이학박사 학위논문

α-Gauss Curvature Flows and Free Boundary Problems

(알파 가우스 곡률 흐름과 자유 경계 문제)

2013년 8월

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α-Gauss Curvature Flows and Free Boundary Problems

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Abstract

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In this dissertation, we study the deformation of the n-dimensional strictly convex hypersurface in $\mathbb{R}^{n+1}$ whose speed at a point on the hypersurface is proportional to an α-power of positive part of Gauss curvature. For $\frac{1}{n} < \alpha \leq 1$, we show that there exist a strictly convex smooth solutions if the initial hypersurface is strictly convex and smooth and the solution hypersurfaces converge to a point. We discuss the asymptotic behavior of the rescaled hypersurfaces, in other words, the rescaled manifold converges to a strictly convex smooth manifold. Moreover, there exists a subsequence whose limit satisfies a certain equation. For the convex surfaces in $\mathbb{R}^3$ with the velocity given by α-Gauss curvature and $\frac{1}{2} < \alpha \leq 1$, by using a certain estimate different from the one that we use in the n-dimensional case, we establish that there are smooth solutions if the initial surface is smooth and strictly convex. In addition, there is a viscosity solution with a $C^{1,1}$-estimate before the collapsing time if the initial surface is only convex. We also discuss that there is a waiting time effect which means a flat spot of the convex surface will persist for a while. Furthermore, we show that the interface between a flat side and a strictly convex side of the surface remains smooth for $0 < t < T_0$ under certain necessary regularity and non-degeneracy initial conditions, where T_0 is the vanishing time of the flat side.

Key words: gauss curvature flows, deformation of hypersurfaces, regularity of α-gauss curvature flows, free boundary problems, nonlinear parabolic partial differential equations

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Chapter 1

Introduction

The theory of Partial Differential Equations and their applications are the study of solutions of equations describing natural phenomena and social phenomena, which use multivariable calculus as the tool of study. In particular, nonlinear parabolic partial differential equations are one type of second order partial differential equations, which contains nonlinear terms and describes the objects changing over time in the field of science and mathematics. Curvature flows are such nonlinear parabolic partial differential equations which describe the deformation of manifolds, which is an object of geometry. The study for these geometric flows not only contributes to the analysis of the existence, uniqueness and regularity that are typically dealt within the field of partial differential equations, but also has close academic connections with other fields like convex geometry, affine geometry, topology and many more. For example, Ricci curvature flows have provided important ideas to solve the *Poincaré conjecture* and the research of minimal surfaces using mean curvature flows and the classification of singularities using geometric flows are actively being studied in the fields of mathematics. Also, the contribution of curvature flows extends to other fields. Many of the basic issues in Image Analysis are being approached by utilizing curvature flows. For instance, noise can be removed by controlling a level set using the curvature flow expressed as a function describing given image and this curvature function can be used in order to obtain a better image. In addition, curvature flows are widely used in the study of the description of rolling stones at the

beach, the strengthening of alloys, flame propagation, and in the theory of relativity.

Basically, curvature flows are written in the form of a parabolic partial differential equation of the following form:

$$\begin{aligned} \frac{\partial X(x,t)}{\partial t} &= -f(\lambda(\mathcal{W}))\,\nu \\ X(x,0) &= X_0(x)\,. \end{aligned}$$

In other words, when an n-dimensional hypersurface is represented by an embedding $X(\cdot,\cdot) : \Sigma \times [0,T) \to \mathbb{R}^{n+1}$ and ν is the unit outward normal vector, a family of hypersurfaces evolves under curvature flow if the velocity of which a point on the hypersurface moves is given by the curvature of the hypersurface. Then, if a smooth symmetric function f is given by the sum of the eigenvalues of the Weingarten map $\mathcal{W}$, where the eigenvalues are denoted by $\lambda_1, \lambda_2, \cdots, \lambda_n$, this flow becomes a mean curvature flow, or if f is given by the product of λ_i, then this flow will be a gauss curvature flow. Also, f can be provided by scalar curvature, harmonic mean curvature, and so on. Especially, Gauss curvature flows describe the deformation of a compact convex body moving under impact from any random angle. A stone hit by waves is one such example. W. Firey who introduced Gauss curvature flows suggested the following conjecture in 1974 [22]:

> Convex surfaces moving by their gauss curvature become spherical as they contract to points.

In this dissertation, we are concerned with the regularity of the α-Gauss curvature flow, which is the curvature flow of a generalized version of the conjecture formed by W. Firey. This flow explains the deformation of an n-dimensional compact convex body Σ in $\mathbb{R}^{n+1}$ moving with collision from any random angle. An example can be a stone on a beach impacted by the sea, where the probability of impact at any point on the hypersurface Σ is in proportion to the α-Gauss curvature K^α. Let $X(\cdot,\cdot) : \Sigma \times [0,T) \to \mathbb{R}^{n+1}$ be an embedding and set $\Sigma_t = X(\Sigma, t)$. Then the hypersurface evolves by the

following flow:

$$\begin{aligned} \frac{\partial X}{\partial t}(x,t) &= -K^{\alpha}(x,t)\,\nu(x,t) \\ X(x,0) &= X_0(x) \end{aligned} \tag{1.0.1}$$

where ν denotes the unit outward normal to Σ_t and $\alpha > 0$.

Now we shall summarize the known results for the evolution of the strictly convex hypersurfaces following (1.0.1). Let $(0, T^*]$ be the maximal interval in which $vol(\Sigma_t)$ is nonzero.

For the case $\alpha = 1$, if the initial surface in $\mathbb{R}^3$ is smooth and strictly convex and has central symmetry, then the solution Σ_t converges to a point as spherical shapes [22]. Also Tso, [37], showed existence and regularity of the solution when the initial hypersurface embedded in $\mathbb{R}^{n+1}$ is smooth and strictly convex. In other words, the solution Σ_t preserves the smoothness and convexity in the time interval $(0, T^*]$. For a smooth, compact, and strictly convex initial surface in $\mathbb{R}^3$, the solution surface Σ_t converges to a point and the rescaled solution surface $\tilde{\Sigma}_t$ approaches the round sphere with normalized volume and for a non-smooth initial surface, the viscosity solution has $C^{1,1}$-regularity in the time interval $(0, T^*)$ and C^{∞} regularity for $t \geq t_0$ where t_0 depends on the volume and diameter of the initial surface Σ_0 [1].

For $\alpha = \frac{1}{n+2}$, the solution, Σ_t, is known as an affine normal flow. There exists a unique, smooth and convex solution such that the hypersurfaces Σ_t converge to a point and the rescaled solution converges to an ellipsoid if the initial hypersurface is a compact, smooth, and strictly convex [4].

For $\frac{1}{n+2} < \alpha \leq \frac{1}{n}$ or $0 < \alpha \leq \frac{1}{n}$ under the assumption that the isoperimetric ratios are bounded, there exist a smooth, strictly convex solution converging to a point and a rescaled solution satisfying a certain equation [2]. In addition, for $\alpha = \frac{1}{n}$, the rescaled solution converges to a sphere and this holds for $\alpha \geq \frac{1}{n}$ if the initial hypersurface is very close to a sphere [10]. Various applications of (1.0.1) have been studied: the affine normal flows ($\alpha = \frac{1}{n+2}$ [35, 36]), the gradient flows of the mean width in L^p-norm ($\alpha = \frac{1}{p-1}$ [2]), and image process ($\alpha = \frac{1}{4}$ [5]).

We also study the regularity of the α-Gauss curvature flow with flat sides, which is associated to the free boundary problem. As a type of partial differential equations, free boundary problems describe various situations in the

fields of mathematics and science. Many problems such as phase transitions, fluid dynamics, and finance problems can be modeled as free boundary problems. Now we discuss the deformation of the hypersurface Σ described by the flow (1.0.1) for the case when the initial hypersurface Σ_0 is convex and smooth. For $\alpha > 0$, there is a viscosity solution, Σ_t, for $0 < t < T_0$ which has a uniform Lipschitz bound [2]. The convex viscosity solution, Σ_t, has a uniform $C^{1,1}$-estimate for $0 < t < T_0$, for $\alpha = 1$ and $n = 2$ [1], or for $\frac{1}{2} < \alpha \leq 1$ and $n = 2$ [29]. For $\alpha = 1$ and $n = 2$, the C^{∞}_{δ}-regularity of the strictly convex part of the surface and the smoothness of the interface between the strictly convex part and flat spot have been proved in [19].

The dynamics and degeneracy of the diffusion vary depending on α. If α is smaller than $\frac{1}{n}$, hypersurface becomes more singular and the solution gets regular instantaneously. On the other hand, if α is greater than $\frac{1}{n}$, it becomes degenerate and has a waiting time effect which means that the flat spot of the hypersurface stays for a while [2, 9]. Waiting time and finite speed of propagation caused by the degeneracy have been studied in other well-known degenerate equations: the Porous Medium Equation

$$u_t = \Delta u^m \quad (u \geq 0,\, m > 1),$$

and Parabolic p-Laplace Equation

$$u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u).$$

For strictly convex and smooth initial hypersurfaces and $\frac{1}{n} < \alpha \leq 1$, we establish the regularity of solutions of the flow (1.0.1) and the asymptotic behavior of the rescaled hypersurfaces. Also, for the convex surfaces in $\mathbb{R}^3$ and $\frac{1}{2} < \alpha \leq 1$, we show the regularity of the solutions before the collapsing time and the interface between the flat side and the strictly convex side. Each chapter in this dissertation will be organized as follows. In Chapter 1, we introduce Gauss curvature flow and the known results. In Chapter 2, we state the definitions of a metric, the second fundamental form, some curvatures and the support function. In addition, we obtain the evolution equations for the geometric quantities. In Chapter 3, we discuss α-Gauss curvature flows of an n-dimensional compact strictly convex hypersurfaces. We prove that the hypersufaces preserve the strict convexity and we also

get the uniform bound of curvatures of hypersurface Σ. An integral quantity plays the key role in getting the asymptotic behavior of hypersurface and $C^{1,1}$-regularity of the rescaled solution. Also, the curvature bounds of the rescaled hypersurfaces will be introduced. In the last part of this chapter, we shall discuss the existence of solutions and the asymptotic behavior of the rescaled hypersurfaces. In Chapter 4, we consider the deformation of the 2-dimensional convex surfaces moving under the α-Gauss curvature flows. We show that the solution is smooth away from the flat spot and the flat spot has a waiting time effect. We also establish that the free boundary has non-degenerate and finite speed and the second derivatives have the bounds. In addition, Aronson-Bénilan type estimate and global optimal regularity are derived. Finally, we shall show that the interface between a strictly convex part and a flat part is smooth for all time. Throughout the whole chapter, we consider the case $\frac{1}{n} < \alpha \leq 1$ unless there is some explicit assumption on α. We will also assume that Σ_t is smooth whenever we prove a priori-estimates.

Chapter 2

Preliminaries

2.1 Definitions and terminology

2.1.1 Metric, the second fundamental form and curvature

Let $\{x_1, \cdots, x_n\}$ be the local coordinates of Σ_t and ν be the outward unit normal vector to Σ_t. Then the induced metric and the second fundamental form are defined by

$$g_{ij} = \left\langle \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \right\rangle \quad \text{and} \quad h_{ij} = -\left\langle \frac{\partial^2 X}{\partial x^i \partial x^j}, \nu \right\rangle.$$

Also the Weingarten map $\mathcal{W}_p : T_pM \to T_pM$ for the hypersurface $M \subset \mathbb{R}^{n+1}$ can be given by

$$h^i_j = g^{ik} h_{kj},$$

where (g^{ij}) denotes the inverse matrix of (g_{ij}), and then

- $\sigma_k = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$,
- $H = \mathrm{trace}(h^i_j) = \sigma_1 = \sum_{1 \le i \le n} \lambda_i$,
- $K = \det(h^i_j) = \sigma_n = \lambda_1 \lambda_2 \cdots \lambda_n$, and
- $|A|^2 = h_{ij} h^{ij} = \lambda_1^2 + \cdots + \lambda_n^2$,

where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of the Weingarten map at p.

2.1.2 Support function

The support function $S(z,t)$ of the strictly convex surface is given by

$$S(z) = \langle z, X(\nu^{-1}(z), t)\rangle, \quad \text{for} \quad z \in \mathbb{S}^n, \tag{2.1.1}$$

where $\mathbb{S}^n$ denotes a unit sphere. Then $X(z)$ can be written as

$$X(z) = S(z)z + \overline{\nabla} S(z)$$

from the definition of the support function, (2.1.1), and $\overline{\nabla}_i S(z) = \langle X(z), \overline{\nabla}_i z\rangle$ for the connection of the standard metric $\overline{g}$ on $\mathbb{S}^n$. We also have

$$\frac{\partial z}{\partial x^i} = h_{ik} g^{kl} \frac{\partial X}{\partial x^l} \tag{2.1.2}$$

from the relationship between the tangent vector and the normal vector and the definition of the second fundamental form. In addition,

$$h_{ij} = \overline{\nabla}_i \overline{\nabla}_j S + S\overline{g}_{ij} \tag{2.1.3}$$

where $\overline{g}_{ij}$ is the metric on $\mathbb{S}^n$, which this can be obtained by taking covariant derivatives of (2.1.1), [40].

We define the width, the inner radius and the outer radius of the convex hypersurface as follows:

- the inner radius $r_{in} = \sup\{r : B_r(y)$ is enclosed by X for some $y \in \mathbb{R}^{n+1}\}$
- the outer radius $r_{out} = \inf\{r : B_r(y)$ encloses X for some $y \in \mathbb{R}^{n+1}\}$
- the width of the convex surface $w(z) = S(z) + S(-z)$ for $z \in \mathbb{S}^n$
- the maximum width $w_{\max} = \max_{z\in\mathbb{S}^n} w(z)$
- the minimum width $w_{\min} = \min_{z\in\mathbb{S}^n} w(z)$
- the maximum support $S_{max} = \max_{z\in\mathbb{S}^n} S(z)$
- the minimum support $S_{min} = \min_{z\in\mathbb{S}^n} S(z)$

2.2 Evolutions of the geometric quantities

2.2.1 Evolutions of metric, the second fundamental form, and curvature

The evolutions of the metric, the second fundamental form, and curvature are the following. Throughout this dissertation, the symbol $\Box$ will be used in place of the operator $K^\alpha (h^{-1})^{kl}\nabla_k\nabla_l$. The proofs follow the same line as those in Chapter 2 of [40].

Lemma 2.2.1. *Let Σ_0 be convex and $\Sigma_t = X(\Sigma, t)$ be smooth. For the α-Gauss curvature flow, we have*

(i) $$\frac{\partial g_{ij}}{\partial t} = -2K^\alpha h_{ij}$$

(ii) $$\frac{\partial \nu}{\partial t} = g^{ij}\frac{\partial K^\alpha}{\partial x^i}\frac{\partial X}{\partial x^j} = \nabla^j K^\alpha \frac{\partial X}{\partial x^j}$$

(iii) $$\begin{aligned}\frac{\partial h_{ij}}{\partial t} &= \nabla_i\nabla_j K^\alpha - K^\alpha h_{jk}h_i^k \\ &= \alpha\Box h_{ij} + \alpha^2 K^\alpha (h^{-1})^{kl}(h^{-1})^{mn}\nabla_i h_{kl}\nabla_j h_{mn} \\ &\quad -\alpha K^\alpha (h^{-1})^{km}(h^{-1})^{nl}\nabla_i h_{mn}\nabla_j h_{kl} + \alpha K^\alpha H h_{ij} - (1+n\alpha)K^\alpha h_{jl}h_i^l\end{aligned}$$

(iv) $$\frac{\partial K}{\partial t} = \alpha\Box K + \alpha(\alpha-1)K^{\alpha-1}(h^{-1})^{ij}\nabla_i K\nabla_j K + K^{\alpha+1}H$$

(v) $$\frac{\partial K^\alpha}{\partial t} = \alpha\Box K^\alpha + \alpha K^{2\alpha}H$$

(vi) $$\begin{aligned}\frac{\partial H}{\partial t} &= \alpha\Box H + \alpha^2 K^{\alpha-2}g^{ij}\nabla_i K\nabla_j K - \alpha K^\alpha g^{ij}(h^{-1})^{km}(h^{-1})^{nl}\nabla_i h_{mn}\nabla_j h_{kl} \\ &\quad + \alpha K^\alpha H^2 + (1-n\alpha)K^\alpha |A|^2\end{aligned}$$

(vii) $$\frac{\partial |X|^2}{\partial t} = \Box|X|^2 - 2K^\alpha (h^{-1})^{kl}g_{kl} + 2(n-1)K^\alpha\langle X, \nu\rangle$$

2.2.2 Evolutions with respect to the standard metric $\overline{g}_{ij}$ on $\mathbb{S}^n$

Now we have the following relationships and the evolution equations.

Lemma 2.2.2. *Let Σ_0 be strictly convex and $X(\nu^{-1}(z),t)$ be smooth, where $z \in \mathbb{S}^n$. For the α-Gauss curvature flow, we have*

(i) $\overline{g}_{ij} = h_{ik}g^{kl}h_{lj}$ *and* $\overline{g}^{ij} = (h^{-1})^{ik}g_{kl}(h^{-1})^{lj}$

(ii) $h^i_j = (h^{-1})^{ik}\overline{g}_{kj}$

(iii) $H = \overline{g}_{ik}(h^{-1})^{ki}$, $|A|^2 = g^{kl}\overline{g}_{kl}$ *and* $K = \det(h^i_j) = \dfrac{\det(\overline{g}_{ij})}{\det(\overline{\nabla}_i\overline{\nabla}_j S + S\overline{g}_{ij})}$

(iv) Set $\mathcal{S}_k$ be the k-th symmetric polynomial of h_{ij} while σ_k is the k-th symmetric polynomial of h^i_j. Then $\mathcal{S}_n = K^{-1}$.

The following lemma gives us the evolution equations of the support function, second fundamental form, and curvatures for the standard metric $\overline{g}_{ij}$ on $\mathbb{S}^n$.

Lemma 2.2.3. *Let Σ_0 be strictly convex and $X(\nu^{-1}(z),t)$ be smooth, where $z \in \mathbb{S}^n$. For the α-Gauss curvature flow, we have*

(i) $\dfrac{\partial S}{\partial t} = -K^\alpha = -\mathcal{K}^{-\alpha}$ *or* $\Big(-\dfrac{\partial S}{\partial t}\Big)\mathcal{K}^\alpha = 1\,,$ *where* $\mathcal{K} = K^{-1}$

(ii) $\dfrac{\partial h_{ij}}{\partial t} = -\overline{\nabla}_i\overline{\nabla}_j K^\alpha - K^\alpha\overline{g}_{ij} = -\big(\overline{\nabla}_i\overline{\nabla}_j\mathcal{K}^{-\alpha} + \mathcal{K}^{-\alpha}\overline{g}_{ij}\big)$

(iii) $\dfrac{\partial H}{\partial t} = g^{ij}\big(\overline{\nabla}_i\overline{\nabla}_j K^\alpha + K^\alpha\overline{g}_{ij}\big)$

(iv) $\dfrac{\partial |A|^2}{\partial t} = 2\overline{g}_{ij}h^{ij}K^\alpha$

(v) $\dfrac{\partial K}{\partial t} = K(h^{-1})^{ij}\big(\overline{\nabla}_i\overline{\nabla}_j K^\alpha + K^\alpha \overline{g}_{ij}\big) = K(h^{-1})^{ij}\overline{\nabla}_i\overline{\nabla}_j K^\alpha + K^{\alpha+1}H$

(vi) $\dfrac{\partial K^\alpha}{\partial t} = \alpha K^\alpha (h^{-1})^{ij}\overline{\nabla}_i\overline{\nabla}_j K^\alpha + \alpha K^{2\alpha}H$

(vii) $\dfrac{\partial \mathcal{K}}{\partial t} = -\mathcal{K}(h^{-1})^{ij}\left(\overline{\nabla}_i\overline{\nabla}_j \mathcal{K}^{-\alpha} + \mathcal{K}^{-\alpha}\overline{g}_{ij}\right)$

Proof. Taking the time derivative of (2.1.1) gives us

$$\frac{\partial S}{\partial t} = \left\langle z, \nabla X \cdot \frac{\partial \nu^{-1}}{\partial t} + \frac{\partial X}{\partial t}\right\rangle$$

and then (i) comes from (1.0.1). Also we can obtain (ii) and (iv) by the definitions of h_{ij} and $|A|^2$, respectively.
We know that

$$\begin{aligned}\frac{\partial}{\partial t}H &= \overline{g}_{ij}(h^{-1})^{ik}(h^{-1})^{lj}(\overline{\nabla}_k\overline{\nabla}_l K^\alpha + \overline{g}_{kl}K^\alpha)\\ &= g^{kl}(\overline{\nabla}_k\overline{\nabla}_l K^\alpha + \overline{g}_{kl}K^\alpha)\end{aligned}$$

by (ii). From the evolution equation of the second fundamental form, we get the evolution equation of K:

$$\begin{aligned}\frac{\partial K}{\partial t} &= -K\overline{g}^{jm}h_{mi}\overline{g}_{jn}(h^{-1})^{nk}(h^{-1})^{li}\frac{\partial}{\partial t}h^{kl}\\ &= K(h^{-1})^{kl}\overline{\nabla}_k\overline{\nabla}_l K^\alpha + K^{\alpha+1}H,\end{aligned}$$

which implies (vi). Also (vii) is obtained directly from the definition of $\mathcal{K}$. □

In addition, S satisfies, as in [37],

$$-S_t(z,t)\big[\det(\bar{\nabla}_i\bar{\nabla}_j S(z,t) + S(z,t)\delta_{ij})\big]^\alpha = 1 \quad \text{for } (z,t) \in \mathbb{S}^n \times (0,T^*), \tag{2.2.1}$$

which comes from Lemma 2.2.2 (iii) and Lemma 2.2.3 (i).

