이학박사 학위논문

Tight Contact Structures on Hyperbolic 3-manifolds

(3 차원 쌍곡 다양체 위의 타이트한 접촉 구조들)

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지도교수 Otto Van Koert

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이 주 현

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위 원 장	김 혁	(인)
부 위 원 장	Otto Van Koert	(인)
위 원	박 종 일	(인)
위 원	김 인 강	(인)
위 원	임 선 희	(인)

Tight Contact Structures on Hyperbolic 3-manifolds

A dissertation
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Juhyun Lee

Dissertation Director : Professor Otto Van Koert

Department of Mathematical Sciences
Seoul National University

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Abstract

In 2003, Honda, Kazez and Matić proved the existence and uniqueness of tight contact structures on a surface bundle over the circle with pseudo-Anosov monodromy in the "extremal" case using bypasses and the curve complex. To extend this result, we will classify tight contact structures on a thickened higher genus surface, $\Sigma_g \times I$ with specific dividing curves on the boundary components. We fix one arbitrary separating dividing curve on $\Sigma_g \times \{0\}$ and consider an arbitrary integer power of a Dehn twist along a nonseparating curve intersecting at two points with the dividing curve of $\Sigma_g \times \{0\}$ as a dividing curve on $\Sigma \times \{1\}$. Then we investigate the upper bound of extensions of tight contact structures inside the whole manifold.
This problem is related to Honda's conjecture, which asserts that every hyperbolic 3-manifold admits a tight contact structure. In this thesis, we will try to explain some general theory related to Honda's conjecture, which lies on the intersection between two different streams of geometry, contact geometry and hyperbolic geometry. The main body of this thesis consists of the proof of the above problem in detail.

Key words: contact geometry, tight contact structure, hyperbolic 3-manifold
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Chapter 1

Introduction : Honda's conjecture

About ten years ago, Honda, who is the one of leading experts of contact geometry, conjectured that every hyperbolic 3-manifold admits a tight contact structure. The main result of this thesis has its root in this conjecture.

A contact manifold (M, ξ) is a closed oriented $(2n+1)$-dimensional manifold with a maximally nonintegrable hyperplane field ξ. By Darboux's theorem which will be introduced in section 3.1, a contact manifold is locally contactomorphic to the open subset of standard contact manifold. In the 1970's, Lutz and Martinet [41] showed the existence of contact structures on arbitrary closed, orientable three-manifolds.

In the case of contact 3-manifolds, there is a dichotomy between so-called overtwisted contact structures and tight contact structures. Overtwisted means that the contact manifold has an overtwisted disk, i.e. an embedded 2-dimensional disk D such that $T_pD = \xi_p$ for every point $p \in \partial D$. Other contact manifolds are called tight contact manifolds. In 1980's, Bennequin [2] gave the first proof of the existence of tight contact manifolds by showing that $(\mathbb{R}^3, \xi_{std})$ does not contain an overtwisted disk using a Legendrian curve theory. About ten years later, Eliashberg [9] showed that there is a one-to-one correspondence between isotopy classes of overtwisted contact structures and homotopy classes of 2-plane fields. Hence there is a complete classification for overtwisted tight contact structures. However, even the existence of tight contact structures has not been fully understood.

An other celebrated result of Eliashberg is the existence of unique tight contact structure of B^3. Using this result, Eliashberg can show the existence of unique tight contact structures of $S^3, S^2 \times S^1$ and $\mathbb{R}^3$ [10]. After Eliashberg's foundational results, classification results for tight contact structures on the 3-torus [36], lens spaces [14, 22, 27], solid tori, $T^2 \times I$ [27, 40], torus bundle over circles [22, 28], circle bundles over closed surfaces [23, 28] and many others have been obtained. Almost all known results concern spherical or toroidal manifolds. On the other hand, Etnyre and Honda [15] gave an example of a manifold which carries no tight contact structure.

Ten years ago, Honda conjectured that every hyperbolic 3-manifolds admits a tight contact structure. There are very few known results on this conjecture though. Colin [5], Honda, Kazez and Matić [31] independently showed that every nontrivial tight contact 3-manifold is contactomorphic to a connected sum of finitely many prime contact 3-manifolds. In 2002, Honda, Kazez and Matić showed that every toroidal 3-manifold carries infinitely many nonisotopic tight contact structures. Hence the problem comes down to atoroidal 3-manifolds and Seifert fibered manifolds. In 2007, Lisca and Stipsicz [39] completely classified the condition for the existence of tight contact structures on Seifert fibered spaces. In 2006, Colin, Giroux and Honda [7, 8, 31] announced that if M is a closed, oriented, irreducible 3-manifold, then M carries finitely many isotopy classes of tight contact structures if and only if M is atoroidal. But this does not give an accurate bound and, in fact, does not say anything about the existence of tight contact structure.

In order to know the tight contact structures on atoroidal manifolds with infinite fundamental group, Honda, Kazez and Matić studied fibered hyperbolic closed 3-manifolds with an extremal condition [32]. Let Σ_g be a closed oriented surface of genus $g > 1$ and $f : \Sigma \to \Sigma$ be a pseudo-Anosov diffeomorphism. Given a tight contact structure ξ, the Euler class $e(\xi)$ satisfies the Thurston-Bennequin inequality [2],

$$|\langle e(\xi), \Sigma\rangle| \leq -\chi(\Sigma)$$

. Here, the extremal condition means the equality holds. Then $M = \Sigma_g \times I/\sim$ with extremal condition has a unique tight contact structure, where $(x, 0) \sim (f(x), 1)$. Hence, this thesis has as its goal to extend this result to the case of the general Thurston-Bennequin inequality. Cofer [4] showed that $\Sigma_2 \times I$ with identical separating dividing curves on each boundary components has two tight contact structures, one of which is the invariant tight contact structure. This is the case $\langle e(\xi), \Sigma_2\rangle = 0$. On the other hand, Etgü [13] recently showed that there exist infinitely many closed hyperbolic 3-manifolds which contain no essential laminations but admit tight contact structures and Stipsicz [51] showed that the Weeks manifold admits tight contact structures. We will discuss the results of tight contact structures for hyperbolic 3-manifolds in chapter 4 in more detail.

Tight contact structures of 3-manifolds tend to be deeply related to the topology of given manifolds. Hence, we need to study some topological decomposition theory of 3-manifolds and their properties. We will see this in section 2.1. In section 2.2, we will sketch how surface bundles over a circle with pseudo-Anosov monodromy can admit a hyperbolic geometry.

In chapter 3, we will introduce contact geometry with a partially focus on contact 3-manifolds. We will deal with general theory of contact structures in section 3.1 and section 3.2. In section 3.3, we will see the dichotomy between overtwisted contact structures and tight contact structures in contact 3-manifolds. From section 3.4 to section 3.6, we will focus on convex decomposition theory developed by Colin, Giroux, Honda, Kazez and Matić, which is very useful to classify tight contact structures. Since our main interest is the global structure of tight contact structures, we will introduce the well known results for tight contact structures of 3-manifolds in section 3.7.

Our main result is the following.

Theorem 1.0.1. Let Σ be a closed oriented surface of genus > 1 and $M = \Sigma \times I, I = [0,1]$. Fix dividing sets $\Gamma_{\Sigma_i} = \gamma_i$, $i = 0, 1$, where γ_0 is an arbitrary separating curve and $\gamma_1 = \tau_\epsilon^n \circ \gamma_0$, $n \in \mathbb{Z}$, and $\chi((\Sigma_0)_+) - \chi((\Sigma_0)_-) = \chi((\Sigma_1)_+) - \chi((\Sigma_1)_-)$. Let ϵ be a nonseparating closed curve intersecting γ_0 at two points and $(\Sigma_i)_+$, $(\Sigma_i)_-$ are the positive and negative regions of $\Sigma \backslash \Gamma_{\Sigma_i}$. Choose a characteristic foliation $\mathcal{F}$ on ∂M which is adapted to $\Gamma_{\Sigma_0} \sqcup \Gamma_{\Sigma_1}$. Then the following holds.

(1) If $n \in \mathbb{Z}_{>0}$, then

$$\sharp\pi_0(Tight(M, \mathcal{F})) \leq \begin{cases} 2 \times 3^n & \text{if } g \geq 3, \\ 3^n + 5 & \text{if } g = 2,\ n = 2, \\ 3^n + 16 + 1 & \text{if } g = 2,\ n = 3, \\ 3^n + 3^{n-3} + \left[\frac{n}{2}\right] & \text{if } g = 2,\ n \geq 4. \end{cases} \tag{1.0.1}$$

(2) If $n = 1$, then $\sharp\pi_0(Tight(M, \mathcal{F})) \leq 4$

(3) If $n = 0$, then $\sharp\pi_0(Tight(M, \mathcal{F}) = 2$

(4) If $n = -1$, then $\sharp\pi_0(Tight(M, \mathcal{F}) \leq 4$

(5) If $n < -1$, then

$$\sharp\pi_0(Tight(M, \mathcal{F})) \leq \begin{cases} 4 + 32 + 16 = 52 & \text{if } g(\Sigma_+) \neq 0 \text{ and } g(\Sigma_-) \neq 0, \\ 4 + 32 + 8 = 44 & \text{otherwise}, \\ 4 + 48 + 6 = 58 & \text{if } g = 2. \end{cases} \tag{1.0.2}$$

In chapter 4, we will prove this theorem. In section 4.1, we briefly introduce the result of Honda, Kazez and Matić [32] and its meaning. Through sections 4.2 to 4.7 we will show the above inequalities in the order of (3), (2), (4), (1), (5).

Chapter 2

3-dimensional topology and geometry

Decomposition theory of 3-dimensional tight contact manifolds is deeply related to the decomposition theory of 3-manifolds. In this chapter, we give some basic background in 3-manifolds and hyperbolic 3-manifolds. In section 3.1, we will see the topological classification theories of 3-manifolds. The existence of tight contact structures seems to be related with geometry rather than topology. Hence we will briefly give a few facts for hyperbolic 3-manifolds and the proof of Thurston's hyperbolization theorem in section 3.2.

We always assume that every 3-manifold M is closed, connected and orientable unless mentioned otherwise. The notions which are introduced in here will be useful in chapter 3. For further understanding, we refer the books of Bonahon [3], Farb [16], Hatcher [25], Hempel [26], Otal [45] and Thurston [**?**].

2.1 Topology of 3-manifolds

If a closed, connected and oriented 3-manifold is not diffeomorphic to S^3, then we call this manifold non-trivial. A non-trivial 3-manifold P is called prime if for every decomposition $P = P_0 \sharp P_1$, one of the summands P_0, P_1 is homeomorphic to S^3. In 1924, Alexander showed that if $S^3 = P_0 \sharp P_1$, then P_0 or P_1 is homeomorphic to S^3.

Theorem 2.1.1 (Kneser [38] , Milnor [42], Prime decomposition of 3-manifold)**.** Every non-trivial 3-manifold M admits a prime decomposition, i.e., M can be written as a connected sum of finitely many prime manifolds. The summands in this prime decomposition are unique up to order and diffeomorphism.

If any embedded 2-sphere in M bounds a ball, then we call manifold M irreducible. If P is a prime 3-manifold, then either it is $S^2 \times S^1$ or the non-orientable S^2 bundle over S^1, or it is irreducible. $S^2 \times S^1$ is the only prime closed orientable but not irreducible manifold.

Theorem 2.1.2 (Jaco, Shalen [34], Johansson [35], JSJ decomposition)**.** Let M be an arbitrary oriented closed irreducible 3-manifold. Then M has a unique (up to isotopy) minimal collection of disjointly embedded incompressible tori such that each component of the 3-manifold M obtained by cutting along these tori is either atoroidal or Seifert-fibered.

An incompressible surface $\Sigma \subset M$ is an embedded surface for which $\pi_1(\Sigma)$ injects into $\pi_1(M)$, or equivalently, if for every embedded disk $D \subset M$ with $D \cap \Sigma = \partial D$, there is a disk $D' \subset \Sigma$ such that $\partial D = \partial D'$. If a torus bounds a solid torus or lie in a 3-ball, then we call this torus trivial torus. By the Loop theorem, trivial tori are only compressible tori.

Theorem 2.1.3 (Loop theorem)**.** If $\pi_1(\partial M) \to \pi_1(M)$ is not injective, then there is a properly embedded disk $D \subset M$ with ∂D representing a nontrivial element of the kernel of $\pi_1(\partial M) \to \pi_1(M)$

For further understanding and proofs, you can see Hatcher's lecture notes [25] and Hempel's book [26]. 3-manifold M is called atoroidal if M does not contain any incompressible torus. A 3-manifold M is aspherical if $\pi_i(M) = 0$, for $\forall i \geq 2$. A 3-manifold M is Haken if it is aspherical and M contains an embedded π_1-injective surface, i.e., it contains compact irreducible incompressible surface. Many Haken 3-manifold is obtained by surface bundles over the circle. One year ago, Agol [1] proved the virtually Haken conjecture for closed hyperbolic 3-manifold which states that every compact, orientable, irreducible 3-manifold with infinite fundamental group is virtually Haken, i.e., it has a finite cover which is Haken.

Seifert fibered space is S^1 bundle over 2-dimensional orbifold with finite singular points. Seifert fiber structures on a compact oriented manifold are classified by the following 3 data :

(1) The topological type of the base surface,

(2) The twists p/q at the exceptional fibers,

(3) A rational Euler number.

The Seifert fibering of a Seifert fibered manifold is unique up to isotopy and exceptional cases are listed explicitly.

From now on, we introduce a different aspect of 3-dimensional decomposition theory. A dimension p foliation $\mathcal{F}$ of an n dimensional manifold M is an integrable rank-p subbundle of the tangent bundle TM. Locally, it looks like a decomposition of the manifold as a union of parallel submanifolds of p dimension. By Frobenius' theorem, $\mathcal{F}$ can be written locally as the kernel of 1-form $\alpha = dz$. If we can find a closed transversal curve δ through each leaf of $\mathcal{F}$, then we call this foliation taut. Equivalently, there is a closed 2-form ω such that $\omega|_{\mathcal{F}} > 0$ or there are no generalized Reeb component. A generalized Reeb component is a compact submanifold $N \subset M$ whose boundary ∂N is a union of torus leaves.

Definition 2.1.4 (Sutured manifold)**.** Let M is a compact, oriented irreducible 3-manifold with corners. A sutured manifold (M, γ) consists of the following data :

(1) $\partial M \cap \gamma \neq \emptyset$,

(2) $\gamma = \sqcup$ annuli(which is called sutures)

(3) $\partial \gamma = \cup \partial M$,

(4) $\partial M \setminus \gamma = R_+ \sqcup R_-$.

Definition 2.1.5. Let Σ be a compact oriented surface with connected components $\Sigma_1, \cdots, \Sigma_n$. Then,

$$\beta(\Sigma) = \sum_{i \text{ for } \chi(\Sigma_i)<0} |\chi(\Sigma_i)|$$

is called the Thurston norm.

Definition 2.1.6. A sutured manifold (M, γ) is taut if

(1) $R_\pm$ minimize the Thurston norm in $H_2(M, \gamma)$,

(2) $R_\pm$ are incompressible in M,

(3) No components are $R_\pm$ are disks, unless $(M, \gamma) = (B^3, S^1 \times I)$.

Theorem 2.1.7 (Gabai's Fundamental Theorem [18])**.** The theorem consists of two part.

(1) (Decomposition) Every taut sutured manifold (M_0, γ_0) has a taut sutured manifold hierarchy

$$(M_0, \gamma_0) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{S_2} \cdot \xrightarrow{S_n} (M_n, \gamma_n).$$

The hierarchy stops when every surface in M_n separates, i.e., $H_2(M_n, \partial M_n) = 0$. In particular, every boundary component of M_n is a 2-sphere.

(2) (Reconstruction) If there is a sequence of sutured manifold decompositions, then if (M_n, γ_n) is taut, (N_0, γ_0) is also taut.

Gabai proved this using sutured manifold decomposition theory. By Gabai's fundamental theorem, we know that many hyperbolic 3-manifolds contain taut foliations. Roberts, Shareshian and Stein [49] constructed hyperbolic 3-manifolds without taut foliations. A (codimension one) lamination λ in a 3-manifold M is a foliated, closed subset of M, i.e., λ is covered by a collection of open sets U of the form $\mathbb{R}^2 \times \mathbb{R}$ such that $\lambda \cup U = \mathbb{R}^2 \times C$, where C is a closed set in $\mathbb{R}$, and transition maps preserve the product structures. The coordinate neighborhoods of leaves are of the form $\mathbb{R}^2 \times x$, $x \in C$. Let M_λ be the metric completion of the manifold $M - \lambda$ with the path metric inherited from a metric on M. Let $H = \{(x, y) \in \mathbb{R}^2 | y \geq 0\}$ be the closed upper half plane. An end compression is a proper embedding $d : (H, \partial H) \to (M_\lambda, \partial M_\lambda)$ such that $d|_{\partial H}$ does not extend to a proper embedding $d' : H \to \partial M_\lambda$. The notion of essential lamination is the generalization of incompressible surface and taut foliation. These two notions were introduced by Gabai and Oertel [19]. The definition of essential lamination is the following.

Definition 2.1.8. λ is an essential lamination in M if it satisfies the following condition:

(1) The inclusion of leaves of the lamination into M induces a injection on π_1.

(2) M_λ is irreducible.

(3) λ has no sphere leaves.

(4) λ has no end compression.

Fenley [17] showed that there are infinitely many hyperbolic 3-manifold which do not admit essential laminations.

2.2 Thurston's hyperbolization theorem

A homogeneous Riemannian manifold (X, g) is one whose group of isometries acts transitively on the manifold. A Riemannian manifold M is modeled on a given homogeneous manifold (X, g) if every point of M has a neighborhood isometric to an open set in (X, g). If M is complete Riemannian, then we call M locally homogeneous. The universal cover of locally homogeneous manifold is a homogeneous manifold that which it is modeled on.
In dimension 3, there are 8 model geometry, which are $S^3, \mathbb{R}^3, \mathbb{H}^3, S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}$, Nil, Sol and the universal covering group of $PSL_2(\mathbb{R})$. You can see more detail in Morgan's article [43, 44]. In 1980', Thurston conjectured the following.

Theorem 2.2.1 (Thurston's Geometrization Theorem)**.** Let M be a closed, oriented, prime 3-manifold. Then there is an embedding of a disjoint union of 2-tori and Klein bottles $\sqcup_i T_i^2 \subset M$ such that every component of the complement admits a locally homogeneous Riemannian metric of finite volume.

Thurston [52, 53] originally showed that the conjecture holds for Haken manifolds. More specifically, he showed that closed atoroidal Haken manifolds are hyperbolic. Geometrization theorem proved completely by Perelman [46, 47, 48].

All spherical 3-manifolds are Seifert manifold with base S^2 and at most 3 exceptional fibers. Most "small" 3-manifolds are Seifert fiber spaces and they account for all compact oriented manifolds in 6 of the 8 Thurston geometries of the geometrization theorem.

$Mod(S)$ is a group of isotopy classes of orientation preserving homeomorphisms of surface S. If $\partial S \neq 0$, we consider a homeomorphism restricted to the identity on ∂S. From now on, we introduce Nielsen-Thurston's classification of $Mod(S)$ and Thurston's hyperbolization theorem for surface bundle over the circle. You can see more about $Mod(S)$ in Farb's book [16] and about hyperbolization theorem in Otal's book [45].

A periodic mapping class has the one of finite order. If a mapping class fixes a collection of isotopy classes of simple closed curves that are pairwise disjoint, then we call this reducible. We can decompose reducible mapping class along this collection of curves, which is called reduction system. We call $f \in Mod(S)$ pseudo-Anosov, if there is a pair of transverse measured foliations $(\mathcal{F}^u, \mu_u)$ and $(\mathcal{F}^s, \mu_s)$ on S, a number $\lambda > 1$, and a representative homeomorphism ϕ such that

$$\phi \cdot (\mathcal{F}^u, \mu_u) = (\mathcal{F}^u, \lambda\mu_u) \text{ and } \phi \cdot (\mathcal{F}^s, \mu_s) = (\mathcal{F}^s, \lambda^{-1}\mu_s)$$

The measured foliations $(\mathcal{F}^u, \mu_u)$ and $(\mathcal{F}^s, \mu_s)$ are called the unstable foliation and stable foliation, respectively, and λ is called the dilatation. We call ϕ a pseudo-Anosov homeomorphism. A measured foliation $(\mathcal{F}, \mu)$ on a surface S is a foliation $\mathcal{F}$ of S equipped with a transverse measure μ. $\phi \in Homeo(S)$ acts on the set of measured foliations of S like the following. For $\phi \in Homeo(S)$, we define $\phi \cdot (\mathcal{F}, \mu) = (\phi(\mathcal{F}, \phi_*(\mu))$, where $\phi_*(\mu)(\gamma) = \mu(\phi^{-1}(\gamma))$.

Theorem 2.2.2 (Nielsen-Thurston Classification)**.** Let S be any surface. Each $f \in Mod(S)$ is either

(1) periodic,

(2) reducible, or

(3) pseudo-Anosov.

Further, pseudo-Anosov mapping classes are neither periodic nor reducible.

Idea of the proof of theorem 2.2.2. $Teich(S)$, which is called the Teichmüller space, is the space of equivalence classes $\{(X, \phi)\}/\sim$, where X is a hyperbolic surface, $(X, \phi) \sim (X', \phi')$ if $\phi' \cdot \phi^{-1}$ is homotopic to an isometry. Let $d(\cdot, \cdot)$ denote the Teichmüller distance. For $h \in Isom(Teich(S))$, define the translation distance of h,

$$\tau(h) := \inf_{X \in Teich(S)} d(X, h \cdot X).$$

Then for any $h \in Isom(Teich(S))$, there are following three categories.

(1) Elliptic : $\tau(h) = 0$

(2) Parabolic : $\tau(h)$ does not exist.

(3) Hyperbolic : $\tau(h) > 0$

Since $Mod(S)$ acts by isometries on $Teich(S)$ with the Teichmüller metric, this classification gives a classification of mapping classes. (1) is periodic, (2) is reducible, (3) is pseudo-Anosov in $Mod(S)$. For the detailed proof, you can see in Farb [16]. □

Theorem 2.2.3 (Thurston)**.** Let $S = S_g$, where $g \geq 2$. Let M_f denote the mapping torus for $f \in \mathcal{M}od(S)$. Then

(1) f is periodic if and only if M_f admits a metric locally isometric to $\mathbb{H}^2 \times \mathbb{R}$.

(2) f is reducible if and only if M_f contains an incompressible torus.

(3) f is pseudo-Anosov if and only if M_f admits hyperbolic metric.

From now on, we will see the proof of Theorem 2.2.3 (3) briefly.
Let S be a compact surface and $\phi \in \mathcal{M}od(S)$ a pseudo-Anosov diffeomorphism. Let M_ϕ be a manifold fibered over the circle obtained by suspension of ϕ. $\pi_1(M_\phi)$ is generated by the elements of the group $\pi_1(S)$ and an element t such that $tgt^{-1} = \phi_*(g)$ for all $g \in \pi_1(\Sigma)$ where ϕ_* is the action of ϕ on $\phi_1(S)$. To construct a complete hyperbolic metric in the interior of M_ϕ, we have to find a discrete, faithful representation of the $\phi_1(M_\phi)$ in $PSL_2(\mathbb{C}) = Isom(\mathbb{H}^3)$.

We will first construct the restriction of the desired representation to the subgroup $\pi_1(S)$. After that, we will show that limit set of representation in Lemma 2.2.9 is whole sphere $\overline{\mathbb{C}}$. This will allow us to apply Sullivan's theorem (Theorem 2.2.17) Finally, we will prove the hyperbolization theorem for fibered manifolds.

Before going to propositions, we need to prepare some technical argument. We can identify the interior of the compact surface Σ with the $\mathbb{H}^2/\Gamma$ where Γ is a Fuchsian group. A Kleinian group is a discrete, torsion-free subgroup of $Isom(\mathbb{H}^n)$ of finite type. A Kleinian group acts freely and properly discontinously on $\mathbb{H}^n$. A Kleinian group in $PSL_2(\mathbb{R}) = Isom(\mathbb{H}^2)$ such that the quotient $\mathbb{H}^2/\Gamma$ has finite volume is called a Fuchsian group. For $\phi \in \mathcal{M}od(\Sigma)$, we denote by ϕ^* the action of ϕ on the Teichm$\ddot{u}$ller space $\mathcal{T}(\Gamma)$. A Fuchsian deformation of Γ in $PSL_2(\mathbb{R})$ is a pair (ρ, ϕ), where ρ is a representation of Γ in $PSL_2(\mathbb{R})$ and ϕ is a normalized quasiconformal homermorphism of $\mathbb{H}^2$ to itself that conjugate Γ and $\rho(\Gamma)$. Normalized means that the continuous extension of ϕ to the real axis fixes the three points 0,1, and ∞. We can give a equivalence relation of the set of Fuchsian deformations of Γ by setting $(\rho, \phi) \simeq (\rho', \phi')$ if and only if $\rho = \rho'$. Then the quotient space of the set of Fuchsian deformations of Γ under this equivalence relation is called Teichmüller space.

Similarly, we can define a quasi-Fuchsian space. A quasi-Fuchsian deformations of Γ is a pair (ρ, ϕ), where ρ is a representation of Γ in $PSL_2(\mathbb{C})$ and ϕ is a quasiconformal homeomorphism of $\overline{\mathbb{C}}$ that conjugates Γ and $\rho(\Gamma)$ and fixes the three points 0,1, and ∞. We can give an equivalence relation on the set of quasi-Fuchsian deformations of Γ by setting $(\rho, \phi) \simeq (\rho', \phi')$ if and only if $\rho = \rho'$. The quotient space of the set of quasi-Fuchsian space under this equivalence relation is called by quasi-Fuchsian space and denoted by $\mathcal{QF}(\Gamma)$.

$\mathcal{M}od(S)$ acts naturally on $\mathcal{T}(\Gamma)$ by the following. If f is a diffeomorphism of the compact surface S, then we can get a bi-Lipschitz diffeomorphism by isotoping f. Denote this bi-Lipschitz diffeomorphism f. Then every lift $\widetilde{f}$ of f to the upper half space $\mathbb{H}^2$ is also bi-Lipschitz. Hence $\widetilde{f}$ and $\widetilde{f}^{-1}$ are quasiconformal homeomorphisms of the upper half-space $\mathbb{H}^2$. Let $\sigma = (\rho, \phi)$ be an Fuchsian deformation of Γ. Then $\phi \circ \widetilde{f}^{-1}$ is a quasiconformal homeomorphism of $\mathbb{H}^2$ that conjugates Γ and some subgroup of $PSL_2(\mathbb{R})$. we normalize this quasiconformal map by choosing an element a^{-1} of $PSL_2(\mathbb{R})$ which takes same values of $\phi \circ \widetilde{f}^{-1}$ at the points $0, 1$ and ∞. Then $a \circ \phi \widetilde{f}^{-1}$ conjugates the representation Id of Γ to a representation $\rho_{\widetilde{f}}$. Hence $(\rho_{\widetilde{f}}, a \circ \phi \circ \widetilde{f}^{-1})$ is again a Fuchsian deformation of Γ. We can show that this action is well-defined and $\rho_{\widetilde{f}}$ is conjugate in $PSL_2\mathbb{R}$ to the representation $f^*(\rho) : \gamma \to \rho \circ (f_*)^{-1}(\gamma)$, where f_* is an automorphism of Γ.

Similarly, we can define an action of $\mathcal{M}od(S)$ on the $\mathcal{QF}(\Gamma)$ by $f^*(\sigma_+, \sigma_-) = (f^*(\sigma_+), f^*(\sigma_-))$ using the Ahlfors-Bers coordinates. The representation $f^*(\rho, \sigma)$ is conjugate to the representation $\gamma \to \rho \circ (f^*)^{-1}(\gamma)$.

Let (ρ, ϕ) be an element of $\mathcal{QF}(\Gamma)$. By Riemann uniformization theorem, $\phi(\mathbb{H}^2)$ is conformally equivalent to the upper half-plane $\mathbb{H}^2$ by a conformal isomorphism $\phi_+ : \phi(\mathbb{H}^2) \to \mathbb{H}^2$. By Caratheodory's theorem and the fact that $\partial\phi(\mathbb{H}^2)$ is a Jordan curve, ϕ_+ can be extend to the boundary of upper half-space and we can choose that this extension of $\phi_+ \circ \phi$ fixes the three points 0,1, and ∞. Then $(\phi_+ \circ \rho \circ \phi_+^{-1}, \phi_+ \circ \phi)$ is a Fuchsian deformations of Γ.

Similarly, we can find a normalized conformal isomorphism $\phi_- : \phi(\overline{\mathbb{H}^2}) \to \overline{\mathbb{H}^2}$. Then we can think $(\phi_- \circ \rho \circ \phi^{-1}, \phi_- \circ \phi)$ is also a Fuchsian deformations of Γ considering the action being on the lower half-space $\overline{\mathbb{H}^2}$. We denote $\mathcal{T}(\overline{\Gamma})$ by using this actions. $\mathcal{T}(\overline{\Gamma})$ is naturally isomorphic $\mathcal{T}(\Gamma)$ under complex conjugation.

So far, we define a map $\mathcal{B} : \mathcal{QF}(\Gamma) \to \mathcal{T}(\Gamma) \times \mathcal{T}(\overline{\Gamma})$. This map is called Ahlfors-Bers map and we can show that this map is bijection.

Let $\sigma = (Id, Id)$ denote the origin of $\mathcal{T}(\Gamma)$. Consider the sequence of representations ρ in Quasi-Fuchsian space $\mathcal{QF}(\Gamma)$ such that $\mathcal{B}(\rho_i, \psi) = ((\phi^*)^{-i}(\sigma), (\phi^*)^i(\sigma))$ where $\mathcal{B}$ is the Ahlfors-Bers map. For quasi-Fuchsian deformation (ρ, ψ) of Γ, we can find Fuchsian deformation $(\psi_+^i \circ \rho \circ \psi_+^{-i}, \psi_+ \circ \psi)$ where $\psi_+ : \psi(\mathbb{H}^2) \to \mathbb{H}^2$ is a conformal isomorphism. The Ahlfors-Bers map is a bijection map.

The Teichmüller space is a metric space and topologically $6g-6$ dimensional open ball. Thurston compactified the Teichm$\ddot{u}$ller space using projective measured laminations. Let $\mathcal{C}$ be the set of conjugacy classes of non-parabolic elements of $\Gamma \setminus Id$ which can be identified with the set of closed geodesics of the surface $\mathbb{H}^2/\Gamma$ and $\mathbb{R}_+$ be the positive real values. The map $\mathcal{L} : \mathcal{T}(\Gamma) \to \mathbb{R}_+^{\mathcal{C}}$ is defined by the following. For $\sigma = (\rho, \phi) \in \mathcal{T}(\Gamma)$, $\mathcal{L}(\sigma)$ is the translation distance of $\rho_\sigma(\gamma)$ of $\rho(\gamma)$ in $\mathbb{H}^2$, where $[\gamma] \in \mathcal{C}$. Let $\mathcal{P}((\mathbb{R}_+)^{\mathcal{C}})$ be the projective space of $(\mathbb{R}_+)^{\mathcal{C}}$ and $\pi : (\mathbb{R}_+)^{\mathcal{C}} \to \mathcal{P}((\mathbb{R}_+)^{\mathcal{C}})$ be the projection map. Thurston showed that $\pi \circ \mathcal{L}(\mathcal{T}(\Gamma))$ is relatively compact.

A geodesic lamination on $\mathbb{H}^2/\Gamma$ is a compact set which is the disjoint union of embedded complete geodesics. These geodesics are called the leaves of the lamination. A measured geodesic lamination is a geodesic lamination on $\mathbb{H}^2/\Gamma$ with a transverse measure. For each interval $I \simeq [0,1]$ embedded in $\mathbb{H}^2/\Gamma$ transversely to λ, we can give a finite positive Borel measure λ_I with the following properties. (1) The support of λ_I is $\lambda \cap I$ and (2) If I and I' are two arcs that are homotopic through embedded arcs transverse to λ, by a homotopy such that the endpoints stay in the same leaf of λ or in the same component of the complement of λ, then $\lambda_I(I) = \lambda_{I'}(I')$. in more detail, You can see in appendix in Otal's book [45]. The space of measured lamination is denoted by $\mathcal{ML}(S)$. An oriented geodesic in $\mathbb{H}^2$ is uniquely determined by its two ends, which is a pair of distinct points in $\partial\mathbb{H}^2$. Let $M(S)$ be the space of pairs of distinct points in $\partial\mathbb{H}^2$. We can think that a measured geodesic lamination is a positive Γ-invariant measure on $M(S)$ whose support is a lamination.

For every $\gamma \in \mathcal{C}$ and every measured geodesic lamination λ, we can define the intersection number $i(\lambda, \gamma)$. See more details in Otal [45]. In other words, there is a map $\mathcal{I} : \mathcal{ML} \to (\mathbb{R}_+)^{\mathcal{C}}$ which assigns each measured geodesic lamination λ to a point whose coordinate on a $[\gamma]$ is $i(\gamma, \lambda)$. Since the positive real numbers naturally acts on $\mathcal{ML}(S)$, we denote the quotient space of $\mathcal{ML}(S)$ by this action by $\mathcal{PML}(S)$. Since the map $\mathcal{I}$ is homogeneous, we can find the map $\pi \circ \mathcal{I} : \mathcal{PML}(S) \to \mathcal{P}((\mathbb{R}_+)^{\mathcal{C}})$.

Theorem 2.2.4. The map $\pi \circ \mathcal{L}$ and $\pi \circ \mathcal{I}$ are embeddings of $\mathcal{T}(\Gamma)$ and $\mathcal{PML}(S)$, respectively, in $\mathcal{P}((\mathbb{R}_+)^{\mathcal{C}})$, with disjoint images. The union of their images homeomorphic to a ball.

So far, we see that an element of $\mathcal{M}od(S)$ naturally acts on $\mathcal{T}(\Gamma)$ and this action can extends continuously to a homeomorphism ϕ^* of Thurston's compactification $\overline{\mathcal{T}(\Gamma)}$. Since $\overline{\mathcal{T}(\Gamma)}$ is a closed ball, ϕ^* has a fixed point by Brower's fixed point theorem. $\mathcal{M}od(S)$ has trichotomy according to this fixed point.

(1) (Periodic case) If ϕ^* has a fixed point in $\mathcal{T}(\Gamma)$, then

(2) ϕ^* has at least one fixed point on the boundary of $\mathcal{T}(\Gamma)$.

(3) A fixed point of ϕ^* is an arational measured lamination. Then ϕ^* fixed two arational measured lamination λ^+, and λ^-, which are called stabled lamination and the unstable lamination of ϕ, respectively. These two laminations intersect each other and there is a constant $k > 1$ such

that

$$i(\phi_*(\gamma), \lambda^+) = \frac{1}{2} i(\gamma, \lambda^+), i(\phi_*(\gamma), \lambda^-) = k i(\gamma, \lambda^-) \qquad (2.2.1)$$

for every $\gamma \in \mathcal{C}$.

The dynamics of a pseudo-Anosov diffeomorphism on $\overline{\mathcal{T}(\Gamma)}$ is similar to the dynamics of a hyperbolic element of $Isom(\mathbb{H}^2)$ on hyperbolic space.

Proposition 2.2.5. Let $\phi \in \mathcal{M}od(S)$ be a pseudo-Anosov diffeomorphism. Then

(1) For every $\sigma \in \mathcal{T}(\Gamma)$,

$$\lim_{i \to \infty} (\phi^*)^i(\sigma) = \lambda^+ \text{ and } \lim_{i \to -\infty} (\phi^*)^i(\sigma) = \lambda^-$$

in the sense of Thurston.

(2) For every $\gamma \in \mathcal{C}$ represented by a simple curve,

$$\lim_{i \to \infty} \frac{(\phi_*)^i(\gamma)}{l((\phi_*)^i(\gamma))} = \lambda^+ \text{ and } \lim_{i \to -\infty} \frac{(\phi_*)^i(\gamma)}{l((\phi_*)^i(\gamma))} = \lambda^-$$

in the topology of $\mathcal{ML}(S)$, where $l((\phi_*)^i(\gamma))$ denotes the length of the geodesic in the conjugacy class of $(\phi_*)^i(\gamma)$ with respect to the metric on $\mathbb{H}^2/\Gamma$.

Let (σ_i) be a sequence in $\mathcal{T}(\Gamma)$. Then possibly after passing to a subsequences, it converges or there exist a measured geodesic laminations λ and q sequence (ϵ_i) tending to 0 such that, $\epsilon_i l_{\sigma_i}(\gamma) \to i(\lambda, \gamma)$ for every $\gamma \in \mathcal{C}$. The second case is called that the sequence (σ_i) converges to the measured geodesic lamination λ in the sense of Thurston.

Let ϕ be a pseudo-Anosov diffeomorphism. Then by Thurston's compactification of Teichmüller space, we have

$$\lim_{i \to \infty} (\phi^*)^{-i}(\sigma) = \lambda^- \text{and} \lim_{i \to \infty} (\phi^*)^i(\sigma) = \lambda,$$

where λ^+ and λ^- are the stable and unstable measured geodesic laminations of ϕ respectively. λ^+ and λ^- are arational and they intersect each other. By the following theorem, the sequence (ρ_i) has a subsequence in the set of representations of Γ, $\mathcal{R}(\Gamma)$, converging to ρ_∞. Since ρ_i's are discrete, faithful representations,ρ_∞ is discrete and faithful.

Theorem 2.2.6. Let $\rho = (\sigma_i^+, \sigma_i^-)$ be a sequence in $\mathcal{QR}(\Gamma)$ such that $\sigma_i^+ \to \lambda^+$ and $\sigma_i^- \to \lambda^-$ in the thurston compactification of Teichmüller space. Suppose that the measured laminations λ^+ and λ^- are arational and that they intersect. Then the sequence (ρ_i) has a converging subsequence in $\mathcal{R}(\Gamma)$.

Proposition 2.2.7. Let $G \subset Isom(\mathbb{H}^n)$ be a nonelementary Kleinian group. Then the set of discrete, faithful representations of Γ, $\mathcal{DF}(\Gamma)$, is closed in $\mathcal{R}(\Gamma)$.

Choose a representative $\phi_* : \Gamma \to \Gamma$ for the action of ϕ on $\pi_i(S) \simeq \Gamma$.

Lemma 2.2.8. There exist a constant K and a K quasi-conformal homeomorphism Ψ_i of $\overline{\mathbb{C}}$ such that $\Psi_i(\rho_i(\gamma)z) = \rho_i(\phi_*(\gamma))\Psi_i(z)$, $\forall \gamma \in \Gamma, \forall \imath$.

Proof. See Lemma 6.1.1. in Otal's book [45]. ∎

Lemma 2.2.9. Perhaps after passing to a subsequence, the homeomorphisms Ψ_i converges uniformly on $\overline{\mathbb{C}}$ to a K-quasiconformal homeomorphism Ψ_∞ such that

$$\forall \gamma \forall z, \psi_\infty(\rho_\infty(\gamma)(z)) = \rho(\phi_*(\gamma))\psi_\infty(z)$$

So far, we construct quasiconformal homeomorphism ψ_∞. From now on, we will show that ψ_∞ is conformal. For this, we need to look at the limit set of the group $\rho_\infty(\Gamma)$.

Theorem 2.2.10. The limit set of the group ρ_∞ is the entire sphere $\overline{\mathbb{C}}$

Sketch of the proof of Theorem 2.2.10. We argue by contradiction. Let Ω_0 be a component of the domain of discontinuity Ω of $\rho_\infty(\Gamma)$.

Proposition 2.2.11. The subgroup G_0 of $\rho_\infty(\Gamma)$ that stabilizes Ω_0 has finite index.

We omit the proof of this.

Let $\Gamma' \subset \Gamma$ be the subgroup of finite index such that $G_0 = \rho_\infty(\Gamma')$; Let S' be the compact surface that is the finite cover of S with fundamental group Γ'. By restriction to Γ', the representations ρ_i define the elements of $\mathcal{QF}(\Gamma')$ that converge to ρ_∞ in the representation space $\mathcal{R}(\Gamma')$. The limit set of $\rho_\infty(\Gamma')$ equals the limit set of $\rho_\infty(\Gamma)$ because Γ' has finite index in Γ. Hence Ω_0 is also a component of the domain of discontinuity of $\rho_\infty(\Gamma')$ and invariant. Ω_0 is simply connected (See Lemma 6.2.4 in Otal's book). Hence the Riemann surface $\Omega_0/\rho_\infty(\Gamma')$ has the same homotopy type as $\mathbb{H}^2/\Gamma'$. The representation ρ_∞ sends parabolic elements of Γ to parabolic elements. Hence, if $h \in \Gamma'$ is parabolic, so is $\rho_\infty(h)$: thus one can associate with $\rho_\infty(h)$ a cusp of type $\mathbb{Z}$ in $M(\rho_\infty)$. The intersections of the convex core of $M(\rho_\infty(\Gamma')$ has the same proper homotopy type as $\mathbb{H}^2/\Gamma'$. Since the latter surface is of finite type, there is a quasiconformal homeomorphism between the surface $\mathbb{H}^2/\Gamma'$ and $\Omega_0/\rho_\infty(\Gamma')$; this homeomorphism lifts to a quasiconformal homeomorphism from $\mathbb{H}^2$ onto Ω that conjugates the actions of Γ' and $\rho_\infty(\Gamma')$.
Fix a constant $\delta > 0$. The image of the surface $\Omega_0/\rho_\infty(\Gamma')$ under the retraction r_δ is a surface Σ_δ of class C^1 that lies in $M(\rho_\infty(\Gamma'))$ and is a boundary component of a convex manifold N_δ. All the closed geodesics in $M(\rho_\infty(\Gamma'))$ are contains in N_δ and the inclusion of Σ_δ into N_δ induces an isomorphism at the level of the fundamental group. If ϵ is chosen sufficiently small, then the only components of the ϵ-thin part of $M(\rho_\infty(\Gamma'))$ that Σ_δ intersects are the cusps of type $\mathbb{Z}$ corresponding to the boundary components of S'.

Let $L(c)$ be the length of a curve $c \subset M(\rho_\infty(\Gamma'))$. If $\gamma \in \Gamma'$, we also let $L(c) = l_\infty(\gamma)$ be the length of the conjugacy class of γ for the representation ρ_∞, i.e., $L(c) = 0$ if $\rho_\infty(\gamma)$ is parabolic, and $L(c)$ is the translations distance of $\rho_\infty(\gamma)$ if $\rho_\infty(\gamma)$ is hyperbolic.

By construction, the power ϕ_*^k of the automorphism ϕ_* corresponding to the pseudo-Anosov diffeomorphism ϕ leaves the group Γ' invariant up to conjugation. Let $\psi \in \mathcal{M}od(S')$ for the lift of ϕ^k to S'. ψ is also pseudo-Anosov. Let λ'_+ and λ'_- denote its two invariant laminations and $\mathcal{C}'$ be the set of closed geodesics in the surface $\mathbb{H}^2/\Gamma'$. We have the following result.

