions of (4.5). Then we obtain

$$(u-v,\psi) = k\kappa_\mu(\nabla^\mu(u-v),\psi) + k(f(u)-f(v),\psi),\ \forall\psi \in V_h \subset H_0^{\frac{\mu}{2}}(\Omega).$$

Replacing $\psi = u - v$ in the above equation and applying Lemma 3.1, we obtain

$$\begin{aligned}
\|u-v\|^2_{L^2(\Omega)} &\leq -k\kappa_\mu|\cos(\pi\cdot\frac{\mu}{2})||u-v|_{H_0^{\frac{\mu}{2}}(\Omega)} \\
&\quad +k\|f(u)-f(v)\|_{L^2(\Omega)}\|u-v\|_{L^2(\Omega)} \\
&\leq k\|f(u)-f(v)\|_{L^2(\Omega)}\|u-v\|_{L^2(\Omega)} \\
&\leq kC_l\|u-v\|^2_{L^2(\Omega)}.
\end{aligned}$$

This implies $u - v = 0$ since $u(0) = v(0)$. □

The following theorem presents the unconditional stability for (4.4).

Theorem 4.2. *The fully discrete scheme (4.4) is unconditionally stable. In fact, for any m*

$$\|u^{m+1}\|_{L^2(\Omega)} \leq C\|u_0\|_{L^2(\Omega)}.$$

Proof. It follows from (1.5), Lemma 3.1 and Young's inequality that by taking $v = u^{m+1}$ in (4.4), we obtain

$$\begin{aligned}
0 &= (u^{m+1},u^{m+1}) - k(\kappa_\mu\nabla^\mu u^{m+1},u^{m+1}) - k(f(u^{m+1}),u^{m+1}) \\
&\quad -(u^m,u^{m+1}) \\
&\geq \|u^{m+1}\|^2_{L^2(\Omega)} + k\kappa_\mu\big|\cos(\pi\cdot\frac{\mu}{2})\big||u^{m+1}|^2_{H_0^{\frac{\mu}{2}}(\Omega)} - C_f k\|u^{m+1}\|^2_{L^2(\Omega)} \\
&\quad -\|u^m\|_{L^2(\Omega)}\|u^{m+1}\|_{L^2(\Omega)} \\
&\geq \frac{1}{2}\|u^{m+1}\|^2_{L^2(\Omega)} + k\kappa_\mu\big|\cos(\pi\cdot\frac{\mu}{2})\big||u^{m+1}|^2_{H_0^{\frac{\mu}{2}}(\Omega)} - C_f k\|u^{m+1}\|^2_{L^2(\Omega)} \\
&\quad -\frac{1}{2}\|u^m\|^2_{L^2(\Omega)}.
\end{aligned}$$

Using Young's inequality,

$$\begin{aligned}\frac{1}{2}\|u^{m+1}\|^2_{L^2(\Omega)} &\leq \frac{1}{2}\|u^{m+1}\|^2_{L^2(\Omega)} + k\kappa_\mu\Big|\cos(\pi\cdot\frac{\mu}{2})\Big||u^{m+1}|^2_{H_0^{\frac{\mu}{2}}(\Omega)} \\ &\leq C_f k\|u^{m+1}\|^2_{L^2(\Omega)} + \frac{1}{2}\|u^m\|^2_{L^2(\Omega)}.\end{aligned}$$

Adding the above inequality from $m=0$ to m, we obtain

$$(1-2C_f k)\|u^{m+1}\|^2_{L^2(\Omega)} \leq \|u_0\|^2_{L^2(\Omega)} + 2C_f k\sum_{j=1}^{m}\|u^j\|^2_{L^2(\Omega)}.$$

Applying the discrete Gronwall's inequality with sufficiently small k such that $k < 1/2C_f$, we obtain the desired result. $\square$

4.2 Error estimates

In this section, we discuss the error estimates of solutions for the backward Euler fully discrete scheme.

The following theorem is an error estimate for the fully discrete problem (4.4).

Theorem 4.3. *Let u be the exact solution of (1.1) and u^m the solution of (4.4). Then there is a constant C such that*

$$\|u(t_m) - u^m\|_{L^2(\Omega)} \leq Ck.$$

Proof. Let $e^m = u(t_m) - u^m$ be the error at t_m. It follows from from (1.1) and (4.4) that for any $v \in H_0^{\frac{\mu}{2}}(\Omega)$

$$\begin{aligned}(e^{m+1}, v) - k(\kappa_\mu\nabla^\mu e^{m+1}, v) &= k(f(u(t_{m+1})) - f(u^{m+1}), v) \\ &\quad + (e^m, v) + (kr^{m+1}, v),\end{aligned}$$

where $r = O(k)$. Taking $v = e^{m+1}$,

$$\begin{aligned}\|e^{m+1}\|^2_{L^2(\Omega)} &\leq \|e^{m+1}\|^2_{L^2(\Omega)} + k\kappa_\mu\Big|\cos(\pi\cdot\frac{\mu}{2})\Big||e^{m+1}|^2_{H_0^{\frac{\mu}{2}}(\Omega)} \\ &\leq k\|f(u(t_{m+1})) - f(u^{m+1})\|_{L^2(\Omega)}\|e^{m+1}\|_{L^2(\Omega)} \\ &\quad + \|e^m\|_{L^2(\Omega)}\|e^{m+1}\|_{L^2(\Omega)} + \|kr^{m+1}\|_{L^2(\Omega)}\|e^{m+1}\|_{L^2(\Omega)}.\end{aligned}$$