Chapter 3

α-Gauss Curvature Flows of an n-Dimensional Compact Strictly Convex Hypersurface

3.1 Main theorem

Let us denote the rescaled Σ and a support function S by $\tilde{\Sigma}$ and $\tilde{S}$ respectively so that the volume enscribed becomes normalized. We state the first main theorem.

Theorem 3.1.1.

Let $\Sigma_0 = X(\Sigma, 0)$ be a compact, connected, strictly convex smooth manifold in $\mathbb{R}^{n+1}$. Assume $\frac{1}{n} < \alpha \leq 1$. Then

(i) there exist a time T^* and a strictly convex smooth solution $\{\Sigma_t = X(\Sigma, t)\}$ satisfying (1.0.1) for $t \in [0, T^*)$, and Σ_t converges to a point as t approaches to T^*.

(ii) The principal curvatures of the rescaled hypersurfaces $\tilde{\Sigma}$ have the uniform upper and lower bounds. In other words, let us denote the eigenvalues of $(\tilde{h}^i_j)$ by $\tilde{\lambda}_k$ for $k = 1, \cdots, n$ and the smallest and largest one by $\tilde{\lambda}_{min}$ and $\tilde{\lambda}_{max}$, respectively. Then we have

$$\frac{1}{M} \leq \tilde{\lambda}_{min} \leq \tilde{\lambda}_{max} \leq M$$

for some constant $0 < M < \infty$.

(iii) For any sequence $\tau_i \to \infty$*, there exists a subsequence* τ_{i_k} *such that the rescaled manifold* $\tilde{\Sigma}_{\tau_{i_k}}$ *converges to a strictly convex manifold* $\tilde{\Sigma}_{T^*}$ *uniformly in* C^∞*-norm.*

(iv) In addition, the limit, $\tilde{S}_*(\cdot)$*, of the volume normalized solution* $\tilde{S}(\cdot, \tau_{i_k})$ *satisfies the equation* $\tilde{K}_*^\alpha = \tilde{C}_* \tilde{S}_*$ *a.e. for some positive constant* $\tilde{C}_*$*, where* $\tilde{K}_*$ *is the gauss curvature of* $\tilde{\Sigma}_{T^*}$.

3.2 Curvature estimates

Now we shall show that the strict convexity of Σ_t will be preserved under the flow.

Lemma 3.2.1.

If Σ_0 *is strictly convex,* $\Sigma_t = X(\Sigma, t)$ *is also strictly convex for* $t > 0$ *as long as it is smooth. We also have*

$$\inf_{x\in\Sigma} K(x,t) \geq \inf_{x\in\Sigma} K(x,0) > 0.$$

Proof. Let $Z(t) = \inf_{x\in\Sigma} K(x,t)$ and assume that the minimum is achieved at $X = X(x,t)$. Then, at X, we have

$$\nabla_i \nabla_j K \geq 0 \quad \text{and} \quad \nabla_i K = 0\,,$$

and hence we get

$$\begin{aligned} \frac{\partial K}{\partial t} &= \alpha \Box K + \alpha(\alpha-1) K^{\alpha-1} (h^{-1})^{ij} \nabla_i K \nabla_j K + K^{\alpha+1} H \\ &\geq K^{\alpha+1} H. \end{aligned} \tag{3.2.1}$$

Now, $H \geq n K^{1/n}$ implies

$$\frac{\partial Z}{\partial t} \geq n Z^{\alpha+1+1/n}.$$

By the maximum principle, we can get $Z(t) \geq Z(0) > 0$ which gives the positive lower bound of K for $t > 0$, and then the strict convexity of Σ_t.

□

We have the following lemma (cf. [2]). We shall use the idea of Lemma 3.5 in [40].

Lemma 3.2.2.

Let Σ_0 be convex, $\Sigma_t = X(\mathbb{S}^n, t)$ be smooth for t in $[0, T^*)$, and $\alpha > 0$. Also let us consider the sphere with radius $r_{in}(T^* - \delta)$ and center at the origin contained in $\Sigma_{T^*-\delta}$ and set $\rho_0 = \frac{1}{2} r_{in}(T^* - \delta)$ where δ is any positive constant satisfying $\delta < T^*$. Then there is a constant $C > 0$ such that

$$\sup_{z \in \mathbb{S}^n,\ 0 \le t \le T^* - \delta} K^\alpha(z,t) \le C = \max\left(\sup_{z \in \mathbb{S}^n} K^\alpha(z,0), \left(\frac{n\alpha + 1}{n\alpha\rho_0}\right)^{n\alpha} \right).$$

Proof. We consider the function $\varphi = \frac{K^\alpha}{S - \rho_0}$, where S is the support function. Here $S(z,t) = (z, X(\nu^{-1}(z), t))$ and then

$$\frac{\partial S}{\partial t} = \left(z, \frac{\partial X}{\partial t} \right) = \left(z, -K^\alpha \nu \right) = -K^\alpha.$$

Let us assume that φ has its maximum at $(z_0,\ t_0)$ for $t_0 \le T^* - \delta$. Then, at $(z_0,\ t_0)$, we get

$$\varphi_t \ge 0,\ \overline{\nabla}_i \varphi = 0 \text{ and } \overline{\nabla}_i \overline{\nabla}_j \varphi \le 0.$$

Now we have $0 = \overline{\nabla}_i \varphi = \frac{(S-\rho_0)\overline{\nabla}_i K^\alpha - K^\alpha \overline{\nabla}_i S}{(S-\rho_0)^2} = \frac{\overline{\nabla}_i K^\alpha}{S-\rho_0} - \frac{K^\alpha \overline{\nabla}_i S}{(S-\rho_0)^2}$, so $\overline{\nabla}_i K^\alpha = \frac{K^\alpha \overline{\nabla}_i S}{S - \rho_0}$. Since

$$\begin{aligned}
0 \ge \overline{\nabla}_i \overline{\nabla}_j \varphi &= \overline{\nabla}_i \Big(\frac{\overline{\nabla}_j K^\alpha}{S - \rho_0} - \frac{K^\alpha \overline{\nabla}_j S}{(S - \rho_0)^2} \Big) \\
&= \frac{\overline{\nabla}_i \overline{\nabla}_j K^\alpha}{S - \rho_0} - \frac{\overline{\nabla}_j K^\alpha \overline{\nabla}_i S}{(S - \rho_0)^2} - \frac{\overline{\nabla}_i K^\alpha \overline{\nabla}_j S + K^\alpha \overline{\nabla}_i \overline{\nabla}_j S}{(S - \rho_0)^2} \\
&\quad + \frac{2K^\alpha \overline{\nabla}_j S \overline{\nabla}_i S}{(S - \rho_0)^3} \\
&= \frac{\overline{\nabla}_i \overline{\nabla}_j K^\alpha}{S - \rho_0} - \frac{K^\alpha \overline{\nabla}_i \overline{\nabla}_j S}{(S - \rho_0)^2},
\end{aligned}$$

we also have $\overline{\nabla}_i\overline{\nabla}_j K^\alpha \leq \frac{K^\alpha \overline{\nabla}_i\overline{\nabla}_j S}{S-\rho_0}$. Therefore φ satisfies, at $(z_0,\ t_0)$,

$$\begin{aligned} 0 \leq \frac{\partial}{\partial t}\varphi &= \frac{1}{S-\rho_0}\Big(\alpha K^\alpha (h^{-1})^{ij}\overline{\nabla}_i\overline{\nabla}_j K^\alpha + \alpha K^{2\alpha} H + \frac{K^{2\alpha}}{S-\rho_0}\Big) \\ &\leq \frac{1}{S-\rho_0}\Big\{\alpha K^\alpha (h^{-1})^{ij}\Big(\frac{K^\alpha \overline{\nabla}_i\overline{\nabla}_j S}{S-\rho_0}\Big) + \alpha K^{2\alpha} H + \frac{K^{2\alpha}}{S-\rho_0}\Big\}. \end{aligned}$$

From Lemma 2.2.2, we can derive

$$\begin{aligned} 0 &\leq \frac{\alpha K^\alpha (h^{-1})^{ij} K^\alpha}{S-\rho_0}(h_{ij} - S\overline{g}_{ij}) + \alpha K^{2\alpha} H + \frac{K^{2\alpha}}{S-\rho_0} \\ &= \frac{K^{2\alpha}}{S-\rho_0}\Big(n\alpha - \alpha\rho_0 H + 1\Big), \end{aligned}$$

which means

$$0 \leq (n\alpha + 1) - \alpha\rho_0 H.$$

If $\frac{n\alpha+1}{\alpha\rho_0} < H$, that is, $H > \frac{C}{\rho_0} > \frac{C}{r_{in}} > \frac{C}{r_{out}}$, where $C = \frac{n\alpha+1}{\alpha}$, we get a contradiction. Therefore H is bounded, so K^α is bounded since $K^\alpha \leq n^{-n\alpha}H^{n\alpha}$. Now we conclude that

$$\sup_{z\in\mathbb{S}^n,\ 0\leq t\leq T^*-\delta} K^\alpha(z,t) \leq C = \max\left(\sup_{z\in\mathbb{S}^n} K^\alpha(z,0), \Big(\frac{n\alpha+1}{n\alpha\rho_0}\Big)^{n\alpha}\right).$$

□

Now we consider the eigenvalues of the reverse second fundamental form.

Lemma 3.2.3.

Let Σ_0 be strictly convex, $\Sigma_t = X(\Sigma, t)$ be smooth, and $\frac{1}{n} < \alpha \leq 1$. Also set $\mathcal{H} = (h^{-1})^{ij} g_{ij}$. Then there is a constant $C > 0$ such that

$$\sup_{x\in\Sigma} \mathcal{H} \leq C = \max\left(\frac{n\alpha - 1}{\alpha} K^{-1/n}, \sup_{x\in\Sigma} \mathcal{H}(x,0)\right)$$

for $t > 0$ as long as it is smooth.

Proof. First we have the evolution equation for $\mathcal{H}$:

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial t} &= \alpha\Box\mathcal{H} - 2\alpha K^\alpha (h^{-1})^{\gamma\beta}(h^{-1})^{ip}(h^{-1})^{kq}(h^{-1})^{lj} g_{ij}\nabla_\gamma h_{kl}\nabla_\beta h_{pq} \\ &\quad - \alpha^2 K^\alpha (h^{-1})^{ik}(h^{-1})^{lj}(h^{-1})^{mn}(h^{-1})^{pq} g_{ij}\nabla_k h_{mn}\nabla_l h_{pq} \\ &\quad + \alpha K^\alpha (h^{-1})^{ik}(h^{-1})^{lj}(h^{-1})^{mp}(h^{-1})^{nq} g_{ij}\nabla_k h_{mn}\nabla_l h_{pq} \\ &\quad - \alpha K^\alpha H\mathcal{H} + n(1+n\alpha)K^\alpha - 2nK^\alpha\end{aligned}$$

since we can obtain

$$\begin{aligned}\alpha\Box\mathcal{H} = &- \alpha (h^{-1})^{ik}(h^{-1})^{lj} g_{ij}\Box h_{kl} \\ &+ 2\alpha K^\alpha (h^{-1})^{\gamma\beta}(h^{-1})^{ip}(h^{-1})^{kq}(h^{-1})^{lj} g_{ij}\nabla_\gamma h_{kl}\nabla_\beta h_{pq}\end{aligned} \tag{3.2.2}$$

from the second derivatives of $\mathcal{H}$ and we also have the Codazzi identity and symmetry of h_{ij}. Then at a maximum point we get

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial t} &\le -2\alpha K^\alpha (h^{-1})^{kl}(h^{-1})^{ip}(h^{-1})^{mq}(h^{-1})^{nj} g_{ij}\nabla_k h_{mn}\nabla_l h_{pq} \\ &\quad + \alpha K^\alpha (h^{-1})^{ik}(h^{-1})^{lj}(h^{-1})^{mp}(h^{-1})^{nq} g_{ij}\nabla_k h_{mn}\nabla_l h_{pq} \\ &\quad - \alpha K^\alpha H\mathcal{H} + n(n\alpha-1)K^\alpha \\ &\le -\alpha K^\alpha (h^{-1})^{kl}(h^{-1})^{ip}(h^{-1})^{mq}(h^{-1})^{nj} g_{ij}\nabla_k h_{mn}\nabla_l h_{pq} \\ &\quad - \alpha K^\alpha H\mathcal{H} + n(n\alpha-1)K^\alpha \\ &\le -\Big(\alpha H\mathcal{H} - n(n\alpha-1)\Big)K^\alpha \\ &\le -\Big(\alpha n K^{1/n}\mathcal{H} - n(n\alpha-1)\Big)K^\alpha.\end{aligned}$$

since $H \ge n(K)^{1/n} \ge c_0 > 0$ for some positive constant c_0. On the other hand, we have a contradiction if $\mathcal{H} > \frac{n\alpha-1}{\alpha K^{1/n}}$ at a maximum point. Hence $\mathcal{H} \le \frac{n\alpha-1}{\alpha}K^{-1/n}$. Then the result follows. $\Box$

3.3 Integral quantity and asymptotic behavior of hypersurface

We shall define the volume $V(t)$ and the area $\mathcal{A}(t)$ enclosed by convex surface Σ as follows:

- the volume function $V(t) = \frac{1}{n+1}\int_\Sigma \langle X, \nu\rangle \, d\sigma_\Sigma = \frac{1}{n+1}\int_{\mathbb{S}^n} \frac{S}{K}\, d\sigma_{\mathbb{S}^n}$
- the area function $\mathcal{A}(t) = \int_\Sigma d\sigma_\Sigma = \int_{\mathbb{S}^n} \frac{1}{K}\, d\sigma_{\mathbb{S}^n}$

Lemma 3.3.1. *For the strictly convex and smooth solution $\Sigma_t = X(\mathbb{S}^n, t)$ of α-Gauss curvature flow* (1.0.1)*, we have*

$$\frac{\partial}{\partial t} V(t) = -\int_{\mathbb{S}^n} \frac{1}{K^{1-\alpha}}\, d\sigma_{\mathbb{S}^n}\,.$$

Proof. First observe that from Lemma 2.2.2 and Lemma 2.2.3, we have

$$\begin{aligned}
\int_{\mathbb{S}^n} S\mathcal{K}_t \, d\sigma_{\mathbb{S}^n} &= \int_{\mathbb{S}^n} S\mathcal{K}(h^{-1})^{ij}(\overline{\nabla}_i\overline{\nabla}_j S_t + S_t\overline{g}_{ij})\, d\sigma_{\mathbb{S}^n} \\
&= \int_{\mathbb{S}^n} S_t\mathcal{K}(h^{-1})^{ij}(\overline{\nabla}_i\overline{\nabla}_j S + S\overline{g}_{ij})\, d\sigma_{\mathbb{S}^n} \\
&= \int_{\mathbb{S}^n} S_t\mathcal{K}(h^{-1})^{ij} h_{ij}\, d\sigma_{\mathbb{S}^n} = n\int_{\mathbb{S}^n} S_t\mathcal{K}\, d\sigma_{\mathbb{S}^n}
\end{aligned}$$

since $\overline{\nabla}_i\mathcal{K}(h^{-1})^{ij} = 0$. Hence we have

$$\begin{aligned}
\frac{\partial}{\partial t} V(t) &= \frac{1}{n+1}\int_{\mathbb{S}^n} (\mathcal{K}S_t + S\mathcal{K}_t)\, d\sigma_{\mathbb{S}^n} \\
&= \int_{\mathbb{S}^n} \mathcal{K}S_t\, d\sigma_{\mathbb{S}^n} = -\int_{\mathbb{S}^n} \frac{1}{K^{1-\alpha}}\, d\sigma_{\mathbb{S}^n}.
\end{aligned}$$

□

Now let us consider the rescaled solution

$$\tilde{X}(\tau) := \frac{X(t)}{V(t)^{1/(n+1)}} \tag{3.3.1}$$

and also assume that the normalized volume $\tilde{V}(\tau) = \frac{1}{n+1}\left(\int_{\mathbb{S}^n} \frac{\tilde{S}}{\tilde{K}}\, d\sigma_{\mathbb{S}^n}\right) = 1$ where $\tau(t) = -\log\left(\frac{V(t)}{V(0)}\right)$. For the rescaled solution, we have the rescaled metric, second fundamental form, and curvatures as follows:

- $\widetilde{g}_{ij} = V(t)^{-\frac{2}{n+1}} g_{ij}$ and $\widetilde{h}_{ij} = V(t)^{-\frac{1}{n+1}} h_{ij}$

- $\widetilde{H} = V(t)^{\frac{1}{n+1}} H$ and $\widetilde{K} = V(t)^{\frac{n}{n+1}} K$

- $\widetilde{S} = V(t)^{-\frac{1}{n+1}} S$ and $\widetilde{\eta} = V(t)^{\frac{n(\alpha-1)}{n+1}} \eta$ where $\eta(t) = \displaystyle\int_{\mathbb{S}^n} \frac{1}{K^{1-\alpha}}\, d\sigma_{\mathbb{S}^n}$.

Then we obtain the following corollary.

Corollary 3.3.2. *For the strictly convex and smooth rescaled solution $\tilde{\Sigma}_\tau = \tilde{X}(\mathbb{S}^n, \tau)$ of α-Gauss curvature flow, we have the evolution equation of $\tilde{X}$:*

$$\frac{\partial \tilde{X}}{\partial \tau} = -\frac{\tilde{K}^\alpha}{\tilde{\eta}} \tilde{\nu} + \frac{1}{n+1} \tilde{X} \quad \text{on } \mathbb{S}^n \times [0, +\infty),$$

where $\tilde{K}$ is the gauss curvature and $\tilde{\nu}$ is the unit outward normal of $\tilde{\Sigma}_\tau$.

Proof. Lemma 3.3.1 implies

$$V(t) = V(0) - \int_0^t \eta(s)\, ds.$$

Since $\frac{\partial \tilde{X}}{\partial \tau} = \frac{\partial \tilde{X}}{\partial t} \frac{dt}{d\tau} = -\frac{K^\alpha V}{\eta V^{1/(n+1)}} \tilde{\nu} + \frac{1}{n+1} \tilde{X}$, we get the result

$$\frac{\partial \tilde{X}}{\partial \tau} = -\frac{\tilde{K}^\alpha}{\tilde{\eta}} \tilde{\nu} + \frac{1}{n+1} \tilde{X} \,. \tag{3.3.2}$$

□

Now we introduce an integral quantity to analyze the asymptotic behavior of the rescaled hypersurface $\tilde{\Sigma}$.

Lemma 3.3.3. *Let us define the integral quantity $\tilde{\mathcal{I}}$ as follows:*

$$\tilde{\mathcal{I}}(\tau) = \begin{cases} \left(\displaystyle\int_{\mathbb{S}^n} \frac{1}{\tilde{S}^{\frac{1}{\alpha}-1}}\, d\sigma_{\mathbb{S}^n} \right)^{\operatorname{sgn}(\alpha-1)} & \text{for } \alpha > 0 \text{ and } \alpha \neq 1 \\ \displaystyle\int_{\mathbb{S}^n} \log \tilde{S}\, d\sigma_{\mathbb{S}^n} & \text{for } \alpha = 1 \end{cases} \tag{3.3.3}$$

Then it satisfies

$$\frac{\partial}{\partial\tau}\tilde{\mathcal{I}}(\tau) \leq 0. \tag{3.3.4}$$

Moreover, the equality holds if and only if $\tilde{K}^{\alpha} = C\tilde{S}$ a.e. for some positive constant C.

Proof.