Proposition 2.2.12. For every $\gamma \in \mathcal{C}'$, there is a constant $C = C(\gamma)$ such that

$$\forall \in \mathbb{Z}, L((\phi_*)^n(\gamma)) \leq C$$

.

Choose a simple closed geodesic $\gamma \subset \mathbb{H}^2/\Gamma'$. If the conjugacy class of $(\phi_*)^n(\gamma)$ in $\rho_\infty(\Gamma')$ is represented by a hyperbolic element, γ_n^* denote the geodesic in N_δ corresponding to this conjugacy class. Let γ_n be the geodesic in the surface Σ_δ corresponding to the conjugacy class of $(\phi_*)^n(\gamma)$ with the induced metric. We begin showing that all but finitely many of the curves γ_n are homotopic to geodesics in N_δ.

Lemma 2.2.13. Only finitely many curves in the sequence (γ_n) are represented by parabolic elements of $\rho_\infty(\Gamma')$.

Corollary 2.2.14. There are only finitely many geodesics γ_n^* at distance less than C from the surfacee Σ_δ for every constant $C > 0$.

From now on, we will obtain a contradiction by using Thurston's intersections number lemma, which says that if two geodesics are "far enough out" in the same end of a hyperbolic manifold which has the homotopy type of a surface, then their homotopic intersections number is "small" relative to their length. If $|n|$ is sufficiently large, then all the curves γ_n are homotopic to closed geodesics γ_n^* in N_δ by Lemma 2.2.13.

Proposition 2.2.15. Let γ_i and γ_j be two curves in the sequence (γ_n) that are homotopic to geodesics γ_i^* and γ_j^* such that

(1) γ_i^* and γ_j^* are at distance at least D from the surface Σ_δ

(2) γ_i^* and γ_j^* are at distance at least 1 from each other. Then

$$i(\gamma_i, \gamma_j) \leq ce^{-D} L(\gamma_i) L(\gamma_j)$$

for some constant c independent of D and the curves γ_i and γ_j.

Proof. See the Lemma 6.2.10 in the book of Otal [45]. □

Hence by Proposition 2.2.15 and Corollary 2.2.14, it holds that

$$\liminf_{i\to\infty, j\to-\infty} \frac{i(\gamma_i,\gamma_j)}{L(\gamma_i)L(\gamma_j)} = 0$$

, where $L(\gamma_i)$ is the length of the geodesic in Σ that represents the conjugacy class of $\gamma_i = (\phi_*)^i(\gamma)$. The induced metric on Σ is equivalent to the reference metric on $\mathbb{H}^2/\Gamma'$. Hence

$$\liminf_{i\to\infty, j\to-\infty} \frac{i(\gamma_i,\gamma_j)}{l(\gamma_i)l(\gamma_j)} = 0$$

. We know that

$$\lim_{i\to\infty} \frac{\gamma_i}{l(\gamma_i)} = \lambda'_+ \text{ and } \lim_{i\to-\infty} \frac{\gamma_i}{l(\gamma_i)} = \lambda'_-$$

in the topology of measured lamination space $\mathcal{ML}(S')$. Since λ'_+ and λ'_- intersect each other,

$$\liminf_{i\to\infty, j\to-\infty} \frac{i(\gamma_i,\gamma_i)}{l(\gamma_i)l(\gamma_j)} \neq 0$$

. Hence this gives a contradiction and implies that the domain of discontinuity of $\rho_\infty(\Gamma)$ should be empty.

Proposition 2.2.16. The homeomorphism Φ_∞ constructed in Lemma 2.2.9 is actually a Möbius transformation.

Proof. Above proposition is equivalent to saying that its Beltrami coefficient is zero almost everywhere. We can show this by contradiction. Let $\mathcal{E}$ be the support of $\mu(\Psi_\infty) = \frac{\bar\partial\Psi_\infty}{\partial\Psi_\infty}$. Suppose its Lebesgue measure is nonzero. By the commutation relation,

$$\Phi_\infty(\rho_\infty(\gamma)(z)) = \rho_\infty(\phi_*(\gamma))(\Psi_\infty(z))$$

, $\mathcal{E}$ is invariant under $\rho_\infty(\Gamma)$ and

$$\mu(\Psi_\infty \circ g(z)) = \mu(\Psi_\infty(z))$$

for every $g \in \rho_\infty(\Gamma)$. Hence

$$\mu(\Psi_\infty \circ g(z))\frac{\overline{g'(z)}}{g'(z)} = \mu(\Psi_\infty(z))$$

for every $g \in \rho_\infty(\Gamma)$. We define a measurable line field $\mathcal{P}$ on $\mathcal{E}$ by letting $\mathcal{P}(z)$ be the line through $z \in \mathcal{E}$ that makes an angle $\frac{1}{2}\arg\mu(\Psi_\infty(z))$ with the x-axis. Then $\mathcal{P}$ is a measurable line field on $\mathcal{E}$ invariant under the group $\rho_\infty(\Gamma)$.

Theorem 2.2.17. Let $G \subset PSL_2(\mathbb{C})$ be a Kleinian group whose limit set is the entire sphere $\overline{\mathbb{C}}$. Then there is no measurable line field that is defined on a Borel set in $\mathbb{C}$ with nonzero measure and is invariant under G.

We leave the proof to the reader. See chapter 7 in Otal's book [45].
This theorem gives a contradiction. Hence the homeomorhpism Ψ_∞ is a Möbius transformation. $\square$

Let $\rho_\infty(t)$ denote the Möbius transformation Φ_∞. Then

$$\rho_\infty(t) \circ \rho_\infty(\gamma) \circ \rho_\infty(t)^{-1} = \rho_\infty(\phi_*(\gamma))$$

for all $\gamma \in \Gamma \simeq \pi_1(S)$. Hence the map on the free product $\pi_1(S) * < t >$ that coincides with ρ_∞ on $\pi_1(S)$ and equals $\rho_\infty(t)$ on the generator t of the cyclic group $< t >$ induces a representation of the fundamental group $\pi_1(M_\phi)$. The representation ρ_∞ is discrete and faithful. We leave the proof of this to the reader.

Let M be a compact, irreducible, orientable and atoroidal 3-manifold. Suppose M is a Haken 3-manifold, i.e., it contains an compact, connected, orientable and properly embedded incompressible surface S, which has negative Euler characteristic and cuts M into an interval bundle. We want to show that the interior of M carries a complete hyperbolic metric with finite volume. There are 2 cases, according to whether or not the surface bundle S disconnects M.

(1) S does not disconnect M.
Then the manifold obtained by cutting $(M, \partial M)$ along $(S, \partial S)$ is an interval bundle that contains two copies of S in its boundary. Since M is orientable, this bundle is diffeomorphic to $S \times [0,1]$. Hence the manifold M is diffeomorphic to the suspension M_ϕ of some orientation-preserving diffeomorphism ϕ of S. Up to a bundle diffeomorphism, M_ϕ is independent of the isotopy class of the diffeomorphism ϕ. We say that, in Theorem 2.2.2, every diffeomorphism ϕ of S is one of three types, (a) ϕ is reducible, (b) periodic or (c) is pseudo-Anosov. (a) and (b) cannot be atoroidal. The monodromy can only be pseudo-Anosov. We say before that there exists a discrete, faithful representation ρ of $\pi_1(M_\phi)$ in $PSL_2(\mathbb{C})$. Hence there is an isomorphism between $\pi_1(M)$ and $\pi_1(M(\rho(\pi_1(M\phi)))$ which is realized by homotopy equivalence f, since M_ϕ and $\pi_1(M(\rho(\pi_1(M\phi)))$ are $K(\pi,1)$'s.

If a manifold M is closed, then the homotopy equivalence f is homotopic to a diffeomorphism by a theorem of Stallings [50]. If a boundary of M is nonempty, then each component of ∂M is a torus, the image of the fundamental group of this torus under the representation ρ is a parabolic subgroup of $PSL_2(\mathbb{C})$, since it is a discrete, abelian, rank-2 subgroup of $PSL_2(\mathbb{C})$. For a constant ϵ less than or equal to the Margulis constant, consider the manifold $M^{[\epsilon,\infty)}$ equal to the ϵ-thick part of the manifold $M(\rho(\pi_1(M_\phi)))$. This is a manifold with tori boundary $T_1, \cdots T_k$, with respective fundamental groups corresponding to those of the boundary components of M_ϕ. Choose a homotopy equivalence between $(M_\phi, \partial M_\phi)$ and $(M^{[\epsilon,\infty)}, \cup_{i\leq k} T_i)$ that sends ∂M_ϕ to $\cup_{i\leq k} T_i$. We can homotope this homotopy equivalence such that it induces a diffeomorphism on the boundary.

$\cup_{i\leq k} T_i$ is null-homologous in $M^{[\epsilon,\infty)}$ because the tori make up the image of the boundary of M_ϕ, which is null-homologous since M_ϕ is compact. Hence $M^{[\epsilon,\infty)}$ is compact. Furthermore, M_ϕ does not contains no essential annuli ; Since curves traced on the boundary of $M^{[\epsilon,\infty)}$ act as parabolic elements in hyperbolic space, every incompressible annulus properly embedded in $M^{[\epsilon,\infty)}$ is homotopic into the boundary. Hence, by a theorem of Johannson [35, 34], the homotopy equivalence between M_ϕ and $M^{[\epsilon,\infty)}$ is homotopic to a diffeo-

morphism.
The interior of the manifold of $M^{[\epsilon,\infty)}$ is diffeomorphic to $M(\rho(\pi_1(M_\phi)))$. Hence $M(\rho(\pi_1(M_\phi)))$ is a manifold diffeomorphic to the interior of M with finite volume as the union of a compact manifold and finitely many cusps of type $\mathbb{Z}+\mathbb{Z}$.

(2) S disconnects M.
We leave this out to the reader.

Chapter 3

Contact 3-manifolds

In this chapter, we will discuss some basic theory about contact 3-manifolds. In section 3.1 and 3.2, we will give general theory for contact manifolds. A contact 3-manifold has a natural dichotomy between overtwisted contact structure and tight one. In section 3.3, we will see the whole classification of overtwisted contact structures up to contact isotopy which is proved by Eliashberg. However, even the existence of tight contact structures has not been fully understood.

In 1980's, Giroux studied the embedded surfaces in tight contact 3-manifolds and found many properties of embedded surfaces in tight contact 3-manifolds, one of which is the convex surface theory. We will see these briefly in section 3.4. At the end of 1990's, Honda invented the bypass theory which captures bifurcations when moving the convex surface along the contact vector field. The bypass theory gives some essential tools to decompose a given contact manifold. We introduce a variety of bypasses in section 3.5. When decomposing a given manifold, one of the most useful theorem is Colin's gluing theorem. We introduce this theorem in section 3.6 and give the alternative proof by Honda, Kazez and Matić. Finally, we introduce some important results for tight contact 3-manifolds in section 3.7.

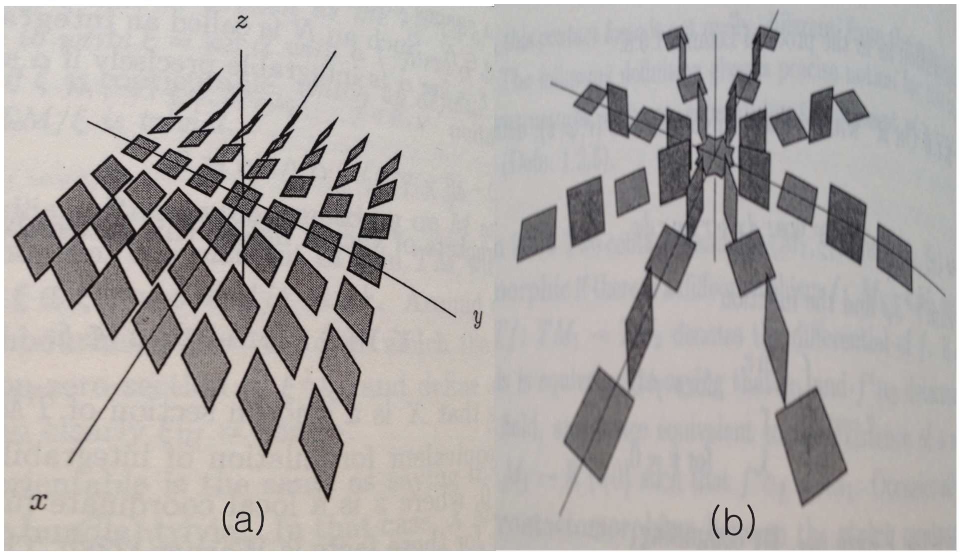


Figure 3.1: Tight vs. Overtwisted

3.1 Prelimenaries

A contact manifold (M, ξ) is a closed oriented $(2n+1)$-dimensional manifold which has a maximally nonintegrable hyperplane field ξ. If TM/ξ is a trivial line bundle, then we call ξ coorientable. If ξ is a coorientable contact hyperplane field, then we can find a global 1-form α such that $\xi = \ker \alpha$ and $\alpha \wedge (d\alpha)^n > 0$. From now on, we always assume that the ambient manifold is orientable and contact structure ξ is coorientable.

Example 3.1.1. We consider a contact manifold $(\mathbb{R}^3, \xi_{std} = \ker(\alpha_{std} = dz + xdy))$. Then contact structure ξ is generated by $\{\partial_x, -x\partial_z + \partial_y\}$. Hence, contact hyperplane field rotates along $y = constant$ line and asymptotes vertical plane field. We call this manifold the standard contact manifold.

Example 3.1.2. $(\mathbb{R}^3, \xi_{ot} = \ker \alpha_{ot})$ with cylindrical coordinates (r, ϕ, z), where $\alpha_{ot} = \cos r dz + r \sin r d\phi$ looks different from Example 3.1.1. The hyperplane field rotates fully along a radial direction and at $r = 2\pi$, hyperplane fields are tangent to the xy-plane. We will explain this phenomenon later.

Definition 3.1.3. If a diffeomorphism $\phi : (M_1, \xi_1) \to (M_2, \xi_2)$ satisfies $\phi_*(\xi_1) = \xi_2$, then we call ϕ a contactomorphism.

Theorem 3.1.4 (Gray stability theorem)**.** Let $\{\xi_t\}(t \in [0,1])$ be a smooth family of contact structures on M. Then there exists a isotopy $\psi_t(t \in [0,1])$ of M s.t. $(\psi_t)_*(\xi_0) = \xi_t$.

Proof. Let α_t be a smooth family of 1-forms with $\ker \alpha_t = \xi_t$. We can use Moser trick to find a isotopy ψ_t of a neighborhood of p such that $\psi_t^* \alpha_t = \alpha_0$. The Moser's technique is that we assume ψ_t is the flow of a time-dependent vector field X_t and then translate into an equation of X_t. If we can find a solution of the equation of X_t, then we can get a isotopy ψ_t by integrating X_t.
Assume that there is an isotopy ψ_t. Then it must satisfies the following equation

$$\psi_t^* \alpha_t = \lambda_t \alpha_0, \tag{3.1.1}$$

where $\lambda_t : M \to \mathbb{R}^+$ is a smooth family of smooth functions. Differentiate this equation with respect to t. Then we can get a following equation by the below lemma,

$$\psi_t^*(\dot{\alpha}_t + \mathcal{L}_{X_t} \alpha_t) = \dot{\lambda}_t \alpha_0 \tag{3.1.2}$$

We put $\mu_t := \frac{d}{dt}(log \lambda_t) \cdot \psi_t^{-1}$. Then by the equation (3.1.1), (3.1.2) and Cartan formula for Lie derivative, we can get the following,

$$\psi_t^*(\dot{\alpha}_t + d(\alpha_t(X_t)) + i_{X_t} d\alpha_t) = \psi_t^*(\mu_t \alpha_t) \tag{3.1.3}$$

If we choose $X_t \in \xi_t$ then it satisfies the equation (3.1.3) and it translates into the equation $\dot{\alpha}_t + i_{X_t} d\alpha_t = \mu_t \alpha_t$. For Reeb vector field R_t of α_t, R_t must satisfy $\dot{\alpha}_t(R_t) = \mu_t$, i.e., we can find μ_t by this equation. Hence we can find a unique solution $X_t \in \xi_t$ satisfying $\dot{\alpha}_t + i_{X_t} d\alpha_t = \mu_t \alpha_t$. The flow of X_t gives the required isotopy. □

Theorem 3.1.5 (Darboux)**.** Let (M^{2n+1}, ξ) be a contact manifold. Then Every point p in M has an open neighborhood U such that $(U, \xi|_U)$ is contactomorphic to an open set in $(\mathbb{R}^{2n+1}, \xi_0)$.

Proof. It suffices to show that there are coordinates $x_1, \cdots, x_n, y_1, \cdots, y_n, z$ on a neighborhood $U \subset M$ of p such that $p = (0, \cdots, 0)$ and $\alpha|_U = dz + \sum_{i=1}^n x_i dy_i$. First we can choose a linear independent vectors $x_1, \cdots, x_n, y_1, \cdots, y_n, z$ on $T_0\mathbb{R}^{2n+1}$ such that $\alpha(\partial_z) = 1, i_{\partial_z} d\alpha = 0, \partial_{x_i}, \partial_{y_i} \in \ker\alpha$, where $i = 1, \cdots, n$ and $d\alpha = \sum_{i=1}^n dx_i \wedge dy_i$. Then put $\alpha_0 = dz + \sum_{i=1}^n x_i dy_i$ and $\alpha_t = (1-t)\alpha_0 + t\alpha, t \in [0,1]$. α_t is a 1-parameter family of contact 1-forms on $\mathbb{R}^{2n+1}$ satisfying $\alpha_t = \alpha, d\alpha_t = d\alpha$ for all t at p.
The idea is the same as the proof of Gray stability theorem. Differentiate $\psi_t^*\alpha_t = \alpha_0$, then we can get $\psi_t^*(\dot{\alpha}_t) = \alpha_0$. It suffices to find the vector field X_t satisfying

$$\dot{\alpha}_t(R_t) + d(\alpha_t(X_t)) + i_{X_t} d\alpha_t = 0 \tag{3.1.4}$$

Put $X_t = H_t R_t + Y_t$ where R_t is the Reeb vector field. The Reeb vector field R is that the vector field uniquely defined by $d\alpha(R, \dot{}) \equiv 0$ and $\alpha(R) \equiv 1$. If α_t and $Y_t \in \ker\alpha_t$. Plug X_t into the equation (3.1.4). Then we can get the following equation for H_t

$$\dot{\alpha}_t(R_t) + dH_t(R_t) = 0 \tag{3.1.5}$$

We can choose small neighborhood of origin such that R_t has any closed orbits there. Then we can find H_t by integration near the origin. Since $\dot{\alpha}_t$=0 at $p = 0$, $H_t(p) = 0$ and $dH_t|_0 = 0$ for all $t \in [0,1]$. Then we plug X_t into the equation (3.1.4) again. Then we can get $\dot{\alpha}_t + dH_t + i_{Y_t} d\alpha_t = 0$. By this equation, we can find the unique Y_t. Hence we can find the X_t which gives an isotopy ϕ_t. □

Lemma 3.1.6. Let $\omega_t, t \in [0,1]$, be a smooth family of differential k-forms on a manifold M and $\psi_t, t \in [0,1]$, an isotopy of M. Let X_t be a time-dependent vector field on M such that $X_t \dot{\psi}_t = \dot{\psi}_t$. Then the following holds,

$$\frac{d}{dt}(\psi_t^*\omega_t)|_{t=t_0} = \psi_{t_0}^*(\dot{\omega}_t|_{t=t_0} + \mathcal{L}_{X_{t_0}}\omega_{t_0}) \tag{3.1.6}$$

By Darboux's theorem, every contact manifold is locally contactomorphic to a subset of $(\mathbb{R}^{2n+1}, \xi_0)$. Hence, we have to look at the global structures of contact manifolds.

3.2 Existence

Theorem 3.2.1 (Martinet [41])**.** Every closed orientable 3-manifold M admits a contact structure.

Proof. S^3 admits a contact structure. By the theorem of Likorish and Wallace, Every closed 3-manifold can be obtained from S^3 by a Dehn surgery along a link K. Let ξ_0 be a contact structure of S^3. We can obtain a transverse knot K by C^0-approximation. A transverse knot is an S^1 embedding $\gamma : S^1 \hookrightarrow M$ such that $\dot{\gamma} \pitchfork \xi$. If contact hyperplane is cooriented and $\alpha(K'(t)) > 0$ for $t \in S^1$, then we call this positively transverse knot. For positively transverse knot K, we can find a standard tubular neighborhood νK of K which is diffeomorphic to $S^1 \times D_{\delta_0}$ such that K is identified with $S^1 \times 0$, D_{δ_0} is a disc of radius δ_0 and $\xi_0 = \ker d\theta_1 + r_1^2 d\phi_1$ with a coordinate $(\theta_1, r_1, \phi_1) \in S^1 \times D_{\delta_0}$.
We perform a Dehn surgery inside this tubular neighborhood νK. Cut out $S^1 \times D_\delta$ for $\delta < \delta_0$ and glue back a copy of $S^1 \times D_\delta$ by gluing map $\mu_0 \mapsto p\mu + q\lambda$, $\lambda_0 \mapsto m\mu + n\lambda$, where μ_0 is meridian of $S^1 \times D_\delta$, λ_0 is a longitude of $S^1 \times D_{delta}$ and $pm - qm \neq 0$, $p, q, m, n \in \mathbb{Z}$. Let (θ, r, ϕ) be a coordinate of a copy of $S^1 \times D_\delta$. Then the contact form $d\theta_1 + r_1^2 d\phi_1$ in $S^1 \times D_{\delta_0}$ pulls back to the contact form $d(n\theta + q\phi) + r^2 d(m\theta + p\phi)$ which is defined on $S^1 \times D_\delta$ except $S^1 \times 0$. The following lemma complete the proof. □

Lemma 3.2.2. There is a contact form on $S^1 \times D_\delta$ which is coincide with $d(n\theta + q\phi) + r^2 d(m\theta + p\phi)$ near $r = \delta$ and with $d\theta + r^2 d\phi$ near $r = 0$.

Proof. Put $\alpha = h_1(r)d\theta + h_2(r)d\phi$ for smooth function $h_1(r), h_2(r)$. To satisfy contact condition $\alpha \wedge d\alpha > 0$, it is enough to find a curve $(h_1(r), h_2(r)), 0 \leq r \leq \delta$ satisfying the following condition :

(1) $h_1(r) = 1$ and $h_2(r) = r^2$ near $r = 0$,

(2) $h_1(r) = n + mr^2$ and $h_2(r) = q + pr^2$ near $r = \delta$,

(3) $h_1 h_2' - h_2 h_1' \neq 0$ for $0 < r \leq \delta$

Then, the function $h_1(r) = \cos(r)$, $h_2(r) = r \sin r$ satisfies the equation, where $r \in [0, 2\pi]$. □

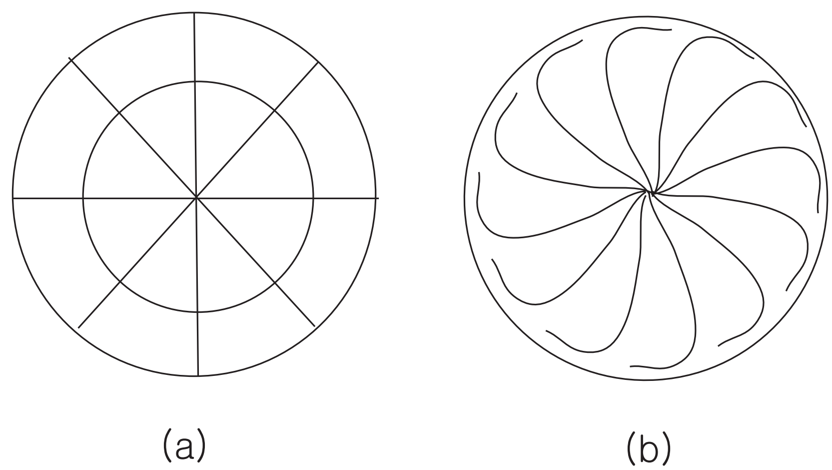


Figure 3.2: Overtwisted disk

3.3 Global structures of overtwisted(OT) contact 3-manifold

In contact 3-manifold case, there are two kinds of contact structures, one of which contains a overtwisted disk (called overtwisted contact structures), the other does not (called tight contact structures). To define overtwisted(OT) disk, we need some concepts.

Definition 3.3.1. A submanifold $L \subset (M^{2n+1}, \xi)$ is Legendrian if $\dim L = n$ and $T_xL \subset \xi_x$ for all $x \in L$, i.e., A Legendrian submanifold is the maximal isotropic submanifold.

Definition 3.3.2. For Legendrian knot L, the twisting number, which is called sometimes relative Thurston-Bennequin invariant, $t(L, \mathcal{F})$ is the integer difference in the number of twists between the normal framing and $\mathcal{F}$, where $\mathcal{F}$ is some fixed framing for Legendrian knot L and normal framing is induced from ξ by taking $v_p \in \xi_p$ so that $(v_p, \dot{L}(p))$ form an oriented basis for ξ_p. For homological zero Legendrian knot L, The Thurston-Bennequin number $tb(L)$ is defined as a twisting number of the contact framing relative to the surface framing of L, which is given by the Seifert surface of L.

Since Thurston-Bennequin number does not depend on the Seifert surface, it is an invariant of Legendrian knot.

Definition 3.3.3. An embedded disk $D \subset (M, \xi)$ is called Overtwisted(OT) disk if $\xi_p = T_pD$ for all $p \in \partial D$ and characteristic foliation D_ξ contains a unique singular point in the interior of D.

Remark 3.3.4. If we consider an embedded surface Σ in contact 3-manifold (M, ξ), ξ induces natural oriented singular foliation Σ_ξ on Σ, which is called characteristic foliation, given by $\Sigma_\xi(p) = \xi_p \cap T_p\Sigma$, for all $p \in \Sigma$. The singular points are $p \in \Sigma$ such that $\xi_p = T_p\Sigma$. The characteristic foliation of OT disk look like the Figure 3.1 (a). We call this standard OT disk. We can also get a disk which looks like the Figure 3.1.(b), by perturbing standard OT disk. We allow that this is also OT disk. Then $tb(\partial D)$ also holds for other disks.

Definition 3.3.5. If (M, ξ) has an OT disk, then we call (M, ξ) OT contact manifold and otherwise, tight contact manifold.

Remark 3.3.6 (Bennequin [2])**.** Example 3.1.1 is tight and Example 3.1.2 is overtwisted contact manifold. In Example 3.1.2, we take the OT disk as a set $\{(r, \phi, z) | r \leq 2\pi, z = 0\}$.

Let M be a closed, orientable 3-manifold. Fix an orientation of M, an embedded orientable 2-disk $\Delta \subset M$ and a center point $0_\Delta \in \Delta$. We call a contact structure ξ on M overtwisted along Δ if (M, ξ) contains Δ as a standard overtwited disk with a center 0_Δ. We denote the space of overtwisted contact structures along Δ by $\mathcal{C}^{OT}(M, \Delta)$ and the space of cooriented 2-plane distributions on M which are tangent to Δ at 0_Δ by $Dist(M, \Delta)$. Then the following holds.

Theorem 3.3.7 (Eliashberg [9])**.** $i_\Delta : \mathcal{C}^{OT}(M, \Delta) \to Dist(M, \Delta)$ is a weak homotopy equivalence.

Corollary 3.3.8. $i_* : \pi_0(\mathcal{C}^{OT}(M)) \to [M : S^2]$ is a bijection.

Lutz and Martinet showed that i_* is surjective and Eliashberg proved the injectivity. We need the following two theorems for surjectivity.

Theorem 3.3.9. For every closed, orientable 3-manifold M, TM is the trivial tangent bundle.

There are many proofs, but in here, we introduce the proof using obstruction theory.

Proof. we first show that the second-Stiefel-Whitney class $w_2(M) = w_2(TM) \in H^2(M;\mathbb{Z}_2)$ vanishes. Wu classes $v_i \in H^i(M;\mathbb{Z}_2)$ are defined by $< Sq^i(u), [M] > =< v_i \cup u, [M] >$ for all $u \in H^{3-i}(M;\mathbb{Z}_2)$, where Sq denotes the Steenrod squaring operations. Since M is 3-manifold, $Sq^i(u) = 0$ if $i > 3 - i$ by definition. Hence the only nonzero Wu classes are $v_0 = 1$ and v_1. Since $w_i = \sum_j Sq^{i-j}(v_j)$ and M is orientable, $v_1 = Sq(v_1) = w_1 = 0$. Hence, $w_2 = Sq^2(v_0) + Sq^1(v_1) + Sq^0(v_0) = 0$.

Let $V_2(\mathbb{R}^3) = SO(3)/SO(1) = SO(3)$ be the Stiefel manifold of oriented, orthonormal 2-frames in $\mathbb{R}^3$. Since $SO(3)$ is connected, there exists a section of the 2-frame bundle $V_2(TM)$ over the 1-skeleton of M. Since the obstruction to extending this section over the 2-skeleton is w_2 and we have seen above that this vanishes, we can extend this section over 2-skeleton. The obstruction to extending the section over all of M lies in $H^3(M;\pi_2(V_2(\mathbb{R}^3)))$, but this is zero because of $\pi_2(SO(3)) = 0$.

So far, we showed that M always has a trivial 2-dimensional subbundle ϵ^2. Define $TM/\epsilon^2 =: \lambda$. $w_1(\lambda) = w_1(\epsilon^2) + w_1(\lambda) = w_1(TM) = 0$. So λ is orientable. Thus TM is trivial. $\square$

Therefore, if we fix a trivialization, there is a one-to-one correspondence between homotopy classes of cooriented 2-plane field distribution ξ in TM and homotopy classes of maps $f : M \to S^2$.

Theorem 3.3.10 (Lutz-Martinet)**.** Every cooriented tangent 2-plane field on a closed, orientable 3-manifold is homotopic to a contact structure.

Proof. Let η be an cooriented 2-plane distribution on an oriented 3-manifold M. And we can choose a contact structure ξ_0 with zero Euler class. For closed, oriented 3-manifold, there is a knot K which represents an arbitrary first homology class. For this knot, we can get a knot K positively transvers to ξ_0 by C^0 perturbation. Choose knot K positively transverse to ξ_0 with $-PD[K] = d^2(\eta, \xi_0)$. Since $-PD[K] = d^2(\xi_0^K, \xi_0)$ and d^2 is additive,

$d^2(\xi_0^K, \eta) = 0$. ξ_0^K means that we perform a full Lutz twist of ξ along K. Since two distributions are homotopic over the 2-skelecton of M if and only if $d^2 = 0$, therefore we may assume that η is homotopic to ξ_0^K except a small Darboux ball. Then we can find a link K' in Darboux ball transverse to ξ_0^K with self-linking number $sl(K')$ equal to $d^3(\eta, \xi_0^K)$.

Then we can obtain a new contact structure xi from ξ_0^K by a Lutz twist along K'. Since this Lutz twists occur inside a Darboux ball, $d^2(\xi, \eta) = 0$. Let $f : S^3 \to S^2$ be the map to compute $d^3(\xi, \xi_0^K)$. Then $H(f) = d^3(\xi, \xi_0^K) = sl(K') = d^3(\eta, \xi_0^K)$. Therefore, $d^3(\xi, \eta) = 0$. Hence, ξ and homotopic to η over M. $\square$

3.4 Convex surface

A contact vector field v on (M^{2n+1}, ξ) is a vector field with the property $(\psi_t)_*(\xi) = \xi$, $t \in \mathbb{R}$ where ψ_t is a local flow of v. A convex surface Σ in (M^3, ξ) is an embedded surface with contact vector field v near Σ such that $v \pitchfork \Sigma$. By Peixoto's theorem, arbitrary smooth surface in contact 3-manifold (M, ξ) is C^∞-generically convex. By the flow of a contact vector field, we can give a contact structure on the neighborhood of convex surface. We call this an I-invariant neighborhood.

Let Σ be a convex surface and v be a contact vector field defined near Σ. We can consider Hamiltonian function $H := \alpha(v)$. Then by multiplying bump function, we can make globally defined contact vector field. We call again this v and put ψ_t as a flow of v. Then this flow gives an embedding $f : \Sigma \times \mathbb{R} \to M$ by $f(p, t) = \psi_t(p)$. $\ker(f^*\alpha)$ is called (vertically or I-) invariant contact structure on $\Sigma \times \mathbb{R}$. We can take the Lie derivative of $\lambda(f^*\alpha)$ along ∂_t for positive smooth function λ. Then $\lambda(f^*\alpha)$ is ∂_t invariant.

Hence we can write $\lambda(f^*\alpha) = \beta_t + u_t dt$ and calculate Lie derivation again. Then we can get $\dot{\beta}_t \equiv 0, \dot{u}_t \equiv 0$. Hence $\lambda(f^*\alpha) = \beta + udt$ for 1-form β on Σ and real-valued function u.Then we can get the following properties of convex surfaces.

A convex surface $\Sigma \subset (M, \xi)$ has a natural set of curves Γ_Σ, called the dividing curves, which is defined as the set of points $p \in \Sigma$ such that $v_p \in \xi_p$ with respect to a given contact vector field v. The characteristic foliation Σ_ξ of hypersurface Σ in (M, ξ) is the singular 1-dimensional foliation of Σ defined by the distribution $(T\Sigma \cap \xi|_\Sigma)^\perp$. Positive (negative) region $R_+(\Gamma_\Sigma) \subset \Sigma$ is the set of points where the orientation of ξ agrees (disagrees) with the orientation of Σ.

Proposition 3.4.1. The dividing set has the following properties.

1. Γ is a nonempty set and a union of closed curves.
2. $\Gamma \pitchfork \Sigma_\xi$.
3. Isotopy class of Γ_Σ does not depend on the choice of v.
4. $\Sigma \backslash \Gamma_\Sigma = R_+(\Gamma_\Sigma) \sqcup R_-(\Gamma_\Sigma)$.

Proof. Nonemptiness can be showed by Stokes' theorem. A contact 1-form α can be written as $\alpha = f dt + \beta$ in a invariant neighborhood, where f is a real valued function and β is a 1-form on Σ. Then the dividing set Γ is a $f^{-1}(0)$ and 0 is a regular value by contact condition. $\ker \beta$ gives a characteristic foliation and $\ker df = \dot{\Gamma}$. $\square$

Definition 3.4.2. A vector field X on a closed, orientable surface S is of Morse-Smale type if it satisfies the following

(1) $\exists$ Finitely many non-degenerate singularities and closed orbits

(2) $\forall$ $\alpha-$ &$\omega-$limit set is either a singular point or a closed orbit

(3) $\nexists$ Trajectories connecting hyperbolic points

Theorem 3.4.3 (Peixoto)**.** A vector field on a closed orientable surface is C^∞-generically Morse-Smale.

Lemma 3.4.4. For a C^∞-generic closed oriented surface $\Sigma \subset (M, \xi)$, its characteristic foliation Σ_ξ is of Morse-Smale type.

Proof. By the theorem of Peixoto, we can choose a perturbation of given contact 1-form which is compactly supported near Σ. Then we can get a 1-parameter family of contact structures. Then use a Gray stability theorem. $\square$

Proposition 3.4.5. Let ξ_0 and ξ_1 be two contact structures which induce the same characteristic foliation on an oriented surface. Then there is an isotopy $\phi_t, t \in [0,1]$ with $\phi_0 = id$ and $(\phi_1)_*\xi_0 = \xi_1$ relative to the surface.

Proof. Let $\Sigma \times [-\epsilon, \epsilon]$ be a neighborhood of surface Σ and $\alpha_i = \beta_i + f_i dt$, $i = 0, 1$. Since $\Sigma_{\xi_0} = \Sigma_{\xi_1}$, we can put $\beta_0 = u\beta_1$ at $t = 0$ for nowhere vanishing function u. Since f_i cannot be zero on the set $\{\beta_0 = 0\} = \{\beta_1 = 0\}$, we can put $f_i = 1$ near the set. Put $\alpha_t = dt + (1-t)\beta_0 + t\beta_1$ and use Moser technique. Plug vector field Y into the equation $\mathcal{L}_Y \alpha_s = i_Y d\beta_s + d(\alpha(Y)) = \frac{d\alpha_s}{ds} = \beta_1 - \beta_0$. If Y satisfies $i_Y d\beta_s = \beta_1 - \beta_0$, then $d\beta_s(Y, Y') = 0$ for any Y' which gives a characteristic foliation. Hence, Y must be tangent to the characteristic foliation. Hence we can put $u = 1$ near the zero set $\{\beta_0 = 0\}$. By dividing α_0 by u, we can assume that $\beta_0 = \beta_1$ at $t = 0$. Put $\alpha_s = (1-s)\alpha_0 + s\alpha_1, s \in [0,1]$, then α_s is 1-parameter family of contact 1-forms. Use the Moser technique again. Put $X_s = g_s R_s + y_s$ where R_s is the Reeb vector field for α_s and $Y_s \in \xi_s$ and ϕ_s as a flow of X_s. Then $\phi(\Sigma) = \Sigma$. We take $g_s = 0$. Then Y_s have to satisfies $i_{Y_s} d\alpha_s = \frac{d\alpha_s}{ds} + \phi_s \alpha_s$. If X' is a vector field tangent to the characteristic foliation, then $d\alpha_s(Y_s, X') = 0$. Therefore, we can $Y_s \in \xi_s \cap T\Sigma$ satisfies the equation. $\square$

Theorem 3.4.6. A C^∞-generic closed embedded surface Σ is convex.

Proof. By Lemma 3.4.4 and Proposition 3.4.5, it suffices to prove that Σ with a Morse-Smale type characteristic foliation is convex, i.e, it is enough to construct a contact structure ξ on $\Sigma \times \mathbb{R}$ with a given characteristic foliation. We have to find a contact 1-form $\alpha = f dt + \beta$ for t-independent β and f.

Let S_+ be a union of a small disk around each positive singular point, a small annuli around each source closed orbit and a band around each orbit δ which flows into a positive hyperbolic point. Similarly we can chooses a subregion S_- with negative regions. we may choose boundaries of S_+ and

S_- is transverse to a given foliation. Then every closed orbit or singularities is contained in S_+ and S_- and every orbit have to come from S_+ to S_-. $\Sigma \setminus (S_+ \cup S_-)$ is a union of annuli.

We can think that $S_\pm = \{p \in S | \pm f(p) > 0\}$. Or equivalently, $\{\pm d\beta > 0\}$. $\pm d\beta$ gives an area form on $S_\pm$. If we rescale it, then $div_{g\Omega}(X) = \frac{X(g)}{g} + div_\Omega(X)$, where X is a smooth vector field which gives a characteristic foliation and g is a positive smooth function. Hence sign of divergence does not change. We can take $f \equiv \pm 1$ on $S_\pm$. Put $d(i_X \Omega_0) = d\beta = \pm\Omega_0$ on $S_\pm$.

Consider a component of Γ_Σ. Use the flow of X to identify a tubular neighborhood of this component with an annulus $A = S^1 \times [-\epsilon, \epsilon]$, where component of dividing curves is identified with $S^1 \times \{0\}$ and $A \cap S_+ = S^1 \times (0, \epsilon]$. (t, s) is a coordinate for $S^1 \times [-\epsilon, \epsilon]$. Then $X = -\partial_s$ and $dt \wedge ds$ gives an area form on A with same orientation of Ω_0. We can choose annulus so wide that $div_{\Omega_0}(X) = \pm 1$ near $S^1 \times \pm\epsilon$.

It is enough to find a global area form Ω. We put $\Omega_A = f(s) dt \wedge ds$ for positive function f. Then $div_{\Omega_A}(-\partial s) = -\frac{f'}{f}$. So we choose f to be strictly increasing on $[-\epsilon, -)$ and strictly decreasing on $(0.\epsilon]$.

Set $\Omega = \Omega_A$ on $S^1 \times (-\epsilon, \epsilon)$ and $\Omega = \Omega_0$ outside of annulus. Denote $A_\pm = [\pm 2\epsilon/3, \pm\epsilon/3]$. If we rescale the area form, $div_{g\Omega}(X) = \frac{X(g)}{g} + div_\Omega(X)$. Since absolute value of divergence of $A_\pm$, we can find g satisfying $\pm(\frac{X(g)}{g} + div_\Omega(X)) > 0$ on $A_\pm$. We can take $g \equiv 1$ outside A and $g \equiv K$ for large constant K on $S^1 \times (-\epsilon/3, \epsilon/3)$.

□

By following theorem, we can think that the isotopy type of the dividing sets of convex surface determines a contact structure of neighborhood of convex surface up to contact isotopy.

Theorem 3.4.7 (Giroux's Flexibility Theorem [20]). Let Σ be convex with characteristic foliation Σ_ξ, contact vector field v, and dividing set Γ_Σ. Let $\mathcal{F}$ be another singular foliation on Σ which is adapted to Γ_Σ, i.e., there is a contact structure ξ' in a neighborhood of Σ such that $\Sigma_{\xi'} = \mathcal{F}$ and Γ_Σ is also a dividing set for ξ'. Then there is an isotopy ϕ_s, $s \in [0,1]$, of Σ in (M, ξ) such that:

1. ϕ_0=id and $\phi_s|_{\Gamma_\Sigma}$=id for all s.
2. $\phi_s(\Sigma) \pitchfork v$ for all s.
3. $\phi_1(\Sigma)$ has characteristic foliation $\mathcal{F}$.

Proof. By the above discussion, neighborhood of Σ is contactomorphic to $\Sigma \times \mathbb{R}$ with an invariant contact form $\xi_0 = \ker \alpha_0$ and $\alpha_0 = \beta_0 + u_0 dz$ where $\Sigma \equiv \Sigma \times \{0\}$, β is a 1-form on Σ and u is a smooth function on Σ.

Let Ω be an area form on Σ and X_0 be a vector field defining Σ_{ξ_0}. We may assume that $u_0 \equiv \pm 1$ on $\Sigma_\pm$ and $u_0 = 0$ along Γ_Σ. Since $\mathcal{F}$ is also divided by Γ_Σ, we can assume that for a vector field X_1' defining $\mathcal{F}$, $\pm div_{g\Omega}(X_1') > 0$ on $\Sigma_\pm$ for a positive smooth function g. Put $X_1 := gX_1'$. Then $X_t := (1-t)X_0 + tX_1, t \in [0,1]$ gives a small isotopy near Γ_Σ such that two characteristic foliations agree on a small annular neighborhood of Γ_Σ and contact structures also agree on there. $\pm div_\Omega(X_t) > 0$ on $\Sigma_\pm$. We find a family of contact forms $\alpha_t = i_{X_t}\Omega + u_t dz, t \in [0,1]$ with $u_t \equiv \pm 1$ on $\Sigma_\pm$ and $u_t = 0$ on Γ_Σ for all t.

Then we can use a Moser technique and solve for a vector field $X = g\partial_t + Y$ for Y tangent to Σ. Then Y is determined by $i_Y = i_{X_1}\Omega - i_{X_0}\Omega$ and g is determined by $-i_{X_s}\Omega(Y)$. Since X does not depend on z, the flow of X gives the isotopy we want. $\square$

The following theorem specifies the isotopy type of dividing curves of convex surface in a tight manifold.