Applying the locally Lipschitz continuity of f and Young's inequality, we obtain

$$\begin{aligned}\|e^{m+1}\|^2_{L^2(\Omega)} &\le kC_l\|e^{m+1}\|^2_{L^2(\Omega)} + \|e^m\|_{L^2(\Omega)}\|e^{m+1}\|_{L^2(\Omega)} \\ &\quad +\|kr^{m+1}\|_{L^2(\Omega)}\|e^{m+1}\|_{L^2(\Omega)} \\ &\le kC_l\|e^{m+1}\|^2_{L^2(\Omega)} + \varepsilon_1\|e^m\|^2_{L^2(\Omega)} + \frac{1}{4\varepsilon_1}\|e^{m+1}\|^2_{L^2(\Omega)} \\ &\quad +\varepsilon_2\|kr^{m+1}\|^2_{L^2(\Omega)} + \frac{1}{4\varepsilon_2}\|e^{m+1}\|^2_{L^2(\Omega)}.\end{aligned}$$

That is,

$$\begin{aligned}(1-\frac{1}{4\varepsilon_1}-\frac{1}{4\varepsilon_2})\|e^{m+1}\|^2_{L^2(\Omega)} \le\ & kC_l\|e^{m+1}\|^2_{L^2(\Omega)} + \varepsilon_1\|e^m\|^2_{L^2(\Omega)} \\ & +\varepsilon_2\|kr^{m+1}\|^2_{L^2(\Omega)}\quad .\end{aligned} \tag{4.8}$$

Denoting $\varepsilon_0 = 1 - \frac{1}{4\varepsilon_1} - \frac{1}{4\varepsilon_2}$ and summing the above equation on m, we obtain

$$\begin{aligned}(\varepsilon_0 - kC_l)\|e^{m+1}\|^2_{L^2(\Omega)} &\le \varepsilon_1\|e^0\|^2_{L^2(\Omega)} + (kC_l + \varepsilon_1 - \varepsilon_0)\sum_{i=1}^{m}\|e^i\|^2_{L^2(\Omega)} \\ &\quad +\varepsilon_2\sum_{i=1}^{m+1}\|kr^i\|^2_{L^2(\Omega)}.\end{aligned}$$

Applying the discrete Gronwall's inequality with sufficiently small k such that $(\varepsilon_0-\varepsilon_1)/C_l < k < \varepsilon_0/C_l$, we obtain the desired result, since $\sum_{i=1}^{m+1}\|kr^i\|_{L^2(\Omega)} \le Ck$ and $\|e^0\|_{L^2(\Omega)} = \|u(0) - u^0\|_{L^2(\Omega)} = 0$. $\square$

As in the previous chapter, denote $\theta^{m+1} = u_h^{m+1} - \tilde{u}_h^{m+1}$ and $\rho^{m+1} = \tilde{u}_h^{m+1} - u(t_{m+1})$, where $\tilde{u}_h^{m+1}$ the elliptic projection of $u(t_{m+1})$ defined in (3.11). Then

$$e_h^{m+1} = \theta^{m+1} + \rho^{m+1}.$$

Theorem 4.4. *Let u be the exact solution of (1.1) – (1.3) and $\{u_h^m\}_{m=0}^M$ be the solution of (4.5) – (4.7). Then we have, when $\mu \neq \frac{3}{2}$*

$$\|u(t_{m+1}) - u_h^{m+1}\|_{L^2(\Omega)} \le Ck + Ch^\mu\|u(t_{m+1})\|_{H^\mu(\Omega)}$$

and, when $\mu = \frac{3}{2},\ 0 < \epsilon < \frac{1}{2}$

$$\|u(t_{m+1}) - u_h^{m+1}\|_{L^2(\Omega)} \leq Ck + Ch^{\mu-\epsilon}\|u(t_{m+1})\|_{H^{\mu-\epsilon}(\Omega)}.$$

Proof. Since we know the estimates on ρ from Lemma 3.3, we have only to show boundedness of θ^{m+1}. Using the property (3.11), we obtain for $v \in V_h$

$$\begin{aligned}
(\theta^{m+1}, v) - k(\kappa_\mu \nabla^\mu \theta^{m+1}, v) &= k(f(u_h^{m+1}), v) + (u_h^m, v) - (\tilde{u}_h^{m+1}, v) \\
&\quad + k\kappa_\mu(\nabla^\mu \tilde{u}_h^{m+1}, v) \\
&\quad + k\kappa_\mu(\nabla^\mu(u(t_{m+1}) - \tilde{u}_h^{m+1}), v) \\
&= k(f(u_h^{m+1}), v) + (u_h^m, v) - (\tilde{u}_h^{m+1}, v) \\
&\quad + k\kappa_\mu(\nabla^\mu u(t_{m+1}), v) \\
&= k(f(u_h^{m+1}), v) + (u_h^m, v) \\
&\quad - k(f(u(t_{m+1})), v) - (u(t_m), v) \\
&\quad - (kr^{m+1}, v) + (u(t_{m+1}) - \tilde{u}_h^{m+1}, v),
\end{aligned}$$