Case 1. Let us assume that $0 < \alpha < 1$.
By the definition of the rescaled support function $\tilde{S}$ and (3.3.2), we know that

$$\tilde{K}^{-\alpha}\left(\frac{\partial\tilde{S}}{\partial\tau} - \frac{1}{n+1}\tilde{S}\right) = -\frac{1}{\tilde{\eta}}\,. \tag{3.3.5}$$

Multiplying both sides of the equation (3.3.5) by $\tilde{S}^{-\beta}$, where β will be chosen later on, implies

$$\frac{1}{\tilde{S}^{\beta}}\left(\frac{\partial\tilde{S}}{\partial\tau} - \frac{1}{n+1}\tilde{S}\right) = -\frac{\tilde{K}^{\alpha}}{\tilde{\eta}\tilde{S}^{\beta}}\,,$$

from the derivation of $\tilde{\mathcal{I}}(\tau)$ with respect to τ, we have

$$\begin{aligned}\frac{\alpha}{1-\alpha}\left(\mathcal{I}(\tau)\right)^{-2}\frac{\partial}{\partial\tau}\tilde{\mathcal{I}}(\tau) &= \int_{\mathbb{S}^n}\frac{\tilde{S}_{\tau}}{\tilde{S}^{\beta}}\,d\sigma_{\mathbb{S}^n}\\ &= \frac{1}{n+1}\int_{\mathbb{S}^n}\tilde{S}^{1-\beta}\,d\sigma_{\mathbb{S}^n} - \int_{\mathbb{S}^n}\frac{\tilde{K}^{\alpha}}{\tilde{\eta}\tilde{S}^{\beta}}\,d\sigma_{\mathbb{S}^n} \leq 0\,.\end{aligned} \tag{3.3.6}$$

Since $\tilde{\eta}(\tau)$ is positive, (3.3.6) is non-positive if

$$\frac{1}{n+1}\left(\int_{\mathbb{S}^n}\tilde{S}^{1-\beta}\,d\sigma_{\mathbb{S}^n}\right)\left(\int_{\mathbb{S}^n}\frac{1}{\tilde{K}^{1-\alpha}}\,d\sigma_{\mathbb{S}^n}\right) \leq \int_{\mathbb{S}^n}\frac{\tilde{K}^{\alpha}}{\tilde{S}^{\beta}}\,d\sigma_{\mathbb{S}^n}, \tag{3.3.7}$$

which implies non-positivity of the evolution equation of $\tilde{\mathcal{I}}(\tau)$. Hence it will suffice to show that inequality (3.3.7) holds. First, notice that we have

$$\begin{aligned}\int_{\mathbb{S}^n}\tilde{S}^{1-\beta}\,d\sigma_{\mathbb{S}^n} &= \int_{\mathbb{S}^n}\left(\tilde{S}^{-\beta}\tilde{K}^{\alpha}\right)^{\frac{\beta-1}{\beta}}\left(\tilde{K}^{-\alpha}\right)^{\frac{\beta-1}{\beta}}\,d\sigma_{\mathbb{S}^n}\\ &\leq \left(\int_{\mathbb{S}^n}\tilde{K}^{\alpha}\tilde{S}^{-\beta}\,d\sigma_{\mathbb{S}^n}\right)^{\frac{\beta-1}{\beta}}\left(\int_{\mathbb{S}^n}\frac{1}{\tilde{K}^{\alpha(\beta-1)}}\,d\sigma_{\mathbb{S}^n}\right)^{\frac{1}{\beta}}\end{aligned} \tag{3.3.8}$$

for $\alpha(\beta-1)=1-\alpha$. That is, $\beta=\frac{1}{\alpha}$ from the Hölder inequality, which implies

$$\int_{\mathbb{S}^n}\frac{1}{\tilde{S}^{\frac{1}{\alpha}-1}}\,d\sigma_{\mathbb{S}^n}\leq\left(\int_{\mathbb{S}^n}\tilde{K}^{\alpha}\tilde{S}^{-\frac{1}{\alpha}}\,d\sigma_{\mathbb{S}^n}\right)^{1-\alpha}\left(\int_{\mathbb{S}^n}\frac{1}{\tilde{K}^{1-\alpha}}\,d\sigma_{\mathbb{S}^n}\right)^{\alpha}. \tag{3.3.9}$$

We also have

$$\begin{aligned}\int_{\mathbb{S}^n}\frac{1}{\tilde{K}^{1-\alpha}}\,d\sigma_{\mathbb{S}^n}&=\int_{\mathbb{S}^n}\left(\frac{\tilde{S}}{\tilde{K}}\right)^{1-\alpha}\tilde{S}^{-(1-\alpha)}\,d\sigma_{\mathbb{S}^n}\\&\leq\left(\int_{\mathbb{S}^n}\frac{\tilde{S}}{\tilde{K}}\,d\sigma_{\mathbb{S}^n}\right)^{1-\alpha}\left(\int_{\mathbb{S}^n}\frac{1}{\tilde{S}^{\frac{1}{\alpha}-1}}\,d\sigma_{\mathbb{S}^n}\right)^{\alpha}.\end{aligned} \tag{3.3.10}$$

Now from (3.3.8) and (3.3.10), we get

$$\begin{aligned}&\left(\int_{\mathbb{S}^n}\frac{1}{\tilde{S}^{\frac{1}{\alpha}-1}}\,d\sigma_{\mathbb{S}^n}\right)\left(\int_{\mathbb{S}^n}\frac{1}{\tilde{K}^{1-\alpha}}\,d\sigma_{\mathbb{S}^n}\right)^{1-\alpha}\\&\leq\left(\int_{\mathbb{S}^n}\tilde{K}^{\alpha}\tilde{S}^{-\frac{1}{\alpha}}\,d\sigma_{\mathbb{S}^n}\right)^{1-\alpha}\left(\int_{\mathbb{S}^n}\frac{1}{\tilde{K}^{1-\alpha}}\,d\sigma_{\mathbb{S}^n}\right)\\&\leq\left(\int_{\mathbb{S}^n}\tilde{K}^{\alpha}\tilde{S}^{-\frac{1}{\alpha}}\,d\sigma_{\mathbb{S}^n}\right)^{1-\alpha}\left(\int_{\mathbb{S}^n}\frac{\tilde{S}}{\tilde{K}}\,d\sigma_{\mathbb{S}^n}\right)^{1-\alpha}\left(\int_{\mathbb{S}^n}\frac{1}{\tilde{S}^{\frac{1}{\alpha}-1}}\,d\sigma_{\mathbb{S}^n}\right)^{\alpha}\end{aligned}$$

and then

$$\begin{aligned}\left(\int_{\mathbb{S}^n}\frac{1}{\tilde{S}^{\frac{1}{\alpha}-1}}\,d\sigma_{\mathbb{S}^n}\right)\left(\int_{\mathbb{S}^n}\frac{1}{\tilde{K}^{1-\alpha}}\,d\sigma_{\mathbb{S}^n}\right)&\leq\left(\int_{\mathbb{S}^n}\tilde{K}^{\alpha}\tilde{S}^{-\frac{1}{\alpha}}\,d\sigma_{\mathbb{S}^n}\right)\left(\int_{\mathbb{S}^n}\frac{\tilde{S}}{\tilde{K}}\,d\sigma_{\mathbb{S}^n}\right)\\&=(n+1)\left(\int_{\mathbb{S}^n}\tilde{K}^{\alpha}\tilde{S}^{-\frac{1}{\alpha}}\,d\sigma_{\mathbb{S}^n}\right)\end{aligned} \tag{3.3.11}$$

since the normalized volume $\tilde{V}(\tau)=\frac{1}{n+1}\left(\int_{\mathbb{S}^n}\frac{\tilde{S}}{\tilde{K}}\,d\sigma_{\mathbb{S}^n}\right)=1$. The last inequality (3.3.11) completes the proof of the desired result.

Case 2. Assume $\alpha=1$.
Since $\tilde{S}$ satisfies the equation $\tilde{S}_\tau=\frac{\tilde{K}}{|\mathbb{S}^n|}+\frac{1}{n+1}\tilde{S}$ where $|\mathbb{S}^n|$ means the volume of $\mathbb{S}^n$, $\frac{\partial\tilde{\mathcal{I}}(\tau)}{\partial\tau}=\int_{\mathbb{S}^n}\frac{\tilde{S}_\tau}{\tilde{S}}\,d\sigma_{\mathbb{S}^n}\leq 0$ is equivalent to

$$\frac{|\mathbb{S}^n|^2}{n+1}\leq\int_{\mathbb{S}^n}\frac{\tilde{K}}{\tilde{S}}\,d\sigma_{\mathbb{S}^n}\,. \tag{3.3.12}$$

Then, we know that

$$|\mathbb{S}^n| \leq \left(\int_{\mathbb{S}^n} \frac{\tilde{K}}{\tilde{S}} \, d\sigma_{\mathbb{S}^n} \right)^{\frac{1}{2}} \left(\int_{\mathbb{S}^n} \frac{\tilde{S}}{\tilde{K}} \, d\sigma_{\mathbb{S}^n} \right)^{\frac{1}{2}} = (n+1)^{\frac{1}{2}} \left(\int_{\mathbb{S}^n} \frac{\tilde{K}}{\tilde{S}} \, d\sigma_{\mathbb{S}^n} \right)^{\frac{1}{2}} \tag{3.3.13}$$

from the Hölder inequality and $\tilde{V}(\tau) = 1$. This implies (3.3.12) directly. In addition, the equality in (3.3.4) holds if and only if the equalities hold in (3.3.9) and (3.3.13), which implies the equation $\tilde{K}^\alpha = C\tilde{S}$ *a.e.* for some positive constant C.

□

We can observe that $\tilde{\mathcal{I}}$ is bounded below from [22] for $\alpha = 1$ and $\tilde{\mathcal{I}} \geq 0$ for $\alpha > 0$ and $\alpha \neq 1$. Lemma 3.3.3 for the evolution equation of $\tilde{\mathcal{I}}$ gives us the following convergence.

Corollary 3.3.4. *For the integral quantity $\tilde{\mathcal{I}}(\tau)$ given by* (3.3.3)*, we have*

$$\lim_{\tau \to \infty} \tilde{\mathcal{I}}(\tau) = \tilde{\mathcal{I}}_0$$

for some constant $\tilde{\mathcal{I}}_0$, moreover

$$\lim_{\tau \to \infty} \frac{\partial}{\partial \tau} \tilde{\mathcal{I}}(\tau) = 0.$$

Lemma 3.3.5. *Let us assume that $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are n-dimensional hypersurfaces embedded in $\mathbb{R}^{n+1}$ and monotone quantities of $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are $\tilde{\mathcal{I}}_1$ and $\tilde{\mathcal{I}}_2$, respectively. If $\tilde{\Sigma}_1 \subset \tilde{\Sigma}_2$, then we have*

$$\tilde{\mathcal{I}}_1 \leq \tilde{\mathcal{I}}_2.$$

Proof. Let us $\tilde{S}_1(z)$ and $\tilde{S}_2(z)$ be the support functions of $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$, respectively. We know that if $\tilde{\Sigma}_1 \subset \tilde{\Sigma}_2$, then $\tilde{S}_1(z) \leq \tilde{S}_2(z)$. Then by definition of $\tilde{\mathcal{I}}$, we have

$$\begin{aligned}\tilde{\mathcal{I}}_1^{-1} &= \int_{\mathbb{S}^n} \frac{1}{\tilde{S}_1^{\frac{1}{\alpha}-1}} d\sigma_{\mathbb{S}^n} = \int_{\mathbb{S}^n} \frac{1}{\langle z, \tilde{X}_1(\nu^{-1}(z))\rangle^{\frac{1}{\alpha}-1}} d\sigma_{\mathbb{S}^n} \\ &\geq \int_{\mathbb{S}^n} \frac{1}{\langle z, \tilde{X}_2(\nu^{-1}(z))\rangle^{\frac{1}{\alpha}-1}} d\sigma_{\mathbb{S}^n} = \int_{\mathbb{S}^n} \frac{1}{\tilde{S}_2^{\frac{1}{\alpha}-1}} d\sigma_{\mathbb{S}^n} = \tilde{\mathcal{I}}_2^{-1} \quad \text{for } z \in \mathbb{S}^n .\end{aligned}$$

□

Now we shall show that $\tilde{\Sigma}(\tau)$ has a finite width.

Lemma 3.3.6. *Let us consider an ellipsoid $E(\tau)$ such that $r_{\min}(\tau)$ is equal to half of the minor axis and $r_{\max}(\tau)$ is equal to half of the major axis. Assume that $E(\tau)$ has a fixed volume $V(\tau)$. If $r_{\max}(\tau)$ goes to infinity, then $\tilde{\mathcal{I}}(\tau)$ is also infinity.*

Proof. Set $r_1 \cdots r_{n+1} = C$ where C is some positive constant. The equation for the ellipsoid is

$$g(x_1, \ldots, x_n) : \frac{x_1^2}{r_1^2} + \cdots + \frac{x_{n+1}^2}{r_{n+1}^2} = 1$$

where $r_1 = r_{\min}$, $r_{n+1} = r_{\max}$ and $r_1 \leq r_2 \leq \cdots \leq r_{n+1}$. Then an ellipsoid can be parameterized by:

$$X = (r_1 q_1, \ldots, r_{n+1} q_{n+1})$$

where $q = (q_1, \ldots, q_{n+1}) \in \mathbb{S}^n$. We also can obtain a normal vector $\boldsymbol{N} = \frac{1}{2}\nabla g = \left(\frac{x_1}{r_1^2}, \ldots, \frac{x_{n+1}}{r_{n+1}^2}\right)$, a unit normal vector $\nu = \frac{\boldsymbol{N}}{\|\boldsymbol{N}\|}$, and the support function $\tilde{S} = \tilde{X} \cdot \tilde{\nu} = \frac{1}{\|\boldsymbol{N}\|}$. Now we have $\frac{x_i}{r_i^2} = \boldsymbol{N}_i = \|\boldsymbol{N}\|\nu_i = \|\boldsymbol{N}\| z_i = \frac{1}{\tilde{S}} z_i$ where $z = (z_1, \ldots, z_n) \in \mathbb{S}^n$, and then

$$\frac{x_i}{r_i} = \frac{r_i}{\tilde{S}} z_i.$$

Since $1 = \frac{x_1^2}{r_1^2} + \cdots + \frac{x_{n+1}^2}{r_{n+1}^2} = \frac{r_1^2}{\tilde{S}^2} z_1^2 + \cdots + \frac{r_{n+1}^2}{\tilde{S}^2} z_{n+1}^2$, we get

$$\tilde{S}^2 = r_1^2 z_1^2 + \cdots + r_{n+1}^2 z_{n+1}^2$$

and we also have

$$\tilde{\mathcal{I}}^{-1} = \int_{\mathbb{S}^n} \frac{1}{\tilde{S}^{\frac{1}{\alpha}-1}} \, d\sigma_{\mathbb{S}^n} = \int_{\mathbb{S}^n} \frac{1}{\left(\sqrt{r_1^2 z_1^2 + \cdots + r_{n+1}^2 z_{n+1}^2}\right)^{\frac{1}{\alpha}-1}} \, d\sigma_{\mathbb{S}^n} \, .$$

We consider the following case in general: there is $1 \leq k \leq n+1$ such that $r_{n+1} \geq \cdots \geq r_k \gg r_{k+1} \geq \cdots \geq r_1$ with $r_1 \cdots r_{n+1} = C$, where C is some positive constant. Then we have

$$\begin{aligned} C_1 r_{n+1}^{1-\frac{1}{\alpha}} &\leq \int_{\mathbb{S}^n \cap \{\frac{1}{2} \leq z_{n+1} \leq 1\}} \frac{1}{\left(\sqrt{n+1}\sqrt{r_{n+1}^2 z_{n+1}^2}\right)^{\frac{1}{\alpha}-1}} \, d\sigma_{\mathbb{S}^n} \\ &\leq \tilde{\mathcal{I}}^{-1} \leq \int_{\mathbb{S}^n} \frac{1}{\left(\sqrt{r_{n+1}^2 z_{n+1}^2}\right)^{\frac{1}{\alpha}-1}} \, d\sigma_{\mathbb{S}^n} \leq C_2 r_{n+1}^{1-\frac{1}{\alpha}}, \end{aligned}$$

where C_1 and C_2 are positive constants. Since $C r_{n+1}^{1-\frac{1}{\alpha}}$ goes to zero for $\alpha < 1$ as $r_{max}(\tau)$ goes to infinity, $\tilde{\mathcal{I}}(\tau)$ is also infinite. Similarly, for $\alpha = 1$, since

$$c_1 \log r_{n+1} \leq \tilde{\mathcal{I}} = \int_{\mathbb{S}^n} \log \tilde{S} \, d\sigma_{\mathbb{S}^n} \leq c_2 \log r_{n+1}$$

for some positive constant c_1 and c_2, $\tilde{\mathcal{I}}(\tau) \to \infty$ as $r_{max}(\tau) \to \infty$. □

Now we shall introduce a theorem called John's Theorem.

Theorem 3.3.7 (John's Theorem, [6])**.** *Let K be a convex body in $\mathbb{R}^n$. Then there exists a unique ellipsoid E of maximal volume which is contained in each K. This ellipsoid E is $B_2^n = \left\{x \in \mathbb{R}^n : \sum_1^n x_i^2 \leq 1\right\}$ if and only if the following conditions are fulfilled:*

(i) $B_2^n \subset K$.

(ii) There are positive numbers $(c_i)_1^m$ and Euclidean unit vectors $(u_i)_1^m$ on the boundary of K such that $\sum_{i=1}^m c_i u_i = 0$ and $\sum_{i=1}^m c_i \langle x, u_i \rangle^2 = |x|^2$ for all $x \in \mathbb{R}^n$.

We define the width of the convex surface by the function $\tilde{w}(z) = \tilde{S}(z) + \tilde{S}(-z)$ for $z \in \mathbb{S}^n$ and let $\tilde{w}_{\max} = \max_{z\in\mathbb{S}^n} \tilde{w}(z)$ and $\tilde{w}_{\min} = \min_{z\in\mathbb{S}^n} \tilde{w}(z)$. Similarly, set $\tilde{S}_{max} = \max_{z\in\mathbb{S}^n} \tilde{S}(z)$ and $\tilde{S}_{min} = \min_{z\in\mathbb{S}^n} \tilde{S}(z)$. Then we have the following.

Corollary 3.3.8. *For the rescaled hypersurface $\tilde{\Sigma}$ with the normalized volume, there exist some positive constants $0 < c \leq C < \infty$ such that*

$$c \leq \tilde{w}_{\min} \leq \tilde{w}_{\max} \leq C$$

for all $\tau \in [0, \infty)$.

Proof. We know that there exists a unique ellipsoid E_n of maximal volume enclosed by the given convex body $\tilde{\Sigma}$ by Theorem 3.3.7. Thus we can set up $\tilde{\Sigma}$ between two ellipsoids by using an affine transformation. In other words,

$$E_n \subset \tilde{\Sigma} \subset \sqrt{n}\, E_n\,.$$

Then if the maximum radius of ellipsoid E_n is infinite, the monotone quantity $\tilde{\mathcal{I}}$ for E_n is also infinite by Lemma 3.3.6. This fact and Lemma 3.3.5 give us that $\tilde{\Sigma}$ does not have the finite monotone quantity $\tilde{\mathcal{I}}$. It is a contradiction to Corollary 3.3.4. Then this implies the desired conclusion for the rescaled hypersurface $\tilde{\Sigma}$ with the normalized volume.

□

Corollary 3.3.9. *For the rescaled hypersurface $\tilde{\Sigma}$ with the normalized volume, we have*

$$\tilde{c} \leq \tilde{S}_{min} \leq \tilde{S}_{max} \leq \tilde{C}$$

for some constants $0 < \tilde{c} \leq \tilde{C} < \infty$ and all $\tau \in [0, \infty)$.

Proof. From Corollary 3.3.8, we get $\tilde{S}_{max} \leq \tilde{C}$ for some positive constant $\tilde{C}$, which implies $\tilde{S}_{min} \geq \tilde{c} > 0$ for some constant $\tilde{c}$ since $\tilde{V}(\tau) = 1$.

□

Lemma 3.3.10.

If $\tilde{\Sigma}_0$ is strictly convex, then there is a constant $C > 0$ such that

$$\sup_{z\in\mathbb{S}^n,\ 0\leq\tau} \tilde{K}^\alpha(z,\tau) \leq C = \max\left(\sup_{z\in\mathbb{S}^n} \tilde{K}^\alpha(z,0), \Big(\frac{n\alpha+1}{n\alpha\tilde{\rho}_0}\Big)^{n\alpha}\right)$$

where $\tilde{\rho}_0 = \frac{1}{4}\tilde{w}_{\min}$.