Theorem 3.4.8 (Giroux's Criterion [20])**.** Suppose $S \neq S^2$ is a convex surface in a contact manifold (M, ξ). There exists a tight neighborhood for S if and only if Γ_S contains no homotopically trivial closed curves. If $S = S^2$, then S has a tight neighborhood if and only if $\#\Gamma_{S^2}$=1.

Lemma 3.4.9. $\langle e(\xi), \Sigma \rangle = \chi(R_+(\Gamma_\Sigma)) - \chi(R_-(\Gamma_\Sigma))$

□

Theorem 3.4.10 (Thurston-Bennequin inequality [2])**.** Let Σ be an embedded surface in a tight (M, ξ). If $\Sigma \neq S^2$ is closed, then

$$|\langle e(\xi), \Sigma \rangle| \leq -\chi(\Sigma) \tag{3.4.1}$$

Proof. This inequality easily comes from the Lemma 3.4.9 and Giroux's criterion(Theorem 3.4.8). □

Theorem 3.4.11 (Semi-Local Thurston-Bennequin inequality)**.** Let ξ be an I-invariant contact structure on the $\Sigma \times I$, where $I = [-1, 1]$. Suppose C is a simple closed curve on $\Sigma = \Sigma \times \{0\}$, and Γ is a dividing set on Σ which is adapted to ξ. Then, for all isotopies ϕ_t of $\Sigma \times I$ which takes C as a Legendrian curve $\phi_1(C)$, the number of twists ξ makes along $\phi(C)$ relative to the tangent planes to $\phi_1(S)$ satisfies the inequality

$$f(\xi, Fr_{\phi_1(\Sigma)}, \phi_1(C)) \leq -\frac{1}{2}\sharp(\Gamma, C) \tag{3.4.2}$$

where $f(\xi, Fr_{\phi_1(\Sigma)}), \phi_1(C))$ is the twisting of ξ along $\phi_1(C)$ relative to $Fr_{phi_1(\Sigma)}$ and $\sharp(\Gamma, C)$ minimal intersection numbers. Moreover, there is an isotopy which realizes equality.

3.5 Bypass

A submanifold $L \subset (M^{2n+1}, \xi)$ is Legendrian if $\dim L = n$ and $T_x L \subset \xi_x$ for all $x \in L$. The Thurston-Bennequin number $t(L, \mathcal{F})$ is the integer difference in the number of twists between the normal framing and $\mathcal{F}$, where $\mathcal{F}$ is some fixed framing for Legendrian knot L and normal framing is induced from ξ by taking $v_p \in \xi_p$ so that $(v_p, \dot{L}(p))$ form an oriented basis for ξ_p. Let C be a collection of closed curves and arcs on a convex surface Σ with Legendrian boundary. We call C nonisolating if:

1. A collection of curves and arcs C is transverse to the dividing set Γ_Σ.

2. Every arc of C begins and ends on Γ_Σ.

3. The elements of C are pairwise disjoint.

4. If we cut Σ along C, each component intersects the dividing set Γ_Σ.

The following theorem is useful for some closed curves or arcs to make Legendrian.

Theorem 3.5.1 (Legendrian Realization Principle [27])**.** Let C be a nonisolating graph on a convex surface S and v a contact vector field transverse to S. Then there exists an isotopy $\phi_s, s \in [0,1]$ so that :

(1) $\phi_0 = id$, $\phi_s|_{\gamma_S} = id$.

(2) $\phi_s(S) \pitchfork$ (and hence $\phi_s(S)$ are all convex),

(3) $\phi_1(\Gamma_S) = \Gamma_{\phi_1(S)}$,

(4) $\phi_1(C)$ is Legendrian.

Proof. By the Giroux's flexibility theorem(Theorem 3.4.7), it is enough to construct a characteristic foliation which has C as Legendrian curve. $\square$

Using dividing curves of a convex surface, we can decompose a contact Haken manifold by cutting and pasting along the incompressible convex surfaces. At this time, the key ingredients are edge-rounding and bypasses. Bypasses were originally developed by Honda [27] and studied further by Honda, Kazez and Matić [31, 32]. Given any Legendrian curve $\delta = \partial S$ with non-positive Thurston-Bennequin number, the following theorem guarantees that we can make S convex by perturbation, which is proved by Honda [27].

Theorem 3.5.2. Let $S \subset M$ be a compact, oriented, properly embedded surface with Legendrian boundary such that $t(\delta, \mathcal{F}_S) \leq 0$ for all components δ of ∂S. There exists a C^0 - small perturbation near the boundary which fixes ∂S and puts an annular neighborhood A of ∂S into standard form. Then, there is a further C^∞ - small perturbation (of the perturbed surface, fixing A) which makes S convex. Moreover, if v is a contact vector field on a neighborhood of A and transverse to A, then v can be extended to a vector field transverse to all of S.

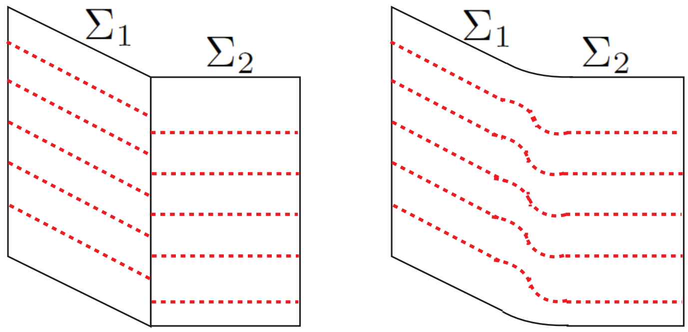


Figure 3.3: Edge rounding

Lemma 3.5.3 (Edge Rounding [27, 29])**.** Let Σ_1 and Σ_2 be convex surfaces with collared Legendrian boundary L which intersect transversely inside the ambient contact manifold along a common boundary Legendrian curve. Then the neighborhood of the common boundary Legendrian is locally isomorphic to the neighborhood $N_\epsilon = x^2 + y^2 \leq \epsilon$ of $M = \mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$ with coordinates (x, y, z) and contact 1-form $\alpha = \sin(2\pi nz)dx + \cos(2\pi nz)dy$, for some $n \in \mathbb{Z}^+$, and that $\Sigma_1 \cap N_\epsilon = x = 0, 0 \leq y \leq \epsilon$ and $\Sigma_2 \cap N_\epsilon = y = 0, 0 \leq x \leq \epsilon$. If we join Σ_1 and Σ_2 along $x = y = 0$ and round the common edge, the resulting surface is convex, and the dividing curve $z = \frac{k}{2n} - \frac{1}{4n}$ on Σ_2, where $k = 0, \cdots, 2n-1$. See Figure 3.3.

Proof. We take the surface $\Sigma = ((\sigma_1 \cup \Sigma_2) \setminus \{x^2 + y^2 \leq \delta^2\} = N_\delta) \cup (\{(x - \delta)^2 + (y - \delta)^2 = \delta^2\} \cap \{x^2 + y^2 \leq \delta^2\})$, where $\delta < \epsilon$ and take the transverse contact vector field for Σ_1 to be $\frac{\partial}{\partial x}$ on N_ϵ and the transverse contact vector field for Σ_2 to be $\frac{\partial}{\partial y}$ on N_ϵ. Then take the transverse contact vector field for $\{(x - \delta)^2 + (y - \delta)^2 = \delta^2\} \cap N_\delta$ to be an inward-radial vector $-\frac{\partial}{\partial r}$ for the circle $\{(x - \delta)^2 + (y - \delta)^2 = \delta^2\}$ $\square$

A bypass half-disk for a convex surface $\Sigma \subset M$ (closed or compact with Legendrian boundary) is an oriented convex half-disk D with a Legendrian boundary satisfying that $\partial D = \alpha \cup \beta, D \cap \Sigma = \alpha, D \pitchfork \Sigma$ and $tb(\partial D) = -1$. We call α bypass attachment arc. The following lemma shows that bypasses are very useful for dealing with convex surfaces.

Lemma 3.5.4. (Bypass Attachment Lemma [27])

Let D be a bypass for Σ and α be a Legendrian bypass attachment arc. There is a neighborhood of $\Sigma \cup D$ in M which is diffeomorphic to $\Sigma \times [0,1]$ with $\Sigma_i = \Sigma \times i, i \in 0, 1$ convex, $\Sigma \times [0,\epsilon]$ is I-invariant and Γ_{Σ_1} is obtained by bypass move of Γ_{Σ_0}. Bypass move is illustrated in Figure 3.4.

Proof. We can extend α to a closed Legendrian curve γ on Σ by Legendrian realization principle. We can also assume that γ has a standard annular neighborhood on Σ and D is a convex half-disk transverse to Σ. Take a one-sided i-invariant neighborhood $\Sigma \times [0,\epsilon]$ of Σ. Then $A' = \gamma \times [0,\epsilon] \subset \Sigma \times [0,\epsilon]$ is an annulus transverse to $\Sigma \times \{0\}$. Put $A = A' \cup D$. Then since A is convex, we can take an I-invariant neighborhood $N(A)$ of A. To be smooth out ∂A, we use the following Pivot lemma. Then we get a new convex surface $\Sigma \times \{0\}) \cup N(A)$ what we want by Edge-rounding lemma. □

Lemma 3.5.5 (Fraser [12], Pivot Lemma)**.** Let D be an embedded disk in a contact manifold (M, ξ) with a characteristic foliation $\xi|_D$ which consists only of one positive elliptic singularity p and unstable orbits from p which exit transversely from ∂D. If δ_1, δ_2 are two unstable orbits meeting at p, and $\delta_i \cap \partial D = p_i$, then, after C^∞-small perturbation of D fixing ∂D, we obtain D' whose characteristic foliation has exactly one positive elliptic singularity p' and unstable orbits from p' exiting transversely form ∂D, and for which the orbits passing through p_1, p_2 meet tangentially at p'.

If we consider the half-elliptic singular points q_1, q_2 on D which are also the expoints of α. Modify D near q_i to replace q_i by a pair of q_i^e, q_i^h, where q_i^e is a full elliptic point and q_i^h is a half-hyperbolic point. Use Pivot lemma to smooth the corners of A, and then A becomes a convex with Legendrian boundary.

If we cut a manifold M along a convex surface Σ, then the resulting manifold has two components of boundary isotopic to Σ, which are Σ_+ and Σ^-. When we attach a bypass on Σ^+, Σ can be isotoped along a bypass. Let $\Sigma' = \Sigma \times \{1\}$ be the isotoped surface along a bypass. Then the dividing set of $(\Sigma')^+$ is obtained by a attaching bypass move which is indicated in Figure 3.4 (a). We also think of that the dividing set of $(\Sigma')^-$ can be obtained by digging-out the bypass on Σ^-. We can think this procedure as an abstract bypass move shown in Figure 3.4(b). This is called attaching-digging principle.

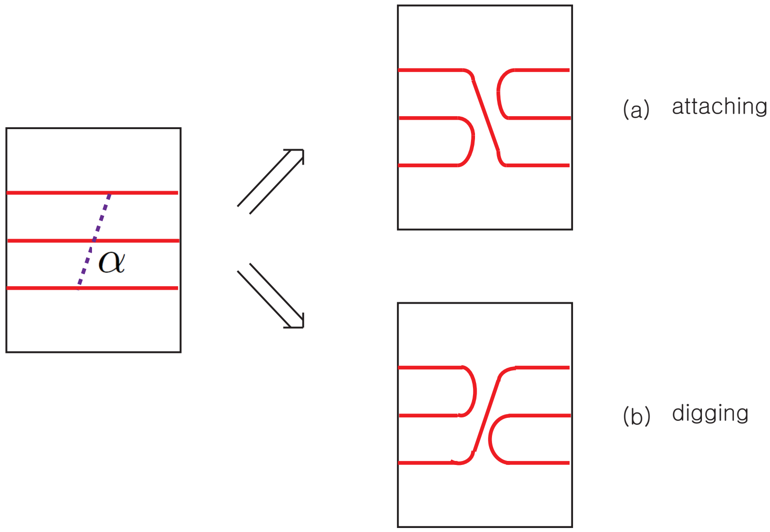


Figure 3.4: (a) The change of dividing set after attaching bypass, (b) The abstract change of dividing set after digging bypass

Since the attachment bypass on a given convex manifold increase the Thurston-Bennequin number, it is, in general, not easy to assure the existence of bypass. There are 4 types of bypasses, regular, folding(long), trivial and degenerate[27] bypasses. Since the role of degenerate bypasses is not important in this thesis, we do not discuss that notion. We always can find folding and trivial bypasses inside an invariant neighborhood. We say δ is a trivial arc if there exists a disk $D \subset \Sigma$ which contains δ in its interior so that $\partial D \pitchfork \Gamma_\Sigma$ and an abstract bypass move does not change the isotopy type of $\Gamma_\Sigma \cup D$ relative to ∂D. Here, by an abstract bypass move we simply mean a modification of the multicurve $\Gamma_\Sigma \subset \Sigma$ in a neighborhood of δ which would theoretically arise form attaching a bypass along δ. The following lemma guarantees the existence of a bypass along a trivial arc as a bypass attachment arc.

Lemma 3.5.6 (Existence of Trivial Bypasses [32])**.** Let Σ be a convex surface in a tight contact manifold (M, ξ) and let δ be a trivial Legendrian arc, as described in the previous paragraph. Then there exists an actual bypass half-disk B along δ (from the proper side) contained in the I-invariant neighborhood of Σ.

Proof. Suppose that δ intersects Γ_Σ successively along p_1, p_2, p_3. Since δ is a trivial arc, we may assume that the subarc from p_1 to p_2 and the subarc of Γ bound a disk D'. Let $D \subset \Sigma$ be a disk which contains $D' \cup \delta$. By LeRP, we can assume that ∂D is Legendrian and $tb(\partial D) = -2$ fixing δ. Then $\Gamma_D = \gamma_1 \cup \gamma_2$, where γ_1 contains p_1, p_2 and γ_2 contains p_3.
Let p_0 be a point on γ_2 which is not p_3 and by which there exists a Legendrian arc $\delta' \subset D$ from p_1 to p_0. δ' does not intersect Γ_D except at p_0 and does not intersect δ except at p_1. Let $\delta_0 = \delta \cup \delta'$ and $\delta_1 \subset D$ to be a Legendrian arc from p_3 to p_0 which lies on the same side of γ_2 and no other intersections with Γ_D. And let $\epsilon \subset D$ be an arc from p_0 to p_3 which lies on the opposite side of γ_2 and has no other intersections with Γ_D. Let $A \subset D \times [0,1]$ be a convex annulus such that $\partial A = (\delta \cup \epsilon) \times \{0\} \cup (\delta_1 \cup \epsilon) \times \{1\}$ and $(\epsilon \times [0,1]) \subset A$. The dividing set of Γ_A has two possibilities. The one intersecting $\{p_1, p_2, p_3\} \times \{0\}$ is what we want. $\square$

From now on, we explain folding bypasses. You can see this in [31] in more detail. We say that a closed curve γ is nonisolating if every component of $\Sigma \setminus \gamma$ intersects Γ_Σ. A Legendrian divide is a Legendrian curve such that all the points of γ are tangencies. We pick a nonisolating curve γ on a convex surface which does not intersect Γ_Σ. By a strong form of Legendrian realization, we can make γ into a Legendrian divide. Take a local model for γ as $(N = S^1 \times [-\epsilon, \epsilon] \times [-1, 1], \alpha = dz - y d\theta)$ with coordinates (θ, y, z). Then convex surface Σ in N is $S^1 \times [-\epsilon, \epsilon] \times \{0\}$ and γ is $S^1 \times \{0\} \times \{0\}$. By folding inside N around the Legendrian divide γ, we can get a pair of dividing curves parallel to γ.

We will use following two lemmas to find the regular bypass what we want.

Lemma 3.5.7 (Bypass Sliding Lemma [32])**.** Let R be an embedded rectangle with consecutive sides a, b, c, d in a convex surface Σ such that a is an arc of attachment of a bypass, b and d are subsets of Γ_Σ and c is a Legendrian arc which is efficient (rel endpoints) with respect to Γ_Σ. Then there exists a bypass for which c is its arc of attachment.

Proof. Let $R' \supset R$ be an embedded disk in Σ such that $\partial R'$ is Legendrian, $\partial R' \pitchfork \Gamma_\Sigma$, $tb(\partial R') = -3$, $\Gamma_{R'}$ consists of 3 parallel dividing arcs and $\partial R' = a' \cup b' \cup c' \cup d'$ where a', c' are Legendrian arcs parallel and close to a, c and the four arcs are consecutive sides of a rectangle whose corners. If we attach the bypass along a, then we can obtain a convex surface R'' isotopic to R' relative to $\partial R'$. R'' and R' are identical away from a. Then trivial bypass move on R'' along c gives isotopic dividing curves $\Gamma_{R''}$. This gives a bypass on R along c. $\square$

We could not get a regular bypass for free because they increase the Thurston-Bennequin number except for trivial bypass. We usually find a bypass inside given manifold by decomposing it along a irreducible surface. The following lemma is useful when doing this work.

Lemma 3.5.8 (Imbalance Principle [27])**.** Let $S^1 \times [0,1]$ be convex with Legendrian boundary inside a tight contact manifold. If $t(S^1 \times \{0\}) < t(S^1 \times \{1\}) \leq 0$, then there exists a bypass along $S^1 \times \{0\}$.

3.6 Gluing of tight contact manifolds

The following theorem allows us to get a new tight contact manifold by pasting along a convex surface. This is originally proved by Colin and reproved by Honda, Kazez and Matić [29, 31].

Theorem 3.6.1 (Colin [6])**.** Let (M, ξ) be an oriented, compact, connected, irreducible contact manifold with nonempty convex boundary, and $S \subset M$ be a properly embedded compact convex surface with nonempty Legendrian boundary such that (1) S is incompressible in M, (2) $t(\gamma, Fr_S) < 0$ for each connected component $\gamma \subset \partial S$ and (3) Γ_S is ∂-parallel. Consider a decomposition of (M, ξ) along S. If $(M \setminus S, \xi|_{M \setminus S})$ is universally tight, then (M, ξ) is universally tight.

In this book, we introduce briefly the alternative proof given by Honda, Kazez and Matić. This theorem holds in the case that splitting surface is a disk with boundary parallel dividing curves.

Proof of Theorem 3.6.1. we will first show that if ξ is universally tight on M/S, then ξ is tight on M. Assume, on the contrary, that there exists a overtwisted disk $D \subset M$. We can assume that D intersects S transversally along Legendrian curves and arcs and $\partial D \cap S \subset \Gamma_S$ after a contact isotopy. See Lemma 2.7 in [29]. Closed curves in $D \cap S$ are homotopically trivial because S is incompressible. Lemma 3.6.2 tells us that we can eliminate closed curves starting from innermost one by pushing S across D. Let δ be a innermost closed curve in $D \cap S$. Since M is irreducible, the 2-sphere formed by a disk on D and one on S bounding an δ bounds a ball across which S can be isotoped.

Lemma 3.6.2. We can push S across D to eliminate δ in a finite number of steps, each of which is a bypass along an arc of the circle δ.

Proof. Let D_δ be a subdisk of D bounded by δ. Since δ is a homotopically trivial Legendrian curve on S, $t(\delta, Fr_S)$ must be negative. We can make D_δ convex satisfying $t(\partial D) < 0$ by perturbation relative to ∂D. Since interior of D_δ lies in the tight manifold $M \setminus S$, $\Gamma_{D\delta}$ consists of properly embedded arcs with endpoints on ∂D_δ and no closed components. Hence there must exist a ∂-parallel dividing curve on D_δ, so we can make a bypass attaching on S which is a subdisk of D_δ using this ∂-parallel dividing curve. We can continue until there is only one arc left. the last step is also contact isotopy. $\square$

Similarly, we can eliminate the outermost arcs of $D \cap S$, Therefore, we can reduce the number of arcs of $D \cap S$ by modifying the dividing curve configuration on S. We have seen that changing the dividing set on S can be taken a sequence of bypass attachment on S.

Lemma 3.6.3. Let S be a convex surface with Legendrian boundary in a contact manifold (M, ξ), such that Γ_S is a ∂-parallel and $(M \setminus S, \xi)$ tight. Then any convex surface S' obtained from S by a sequence of bypasses will have $\Gamma_{S'}$ obtained from Γ_S by possibly adding pairs of parallel nontrivial curves (up to isotopy relative to boundary).

Proof. This comes from examining all possible bypasses on ∂-parallel dividing set. Trivial bypass does not change the isotopy class of dividing set. Some bypass which does not give an overtwisted disk introduce a pair of parallel curves, other possible bypass may change one pair of parallel dividing curves into another pair or remove a pair of parallel curves. $\square$

Lemma 3.6.4. Let S be a convex surface with Legendrian boundary in a contact manifold (M, ξ), such that Γ_S is ∂-parallel. If a convex surface S' is obtained from S by a bypass such that $\Gamma_{S'}$ is isotopic to Γ_S, then S and S' are contact isotopic, and in particular $(M \setminus S, \xi)$ is tight if and only if $(M \setminus S', \xi)$ is.

Consider a single bypass move on S. It is either a trivial bypass or increases $\sharp\Gamma_S$ by 2. In the case that the number of $\sharp\Gamma_S$ increases, the bypass attachment arc δ starts from an arc $l \subset \Gamma_S$ comes back to l. Therefore, it generates a non-trivial element of $\pi_1(S, l)$. Since S is incompressible and a Haken manifold has residually finite π_1, there exists a large enough finite cover $\pi : \widetilde{M} \to M$ which expands S to $\widetilde{S} = \pi^{-1}S$, such that a lift of δ becomes a trivial bypass attachment which means it connects two different components of $\Gamma_{\widetilde{S}}$. Therefore through a finite succession of covers, we can construct a cover $\widetilde{M}$, $\widetilde{S} = \pi^{-1}S$ and a lift $\widetilde{D}$ of D, in which all bypasses to isotope $\widetilde{S}$ across $\widetilde{D}$ are trivial. Since $\widetilde{S}'$ is obtained from $\widetilde{S}$ via trivial bypass attachments and $\widetilde{S}' \cap \widetilde{D} = \varnothing$, $\widetilde{S}'$ and $\widetilde{S}$ are contact isotopic, i.e., $(\widetilde{M} \setminus \widetilde{S}', \pi^*\xi)$ is tight. This contradict the assumption, hence completes the proof of Theorem 3.6.1. $\square$

By the Haken manifold decomposition developed by Honda, Kazez and Matić [31, 32], we can decompose given contact Haken manifold by cutting along proper convex surfaces to a union of 3-balls with one dividing curve on each ball which admits a unique tight contact structure due to the Eliahberg's theorem and glue back using the above theorem to show the existence of tight contact structures.

3.7 Global structures of tight contact 3-manifold

In this section, we introduce important results for tight contact structures.

Theorem 3.7.1 (Colin [5], Honda, Kazez and Matić [31])**.** Every non-trivial tight contact 3-manifold (M, ξ) is contactomorphic to a connected sum

$$(M_1, \xi_1)\# \cdots \#(M_k, \xi_k)$$

of finitely many prime tight contact 3-manifolds. The summands are unique up to order and contactomorphism.

Theorem 3.7.2 (Eliashberg's Uniqueness Theorem [10])**.** If ξ is a contact structure in a neighborhood of ∂B^3 that makes ∂B^3 convex and the dividing set on ∂B^3 consists of a single closed curve, then there is a unique extension of ξ to a tight contact structure on B^3 (up to isotopy that fixes the boundary).

Corollary 3.7.3. S^3, $\mathbb{R}^3$ and $S^2 \times S^1$ admit a unique tight contact structure up to isotopy.

Theorem 3.7.4 (Gabai, Eliashberg, Thurston [11, 18])**.** If M is a closed, orientable and irreducible 3-manifold with $rkH_2(M;\mathbb{Z}) \neq 0$, then M carries a tight contact structure.

Theorem 3.7.5 (Honda, Kazez, Matić [30, 31])**.** Let (M, γ) be a sutured manifold. Then the following are equivalent.

1. (M, γ) is taut.

2. (M, γ) carries a taut foliation.

3. (M, Γ) carries a universally tight contact structure.

4. (M, Γ) carries a tight contact structure.

Theorem 3.7.6 (Honda, Kazez, Matić [31])**.** Every toroidal 3-manifold carries infinitely many nonisotopic, nonisomorphic tight contact structures.

Theorem 3.7.7 (Honda, Colin, Giroux [8])**.** Let M be a closed, orientable, irreducible 3-manifold. Then M carries finitely many isotopy classes of tight contact structures if and only if M is atoroidal.

Theorem 3.7.8 (Lisca, Stipsicz [39])**.** A closed oriented Seifert fibered space carries a tight contact structure if and only if it is not obtained by $(2n-1)$-surgery along the $(2, 2n+1)$ torus knot in S^3 for $n \in \mathbb{Z}_+$.

Theorem 3.7.9 (Stipsicz [51])**.** The Weeks manifold admits tight contact structures.

Theorem 3.7.10 (Etg$\ddot{u}$ [13])**.** There exist infinitely many closed hyperbolic 3-manifolds which contain no essential lamination but admit tight contact structures.

Chapter 4

Tight contact structures on hyperbolic 3-manifolds

In this chapter, we will discuss the recent results for the tight contact structures on hyperbolic 3-manifolds. In 2006, Giroux, Colin and Honda [7, 8, 31] announced that if M is a closed, oriented, irreducible 3-manifold, then M carries finitely many isotopy classes of tight contact structures if and only if M is atoroidal. However, this result gives neither an accurate bound nor even the existence of tight contact structures.

There are rarely known results for the tight contact structures on atoroidal manifolds with infinite fundamental group, one of which is the result by Honda, Kazez and Matić [32] in 2003. They proved that every fibered hyperbolic 3-manifold with the extremal condition admits exactly one tight contact structure. Soon after that, Cofer [4] showed the existence of the unique non-invariant tight contact structure for $\Sigma_2 \times I$ with identical separating curves on each boundary components. These two results use the convex decomposition theory. As an another approach for the existence of tight contact structures on hyperbolic 3-manifolds, Etgü [13] and Stipsicz [51] used Heegaard Floer homology.

We start from introducing the result of Honda, Kazez and Matić in section 4.1 and give the detailed proof of our result in the whole remaining chapter.

4.1 Tight contact structures on hyperbolic surface bundles over the interval

Let Σ be a closed, oriented surface of genus $g > 1$ and $f : \Sigma \to \Sigma$ be a pseudo-Anosov diffeomorphism. Then $M = \Sigma_g \times I/\sim$ where $(x, 0) \sim (f(x), 1)$ admits hyperbolic metric by Theorem 2.2.3(3). Given tight contact structure ξ, Euler class $e(\xi)$ satisfies Thurston-Bennequin inequality [2] by Lemma 3.3. Here, extremal condition means that absolute value of evaluation of $e(\xi)$ on each fiber is maximal. Honda, Kazez and Matić first showed that the following theorem, and using this theorem, they showed Theorem 4.1.2. The main tools of the proof are the convex decomposition theory using bypasses and curve complex. $\sharp\pi_0(Tight(M, \mathcal{F}))$ denotes the number of connected components of tight contact 2-plane fields adapted to $\mathcal{F}$

Theorem 4.1.1 (Honda, Kazez, Matic [32]). Let Σ be a closed oriented surface of genus > 1 and $M = \Sigma \times I$. Fix dividing sets $\Gamma_{\Sigma_i} = 2\gamma_i$ (2 parallel disjoint copies of γ_i), $i = 0, 1$, so that γ_0, γ_1 are nonseparating curves. Then choose a characteristic foliation $\mathcal{F}$ on ∂M which is adapted to $\Gamma_{\Sigma_0} \sqcup \Gamma_{\Sigma_1}$. We have the following :

1. All the tight contact structures which satisfy the boundary condition $\mathcal{F}$ are universally tight.

2. If $\gamma_0 \neq \gamma_1$, then $\#\pi_0(Tight(M, \mathcal{F})) = 4$.

3. If $\gamma_0 = \gamma_1$, then $\#\pi_0(Tight(M, \mathcal{F})) = 5$.

If we choose a characteristic foliation $\mathcal{F}$ on the boundaries adapted to fixed dividing curves, we can get a unique tight contact structure in the neighborhood of boundary with chosen characteristic foliation up to contact isotopy by Giroux's flexibility theorem [20], i.e., we can make boundaries convex and fix a tight contact structure on boundary. Hence this theorem tells us how many tight contact structures can be extended inside given manifold with fixed convex boundaries up to isotopy. By using the above theorem, they showed the existence of unique tight contact structure on an arbitrary surface bundle over the circle with pseudo-Anosov monodromy, which is the following.

Theorem 4.1.2. Let M be a closed, oriented, hyperbolic 3-manifold which fibers over S^1, where the fiber is a closed oriented surface Σ of genus $g > 1$ and the monodromy map is pseudo-Anosov. Then there exists a unique tight contact structure up to isotopy in each of the two extremal cases. This contact structure is universally tight and weakly symplectically fillable.

However, above two results have an assumption, extremal condition, which makes easier to control the dividing curves on each fiber than the one of general case. After this result, Cofer [4] showed that $\Sigma_2 \times I$ with an identical separating dividing curve on each boundary has two tight contact structures, one of which is a product type(i.e., invariant type) of tight contact structure. This is the case $\langle e(\xi), \Sigma_2 \rangle = 0$ and Σ_2 is genus 2 surface. Lately, Etgü [13] showed that there exist infinitely many closed hyperbolic 3-manifolds which contain no essential lamination but admit tight contact structures. He showed this using different approach which is handle decomposition and Heegaard Floer homology.

Our main goal is to eliminate the extremal condition in Theorems 4.1.1 and 4.1.2 and consider a closed surface fiber with arbitrary genus higher than one. Hence for a first attempt to do that, we choose a dividing curve on $\Sigma \times \{0\}$ as an arbitrary separating curve and a dividing curve on $\Sigma \times \{1\}$ with a specific shape determined by a dividing curve on $\Sigma \times \{0\}$ in Theorem 1.0.1. Then we can express all possible values of $\langle e(\xi), \Sigma \times \{t\} \rangle$. Recall fist the assumptions in Theorem 1.0.1. See Figure 4.1.

Statements of the results

Let Σ be a closed oriented surface of genus > 1 and $M = \Sigma \times I, I = [0, 1]$. Fix dividing sets $\Gamma_{\Sigma_i} = \gamma_i$, $i = 0, 1$, so that γ_0 is an arbitrary separating curve,

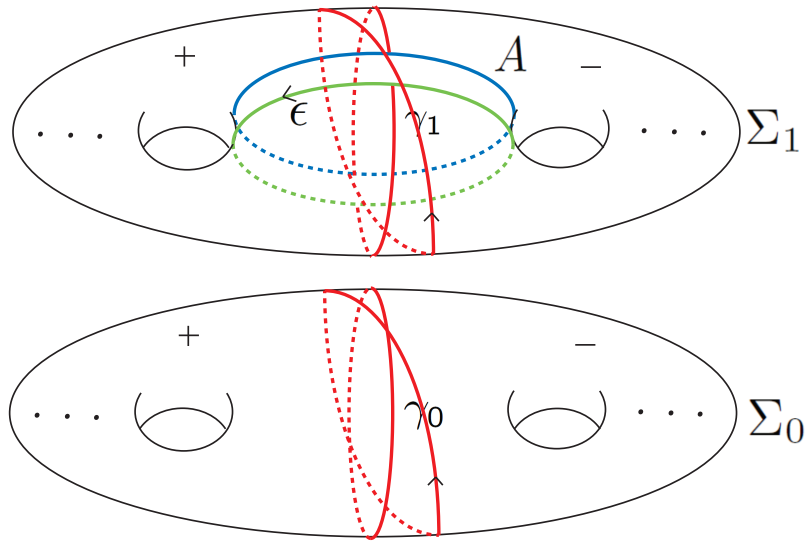


Figure 4.1: Base case

$\gamma_1 = \tau_\epsilon^n \circ \gamma_0$, $n \in \mathbb{Z}$, and $\chi((\Sigma_0)_+) - \chi((\Sigma_0)_-) = \chi((\Sigma_1)_+) - \chi((\Sigma_1)_-)$. ϵ is a nonseparating closed curve intersecting γ_0 at two points and $(\Sigma_i)_+$, $(\Sigma_i)_-$ are the positive and negative regions of $\Sigma \setminus \Gamma_{\Sigma_i}$. Choose a characteristic foliation $\mathcal{F}$ on ∂M which is adapted to $\Gamma_{\Sigma_0} \sqcup \Gamma_{\Sigma_1}$. By Giroux's flexibility theorem [20] in chapter 3, we can find a unique tight contact structure on the neighborhood of boundary of $M = \Sigma \times I$ adapted to given dividing curves of ∂M. Then the following five inequalities hold. A series of proofs of propositions through section 4.2 to 4.6 gives an answer to the question whether or not the given tight contact structure on the neighborhood of boundaries can extend to the inside of the manifold and how many there are.

Proposition 4.1.3. If $n > 1$, then

$$\sharp\pi_0(Tight(M, \mathcal{F})) \leq \begin{cases} 2 \times 3^n & \text{if } g \geq 3, \\ 3^n + 5 & \text{if } g = 2,\ n = 2, \\ 3^n + 16 + 1 & \text{if } g = 2,\ n = 3, \\ 3^n + 3^{n-3} + \left[\frac{n}{2}\right] & \text{if } g = 2,\ n \geq 4. \end{cases} \tag{4.1.1}$$

Proposition 4.1.4. If $n = 1$, then $\sharp\pi_0(Tight(M, \mathcal{F})) \leq 4$.

Proposition 4.1.5. If $n = 0$, then $\sharp\pi_0(Tight(M, \mathcal{F})) = 2$.

Proposition 4.1.6. If $n = -1$, then $\sharp\pi_0(Tight(M, \mathcal{F})) = 4$.

Proposition 4.1.7. If $n < -1$, then

$$\sharp\pi_0(Tight(M, \mathcal{F})) \leq \begin{cases} 4 + 32 + 16 = 52 & \text{if } g(\Sigma_+) \neq 0 \text{ and } g(\Sigma_-) \neq 0, \\ 4 + 32 + 8 = 44 & \text{otherwise}, \\ 4 + 48 + 6 = 58 & \text{if } g = 2. \end{cases} \tag{4.1.2}$$

To prove these propositions, we can assume that the Dehn twists occur inside a small annular neighborhood of curve ϵ. Rename the green colored boundary ϵ. See Figure 4.1. By Theorem 3.4.1, one of which is positive and the other is negative. For convenience, we assume that the sign of the left hand side subsurface in $\Sigma \times \{i\}$ in Figure 4.1 is positive, where $i = 0, 1$. Denote $\epsilon \times I$ by A, the left hand side subsurface in $\Sigma \times \{i\}$ by Σ_i^+ and the right hand side subsurface in $\Sigma \times \{i\}$ by Σ_i^-, where $i = 0, 1$. We can make $\epsilon \times \{0\}$ and $\epsilon \times \{1\}$ Legendrian by Legendrian realization principle (Theorem 3.5.1) and A convex surface rel to ∂A by Theorem 3.5.2. Then cut $\Sigma_g \times I$ along a convex annulus $A = \epsilon \times I$. Denote $M^1 = M \setminus A$. Then we can get a thickened surface which has genus $(g-1)$ and two punctures as in Figure 4.3 (a).

Give an orientation to ϵ as shown in Figure 4.1. Let A^+ denote the copy of A in $M \setminus A$ where the outward orientation on A agrees with the orientation induced from the orientation of ϵ. Then all possible configurations of dividing curves of annulus A^+ are in Figure 4.2. The $I_{k \in Z}$ case has two parallel dividing curve components starting from one boundary and ending another boundary. Here k is the number of Dehn twists along the core curve. In case $II^{\pm}_{k \in \mathbf{Z}_{\geq 0}}$ there are two boundary parallel components and k core curves. $+$(resp. $-$) means that the coorientation of contact structure of disk region surrounded by outer boundary parallel component coincides with the orientation of A^+_+(it means positive region of a convex annulus A^+). We can reduce all these possibilities to a few cases through the following lemmas. The setup is similar to that of Theorem 4.1.1 and the result by Cofer. For further understanding, see Honda, Kazez, Matić[32] and Cofer [4]. We first prove the case $n = 0$. The strategy is the following.

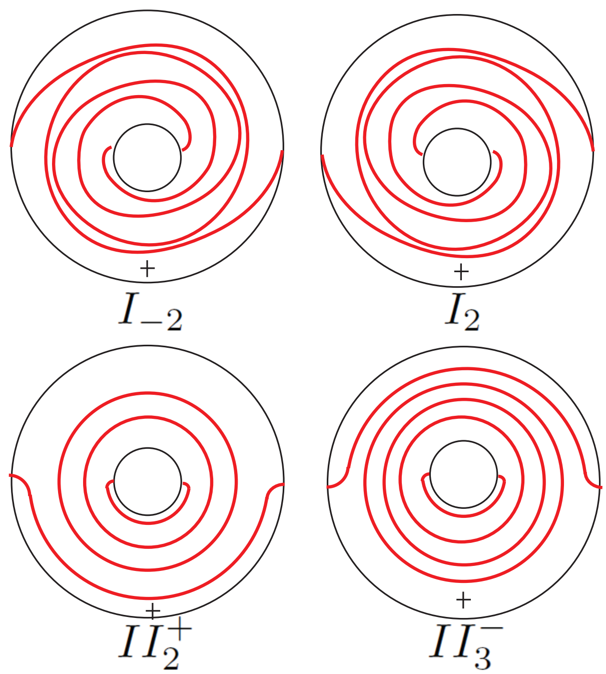


Figure 4.2: All possibilities of dividing curves on A_+

Strategy of the proof of Proposition 4.1.5

Step 1 If $k > 0$, then I_k can be reduced to $II_1^{\pm}$ case. (cf. Lemma 4.2.1).

Step 2 If $k < -1$, then I_k can be reduced to $I_{k-1 \in \mathbb{Z}_{<0}}$ case. (cf. Lemma 4.2.2).

Step 3 If $k > 0$, then $II_{2k}^{\pm}$ can be reduced to $II_{2k-2}^{\pm}$ case and $II_{2k+1}^{\pm}$ can be reduced to $II_{2k-1}^{\pm}$. (cf. Lemma 4.2.3).

Step 4 The II_0^+ (resp. -) case cannot admit a tight contact structure. (cf. Lemma 4.2.4).

Step 5 The manifold M with the convex annulus A of dividing curves I_0 admits at most two tight contact structures. (cf. Lemma 4.2.6).

Step 6 There is one non-invariant tight contact structure in each of $II_1^{\pm}$ and I_{-1} and there are state transitions among these cases. (cf. Lemma 4.2.7).

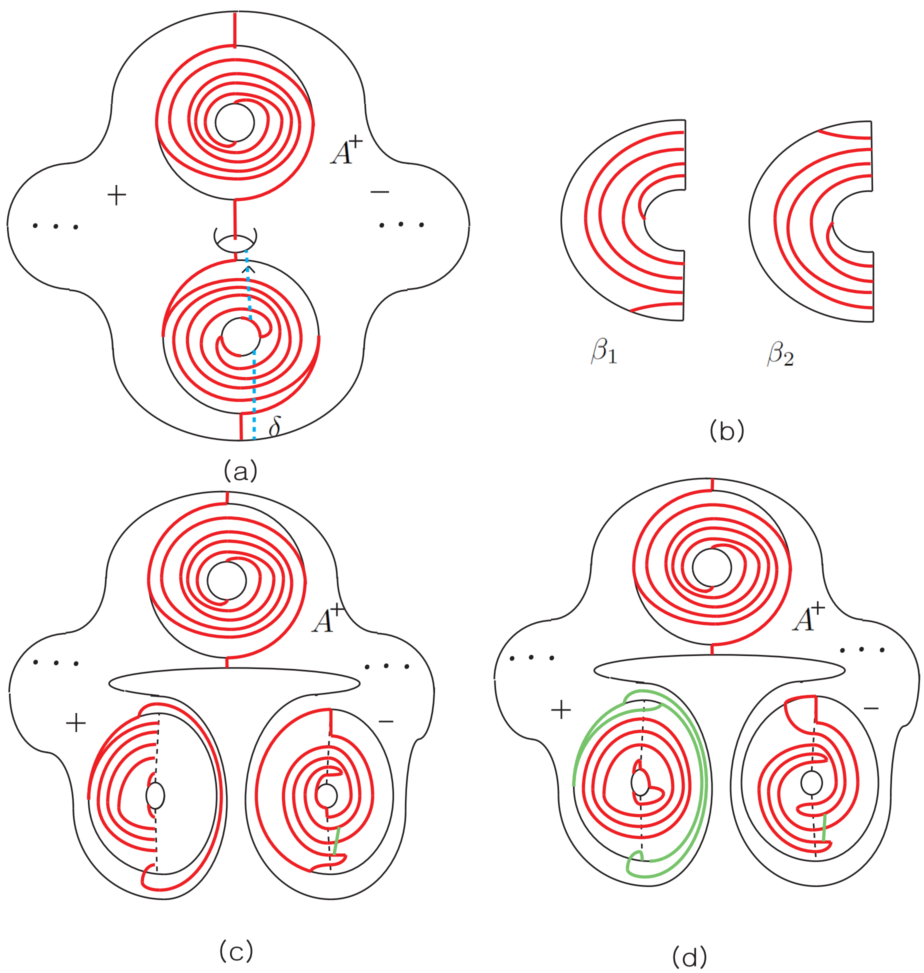


Figure 4.3: $I_{k\in\mathbb{Z}_{>0}}$ case

4.2 Proof of Proposition 4.1.5

Lemma 4.2.1. If $k > 0$, then I_k can be reduced to $II_1^{\pm}$ case.

Proof. The proof does not depend on the genus. Take a small neighborhood in ∂M^1 of an subarc of γ_1 which intersects with A^-. Denote one of boundaries of this disk neighborhood in $(\partial M^1)_-$ by δ. See Figure 4.3 (a). By Legendrian realization principle (Lemma 3.5.1), we can make this arc Legendrian rel to boundary points. Furthermore, we can make $\delta \times I$ convex surface rel to boundaries by Theorem 3.4.6. Cut $M^1 = M \setminus A$ along this convex surface $\delta \times I$. Let M^2 be the resulting cut-open manifold. There are two cases of dividing curve configurations of $(\delta \times I)^+$ which does not give a digging bypass on A^- directly. These cases are indicated in Figure 4.3 (b). If $(\delta \times I)^+$ has a boundary parallel dividing curve on A^- side, it can give a state transition to II_1^+ or II_1^- depending on the location of bypass attachment. M^2 has two new annular boundaries. For the β_1 case, we can get one component of the dividing curve on the right hand side annular boundary which is homotopic to a core curve. Then we can find a long digging bypass which is indicated as the green line in Figure 4.3 (c). This bypass gives a state transition to II_1^+. For the β_2 case, we can get a OT disk after edge-rounding. Following the proof, we can see that this holds for an arbitrary positive integer k and an arbitrary integer number of Dehn twists, n.

□

Lemma 4.2.2. If $k < -1$, then I_k can be reduced to $I_{k-1\in\mathbb{Z}_{<0}}$ case.