we obtain

$$\begin{aligned}
(\theta^{m+1}, v) - k(\kappa_\mu \nabla^\mu \theta^{m+1}, v) &= k(f(u_h^{m+1}) - f(u(t_{m+1})), v) \\
&\quad + (u_h^m - u(t_m), v) - (kr^{m+1}, v) \\
&\quad - (\rho^{m+1}, v),
\end{aligned}$$

where $r = O(k)$.

Taking $v = \theta^{m+1}$ and applying Lemma 3.1, the locally Lipschitz continuity of f, Young's inequality and the triangular inequality, we obtain

$$\begin{aligned}
\|\theta^{m+1}\|^2_{L^2(\Omega)} &\leq \|\theta^{m+1}\|^2_{L^2(\Omega)} + k\kappa_\mu\big|\cos(\pi\cdot\frac{\mu}{2})\big||\theta^{m+1}|^2_{H_0^{\frac{\mu}{2}}(\Omega)} \\
&\leq k\|f(u_h^{m+1}) - f(u(t_{m+1}))\|_{L^2(\Omega)}\|\theta^{m+1}\|_{L^2(\Omega)} \\
&\quad +\|e_h^m\|_{L^2(\Omega)}\|\theta^{m+1}\|_{L^2(\Omega)} + \|kr^{m+1}\|_{L^2(\Omega)}\|\theta^{m+1}\|_{L^2(\Omega)} \\
&\quad +\|\rho^{m+1}\|_{L^2(\Omega)}\|\theta^{m+1}\|_{L^2(\Omega)} \\
&\leq kC_l\|e_h^{m+1}\|_{L^2(\Omega)}\|\theta^{m+1}\|_{L^2(\Omega)} \\
&\quad +\big(\|\theta^m\|_{L^2(\Omega)} + \|\rho^m\|_{L^2(\Omega)}\big)\|\theta^{m+1}\|_{L^2(\Omega)} \\
&\quad +\|kr^{m+1}\|_{L^2(\Omega)}\|\theta^{m+1}\|_{L^2(\Omega)} + \|\rho^{m+1}\|_{L^2(\Omega)}\|\theta^{m+1}\|_{L^2(\Omega)} \\
&\leq kC_l\big(1+\frac{1}{4\varepsilon_6}\big)\|\theta^{m+1}\|^2_{L^2(\Omega)} \\
&\quad +\big(\frac{1}{4\varepsilon_3}+\frac{1}{4\varepsilon_4}+\frac{1}{4\varepsilon_5}+\frac{1}{4\varepsilon_6}\big)\|\theta^{m+1}\|^2_{L^2(\Omega)} \\
&\quad +\varepsilon_3\|\theta^m\|^2_{L^2(\Omega)} + \varepsilon_4\|kr^{m+1}\|^2_{L^2(\Omega)} + \varepsilon_5\|\rho^m\|^2_{L^2(\Omega)} \\
&\quad +(1+kC_l)\varepsilon_6\|\rho^{m+1}\|^2_{L^2(\Omega)}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\big(1-\frac{1}{4\varepsilon_3}-\frac{1}{4\varepsilon_4}-\frac{1}{4\varepsilon_5}-\frac{1}{4\varepsilon_6}\big)\|\theta^{m+1}\|^2_{L^2(\Omega)} \\
\leq\ & kC_l\big(1+\frac{1}{4\varepsilon_6}\big)\|\theta^{m+1}\|^2_{L^2(\Omega)} + \varepsilon_3\|\theta^m\|^2_{L^2(\Omega)} + \varepsilon_4\|kr^{m+1}\|^2_{L^2(\Omega)} \\
&+\varepsilon_5\|\rho^m\|^2_{L^2(\Omega)} + (1+kC_l)\varepsilon_6\|\rho^{m+1}\|^2_{L^2(\Omega)}.
\end{aligned}$$

Denote $\varepsilon_7 = 1 - \frac{1}{4\varepsilon_3} - \frac{1}{4\varepsilon_4} - \frac{1}{4\varepsilon_5} - \frac{1}{4\varepsilon_6}$ and $\varepsilon_8 = 1 + \frac{1}{4\varepsilon_6}$. Then adding the above inequality from $m = 0$ to m, we obtain

$$\begin{aligned}
(\varepsilon_7 - kC_l\varepsilon_8)\|\theta^{m+1}\|^2_{L^2(\Omega)} &\leq \varepsilon_3\|\theta^0\|^2_{L^2(\Omega)} + (kC_l\varepsilon_8 + \varepsilon_3 - \varepsilon_7)\sum_{i=1}^{m}\|\theta^i\|^2_{L^2(\Omega)} \\
&\quad +\varepsilon_4\sum_{i=1}^{m+1}\|kr^i\|^2_{L^2(\Omega)} + \varepsilon_5\sum_{i=0}^{m}\|\rho^i\|^2_{L^2(\Omega)} \\
&\quad +(1+kC_l)\varepsilon_6\sum_{i=1}^{m+1}\|\rho^i\|^2_{L^2(\Omega)}.
\end{aligned}$$