Proof. From the evolution equation of K^α, we have

$$\frac{\partial \tilde{K}^\alpha}{\partial \tau} = \frac{\alpha}{\tilde{\eta}}\tilde{\Box}\tilde{K}^\alpha + \frac{\alpha}{\tilde{\eta}}\tilde{K}^{2\alpha}\alpha\tilde{H} - \frac{n\alpha}{n+1}\tilde{K}^\alpha$$

where $\tilde{\Box} = \tilde{K}^\alpha(\widetilde{h^{-1}})^{ij}\overline{\nabla}_i\overline{\nabla}_j$. By Corollary 3.3.8, we can consider $\tilde{\rho}_0 = \frac{1}{4}\tilde{w}_{\min}$ and then apply the maximum principle to the function $\tilde{\varphi} = \frac{\tilde{K}^\alpha}{\tilde{S}-\rho_0}$. Let us assume that the maximum of $\tilde{\varphi}$ is achieved at the interior point $\tilde{P}_0$ of $\tilde{X}$. Then we have the following properties

$$\tilde{\varphi}_\tau \geq 0,\ \overline{\nabla}_i\tilde{\varphi} = 0 \text{ and } \overline{\nabla}_i\overline{\nabla}_j\tilde{\varphi} \leq 0$$

at P_0. Using the evolution equations of $\tilde{K}^\alpha$ and $\tilde{S}$ and calculating by the similar way to Lemma 3.2.2 implies

$$\begin{aligned} 0 &\leq \frac{\alpha\tilde{K}^{2\alpha}(n-\tilde{S}\tilde{H}}{\tilde{\eta}(\tilde{S}-\tilde{\rho}_0)} + \frac{\alpha}{\tilde{\eta}}\tilde{K}^{2\alpha}\tilde{H} + \frac{\tilde{K}^{2\alpha}}{\tilde{\eta}(\tilde{S}-\tilde{\rho}_0)} - \frac{n\alpha\tilde{K}^\alpha}{n+1} - \frac{\tilde{K}^\alpha\tilde{S}}{(n+1)(\tilde{S}-\tilde{\rho}_0)} \\ &\leq \frac{\tilde{K}^{2\alpha}}{\tilde{\eta}(\tilde{S}-\tilde{\rho}_0)}\Big(n\alpha - \alpha\tilde{\rho}_0\tilde{H} + 1\Big) \end{aligned}$$

at P_0, which gives us that

$$0 \leq (n\alpha+1) - \alpha\tilde{\rho}_0\tilde{H} + 1.$$

Following the same line of the last argument in Lemma 3.2.2, we get

$$\sup_{z\in\mathbb{S}^n,\ \tau\geq 0} \tilde{K}^\alpha(z,\tau) \leq C = \max\left(\sup_{z\in\mathbb{S}^n} \tilde{K}^\alpha(z,0), \Big(\frac{n\alpha+1}{n\alpha\tilde{\rho}_0}\Big)^{n\alpha}\right).$$

□

Lemma 3.3.11. *There is a uniform constant* $0 < \Lambda < \infty$ *such that*

(i) $\dfrac{1}{\Lambda} \leq \tilde{S} \leq \Lambda\,,$

(ii) $\dfrac{1}{\Lambda} \leq \tilde{\eta} \leq \Lambda$ *and*

(iii) $\dfrac{1}{\Lambda^n} \leq \tilde{K} \leq \Lambda^n.$

Proof. (i) $\frac{1}{\Lambda_1} \leq \tilde{S} \leq \Lambda_1$ comes from Corollary 3.3.9 for some $\Lambda_1 > 0$.
(ii) From Lemma 3.3.10, we can derive that $\tilde{\eta}(\tau) \geq \frac{1}{\Lambda_l}$ for some positive constant $\Lambda_l > C^{\frac{1-\alpha}{\alpha}}|\mathbb{S}^n|^{-1}$, where $|\mathbb{S}^n|$ is the volume of $\mathbb{S}^n$ and C is the upper bound of $\tilde{K}^\alpha$. In addition, by the Hölder inequality and $\tilde{V} = 1$, we have

$$\tilde{\eta} = \int_{\mathbb{S}^n} \tilde{K}^{\alpha-1}\, d\sigma_{\mathbb{S}^n} \leq \left(\int_{\mathbb{S}^n} \frac{1}{\tilde{K}}\, d\sigma_{\mathbb{S}^n} \right)^{1-\alpha} \cdot |\mathbb{S}^n|^\alpha \leq \big((n+1)\Lambda_1\big)^{1-\alpha}|\mathbb{S}^n|^\alpha < \Lambda_u \tag{3.3.14}$$

for some positive constant Λ_u. Then we get $\frac{1}{\Lambda_2} \leq \tilde{\eta} \leq \Lambda_2$ by selecting $\Lambda_2 = \max\big(\Lambda_l,\, \Lambda_u\big)$.
(iii) Let us consider the evolution of $\overline{S} = \mu\tilde{S}$ for $\mu > 0$. Let $\overline{K}$ and $\overline{H}$ be the gauss curvature and mean curvature of the hypersurface given by the support function $\overline{S}$, respectively. Then $\overline{K} = \frac{1}{\mu^n}\tilde{K}$, $\overline{H} = \frac{1}{\mu}\tilde{H}$, $\overline{(h^{-1})}^{ij} = \frac{1}{\mu}\widetilde{(h^{-1})}^{ij}$, and $\overline{\eta} = \mu^{(1-\alpha)n}\tilde{\eta}$. Let $\overline{Z}(\tau) = \inf_{z\in\mathbb{S}^n} \overline{K}(z,\tau)$. Then we assume that the interior minimum of $\overline{Z}(\tau)$ is achieved at $\tilde{P}_0 = (z_0, \tau_0)$. From the evolution equation of $\overline{Z}(\tau)$, we have, at $\tilde{P}_0$,

$$\begin{aligned}
\frac{\partial \overline{Z}}{\partial \tau} &= \frac{\alpha\mu^{n+1}}{\overline{\eta}}\overline{Z}^\alpha\overline{(h^{-1})}^{ij}\overline{\nabla}_i\overline{\nabla}_j\overline{Z} + \frac{\alpha(\alpha-1)\mu^{n+1}}{\overline{\eta}}\overline{Z}^{\alpha-1}\overline{(h^{-1})}^{ij}\overline{\nabla}_i\overline{Z}\,\overline{\nabla}_j\overline{Z} \\
&\quad + \frac{\mu^{n+1}}{\overline{\eta}}\overline{Z}\,\overline{H} - \frac{n}{n+1}\overline{Z} \\
&\geq \frac{\mu^{n+1}}{\overline{\eta}}\overline{Z}\,\overline{H} - \frac{n}{n+1}\overline{Z} \\
&\geq \frac{n\mu^{n+1}}{\overline{\eta}}\overline{Z}^{1+1/n} - \frac{n}{n+1}\overline{Z} \\
&\geq n\overline{Z}\Big(\frac{\mu^{n+1}}{\overline{\Lambda}_2}\overline{Z}^{1/n} - \frac{1}{n+1}\Big),
\end{aligned}$$

where $\overline{\Lambda}_2 = \mu^{n(1-\alpha)}\Lambda_2$. Set $Q(\tau) = \frac{\mu^{n+1}}{\overline{\Lambda}_2}\overline{Z}^{1/n}(\tau) - \frac{1}{n+1}$ and choose $\mu > \left(\frac{\Lambda_2}{n+1}\right)^{\frac{1}{n\alpha}}\left(\tilde{Z}(0)\right)^{-\frac{1}{n^2\alpha}}$ for $\tilde{Z}(\tau) = \inf_{z\in\mathbb{S}^n}\tilde{K}(z,\tau)$ and $\overline{Z}(\tau) = \frac{1}{\mu^n}\tilde{Z}(\tau)$, which tells us $Q(0) > 0$. Then the evolution equation of $Q(\tau)$ is

$$\begin{aligned}
\frac{\partial Q}{\partial \tau} &= \frac{\alpha\mu^{n+1}}{\overline{\eta}}\overline{Z}^{\alpha}\overline{(h^{-1})}^{ij}\overline{\nabla}_i\overline{\nabla}_j Q + \left(\alpha - \frac{1}{n}\right)\frac{\alpha n\overline{\Lambda}_2}{\overline{\eta}}\overline{Z}^{\alpha-\frac{1}{n}}\overline{(h^{-1})}^{ij}\overline{\nabla}_i Q\,\overline{\nabla}_j Q \\
&\quad + \frac{\overline{\Lambda}_2^{n-1}}{\mu^{(n+1)(n-1)}}\left(Q + \frac{1}{n+1}\right)^n\left(\frac{\mu^{n+1}}{n\overline{\eta}}\overline{H} - \frac{1}{n+1}\right) \\
&\geq \frac{\overline{\Lambda}_2^{n-1}}{\mu^{(n+1)(n-1)}}\left(Q + \frac{1}{n+1}\right)^n\left(\frac{\mu^{n+1}}{n\overline{\eta}}\overline{H} - \frac{1}{n+1}\right) \\
&\geq \frac{\overline{\Lambda}_2^{n-1}}{\mu^{(n+1)(n-1)}}\left(Q + \frac{1}{n+1}\right)^n Q
\end{aligned}$$

at the interior minimum point since $n\overline{Z}^{\frac{1}{n}} \leq n\overline{K}^{\frac{1}{n}} \leq \overline{H}$. By the maximum principle, we have

$$Q(\tau) \geq Q(0) > 0$$

for all $\tau > 0$, which implies $\frac{\partial\overline{Z}}{\partial\tau} > 0$ at $\tilde{P}_0$ and then it gives us contradiction. Hence we obtain

$$\inf_{z\in\mathbb{S}^n}\overline{K}(z,\tau) \geq \inf_{z\in\mathbb{S}^n}\overline{K}(z,0) > 0,$$

and we also have the desired result $\inf_{z\in\mathbb{S}^n}\tilde{K}(z,\tau) \geq \inf_{z\in\mathbb{S}^n}\tilde{K}(z,0) > 0$ for all τ. Combining with Lemma 3.3.10 implies $\frac{1}{\Lambda_3^n} \leq \tilde{K} \leq \Lambda_3^n$ for some positive constant Λ_3. Now we select $\Lambda = \max_{i=1,2,3}\Lambda_i$.

□

To obtain the regularity of the solution around the maximal time T^*, let us consider the evolution equation (3.3.2). Then the evolution equation for $\tilde{S}$ is

$$\frac{\partial\tilde{S}}{\partial\tau} = -\frac{\tilde{K}^{\alpha}}{\tilde{\eta}} + \frac{1}{n+1}\tilde{S}, \tag{3.3.15}$$

so

$$\left(\frac{1}{n+1}\tilde{S} - \tilde{S}_\tau\right)\left[\det\left(\overline{\nabla}_i\overline{\nabla}_j\tilde{S} + \tilde{S}\delta_{ij}\right)\right]^{\alpha} = \frac{1}{\tilde{\eta}}. \tag{3.3.16}$$

Now we shall derive $C^{1,1}$-estimates for the solution of (3.3.16) as in [23] and [25].

Lemma 3.3.12. *Suppose that $\tilde{S} \in C^4$ is a solution of the equation* (3.3.16). *Then we have*

$$\left|\overline{\nabla}^2 \tilde{S}\right| \leq C \quad on \quad \mathbb{S}^n \times [0, \infty)$$

where C is a positive constant depending on $\tilde{S}$ and the first derivative of $\tilde{S}$ in time and space.

Proof. Let

$$v(z,\tau) = \left|\tilde{S}(z,\tau)\right|\left(\overline{\nabla}_\zeta \overline{\nabla}_\zeta \tilde{S}(z,\tau) + \tilde{S}(z,\tau)\right) \exp\left(\frac{1}{2}\mu\left|\overline{\nabla}_\zeta \tilde{S}(z,\tau)\right|^2 - \rho \tilde{S}(z,\tau)\right),$$

where μ and ρ are positive constants. If the maximum of v is achieved at the initial time, we are done. So we assume that v has its space-time maximum at some interior point $P_0 = (z_0, \tau_0)$ and for some unit vector ζ. We can assume $\zeta = (1, 0, \ldots, 0)$ by choosing an orthonormal frame about z_0 and then the matrix $\{\overline{\nabla}_i \overline{\nabla}_j \tilde{S}(z_0)\}$ is diagonal. Then

$$v(z,\tau) = \left|\tilde{S}(z,\tau)\right|\left(\overline{\nabla}_1 \overline{\nabla}_1 \tilde{S}(z,\tau) + \tilde{S}(z,\tau)\right) \exp\left(\frac{1}{2}\mu\left|\overline{\nabla}_1 \tilde{S}(z,\tau)\right|^2 - \rho \tilde{S}(z,\tau)\right).$$

Let $\mathcal{L}$ be the linearized operator at P_0

$$\mathcal{L} = \frac{1}{\left(\tilde{S}_\tau - \frac{1}{n+1}\tilde{S}\right)(P_0)} \frac{\partial}{\partial \tau} + \alpha F_{ij}(\overline{\nabla}_k \overline{\nabla}_l \tilde{S} + \tilde{S}\delta_{kl})\overline{\nabla}_i \overline{\nabla}_j.$$

Then $\{\overline{\nabla}_k \overline{\nabla}_l \tilde{S} + \tilde{S}\delta_{kl}\}$ is diagonal.
We know that for $F(M) = \log(\det M)$ where M is a positive definite matrix,

$$(F_{ij}) = \frac{\partial F}{\partial M_{ij}} = M^{-1} \quad \text{and} \quad \frac{\partial^2 F}{\partial M_{ij} \partial M_{kl}} = F_{ij,kl} = -F_{ik}F_{jl}.$$

Let

$$\begin{aligned} w &= \log v(z,\tau) \\ &= \log\left|\tilde{S}(z,\tau)\right| + \log\left(\overline{\nabla}_1 \overline{\nabla}_1 \tilde{S}(z,\tau) + \tilde{S}(z,\tau)\right) + \frac{1}{2}\mu\left|\overline{\nabla}_1 \tilde{S}(z,\tau)\right|^2 - \rho \tilde{S}(z,\tau) \end{aligned}$$

which also attains its maximum at P_0, so $\tilde{\nabla} w(P_0) = 0$, $\overline{\nabla}_i \overline{\nabla}_j w(P_0) \leq 0$, and $w_\tau(P_0) \geq 0$.
Since $\overline{\nabla}_i \overline{\nabla}_j \tilde{S} + \tilde{S}\delta_{ij} > 0$ from the strict convexity of $\tilde{\Sigma}$, $(F_{ij}((\overline{\nabla}_i \overline{\nabla}_j \tilde{S} + \tilde{S}\delta_{ij})(z_0)))$ is diagonal, so

$$\begin{aligned}\mathcal{L}(w)(P_0) = &\frac{1}{(\tilde{S}_\tau - \frac{1}{n+1}\tilde{S})(P_0)} \frac{\partial w}{\partial \tau}(P_0) + \alpha F_{ii}((\overline{\nabla}_k \overline{\nabla}_l \tilde{S} + \tilde{S}\delta_{kl})(P_0))\overline{\nabla}_i \overline{\nabla}_j w(P_0) \\ &\leq 0.\end{aligned}$$

From now on, we will use the notation $\overline{\nabla}_{ij}$ in place of $\overline{\nabla}_i \overline{\nabla}_j$ for convenience. We have that at P_0

$$\overline{\nabla}_i w = \frac{\overline{\nabla}_i \tilde{S}}{\tilde{S}} + \frac{\overline{\nabla}_{i11}\tilde{S} + \overline{\nabla}_i \tilde{S}}{\overline{\nabla}_{11}\tilde{S} + \tilde{S}} - \rho \overline{\nabla}_i \tilde{S} = 0 \quad \text{for } i = 2, \ldots, n\,. \tag{3.3.17}$$

In addition, we get

$$\begin{aligned}\overline{\nabla}_i \overline{\nabla}_i w = &\frac{\overline{\nabla}_{ii}\tilde{S}}{\tilde{S}} - \frac{\left(\overline{\nabla}_i \tilde{S}\right)^2}{\tilde{S}^2} + \frac{\overline{\nabla}_{ii11}\tilde{S} + \overline{\nabla}_{ii}\tilde{S}}{\overline{\nabla}_{11}\tilde{S} + \tilde{S}} - \frac{(\overline{\nabla}_{i11}\tilde{S} + \overline{\nabla}_i \tilde{S})^2}{(\overline{\nabla}_{11}\tilde{S} + \tilde{S})^2} + \mu(\overline{\nabla}_{i1}\tilde{S})^2 \\ &+ \mu \overline{\nabla}_1 \tilde{S} \overline{\nabla}_{ii1}\tilde{S} - \rho \overline{\nabla}_{ii}\tilde{S} \leq 0 \quad \text{for all } i\,,\end{aligned} \tag{3.3.18}$$

and

$$w_\tau = \frac{\tilde{S}_\tau}{\tilde{S}} + \frac{\overline{\nabla}_{11}\tilde{S}_\tau + \tilde{S}_\tau}{\overline{\nabla}_{11}\tilde{S} + \tilde{S}} + \mu \overline{\nabla}_1 \tilde{S}\, \overline{\nabla}_1 \tilde{S}_\tau - \rho \tilde{S}_\tau \geq 0 \tag{3.3.19}$$

at the point P_0. Then

$$\begin{aligned}\mathcal{L}(w)(P_0) = &\frac{1}{(\tilde{S}_\tau - \frac{1}{n+1}\tilde{S})(P_0)} \left(\frac{\tilde{S}_\tau}{\tilde{S}} + \frac{\overline{\nabla}_{11}\tilde{S}_\tau + \tilde{S}_\tau}{\overline{\nabla}_{11}\tilde{S} + \tilde{S}} + \mu \overline{\nabla}_1 \tilde{S}\, \overline{\nabla}_1 \tilde{S}_\tau - \rho \tilde{S}_\tau \right) \\ &+ \alpha F_{ii}((\overline{\nabla}_k \overline{\nabla}_l \tilde{S} + \tilde{S}\delta_{kl})(P_0)) \left(\frac{\overline{\nabla}_{ii}\tilde{S}}{\tilde{S}} - \frac{\left(\overline{\nabla}_i \tilde{S}\right)^2}{\tilde{S}^2} + \frac{\overline{\nabla}_{ii11}\tilde{S} + \overline{\nabla}_{ii}\tilde{S}}{\overline{\nabla}_{11}\tilde{S} + \tilde{S}} \right. \\ &\left. - \frac{(\overline{\nabla}_{i11}\tilde{S} + \overline{\nabla}_i \tilde{S})^2}{(\overline{\nabla}_{11}\tilde{S} + \tilde{S})^2} + \mu(\overline{\nabla}_{i1}\tilde{S})^2 + \mu \overline{\nabla}_1 \tilde{S} \overline{\nabla}_{ii1}\tilde{S} - \rho \overline{\nabla}_{ii}\tilde{S} \right) \\ &\leq 0.\end{aligned} \tag{3.3.20}$$

Since $\left(\frac{1}{n+1}\tilde{S}-\tilde{S}_\tau\right)\left[\det\left(\overline{\nabla}_i\overline{\nabla}_j\tilde{S}+\tilde{S}\delta_{ij}\right)\right]^\alpha=\frac{1}{\tilde{\eta}}$, after differentiation and then some calculations, we have

$$\frac{\frac{1}{n+1}\overline{\nabla}_1\tilde{S}-\overline{\nabla}_1\tilde{S}_\tau}{\frac{1}{n+1}\tilde{S}-\tilde{S}_\tau}+\alpha F_{ii}\left\{\overline{\nabla}_{1ii}\tilde{S}+\overline{\nabla}_1\tilde{S}\right\}=0 \quad \text{at} \quad P_0. \tag{3.3.21}$$

Once again the differentiation implies

$$\begin{aligned}\frac{\frac{1}{n+1}\overline{\nabla}_{11}\tilde{S}-\overline{\nabla}_{11}\tilde{S}_\tau}{\frac{1}{n+1}\tilde{S}-\tilde{S}_\tau}-\frac{(\frac{1}{n+1}\overline{\nabla}_1\tilde{S}-\overline{\nabla}_1\tilde{S}_\tau)^2}{(\frac{1}{n+1}\tilde{S}-\tilde{S}_\tau)^2}+\frac{\alpha(\overline{\nabla}_{11ii}\tilde{S}+\overline{\nabla}_{11}\tilde{S})}{\overline{\nabla}_{ii}\tilde{S}+\tilde{S}}\\ -\frac{\alpha(\overline{\nabla}_{1ij}\tilde{S}+\overline{\nabla}_1\tilde{S}\,\delta_{ij})^2}{(\overline{\nabla}_{ii}\tilde{S}+\tilde{S})(\overline{\nabla}_{jj}\tilde{S}+\tilde{S})}=0 \quad \text{at} \quad P_0.\end{aligned} \tag{3.3.22}$$

We use the properties of covariant derivatives:

$$\overline{\nabla}_{kji}\tilde{S}=\overline{\nabla}_{jik}\tilde{S}+\delta_{ik}\overline{\nabla}_j\tilde{S}-\delta_{ij}\overline{\nabla}_k\tilde{S} \tag{3.3.23}$$

and

$$\overline{\nabla}_{lkji}\tilde{S}=\overline{\nabla}_{jilk}\tilde{S}+2\delta_{kl}\overline{\nabla}_{ji}\tilde{S}-2\delta_{ij}\overline{\nabla}_{lk}\tilde{S}+\delta_{il}\overline{\nabla}_{jk}\tilde{S}-\delta_{kj}\overline{\nabla}_{li}\tilde{S}. \tag{3.3.24}$$

After using the formulas (3.3.21)-(3.3.24) and the following properties

$$\begin{aligned}&\frac{\alpha(\overline{\nabla}_{ij1}\tilde{S}+\delta_{j1}\overline{\nabla}_i\tilde{S})^2}{(\overline{\nabla}_{ii}\tilde{S}+\tilde{S})(\overline{\nabla}_{jj}\tilde{S}+\tilde{S})}\\ &\quad=\frac{\alpha(\overline{\nabla}_{i11}\tilde{S}+\overline{\nabla}_i\tilde{S})^2}{(\overline{\nabla}_{ii}\tilde{S}+\tilde{S})(\overline{\nabla}_{11}\tilde{S}+\tilde{S})}+\sum_{i=1}^{n}\sum_{j=2}^{n}\frac{\alpha\left(\overline{\nabla}_{ij1}\tilde{S}\right)^2}{(\overline{\nabla}_{ii}\tilde{S}+\tilde{S})(\overline{\nabla}_{jj}\tilde{S}+\tilde{S})}\end{aligned}$$

and

$$\frac{\alpha\mu(\overline{\nabla}_{11}\tilde{S}+\tilde{S})\overline{\nabla}_1\tilde{S}\,\delta_{i1}\overline{\nabla}_i\tilde{S}}{\overline{\nabla}_{ii}\tilde{S}+\tilde{S}}=\alpha\mu(\overline{\nabla}_1\tilde{S})^2$$

and several computations, (3.3.20) can be simplified to

$$
\begin{aligned}
0 \geq & -\frac{\tilde{S}_\tau(\overline{\nabla}_{11}\tilde{S}+\tilde{S})}{\tilde{S}\big(\frac{1}{n+1}\tilde{S}-\tilde{S}_\tau\big)} - \frac{\tilde{S}_\tau+\overline{\nabla}_{11}\tilde{S}}{\frac{1}{n+1}\tilde{S}-\tilde{S}_\tau} + \frac{\big(\frac{1}{n+1}\overline{\nabla}_1\tilde{S}-\overline{\nabla}_1\tilde{S}_\tau\big)^2}{\big(\frac{1}{n+1}\tilde{S}-\tilde{S}_\tau\big)^2} + \frac{\alpha(\overline{\nabla}_{11}\tilde{S}-\overline{\nabla}_{ii}\tilde{S})}{\overline{\nabla}_{ii}\tilde{S}+\tilde{S}} \\
& + \sum_{i=1}^{n}\sum_{j=2}^{n}\frac{\alpha\big(\overline{\nabla}_{ij1}\tilde{S}\big)^2}{(\overline{\nabla}_{ii}\tilde{S}+\tilde{S})(\overline{\nabla}_{jj}\tilde{S}+\tilde{S})} + \frac{\alpha\overline{\nabla}_{ii}\tilde{S}(\overline{\nabla}_{11}\tilde{S}+\tilde{S})}{\tilde{S}(\overline{\nabla}_{ii}\tilde{S}+\tilde{S})} \\
& - \frac{\alpha\big(\overline{\nabla}_i\tilde{S}\big)^2(\overline{\nabla}_{11}\tilde{S}+\tilde{S})}{\tilde{S}^2(\overline{\nabla}_{ii}\tilde{S}+\tilde{S})} - \alpha\mu(\overline{\nabla}_1\tilde{S})^2 - \frac{\frac{\mu}{n+1}(\overline{\nabla}_{11}\tilde{S}+\tilde{S})(\overline{\nabla}_1\tilde{S})^2}{\frac{1}{n+1}\tilde{S}-\tilde{S}_\tau} \\
& + \frac{\rho(\overline{\nabla}_{11}\tilde{S}+\tilde{S})\tilde{S}_\tau}{\frac{1}{n+1}\tilde{S}-\tilde{S}_\tau} + \frac{\alpha\mu(\overline{\nabla}_{11}\tilde{S}+\tilde{S})(\overline{\nabla}_{i1}\tilde{S})^2}{\overline{\nabla}_{ii}\tilde{S}+\tilde{S}} - \frac{\alpha\rho(\overline{\nabla}_{11}\tilde{S}+\tilde{S})\overline{\nabla}_{ii}\tilde{S}}{\overline{\nabla}_{ii}\tilde{S}+\tilde{S}}.
\end{aligned}
\tag{3.3.25}
$$