Proof. Cut M' by convex surface $\delta_1 \times I$ where δ_1 is a Legendrian curve which starts from A^- ends at A^- and is homotopic to a component of γ_0. If $\delta_1 \times I$ has a boundary parallel dividing curve component, this gives digging bypass on A^- side reducing to I_{k+1} case. The remaining possibilities are two cases in Figure 4.4 (b). After edge rounding, the β_1 case has one dividing curve homotopic to a core curve on each new annular boundary. Then we can find a long bypass whose attachment arc is the dotted blue line in Figure 4.4 (c). Then this bypass gives a digging bypass on A^- and reduces the number of

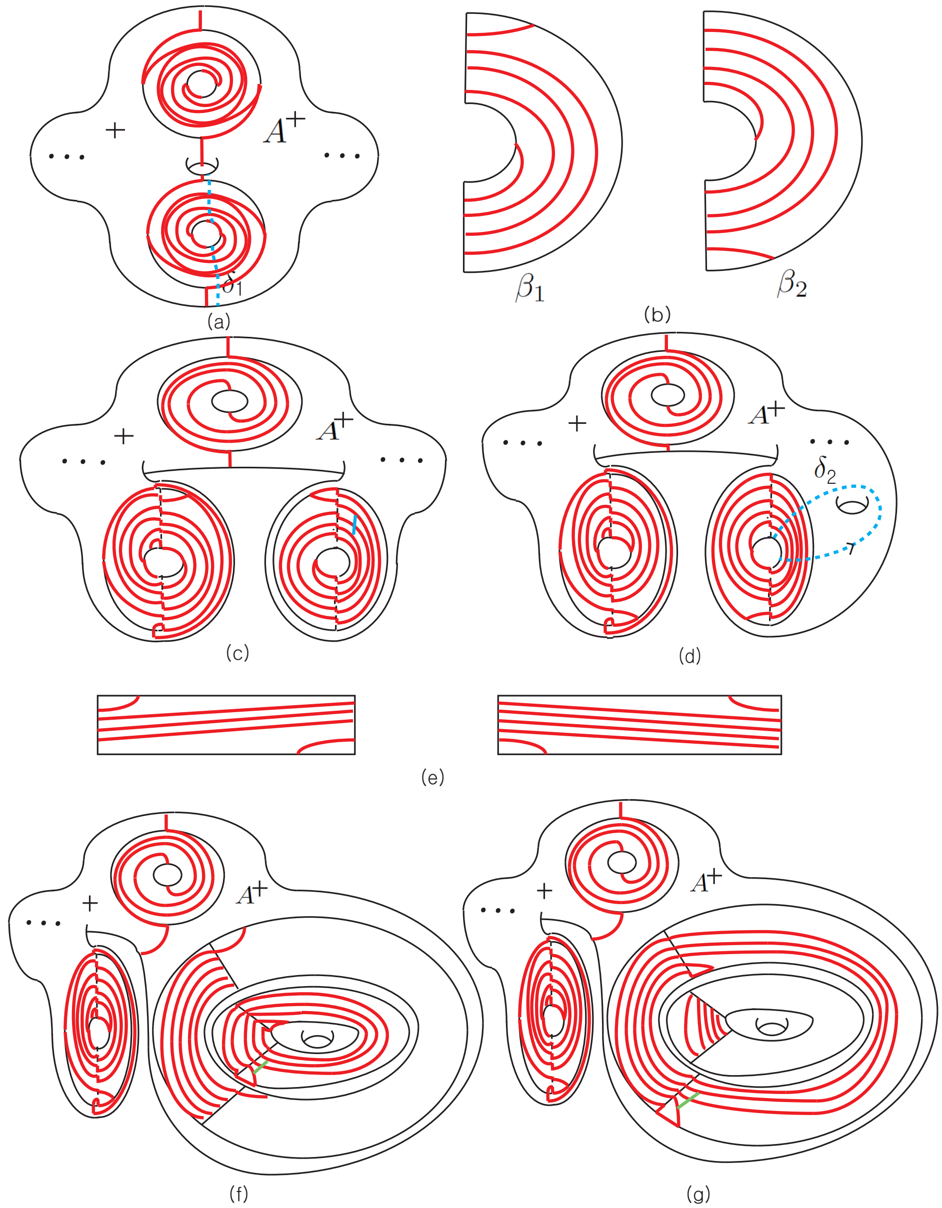


Figure 4.4: $I_{k\in\mathbb{Z}_{<0}}$, $g > 2$ case

Dehn twists.

For the β_2 case, the proof can be divided into two cases depending on the genus. First suppose that genus $g = g(\Sigma)$ is greater than 2. Let a be the genus of Σ_i^+ and b be the genus of Σ_i^-. Then a or b must be nonzero. Without loss of generality, we can assume that b is nonzero. Take a Legendrian arc δ_2 in Σ_1^- which goes around genus and join two points in $\Sigma_1^- \cap A^-$. Then by cutting M^1 along a convex surface $\delta_2 \times I$, we can reduce the number of genus. There are two possibilities η_1, η_2 for $\Gamma_{(\delta_2 \times I)^+}$ which does not have a boundary parallel dividing curve component on A^- side. They are indicated in Figure 4.4 (e). In either case, we can find a long digging bypass which can be slided into A^-. See Figures 4.4 (f) and (g). Hence we can reduce the number of Dehn twists k.

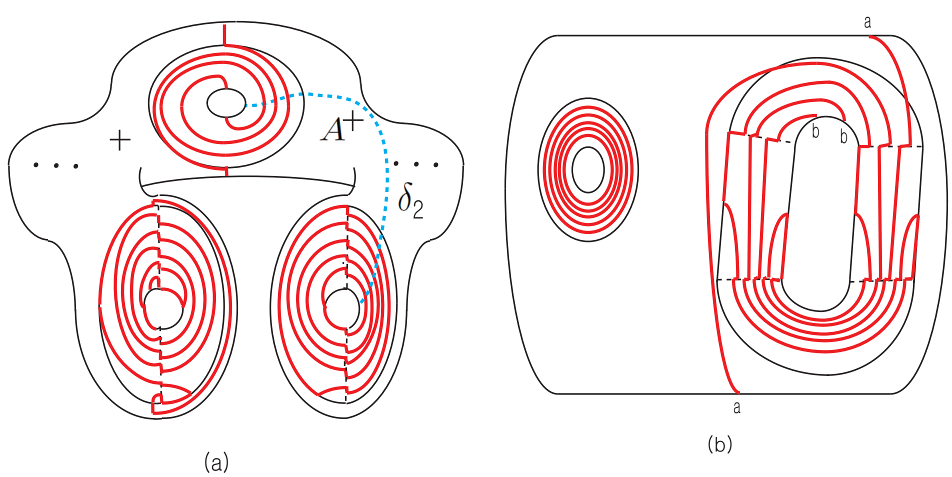


Figure 4.5: $I_{k \in \mathbb{Z}_{<0}}$, $g = 2$ case

Consider the $g = 2$ case. Cut $M^1 \setminus (\delta_1 \times I)$ along this convex surface.

Then $\sharp(\delta_2 \times I) \cap A^+ = 2|k| - 1$ and $\sharp(\delta_2 \times I) \cap A^- = 2|k| + 1$ hold. Hence for all possibilities of the dividing curves on $\delta_2 \times I$ except one case have at least one boundary parallel component of dividing curves on A^- side which gives a digging bypass on A^- side by bypass sliding lemma(cf. Lemma 3.5.7). Remaining possibility for dividing curves on $\delta_2 \times I$ is represented in Figure 4.5 (a). We can get a solid torus with two annular boundaries after cutting along the convex surface $\delta_2 \times I$ and edge-rounding. Since the number of dividing curves on left hand side annulus is $2|k|+1$ (the figure represents the $k = -2$ case) and the one of right hand side is 1, we can find a digging bypass whose attachment arc is inside left hand side annulus. By tracing back the figures, we know that this digging bypass has to be attached on inside A^-. Hence, it reduces the number of Dehn twists. □

Lemma 4.2.3. If $k > 0$, then $II^{\pm}_{2k}$ can be reduced to $II^{\pm}_{2k-2}$ case and $II^{\pm}_{2k+1}$ can be reduced to $II^{\pm}_{2k-1}$.

Proof. First consider the II^{+}_{2k} case. Take a similar arc as in the proof of Lemma 4.2.1. At this time, we take a boundary of disk neighborhood in $(\partial M^1)_+$ as δ. See Figure 4.6 (a). We can make this arc δ Legendrian by LeRP (Theorem 3.5.1) and $\delta \times I$ convex by Theorem 3.4.6. Cut manifold M^1 along a convex surface $\delta \times I$. Then there is one exceptional case of $\Gamma_{\delta\times I}$ which does not give a bypass reducing number of core curves. It is indicated in Figure 4.6 (b). Then after rounding edges, we can get a manifold two new annular boundaries. See Figure 4.6 (c). We can find a long digging bypass which is the green line in left hand side annulus. The left half part originally comes from A^- side, we can always find a digging bypass on A^- which reduces the number of core curves. Other remaining cases are all similar. Figure 4.6 (d) represents the II^{-}_{2k} case and Figure 4.7 represents the $II^{\pm}_{2k+1}$ case. □

Lemma 4.2.4. The II^{+}_{0} (resp. -) case cannot admit a tight contact structure.

Proof. Consider the II^{+}_{0} case. See Figure 4.8 (a). If $g > 2$, we can reduce all genus in positive region and negative region by using similar argument in

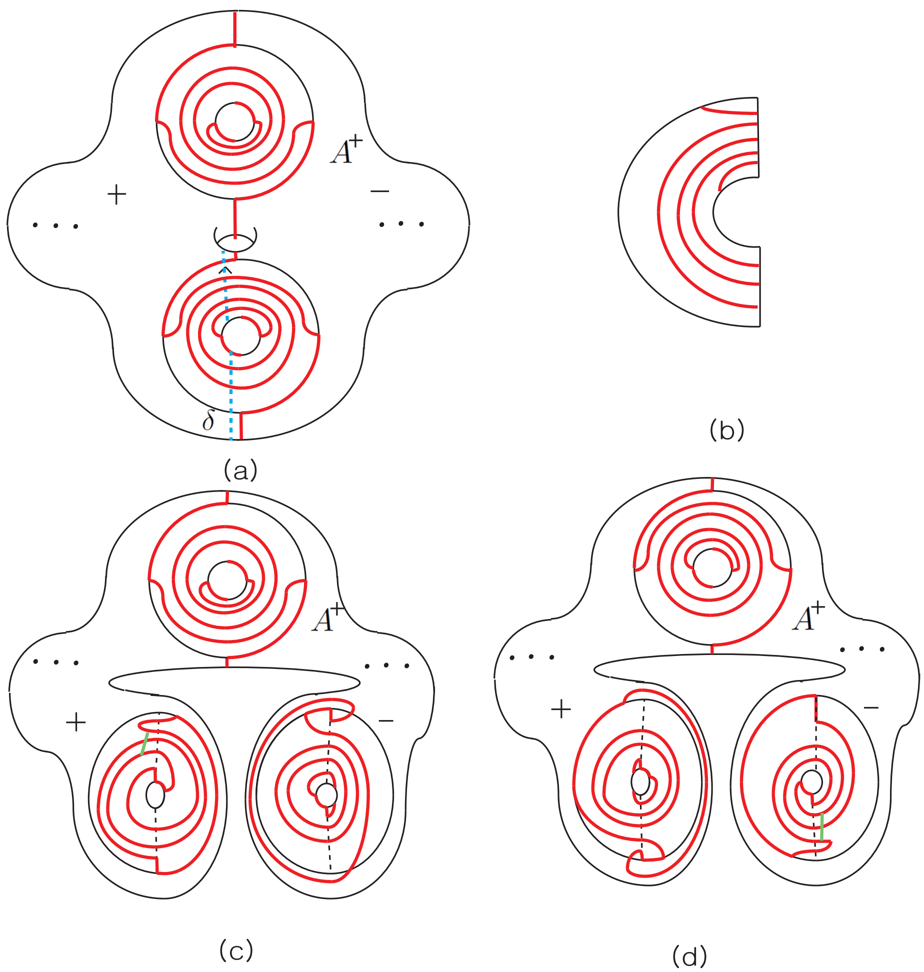


Figure 4.6: $II^{\pm}_{2k}, k \in \mathbb{N}$ case

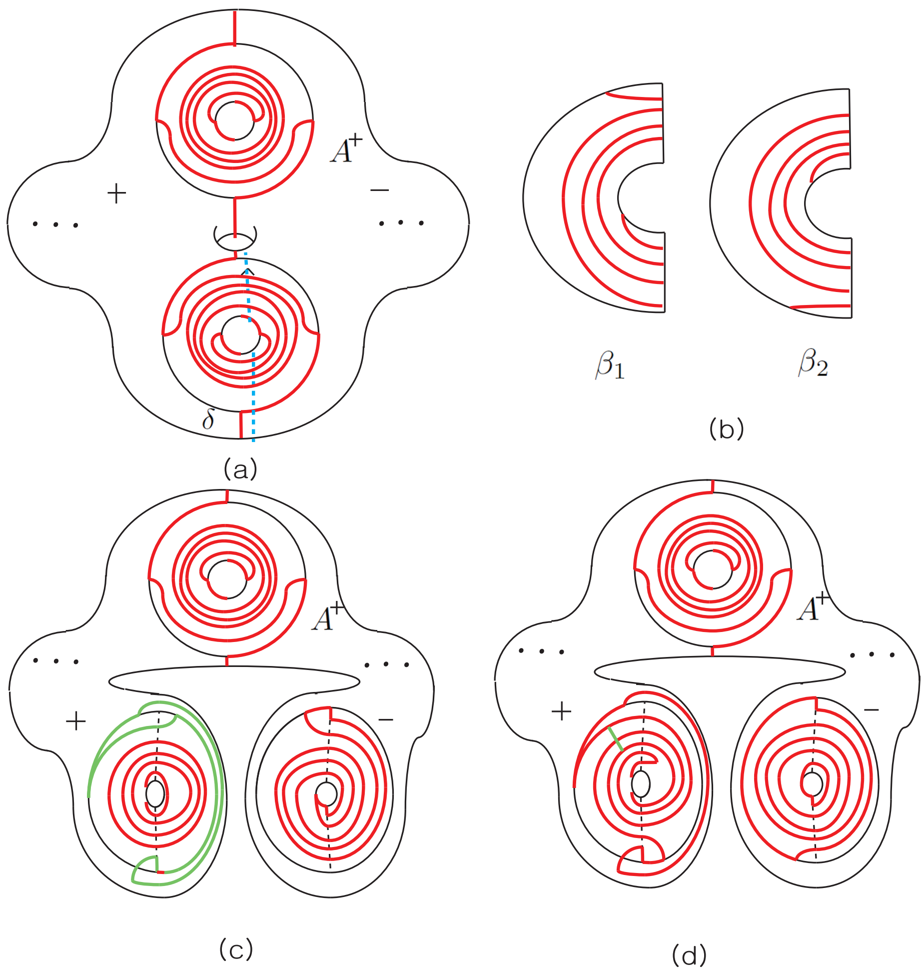


Figure 4.7: $II^{+}_{2k+1}, k \in \mathbb{N}$ case

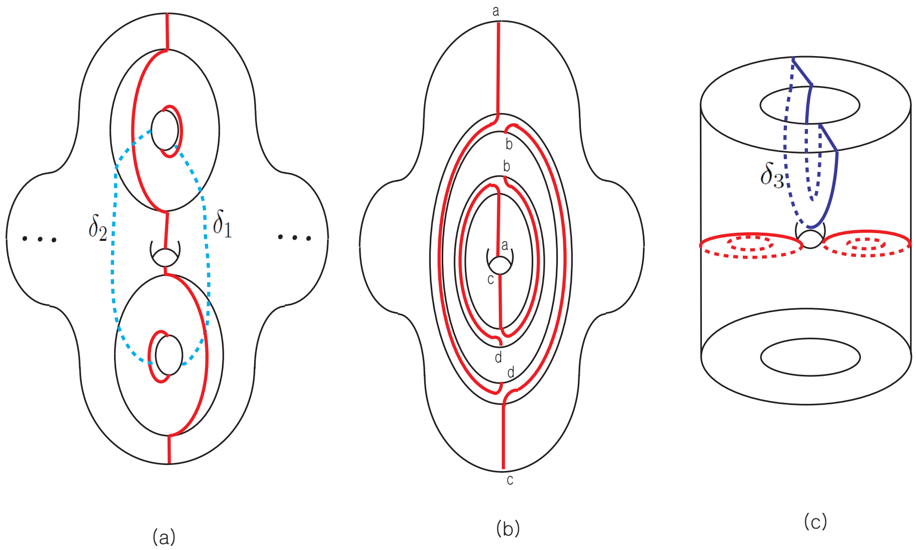


Figure 4.8: II_0 case

the proof of Lemma 4.2.7. Hence, it suffices to show for the case $g = 2$. We can make $\delta_1 \times I$, $\delta_2 \times I$ convex surfaces and cut along these convex surfaces. Then we can get a thickened two punctured torus in Figure 4.8 (b). This manifold has a overtwisted disk since we can make a curve δ_3 Legendrian which has a $tb = 0$. Hence II_0^+ cannot admit a tight contact structure. $\square$

Remark 4.2.5. The convex decompositions in the proofs of Lemmas 4.2.1 and 4.2.3 occur only in the neighborhood of A^- side, i.e., it does not touch A^+ side, we can extend these proofs for all n. Furthermore, for arbitrary n with $II_0^\pm$ case, we can homotope the dividing curves on $\Gamma_{M^1(n)}$ into the M^1 case with $n = 0$. Hence $II_0^\pm$ case is overtwisted for all n. We omit this part from now on.

Lemma 4.2.6. The manifold M with the convex annulus A of dividing curves I_0 admits at most two tight contact structures.

Proof. See Figure 4.9 (a). Cut M' along a convex surface $\delta_1 \times I$, then we can get a manifold in Figure 4.9 (b). Then we can find a disjoint disk system $\{D_i\}_{i=1}^{i=n}$ which is indicated in Figure 4.9 (b), where $tb(\partial D_i) = -1$ for all

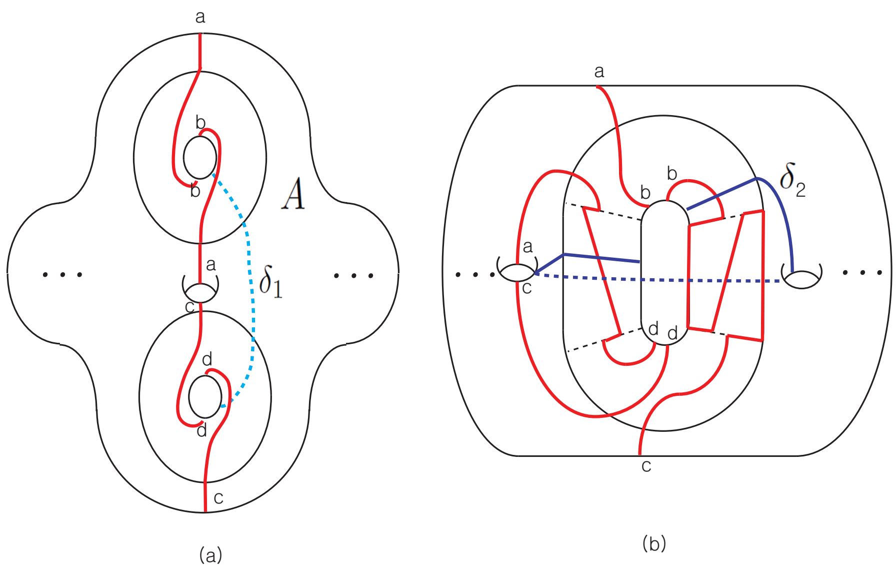


Figure 4.9: I_0 case

i, $M' \setminus \cup_{i=1}^{i=n} D_i$ is 3-ball with one dividing curve. Then this 3-ball admits a unique tight contact structure by Eliashberg's theorem(Theorem 3.7.2). Since all D_i have only boundary parallel dividing curve components, we can conclude that M' admits a unique (universally) tight contact structure by Colin's theorem (Theorem 3.6.1). Hence M has at most unique tight contact structure. Here, the dividing curves of I_0 type are not boundary parallel, we cannot use Colin's theorem for M. However, we know that the invariant tight contact structure always exists, hence, I_0 should be this tight contact structure. Hence I_0 case admits a unique tight contact structure. □

Lemma 4.2.7. There is one non-invariant tight contact structure in each of $II_1^{\pm}$ and I_{-1} and there are state transitions among these cases.

Proof. Since the $g = 2$ case is already treated by Cofer [4], we assume that $g > 2$.

(1) Consider the II_1^+ case. Let the number of genus of negative region be nonzero. We can pick two points p, q on $\partial A^+ \cap \Sigma_1^-$ and an arc δ_1 joining two points and going around genus of negative region. See Figure 4.10 (a).

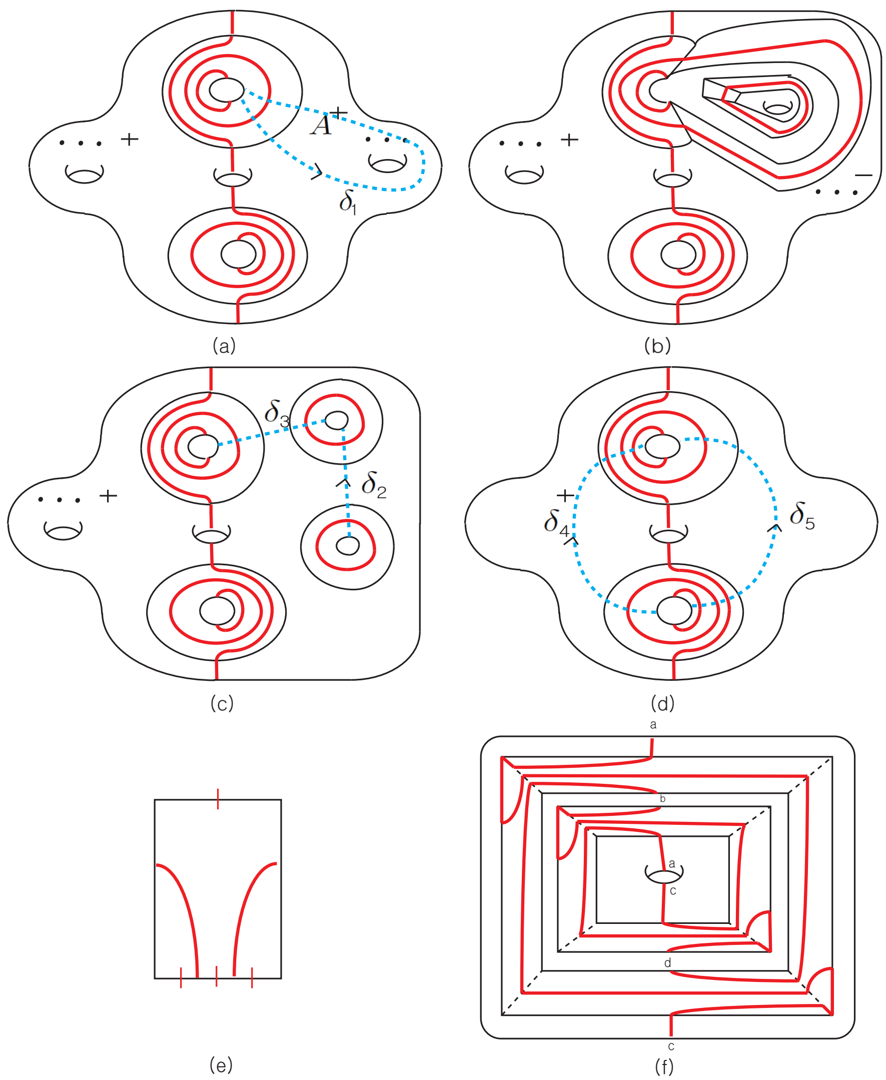


Figure 4.10: Reducing the number of genus and the unique tight contact structure

We can make δ_1 Legendrian and $\delta_1 \times I$ convex rel to boundary. Then, since $tb(\partial(\delta_1 \times I)) = -1$, there is only one possibility for $\Gamma_{\delta_1 \times I}$. Hence after cutting M^1 along $\delta_1 \times I$, we can get a thickened $g-2$ surface with 3 annular boundaries and dividing set like as the one in Figure 4.10 (b). Let b be the number of genus of negative region. If we cut all genus using the previous method, we can get new b components of annuli with one dividing core curve. Then we can find an arc connecting each annuli as shown in Figure 4.10 (c) by which we can reduce all annuli. We also eliminate the genus of positive region using the dividing core curve on A^-. By this procedure, we can get a thickened two-punctured torus, call this M^2, with II_1^+ type dividing curve. This is just the Cofer's case.
Cofer [4] showed that M^2 which does not have a state transition to I_0 admits a unique tight contact structure. This is due to Eliashberg and Colin (Theorems 3.7.2 and 3.6.1). Furthermore, the disks $\delta_i \times I$ used to eliminate genus satisfy all $tb(\partial(\delta_i \times I)) = -1$. Hence M^1 admits unique (universally) tight contact structure by Colin's theorem. Hence M with II_1^+ admits at most unique tight contact structure which is not isotopic to I_0 type. Unfortunately, we cannot apply Colin's theorem for the last gluing since the dividing set of II_1^+ is not boundary parallel. We leave the discussion for the last step.

(2) We can argue similarly for II_1^+ case.

(3) Consider the I_{-1} case. Take a small disk neighborhood in ∂M^1 of subarc of γ_1 which intersects with A^+ and denote one of boundaries of this disk neighborhood in $(\partial M^1)_-$ by δ_1. See Figure 4.11 (a). First cut M^1 along $\delta_1 \times I$ inefficiently. Let M^2 be $M^1 \setminus (\delta_1 \times I)$. Then there are two possibilities for $\Gamma_{(\delta_1 \times I)^+}$, β_1, β_2. For the β_2 case, we can get a OT disk after edge-rounding. For the β_1 case, we can get two new annular boundaries with one dividing core curve on each boundary. Then we can find long digging bypass on right hand side annulus whose attachment arc is indicated in Figure 4.11 (c). Since this bypass attachment is the same as the one on A^+ as shown in Figure 4.11 (a), this gives a state transition to the II_1^-. Since there is an annulus with one dividing curve component on each region, we can eliminate

genus using similar manner to the above. By cutting along $\delta_2 \times I$ and $\delta_3 \times I$, we can show that I_{-1}, which does not have a state transition to I_0, admits at most unique tight contact structure. See Figure 4.11 (e)

(4) State transition can be argued similarly to the one of Cofer's. See section 5 in [4]. □

Proof of Proposition 4.1.5. We need the last gluing step to complete the proof. For the $g = 2$ case, Cofer [4] cut the M along $\gamma_0 \times I$ as shown in Figure 4.1 and then she showed that there is at most one non-invariant tight contact structure on each submanifold. The remaining dividing curve configuration is II_1^+ on $\Gamma_{\gamma_0 \times I}$. She showed that there is no nontrivial bypass on $\Gamma_{\gamma_0 \times I}$. We can apply this to the arbitrary genus greater than 1 case. Hence $\sharp\pi_0(Tight(M, \mathcal{F})) = 2$. □

4.3 Proof of Proposition 4.1.4

Strategy of the proof of Proposition 4.1.4

Step 1 If $k > 0$, then the I_k case can be reduced to the I_{k-1} case. (cf. Lemma 4.3.1).

Step 2 If $k < -1$, then the I_k case can be reduced to the I_{k+1} case. (cf. Llemma 4.3.2).

Step 3 If $k > 0$, then $II_{2k}^{\pm}$ can be reduced to $II_{2k-2}^{\pm}$ case and $II_{2k+1}^{\pm}$ can be reduced to $II_{2k-1}^{\pm}$. (cf. Lemma 4.3.3)

Step 4 The II_0^+ (resp. -) case cannot admit a tight contact structure. (This is same as the one of the case $n = 0$. We omit the proof)

Step 5 The manifold M with the convex annulus A of dividing curves I_0 type admits at most 3 tight contact structure (cf. Lemma 4.3.4).

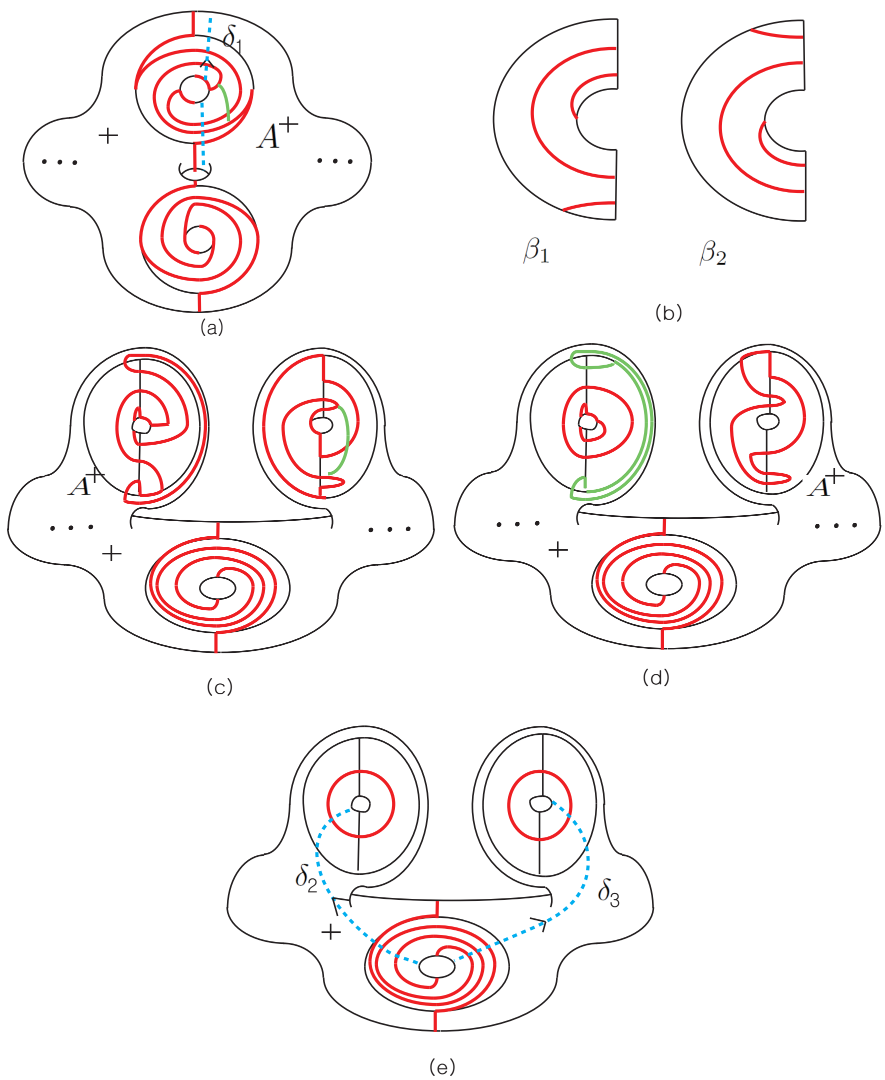


Figure 4.11: State transition from I_{-1} to I_0

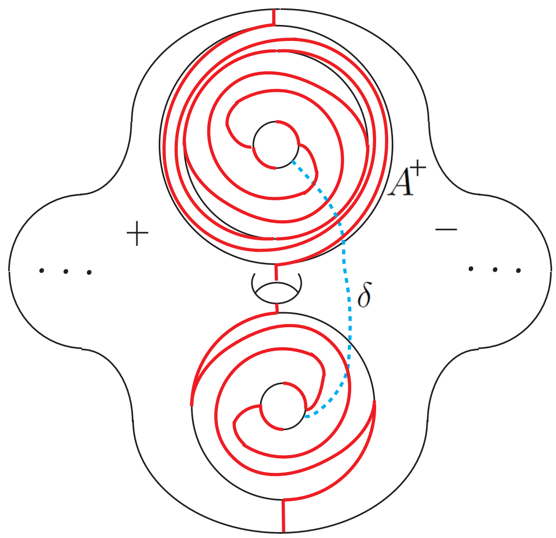


Figure 4.12: $I_{k\in\mathbb{Z}_{>0}}$ case

Step 6 The manifold M with the convex annulus A of dividing curves $II_1^{\pm}$ can be reduced to the I_0 case. The manifold M with the convex annulus A of dividing curves I_{-1} case, which does not transforms to I_0 case, admits at most one tight contact structures. (cf. Lemma 4.3.5).

Lemma 4.3.1. If $k > 0$, then the I_k case can be reduced to the I_{k-1} case.

Proof. We can make δ in Figure 4.12 Legendrian by Legendrian realization principle (Lemma 3.5.1) and $\delta \times I$ convex by Theorem 3.4.6. Cut M' along $\delta \times I$. Then $\sharp(\delta \times I) \cap \Gamma_{A^-} = 2k - 1$ and $\sharp(\delta \times I) \cap \Gamma_{A^+} = 2k + 3$ by Edge rounding (Lemma 3.5.3). By Imbalance principle (Lemma 3.5.8) and bypass sliding lemma (Lemma 3.5.7), $\delta \times I$ must have one boundary parallel dividing curve on A^+ side and this boundary parallel dividing curve gives a digging bypass on A^+ side. Hence we can reduce the number of Dehn twists. By looking at the proof. we see that it does not depend on the genus. $\square$

Lemma 4.3.2. If $k < -1$, then the I_k case can be reduced to the I_{k+1} case.

Proof. It suffices to show for the case $a = 0$, $b = 0$ and $k = -2$. Cut M' by convex surface $\delta_1 \times I$ where δ_1 is a Legendrian curve homotopic to a part of

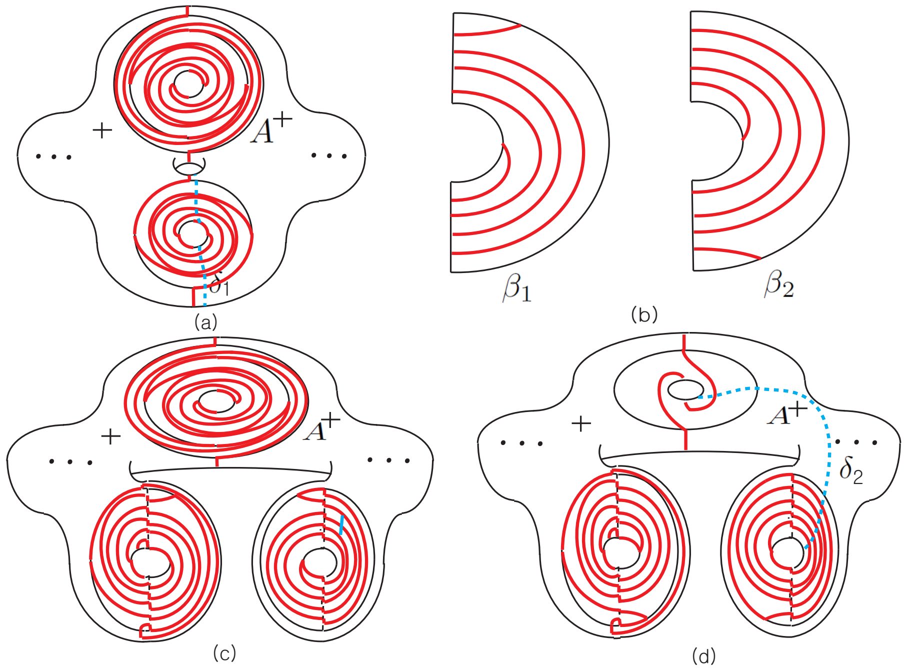


Figure 4.13: $I_{k\in\mathbb{Z}_{<0}}$ case

γ_0 which is indicated in Figure 4.13 (a). $\sharp\delta_1 \times I \cap \Gamma_{A^-} = 2(2k+1)$. If $\delta_1 \times I$ has a boundary parallel dividing curve component inside A^-, this gives a digging bypass on A^- side reducing to I_{k+1} case. Remaining possibilities are two cases as in Figure 4.13 (b). After edge rounding, β_1 case has one dividing curve homotopic to a core curve on each new annular boundary. Then we can find a long bypass whose attachment arc is blue line in the figure. Then this bypass gives a digging bypass on A^- and reduces the number of Dehn twists. For β_2 case, we can get $2|k|+1$ parallel dividing curves homotopic to a core curve on each boundary after edge rounding. After homotoping dividing curves of A^+, we can get a Figure 4.13 (d). Take convex surface $\delta_2 \times I$ in Figure 4.13 (d). Then, since $\sharp A^+ \cap (\delta_2 \times I) = |2-2k+1| = 2|k|-3$ and $\sharp(\delta_2 \times I) \cap \Gamma_{\text{bottom annulus}} = 2|k|+1$, $\delta_2 \times I$ should have a boundary parallel dividing curve on A^- side by Imbalance principle. This digging bypass reduces the number of Dehn twists. □

Lemma 4.3.3. If $k > 0$, then $II^{\pm}_{2k}$ can be reduced to $II^{\pm}_{2k-2}$ case and $II^{\pm}_{2k+1}$ can be reduced to $II^{\pm}_{2k-1}$.

Proof. The proof of Lemma 4.2.3 can be applied for this lemma. We introduce the alternate proof here. We show for the case II^{+}_{2k}. The proof of other case is similar.
(1) We assume that the genus $g = 2$. Consider inefficient cut for M' along $\delta_1 \times I$ as shown in Figure 4.14 (a). There are 4 possible dividing curve configurations of $(\delta_1 \times I)^+$ which does not give a bypass on A^+ or A^- directly. Since a boundary parallel dividing curve straddling $2k+1$, 1 and 2, which is the red curve in Figure 4.14 (b), gives a overtwisted disk, it has to be excluded. For β_1 case, we first homotope the dividing curves and cut along $\delta_2 \times I$. The dividing curve configuration of $\delta_2 \times I$ which does not give a digging bypass on A^+ or A^- is only one case. However, we can see this gives overtwisted disk by edge rounding. See from a Figure 4.14 (c) to (e).

Figure 4.14 (f) expresses β_2 case. After homotope the dividing curves a little bit, we can get Figure 4.15 (a). Then since $\sharp(\delta_2 \times I)^- \cap \Gamma_{M'\setminus(\delta_1 \times I)} - \sharp(\delta_2 \times I)^+ \cap \Gamma_{M'\setminus(\delta_1 \times I)} = (2k+1)-(2k-3) = 4$, we can always find a digging bypass on A^- side. Other case is similar. Figures from 4.15 (b) to 4. 15

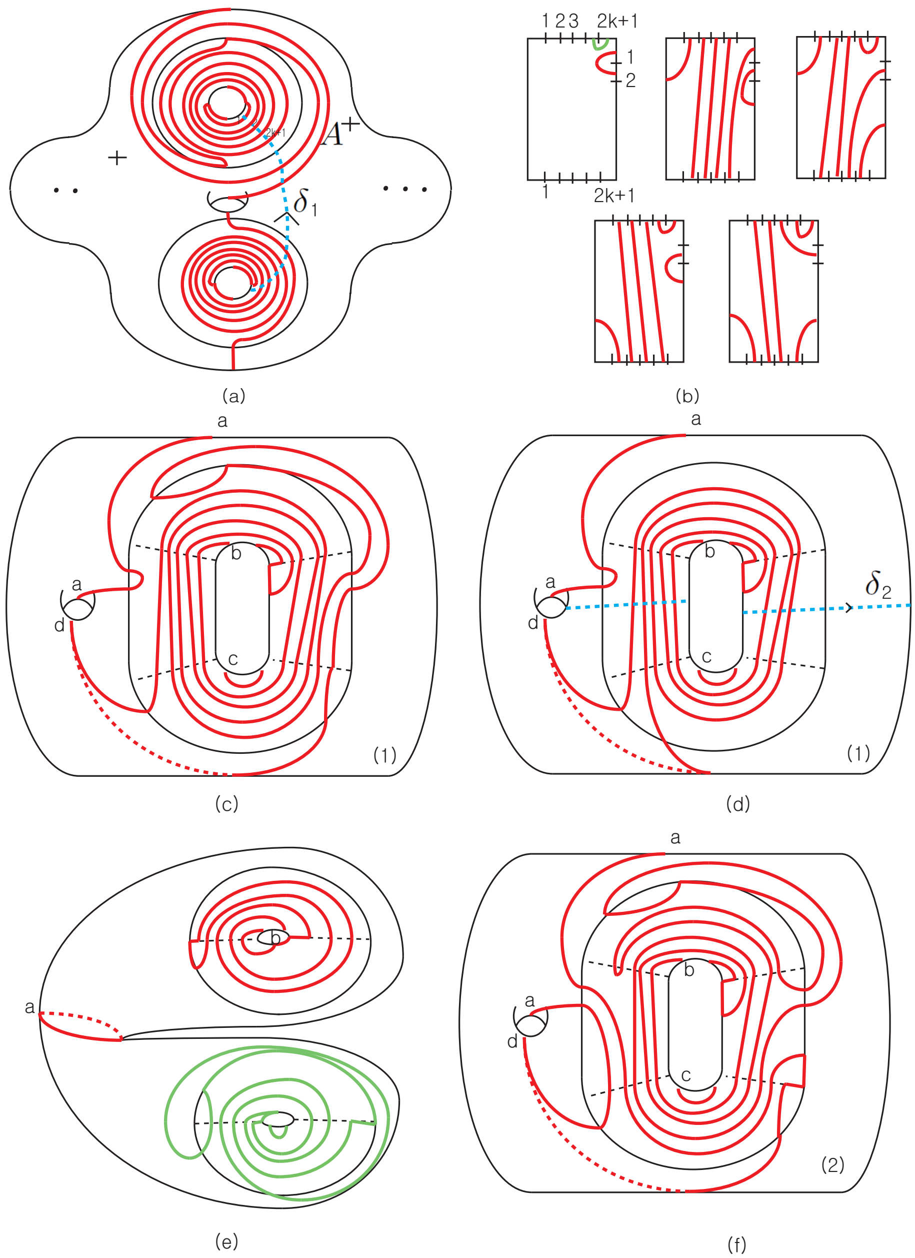


Figure 4.14: II^+_{2k}, $g = 2$ case

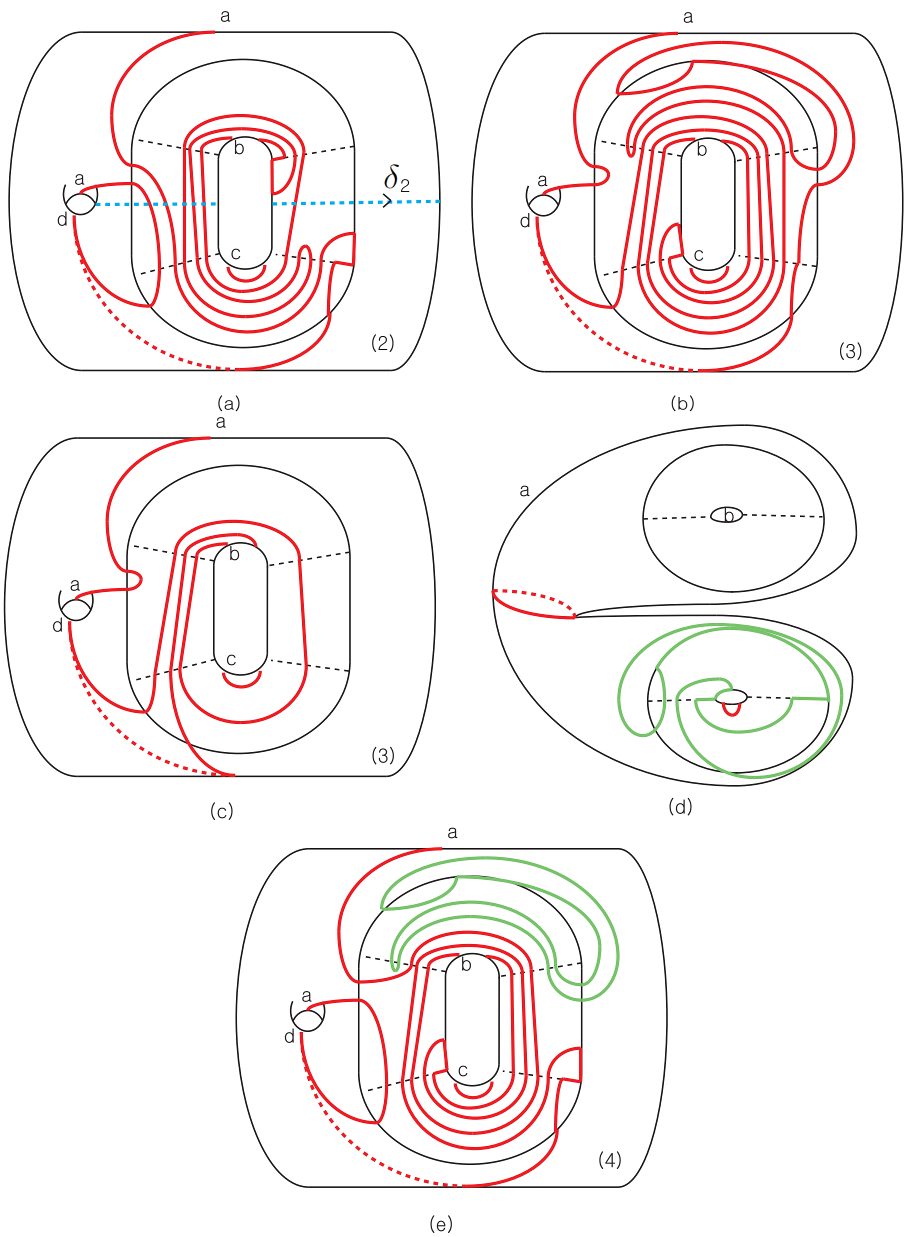


Figure 4.15: II^+_{2k}, $g = 2$ case

(d) represent the β_3 case and Figure 4.15 (e) represents the β_4 case. Hence we can always reduce the number of core curves by 2 or the case admits an overtwisted contact structure.