Applying the discrete Gronwall's inequality with sufficiently small k such that $(\varepsilon_7 - \varepsilon_3)/\varepsilon_8 C_l < k < \varepsilon_7 / C_l \varepsilon_8$,

$$\|\theta^{m+1}\|^2_{L^2(\Omega)} \leq C_1 \|\theta^0\|^2_{L^2(\Omega)} + C_2 \sum_{i=1}^{m+1} \|kr^i\|^2_{L^2(\Omega)} + C_3 \sum_{i=0}^{m+1} \|\rho^i\|^2_{L^2(\Omega)}.$$

Also, using Lemma 3.3 and the initial conditions (1.2) and (4.6), we obtain

$$\begin{aligned} \|\theta^0\|_{L^2(\Omega)} &\leq \|u_h^0 - u(0)\|_{L^2(\Omega)} + \|\tilde{u}_h^0 - u(0)\|_{L^2(\Omega)} \\ &\leq Ch^{\tilde{\gamma}} \|u_0\|_{H^\gamma(\Omega)}. \end{aligned}$$

Since $\sum_{i=1}^{m+1} \|kr^i\|_{L^2(\Omega)} \leq Ck$, we get

$$\|\theta^{m+1}\|_{L^2(\Omega)} \leq Ck + C(u) h^{\tilde{\gamma}},$$

where $\tilde{\gamma} = \mu$ if $\mu \neq \frac{3}{2}$ and $\tilde{\gamma} = \mu - \epsilon$, $0 < \epsilon < \frac{1}{2}$, if $\mu = \frac{3}{2}$. Thus we obtain the desired result. $\square$

Chapter 5

Numerical experiments

In this chapter, we present numerical results for the Galerkin approximations which supports the theoretical analysis derived in the previous chapter.

Let S_h denote a uniform partition of Ω and V_h the space of continuous piecewise linear functions defined on S_h. In order to implement the Galerkin finite element approximation, we adapt finite element discretization on the spatial axis and the backward Euler finite difference scheme along the temporal axis. We associate shape functions of space V_h with the standard basis of hat functions on the uniform interval with length h.

5.1 Linear source term case

In this section, we will present numerical results for the problem with a linear source term.

Example 5.1.1 We first consider a space fractional linear diffusion equation

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} = \nabla^{\mu} u(x,t) + \frac{2t}{t^2+1} u(x,t) - (t^2+1)\Bigg(&\frac{\{x^{2-\mu} + (1-x)^{2-\mu}\}}{\Gamma(3-\mu)} \\ &- \frac{6\{x^{3-\mu} + (1-x)^{3-\mu}\}}{\Gamma(4-\mu)} + \frac{12\{x^{4-\mu} + (1-x)^{4-\mu}\}}{\Gamma(5-\mu)}\Bigg)\end{aligned} \tag{5.1}$$

with an initial condition

$$u(x,0) = x^2(1-x)^2,\ 0 < x < 1. \tag{5.2}$$

and boundary conditions

$$u(0, t) = u(1, t) = 0, \ t > 0. \tag{5.3}$$

In this case, the exact solution of the equation is

$$u(x, t) = (t^2 + 1)x^2(1 - x)^2.$$

Table 5.1 and Table 5.2 show the order of convergence and L^2-error between the exact solution and the Galerkin approximate solution of the fully discrete backward Euler method for (5.1) when $\mu = 1.6$ and $\mu = 1.8$, respectively. For numerical computation, the temporal step size $k = 0.001$ is used in both cases. Table 5.3 shows L^2-errors and orders of convergence for the Galerkin approximate solution when $\mu = 1.8$. Numerical computation is performed using the spatial step size $h = 0.0625$.

According to Tables 5.1 – 5.3 that we may find the order of convergence of $O(k + h^\mu)$ in this linear fractional diffusion problem (5.1) – (5.3).

	$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}$	
h	Error	Order
1/4	8.37811e-03	
1/8	2.73537e-03	1.615
1/16	8.75752e-04	1.643
1/32	2.83167e-04	1.629

Table 5.1: L^2-error and order of convergence in x when $\mu = 1.6$.

We plot the exact solution and approximate solutions obtained by the backward Euler Galerkin method using $h = 1/32$ and $k = 1/1000$ for (5.1) with $\mu = 1.6$ and $\mu = 1.8$. Figure 5.1 shows the contours of exact solution and numerical solutions at $t = 1$ and Figure 5.2 shows log-log graph for the order of convergence.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$ Error	Order
1/4	8.03045e-03	
1/8	2.28959e-03	1.810
1/16	6.32962e-04	1.855
1/32	1.76406e-04	1.843

Table 5.2: L^2-error and order of convergence in x when $\mu = 1.8$.

k	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$ Error	Ratio
1/20	4.20420e-03	
1/30	2.94873e-03	0.951
1/40	2.31793e-03	0.954
1/50	1.93046e-03	0.961

Table 5.3: L^2-error and order of convergence in t when $\mu = 1.8$.