In addition, since

$$
\begin{aligned}
& \sum_{i=1}^{n}\sum_{j=2}^{n}\frac{\alpha\big(\overline{\nabla}_{ij1}\tilde{S}\big)^2}{(\overline{\nabla}_{ii}\tilde{S}+\tilde{S})(\overline{\nabla}_{jj}\tilde{S}+\tilde{S})} - \frac{\alpha\big(\overline{\nabla}_i\tilde{S}\big)^2(\overline{\nabla}_{11}\tilde{S}+\tilde{S})}{\tilde{S}^2(\overline{\nabla}_{ii}\tilde{S}+\tilde{S})} \\
& \qquad = \sum_{i=2}^{n}\sum_{j=2}^{n}\frac{\alpha\big(\overline{\nabla}_{ij1}\tilde{S}\big)^2}{(\overline{\nabla}_{ii}\tilde{S}+\tilde{S})(\overline{\nabla}_{jj}\tilde{S}+\tilde{S})} + \sum_{i=2}^{n}\frac{2\alpha\rho\big(\overline{\nabla}_{1i1}\tilde{S}\big)\overline{\nabla}_i\tilde{S}}{\overline{\nabla}_{ii}\tilde{S}+\tilde{S}} \\
& \qquad - \sum_{i=2}^{n}\frac{\alpha\rho^2(\overline{\nabla}_{11}\tilde{S}+\tilde{S})\big(\overline{\nabla}_i\tilde{S}\big)^2}{\overline{\nabla}_{ii}\tilde{S}+\tilde{S}} - \frac{\alpha\big(\overline{\nabla}_1\tilde{S}\big)^2}{\tilde{S}^2}
\end{aligned}
\tag{3.3.26}
$$

from (3.3.23) and (3.3.17), we have

$$
\begin{aligned}
0 \geq & -\frac{\tilde{S}_\tau(\overline{\nabla}_{11}\tilde{S}+\tilde{S})}{\tilde{S}\big(\frac{1}{n+1}\tilde{S}-\tilde{S}_\tau\big)} - \frac{\tilde{S}_\tau+\overline{\nabla}_{11}\tilde{S}}{\frac{1}{n+1}\tilde{S}-\tilde{S}_\tau} + \frac{\big(\frac{1}{n+1}\overline{\nabla}_1\tilde{S}-\overline{\nabla}_1\tilde{S}_\tau\big)^2}{\big(\frac{1}{n+1}\tilde{S}-\tilde{S}_\tau\big)^2} + \frac{\alpha\overline{\nabla}_{11}\tilde{S}}{\tilde{S}} \\
& + \sum_{i=2}^{n}\sum_{j=2}^{n}\frac{\alpha\big(\overline{\nabla}_{ij1}\tilde{S}\big)^2}{(\overline{\nabla}_{ii}\tilde{S}+\tilde{S})(\overline{\nabla}_{jj}\tilde{S}+\tilde{S})} + \sum_{i=2}^{n}\frac{2\alpha\rho\big(\overline{\nabla}_{1i1}\tilde{S}\big)\overline{\nabla}_i\tilde{S}}{\overline{\nabla}_{ii}\tilde{S}+\tilde{S}} \\
& - \sum_{i=2}^{n}\frac{\alpha\rho^2(\overline{\nabla}_{11}\tilde{S}+\tilde{S})\big(\overline{\nabla}_i\tilde{S}\big)^2}{\overline{\nabla}_{ii}\tilde{S}+\tilde{S}} - \frac{\alpha\big(\overline{\nabla}_1\tilde{S}\big)^2}{\tilde{S}^2} - \frac{\frac{\mu}{n+1}(\overline{\nabla}_{11}\tilde{S}+\tilde{S})(\overline{\nabla}_1\tilde{S})^2}{\frac{1}{n+1}\tilde{S}-\tilde{S}_\tau} \\
& - \alpha\mu(\overline{\nabla}_1\tilde{S})^2 + \frac{\rho(\overline{\nabla}_{11}\tilde{S}+\tilde{S})\tilde{S}_\tau}{\frac{1}{n+1}\tilde{S}-\tilde{S}_\tau} + \alpha\mu\big(\overline{\nabla}_{11}\tilde{S}\big)^2 - \frac{\alpha\rho(\overline{\nabla}_{11}\tilde{S}+\tilde{S})\overline{\nabla}_{ii}\tilde{S}}{\overline{\nabla}_{ii}\tilde{S}+\tilde{S}}.
\end{aligned}
\tag{3.3.27}
$$

Let $\gamma_i = \overline{\nabla}_{ii}\tilde{S} + \tilde{S}$. Then (3.3.27) can be written as

$$\begin{aligned} 0 \geq & -\alpha - \frac{\tilde{S}_\tau \gamma_1}{\tilde{S}\left(\frac{1}{n+1}\tilde{S} - \tilde{S}_\tau\right)} - \frac{\gamma_1}{\frac{1}{n+1}\tilde{S} - \tilde{S}_\tau} + \frac{\alpha\gamma_1}{\tilde{S}} + \sum_{i=2}^{n} \frac{\alpha\rho\left(\overline{\nabla}_i\tilde{S}\right)^2\gamma_1}{\gamma_i}\left(\rho - \frac{2}{\tilde{S}}\right) \\ & - \frac{\alpha\left(\overline{\nabla}_1\tilde{S}\right)^2}{\tilde{S}^2} - \alpha\mu(\overline{\nabla}_1\tilde{S})^2 - \frac{\frac{\mu}{n+1}(\overline{\nabla}_1\tilde{S})^2\gamma_1}{\frac{1}{n+1}\tilde{S} - \tilde{S}_\tau} + \frac{\rho\tilde{S}_\tau\gamma_1}{\frac{1}{n+1}\tilde{S} - \tilde{S}_\tau} + \alpha\mu{\gamma_1}^2 \\ & - 2\alpha\mu\tilde{S}\gamma_1 + \alpha\mu\tilde{S}^2 - \alpha\rho\gamma_1 + \frac{\alpha\rho\tilde{S}\gamma_1}{\gamma_i} \end{aligned} \tag{3.3.28}$$

at P_0. We obtained the lower and upper bounds of $\tilde{S}$ on $[0, \infty)$ in Lemma 3.3.11, and $|\overline{\nabla}_i\tilde{S}|$ also is bounded for $i = 1, \ldots, n$ since $\tilde{\Sigma}$ is strictly convex. In addition, since $\frac{1}{n+1}\tilde{S} - \tilde{S}_\tau$ has the positive lower bound from Lemma 3.3.11, choosing μ and ρ such that

$$0 \geq A{\gamma_1}^2 + B\gamma_1 + C_1,$$

where A is a positive constant and B and C_1 are some constants, give us the desired result.

□

Corollary 3.3.13. *There exist some positive constants C such that*

$$\sup_{x\in\tilde{\Sigma},\tau\geq 0} \tilde{\mathcal{H}} \leq C.$$

Moreover, $\tilde{\lambda}_{min} \geq C_1 > 0$ for some constant C_1. Here $\tilde{\lambda}_{min} = \tilde{\lambda}_1 \leq \cdots \leq \tilde{\lambda}_n = \tilde{\lambda}_{max}$ where $\tilde{\lambda}_i's$ are the eigenvalues of $(\tilde{h}^i_j)$.

Furthermore, combining Lamma 3.3.10 and Corollary 3.3.13 implies the following Corollary.

Corollary 3.3.14. *All curvatures on the rescaled hypersuface $\tilde{\Sigma}$ are bounded above and below by the uniform constants. In other words there exists some constant $0 < M < \infty$ such that*

$$\frac{1}{M} \leq \tilde{\lambda}_{min} \leq \tilde{\lambda}_{max} \leq M.$$

3.4 Existence of solutions and proof of main theorem

3.4.1 Short time existence

Let us assume that Σ_t is smooth. Then we get the uniform $C^{1,1}$ estimates of the coefficient of our equation (2.2.1) and this equation becomes uniformly parabolic. Thus the regularity theory of uniform parabolic equations and application of the implicit function theorem give us the short time existence as in [33].

3.4.2 Long time existence

Let λ_i be the eigenvalues of (h^i_j). We know that λ_i is positive by the strict convexity. Also we have $K = \lambda_1 \cdots \lambda_n \leq C_1$ and $\mathcal{H} = \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n} \leq C_2$ from Lemma 3.2.2 and Lemma 3.2.3, where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and C_1 and C_2 are some positive constants. These give us, for each $i = 1, \ldots, n$,

$$0 < \frac{1}{C_2} \leq \lambda_i$$

from $\frac{1}{\lambda_i} < \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n} \leq C_2$ and also

$$0 < \lambda_i \leq \frac{C_1}{\Pi_{j \neq i} \lambda_j} \leq C_1 C_2^{n-1},$$

which imply there are $0 < \lambda \leq \Lambda < \infty$ satisfying

$$\lambda |\xi|^2 \leq K^\alpha (h^{-1})^{ij} \xi_i \xi_j \leq \Lambda |\xi|^2.$$

Then we know that the support function $S(z,t)$ satisfies a uniformly parabolic equation in Σ_t. Hence $S(z,t)$ is $C^{2,\gamma}$ and then C^∞ in Σ_t through the standard bootstrap argument using the Schauder theory. If there is a $0 < T_1 < T^*$ such that Σ_t is smooth on $[0, T_1)$ but not smooth after T_1, the uniform $C^{2,\gamma}$-estimate for $S(z,t)$ implies that Σ_{T_1} is $C^{2,\gamma}$, and therefore C^∞. From the short time existence and uniqueness, Σ_t is C^∞ on $[0, T_1 + \delta)$. It is a contradiction. Therefore $T_1 = T^*$ for some small $\delta > 0$ and there is a smooth solution Σ_t

on $[0, T^*)$. Also, the solution Σ_t will be strictly convex by Lemma 3.2.1.

Proof of Theorem 3.1.1. We have the uniform bounds of curvature and all of the higher derivatives of the second fundamental form to the rescaled manifold by Corollary 3.3.14 and then the equation (3.3.16) will be uniformly parabolic. In addition, we have $C^{1,1}$-regularity of the solution $\tilde{S}$ from Lemma 3.3.12. By applying the Harnack inequality to the linearized equation satisfied by $\tilde{S}_\tau$, we obtain that $\tilde{S}_\tau$ is Hölder continuous through a similar argument as in [23]. We can apply Evans-Krylov theorem and Schauder estimates (see [7]) to the concave operator obtained by taking exponent $\frac{1}{n\alpha}$ to the equation (3.3.16), which implies $C^{2,\gamma}$-regularity of $\tilde{S}$ for $0 < \gamma < 1$. And then we have the smooth and strictly convex rescaled solution by the standard bootstrap argument using Schauder theory and Corollary 3.3.14. In other words, for every sequence of $\tau_k \to \infty$, we can find a subsequence τ_{k_i} such that $\tilde{S}(\cdot, \tau_{k_i}) \to \tilde{S}_*(\cdot)$. Also the integral quantity

$$\tilde{\mathcal{I}}(\tau) \begin{cases} \left(\displaystyle\int_{\mathcal{S}^n} \frac{1}{\tilde{S}^{\frac{1}{\alpha}-1}} \, d\sigma_{\mathcal{S}^n} \right)^{\operatorname{sgn}(\alpha-1)} & \text{for } \alpha > 0 \text{ and } \alpha \neq 1, \\ \displaystyle\int_{\mathcal{S}^n} \log \tilde{S} \, d\sigma_{\mathcal{S}^n} & \text{for } \alpha = 1 \end{cases}$$

satisfies the monotonicity $\frac{d}{d\tau}\tilde{\mathcal{I}}(\tau) \leq 0$, and equality holds if and only if $\tilde{K}^\alpha = C\tilde{S}$, for some positive constant C, holds for a choice of origin. For the limit manifold $\tilde{\Sigma}_{T^*}$ of the volume rescaled manifold $\tilde{\Sigma}_{\tau_{i_k}}$, following the same argument as in Theorem 16, [2], $\tilde{\mathcal{I}}(\tau) \to -\infty$ if $\tilde{\Sigma}_{T^*}$ does not satisfy $\tilde{K}_*^\alpha = \tilde{C}_* \tilde{S}_*$ a.e. for some positive constant $\tilde{C}_*$, which gives a contradiction. Therefore the proof is complete.

□

Chapter 4

α-Gauss Curvature Flows with Flat Sides and Free Boundary Problems

In this chapter, we shall study the regularity of the solutions Σ_t of α-Gauss curvature flows (1.0.1) for the initial surface Σ_0 in $\mathbb{R}^3$ with a flat spot and $\frac{1}{2} < \alpha \leq 1$.

4.1 Preliminaries

4.1.1 The balance of terms

We will assume for simplicity that the initial surface Σ_0 has only one flat spot, namely that at t we have $\Sigma_t = \Sigma_t^1 \cup \Sigma_t^2$ where Σ_t^1 is the flat spot and Σ_t^2 is strictly convex part of Σ_t. The intersection between two regions is the free boundary $\Gamma_t = \Sigma_t^1 \cap \Sigma_t^2$. The lower part of the surface Σ_0 can be written as a graph $z = f(x)$. And similarly we can write the lower part of Σ_t as $z = f(x,t)$ for $x \in \Omega \subset \mathbb{R}^n$ where Ω is an open subset of $\mathbb{R}^n$.

The function $f(x,t)$ satisfies α-Gauss Curvature flow:

$$f_t = \frac{[\det(D^2 f)]^\alpha}{(1+|\nabla f|^2)^{\frac{\alpha(n+2)-1}{2}}}. \tag{4.1.1}$$

Let us first consider the rotationally symmetric case to see the balance between terms for $n = 2$. If $f = f(r)$ is rotationally symmetric, (4.1.1) can be written as

$$f_t = \frac{f_r^\alpha f_{rr}^\alpha}{r^\alpha (1 + f_r^2)^{\frac{4\alpha-1}{2}}} \tag{4.1.2}$$

Let $r = \gamma(t)$ be the equation of the free boundary $\Gamma(f) = \partial\{f = 0\}$. The speed of the boundary is given by

$$\gamma_t = -\frac{f_t}{f_r} = -\frac{f_r^{\alpha-1} f_{rr}^\alpha}{r^\alpha (1 + f_r^2)^{\frac{4\alpha-1}{2}}}.$$

The regularity comes from the non-degenerate finite speed of the free boundary before the flat spot converges to a lower dimensional singularity at a focusing time. When $f = (r-1)_+^\beta$ at a given time t, for $r \approx 1$,

$$|\gamma_t| = s^{(\alpha-1)(\beta-1)} s^{\alpha(\beta-2)} \approx 1$$

for $s = r - 1$, which implies $\beta = \frac{3\alpha-1}{2\alpha-1}$.

For a general $f = f(x, y, t)$, let $f = \frac{1}{\beta} g^\beta$ for $\beta = \frac{3\alpha-1}{2\alpha-1}$. The equation for this pressure g will be

$$g_t = \frac{[g \det(D^2 g) + \theta(\alpha)(g_x^2 g_{yy} + g_y^2 g_{xx} - 2 g_x g_y g_{xy})]^\alpha}{(1 + g^{2\beta-2} |\nabla g|^2)^{\frac{4\alpha-1}{2}}} \tag{4.1.3}$$

for $\theta(\alpha) = \beta - 1 = \frac{\alpha}{2\alpha-1}$.

Assuming $g_\tau = 0$ at the boundary, the speed of the boundary will be

$$\gamma_t = -\frac{g_t}{g_\nu} = -\theta(\alpha)^\alpha \, g_\nu^{2\alpha-1} g_{\tau\tau}^\alpha \tag{4.1.4}$$

for a tangential direction τ and a normal direction ν to $\partial\Omega$.

4.1.2 Conditions for f

Condition 4.1.1. Set $\Lambda(f) = \{f = 0\}$ and $\Gamma(f) = \partial\Lambda(f)$.

(I) (*Non-degeneracy Condition*) Our basic assumption on the initial surface is that the function f vanishes of the order $\mathrm{dist}(X, \Lambda(f))^{\frac{3\alpha-1}{2\alpha-1}}$ and that the interface $\Gamma(f)$ is strictly convex so that the interface moves with finite non-degenerate speed. Namely, setting $g = (\beta f)^{\frac{1}{\beta}}$, we assume that at time $t = 0$ the function g satisfies the following non-degeneracy condition: at $t = 0$,

$$0 < \lambda < |Dg(X)| < \frac{1}{\lambda} \qquad \text{and} \quad 0 < \lambda^2 < D^2_{\tau\tau} g(X) < \frac{1}{\lambda^2} \tag{4.1.5}$$

for all $X \in \Gamma_0$ and some positive number $\lambda > 0$, where $D^2_{\tau\tau}$ denotes the second order tangential derivative at Γ. Then the initial speed of free boundary has the speed, at $t = 0$,

$$0 < \lambda^{4\alpha-1} < |\gamma_t| < \frac{1}{\lambda^{4\alpha-1}}. \tag{4.1.6}$$

(II) (*Before focusing of flat spot*) Let T be any number on $0 < T < T_0$, so that the flat side Σ^1_t is non-zero. Since the area is non-zero, Σ^1_t contains a disc D_{ρ_0} for some $\rho_0 > 0$. We may assume that

$$D_{\rho_0} = \left\{ X \in \mathbb{R}^2 \,:\, |X| \le \rho_0 \right\} \subset \Sigma^1_t \quad \text{for } 0 \le t \le T_0. \tag{4.1.7}$$

(III) (*Graph on a neighborhood of the flat spot* Σ^1_t) We will also assume, without loss of generality, throughout this chapter that

$$\max_{x \in \Omega(t)} f(\cdot, t) \ge 2, \quad 0 \le t \le T_0 \tag{4.1.8}$$

where $\Omega(t) = \{X = (x, y) \in \mathbb{R}^2 \,:\, |Df|(X, t) < \infty\}$. Set

$$\Omega_P(t) = \left\{ x \in \mathbb{R}^2 \,:\, f(x, y, t) \le f(P) \right\}. \tag{4.1.9}$$

4.1.3 The concept of regularity

Let us assume $P_0 = (x_0, y_0, t_0)$ is an interface point and t_0 is sufficiently small. Then condition (4.1.5) is satisfied at t_0 for a small constant c. We can assume

$$g_x(P_0) \ge c > 0 \quad \text{for some} \quad c > 0 \tag{4.1.10}$$

by rotating the coordinates. Also by transforming the free-boundary to a fixed boundary near P_0, we can obtain the map $x = h(z, y, t)$ where (z, y, t) is around $Q_0 = (0, y_0, t_0)$ and then the free-boundary $g = 0$ is transformed into the fixed boundary $z = 0$. From the calculation on $g(h(z, y, t), y, t) = z$, we have the fully nonlinear degenerate equation

$$h_t = -\frac{\left\{z\left(h_{zz}\, h_{yy} - h_{zy}^2\right) - \theta(\alpha)\, h_z h_{yy}\right\}^{\alpha}}{\left\{z^{2(\beta-1)} + h_z^2 + z^{2(\beta-1)}\, h_y^2\right\}^{\frac{4\alpha-1}{2}}}, \qquad z > 0 \tag{4.1.11}$$

implying that under (4.1.5) and initial regularity conditions, the linearized operator

$$\tilde{h}_t = z\, a_{11}\tilde{h}_{zz} + 2\sqrt{z}\, a_{12}\tilde{h}_{zy} + a_{22}\tilde{h}_{yy} + b_1\tilde{h}_z + b_2\tilde{h}_y \tag{4.1.12}$$

where (a_{ij}) is strictly positive and $b_1 \geq \nu > 0$ for some $\nu > 0$.