(2) We consider the case of that $g \geq 3$. Then the number of negative region genus, a, or the one of positive region genus, b, should be nonzero. Without loss of generality, we can assume that $b \neq 0$ and $k = 2$. The procedure to reduce the number of genus is similar to the one used in Lemma 4.2.2 and Figure 4.4. When reducing the number of genus, 2 cases occur. The first case is the case like as Figure 4.4 (e). For this case, we can reduce the number of core curves. For the second case, we can get Figure 4.16 (a) by cutting along a convex surface which connects two annular boundaries. Since the dividing curve components of bottom annulus is just 2, we can reduce all other genus. Then we can get a thickened 2-punctured torus as shown in Figure 4.20 (a). Since $\delta_1 \times I \cap A^+ + \delta_1 \times I \cap \Gamma_{\Sigma_1} = (2k+1) + 2$ and $\delta_1 \times I \cap \Gamma_{\text{bottom annulus}} = 1$, we always find a digging bypass on A^+, which reduce the number of core curves, except one case which is indicated in Figure 4. 20 (b). However, we can know that this case gives a overtwisted contact structure, which is indicated in Figure 4. 20 (d), by cutting along $\delta_2 \times I$ in Figure 4. 20 (c). □

Lemma 4.3.4. The manifold M with the convex annulus A of dividing curves I_0 type admits at most 3 tight contact structure.

Proof. We can reduce all remaining genus by cutting along finite disk system $\{D_i\}$ with $tb(D_i) = -1$ intersecting with A^- as before. Hence it suffices to show for the case $a = 0$ and $b = 0$. See Figure 4.21 (a). First cut M' by convex surface $\delta_1 \times I$. Then the dividing sets on $(\delta_1 \times I)^+$ have two possibilities, we call these β_1, β_2. See Figure 4.21 (b). For the β_1 case, we can cut $M' \setminus (\delta_1 \times I)$ again along a convex surface $\delta_2 \times I$ in Figure 4.21 (c). Then we can get a solid torus with two longitudal dividing curves like as Figure 4.21 (d). Hence this solid torus admits a unique tight contact structure. Since all dividing curves of the β_1 case are boundary parallel, M^1 admits a unique tight contact structure for β_1 case. For this case, there exists a bypass straddling the purple colored line in Figure 4.21 (a). By bypass sliding lemma,

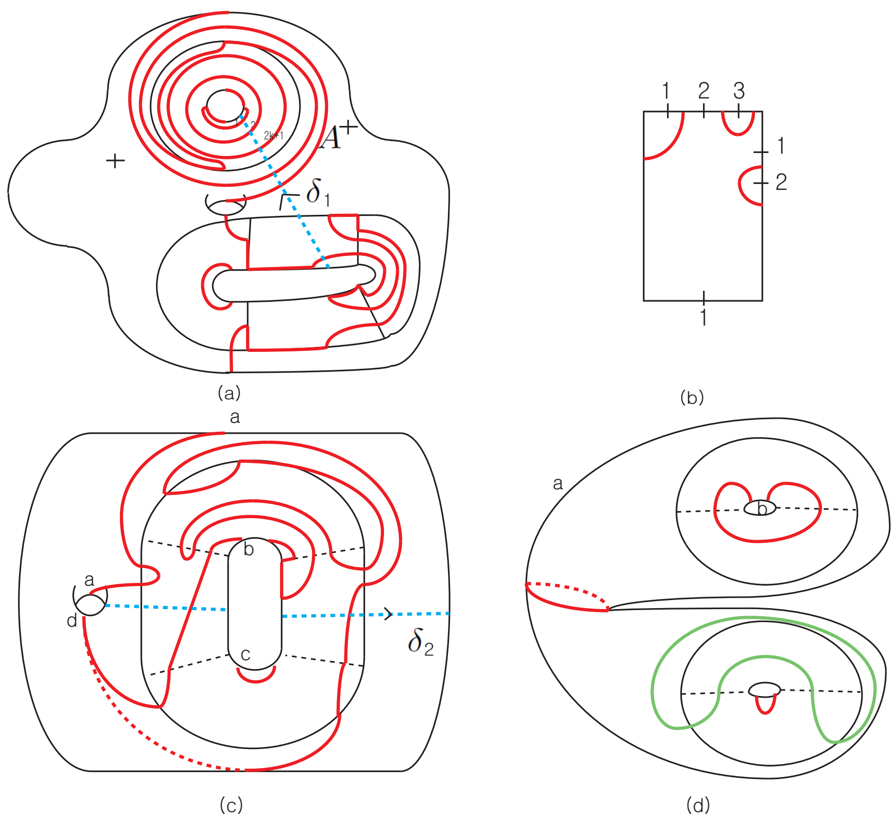


Figure 4.16: II_{2k}^{+} case with nonzero genus on positive region($g > 3$)

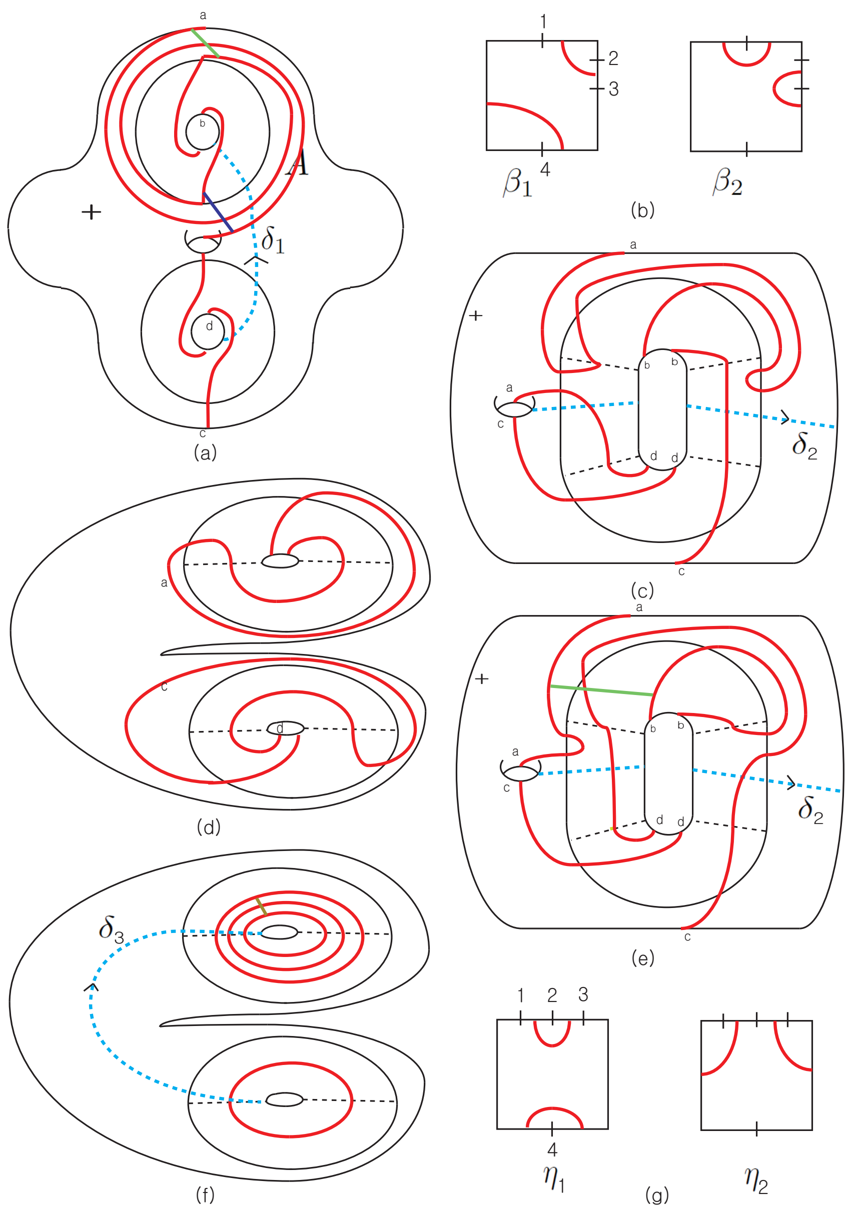


Figure 4.17: I_0 case

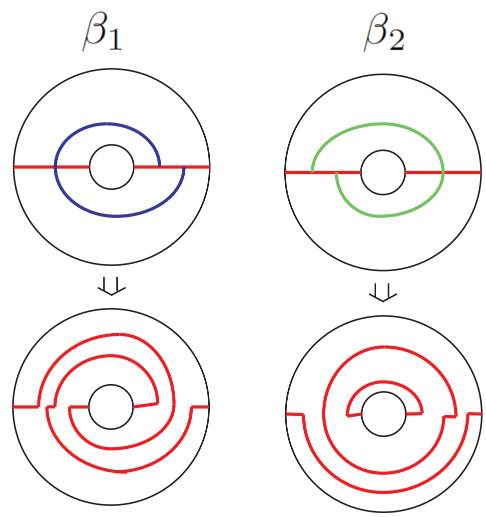


Figure 4.18: I_0 case

we can slide until it becomes purple colored attachment line on A^+ in left Figure 4.22. Then this bypass transforms to I_{-1}^+ case.

Consider β_2 case. Then we can get a thickened one-punctured torus with dividing curves as shown in Figure 4.21 (e) after edge-rounding. After cutting this manifold along $\delta_2 \times I$, then we can get a solid torus with 4 longitudal dividing curves in Figure 4.21 (f). Cut this solid torus along a convex surface $\delta_3 \times I$ efficiently. Then there are two possibilities of dividing curves on $(\delta_3 \times I)^+$. Call these η_1, η_2. See Figure 4.21 (g). For the η_1 case, we can get a 3-ball with one dividing curve after cutting along $\delta_3 \times I$. Hence M' with η_1 case admits a unique tight contact structure. For this case, there exists a digging bypass which straddles on green colored line in Figure 4.21 (e). This green line can be slided to green line inside A^+ as shown in Figure 4.22. Hence this digging bypass transforms to II_1^+ case. η_2 case also gives a unique tight contact structure by similar manner.

We need to show that these 3 cases. There is no state transition between β_1 and β_2, since $\delta_1 \times I$ is a disk and $\Gamma_{\delta_1 \times I}$ is ∂-parallel. Hence β_1 and β_2 are not contact isotopic. The green line and purple line are all possible nontrivial bypass attachment inside A^+. Hence, if there exists such a bypass inside a structure with β_2 and η_2 decomposition, then it gives a overtwisted disk. Hence it can not be isotopic to β_1 and η_1. Consequently, I_0 case admits at last one and at most 3 tight contact structures. □

There are 3 cases left, which are $II_1^{\pm}$ and I_{-1}. We will show the existence of state transitions between each other in the following lemma.

Lemma 4.3.5. The manifold M with the convex annulus A of dividing curves $II_1^{\pm}$ can be reduced to I_0. The manifold M with the convex annulus A of dividing curves I_{-1} case, which does not transforms to I_0 case, admits at most one tight contact structures.

Proof. (1) First look at the case II_1^+. We assume that there is no genus on negative region. See Figure 4.19 (a). Using a dividing curve on A^- homotopic to a core curve, we can eliminate all genus in positive region. Then cut resulting manifold along a convex surface $\delta_1 \times I$. A bypass straddling from 2 to 4 gives a overtwisted disk and a bypass attached from 1 to 3 gives a state transition to I_0. Hence there is one possibility remaining as shown in Figure 4.19 (b). After Edge-rounding, we can get a manifold in Figure 4.19 (c). Cut resulting manifold along $\delta_2 \times I$. Then we can get a solid torus with 4 longitudal dividing curves. In this case, we can find a long digging bypass on upper annulus which is originally from A^+. The bypass attachment is indicated as a purple line in Figure 4.19 (d) which goes back to the purple line in Figure 4. 19 (a). Hence this gives a state transition to I_1. However, since we showed that I_1 can be reduced to the case I_0, II_1^+ case can always be reduced to I_0. If $g > 2$, then we can get a manifold as shown in the Figure 4.19 (e). This case is similar to the one of $g = 2$.

(2) The proof of II_1^- case almost same the one of II_1^+. Refer to Figure 4. 20. There might be a digging bypass on A^+ which gives a state transition to I_0 or to I_1 (Attachment arc of this is purple line in Figure 4.20 (a)). However, since I_1 can be reduced to the case I_0, we can always reduce to the case I_0.

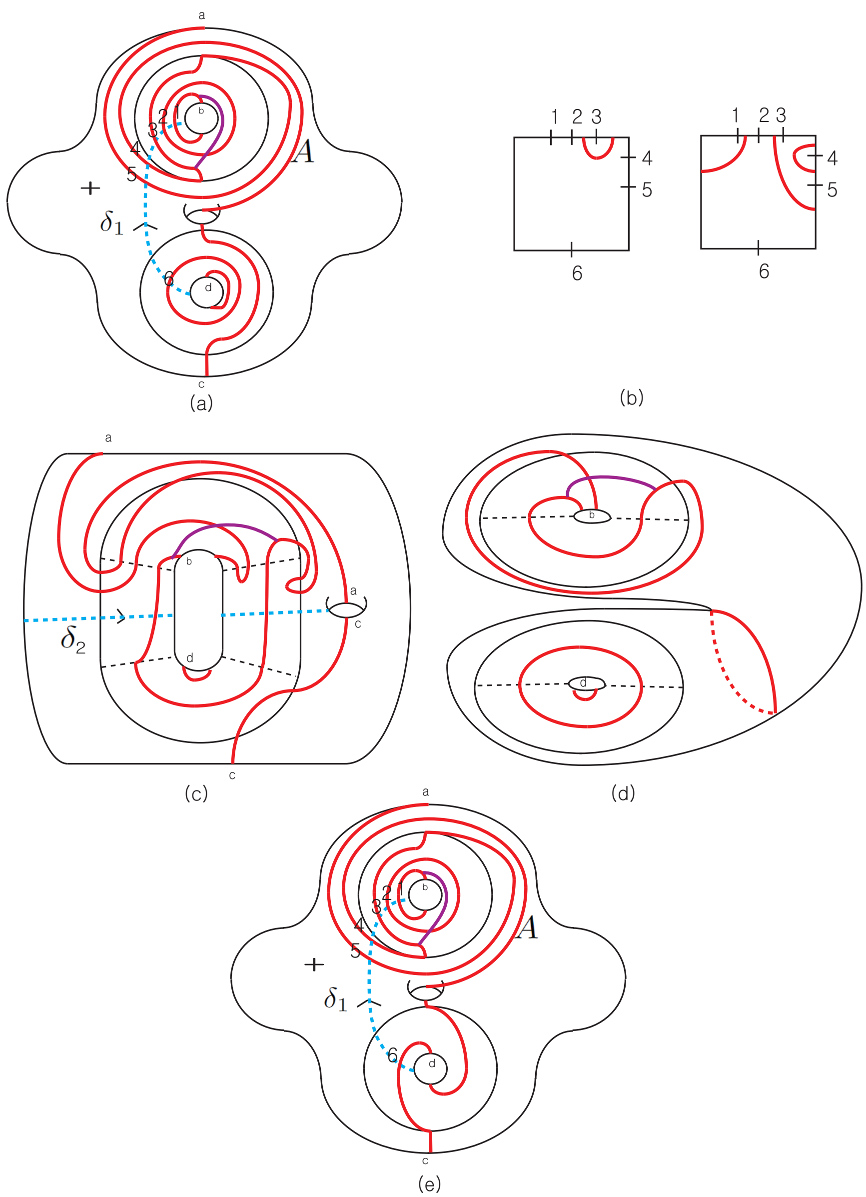


Figure 4.19: II_1^+ case

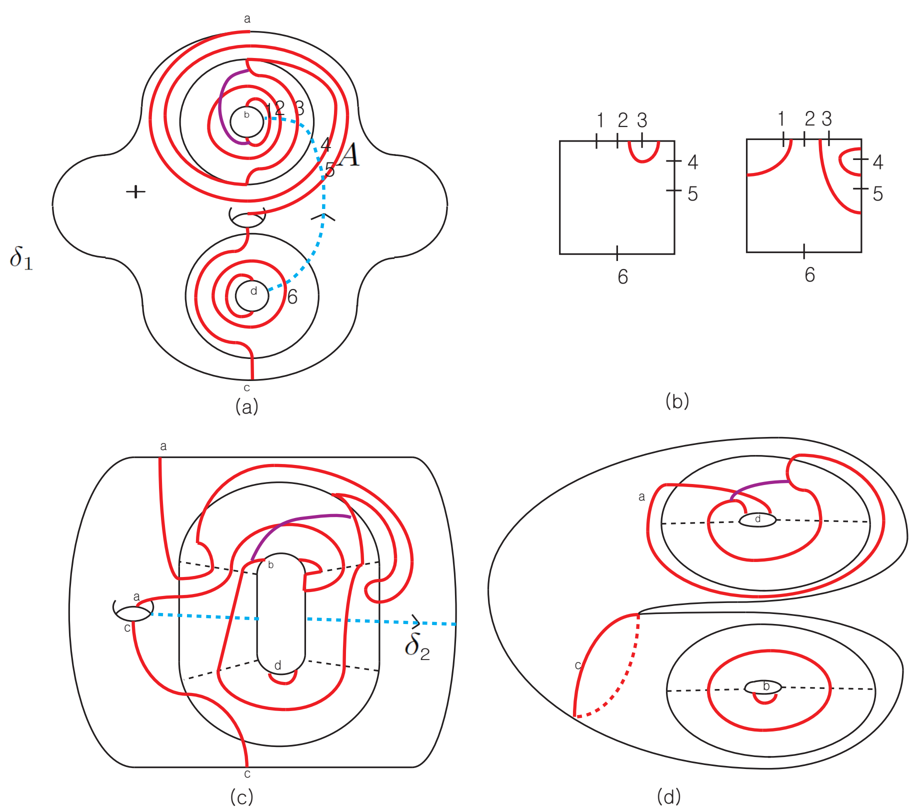


Figure 4.20: II_1^- case

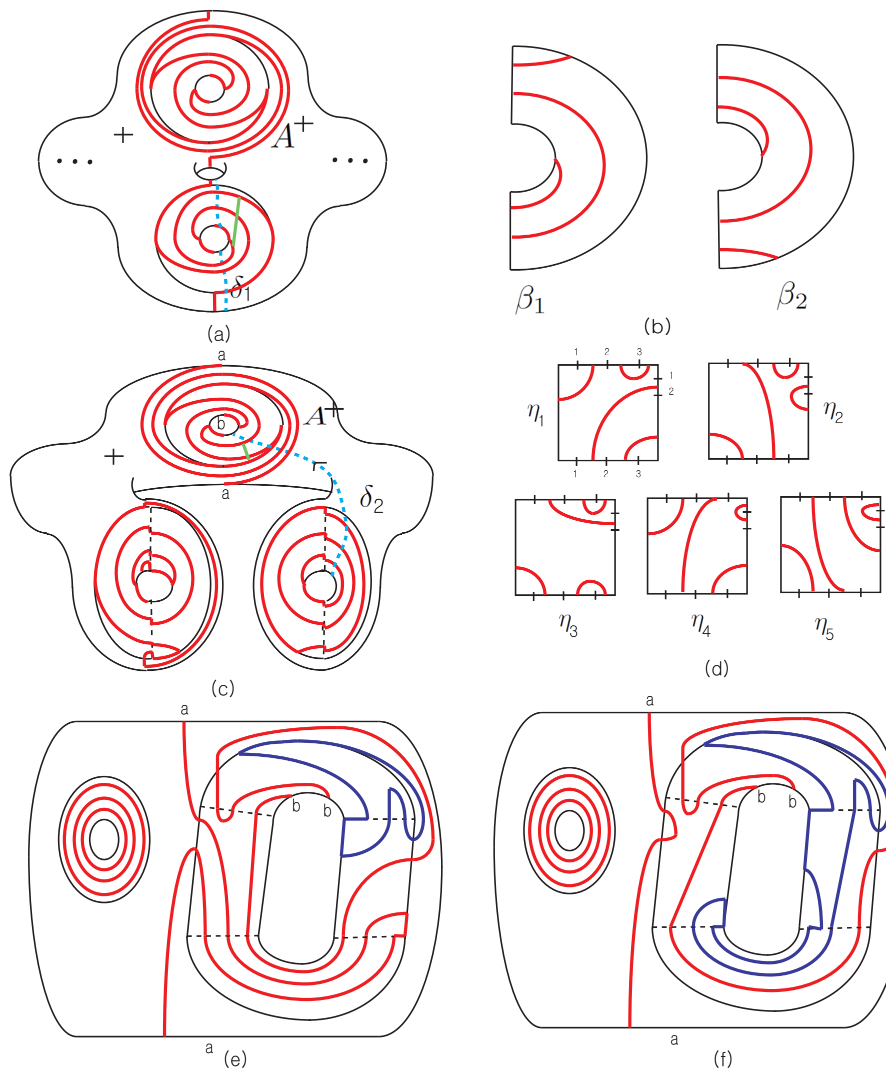


Figure 4.21: I_{-1} case

(3) The Only I_{-1} case is left. See Figure 4.21. The first part of the proof is similar to the part proof of Lemma 4.2.2. Cut first M' along a convex surface $\delta_1 \times I$. Then there are 2 possibilities for $\Gamma_{\delta_1 \times I}$. For β_1 case, we can find a long bypass which can be slided into a digging bypass on A^-, hence, gives a state transition to I_0. We omit the figure for this part. For the β_2 case, we have to consider two cases depending on the number of genus.
First consider $g = 2$ case. Take a point p on (right hand side annulus$\cap \Sigma_1$, q on $\Gamma_{A^+} \cap \Sigma_1$ and an arc $\delta_2 \subset \Sigma_1$ joining two points. See Figure 4.21 (c). Then there are 5 possibilities, $\eta_1, \cdots, \eta_5$, of $\Gamma_{(\delta_2 \times I)^+}$ which does not give a digging bypass on A^- side. Note that the bypass attachment arc on A^+ straddling from 1 to 3 gives a OT disk. Hen we can not allow that. η_1, η_2 give a OT disk which in indicated in Figure 4.21 (e) and (f). We can find a long digging bypass on A^+ side for η_4, η_5 cases which are indicated as the blue line in Figures 4. 22 (c) and (d).
For the η_3 case, we can get a solid torus with 4 logitudal dividing curves. Cut solid torus along a convex surface $\delta_3 \times I$ efficiently. $\Gamma_{(\delta_3 \times I)^+}$ has two possibilities. One of them gives a state transition to I_0. For the other one, we can get a 3-ball with one dividing curve component by cutting along $\delta_3 \times I$ and edge-rounding. Hence I_{-1} admits at most tight contact structure. In fact, we consider another convex decomposition. In Figure 4.21 (a), we homotope the dividing curves on A^+ side. Then we can get a manifold in Figure 4.22 (e). Then we cut this manifold along a convex surface $\delta_4 \times I$ and $\delta_5 \times I$. There is only one possibilities which does not give a state transition to I_0 and this gives a decomposition to 3-ball with one dividing curve component. Hence M^1 admits unique tight contact structure by Eliashberg and Colin's Theorem. However, it can not guarantee the tight contact structure on M. □

Proof of Proposition 4.1.4. So far, we prove that (M^1, I_0) admits 3 tight contact structures which are $\{(\delta_1 \times I, \beta_1)\}$, $\{(\delta_1 \times I, \beta_2), (\delta_2 \times I, \eta_1)\}$, $\{(\delta_1 \times I, \beta_2), (\delta_2 \times I, \eta_2)\}$ (We omit the disks to reduce the number of genus) disk decompositions and they are not contact isotopic to each other. In Lemma 4.3.5, we prove that $II_1^{\pm}$ can always be reduced to the case I_0 and (M^1, I_{-1}) admits a unique tight contact structures which can not be transformed to other case. Hence M^1 admits 4 different tight contact structures. However,

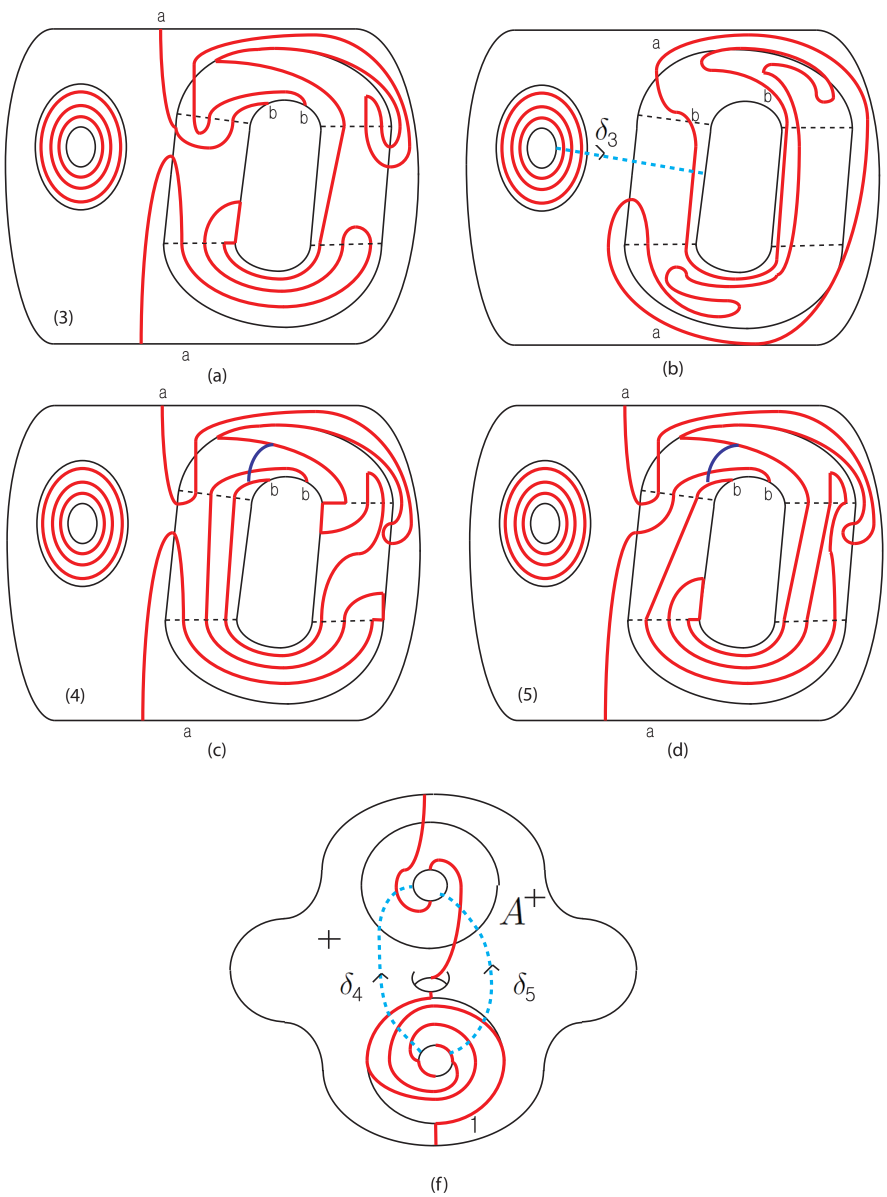


Figure 4.22: I_{-1}, $g = 2$ case

since we cannot apply Colin's theorem to A, M admits at most 4 tight contact structures. □

4.4 Proof of Proposition 4.1.6

Strategy of the proof of Proposition 4.1.6

Step 1 If $k > 0$, then the I_k case can be reduced to the $II_1^{\pm}$ case. (cf. Lemma 4.4.1).

Step 2 If $k < -1$, then the I_k case can be reduced to the I_{k+1} case or $II_1^{\pm}$ case. (cf. Lemma 4.4.2).

Step 3 If $k > 0$, then $II_{2k}^{\pm}$ can be reduced to $II_{2k-2}^{\pm}$ case and $II_{2k+1}^{\pm}$ can be reduced to $II_{2k-1}^{\pm}$.

Step 4 The manifold M with the convex annulus A of dividing curves II_0^{+} (resp. -) case cannot admit a tight contact structure. (This is same as the one of the case $n = 0$.

Step 5 The manifold M^1 with the convex annulus A of dividing curves I_0 type and I_1 type admit a unique tight contact structure. (cf. Lemma 4.4.3).

Step 6 The convex annulus A with dividing curves $II_1^{\pm}$ type can be transformed to the one of with I_1, I_0 or The manifold M^1 with the convex annulus A with dividing curves $II_1^{\pm}$ type admits a unique tight contact structure. The convex annulus A with dividing curves I_{-1} type can be transformed to the one of with I_0 or $II_1^{\pm}$. (cf. Lemma 4.4.4).

Since the proofs of step 3,4 are same as the one of $n = 0$, we omit these.

Lemma 4.4.1. If $k > 0$, then the I_k case can be reduced to the $II_1^{\pm}$ case.

Proof. First cut M^1 along a convex surface $\delta_1 \times I$ in Figure 4.23 (a). Then there are two possible configurations of dividing curves on $(\delta_1 \times I)^+$. For the β_1 case, we can find a long digging bypass after edge-rounding which gives a state transition to $II_1^{\pm}$. For the β_2 case, we can find a overtwisted disk. Hence there cannot exist inside tight contact structures. □

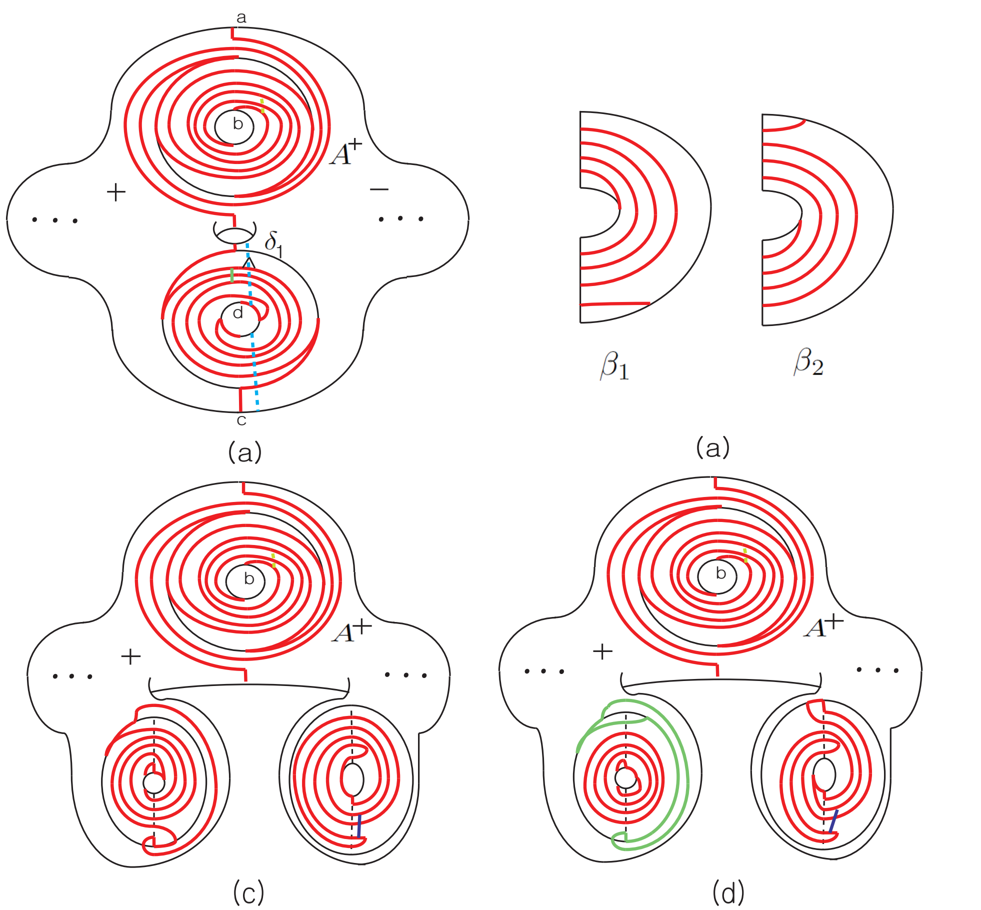


Figure 4.23: $I_{k\in\mathbb{Z}_{>0}}$ case

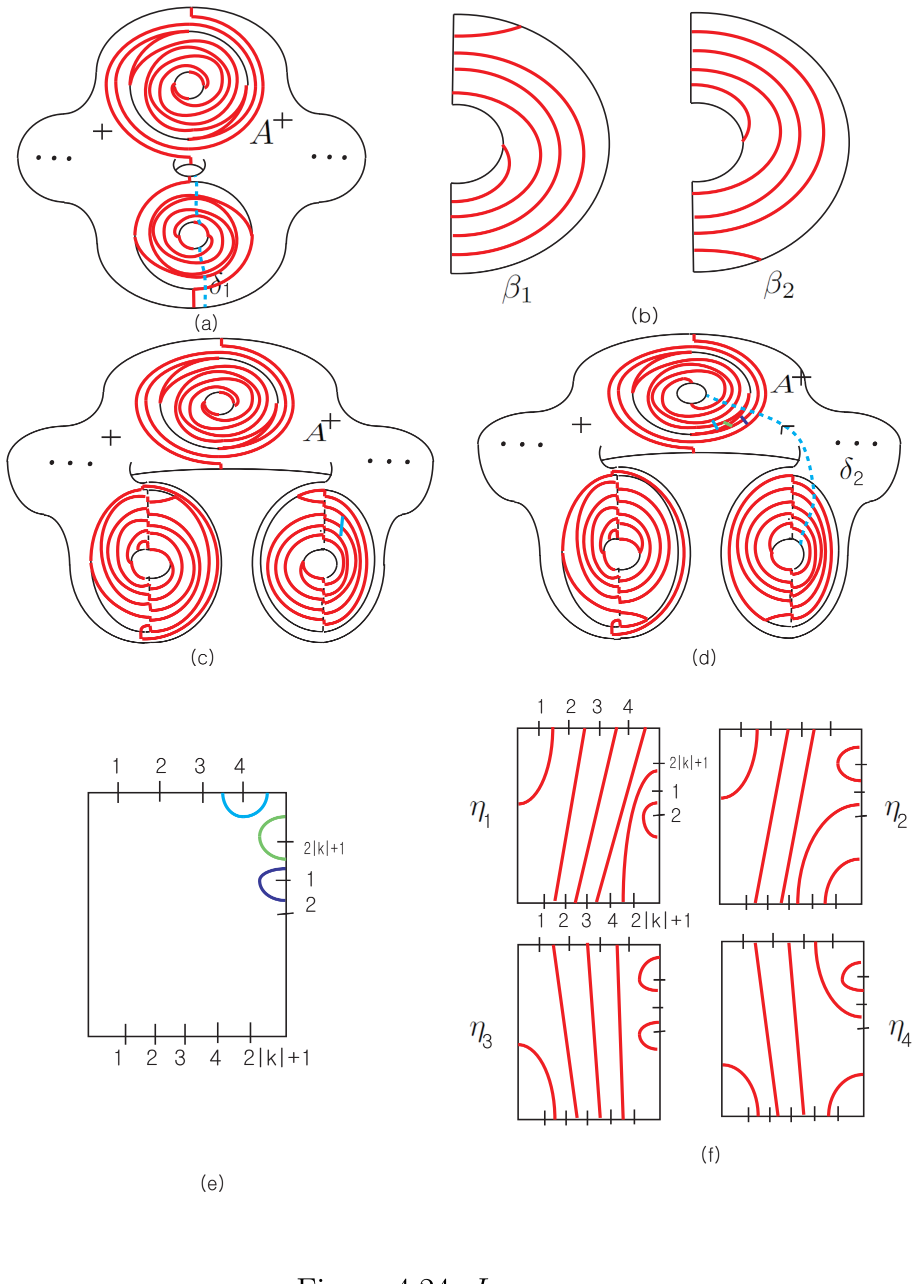


Figure 4.24: $I_{k\in\mathbb{Z}_{<0}}$ case

Lemma 4.4.2. If $k < -1$, then the I_k case can be reduced to the I_{k+1} case or $II_1^{\pm}$ case.

Proof. Cut $M^1 = M \setminus A$ by convex surface $\delta_1 \times I$. This cut is similar to the one in the proof of step 2 of $n = 0$. See Figure 4.24 (a). If $\delta_1 \times I$ has a boundary parallel dividing curve component on $A^- \cap \delta_1 \times I$, this gives digging bypass on A^- side reducing to the I_{k+1} case. Remaining possibilities are two cases in Figure 4.24 (b). After edge rounding, β_1 case has one dividing curve homotopic to a core curve on each new annular boundary. Then we can find a long bypass whose attachment arc is the blue line in Figure 4.24 (c). Then this bypass gives a digging bypass on A^- and reduces the number of Dehn twists. For the β_2 case, we can get $2|k|+1$ parallel dividing curves homotopic to a core curve on each new annular boundary after edge rounding. In this case, we can think divided into 2 cases. $g = 2$ and $g > 2$.
First consider the $g > 2$ case. Then we can find a convex surface to reduce the number of genus which can be thought similarly as before. For each case, we always find a long digging bypass which gives a digging bypass on A^- side. Hence we can reduce the number of core curves on A.

For the $g = 2$ case, we make δ_2 in Figure 4.24 (d) Legendrian and take inefficient cut along the convex surface $\delta_2 \times I$. If a boundary parallel dividing curve on $(\delta_2 \times I)^+$ straddles from $2|k| - 1$ to $2|k| + 1$ or from $2|k| + 1$ to 2, it gives a OT disk. A boundary parallel bypass straddling from $2|k|$ to 1 gives a trivial bypass. Hence there are 4 possibilities of dividing curve configurations on $(\delta_2 \times I)^+$ which does not give a digging bypass on A^+ nor A^- (A digging bypass on A^+ gives a state transition to $II_1^{\pm}$). For η_1 case, we can get a solid torus in Figure 4.25 (a) with two annular boundaries. The number of dividing curves on left annulus is $2|k| + 1 \geq 5$ and the one on right annulus is just 1. Hence there should exist a digging bypass on left annulus. Since this originally comes from A^-, we can reduce the number of Dehn twists.