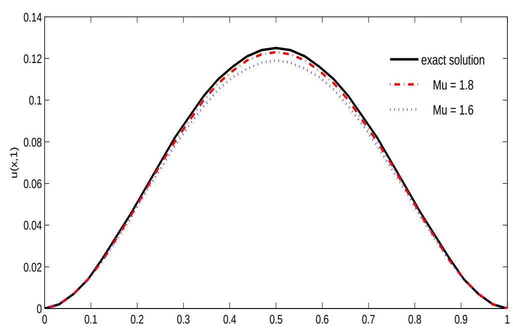

Figure 5.1: Exact and numerical solutions with $\mu = 1.6$ and $\mu = 1.8$.

5.2 Nonlinear source term case

In this section, we will consider numerical solutions for a space fractional diffusion problem with a nonlinear source term.

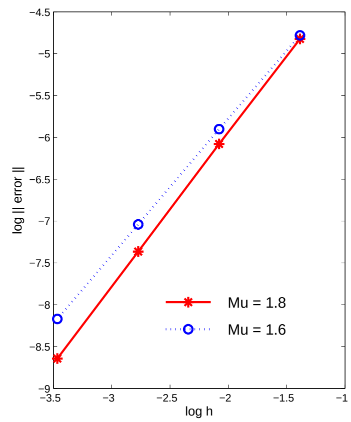

Figure 5.2: Log-log plots of the error for the rate of convergence.

Example 5.2.1 We consider a space fractional diffusion equation with a nonlinear Fisher type source term which is described as

$$\frac{\partial u(x,t)}{\partial t} = \kappa_\mu \nabla^\mu u(x,t) + \lambda u(x,t)(1-\beta u(x,t)) \tag{5.4}$$

with an initial condition

$$u(x,0) = u_0(x), \tag{5.5}$$

and boundary conditions

$$u(-1,t) = u(1,t) = 0. \tag{5.6}$$

In fact, we will consider the case of $\kappa_\mu = 0.1$, $\beta = 1$ in (5.4) with an initial condition

$$u_0(x) = \begin{cases} e^{-10x}, & x \geq 0, \\ e^{10x}, & x < 0, \end{cases} \tag{5.7}$$

For numerical computations, we have to take care of the nonlinear term $f(u) = \lambda u(1-\beta u)$. This gives a complicated nonlinear matrix. In order to avoid the difficulty of solving nonlinear system, we adopted a linearized method. We replaced $\lambda u^{n+1}(1-\beta u^{n+1})$ by $\lambda u^{n+1}(1-\beta u^{n})$. Figure 5.3 shows the numerical solutions at $t=1$ for the equation (5.4) – (5.7) with $\lambda=0.25$. For numerical computations, step sizes $h=0.01$ and $k=0.005$ are used. From the numerical results we may find that the numerical solutions converge to the solution of classical diffusion equation as μ approaches to 2.

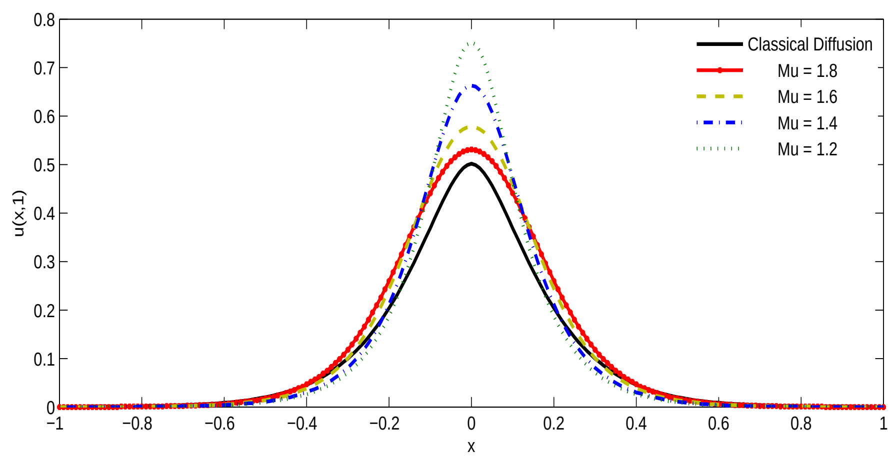

Figure 5.3: Numerical solutions for (5.4) with (5.7).