Definition 4.1.2. *For the Riemannian metric ds with $ds^2 = \frac{dz^2}{z} + dy^2$, let the distance between $Q_1 = (z_1, y_1)$ and $Q_2 = (z_2, y_2)$ in the metric s be $s(Q_1, Q_2) = |\sqrt{z_1} - \sqrt{z_2}| + |y_1 - y_2|$ and the parabolic distance between $Q_1 = (z_1, y_1, t_1)$ and $Q_2 = (z_2, y_2, t_2)$ be $s(Q_1, Q_2) = |\sqrt{z_1} - \sqrt{z_2}| + |y_1 - y_2| + \sqrt{|t_1 - t_2|}$. Then we define C_s^{γ}, $\gamma \in (0, 1)$, as the space of Hölder continuous functions with respect to the metric s and $C_s^{2+\gamma}$ as the space of all functions h with*

$$h, h_z, h_y, h_t, z\, h_{zz}, \sqrt{z}\, h_{zy}, h_{yy} \in C_s^{\gamma}.$$

Remark 4.1.3. When we consider the equation

$$h_t = z\, h_{zz} + h_{yy} + \nu\, h_z \tag{4.1.13}$$

on the half-space with $\nu > 0$, which does not have the other condition of h on $z = 0$, the Riemannian metric ds decides the diffusion of the equation.

Remark 4.1.4. If the transformed function $h \in C_s^{2+\gamma}$, we say that $g \in C_s^{2+\gamma}$ around the interface Γ.

4.2 Main theorems

Now we shall state the main theorems.

Theorem 4.2.1. *Let us assume $\frac{1}{2} < \alpha \leq 1$. If Σ_0 is convex, then any viscosity solution Σ_t of* (1.0.1) *is $C^{1,1}$ for $0 < t < T_0$. Moreover the strictly convex part, Σ_t^2, is smooth for $0 < t < T_0$.*

The following short time existence of C_s^∞-solution with a flat spot has been essentially proved in [15] since the linearized equation for h, (4.6.3), is in the same class of operators considered in [15] because of the conditions, (4.1.5), as in [15]. Therefore the Schauder theory can be applied to (4.6.3) and then the application of the implicit function theorem gives the short time existence as in [15].

Theorem 4.2.2 (Short Time Regularity, [15])**.** *For $\frac{1}{2} < \alpha \leq 1$, assume that $g = (\beta f)^{\frac{1}{\beta}}$ is of class $C^{2+\gamma}$ up to the interface $z = 0$ at time $t = 0$, for some $0 < \gamma < 1$, and satisfies Conditions 4.1.1 for f. Then there exists a time $T > 0$ such that the $\alpha-$Gauss Curvature Flow* (1.0.1) *admits a solution $\Sigma(t)$ on $0 \leq t \leq T$. In addition the function $g = (\beta f)^{\frac{1}{\beta}}$ is smooth up to the interface $z = 0$ on $0 < t \leq T$. In particular the junction $\Gamma(t)$ between the strictly convex and the flat side will be a smooth curve for all t in $0 < t \leq T$.*

One of the main results in this chapter is the following long time regularity of the solution.

Theorem 4.2.3 (Long Time Regularity)**.** *Under the assumptions of Theorem 4.2.2, the function $g = (\beta f)^{\frac{1}{\beta}}$ remains smooth up to the interface $z = 0$ on $0 < t < T$ for all $T < T_0$. And the interface Γ_t between the strictly convex and the flat side will be a smooth curve for all t in $0 < t < T_0$.*

To show Theorem 4.2.3, we follow the main steps in [19]. However, the exponent α creates a large number of nontrivial terms, especially in the estimate of the second derivatives. New quantities have been considered to absorb the

effect of terms depending on $(1-\alpha)$ in Lemma 4.5.3. Optimal regularity and Aronson-Bénilan type estimate have been proved in Lemmas 4.5.4 and 4.5.5.

4.3 Convex surfaces

4.3.1 Curvature estimates

Now we shall show the regularity of Σ_t. The following lemma was proved in [2, 29].

Lemma 4.3.1. *Let Σ_0 be strictly convex and $\alpha > 0$. Then:*

(i) There is a constant $C > 0$ such that

$$\sup_{x\in\Sigma,\ 0<t<T_0} K^\alpha(x,t) \le C = \max\left(\sup_{x\in\Sigma} K^\alpha(x,0), \left(\frac{2\alpha+1}{2\alpha\rho_0}\right)^{2\alpha}\right).$$

(ii) $\inf_{x\in\Sigma,\ 0<t} K^\alpha \ge \inf_{x\in\Sigma} K^\alpha(x,0) > 0$ *as long as it is smooth.*

(iii) There is a unique viscosity solution Σ_t.

Lemma 4.3.2. *Set $\psi(x,t) = \langle x,\nu\rangle$ and let $B_{R_0}(0)$ be a ball of radius R_0 about the origin and $P = \frac{H}{\psi+4R^2-|x|^2}$, where Σ_0 is contained in $B_{R_0}(0)$ and $R^2 = \max({R_0}^2, R_0)$. Then there exists a positive constant C for $\frac{1}{2} < \alpha \le 1$ such that*

$$\sup_{x\in\Sigma,\ 0\le t<T_0} H(x,t) \le C,$$

where $C = C(\sup_{x\in\Sigma,\ 0\le t<T_0} K^\alpha, R) > 0$.

Proof. Since $|x|$ is decreasing, $\psi + 4R^2 - |x|^2$ is positive and then we have

$$\frac{\partial}{\partial t}|x|^2 = \Box|x|^2 + 2K^\alpha\langle x,\nu\rangle - 2K^\alpha(h^{-1})^{kl}g_{kl}.$$

By using $\nabla_i P = 0$ at the maximum point, we can obtain

$$\alpha\Box P = \frac{\alpha\Box H}{\psi + 4R^2 - |x|^2} + \frac{\alpha H\Box|x|^2}{(\psi + 4R^2 - |x|^2)^2} - \frac{\alpha H\Box\psi}{(\psi + 4R^2 - |x|^2)^2}$$

and then since $\nabla_i\nabla_j P \leq 0$ at the maximum point, we get

$$\begin{aligned}\frac{\partial}{\partial t}P \leq{}& \frac{(1-\alpha)H\Box|x|^2}{(\psi + 4R^2 - |x|^2)^2} + \frac{H\big((2\alpha+1)K^\alpha + 2K^\alpha\psi\big)}{(\psi + 4R^2 - |x|^2)^2} - \frac{H^2(\alpha K^\alpha\psi + 2K^{\alpha-1})}{(\psi + 4R^2 - |x|^2)^2}\\ &+ \frac{\alpha K^\alpha}{\psi + 4R^2 - |x|^2}\Big(\alpha g^{ij}(h^{-1})^{kl}(h^{-1})^{mn}\nabla_i h_{kl}\nabla_j h_{mn}\\ &\quad - g^{ij}(h^{-1})^{km}(h^{-1})^{nl}\nabla_i h_{mn}\nabla_j h_{kl}\Big)\\ &+ \frac{1}{\psi + 4R^2 - |x|^2}\big(\alpha K^\alpha H^2 + (1-2\alpha)K^\alpha|A|^2\big)\end{aligned} \tag{4.3.1}$$

at the maximum point. Now, we can estimate the fourth term of (4.3.1) by the following inequality

$$\begin{aligned}&\alpha g^{ij}(h^{-1})^{kl}(h^{-1})^{mn}\nabla_i h_{kl}\nabla_j h_{mn} - g^{ij}(h^{-1})^{km}(h^{-1})^{nl}\nabla_i h_{mn}\nabla_j h_{kl}\\ =&(\alpha-1)\Big[\big((h^{-1})^{11}\nabla_1 h_{11} + (h^{-1})^{22}\nabla_1 h_{22}\big)^2 + \big((h^{-1})^{11}\nabla_2 h_{11} + (h^{-1})^{22}\nabla_2 h_{22}\big)^2\Big]\\ &+ 2(h^{-1})^{11}(h^{-1})^{22}\Big\{\nabla_1 h_{11}\nabla_1 h_{22} + \nabla_2 h_{11}\nabla_2 h_{22}\Big\}\\ &- 2(h^{-1})^{11}(h^{-1})^{22}\Big\{(\nabla_2 h_{11})^2 + (\nabla_1 h_{22})^2\Big\}\\ \leq& 2(h^{-1})^{11}(h^{-1})^{22}\Big\{-(\nabla_1 h_{22})^2 - P\nabla_1 h_{22}(\nabla_1|x|^2 - \nabla_1\psi) - (\nabla_2 h_{11})^2\\ &- P\nabla_2 h_{11}(\nabla_2|x|^2 - \nabla_2\psi)\Big\}\\ \leq& 2(h^{-1})^{11}(h^{-1})^{22}\Big(1 - \frac{\tilde{h}}{2}\Big)^2\Big(|x|^2 - \langle x,\nu\rangle^2\Big)P^2\end{aligned}$$

where $\tilde{h} = \min\{h_{11}, h_{22}\}$. Hence

$$\begin{aligned}\frac{\partial}{\partial t}P \leq \Bigg[&\frac{2\alpha K^{\alpha-1}\Big(1-\frac{\tilde{h}}{2}\Big)^2}{\psi+4R^2-|x|^2}(|x|^2-\psi^2)+(1-2\alpha)K^\alpha\psi \\ &+K^\alpha\big\{4(1-\alpha)R^2-(1-\alpha)|x|^2-2K^{\alpha-1}\big\}\Bigg]P^2+\frac{2(2\alpha-1)K^{\alpha+1}}{\psi+4R^2-|x|^2} \\ &+\frac{1}{\psi+4R^2-|x|^2}\Bigg[(1-\alpha)\Box|x|^2+\big\{2\psi+(2\alpha+1)\big\}K^\alpha\Bigg]P.\end{aligned}$$

For $\frac{1}{2} < \alpha \leq 1$, we can make the coefficient of P^2 negative, which can be achieved if η is small enough. The reason is if we begin with $\eta\Sigma_0$ for any given Σ_0, we can make $K \geq \frac{C_0}{\eta^2}$ where C_0 is some constant depending on initial surface, which comes from Lemma 4.3.1, and $|x|^2 \leq \eta^2$, $R^2 \leq \eta^2$, and $\psi \leq \eta$ for sufficiently small η. Then the first term and second term of coefficient of P^2 are $O(\eta^{2-2\alpha})$ and the third term is negative with $K^\alpha\psi = O(\eta^{1-2\alpha})$ for η small enough. This implies $\frac{\partial P}{\partial t} \leq -\frac{1}{2}P^2 + C$ where $C = C(\sup_{x\in\Sigma,\ 0\leq t<T_0} K^\alpha, R)$ and then if $-\frac{1}{2}P^2 + C < 0$, it is a contradiction. So P is bounded and hence H is bounded before Σ shrinks to a point.

□

4.3.2 Strict convexity away from the flat spot

To apply the Harnack principle, let us introduce new coordinates defined on the sphere S^n. If Σ_t is strictly convex, $\nu(x,t)$ is a one-to-one map from Σ_t to S^n, which means that for each $z \in S^n$, there is $X(x,t) = \nu^{-1}(z,t)$. $K(z,t)$ denotes Gauss curvature K at $\nu^{-1}(z,t)$. If Σ_t is convex, we still use the same coordinates (z,t) for the strictly convex part Σ_t^2 by using strictly convex surfaces as approximations. Also the support function $S(z,t)$ of the strictly convex surface is given as

$$S(z,t) = \langle z, X(\nu^{-1}(z),t)\rangle, \qquad \text{for } \ (z,\, t) \in S^n \times [0,\, T_0].$$

Lemma 4.3.3. *Assume that the flat spot, Σ_0^1, is a part of the plane orthogonal to e_{n+1}. For any $\eta > 0$, there is a constant $c_\eta > 0$ such that*

$$K^\alpha(z,t) \geq c_\eta$$

for $z \in \mathcal{S}_\eta := \{z \in S^n \text{ and} \|z + e_{n+1}\| \geq \eta > 0\}$.

Proof. We can immediately obtain the result from the Harnack estimate in [12]:
For any points $z_1, z_2 \in \mathcal{S}_\eta$ and times $0 \leq t_1 < t_2$

$$\frac{K^\alpha(z_2,t_2)}{K^\alpha(z_1,t_1)} \geq e^{-\Theta/4}\left(\frac{t_2}{t_1}\right)^{-(1+(2\alpha)^{-1})^{-1}}$$

where $\Theta = \Theta(z_1,z_2,t_1,t_2) = \inf_\gamma \int_{t_1}^{t_2} |d_t\gamma(t)|^2_{m(t)} dt$ and the infimum is taken over all paths γ in Σ whose graph $(\gamma(t),t)$ joins (z_1,t_1) to (z_2,t_2). The short time existence of smooth surfaces implies that, for $z \in \mathcal{S}_\eta$, $X(z,t)$ is the strictly convex part, Σ_t^2, for $0 \leq t \leq \delta_0$ for some $\delta_0 > 0$. Therefore we can take $0 < \delta_0 \leq t_1 < t_2 \leq T$, which implies $K^\alpha(z_2,t_2) \geq c_1 K^\alpha(z_1,t_1) \geq c_\eta$ for some $c_1, c_\eta > 0$ and then the conclusion.

□

We finally know (4.1.1) is uniformly parabolic, which comes from Lemmas 4.3.1-4.3.2. Therefore we can show that Σ_t is C^∞ on the point being away from flat spot.

Corollary 4.3.4. *Under the conditions of Lemma 4.3.3, $\tilde{\Sigma}_t^2 := \{X(z,t) \in \Sigma_t^2 : z \in \mathcal{S}_{2\eta}\}$ is smooth.*

Proof. Let λ_i be the eigenvalues of (h^i_j). From the convexity, $\lambda_i \geq 0$. And from the upper bound of Mean Curvature and the lower bound of Gauss Curvature, $\lambda_1 + \cdots + \lambda_n < C_1$ and $K = \lambda_1 \cdots \lambda_n > c_2$. Now we have

$$C_1 \geq \lambda_i \geq \frac{c_2}{\Pi_{j\neq i}\lambda_j} \geq \frac{c_2}{C_1^{n-1}} > 0.$$

It implies there are $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda|\xi|^2 \leq K^\alpha (h^{-1})^{ij}\xi_i\xi_j \leq \Lambda|\xi|^2$$

and the support function $S(z,t)$ satisfies a uniformly parabolic equation in $\tilde{\Sigma}_t^2$. Therefore $S(z,t)$ is $C^{2,\gamma}$ and then C^∞ in $\tilde{\Sigma}_t^2$ through the standard bootstrap argument using the Schauder theory.

□

4.3.3 Proof of Theorem 4.2.1

Recall that $|A|^2$ is the square sum of principle curvatures of a given surface. First, we approximate the initial surface Σ_0 with strictly convex smooth functions, $\Sigma_{0,\varepsilon}$ whose $|A_{0,\varepsilon}|^2$ is uniformly bounded by $2|A_0|^2$ of Σ_0. Then there are smooth solutions $\Sigma_{t,\varepsilon}$ of (1.0.1) [29], and $|A_{0,\varepsilon}|^2 \leq 2H_\varepsilon^2 < 8|A_0|^2 < C$ uniformly. As $\varepsilon \to 0$, $\Sigma_{t,\varepsilon}$ converges to a viscosity solution Σ_t as in [1]. $|A_t|^2$ of Σ_t will be uniformly bounded, which implies that Σ_t is $C^{1,1}$. And for any $X \in \Sigma_t^2$, there is a small $\eta > 0$ such that $\|\nu_X + e_{n+1}\| \geq \eta > 0$ and then $X \in \tilde{\Sigma}_t^2$. Since $\tilde{\Sigma}_t^2$ is smooth at X, so is Σ_t^2.

4.3.4 A waiting time effect

We will now show that the flat spot of the convex surface persists for some time.

Lemma 4.3.5. *Let Σ_0 be convex. For $\frac{1}{2} < \alpha \leq 1$, there is a waiting time of the flat spot: if $P_0 \in int_n(\Sigma_0 \cap \Pi)$ where Π is an n-dimensional plane and $int_n(A)$ is the interior of A with respect to the topology in Π, there is $t_0 > 0$ such that $P_0 \in int_n(\Sigma_t \cap \Pi)$ for $0 < t < t_0$.*

Proof. Let $h^+ = C_+ \frac{|X-P_0|^\mu}{(T-t)^\gamma}$ for $\mu = \frac{4\alpha}{2\alpha-1}$, $\gamma = \frac{1}{2\alpha-1}$, and $C_+ = \left(\frac{\gamma}{\mu^{2\alpha}(\mu-1)^\alpha}\right)^{\frac{1}{2\alpha-1}}$. Then h^+ is a super-solution of (4.1.1). Now we compare the solution f with h^+. From $C^{1,1}$-estimates of f, f_t is bounded and then there is a ball $B_{\rho_0}(P_0) \subset int_n(\Sigma_0 \cap \Pi)$ and $t_0 > 0$ such that $f(X,t) \leq h^+(X,t)$ on $\partial B_{\rho_0}(P_0)$ for $0 \leq t \leq t_0$ and $f(X,0) \leq h^+(X,0)$. From the comparison principle, we have $f(X,t) \leq h^+(X,t)$ for $(X,t) \in B_{\rho_0}(P_0) \times [0,t_0)$, which implies

$f(P_0, t) = 0$ and $P_0 \in \Sigma_t$ for $0 < t < t_0$.

□

4.4 Optimal gradient estimate near free boundary

4.4.1 Finite and non-degenerate speed of level sets

From using the differential Harnack inequalities, we can show that the free-boundary $\Gamma(t)$ has finite and non-degenerate speed as in [19]. As in Theorem 4.2.2, we assume that $z = f(x,t)$ is a solution of (4.1.1) and $C^{1,1}$ on $\Omega(t)$ for all $0 < t \leq T$ and $g = (\beta f)^{\frac{1}{\beta}}$ is smooth up to the interface $\Gamma(t)$ on $0 < t \leq \tau$ for some $\tau < T$.

Let us consider the function

$$f_\epsilon(x,t) = \frac{(1 - A\epsilon)^{(4\alpha-1)/2}\,(1+\epsilon)^{4\alpha}}{(1+B\epsilon)^{2\alpha-1}}\, f((1+\epsilon)x, (1-A\epsilon)t) \tag{4.4.1}$$

and then the results of [19] can be applied to our equation in a similar way. We may assume condition (4.1.7) and let $r = \gamma(\theta, t)$ be the interface $\Gamma(t)$ and $r = \gamma_\varepsilon(\theta, t)$ be the ε-level set of the function f with $0 \leq \theta < 2\pi$ by expressing in polar coordinates. Then

Lemma 4.4.1. *There exist constants $A, B, C > 0$ and $\tilde{A}, \tilde{B}, \tilde{C} > 0$ such that*

$$e^{-\frac{t-t_0}{B+AT}}\,\gamma(\theta, t_0) \geq \gamma(\theta, t) \geq e^{-\frac{t-t_0}{Ct_0}}\,\gamma(\theta, t_0) \tag{4.4.2}$$

and

$$e^{-\frac{t-t_0}{\tilde{B}+\tilde{A}T}}\,\gamma_\varepsilon(\theta, t_0) \geq \gamma_\varepsilon(\theta, t) \geq e^{-\frac{t-t_0}{\tilde{C}t_0}}\,\gamma_\varepsilon(\theta, t_0) \tag{4.4.3}$$

for all $0 < t_0 \leq t \leq T$, $0 \leq \theta < 2\pi$. In particular, the free-boundary $r = \gamma(\theta, t)$ and the ε-level set $r = \gamma_\varepsilon(\theta, t)$ of f for each $\varepsilon > 0$ move with finite and non-degenerate speed on $0 \leq t \leq T$.

4.4.2 Gradient estimates

Throughout this subsection, we will assume that $g = (\beta f)^{\frac{1}{\beta}}$ is a solution of (4.1.3) and smooth up to the interface on $0 \le t \le T$, and satisfies condition (4.1.5) and

$$\max_{x \in \Omega(t)} g(x,t) \ge 2, \qquad \text{for } 0 \le t \le T, \tag{4.4.4}$$

which comes from (4.1.8). We will now show that the gradient $|Dg|$ has a bound from above and below.

Lemma 4.4.2 (Optimal Gradient estimates)**.** *With the same assumptions as in Theorem 4.2.2 and* (4.4.4)*, there is a positive constant C_0 such that*

$$|Dg| \le C_0\,, \qquad \text{on } 0 \le g(\cdot,t) \le 1,\ 0 \le t \le T.$$

Moreover if (4.1.7) *is satisfied and if g is smooth up to the interface on $0 \le t \le T$, then there is a positive constant c_0 such that*

$$|Dg| \ge c_0\,, \qquad \text{on } \ g(\cdot,t) > 0,\ 0 \le t \le T.$$

Proof. (i) First, we shall show the upper bound of ∇g. Suppose that f is approximated by f_ε of (4.1.1) which is a decreasing sequence of solutions satisfying the positivity, strictly convexity and smoothness on $\{\, x \in \mathbb{R}^2 : |Df_\varepsilon(x)| < \infty \,\}$ for $0 \le t \le T$. Set $g_\varepsilon = (\beta f_\varepsilon)^{\frac{1}{\beta}}$. We can choose the f_ε's such that $|Dg_\varepsilon| \le C_0$ at $t = 0$, on $\{\, x : 0 \le g_\varepsilon \le 1 \,\}$ and $|Dg_\varepsilon| \le C_0$ at $g_\varepsilon = 1$, $0 \le t \le T$, for some uniform constant C_0. Then the last estimate comes from (4.1.8) and (4.4.4).