We can argue similarly for the η_2 and η_3 cases which is indicated in Figures 4.25 (b) and (c). For the η_4 case, we can find a OT disk which is blue line in Figure 4.25 (d). □

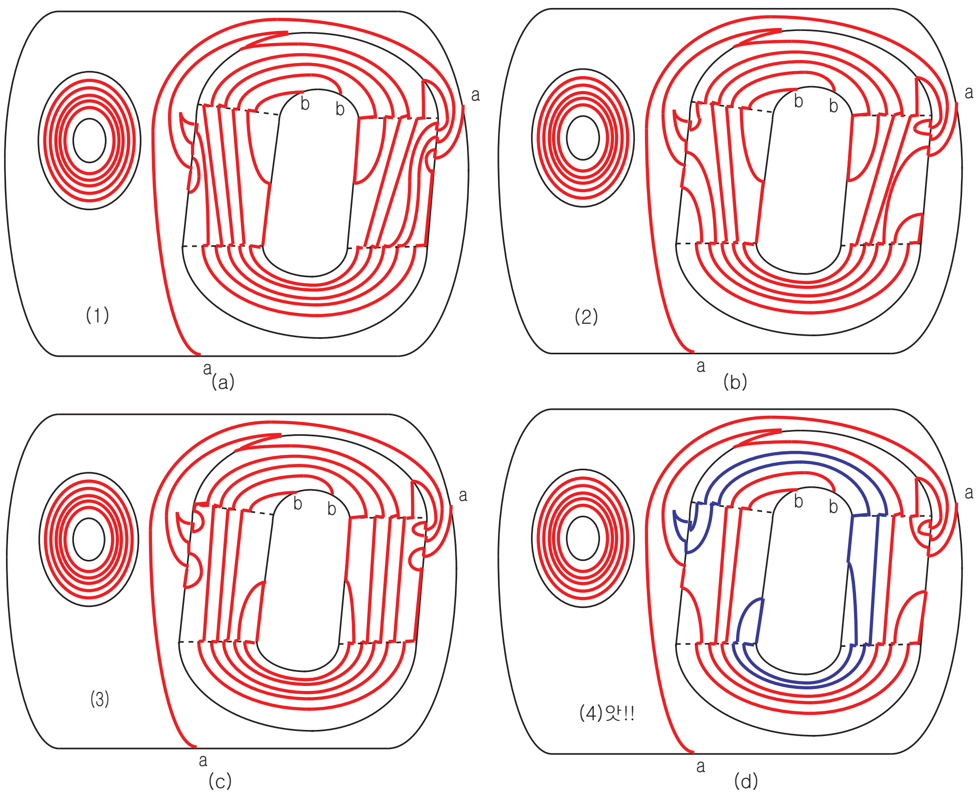


Figure 4.25: $I_{k\in\mathbb{Z}_{<0}}$ case

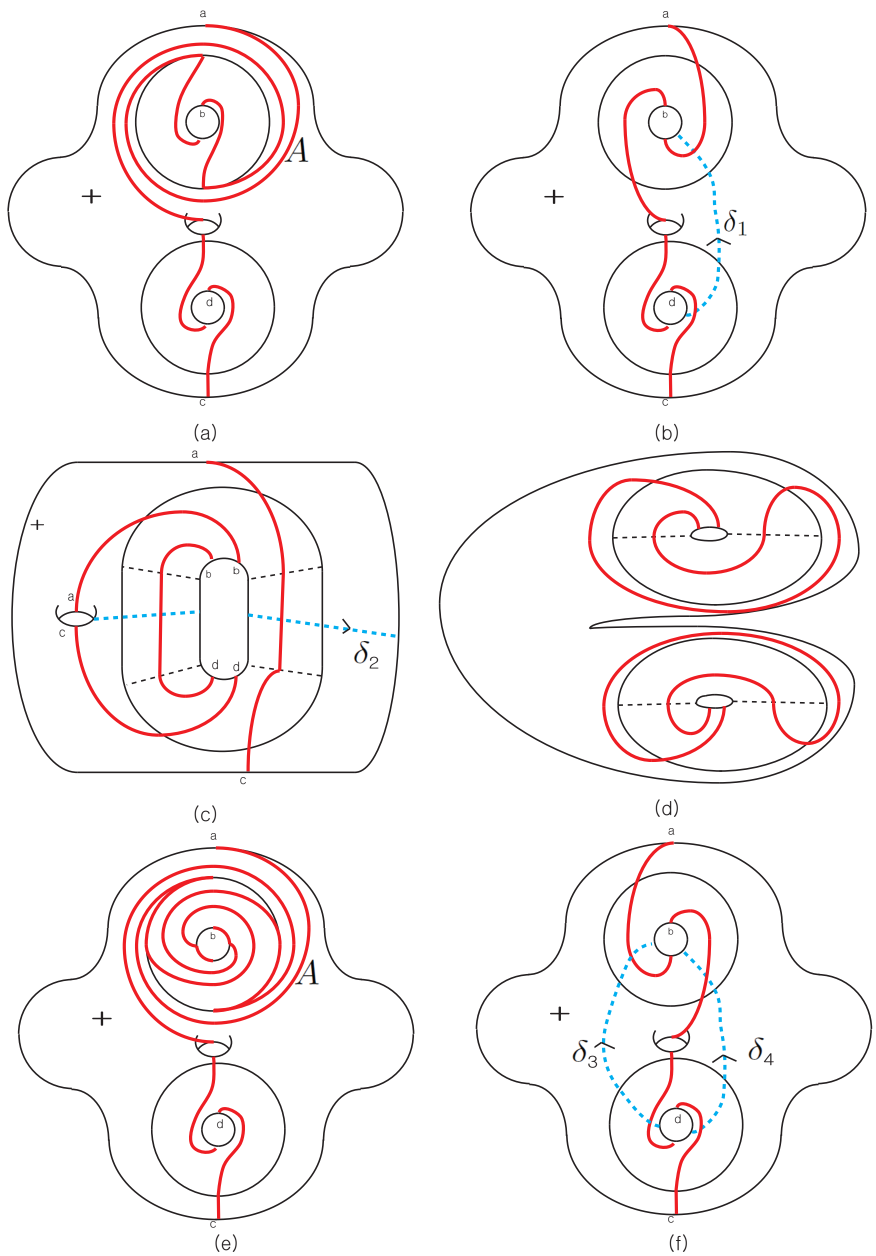


Figure 4.26: I_0, I_1 case

Lemma 4.4.3. The manifold M^1 with the convex annulus A of dividing curves I_0 type and I_1 type admit a unique tight contact structure.

Proof. See Figure 4.26 (a). If $g > 2$, then we can find disk system $\{D_i\}$ intersecting A^- with $tb(D_i) = -1$ to eliminate genus. Homotope the dividing curve on A^+ side. Then we can get a thickened two punctured torus with dividing set as shown in Figure 4.26 (b). After cutting this manifold along convex surfaces $\delta_1 \times I$ and $\delta_2 \times I$, then we can get a solid torus with two longitudal dividing curves. Hence it admits unique tight contact structure. Since all disks used to decompose M^1 to a 3-ball with $tb = -1$, (M^1, I_0) admits a unique tight contact structure. Similarly, we can show that (M^1, I_1) admits a unique tight contact structure. See Figures 4.26 (e) and (f). $\square$

Lemma 4.4.4. The convex annulus A with dividing curves $II_1^{\pm}$ type can be transformed to the one of with I_1, I_0 or The manifold M^1 with the convex annulus A with dividing curves $II_1^{\pm}$ type admits a unique tight contact structure. The convex annulus A with dividing curves I_{-1} type can be transformed to the one of with I_0 or $II_1^{\pm}$.

Proof. Consider the II_1^+ case. First eliminate the genus on negative region using a core dividing curve on A^-. Then cut along $\delta_1 \times I$ as shown in Figure 4.27 (a). There are two possibilities of $\Gamma_{(\delta_1 \times I)^+}$. For the β_1 case, we can find a trivial digging bypass by cutting and edge-rounding which gives a state transition to I_1. See Figure 4.27 (c) For the β_2 case, we can a manifold in Figure 4.27 (e). Then cut this manifold again along $\delta_3 \times I$. Then we can get a solid torus with 4 longitudal dividing curves. Cut solid torus along $\delta_3 \times I$ efficiently. Then we can get a 3-ball with one dividing curve. There are two possibilities on $\Gamma_{(\delta_3 \times I)^+}$, one of which gives a state transition to I_0 and the other does not. Hence (M^1, II_1^+) which cannot be transformed to I_1 or I_0 admits a unique tight contact structure (Since every dividing curve components of β_2 is boundary parallel, we can use Colin's theorem for M^1.The II_1^- type is similar. See Figure 4.28.

For the I_{-1} case, we cut manifold M^1 in a similar way to the former part of the proof of Lemma 4.4.2. Then we can find a digging bypass on A^- or

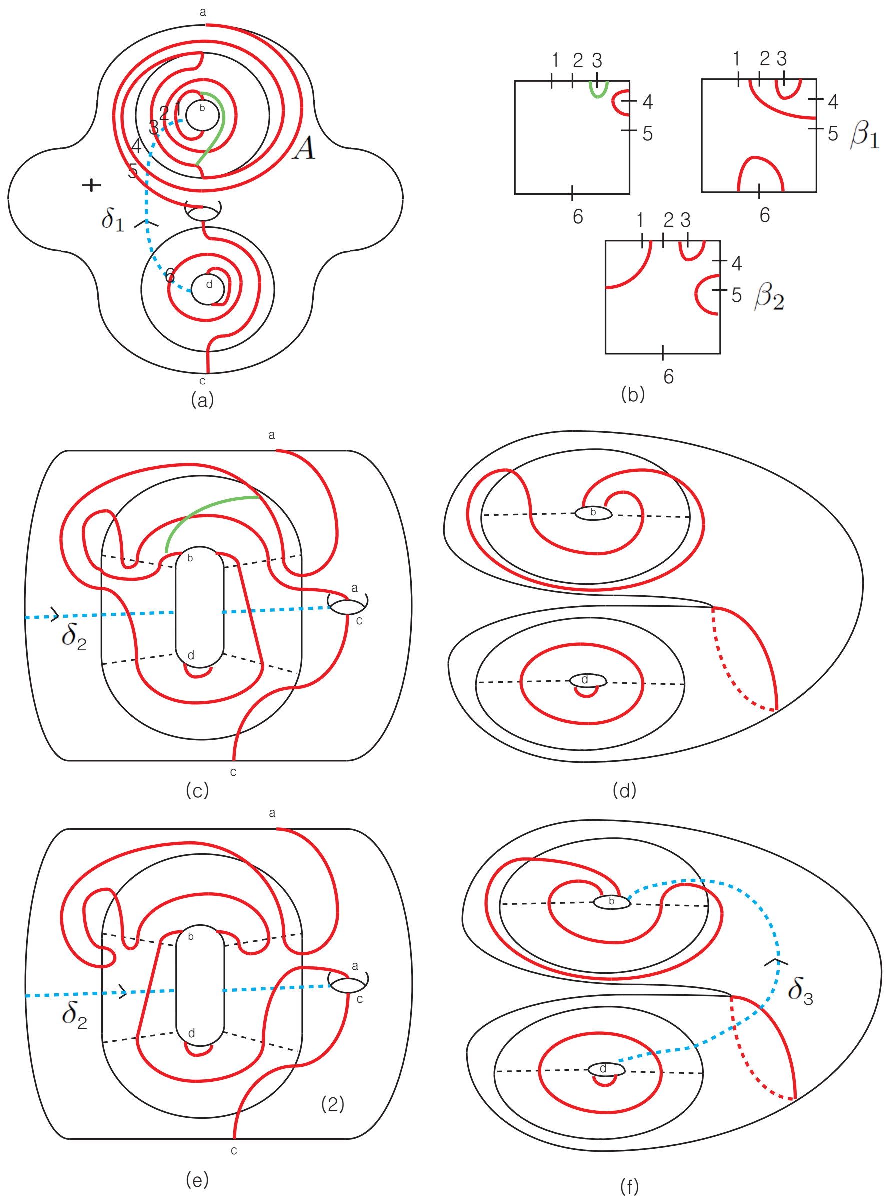


Figure 4.27: II_1^+ case

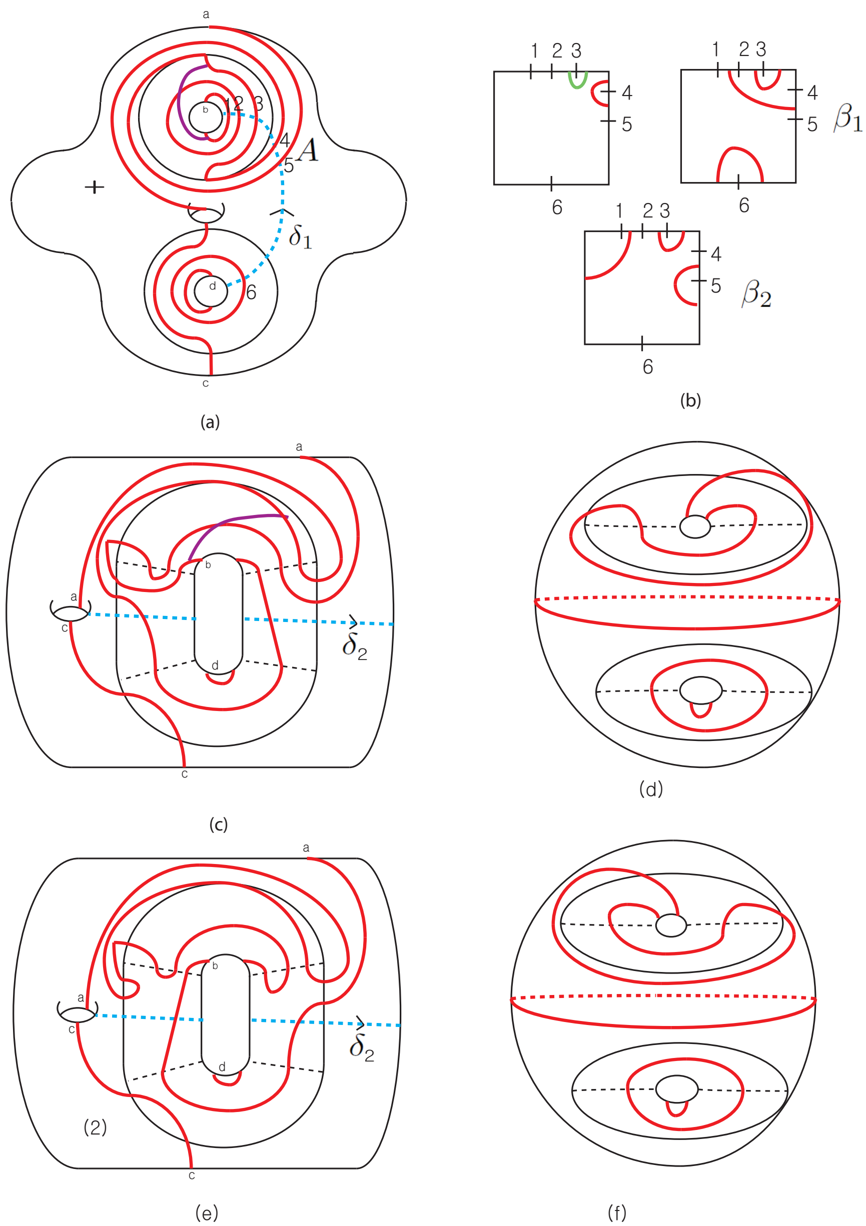


Figure 4.28: II_1^- case

get a manifold like as Figure 4.29 (a). We can think divided into two cases depending on the genus. Figure 4.29(b) indicates the $g > 2$ case. After eliminating genus on negative regions, we can get a annulus with one core dividing curve. Take a Legendrian curve δ_2 as shown in Figure 4.29 (b). Then there are two possibilities of $\Gamma_{(\delta_2 \times I)^+}$, since bypasses straddling from 1 to 3 and from 3 to 5 give OT disks. For the β_1 case, we can find a trivial bypass on A^+ whose attachment arc is the green line in Figure 4.29 (d). This gives a state transition to II_1^-. Similarly, the β_2 case gives a state transition to II_1^+. See Figure 4.29 (e).

For the I_{-1}, $g = 2$ case is similar. There are 4 possibilities of $(\delta_2 \times I)^+$. For β_1 case, we can find a long digging bypass which is green line in Figure 4.30 (c). This gives a state transition to the I_0 case. For the β_2 case, we can find a trivial digging bypass by which A^+ can be transformed to the II_1^- case. Similarly, we can find trivial digging bypasses for β_3, β_4 cases by which A^+ can be transformed to the case II_1^-, II_1^+ respectively.

□

Proof of Proposition 4.1.6. I_0, I_1 and $II_1^{\pm}$ admit at most one tight contact structure respectively. □

4.5 Proof of Proposition 4.1.3

Strategy of the proof of Proposition 4.1.3

Step 1 If $k > 0$, then the I_k case can be reduced to the I_{k-1} case. (cf. Lemma 4.5.1).

Step 2 Suppose $k < -1$. If $g > 2$, then the I_k case can be reduced to the I_{k+1} case. If $g = 2$, thenthe I_k case can be reduced to the I_{k+1} case or $II_1^{\pm}$ case except 4 cases. For each exceptional case, there exist at most one tight contact structure. (cf. Lemma 4.5.2).

Step 3 If $k > 0$, then $II_{2k}^{\pm}$ can be reduced to $II_{2k-2}^{\pm}$ case and $II_{2k+1}^{\pm}$ can be reduced to $II_{2k-1}^{\pm}$.

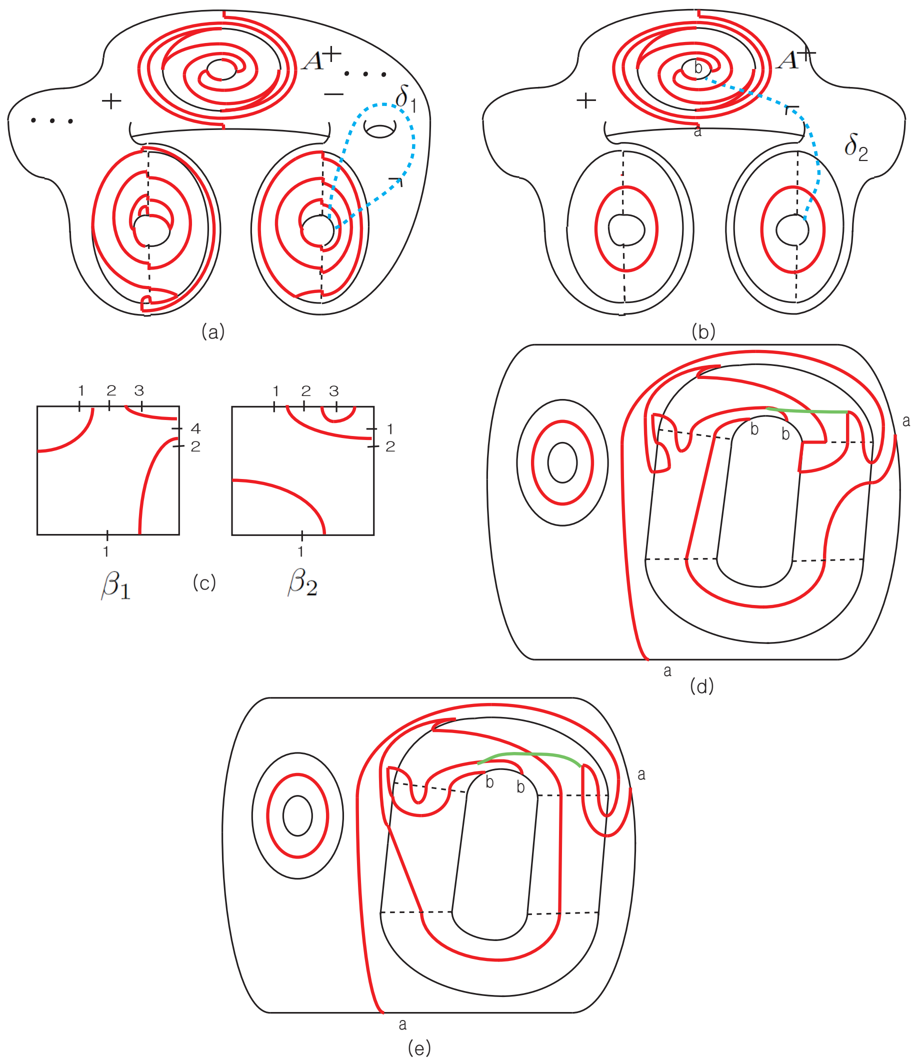


Figure 4.29: I_{-1} case with genus

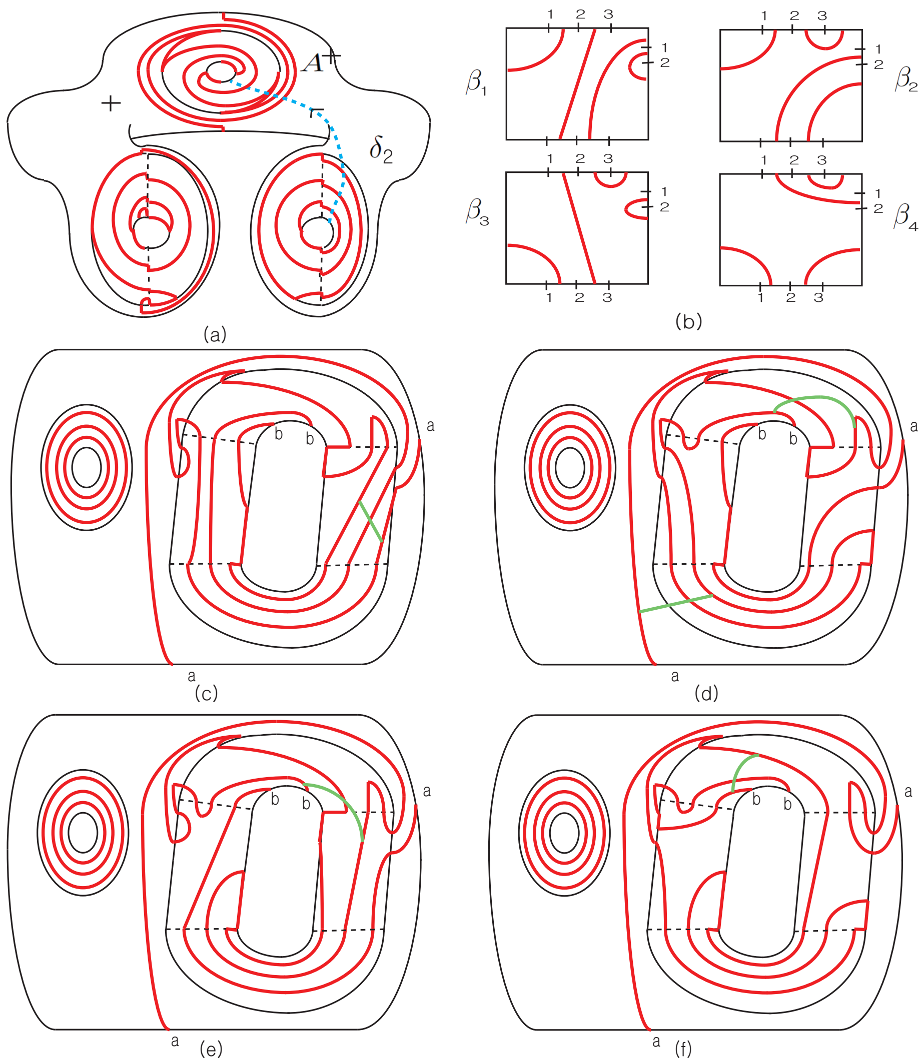


Figure 4.30: I_{-1} case

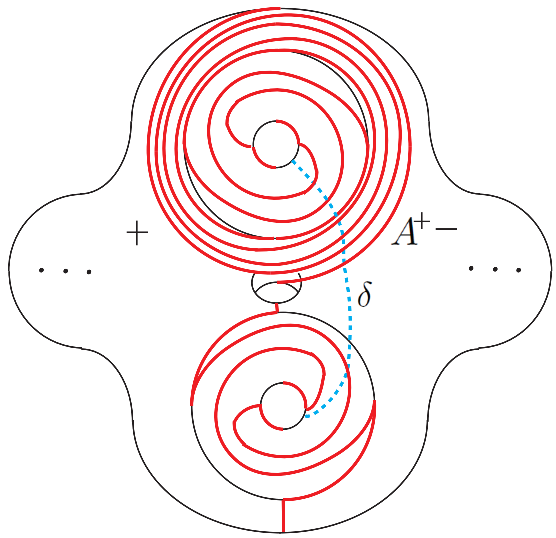


Figure 4.31: $I_{k\in\mathbb{Z}_{>0}}$ case

Step 4 The manifold M with the convex annulus A of dividing curves II_0^+ (resp. -) type cannot admit a tight contact structure.

Step 5 The convex annulus A with dividing curves $II_1^\pm$ type can be transformed to the one of with I_0, if $g > 2$ or $g = 2$ and $n > 2$. If $g = 2$ and $n = 2$, then $II_1^\pm$ type can be transformed to the one of with I_0 or I_{-1}. (cf. Lemma 4.5.3).

Step 6 The manifold M with the convex annulus I_0 admits at most 3^n tight contact structures. (cf. Lemma 4.5.4).

Step 7 The manifold M with the convex annulus I_{-1} admits at most 3^n or 3^{n-1} tight contact structures if $g > 2$. If $g = 2$, then it admits at most 5 if $n = 2$, 16 if $n = 3$ and $16 \cdot 3^{n-3}$ if $n \geq 4$, tight contact structures. (cf. Lemma 4.5.5).

Since the proofs of step 3 and 4 are same as the one of the former case, we leave out these.

Lemma 4.5.1. If $k > 0$, then the I_k case can be reduced to the I_{k-1} case.

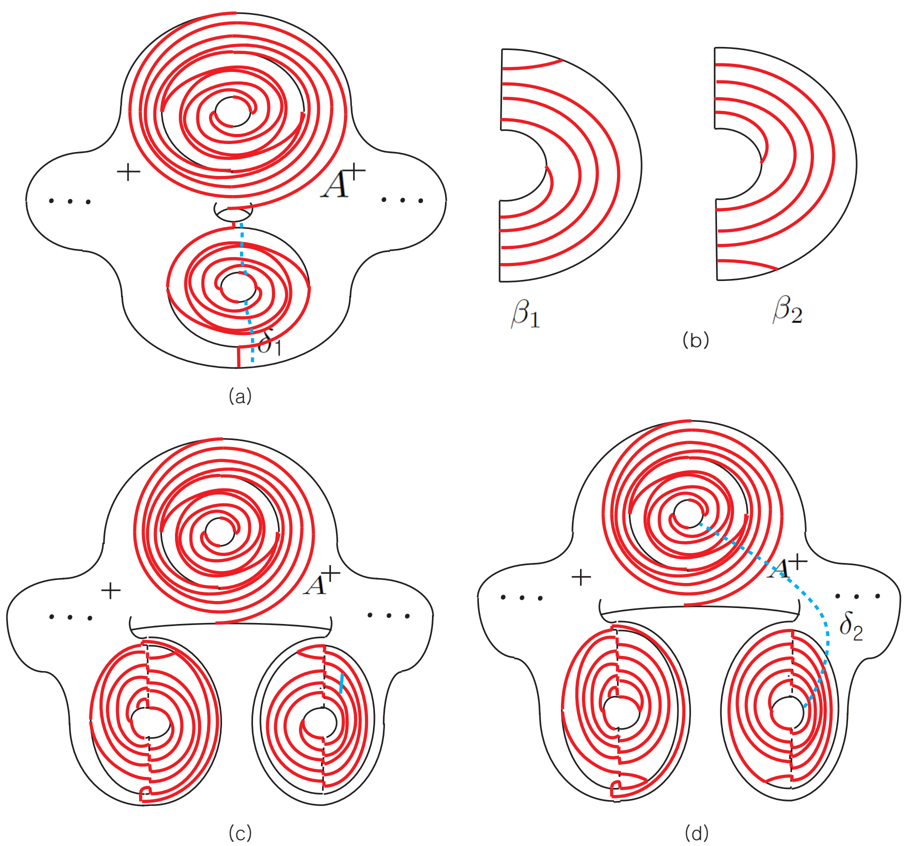


Figure 4.32: $I_{k\in\mathbb{Z}_{<0}}$ case

Proof. It suffices to show that this lemma holds in the case of $a = 0$, $b = 0$ and $k = 1$. Denote $M' = M \setminus A$. We can make δ of Figure 4.31 Legendrian by Legendrian realization principle (Lemma 3.5.1) and $\delta \times I$ convex by Theorem 3.4.6. Cut M' again by $\delta \times I$ and edge-round. Then $\sharp(\delta \times I) \cap \Gamma_{A^+} - \sharp(\delta \times I) \cap \Gamma_{A^-} = (2k+1+2n) - (2k-1) \geq 5$. (In this case, we count the number $\sharp(\delta \times I) \cap \Gamma_{\Sigma_1}$ contained the intersection numbers between ΓA^+ and $(\delta \times I)$.) Then by Imbalance principle (Lemma 3.5.8), the dividing set of $\delta \times I$ has at least one boundary parallel dividing curve on A^+ side (If it straddles on Σ_1, then we can slide this bypass inside A^+ by bypass sling lemma) for arbitrary $n \geq 2$ and $k \in \mathbb{Z}_{>0}$ and this dividing curve gives digging bypass on A^+ which reduces the number of Dehn twists. □

Lemma 4.5.2. Suppose $k < -1$. If $g > 2$, then the I_k case can be reduced to the I_{k+1} case. If $g = 2$, thenthe I_k case can be reduced to the I_{k+1} case or $II_1^{\pm}$ case except 4 cases. For each exceptional case, there exist at most one tight contact structure.

Proof. Cut M' by convex surface $\delta_1 \times I$ where δ_1 is a Legendrian curve which starts from A^- ends at A^- and is homotopic to a component of γ_0. See Figure 4.32. If $\delta_1 \times I$ has a boundary parallel dividing curve component, this gives digging bypass on A^- side reducing to I_{k+1} case. Remaining possibilities are two cases in Figure 4.32 (b). After edge rounding, the β_1 case has one dividing curve homotopic to a core curve on each new annular boundary. Then we can find a long bypass whose attachment arc is blue line in Figure 4.32 (c). Then this bypass gives a digging bypass on A^- and reduces the number of Dehn twists. For the β_2 case, we consider two cases, $g > 2$ and $g = 2$. For the $g > 2$ case, we can find a long digging bypass when we reduce the number of genus. Hence we can find a digging bypass on A^- which reduce the number of Dehn twists.

For the $g = 2$ case, we need a different approach depending on the intersection numbers between dividing curves on A^+ side and convex surface $\delta_2 \times I$. Look at the following table. Each element tells us the intersection number between dividing curves on A^+ side and convex surface $\delta_2 \times I$ when cutting efficiently along $\delta_2 \times I$ in each case. The numbers inside parenthesis mean the intersection numbers given by cutting inefficiently and first row means the intersection numbers between $\delta_2 \times I$ and Γ_{A^-}.

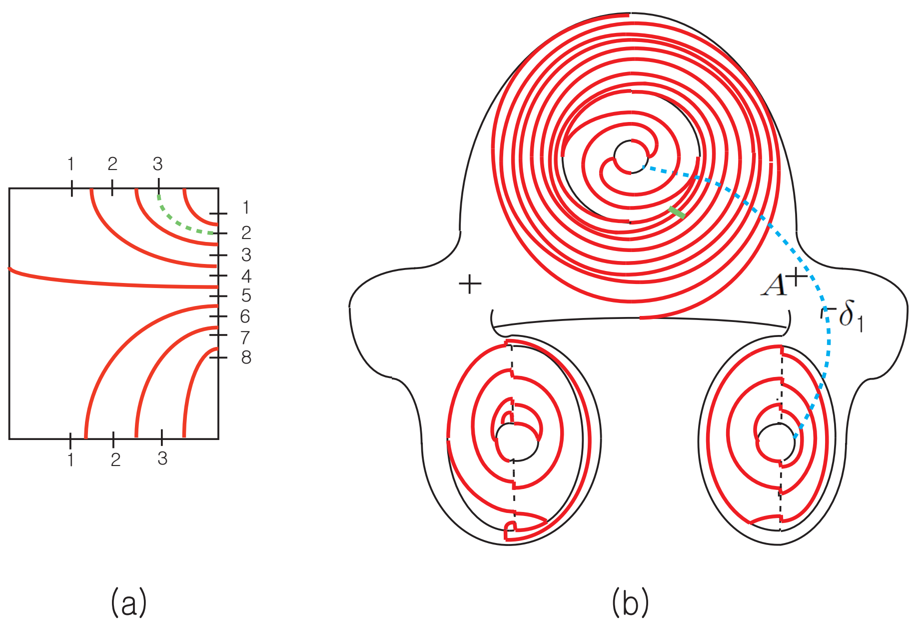


Figure 4.33: $I_{k\in\mathbb{Z}_{<0}}$ case

$\sharp(\delta_2 \times I) \cap \Gamma_{A^-}$	3	5	7	9	11
n$\backslash$ $\lvert k\rvert$	1	2	3	4	5
1	(a) 1 (5)	1 (7)	3	5	7
2	(b) 3 (7)	1 (9)	1	3	5
3	(c) 5 (9)	(a) 3 (11)	1	1	3
4	7 (11)	(b) 5 (13)	3	1	1
5	9 (13)	(c) 7 (15)	(a) 5	3	1
6	11 (15)	9 (17)	(b) 7	5	3
7	13 (17)	11 (19)	(c) 9	(a) 7	5

we have to consider 3 possibilities, (i) yellow colored, (ii) green colored and (iii) red colored case, with respect to the above table :

(i) Yellow colored case : we cut first M' along $\delta_1 \times I$ inefficiently. Since $\sharp(\delta_2 \times I) \cap \Gamma_{\Sigma_1} - \{\sharp(\delta_2 \times I) \cap \Gamma_{A^+} + \sharp(\delta_2 \times I) \cap \Gamma_{A^-}\} \geq 2n - (4|k| + 2) \geq 2$ in this case, There should exist a digging bypass whose attachment arc straddles Γ_{Σ_1} except 1 case. Since this digging bypass does not touch the annulus A, we can find a convex surface $\Sigma_{\frac{1}{2}}$ inside the manifold $[\Sigma_0, \Sigma_1]$ with $\Gamma_{\Sigma_{\frac{1}{2}}} = \tau_\epsilon^{n-1} \circ \gamma_0$ and the I_k type of Γ_{A^+} by attaching this bypass. Hence we can use induction for n. The exceptional case is indicated in Figure 4.33 (a). It is enough to show for the case $n = 4$ and $k = -1$. For this case, we can also make green dashed line a digging bypass which gives layer inside.

(ii) Green colored case : In this case, we take efficient cut along $\delta_2 \times I$. Then, since $\sharp(\delta_2 \times I) \cap \Gamma_{A^-} - \sharp(\delta_2 \times I) \cap \Gamma_{A^+} \geq 4$, there should exist a digging bypass on A^- side. This digging bypass reduce the number of Dehn twists.

(iii) Red colored case : These cases are exceptional cases. We need to consider these cases separately. It is enough to see the following 3 cases, (a) $n = 3$ and $k = -2$ $((2|k| + 1) + (2|k| + 1) - 2n = 4)$, (b) $n = 4$ and $k = -2$ $((2|k| + 1) + (2|k| + 1) - 2n = 2)$ and (c) $n = 5$ and $k = -2$ $((2k + 1) + (2k + 1) = 2n)$, on behalf of all cases :

(a) $n = 3$ and $k = -2$: There are 4 possible configurations of $\Gamma_{(\delta_2 \times I)^+}$ which does not give a digging bypass on A^+, A^- nor Σ_1 side. (Here, attaching a digging bypass on A^+ (A^-) transforms to II_1^+ case (I_{k+1} resp.) and we can find a convex surface inside a manifold whose dividing set is $\tau_\epsilon^{n-1} \circ \gamma_0$ by attaching a bypass on Σ_1.) See Figure 4.34. For the β_1, β_2 cases, we can find a OT disk as shown blue closed curve in Figures 4.34 (b) and (e). For the β_2 case, we can find a long digging bypass on A^+ side. Hence, it can be transformed to $II_1^{\pm}$ case. For the β_3 case, we can find a unique tight contact structure which can not be transformed to other cases. Hence there exist at most one tight contact structure in this case.

(b) $n = 4$ and $k = -2$: See Figure 4.35. In a similar way to the above, we can find a OT disk for the β_3 case, but we cannot find any bypass for the β_1, β_3 cases. Hence there are at most 2 tight contact structures for $n = 4$

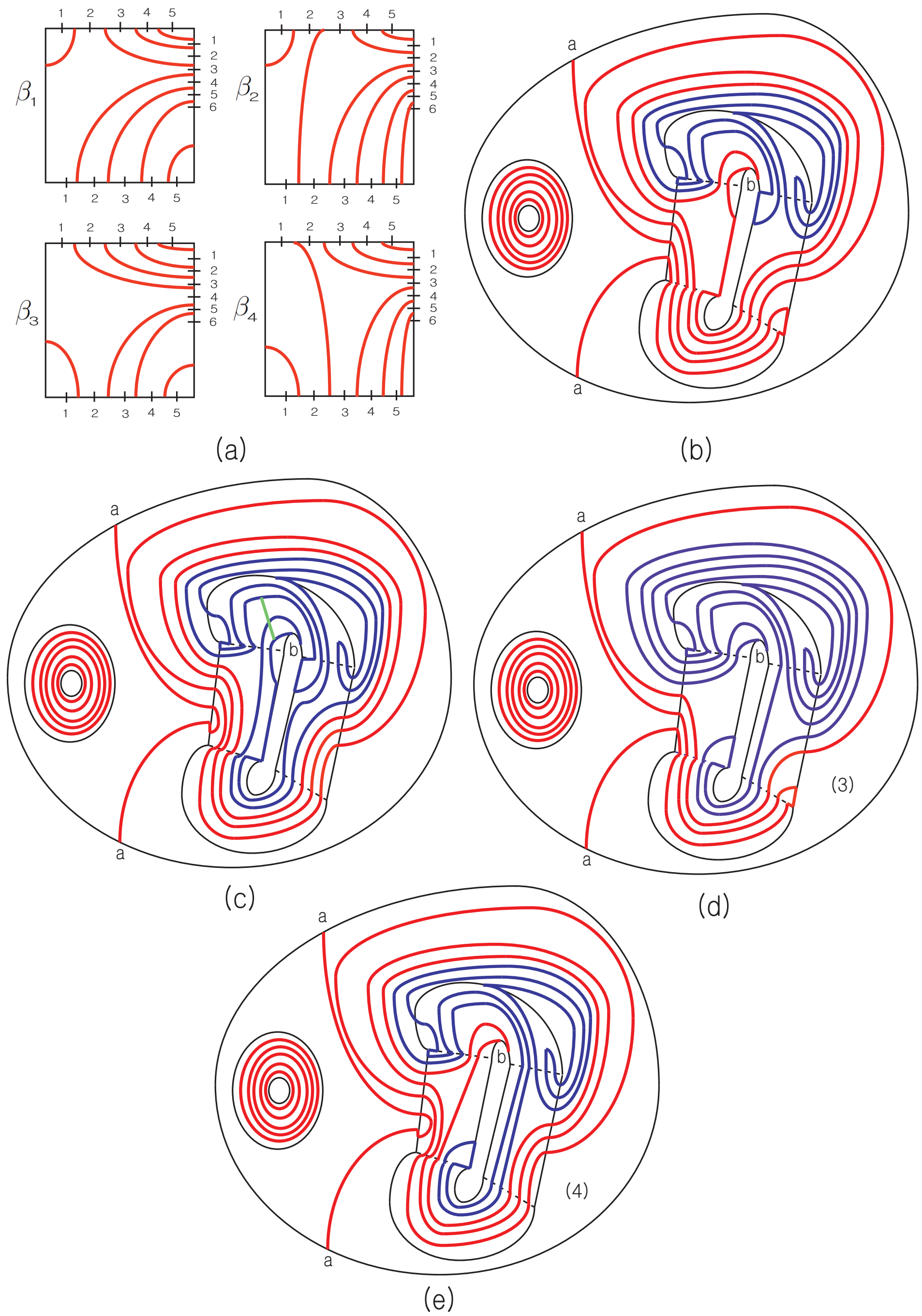


Figure 4.34: $I_{k\in\mathbb{Z}_{<0}}$ exceptional case (a)

and $k = -2$.

(c) $n = 5$ and $k = -2$: See Figure 4.36. There is only one possible dividing curve configuration on $\Gamma_{(\delta_2 \times I)^+}$ which does not give a digging bypass directly on A^+, A^- or Σ_1. We also cannot find a bypass for this case, hence, there exists at most one tight contact structure.

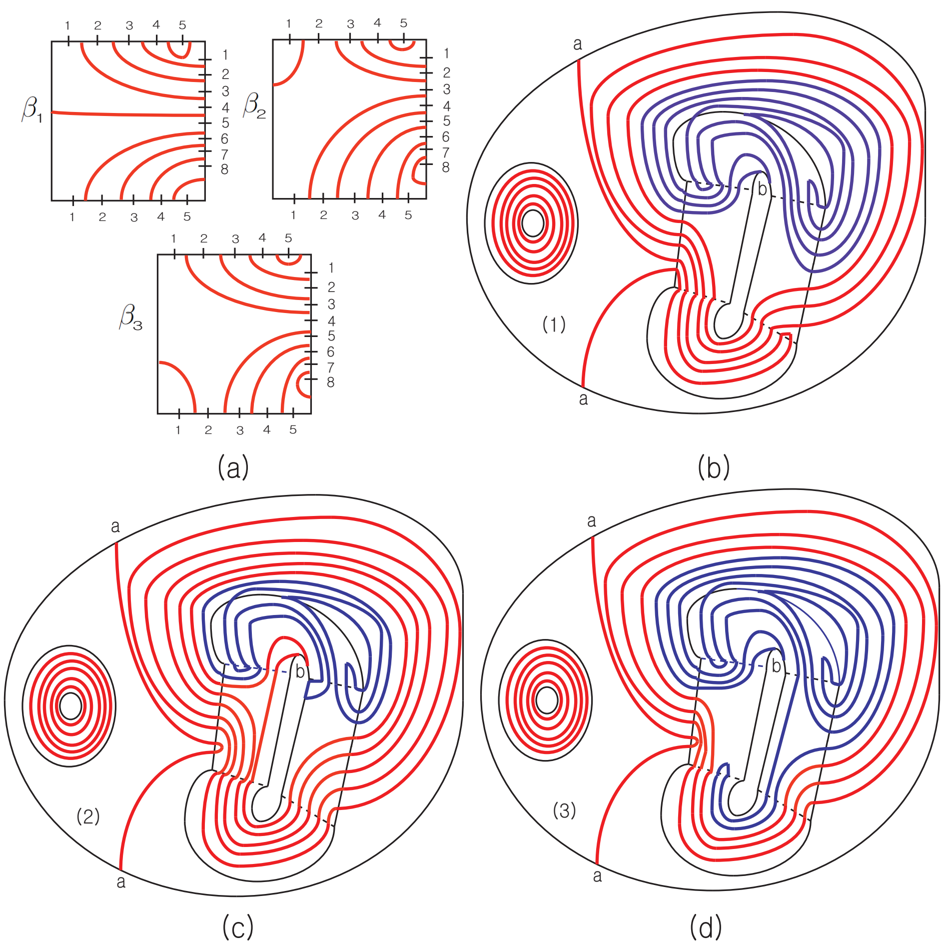
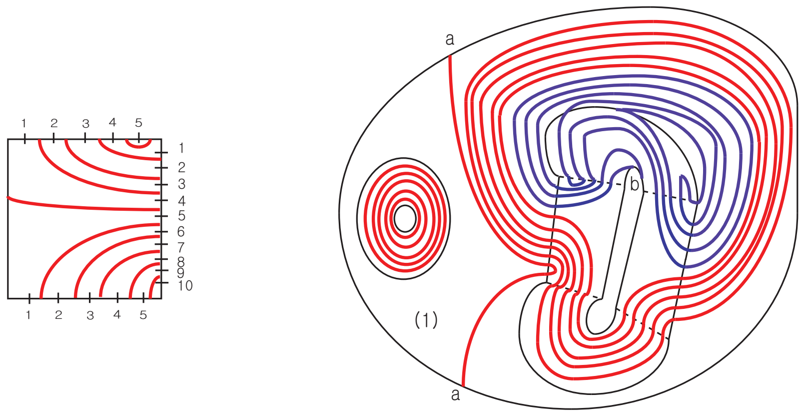


Figure 4.35: $I_{k \in \mathbb{Z}_{<0}}$ exceptional case (b)

We summarize the result for $g = 2$ as the following. See the table :



(a) (b)

Figure 4.36: $I_{k\in\mathbb{Z}_{<0}}$ exceptional case (c)

For $n = 2$, $I_{k<-1}$ can be reduce to the case I_{k+1}.
For $n = 3$,. $I_{k<-2}$ can be reduce to the case I_{k+1}. I_{-2} admits at most unique tight contact structure which does not transformed to I_{k+1} or $II_1^{\pm}$.
For $n = 4$, $I_{k<-2}$ can be reduced to the case I_{k+1}. I_{-2} admits at most two tight contact structure which does not transformed to I_{k+1} or $II_1^{\pm}$.
For $n = 5$, $I_{k<-3}$ can be reduced to the case I_{k+1}. I_{-2}, I_{-3} admit at most two tight contact structure which does not transformed to I_{k+1} or $II_1^{\pm}$, respectively.
For $n = 6$, $I_{k<-4}$ can be reduced to the case I_{k+1}. I_{-2} admits at most 1, I_{-3} admits at most two tight contact structure which does not transformed to I_{k+1} or $II_1^{\pm}$. $\cdots$
Hence, for $n = 2k+1$, there are at most n exceptional tight contact structures, where $k \geq 1$, and for $n = 2k$, there are at most n exceptional tight contact structures, where $k \geq 2$.