Example 5.2.2 We now consider the equation (5.4) with $\kappa_\mu=0.1$, $\beta=1$ and boundary conditions

$$\lim_{|x|\to\infty} u(x,t)=0.$$

We will consider an initial condition with a sharp peak in the middle as

$$u_0(x)=\operatorname{sech}^2(10x), \tag{5.8}$$

and an initial condition with a flat roof in the middle as

$$u_0(x)=\begin{cases} e^{-10(x-1)}, & x>1, \\ 1, & -1<x\le 1, \\ e^{10(x+1)}, & x\le -1. \end{cases} \tag{5.9}$$

Tang and Weber [40] have obtained computational solutions for (5.4) with initial conditions (5.8) and (5.9) using a Petrov-Galerkin method when the equation (5.4) is a classical diffusion problem. We obtain computational results using the method as in Example 5.2.1. Figure 5.4 shows numerical solutions at $t = 1$ for (5.4) with an initial condition (5.8) when $\Omega = (-2, 2)$ and $\lambda = 0.25$. Figure 5.5 shows numerical solutions at $t = 4$ for (5.4) and (5.8) when $\Omega = (-4, 4)$ and $\lambda = 1$. In both cases, step sizes $h = 0.01$ and $k = 0.005$ are used for computation. According to Figures 5.4 – 5.5, we may see that the diffusivity depends on μ but it is far less than that of the classical solution. That is, the fractional diffusion problem keeps the peak in the middle for longer time than the classical one does.

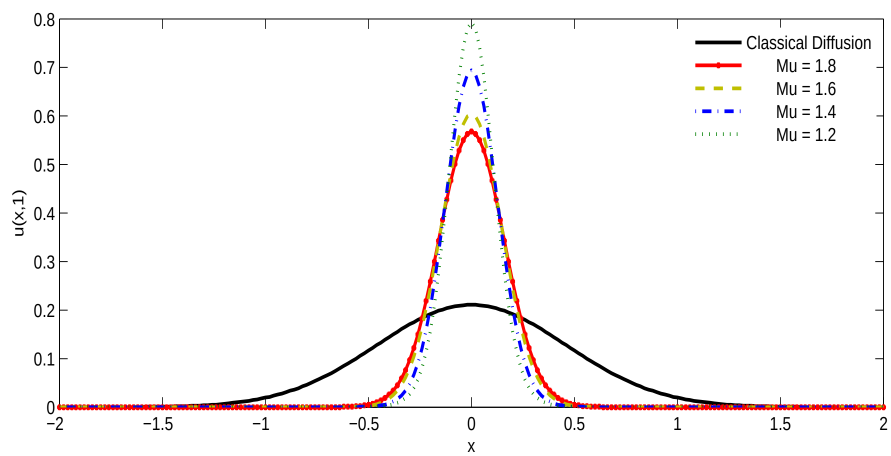

Figure 5.4: Numerical solutions at $t = 1$ for (5.4) and (5.8) with $\lambda = 0.25$.

Figure 5.6 shows contour plots of numerical solutions for (5.4) with an initial condition (5.8) when $\mu = 1.8$, $\Omega = (-2, 2)$ and $\lambda = 1$. In this case, step sizes $h = 0.01$ and $k = 0.005$ are also used for computation. But the period of time is from $t = 0$ to $t = 5$. According to Figure 5.6, we may see that the peak goes down rapidly for a short time and it begins to go up after the contour arrives at the lowest level.

Figure 5.7 shows contour plots of numerical solutions at $t = 1$ for (5.4)

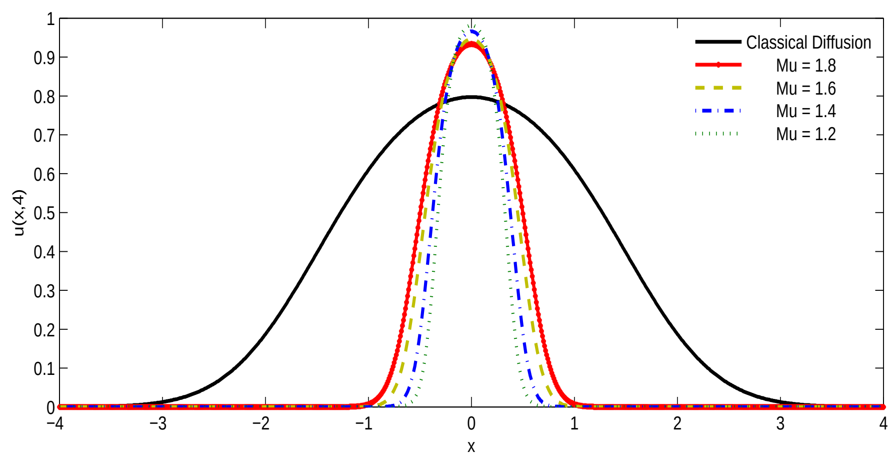

Figure 5.5: Numerical solutions at $t = 4$ for (5.4) and (5.8) with $\lambda = 1$.