Let us denote g_ε by g for convenience of notation, where $g = (\beta f)^{\frac{1}{\beta}}$ is a strictly positive and smooth solution of (4.1.3) with convex f. Let us apply the maximum principle to $X = \frac{|Dg|^2}{2} = \frac{g_x^2 + g_y^2}{2}$ and assume X has an interior maximum at the point $P_0 = (x_0, y_0, t_0)$. By rotating the coordinates, we can assume $g_x > 0$ and $g_y = 0$ at P_0. Then we have $X_t \le 0$ by using the facts that $X_x = X_y = 0$, $X_{xx} \le 0$ and $X_{yy} \le 0$ are satisfied at P_0. On the other hand $|\nabla g|$ is bounded at $t = 0$ from the condition on the initial data and on $\{g = 1\}$, $|\nabla g| = \frac{|\nabla f|}{g^{\beta-1}} = |\nabla f|$ is bounded since f is convex. Hence $X \le \tilde{C}$, on $0 \le g \le 1,\ 0 \le t \le T$, provided that $X \le \tilde{C}$ at $t = 0$ and $g = 1$, $0 \le t \le T$ so that

$$|Dg| \le C_0, \qquad \text{on } 0 \le g(\cdot,t) \le 1,\ 0 \le t \le T.$$

(ii) Now we shall show the lower bound of the gradient. Consider

$$X = x\, g_x + y\, g_y.$$

Using the maximum principle as in (i), we have that

$$X_t \geq -C\, X \tag{4.4.5}$$

where C is a constant depending on ρ_0 and

$$\frac{d}{dt}\, X(\gamma(t), t) \geq -C\, X \tag{4.4.6}$$

at an interior or boundary minimum point P_0 of X. Then

$$\min_{\{g(\cdot,t)>0\}} X(t) \geq \min_{\{g(\cdot,0)>0\}} X(0)\, e^{-Ct}$$

for all $0 \leq t \leq T$ by Gronwall's inequality, and it implies the desired estimate. □

Theorem 4.4.3. *Under the same assumptions as in Lemma 4.4.2, there exist positive constants C_1, C_2 and ε_0, depending only on ρ_0 and the initial data, for which*

$$-C_2 \leq (\gamma_\varepsilon)_t(\theta, t) \leq -C_1 < 0, \qquad \textit{for} \quad 0 \leq t \leq T \ \textit{and} \ \ 0 < \varepsilon < \varepsilon_0. \tag{4.4.7}$$

4.5 Second derivative estimates

Throughout this section, we will assume that $g = (\beta f)^{\frac{1}{\beta}}$ is a solution of (4.1.3) and smooth up to the interface on $0 \leq t \leq T$, and satisfies conditions (4.1.7) and (4.1.8).

4.5.1 Decay rate of α-Gauss curvature

Under the same conditions as in Sections 4.4.1 and 4.4.2, we will show a priori bounds of the Gauss curvature $K = \det(D^2 f)/(1+|Df|^2)$ and the second derivatives of f and g.

Lemma 4.5.1. *With the same hypotheses as in Theorem 4.2.2 and* (4.1.7)*, there exists a positive constant c such that*

$$c \leq \frac{K^\alpha}{g^{\frac{\alpha}{2\alpha-1}}} \leq c^{-1}, \qquad \text{on } 0 \leq t \leq T \tag{4.5.1}$$

for $K = \det D^2 f/(1+|Df|^2)$.

Proof. We will only consider the bound of (4.5.1) around the interface. It suffices to show the bound of g_t from $g_t = K^\alpha / \left((1+|Df|^2)^{\frac{2\alpha-1}{2}} g^{\frac{\alpha}{2\alpha-1}}\right)$ because $|Df|$ is bounded around $\{g=0\}$. For $r = \gamma_\varepsilon(\theta,t)$ which is the ε-level set of g in polar coordinates,

$$g_t = -g_r \cdot \dot{\gamma}_\varepsilon(\theta,t)$$

since $g(\gamma_\varepsilon(\theta,t),\theta,t) = \varepsilon$. Then since the level sets of g is convex, we know that $c < g_r < c^{-1}$ and $-C_2 \leq \dot{\gamma}_\varepsilon(\theta,t) \leq -C_1 < 0$ for $0 \leq t \leq T$ from Lemma 4.4.2 and Theorem 4.4.3 implying that $C_1\, c < g_t < C_2\, c^{-1}$, so the proof is complete.

□

Corollary 4.5.2. *Under the assumptions of Lemma* 4.5.1*, the solution g of* (4.1.3) *satisfies the bound*

$$c \leq g_t \leq c^{-1}. \tag{4.5.2}$$

4.5.2 Upper bound of the curvature of level sets

Lemma 4.5.3. *With the assumptions of Theorem 4.2.2 and condition* (4.1.7)*, there exists a constant $C > 0$ such that*

$$0 < g_{\tau\tau} \leq C$$

with τ denoting the tangential direction of the level sets of g.

Proof. Strict convexity of the level sets of g directly implies $g_{\tau\tau} > 0$. We will obtain the bound from above by using the maximum principle on

$$X = g_y^2 g_{xx} - 2g_x g_y g_{xy} + g_x^2 g_{yy} + \big(g(g_{xx} + g_{yy}) + \theta|\nabla g|^2\big). \tag{4.5.3}$$

Let ν and τ denote the outward normal and tangential direction to the level sets of g respectively. Then we can write X as

$$X = (g + g_\nu^2)\, g_{\tau\tau} + (g\, g_{\nu\nu} + \theta g_\nu^2) \tag{4.5.4}$$

since $g_\tau = 0$. We also know that

$$0 < c \le g_\nu \le c^{-1} \qquad \text{on } g > 0,\ 0 \le t \le T$$

for some $c > 0$, depending on ρ_0 and the initial data. Also, $g(g_{xx}+g_{yy})+\theta|\nabla g|^2$ is bounded since $f \in C^{1,1}$. Hence an upper bound on X will imply the desired upper bound on $g_{\tau\tau}$. We will apply the maximum principle on the evolution of X. The term $\big(g(g_{xx} + g_{yy}) + \theta|\nabla g|^2\big)$ on X will control the sign of error terms. Corollary 4.5.2 implies

$$X \le C \qquad \text{at} \quad g = 0,$$

since we know that $X = \frac{1}{\theta} g_t^{\frac{1}{\alpha}} + \theta|\nabla g|^2$ at the free-boundary $g = 0$. Then we can assume that X has its space-time maximum at an interior point $P_0 = (x_0, y_0, t_0)$. Let us assume that

$$g_\tau = g_y = 0 \qquad \text{and} \qquad g_\nu = g_x > 0 \quad \text{at} \quad P_0 \tag{4.5.5}$$

without loss of generality, since X is rotation invariant. Also let us consider the following transformation

$$\tilde{g}(x, y) = g(\mu, \eta)$$

where $\mu = x$ and $\eta = y - ax$ with $a = \frac{g_{\mu\eta}(x_0,y_0,t_0)}{g_{\mu\mu}(x_0,y_0,t_0)}$, which is similar in the transformation to Proposition 4.1 of [34]. Then we can obtain $\tilde{g}_x = g_\mu - \frac{g_\eta g_{\mu\eta}}{g_{\eta\eta}} = g_\mu > 0$, $\tilde{g}_y = g_\eta = 0$, and

$$(\tilde{g}_{ij}) = \begin{bmatrix} g_{\mu\mu} - \frac{g_{\mu\eta}^2}{g_{\eta\eta}} & 0 \\ 0 & g_{\eta\eta} \end{bmatrix}$$

at P_0. Here $\tilde{g}_{yy} = g_{\eta\eta} > 0$ and $\tilde{g}_{xx} < 0$ at P_0. Hence the equation is unchanged under this change of coordinates. We can also drop the third derivative term of $\tilde{g}$ because it is changed under the perfect square of the third derivative of g. Hence we can assume

$$\tilde{g}_{xy} = 0 \tag{4.5.6}$$

at P_0 without loss of generality. We will proceed with the function g instead of $\tilde{g}$ for notation purposes. From (4.5.3), we get

$$X = (g + g_x^2)g_{yy} + (gg_{xx} + \theta g_x^2), \quad \text{at} \quad P_0.$$

At the maximum point P_0, we also have $X_x = 0$ and $X_y = 0$ implying that

$$g_{xyy} = -\frac{gg_{xxx} + 2g_x \det D^2 g + (2\theta + 1)g_x g_{xx} + g_x g_{yy}}{g + g_x^2} \quad \text{and} \quad g_{yyy} = -\frac{gg_{xxy}}{g + g_x^2}. \tag{4.5.7}$$

We shall compute the evolution equation of X from the evolution equation of g to find a contradiction saying that

$$0 \le X_t < 0 \qquad \text{at} \quad P_0,$$

when $X > C > 0$ for some constant C. This implies that $X \le C$, on $0 \le t \le T$.

First we will consider the following simpler case that f satisfies the evolution

$$f_t = (\det D^2 f)^\alpha$$

for the convenience of the reader. Then $g = (\beta f)^{\frac{1}{\beta}}$ satisfies the equation

$$g_t = \Big(g \det D^2 g + \theta\big(g_y^2 g_{xx} - 2g_x g_y g_{xy} + g_x^2 g_{yy}\big)\Big)^\alpha. \tag{4.5.8}$$

We differentiate (4.5.8) twice to obtain the evolution of X. Set

$$K_g = g \det D^2 g + \theta\big(g_y^2 g_{xx} - 2g_x g_y g_{xy} + g_x^2 g_{yy}\big),$$
$$I = 1 + g^{2\beta-2}|\nabla g|^2, \text{ and } J = g + |\nabla g|^2.$$

Let L denote the operator

$$LX := X_t - \alpha K_g^{\alpha-1}\Big\{ \big(gg_{yy} + \theta g_y^2\big) X_{xx} - 2\big(gg_{xy} + \theta g_x g_y\big) X_{xy} + \big(gg_{xx} + \theta g_x^2\big) X_{yy} \Big\}.$$

Then after many tedious calculations, we have that at the maximum point P_0,

$$LX = A + \frac{1}{(1+2\gamma)^2(g+g_x^2)^2K_g^2}B \tag{4.5.9}$$

where $\gamma = \theta - 1$ and

$$\begin{aligned} A = &-4g^2g_{xxy}^2 - \frac{4g^3}{g_x^2+g}\Big(g_{xxx} + \frac{6g_xg_{xx}+3g_xg_{xx}g_{yy}}{g}\Big)^2 \\ &+ \frac{g_{yy}^2}{g_x^2+g}\Big\{-(2g_x^2-gg_{xx})^2 + 3g_{xx}(g_x^2+g)(g_x^2-gg_{xx})\Big\}. \end{aligned} \tag{4.5.10}$$

In addition, $B = 0$ if $\gamma = 0$, otherwise

$$B = -B_1g^2(g_x^2+g)g_{xxy}^2 - g^3B_1(g_{xxx}+B_{11})^2 + \Big((1+2\gamma)^2(g+g_x^2)^2K_g^2\Big)\frac{E_1}{E_2}. \tag{4.5.11}$$

Here

$$\begin{aligned} B_1 =& 4(1+2\gamma)^2K_g^{\frac{2+3\gamma}{1+2\gamma}}\left(\frac{1+\gamma}{1+2\gamma} - K_g^{\frac{\gamma}{1+2\gamma}}\right) + \gamma(1+\gamma)K_g^{\frac{1+\gamma}{1+2\gamma}}\Bigg((g_x^2+g)g_{yy} \\ &- \Big\{gg_{xx} + (1+\gamma)g_x^2\Big\}\Bigg)^2 \end{aligned}$$

and set $Z = g_x^2g_{yy}$ so that

$$\begin{aligned} E_2 =& 4(1+2\gamma)^3gg_x^6(g_x^2+g)^2K_g^{\frac{6+13\gamma}{1+2\gamma}}Z^4\left(\frac{(1+\gamma)(1+\frac{3}{2}\gamma)}{(1+2\gamma)^2} - K_g^{\frac{\gamma}{1+2\gamma}}\right) \\ &+ g\gamma(1+\gamma)(1+2\gamma)g_x^2(g_x^2+g)^3K_g^{\frac{5+11\gamma}{1+2\gamma}}Z^6 + l.o.t. \\ &\geq g\big(E_{11}Z^4 + \gamma E_{12}Z^6 + l.o.t.\big), \end{aligned}$$

where $l.o.t.$ means lower order terms. We may assume that P_0 lies close to the free-boundary and that $K_g^{\frac{\gamma}{1+2\gamma}} < \frac{1}{2}$ by considering a scaled solution $g_\lambda(x,t) = \lambda^{-\frac{1}{\gamma+2}}g(\lambda x, \lambda^{\frac{4\gamma+3}{2\gamma+1}}t)$ as g at the beginning of the proof with $\lambda^{\frac{4\gamma+5}{\gamma+2}} \leq \frac{1}{||K_g||_{L^\infty}}\left(\frac{1}{2}\right)^{\frac{1+\gamma}{\gamma}}$. Then on $g \leq 1$, we have A is negative in (4.5.10) since g_{xx}

is negative and $E_{11}, E_{12} \geq \delta_0(g_x, K_g) > 0$ uniformly, which implies E_2 is positive. And we also have, in (4.5.11),

$$E_1 = -\gamma(1+\gamma)(g_x^2+g)^2 K_g^5 Z^8 C_8 + \gamma(Z^7 C_7 + l.o.t.)$$

with $C_8 = \Big((1+\gamma)^2(g_x^2+g)(2\gamma g + 3(5+4\gamma)g_x^2) + \gamma(1+2\gamma)K_g^{\frac{\gamma}{1+2\gamma}} g^2$
$- (1+2\gamma)(15+2\gamma(7+2\gamma))K_g^{\frac{\gamma}{1+2\gamma}} g g_x^2 - 3(1+\gamma)(1+2\gamma)(5+4\gamma)K_g^{\frac{\gamma}{1+2\gamma}} g_x^4\Big)$.

Now we can show $C_8 \geq \delta_1(g_x, K_g) > 0$ uniformly and then $E_1 < 0$ for sufficiently large Z. Therefore B is negative. Hence we obtain desired result.

We now return to the case of the α-Gauss Curvature Flow. Let us set $I = 1 + g^{2\beta-2}|Dg|^2$, $J = g + |Dg|^2$ and $Q = (g \det D^2 g + \theta(g_y^2 g_{xx} - 2g_x g_y g_{xy} + g_x^2 g_{yy}))^\alpha$. Also let $C = C(\|g\|_{C^1}, \|f\|_{C^{1,1}})$ denote various constants and $\tilde{L}X$ denote the operator

$$\begin{aligned}\tilde{L}X :=& X_t - \alpha K_g^{\alpha-1} I^{-\frac{4\alpha-1}{2}} \{ (g g_{yy} + \theta g_y^2) X_{xx} - 2(g g_{xy} + \theta g_x g_y) X_{xy} \\ &+ (g g_{xx} + \theta g_x^2) X_{yy} \}.\end{aligned}$$

We find, after several calculations, that at the maximum point P_0, where (4.5.5) and (4.5.6) hold, X satisfies the equality

$$\begin{aligned}\tilde{L}X =& I^{-\frac{4\alpha-1}{2}} LX - \frac{4\alpha-1}{2}(g+g_x^2) I^{-\frac{4\alpha+1}{2}} Q I_{yy} - (4\alpha-1) g_x(\theta + g_{yy}) I^{-\frac{4\alpha+1}{2}} Q I_x \\ &+ g\Big\{ -(4\alpha-1) I^{-\frac{4\alpha+1}{2}} Q_x I_x - \frac{4\alpha-1}{2} I^{-\frac{4\alpha+1}{2}} Q I_{xx} + \frac{16\alpha^2-1}{4} I^{-\frac{4\alpha+3}{2}} Q I_x^2 \Big\} \\ &+ I^{-\frac{4\alpha+1}{2}} Q g_{xx}\end{aligned}$$

and from (4.5.9) and some computation we obtain that

$$\begin{aligned}
\tilde{L}X \leq\ & I^{-\frac{4\alpha-1}{2}}A + \frac{I^{-\frac{4\alpha-1}{2}}}{(1+2\gamma)^2(g+g_x^2)^2K_g^2}\Big\{ - B_1 g^2(g_x^2+g)g_{xxy}^2 \\
& - g^3B_1(g_{xxx}+B_{11})^2 + \big((1+2\gamma)^2(g+g_x^2)^2K_g^2\big)\frac{E_1}{E_2}\Big\} \\
& - \frac{8\gamma(\gamma+1)}{(1+2\gamma)^2}I^{-\frac{4\alpha+1}{2}}K_g^{\alpha-1}g^{2\gamma+3}\frac{g_x}{g+g_x^2}\Big\{((\gamma+1)g_x^2+gg_{xx})^2 \\
& - (g+g_x^2)K_g\Big\}g_{xxx} + g^{2\gamma+1}\cdot l.o.t. + I^{-\frac{4\alpha+1}{2}}K_g^{\alpha}g_{xx} \\
\leq\ & I^{-\frac{4\alpha-1}{2}}A + \frac{I^{-\frac{4\alpha-1}{2}}}{(1+2\gamma)^2(g+g_x^2)^2K_g^2}\Big\{ - B_1 g^2(g_x^2+g)g_{xxy}^2 \\
& - g^3B_1(g_{xxx}+B_{11}+O(g))^2 + \big((1+2\gamma)^2(g+g_x^2)^2K_g^2\big)\frac{E_1}{E_2} + O(g)\Big\} \\
& + O(g) + I^{-\frac{4\alpha+1}{2}}K_g^{\alpha}g_{xx}.
\end{aligned}$$

Here $O(g)$ denotes various terms satisfying $|O(g)| \leq Cg$ with constant C. We know that the first term and the second term are negative as in the case of LX and provided that $X > C$ is sufficiently large. Then $\tilde{L}X < C$ with C depending on $||f||_{C^{1,1}}$ and $||g||_{C^1}$ on $g \leq 1$, which implies that $(X - Ct)_t < 0$. Applying the evolution of $\tilde{X} = X - Ct$ with a simple trick implies the desired contradiction. Hence $\tilde{X} \leq C$ where C is a positive constant.

□

4.5.3 Aronson-Bénilan type estimate

Lemma 4.5.4. *Under the assumptions of Theorem 4.2.2 and condition* (4.1.7)*, there exists a constant $C > 0$ for which*

$$\det(D^2g) \geq -C$$

for a uniform constant $C > 0$.

Proof. To establish the bound of $\det(D^2 g)$ from below, we will use the maximum principle on the quantity

$$Z = \frac{\det D^2 g}{g_x^2 g_{yy} + g_y^2 g_{xx} - 2 g_x g_y g_{xy}} + b \mid \nabla g \mid^2$$

with some positive constant b on $\{ g(\cdot, t) > 0,\, 0 \le t \le T \}$. Let us assume that Z becomes minimum at the interior point P_0. We can assume $g_y = 0$, $g_x > 0$ and $g_{xy} = 0$ at P_0 by using a similar transformation and the change of coordinates as in Lemma 4.5.3 at P_0. Then we have

$$a_{ij} Z_{ij} \le 0$$

for

$$(a_{ij}) = \begin{bmatrix} \alpha K_g^{\alpha-1} g g_{yy} (1 + g^{\frac{2\alpha}{2\alpha-1}} g_x^2)^{\frac{1-4\alpha}{2}} & 0 \\ 0 & \alpha K_g^{\alpha-1} (g g_{xx} + \theta g_x^2)(1 + g^{\frac{2\alpha}{2\alpha-1}} g_x^2)^{\frac{1-4\alpha}{2}} \end{bmatrix} \tag{4.5.12}$$

and $Z_x = Z_y = 0$ at the minimum point P_0 implying that

$$Z_t \ge B_1 (g_{xyy} + B_2)^2 + A_0 + \frac{1}{(\alpha+1)(2\alpha-1)^3} O(g) Z + O(Z^2) \tag{4.5.13}$$

where

$$B_1 = \frac{\alpha}{(1-2\alpha)^2 g_x^2 g_{yy}} \big(1 + g^{\frac{2\alpha}{2\alpha-1}} g_x^2\big)^{\frac{1}{2}-2\alpha} K_g^{\alpha-2} \big[\alpha g_x^2 + (2\alpha-1) g g_{xx}\big] \cdot \Big[(4\alpha-2) K_g + (\alpha-1)\big\{\alpha g_x^2 + (2\alpha-1) g g_{xx}\big\} g_{yy}\Big]$$

and

$$A_0 = A_{0,0} g^2 b^4 + E_0 + A_{0,5} (\alpha-1) g^3 b^5 + (\alpha-1) E_1$$

with

$$A_{0,0} = \frac{g_x^{10} g_{yy} (24 + 12 g^2 g_x^2 - 21 g^4 g_x^4)}{2(1 + g^2 g_x^2)^{7/2}} \tag{4.5.14}$$

and

$$A_{0,5} = -\frac{30g^{1+\frac{1}{2\alpha-1}}(1-2\alpha)^2 g_x^{14}(g^{\frac{2\alpha}{1-2\alpha}} + g_x^2)(1+g^{1+\frac{1}{2\alpha-1}}g_x^2)^{-\frac{1}{2}-2\alpha}K_g^{\alpha-1}g_{yy}^2}{(4\alpha-2)K_g + (\alpha-1)\alpha g_x^2 g_{yy}}$$

where $E_0 = O(b^3, g^2)$ and $E_1 = O(b^4, g^3)$. Here we can also show $A_{0,0} \geq \delta_1(g_x, g_{yy}) > 0$ uniformly and we have

$$\begin{aligned}(4\alpha-2)K_g + (\alpha-1)\alpha g_x^2 g_{yy} &= (4\alpha-2)(gg_{xx}g_{yy} + \theta g_x^2 g_{yy}) + (\alpha-1)\alpha g_x^2 g_{yy} \\ &= (4\alpha-2)gg_{xx}g_{yy} + \{\theta(4\alpha-2) + \alpha^2 - \alpha\}g_x^2 g_{yy} \\ &> 0\end{aligned} \tag{4.5.15}$$

on $g \leq 1$ since $\frac{1}{2} < \alpha \leq 1$. Then $(\alpha-1)A_{0,5}$ is nonnegative so that A_0 is positive for sufficiently large $b \gg 1$. Also, we know that B_1 is positive on $g \leq 1$ from (4.5.15). This implies

$$Z_t > 0 > -aZ$$

with a positive constant a. By Grönwall's inequality, we have

$$Z \geq Z_0 e^{-\tilde{a}t}$$

where Z_0 is initial data of Z at $t = 0$ and $\tilde{a}$ is constant, which concludes the proof.