□

Lemma 4.5.3. The convex annulus A with dividing curves $II_1^{\pm}$ type can be transformed to the one of with I_0, if $g > 2$ or $g = 2$ and $n > 2$. If $g = 2$ and $n = 2$, then $II_1^{\pm}$ type can be transformed to the one of with I_0 or I_{-1}.

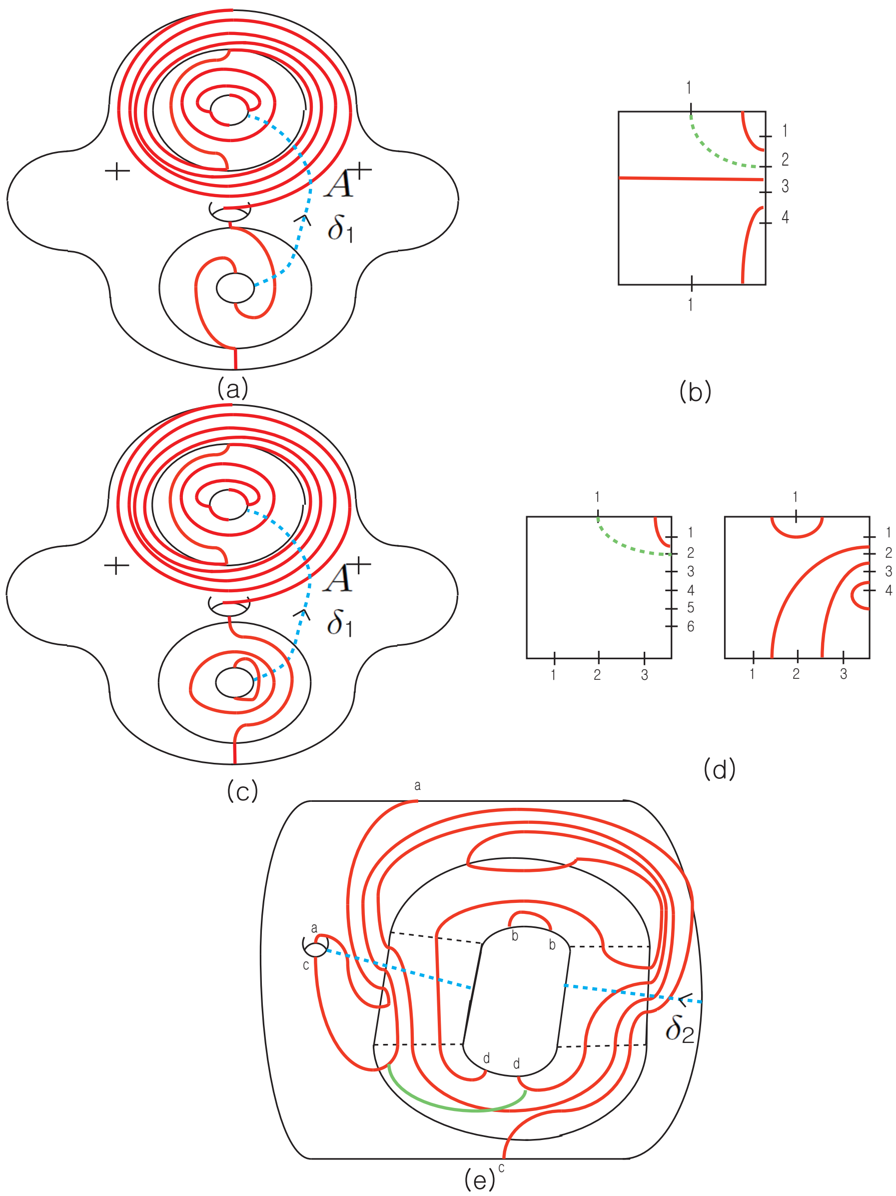


Figure 4.37: II_1^+ case

Proof. Consider the positive case. If $g > 2$, we can get a manifold as shown in Figure 4.37 (a) by genus reducing argument. Then cut resulting manifold along a convex surface $\delta_1 \times I$. Since $(\delta_1 \times I) \cap A^+ = 1$ and $(\delta_1 \times I) \cap A^+ = 1$, we can find a digging bypass inside $\epsilon \times I \subset \Sigma_1$ which does not touch annulus A if $n \geq 3$. If $n = 2$, then only possible configuration which does not give a layer inside manifold is shown in Figure 4. 37 (b). However, for this case, the green dashed line gives a OT disk. Hence we always find a convex surface $\Sigma_{\frac{1}{2}} \subset [\Sigma_0, \Sigma_1]$ with $\Gamma = \tau_\epsilon^{n-1} \circ \gamma_0$ and II_1^+ type. We already showed that II_1^+ type can be transformed to I_0 when $n = 1$ case. Hence II_1^+ case for arbitrary positive integer n can be reduced to I_0 case by induction. The negative case is similar to the positive one.

Consider $g = 2$ case. If $n > 3$, $2n - \{\sharp(\delta_1 \times I) \cap (A^+ \cup A^-)\} = 2n - 4 \geq 4$. Hence there must be exist a digging bypass by which we can find a convex surface $\Sigma_{\frac{1}{2}} \subset [\Sigma_0, \Sigma_1]$ with $\Gamma = \tau_\epsilon^{n-1} \circ \gamma_0$ and II_1^+ type. Therefore, this case always can be transformed to I_0 by induction. For $n = 3$, the only exceptional case which does not give a digging bypass on A^- or Σ_1 must have a boundary parallel like as in the left figure in Figure 4.37 (d) which gives a OT disk. Hence it can be transformed to I_0. If $n = 2$, the right hand side Figure in 4.37 (d) indicates the exceptional case. For this case, we can get a thickened one-punctured torus after cutting and edge-rounding as shown in Figure 4.37 (e). We can find a long digging bypass on A^- indicated as green line in Figure 4.37(e) which gives a state transition to I_{-1}.

□

Lemma 4.5.4. The manifold M^1 with the convex annulus I_0 admits at most 3^n tight contact structures.

Proof. If $n \geq 3$, $2n - \{\sharp(\delta_1 \times I) \cap A^+ + \sharp(\delta_1 \times I) \cap A^-\} \geq 4$. Hence we always find a layer $\Sigma_{\frac{1}{2}}$ with $\Gamma_{\Sigma_{\frac{1}{2}}}$. If $n = 2$, exceptional case of $\Gamma_{(\delta_1 \times I)^+}$ is as shown in figure 4.38 (b). However, this case also gives a digging bypass on Σ_1 by sliding. Hence, by induction, we can find a layer, $[\Sigma_0, \Sigma_{\frac{1}{2^{n-1}}}], \cdots [\Sigma_{\frac{1}{2}}, \Sigma_1]$ with $\Gamma_{\Sigma_{\frac{1}{2^i}}} = \tau_\epsilon^{n-i} \circ \gamma_0$, where $i = 0, \cdots, n$. In section 4.3, we already showed that $([\Sigma_0, \Sigma_{\frac{1}{2^{n-1}}}], I_0)$ admits at most 3 tight contact structures. We also show that

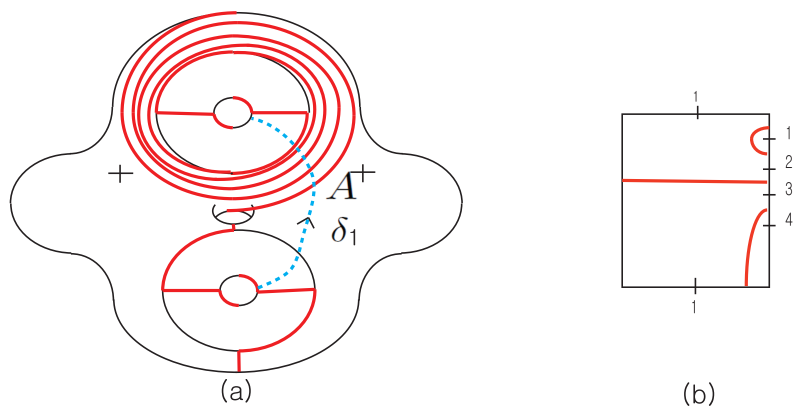


Figure 4.38: I_0 case

$([\Sigma_{\frac{1}{2^i}}, \Sigma_{\frac{1}{2^{i-1}}}], I_0)$ has at most 3 tight contact structure, since we just Dehn-twist $n-i$ times along ϵ in the proof of for the case $([\Sigma_{\frac{1}{2^i}}, \Sigma_{\frac{1}{2^{i-1}}}], I_0)$. Hence $([\Sigma_0, \Sigma_1], I_0)$ has at most 3^n tight contact structures.

□

Lemma 4.5.5. The manifold M with the convex annulus I_{-1} admits at most 3^n or 3^{n-1} tight contact structures if $g > 2$. If $g = 2$, then it admits at most 5 if $n = 2$, 16 if $n = 3$ and $16 \cdot 3^{n-3}$ if $n \geq 4$, tight contact structures.

Proof. We consider divided into three cases, (1) the number of genus in positive and negative region are all nonzero, (2) the number of genus in negative region is zero, (3) $g = 2$ case.
(1) By genus reducing argument, we can get a manifold in Figure 4.39 (b). Cut this manifold along $\delta_1 \times I$. There is one exceptional possibility on $\Gamma_{(\delta_1 \times I)^+}$ which does not give a digging bypass on Σ_1. After edge-rounding, we can get a thickened one-punctured torus as shown in Figure 4.39(d). There are two possibilities for decomposing this manifold into 3-ball with one dividing curve, but one of which has a digging bypass on Σ_1. There is at most one tight contact structure which does not have a layer, we call this structure ξ. Hence by induction, we can have two kind of layering, (a) $([\Sigma_0, \Sigma_{\frac{1}{2^{n-2}}}], \xi), \cdots ([\Sigma_{\frac{1}{2}}, \Sigma_1], I_0)$ with $\Gamma_{\Sigma_{\frac{1}{2^i}}} = \tau_\epsilon^{n-i} \circ \gamma_0$ and (b) $([\Sigma_0, \Sigma_{\frac{1}{2^{n-1}}}], I_{-1}), \cdots ([\Sigma_{\frac{1}{2}}, \Sigma_1], I_0)$ with $\Gamma_{\Sigma_{\frac{1}{2^i}}} = \tau_\epsilon^{n-i} \circ \gamma_0$, where $i =$

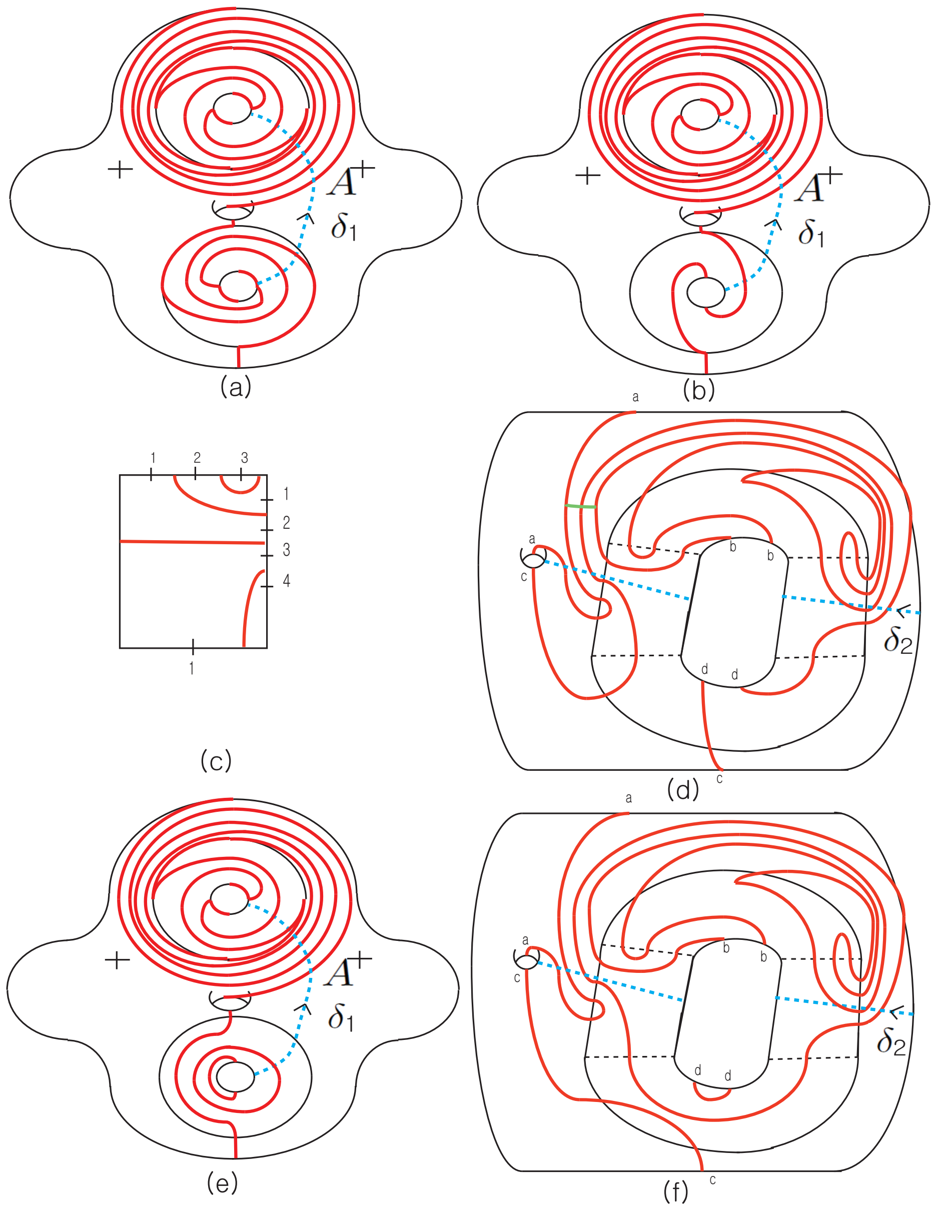


Figure 4.39: I_{-1} case

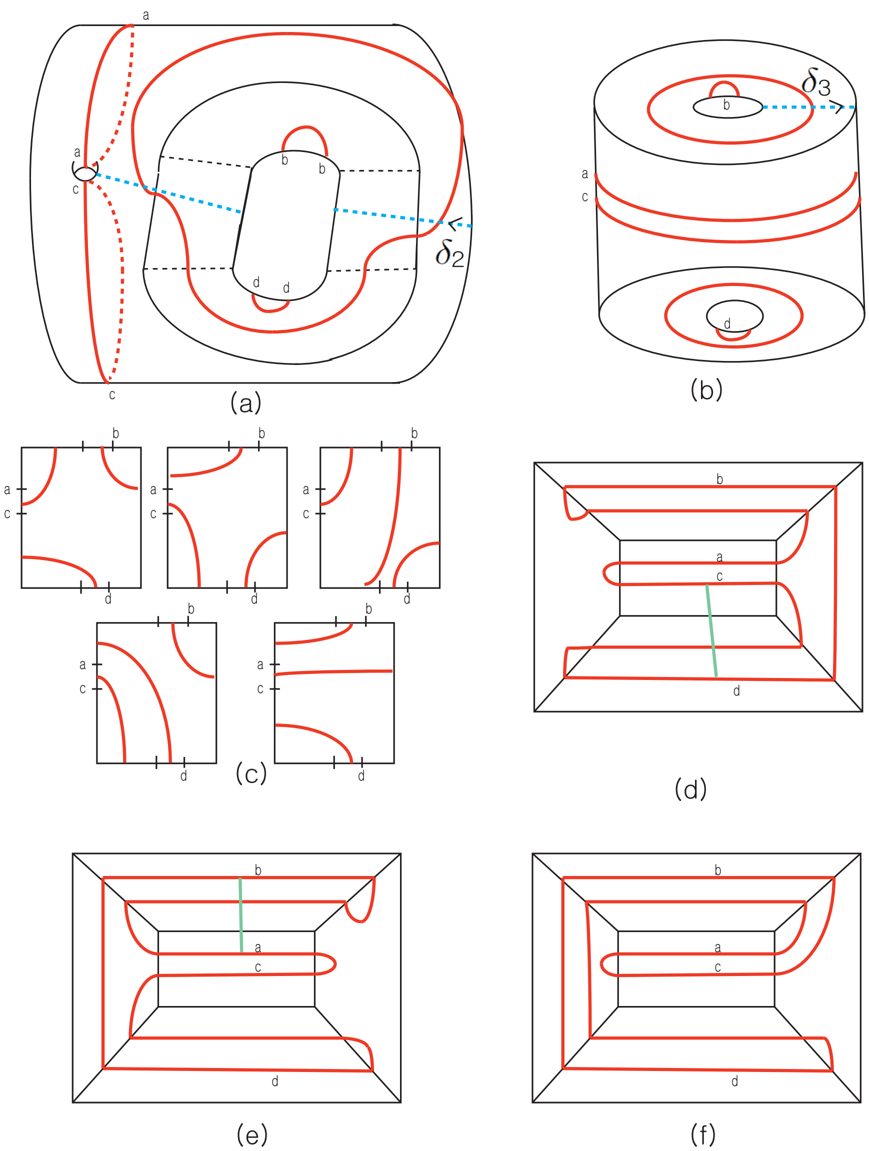


Figure 4.40: I_{-1} case

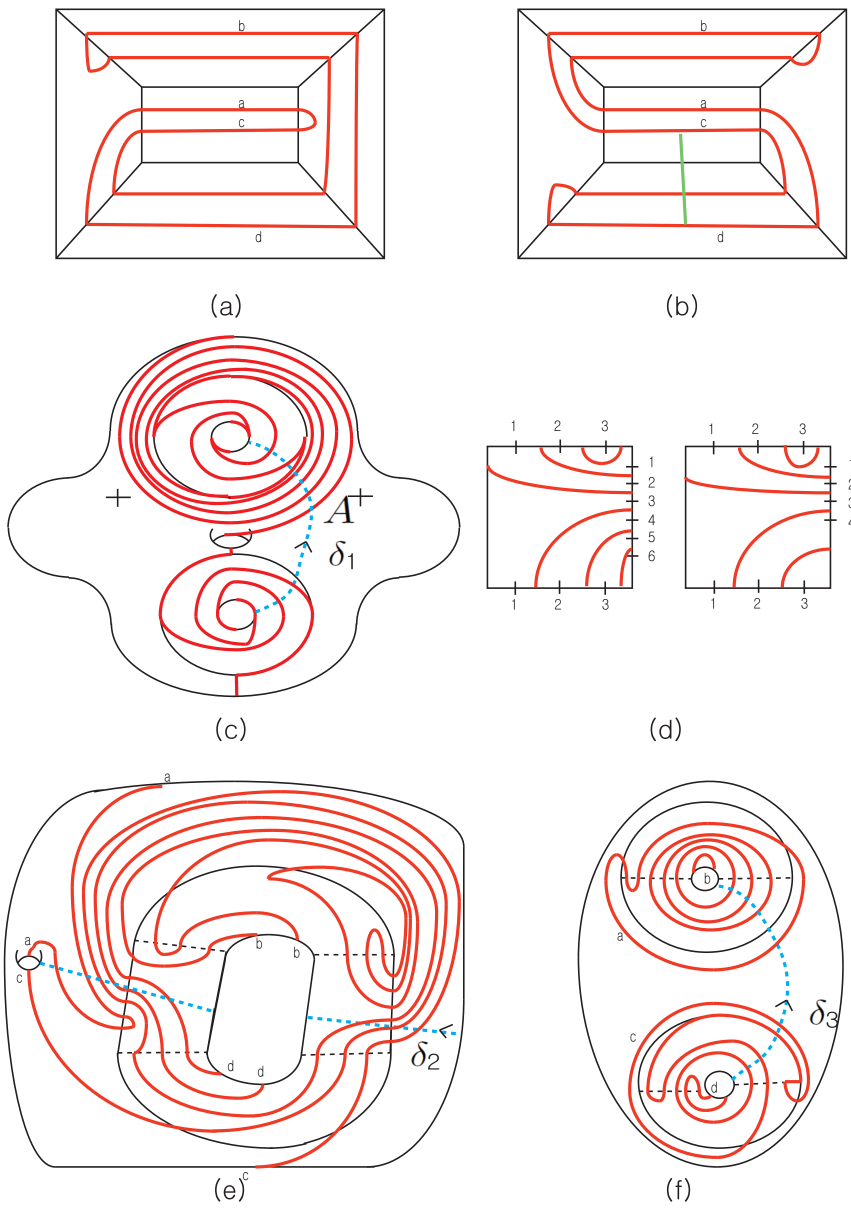


Figure 4.41: I_{-1} case

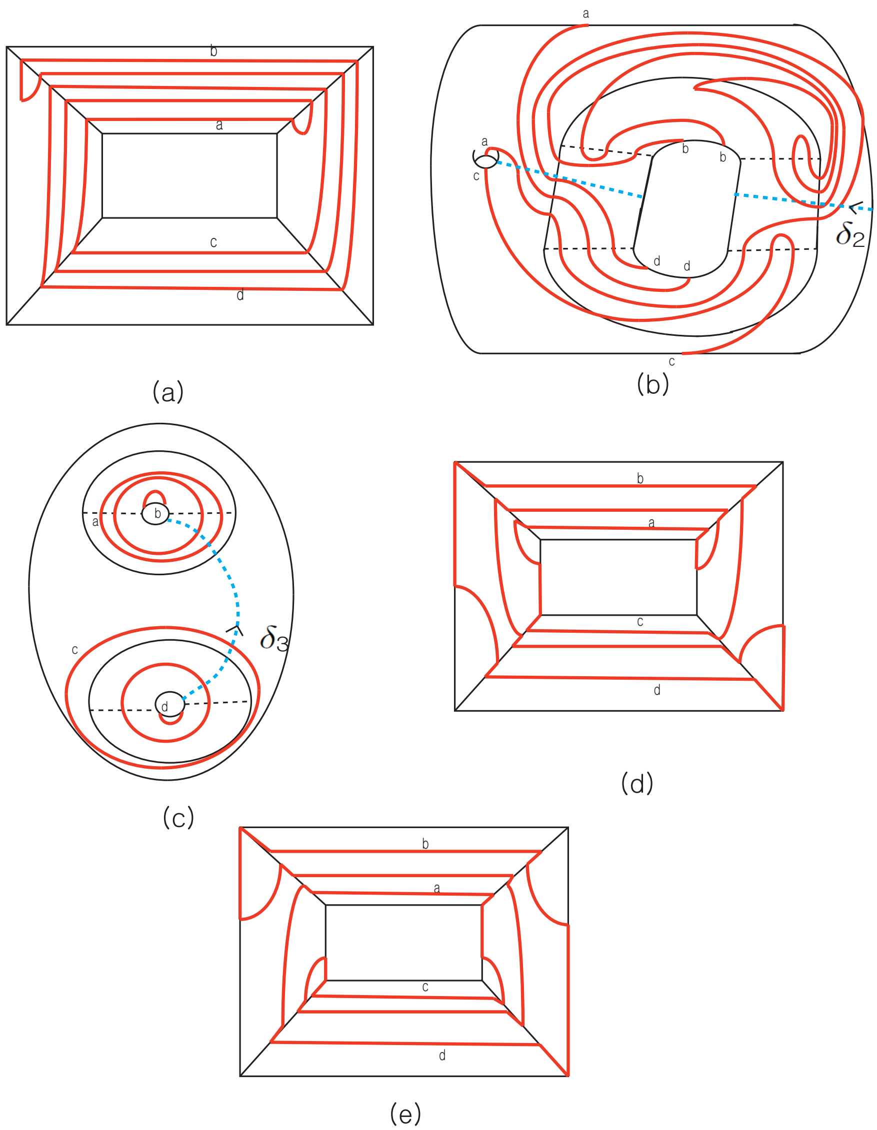


Figure 4.42: I_{-1} case

$1, \cdots, n-1$. For (a) case, M admits at most 3^{n-1} and, for (b) case, admits at most 3^n case.

(2) See Figure 4.39 (e). If $n \geq 3$, there must be a digging bypass which gives a layer. If $n = 2$, there is one exceptional possibility which does not give a digging bypass on Σ_1 which is same as the one of (1). After cutting along $\delta_1 \times I$ and edge-rounding, we can get a figure in Figure 4.39 (f). Cut again efficiently along $\delta_2 \times I$. Then we have a solid torus with 6 longitudal dividing curves. There are 5 possibilities for $\Gamma_{(\delta_3 \times I)^+}$. From Figure 4.40 (d) to 4.41 (b) represents the resulting 3-ball after cutting along $\delta_3 \times I$ with dividing configuration β_i in turn. We can find a digging trivial bypass for $\beta_1, \beta_2, \beta_5$. Trivial bypasses for β_1, β_5 gives a digging bypass on A^-, hence, it can be transformed to I_0. A trivial bypass for β_2 gives a digging bypass for layer. The other 2 cases, we have tight contact structures, we call these ξ_1, ξ_2. Hence, we have the following three kind of layering :
(a) $([\Sigma_0, \Sigma_{\frac{1}{2^{n-2}}}], \xi_1), \cdots ([\Sigma_{\frac{1}{2}}, \Sigma_1], I_0)$ with $\Gamma_{\Sigma_{\frac{1}{2^i}}} = \tau_\epsilon^{n-i} \circ \gamma_0$,
(b) $([\Sigma_0, \Sigma_{\frac{1}{2^{n-2}}}], \xi_2), \cdots ([\Sigma_{\frac{1}{2}}, \Sigma_1], I_0)$ with $\Gamma_{\Sigma_{\frac{1}{2^i}}} = \tau_\epsilon^{n-i} \circ \gamma_0$,
(c) $([\Sigma_0, \Sigma_{\frac{1}{2^{n-1}}}], I_{-1}), \cdots ([\Sigma_{\frac{1}{2}}, \Sigma_1], I_0)$ with $\Gamma_{\Sigma_{\frac{1}{2^i}}} = \tau_\epsilon^{n-i} \circ \gamma_0$,
where $i = 1, \cdots, n-1$. For (a) and (b) case, we have at most 3^{n-1}, and, for (c) case, we have at most 3^n tight contact structures.

(3) See Figure 4.41 (c). If $n \geq 4$, there should exist a digging bypass on Σ_1. If $n = 3$, the only possible dividing configuration which does not give a digging bypass on A^+, A^- or Σ_1 is indicated in left hand side of Figure 4.41 (d). After cutting along $\delta_1 \times I$ and edge-rounding, we can get a Figure in 4.41 (e). Cut this manifold along $\delta_2 \times I$ inefficiently. There is only one possibility for $\Gamma_{(\delta_2 \times I)^+}$ which does not give a digging bypass on A^- or Σ_1. Then we can get a solid torus with 8 longitudal dividing curves after cutting along $\delta_2 \times I$ and edge-rounding. Cut the solid torus along $\delta_3 \times I$ efficiently. Then the only one possible dividing curve on $(\delta_3 \times I)^+$ which does not give a digging bypass on the top annulus or the bottom annulus (They give a state transition to I_0 or layer) can be decomposed to a 3-ball with one dividing curve. Hence it admits tight contact structure by Eliashberg's theorem. Hence, for $g = 2$ and

$n = 3$ case, there exists at most unique tight contact structure which cannot be transformed to the other case or have a layer inside. We call this tight contact structure ξ_3. Similarly, there exist two tight contact structure for the case $g = 2$ and $n = 2$. See Figure from 4.42 (b) to (e). We denote these ξ_4, ξ_5.

We conclude that the following.
(a) If $g = 2$, $n = 2$, then there exits three kind of layering, $([\Sigma_0, \Sigma_1], \xi_4)$, $([\Sigma_0, \Sigma_1], \xi_5)$ and $([\Sigma_0, \Sigma_{\frac{1}{2}}, I_{-1}), ([\Sigma_{\frac{1}{2}}, I_0)$, with $\Gamma_{\Sigma_{\frac{1}{2}}} = \tau_\epsilon^1 \circ \gamma_0$. Hence there are at most 5 contact structures.
(b) If $g = 2, n = 3$, then ehre exist $([\Sigma_0, \Sigma_1], \xi_3)$, or $([\Sigma_0, \Sigma_{\frac{1}{2}}], I_{-1}), ([\Sigma_{\frac{1}{2}}, \Sigma_0], I_0)$, with $\Gamma_{\Sigma_{\frac{1}{2}}} = \tau_\epsilon^2 \circ \gamma_0$. Hence, there are at most 16 tight contact structures.
(c) If $g = 2$, $n \geq 4$, then $([\Sigma_0, \Sigma_{\frac{1}{2^{n-3}}}], I_{-1}), \cdots ([\Sigma_{\frac{1}{2}}, \Sigma_1], I_0)$ with $\Gamma_{\Sigma_{\frac{1}{2^i}}} = \tau_\epsilon^{n-i} \circ \gamma_0$, where $i = 1, \cdots, n-3$. Hence, there are at most $16 \times 3^{n-3}$ tight contact structures.

□

Proof of Proposition 4.1.3. We showed that I_0 admit at most 3^n tight contact structures and I_{-1} admit at most 3^n or 3^{n-1} tight contact structures if $g > 2$and 5 if $n = 2$ and $g = 2$, 16 if $n = 3$ and $g = 2$, and $16 \cdot 3^{n-3}$ if $n \geq 4$ and $g = 2$. Hence we can conclude as the following :

$$\sharp\pi_0(Tight(M, \mathcal{F})) \leq \begin{cases} 2 \times 3^n & \text{if } g \geq 3, \\ 3^n + 5 & \text{if } g = 2,\ n = 2, \\ 3^n + 16 + 1 & \text{if } g = 2,\ n = 3, \\ 3^n + 3^{n-3} + m & \text{if } g = 2,\ n \geq 4,\ n = 2m+1, \\ 3^n + 3^{n-3} + m & \text{if } g = 2,\ n \geq 4,\ n = 2m. \end{cases} \tag{4.5.1}$$

□

4.6 Proof of Proposition 4.1.7

Strategy of the proof of Proposition 4.1.7

Step 1 If $k > 0$, then the I_k case can be reduced to the $II_1^{\pm}$ case.

Step 2 Suppose $k < -1$. If $g > 2$, then the I_k case can be reduced to the I_{k+1} case. If $g = 2$, then then the I_k case can be reduced to the I_{k+1} case or $II_1^{\pm}$ case. (cf. Lemma 4.6.1).

Step 3 If $k > 0$, then $II_{2k}^{\pm}$ can be reduced to $II_{2k-2}^{\pm}$ case and $II_{2k+1}^{\pm}$ can be reduced to $II_{2k-1}^{\pm}$.

Step 4 The manifold M with the convex annulus A of dividing curves II_0^{+} (resp. -) type cannot admit a tight contact structure.

Step 5 The manifold M^1 with the convex annulus A of dividing curves I_0 admits at most 4 tight contact structures. (cf. Lemma 4.6.2).

Step 6 The convex annulus A with dividing curves $II_1^{\pm}$ type can be transformed to the one of with I_0 or The manifold M^1 with the convex annulus A of dividing curves $II_1^{\pm}$ type admits at most 16 tight contact structures if $g > 2$ and 24 if $g = 2$. (cf. Lemma 4.6.3).

Step 7 The manifold M^1 with the convex annulus I_{-1} admits at most 16 if the number of genus on negative and positive region are all nonzero, 8 if one of the either is zero, 6 if $g = 2$. (cf. Lemma 4.6.4)

Since the proofs of step 1, 3 and 4 are same as the one of $n = -1$, we omit these.

Lemma 4.6.1. Suppose $k < -1$. If $g > 2$, then the I_k case can be reduced to the I_{k+1} case. If $g = 2$, then then the I_k case can be reduced to the I_{k+1} case or $II_1^{\pm}$ case.

Proof. The proof is similar to the one of step 2 for the $n > 1$ case. First cut M^1 along $\delta_1 \times I$ like as in Figure 4.32 (a). Then it suffices to consider the second case in (b). Then we can get Figure 4.43 (a). Denotel this manifold M^2. If $g > 2$, then we can use genus reducing argument by which we can reduce the number of genus. We assume that $g = 2$. Then we take an inefficient convex surface when cutting M^2. In a similar way as the one in the proof of step 2 for $n > 1$, we can divide into three zone depending on

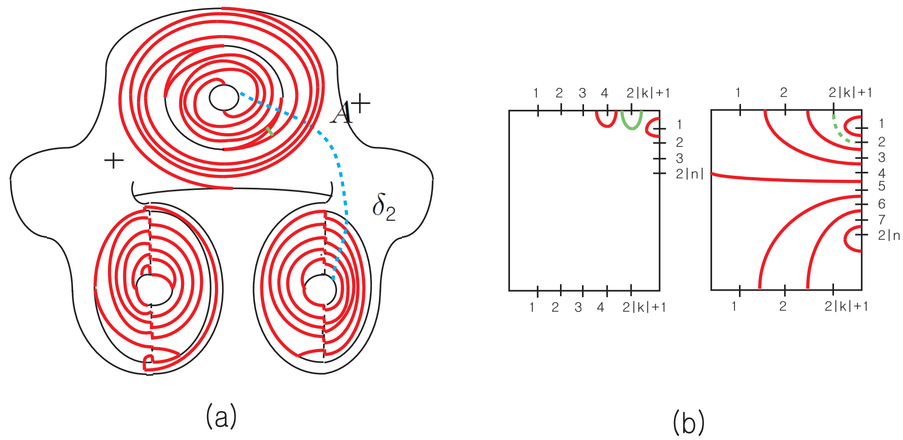


Figure 4.43: $I_{k\in\mathbb{Z}_{<0}}$ case

the intersections numbers. See the following table. The number on each cell means $\sharp\delta_2 \times I \cap (\Gamma_{A^+} \cup \Gamma_{\Sigma_1})$, when we cut efficiently, and the number in parenthesis means $\sharp\delta_2 \times I \cap (\Gamma_{A^+} \cup \Gamma_{\Sigma_1})$, when we cut inefficiently.

$\sharp(\delta_2 \times I) \cap \Gamma_{A^-}$	3	5	7	9	11
n	1	2	3	4	5
1	1 (5)	3 (7)	5 (9)	7 (11)	9 (13)
2	1 (7)	1 (9)	3 (11)	5 (13)	7 (15)
3	3 (9)	(a)11	13	15	17
4	5 (11)	(b)13	15	17	19
5	13	(c)15	17	19	21
6	15	17	19	21	23
7	17	19	21	23	25

We have to consider 3 possibilities, (i) yellow colored, (ii) green colored and (iii) red colored zone, with respect to the above table :
(i) Yellow colored zone : we cut first M' along $\delta_2 \times I$ inefficiently. Since $\sharp(\delta_2 \times I) \cap \Gamma_{\Sigma_1} - \{\sharp(\delta_2 \times I) \cap \Gamma_{A^+} + \sharp(\delta_2 \times I) \cap \Gamma_{A^-}\} \geq 2n - (4|k| + 2) \geq 2$ in this case, There should exist a digging bypass whose attachment arc straddles Γ_{Σ_1} except 1 case. Since a digging bypass on Σ_1 does not touch the

annulus A, we can find a convex surface $\Sigma_{\frac{1}{2}}$ inside the manifold $[\Sigma_0, \Sigma_1]$ with $\Gamma_{\Sigma_{\frac{1}{2}}} = \alpha \cup \beta \cup \gamma$ and the $I_{k<0}$ type of Γ_{A^+} by attaching this bypass. The homotopy class of $\alpha \cup \beta \cup \gamma$ can be divided into two cases. See Figures 4.44 (a) and (b). For either case, we can find a convex surface $\delta_2 \times I$ which intersects with Γ_{A^+} at one points and with Γ_{A^-} at $2|k| + 1$ points by homoping a dividing curve on A^+. Therefore, since the difference between their intersection numbers is greater than 4, we can find a digging bypass on A^- side, which reduce the number of Dehn twists. The exceptional case is indicated in Figure 4.43 (b). However, it gives a OT disk by attaching green dashed line bypass.

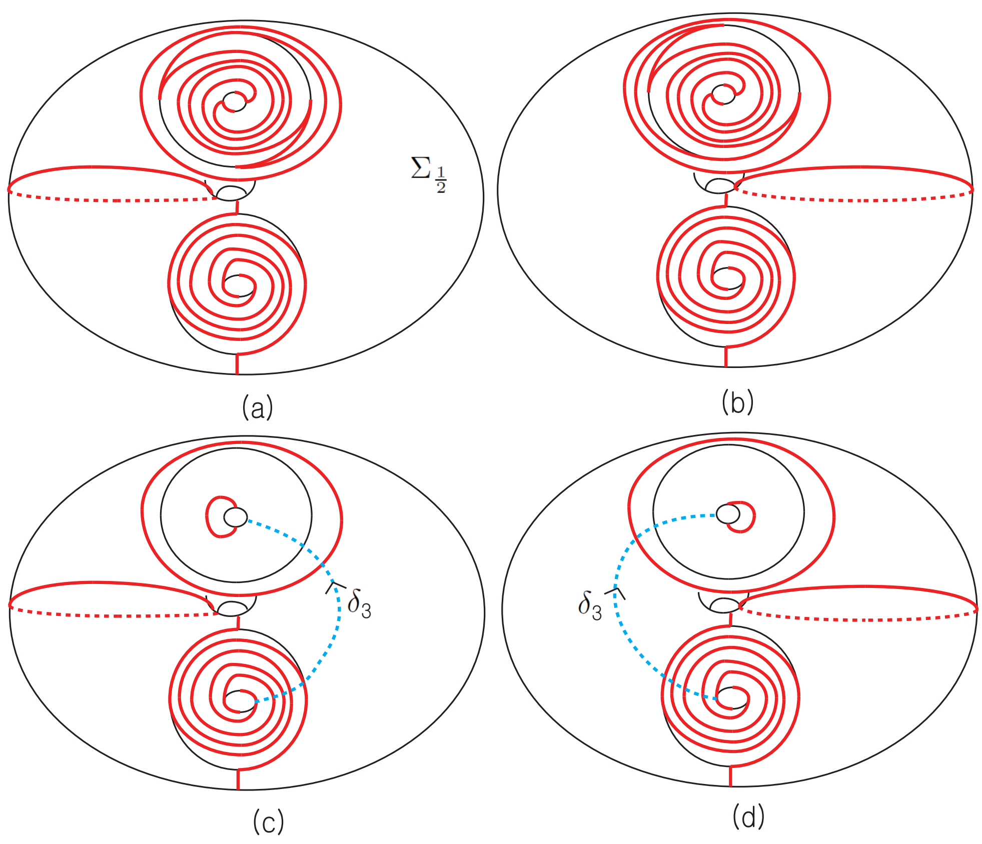


Figure 4.44: Yellow zone

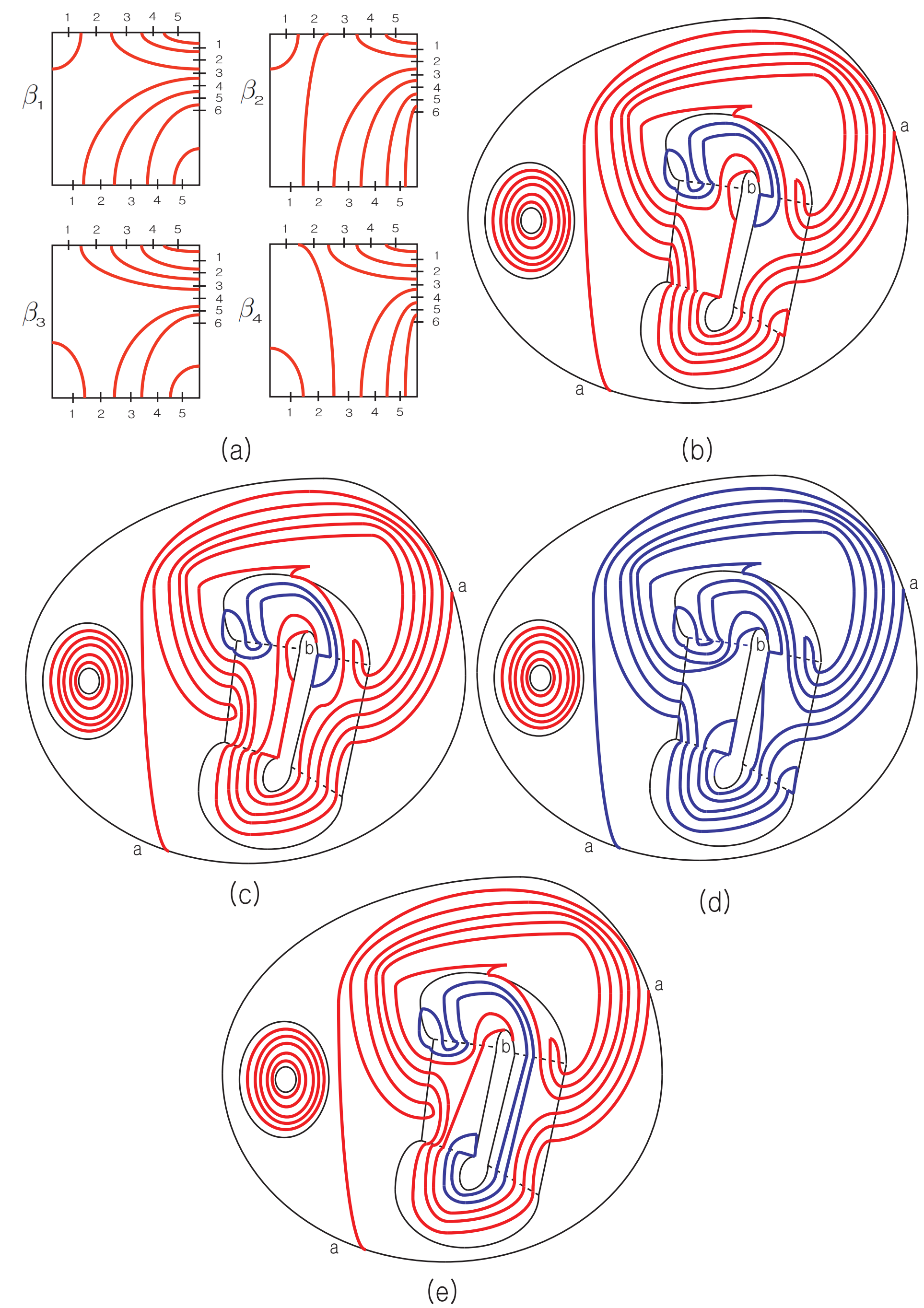


Figure 4.45: $I_{k\in\mathbb{Z}_{<0}}$ exceptional case (a)

(ii) Green colored zone : In this case, we take efficient cut along $\delta_2 \times I$. Then, since $\sharp(\delta_2 \times I) \cap \Gamma_{A^-} + \sharp(\delta_2 \times I) \cap \Gamma_{A^+} - 2|n| \geq 6$, there should exist a digging bypass on A^+ side or A^-. Hence it can be reduced to I_{k+1} or $II_1^{\pm}$.