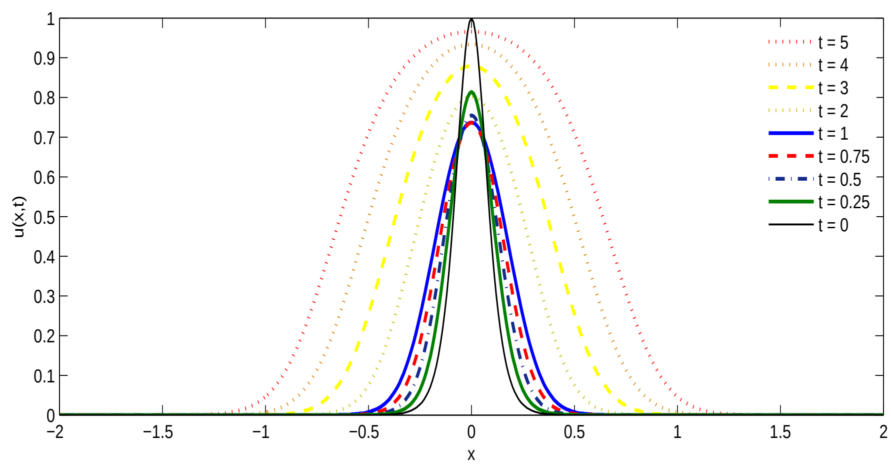

Figure 5.6: Numerical solutions for (5.4) and (5.8) with $\lambda = 1$.

with an initial condition (5.9) when $\Omega = (-4, 4)$ and $\lambda = 0.25$. In this case, step sizes $h = 0.01$ and $k = 0.005$ are also used for computation. According to Figure 5.7, we may find that the fractional diffusion problem keeps the flat roof in the middle for longer time than the classical one does.

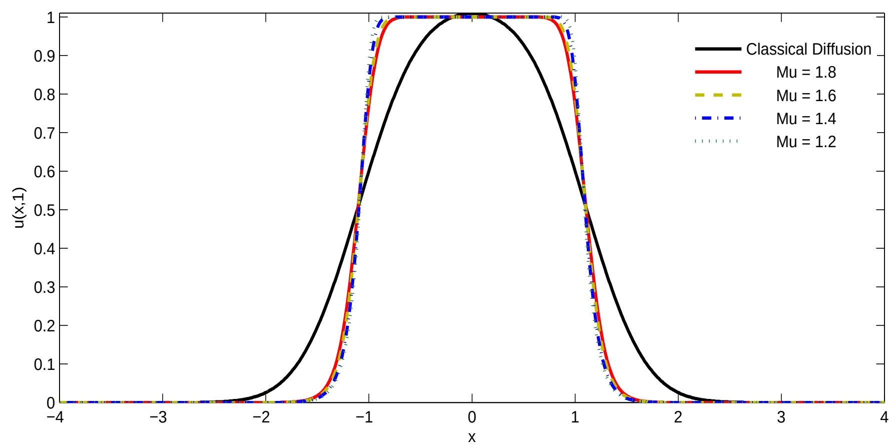

Figure 5.7: Numerical solutions for (5.4) and (5.9) with $\lambda = 0.25$.

Concluding Remarks Galerkin finite element methods are considered for the space fractional diffusion equation with a nonlinear source term. We have derived the variational formula of the semidiscrete scheme by using the Galerkin finite element method in space. We showed existence and stability of solutions for the semidiscrete scheme. Furthermore, we derived the fully time-space discrete variational formulation using the backward Euler method. Existence and uniqueness of solutions for the fully discrete Galerkin method have been discussed. Also we proved that the scheme is unconditionally stable, and it has the order of convergence of $O(k + h^{\tilde{\gamma}})$, where $\tilde{\gamma}$ is a constant depending on the order of fractional derivative. Numerical computations confirm the theoretical results discussed in the previous section for the problem with a linear source term. For the fractional diffusion problem with a nonlinear source term, we may find that the diffusivity depends on the order of fractional derivative, and numerical solutions of fractional order problems are less diffusive than the solution of a classical diffusion problem.

Chapter 6

Educational remarks

Fractional calculus is as old as classical calculus. Interestingly, when Leibniz invented the notation $\frac{d^n}{dx^n}f(x)$ which denotes the nth derivative of a function f with the implicit assumption that $n \in \mathbb{N}$, de l'Hôspital replied: "What does $\frac{d^n}{dx^n}f(x)$ mean if $n = \frac{1}{2}$?" This letter from de l'Hôspital, written in 1695, is nowadays commonly accepted as the first occurrence of a fractional derivative [35].

Recently, fractional calculus has been proven that fractional calculus is a significant tool in the modeling of many phenomena in various fields such as engineering, physics, porous media, economics and biological sciences. In the classical diffusion model, it is assumed that particles are distributed in a normal bell-shaped pattern based on the Brownian motion. For example, the probability density function of the normal Brownian motion is described as

$$p(x,t) = \frac{1}{\sqrt{4\pi\kappa_1 t}} e^{-x^2/4\kappa_1 t}$$

which satisfies the classical diffusion equation(heat equation)

$$\frac{\partial p(x,t)}{\partial t} = \kappa_1 \frac{\partial^2 p(x,t)}{\partial x^2}.$$

In contrast to the Brownian motion, processes deviating from the Gausssian probability distribution are referred to as anomalous diffusion. In many real world problems, it is more adequate to use anomalous diffusion described by fractional derivatives than the normal diffusion. This means that fractional diffusion equations are useful for applications in which a cloud of particles spreads faster than the classical diffusion model. In general, the fractional sub-diffusion corresponds to the divergence of microscopic time scales in random walk schemes. One typical fractional super-diffusion equation arises in chaotic dynamics and turbulent processes, where the usual second derivative in space is replaced by a fractional derivative of order $1 < \mu < 2$ [13, 15, 26, 27, 33, 34].

The mathematical models of science and engineering are mainly expressed in the form of differential equations. The mathematical formulation of problems involving rates of change with respect to independent variables, usually representing time and space, leads to partial differential equations. In many interesting partial differential equation problems, existence and uniqueness of classical solution to the problem have not been proved yet, or it is hard to find the explicit formula of the solution even if one can prove the existence and uniqueness of the solution [10]. Thus, numerical computations to seek a solution of partial differential equations are highly demanded in most research areas. In order to obtain numerical approximate solutions for the problem, we have to consider a finite dimensional vector space with a suitable formulation.

Connecting these mathematical areas, students can describe natural phenomena and social phenomena mathematically by observing and analyzing them. Further, using computer-implemented mathematical models, one can simulate and analyze complicated systems in natural phenomena. Thus, we can find the importance of differential and integral calculus education in high school mathematics to strengthen the basics of learning in the field of natural science and physics by using numerical approach for the differential equations.

According to the above considerations, we can adapt the calculus education which uses the numerical approach for the fractional order differential equation to students by using the following procedures. First, natural phenomena and social phenomena are modeled mathematically after analyzing them. Students confirm that data are distributed in a normal bell-shaped pattern based on the Brownian motion which they learned in school. Therethrough, they realize that another modeling in detail is necessary for analyzing phenomena which have been described as a Brownian motion to elaborate description. Students discuss how to change the mathematical model, in other words, an integer order diffusion equation which has been modeled by normal distribution assumption. Through mathematical discussions, students learn that fractional calculus is a tool in the modeling of phenomena in various fields. For example, for anomalous phenomena, the fractional diffusion equation is more elaborate than the normal heat equation which is modeled as a parabolic diffusion equation under the assumption of normal distribution. Secondly, students understand fundamental concepts of mathematics including difference between integer and fractional derivatives to solve the mathematical model. Thirdly, students learn the basic idea using numerical method for a fractional order differential equation to discretize the given continuous problem. In particular, students apply a finite difference method based on Taylor series and understand a truncation error which is an error between the differential equation and its discretized formula. Fourthly, students make numerical algorithm to find approximate solutions by using computer programs on mathematical models. Finally, students understand the phenomena by analyzing the numerical results presented by a graph of approximate solutions. Then students can learn that fractional calculus is a significant tool in the modeling of phenomena in various fields.

Even though there is no numerical analysis course in school mathematics, we may think of educational significance of finding approximate solutions for differential equations as applications of mathematics. The utilization of derivatives and integral can be found in the differential equation organized structurally. In other words, derivatives constitute the differential equations and integration is a reverse process of derivative to solve the one [11]. Thus, we may consider mathematical concept on differential and integral calculus which are in the structural form, not in the independent form.

The graph of a function is a useful mathematical symbol in understanding and analyzing various phenomena. Unlike in school mathematics, there are many cases which are understood only by using the graph without functional formula. In Chapter 5, we analyzed the anomalous diffusion by using the graph of numerical solutions for nonlinear problems since we don't know the explicit formula of the exact solution for them. In these cases, students should be able to interpret and understand the meaning of the graph in order to understand the physical or biological phenomena [24].

In summary, calculus is a foundation of study for the analysis of various fields such as physics, economics and so on and it is in the structural form. Therefore, it is important to understand the relations between derivatives and integral structurally in school mathematics calculus education as well as in tertiary mathematic educations to solve the differential equation [9]. Since there are many cases to be analyzed only by using the graph without functional formula, we suggest that the meaning of the graph obtained by numerical approximations or numerical simulations should be learned in school mathematics.

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국문초록

변칙적인 확산문제는 최근 다양한 분야에서 중요한 역할을 담당하고 있다. 본 연구에서는, 비선형 소스항을 가지고 공간에서의 실수 차수의 확산 방정식의 해를 논하기 위해 Galerkin 유한 요소법에 대하여 알아본다. 우선, 공간에 대하여 Galerkin 유한 요소법을 적용하여 유도된 반이산 방정식의 해의 존재성과 안정성을 증명한 후, 오차 추정을 통해 수렴 차수를 논한다. 더 나아가 시간에 관하여 후진 Euler 방법을 적용하여 완전 이산 방정식에 관하여 논의를 한다. 본 연구에서는 Brouwer 고정점 정리를 이용하여 후진 Euler 완전 이산문제의 근사해의 존재성을 증명한다. 이를 바탕으로 완전 이산문제의 근사해에 대한 안정성과 오차 추정을 논하여, 시간에 대하여 1차의 수렴성과 공간에 관한 실수 차수에 의존하는 $\tilde{\gamma}$차의 수렴성을 증명한다. 마지막으로 위의 방법을 통해 이론적 수렴 차수를 확인해 보고, $\tilde{\gamma}$가 2에 접근할 때 정수가 아닌 유리수 차수의 확산 방정식의 근사해가 정수 차수인 확산 방정식의 근사해에 가까이 다가감을 확인한다.

주요어휘: 변칙적인 확산, 실수 차수에 대한 미분, 비선형 소스 항, Galerkin 유한요소법
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