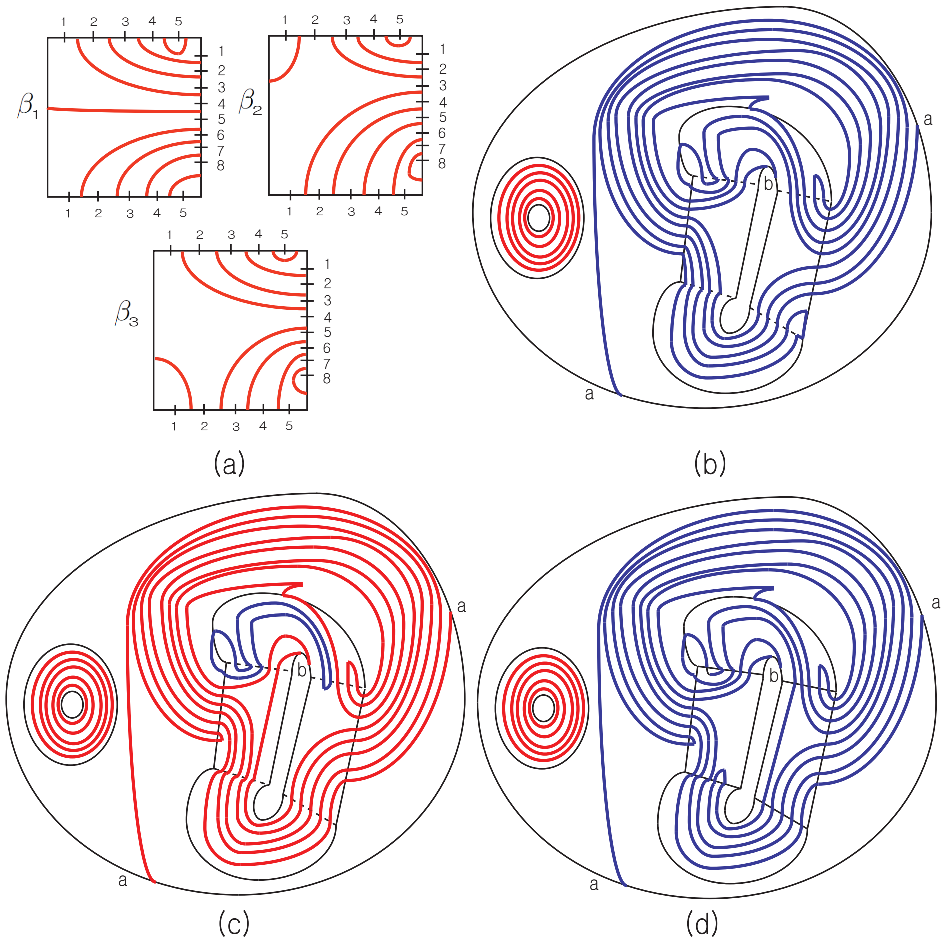
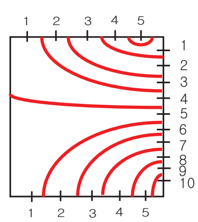


Figure 4.46: $I_{k\in\mathbb{Z}_{<0}}$ exceptional case (b)

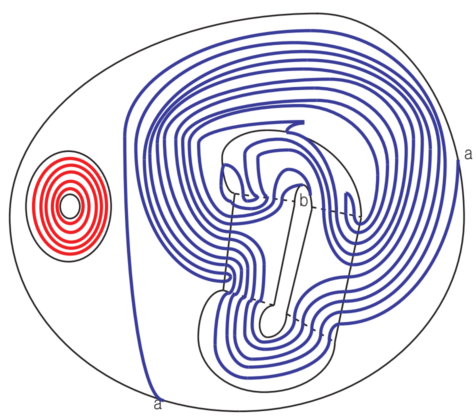
(iii) Red colored zone : These cases are exceptional cases. We need to consider these cases separately. It is enough to see the following 3 cases, (a) $n = 3$ and $k = -2$ $((2|k|+1)+(2|k|+1)-2n = 4)$, (b) $n = 4$ and $k = -2$ $((2|k|+1)+(2|k|+1)-2n = 2)$ and (c) $n = 5$ and $k = -2$

$((2k+1)+(2k+1)=2n)$, on behalf of all cases :

(a) $n=3$ and $k=-2$: There are 4 possible configurations of $\Gamma_{(\delta_2\times I)^+}$ which does not give a digging bypass on A^+, A^- nor Σ_1 side. (Here, attaching a digging bypass on A^+ (A^-) transforms to II_1^+ case (I_{k+1} resp.) and we can find a convex surface inside a manifold whose dividing set is disjoint union of nonseparating closed curve α, β, γ by attaching a bypass on Σ_1.) See Figure 4.45. For the $\beta_1, \beta_2, \beta_3$ cases, we can find a OT disk as shown blue closed curve in Figures 4.45 (b), (c) and (e). For the β_3 case, we can have a solid torus with $\sharp\Gamma_{\text{left annulus}} = 2|k|+1$ and $\sharp\Gamma_{\text{right annulus}} = 1$ after edge-rounding. Hence, we can find a digging bypass on A^- side which reduce the number of core curves.



(a)



(b)

Figure 4.47: $I_{k\in\mathbb{Z}_{<0}}$ exceptional case (c)

(b) $n=4$ and $k=-2$: See Figure 4.46. In a similar way to the above, we can find a OT disk for the β_2 case. For the β_1, β_3 cases, since we have a solid torus with $\sharp\Gamma_{\text{left annulus}} = 2|k|+1$ and $\sharp\Gamma_{\text{right annulus}} = 1$ after edge-rounding, we can reduce the number of Dehn twists.

(c) $n=5$ and $k=-2$: See Figure 4.47. There is only one possible dividing curve configuration on $\Gamma_{(\delta_2\times I)^+}$ which does not give a digging bypass directly on A^+, A^- or Σ_1. In this case, since we can also have a solid torus with $\sharp\Gamma_{\text{left annulus}} = 2|k|+1$ and $\sharp\Gamma_{\text{right annulus}} = 1$ after edge-rounding, we can reduce the number of Dehn twists.

Hence, for $g = 2$ case, we can always reduce to the case I_{k+1} or $II_1^{\pm}$. □

Lemma 4.6.2. The manifold M^1 with the convex annulus A of dividing curves I_0 admits at most 4 tight contact structures.

Proof. If $n > 2$, then we can always find a digging bypass (It does not touch the annulus A) on Σ_1 by which we can find a convex surface $\Sigma_{\frac{1}{2}}$ with $\Gamma_{\Sigma_{\frac{1}{2}}} = \alpha \cup \beta \cup \gamma$ as shown in Figure 4.48 (c). There are two possibilities for the homotopy class of $\alpha \cup \beta \cup \gamma$. For either case, the proof is same. We only consider the case in Figure 4.48 (c). We cut a manifold $([\Sigma_0, \Sigma_{\frac{1}{2}}], I_0)$ along $\delta_2 \times I$. Then we can get Figure 4.48 (e). Since there are two possibilities to decompose resulting manifold into a 3-ball, we have at most 2 tight contact structures for $([\Sigma_0, \Sigma_{\frac{1}{2}}], I_0)$. We can argue similarly for $([\Sigma_{\frac{1}{2}}, \Sigma_0], I_0)$. For this case, we also get at most 2 tight contact structures. Hence, (M, I_0) admits at most 4 tight contact structures. □

Lemma 4.6.3. The convex annulus A with dividing curves $II_1^{\pm}$ type can be transformed to the one of with I_0 or The manifold M^1 with the convex annulus A of dividing curves $II_1^{\pm}$ type admits at most 16 tight contact structures if $g > 2$ and 24 if $g = 2$.

Proof. Consider two cases, $g > 2$ and $g = 2$. For the $g > 2$ case, we can get Figure 4.49 (a) by genus reducing argument. If $n > 2$, then we always find a digging bypass on Σ_1. Hence we can find a convex surface $\Sigma_{\frac{1}{2}}$ with three nonseparating closed dividing curves. We consider this at the end. If $n = 2$, then there is only one possibility for $\Gamma_{(\delta_1 \times I)^+}$ which does not give a digging bypass on Σ_1. Then we can get a OT disk by edge-rounding as shown in Figure 4.49 (c). For the $g = 2$ case, there is 5 exceptional case. First consider $n = 2$ case. Then there are 3 possibilities which does not give a digging bypass on A^- or Σ_1. From Figure 4.49 (f) to 4.50 (b) represents

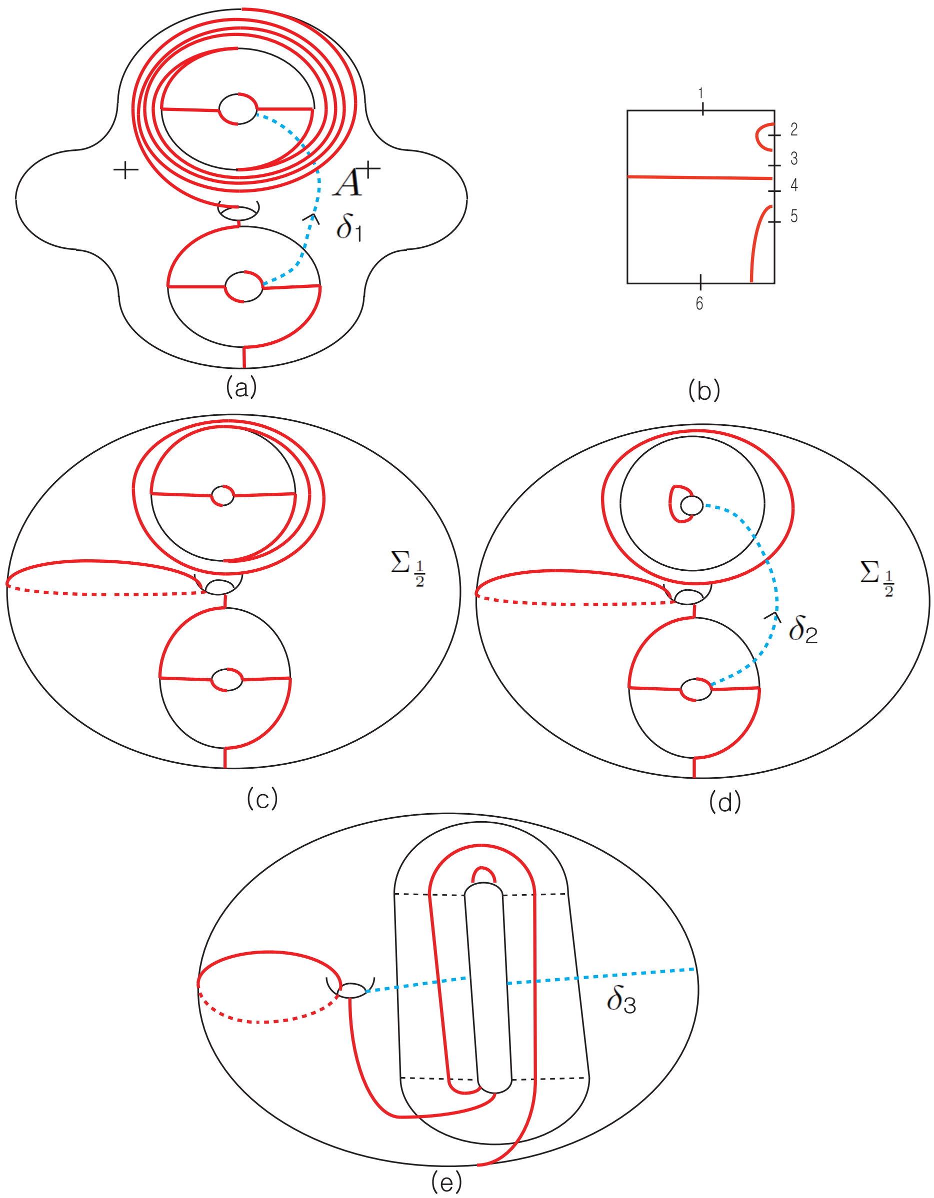


Figure 4.48: I_0 case

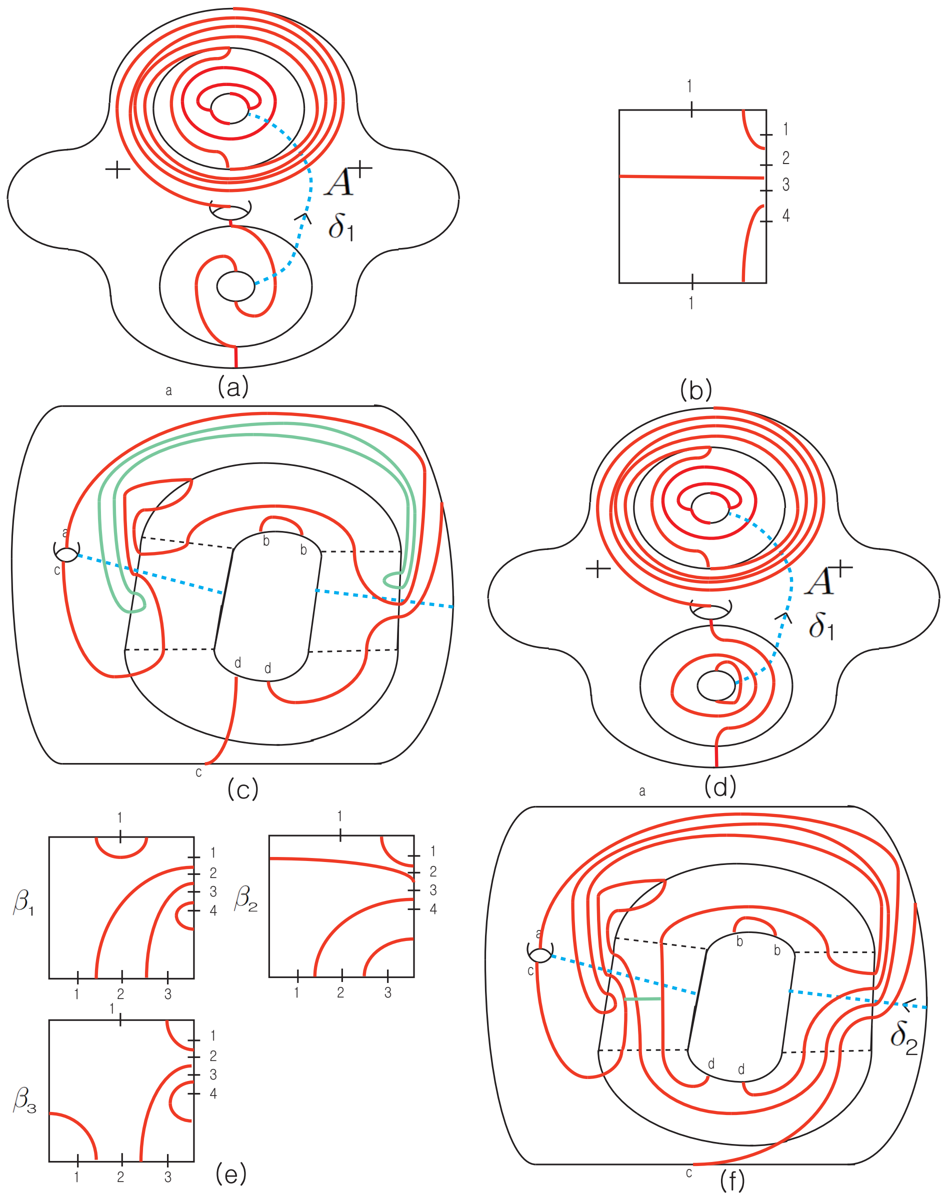


Figure 4.49: II_1^+ case

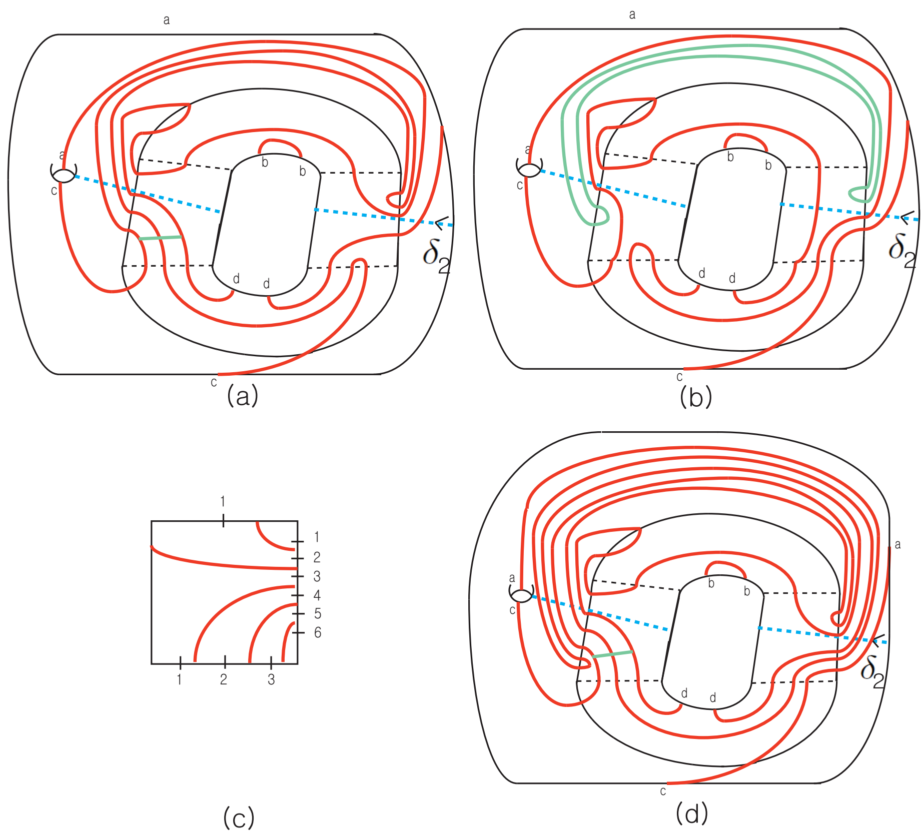


Figure 4.50: II_1^+ case

for the cases $\beta_1, \cdots, \beta_3$ in turn.

For the β_1 case, cut a thickened one-punctured torus along a convex surface $\delta_2 \times I$ inefficiently. Then we can find a trivial digging bypass which gives a state transition digging bypass on A^- to I_0. For the β_2 case, we also take an inefficient cut along $\delta_2 \times I$. Then we can find a state transition bypass to I_0. We can find a OT disk directly for the β_3 case. For $n = 3$, there is only one possible dividing curve configuration of $\delta_1 \times I$. In this case, we can also find a trivial bypass after cutting along $\delta_2 \times I$ inefficiently which can be slided into A^-. Hence it can be transformed to I_0.

From now on, we consider the case that there exists a convex surface $\Sigma_{\frac{1}{2}}$ inside M with three nonseparating closed dividing curves, i.e., there is a layering $([\Sigma_0, \Sigma_{\frac{1}{2}}], I_{-1}), ([\Sigma_{\frac{1}{2}}, \Sigma_1], I_0)$ with $\Gamma_{\Sigma_{\frac{1}{2}}} = \alpha \cup \beta \cup \gamma$. Then $([\Sigma_0, \Sigma_{\frac{1}{2}}], I_{-1})$ has 4 possible disk decomposition into a 3-ball with one dividing curve if $g > 2$ and 6 possible disk decompositions if $g = 2$. We showed that $([\Sigma_{\frac{1}{2}}, \Sigma_1], I_0)$ has 2 possible disk decompositions into a 3-ball with one dividing curve in Lemma 4.6.2 and the possibilities for $\Sigma_{\frac{1}{2}}$ is 2. Hence $II_1^{\pm}$ which cannot be transformed to I_0 admit at most 16 tight contact structures if $g > 2$ and 24 if $g = 2$.

□

Lemma 4.6.4. The manifold M^1 with the convex annulus I_{-1} admits at most 16 if the number of genus on negative and positive region are all nonzero, 8 if one of the either is zero, 6 if $g = 2$.

Proof. If the number of genus of negative and positive region are all nonzero, then we can get Figure 4.51 (b) by genus reducing argument. If $n \geq 3$, then we can always find a digging bypass on Σ_1. Hence there is only one possible dividing curve configuration of $(\delta_1 \times I)^+$ which does not give a bypass on Σ_1 (A bypass on A^+ gives a OT disk). However, we can find OT disk by cutting along $\delta_1 \times I$ and edge-rounding. If one of the number of genus of negative or positive region is zero, then we can get Figure 4.51 (e). This case also one possibility and give a OT disk. If $g = 2$, then we can find a digging bypass on Σ_1 if $n \geq 4$. Hence there are two exceptional cases. For the β_1, we can find a

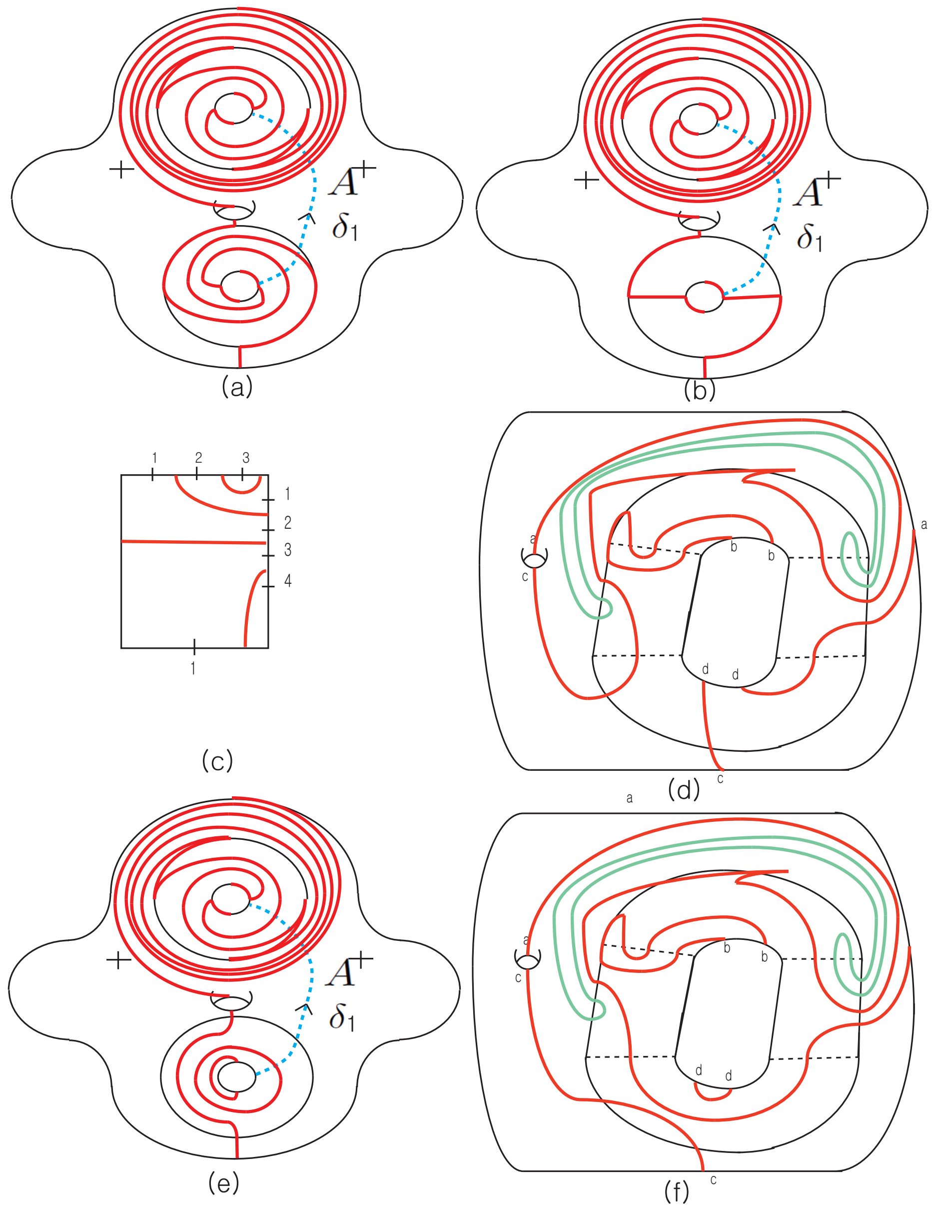


Figure 4.51: I_{-1} case

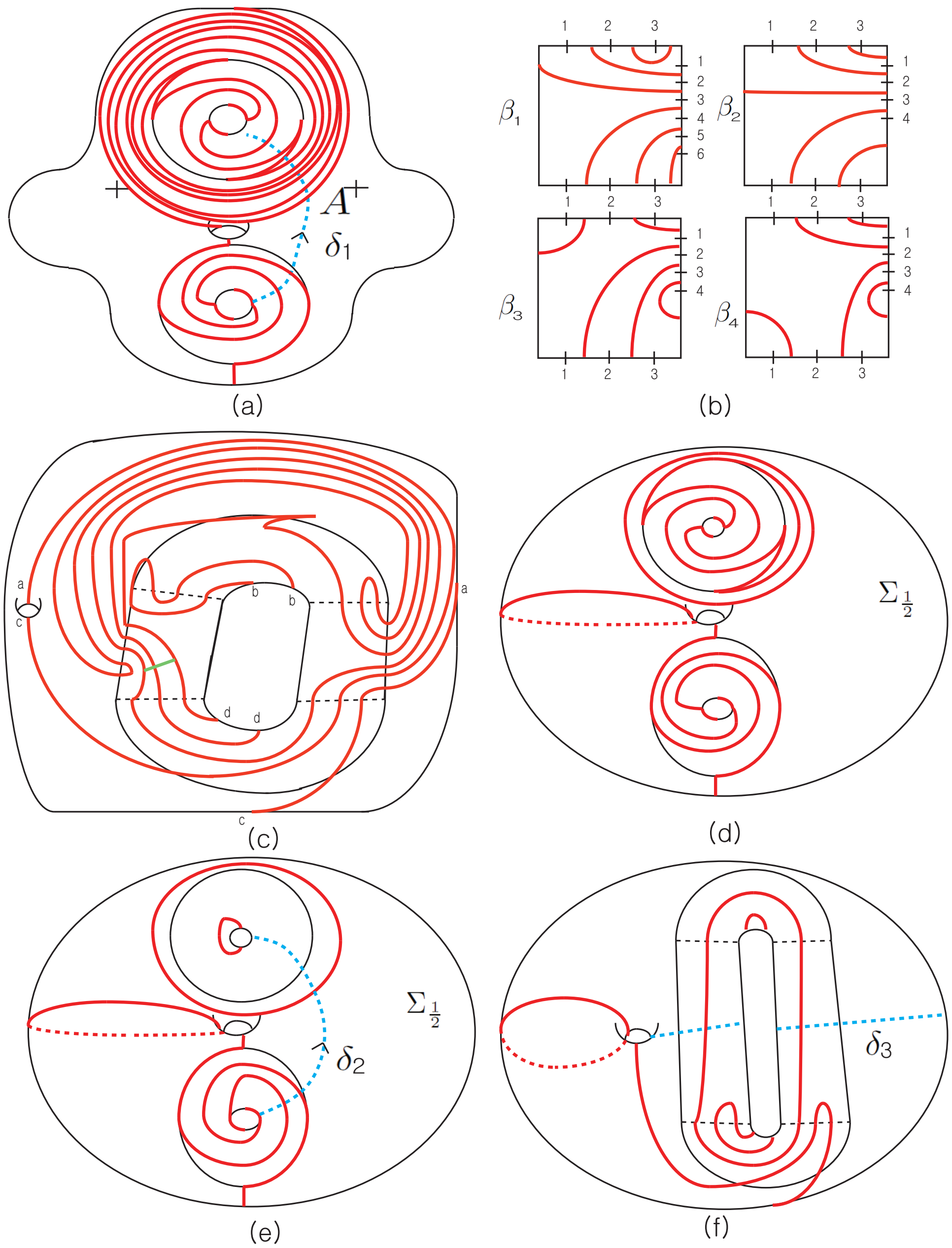


Figure 4.52: I_{-1} case

bypass for state transition to I_0 which is indicated in green line in Figure 4.52 (c). For the β_3 case, we can find a OT disk and, for β_1 and β_4 case, we can always find a state transition bypass into I_0 case in a similar manner to β_1 case. Hence, for $n = 2, 3$, it can be transformed to I_0 or have a layer inside M.

Consider the layer $([\Sigma_0, \Sigma_{\frac{1}{2}}], I_{-1})$ with $\Gamma_{\Sigma_{\frac{1}{2}}} = \alpha \cup \beta \cup \gamma$. $\Gamma_{\Sigma_{\frac{1}{2}}}$ can be homotoped as shown in Figure 4.52 (e). If the number of genus on negative and positive region are all nonzero, there are 8 possible disk decompositions to a 3-ball. If one of the either is zero, there are 4 possible disk decompositions to a 3-ball. If $g = 2$, then 3 possible disk decompositions to a 3-ball. We showed that $([\Sigma_{\frac{1}{2}}, \Sigma_1, I_0)$ admits at most 2 tight contact structure in Lemma 4.6.2. Hence, I_{-1} admits at most 16 if the number of genus on negative and positive region are all nonzero, 8 if one of the either is zero, 6 if $g = 2$.

□

Proof of Proposition 4.1.7. By Lemmas 4.6.2, 4.6.3 and 4.6.4, we can conclude as the following :

$$\sharp\pi_0(Tight(M, \mathcal{F})) \leq \begin{cases} 4 + 32 + 16 = 52 & \text{if } g(\Sigma_+) \neq 0 \text{ and } g(\Sigma_-) \neq 0, \\ 4 + 32 + 8 = 44 & \text{otherwise}, \\ 4 + 48 + 6 = 58 & \text{if } g = 2. \end{cases} \tag{4.6.1}$$

□

Bibliography

[1] I. Agol, *The Virtual Haken Conjecture*. arXiv:1204.2810

[2] D. Bennequin, *Entrelacements et équations de Pfaff*. Third Schnepfenried geometry conference, Vol. 1 (Schnepfenried, 1982), 87-161, Astérisque, 107-108, Soc. Math. France, Paris, 1983. MR0753131 (86e:58070).

[3] F. Bonahon, *Low-dimensional geometry. From Euclidean surfaces to hyperbolic knots.* Student Mathematical Library, 49. IAS/Park City Mathematical Subseries. American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 2009. xvi+384 pp. ISBN: 978-0-8218-4816-6 MR2522946 (2011f:57024)

[4] T. Cofer, *A class of tight contact structures on* $\Sigma_2 \times I$. Algebr. Geom. Topol. 4 (2004), 961-1011 (electronic) MR2100688 (2005g:57053).

[5] V. Colin, *Chirurgies d'indice un et isotopies de sphères dans les variétés de contact tendues.* C.R. Acad. Sci. Paris Sér. I Math. 324 (1997), no. 6, 659-663. MR1447038 (98e:57036)

[6] V. Colin, *Recollement de variétés de contact tendues.* Bull. Soc. Math. France 127 (1999), no. 1, 43-69. MR1700468 (2000e:57042)

[7] V. Colin, *Une infinité de structures de contact tendues sur les variétés toroïdales.* Comment. Math. Helv. 76 (2001), no.2, 353-372. MR1839351 (2002d:53118)

[8] V. Colin, E. Giroux and K. Honda, *Finitude homotopique et isotopique des structures de contact tendues*. Publ. Math. Inst. Hautes Études Sci. No. 109 (2009), 245-293. MR2511589 (2010g:53167)

[9] Y. Eliashberg, *Classification of overtwisted contact structures on* 3-*manifolds*. Invent. Math. 98 (1989), no. 3, 623-637.

[10] Y. Eliashberg, *Contact* 3-*manifolds twenty years since J. Martinet's work*. Ann. Inst. Fourier (Grenoble) 42 (1992), no. 1-2, 165-192.

[11] Y. Eliashberg and W. Thurston, *Confoliations*. University Lecture Series, 13. American Mathematical Society, Providence, RI, 1998. x+66 pp. ISBN: 0-8218-0776-5 MR1483314 (98m:53042)

[12] Y. Eliashberg, M. Fraser, *Classification of topologically trivial Legendrian knots*. Geometry, Topology and Dynamics (Montreal, PQ, 1995), 17-51, CRM Proc. Lecture Notes 15, Amer. Math. Soc., Providence, R1, 1998. MR1619122 (99f:57007)

[13] T. Etgü, *Tight contact structures on laminar free hyperboilc three-manifolds*. Preprint arXiv:1004.2247.

[14] J. Etnyre, *Tight contact structures on lens spaces*. Commum. Contemp. Math. 2 (2000), no. 4, 559-577. MR1806947 (2001k:57017)

[15] J. Etnyre and K. Honda, *On the nonexistence of tight contact structures*. Ann. of Math. (2) 153 (2001), no. 3, 749-766. MR1836287 (2002d:53119)

[16] B. Farb, *A Primer on Mapping Class Groups*. Princeton Mathematical Series, 49. Princeton University Press, Princeton, NJ, 2012. xiv+472 pp. ISBN: 978-0-691-14794-9 MR2850125 (2012h:57032)

[17] S. Fenley, *Laminar free hyperbolic* 3-*manifolds*. Comment. Math. Helv. 82 (2007), no. 2, 247-321. MR2319930 (2008g:57020)

[18] D. Gabai, *Foliations and the topology of* 3-*manifolds*. J. Differential Geom. 18 (1983), no. 3, 445-503. MR0723813 (86a:57009)

[19] C. Gabai, U. Oertel, *Essential laminations in 3-manifolds*. Ann. of Math. (2) 130 (1989) no. 1, 41-73. MR1005607 (90h:57012)

[20] E. Giroux, *Convexité en topologie de contact*. Comment. Math. Helv. 66 (1991), no. 4, 637-677. MR1129802 (93b:57029)

[21] E. Giroux, *Une infinité de structures de contact tendues sur une infinité de variétés*. Invent. Math. 135 (1999), no. 3, 789-802. MR1669264 (2000d:53133)

[22] E. Giroux, *Structures de contact en dimension trois et bifurcations des feuilletages de surfaces*. Invent. Math. 141 (2000), no. 3, 615-689. MR1779622 (2001i:53147)

[23] E. Giroux, *Structures de contact sur les variétés fibrées en cercles audessus d'une surface*. Comment. Math. Helv. 76 (2001), no. 2, 218-262. MR1839346 (2002c:53138)

[24] R. Gompf, A. Stipsicz, *4-manifolds and Kirby calculus*. Graduate Studies in Mathematics, 20. American Mathematical Society, Providence, RI, 1999. xvi+558 pp. ISBN: 0-8218-0994-6 MR1707327 (2000h:57038)

[25] A. Hatcher, *Notes on Basic 3-Manifold Topology*. book in progress. Available at http://www.math.cornell.edu/∼hatcher.

[26] J. Hempel, *3-Manifolds*. Reprint of the 1976 original. AMS Chelsea Publishing, Providence, RI, 2004. xii+195 pp. ISBN: 0-8218-3695-1 MR2098385 (2005e:57053)

[27] K. Honda, *On the classification of tight contact structures. I*. Geom. Topol. 4 (2000), 309-368. MR1786111 (2001i:53148)

[28] K. Honda, *On the classification of tight contact structures. II*. J. Differential Geom. 55 (2000), no. 1, 83-143. MR1849027 (2002g:53155)

[29] K. Honda, *Gluing tight contact structures*. Duke Math. J. 115 (2002), no. 3, 435-478. MR1940409 (2003i:53125)

[30] K. Honda, W. Kazez and G. Matic, *Tight contact structures and taut foliations*. Geom. Topol. 4 (2000) 219-242 (electronic). MR1780749 (2001i:53149)

[31] K. Honda, W. Kazez and G. Matic, *Convex decomposition theory*. Int. Math. Res. Not. 2002, no. 2, 55-88. MR1874319 (2002k:57064)

[32] K. Honda, W. Kazez and G. Matic, *Tight contact structures on fibered hyperbolic 3-manifolds*. J. Differential Geom. 64 (2003), no. 2, 305-358. MR2029907 (2005b:53140)

[33] N. Ivanov, *Mapping class groups*. Handbook of geometric topology, 523-633, North-Holland, Amsterdam, 2002. MR1886678 (2003h:57022)

[34] W. Jaco, P. B. Shalen, *Seifert fibered spaces in* 3*-manifolds*. Mem. Amer. Math. Soc. 21 (1979), no. 220, viii+192 pp. MR0539411 (81c:57010)

[35] K. Johannson, *Homotopy equivalences of* 3*-manifolds with boundaries*. Lecture Notes in Mathematics, vol. 761. Springer, Berlin, 1979. ii+303 pp. ISBN: 3-540-09714-7 MR0551744 (82c:57005)

[36] Y. Kanda, *The classification of tight contact structures on the* 3*-torus*. Comm. Anal. Geom. 5 (1997), no. 3, 413-438. MR1487723 (99c:57054)

[37] Y. Kanda, *On the Thurston-Bennequin invariant of Legendrian knots and nonexactness of Bennequin's inequality*. Invent. Math. 133 (1998), no. 2, 227-242. MR1632790 (99g:57030)

[38] H. Kneser, *Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten*. Jahresber. Deutsch. Math.-Verein. 38 (1929), 248-260.

[39] P. Lisca and A. Stipsicz, *On the existence of tight contact structures on Seifert fibered* 3*-manifolds*. Duke Math. J. 148 (2009), no. 2, 175-209. MR2524494 (2010c:53127)

[40] S. Makar-Limanov, *Tight contact structures on solid tori.* Trans. Amer. Math. Soc. 350 (1998), no. 3, 1013-1044. MR1401526 (98e:53046)

[41] J. Martinet, *Formes de contact sur les variétés de dimension* 3. Proceedings of Leverpool Singularities Symposium, II (1969/1970), pp. 142-163. Lecture Notes in Math., Vol. 209, Springer, Berlin, 1971. MR0350771 (50 ♯3263)

[42] J. W. Milnor, *A unique decomposition theorem for* 3*-manifolds.* Amer. J. Math. 84 1962 1-7. MR0142125 (25 ♯5518)

[43] J. W. Morgan, *Recent progress on the Poincaré conjecture and the classification of* 3*-manifolds.* Bulletin Amer. Math. Soc. (N.S.) 42 (2005), no. 1, 57-78 (electronic). MR2115067 (2005i:57015)

[44] J. W. Morgan and F. Fong, *Ricci flow and geometrization of* 3*-manifolds.* University Lecture Series, 53. American Mathematical Society, Providence, RI, 2010. x+150 pp. ISBN: 978-0-8218-4963-7 MR2597148 (2011d:53160)

[45] J. Otal, *The hyperbolization theorem for fibered* 3*-manifolds.* Translated from the 1996 French original by Leslie D. Kay. SMF/AMS Texts and Monographs, 7. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2001. xiv+126 pp. ISBN: 0-8218-2153-9 MR1855976 (2002g:57035)

[46] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications.* arXiv:math.DG/0211159.

[47] G. Perelman, *Ricci flow with surgery on three-manifolds.* arXiv:math.DG/0303109.

[48] G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds.* arXiv:math.DG/0307245.

[49] R. Roberts, J. Shareshian, M. Stein, *Infinitely many hyperbolic* 3*-manifolds which contain no Reebless foliation.* J. Amer. Math. Soc. 16 (2003), no. 3, 639-679 (electronic). MR1969207 (2004e:57023)

[50] J. Stallings, *On fibering certain 3-manifods*. 1962 Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961) pp. 95–100 Prentice-Hall, Englewood Cliffs, N.J. MR0158375 (28 ♯1600)

[51] A. Stipsicz, *Tight contact structures on the Weeks manifold*. Proceedings of Gökova Geometry-Topology Conference 2007, 82-89, Gökova Geometry/Topology Conference(GGT), Gökova, 2008. MR2509751 (2010g:53170)

[52] W. Thurston, *Three-dimensional geometry and topology. Vol. 1.* Edited by Silvio Levy. Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ, 1997. x+311 pp. ISBN 0-691-08304-5. MR1435975 (97m:57016)

[53] W. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*. Bull. Amer. Math. Soc. (N.S.) 6 (1982), no. 3, 357–381. MR0648524 (83h:57019)

국문초록

2003 년, Honda, Kazez, Matić 은 bypass 와 곡선 복합체를 이용하여 유사 아노소프 모노드로미를 가지는 원 위의 곡면 다발은 Thurston-Bennequin 부등식이 극값을 가지는 경우에 유일한 타이트한 접촉 구조를 가지는 것을 증명하였다. 이 결과를 확장하기 위하여, 우리는 각 경계에 적당한 조건의 dividing 곡선을 준 다음, genus 가 2 이상인 곡면을 두껍게 한 다양체, 즉 $\Sigma \times I$ 의 타이트한 접촉 구조를 분류할 것이다. 우리는 먼저 $\Sigma \times \{0\}$ 에 임의의 separating 곡선 한 개를 dividing 곡선으로 고정하고, 이 곡선과 두 개의 점에서 만나는 nonseparating 곡선을 택한 다음, $\Sigma \times \{0\}$ 의 dividing 곡선을 이 곡선의 주위로 임의의 정수 n 번 Dehn 뒤틀림하여 얻은 곡선을 $\Sigma \times \{1\}$ 의 dividing 곡선으로 생각한다. 이러한 경계 조건 하에서, 우리는 경계의 타이트한 접촉 구조를 내부로 확장한 타이트한 접촉구조 개수의 최소값과 최대값을 조사할 것이다.
이 학위 논문에서 다루는 문제는 Honda의 예상문제와 깊은 관련이 있는데, 이는 모든 쌍곡 3 차원 다양체는 타이트한 접촉 구조를 가진다는 문제로써 두 개의 전혀 다르고 깊은 학문의 줄기인 쌍곡 기하와 접촉 기하의 교차점에 있는 예상문제이다. 우리는 Honda 의 예상문제와 관련된 전체적인 이론들을 간략히 설명한 뒤, 위 문제의 자세한 증명을 다룰 것이다.

주요어휘 : 접촉 기하, 타이트한 접촉 구조, 3 차원 쌍곡 다양체
학번 : 2005-20334

감사의 글

먼저 저를 키워주신 부모님과 항상 친구처럼 고민을 함께 나눈 동생에게 깊은 감사를 드립니다. 그리고 이 논문을 완성하기까지 제가 길을 잃지 않도록 지도해주신 고등과학원의 김인강 교수님께 깊은 감사를 드립니다. 제 미흡한 부분을 친절하고 세심하게 보완해주신 Otto Van Koert 교수님께도 감사드립니다. 대학원 생활을 하는 동안 많은 도움을 주시고 바쁘신 와중에 논문 심사를 해주신 김혁 교수님, 박종일 교수님 정말 감사드립니다. 먼 타지의 낯선 학생인 제게 많은 가르침을 주신 University of Southern California 의 Ko Honda 교수님 정말 감사드립니다. 전공이 다름에도 불구하고 많은 관심과 도움을 주셨던 강석진 교수님 감사드립니다. 그리고 논문 심사를 맡아주신 임선희 교수님께도 감사드립니다.

같은 팀으로써 오랜 기간 함께 공부한 우진오빠, 성운오빠, 준형오빠, 희철오빠 그리고 저와 대학원 생활을 함께 한 승미언니, 경동, 상욱, 지금은 함께 있지 않지만 많은 도움을 준 성은언니, M 에게 고마움을 전하고 싶습니다.