□

4.5.4 Global optimal regularity

Let us consider the quantity

$$\mathcal{Z} = \max_{\gamma}(gD_{\gamma\gamma}g + \theta|D_\gamma g|^2). \tag{4.5.16}$$

Now we will show that $\mathcal{Z}$ is bounded from above in the next lemma.

Lemma 4.5.5. *With the assumptions of Theorem 4.2.2 and condition* (4.1.7)*, there exists a positive constant* $C = C(\theta, \rho, \lambda, \|g\|_{C^2(\partial\Omega)})$ *with*

$$\max_{\Omega(g)} \mathcal{Z} \leq C$$

where $\Omega(g) = \{x | g(x) > 0\}$.

Proof. First, we know that $\mathcal{Z}$ is nonnegative from $\mathcal{Z} = \beta^{\frac{2-\beta}{\beta}} f^{\frac{2-\beta}{\beta}} f_{\gamma\gamma}$ and convexity of f. Also Lemma 4.4.2 implies

$$\mathcal{Z} \leq C(\theta, \rho, \lambda, \|g\|_{C^2(\partial\Omega)}) \quad \text{at } g = 0,$$

since $\mathcal{Z} = \theta |D_\gamma g|^2$ at the free-boundary $g = 0$. Then we can assume that $\mathcal{Z}$ has its maximum at an interior point $P_0 \in \Omega(g)$ and in a direction γ. To show the bound of $\mathcal{Z}$, we consider γ as $\gamma = \lambda_1 \nu + \lambda_2 \tau$ with $\lambda_1^2 + \lambda_2^2 = 1$, where ν, τ denote the outward normal and tangential directions to the level sets of g respectively. Then $\mathcal{Z}(P_0) = g D_{\gamma\gamma} g + \theta |D_\gamma g|^2$ and (4.1.3) can be rewritten as

$$\begin{aligned} &\mathcal{Z}(P_0) = g[\lambda_1^2 g_{\nu\nu} + 2\lambda_1\lambda_2 g_{\nu\tau} + \lambda_2^2 g_{\tau\tau}] + \theta \lambda_1^2 g_\nu^2 \\ \text{and} \quad &(g g_{\nu\nu} + \theta g_\nu^2) g_{\tau\tau} = g g_{\nu\tau}^2 + \{g_t (1 + g^{2\beta-2} g_\nu^2)^{\frac{4\alpha-1}{2}}\}^{\frac{1}{\alpha}}. \end{aligned} \tag{4.5.17}$$

Here, if $g g_{\nu\nu}$ is not sufficiently large at P_0, we have

$$\mathcal{Z}(P_0) \leq C(\theta, \rho, \lambda, \|g\|_{C^2(\partial\Omega)})$$

from Lemma 4.4.2, Corollary 4.5.2 and Lemma 4.5.3 implying the desired result immediately. On the other hand, if $\theta g_\nu^2 \leq g g_{\nu\nu}$ at P_0, then we get

$$g g_{\nu\tau}^2 \leq 2 g g_{\nu\nu} g_{\tau\tau} \leq C(\theta, \rho, \lambda, \|g\|_{C^2(\partial\Omega)}) g g_{\nu\nu}$$

implying $g_{\nu\tau} \leq C(\theta, \rho, \lambda, \|g\|_{C^2(\partial\Omega)}) \sqrt{g_{\nu\nu}}$. Then we know that $\mathcal{Z}(P_0)$ is maximum when $\lambda_2 = 0$ from Lemma 4.5.3 and (4.5.17) so that $\mathcal{Z}(P_0) = g g_{\nu\nu} + \theta g_\nu^2$. Also we get $g g_{\nu\tau} + \theta g_\nu g_\tau = 0$ at P_0 implying that $g_{\nu\tau} = 0$ at P_0. Here by a similar transformation as in Lemma 4.5.3, we can assume

$$g_\tau = g_y = 0, \quad g_\nu = g_x > 0 \quad \text{and} \quad g_{xy} = 0 \quad \text{at} \quad P_0.$$

Then we have

$$a_{ij}\mathcal{Z}_{ij} \leq 0$$

with (4.5.12) at the maximum point P_0. And since $g_{xxx} = \frac{-g_x g_{xx} - 2\theta g_x g_{xx}}{g}$ and $g_{xxy} = 0$ at P_0, we obtain

$$\begin{aligned}
\mathcal{Z}_t &= g_t g_{xx} + g g_{xxt} + 2\theta g_x g_{xt} \\
&\leq -\big(1 + g^{\frac{2\alpha}{2\alpha-1}} g_x^2\big)^{-\frac{1+4\alpha}{2}} \Big[(1-2\alpha)^2(\alpha-1)(g^{\frac{1}{1-2\alpha}} + g g_x^2)\Big]^{-1} \Big[g_{yy}(\theta g_x^2 + g g_{xx})\Big]^{\alpha} \\
&\quad \cdot \Bigg[g^{\frac{2\alpha}{2\alpha-1}} \alpha(4\alpha-1)\Big(2\alpha(2\alpha\theta - 3\theta + \alpha + 1) + 2\theta - 1\Big) g_x^6 \\
&\quad + (1-2\alpha)^2(\alpha-1) g_{xx} \Big\{ g^{\frac{1}{1-2\alpha}}(\alpha-1) + (4\alpha-1) g^2 g_{xx} \Big\} \\
&\quad + (2\alpha-1) g g_x^2 g_{xx} \Big\{ (\alpha-1)(16\alpha^2 - 5\alpha + 1) + 3\alpha(8\alpha^2 - 6\alpha + 1) g^{\frac{4\alpha-1}{2\alpha-1}} g_{xx} \Big\} \\
&\quad + \alpha g_x^4 \Big\{ (4\alpha^2 - 5\alpha + 1)\big(\theta(4\alpha-2) + 1\big) + 6\alpha(10\alpha^2 - 9\alpha + 2) g^{\frac{4\alpha-1}{2\alpha-1}} g_{xx} \Big\} \Bigg]
\end{aligned}$$

at the point P_0. Also from $g_{xx} = \frac{\mathcal{Z} - \theta g_x^2}{g}$, we have

$$\begin{aligned}
\mathcal{Z}_t \leq & \frac{\big(1 + g^{\frac{2\alpha}{2\alpha-1}} g_x^2\big)^{-\frac{3+4\alpha}{2}} (\mathcal{Z} g_{yy})^{\alpha}}{g(\alpha-1)(2\alpha-1)} \Bigg[-\mathcal{Z}(2\alpha^2 - 3\alpha + 1)\Big\{(4\alpha-1)\mathcal{Z} g^{\frac{2\alpha}{2\alpha-1}} + \alpha - 1\Big\} \\
&\quad + g^{\frac{4\alpha}{2\alpha-1}} g_x^2 \Big\{ g^{\frac{4\alpha}{1-2\alpha}}(\alpha-1)^2\alpha - (\alpha-1)(8\alpha^2 - 3\alpha + 1)\mathcal{Z} g^{\frac{2\alpha}{1-2\alpha}} \\
&\quad - 3\alpha(8\alpha^2 - 6\alpha + 1)\mathcal{Z}^2 + \alpha(\alpha-1) g_x^2 (2(\alpha-1) g^{\frac{2\alpha}{1-2\alpha}} - 6\alpha\mathcal{Z} + (\alpha-1) g_x^2) \Big\} \Bigg] \\
\leq & (1-4\alpha)\Big(1 + g^{\frac{2\alpha}{2\alpha-1}} g_x^2\Big)^{-\frac{3+4\alpha}{2}} g_{yy}^{\alpha} \mathcal{Z}^{2+\alpha} g^{\frac{1}{2\alpha-1}} + O\big(\mathcal{Z}^{2+\alpha}\big) g^{\frac{2\alpha+1}{2\alpha-1}} \\
&\quad + O\big(\mathcal{Z}^{1+\alpha}, g^{\frac{1}{2\alpha-1}}\big).
\end{aligned}$$

Then on $g \leq 1$, $\mathcal{Z}_t < 0$ at P_0 since $1 - 4\alpha < 0$. Hence we obtain the desired result.

□

4.5.5 Decay rates of second derivatives

Corollary 4.5.6. *With the hypotheses of Lemma 4.5.3, there exists a positive constant c depending on ρ_0 and the initial data such that*

(i) $c \leq g_{\tau\tau} \leq c^{-1}$,

(ii) $c \leq \dfrac{f_{\nu\nu}}{g^{\beta-2}}, \dfrac{f_{\tau\tau}}{g^{\beta-1}} \leq c^{-1}$ *and* $\dfrac{|f_{\nu\tau}|}{g^{(2\beta-3)/2}} \leq c^{-1}$ *for uniformly small g*

with τ denoting the tangential direction to the level sets of g.

Proof. (i) The upper bound of $g_{\tau\tau}$ comes from Lemma 4.5.3. Now we show the lower bound. From Lemma 4.5.1, we have

$$\det D^2 f \geq c\, g^{\frac{1}{2\alpha-1}} = c\, g^{2\beta-3},$$

which implies

$$f_{\nu\nu} f_{\tau\tau} \geq c\, g^{2\beta-3} + f_{\nu\tau}^2 \geq c\, g^{2\beta-3} \tag{4.5.18}$$

and then

$$f_{\tau\tau} \geq \frac{c\, g^{2\beta-3}}{f_{\nu\nu}} \geq \tilde{c}\, g^{\beta-1}$$

since $f_{\nu\nu} \leq C g^{\beta-2}$ from Lemma 4.5.5. Since $f_{\tau\tau} = g^{\beta-1}\, g_{\tau\tau} + (\beta-1) g^{\beta-2}\, g_\tau^2$, we conclude that

$$g_{\tau\tau} = \frac{f_{\tau\tau}}{g^{\beta-1}} \geq \tilde{c},$$

for some positive constant $\tilde{c}$ depending only on the initial data and ρ_0.

(ii) $f_{\tau\tau} = g^{\beta-1}\, g_{\tau\tau}$ and the bound on $g_{\tau\tau}$ tell us

$$c \leq \frac{f_{\tau\tau}}{g^{\beta-1}} \leq c^{-1}.$$

(iii) From Lemma 4.5.5, we have $f_{\nu\nu} \leq C g^{\beta-2}$ for uniformly small g, so we shall show

$$f_{\nu\nu} + f_{\tau\tau} \geq c g^{\beta-2}.$$

Let us denote by λ_1, λ_2 the two eigenvalues of the matrix $\det D^2 f$ such that $\lambda_1 \geq \lambda_2$. Then, from Lemma 4.5.1, we have

$$c \leq \frac{\lambda_1 \lambda_2}{g^{2\beta-3}} \leq c^{-1}$$

and $\lambda_2 \leq f_{\tau\tau} \leq c^{-1}\, g^{\beta-1}$, implying that $\lambda_1 \geq cg^{\beta-2}$ for some positive constant c. Hence $f_{\nu\nu} + f_{\tau\tau} \geq \lambda_1 + \lambda_2 \geq cg^{\beta-2} > 0$ as desired.

(iv) The convexity of f says $0 \geq f_{\nu\tau}^2 - f_{\nu\nu}\, f_{\tau\tau}$. By using the bound of $f_{\nu\nu}$ and $f_{\tau\tau}$, we obtain

$$f_{\nu\tau}^2 \leq f_{\nu\nu}\, f_{\tau\tau} \leq c^{-1}\, g^{2\beta-3}.$$

□

4.6 Higher regularity

4.6.1 Local change of coordinates

For any point $P_0 = P_0(x_0, y_0, t_0)$ at the interface Γ with $0 < t_0 \leq T$, let us assume that n_0 is the unit vector in the direction of the vector $\mathbf{P_0} = \overline{OP_0}$ and n_0 satisfies

$$n_0 := \frac{\mathbf{P_0}}{|\mathbf{P_0}|} = \mathbf{e_1} \tag{4.6.1}$$

by rotating the coordinates. Then we will have the following lemma as Lemma 4.6 in [19].

Lemma 4.6.1. *There exist positive constants c and η, depending only on the initial data and the constant ρ_0 in* (4.1.7)*, for which*

$$c \leq g_x(P) \leq c^{-1} \quad \textit{and} \quad c \leq f_{xx}(P) \leq c^{-1}$$

at all points $P = (x, y, t)$ with $f(P) > 0$, $|P - P_0| \leq \eta$ and $t \leq t_0$ under (4.6.1)*.*

4.6.2 Class of linearized equation

In this subsection, we shall show that our transformed function h from g near the free boundary satisfies the same class of operators considered in [19] so

that all the results in [19] can be applied to our equation by using similar methods.

Throughout this subsection, we will assume that at time $t = 0$ the function $g = (\beta\, f)^{\frac{1}{\beta}}$ satisfies the hypotheses of Theorem 4.2.3 and that g is smooth up to the interface on $0 \le t \le T$ for $T > 0$ satisfying condition (4.1.7).

We will state the results of uniform $C_s^{1,\gamma}$-estimates in [19], where the reader also can find detailed proofs.

Let $P_0 = (x_0, y_0, t_0)$ be a point on the interface curve $\Gamma(t_0)$ at time $t = t_0$, for $0 < t_0 \le T$. We may assume, without loss of generality, that $\tau \le t_0 \le T$, for some $\tau > 0$. From the short time regularity of Theorem 4.2.2, we know that solutions are smooth up to the interface on $0 \le t \le 2\tau$, for some τ depending only on the initial data. Also we may assume that condition (4.6.1) holds at the point P_0 by rotating the coordinates. By Lemma 4.6.1, $g_x(P) > 0$ for all points $P = (x, y, t)$ with $t \le t_0$, sufficiently close to P_0 and then from (4.1.11) (see in [15], Section II),

$$h_t = -\frac{\left\{ z\,(h_{zz}\, h_{yy} - h_{zy}^2) - \theta(\alpha)\, h_z h_{yy} \right\}^{\alpha}}{\left\{ z^{2(\beta-1)} + h_z^2 + z^{2(\beta-1)}\, h_y^2 \right\}^{\frac{4\alpha-1}{2}}}, \qquad z > 0. \tag{4.6.2}$$

Set $K_h = z\,(h_{zz} h_{yy} - h_{zy}^2) - \theta(\alpha)\, h_z\, h_{yy}$ and $J = z^{2(\beta-1)} + h_z^2 + z^{2(\beta-1)}\, h_y^2$. By linearizing this equation around h, we can obtain the equation

$$\begin{aligned} \tilde{h}_t =& \frac{\alpha\, K_h^{\alpha-1}}{J^{\frac{4\alpha-1}{2}}} \left\{ -z\, h_{yy} \tilde{h}_{zz} + 2z\, h_{zy}\, \tilde{h}_{zy} + \Big(\theta(\alpha)\, h_z - z\, h_{zz} \Big)\, \tilde{h}_{yy} \right\} \\ &+ \frac{(\,4\alpha - 1)\, z^{2(\beta-1)}\, K_h^{\alpha}\, h_y}{J^{\frac{4\alpha+1}{2}}}\, \tilde{h}_y \\ &+ \frac{(\,4\alpha - 1)\, K_h^{\alpha}\, h_z + \alpha K_h^{\alpha-1} \theta(\alpha)(h_z^2 + z^{2(\beta-1)}(1 + h_y^2)) h_{yy}}{J^{\frac{4\alpha+1}{2}}}\, \tilde{h}_z. \end{aligned} \tag{4.6.3}$$

Let us denote by $\mathcal{B}_\eta$ the box

$$\mathcal{B}_\eta = \{\, 0 \le z \le \eta^2,\ |y - y_0| \le \eta,\ t_0 - \eta^2 \le t \le t_0 \,\}$$

around the point $Q_0 = (0, y_0, t_0)$. We can obtain a priori bounds on the matrix

$$A = (a_{ij}) = \alpha K_h^{\alpha-1} \begin{pmatrix} -\,h_{yy} & \sqrt{z}\ h_{zy} \\ \\ \sqrt{z}\ h_{zy} & \big(\theta(\alpha)\, h_z - z\, h_{zz} \big) \end{pmatrix}$$

and the coefficient

$$b = \frac{(4\alpha - 1)\, K_h^{\alpha}\, h_z + \alpha K_h^{\alpha-1}\theta(\alpha)(h_z^2 + z^{2(\beta-1)}(1+h_y^2))h_{yy}}{J^{\frac{4\alpha+1}{2}}}.$$

Therefore we obtain the following.

Lemma 4.6.2. *There exist positive constants η, λ and ν, depending only on the initial data and the constant ρ_0 in* (4.1.7) *such that*

$$\lambda\, |\xi|^2 \le a_{ij}\, \xi_i\, \xi_j \le \lambda^{-1}\, |\xi|^2, \qquad \forall \xi \neq 0$$

and

$$|b| \le \lambda^{-1} \ \text{ and } \ b \ge \nu > 0 \ \text{ on the box } \mathcal{B}_\eta.$$

Notice that $b \ge \nu > 0$ comes from the decay rates of the second derivatives, Corollary 4.5.6, and Aronson-Bénilan type estimate, Lemma 4.5.4. Similarly, we can get the bound of $\tilde{A} := (\tilde{a}_{ij})$ and $\tilde{b}_i$, $i = 1, 2$, to be the coefficients

$$\tilde{a}_{ij} = \frac{a_{ij}}{(z^{2(\beta-1)} + h_z^2 + z^{2(\beta-1)}\, h_y^2)^{\frac{4\alpha-1}{2}}}$$

and

$$\tilde{b}_1 = b - \frac{\alpha\, K_h^{\alpha-1} J h_{yy}}{J^{\frac{4\alpha+1}{2}}} \qquad \text{and} \qquad \tilde{b}_2 = \frac{(4\alpha-1)\, z^{2(\beta-1)}\, K_h^{\alpha}\, h_y}{J^{\frac{4\alpha+1}{2}}}$$

of Eq. (4.6.3).

Lemma 4.6.3. *There exist constants $\eta > 0$, $\lambda > 0$ and $\nu > 0$, depending only on the initial data and the constant ρ_0 in* (4.1.7)*, for which*

$$\lambda\, |\xi|^2 \le \tilde{a}_{ij}\, \xi_i\, \xi_j \le \lambda^{-1}\, |\xi|^2, \qquad \forall \xi \neq 0$$

and

$$|\tilde{b}_i| \le \lambda^{-1} \ \text{ and } \ \tilde{b}_1 \ge \nu > 0 \ \text{ on the box } \mathcal{B}_\eta.$$

4.6.3 Regularity theory

Recall Definition 4.1.2. Then Lemma 4.6.3 tells us the linearized equation (4.6.3) is in the same class of operators considered in Lemma 5.2 of [19] and [18].

We are now in a position to show the uniform Hölder bounds of the first order derivatives h_t, h_y and h_z of h on $\mathcal{B}_\eta$. In [19], the authors have obtained the $C_s^{2,\gamma}$ regularity of h in the box.

Lemma 4.6.4. *There exist numbers γ and μ in $0 < \gamma, \mu < 1$, and positive constants η and C, depending only on the initial data and ρ_0, such that*

$$\|h_y\|_{C_s^{2+\gamma}(\mathcal{B}_{\frac{\eta}{2}})} \leq C, \qquad \|h_t\|_{C_s^{2+\gamma}(\mathcal{B}_{\frac{\eta}{2}})} \leq C, \qquad \text{and} \qquad \|h_z\|_{C_s^{\mu}(\mathcal{B}_{\frac{\eta}{2}})} \leq C.$$

Following the proof Theorem 6.10 of [19], we will have the following theorem.

Theorem 4.6.5. *With the assumptions of Theorem 4.2.2 and condition* (4.1.7) *which satisfies at $T < T_c$, there exist constants $0 < \alpha_0 < 1$, $C < \infty$ and $\eta > 0$, depending only on the initial data and ρ_0, for which $x = h(x, y, t)$ fulfills*

$$\|h\|_{C_s^{2+\gamma}(\mathcal{B}_\eta)} \leq C$$

on $\mathcal{B}_\eta = \{\, 0 \leq z \leq \eta^2,\ |y - y_0| \leq \eta,\ t_0 - \eta^2 \leq t \leq t_0 \,\}$ for $P_0 = (x_0, y_0, t_0)$ with $0 < \tau < t_0 < T$, which is any free-boundary point holding condition (4.6.1).

Proof of Theorem 4.2.3. From the short time existence of Theorem 4.2.2, there exists a maximal time $T > 0$ such that g is smooth up to the interface on $0 < t < T$. Assuming that $T < T_0$, we will show that at time $t = T$, the function $g(\cdot, T)$ is of class $C_s^{2+\gamma}$, up to the interface $z = 0$, for some $\gamma > 0$, and satisfies the non-degeneracy conditions (4.1.5). Hence by Theorem 4.2.2, there exists a time $T' > 0$ such that g is of class $C_s^{2+\gamma}$, for all $\tau < T + T'$, and hence C^∞ up to the interface, by Theorem 9.1 in [15]. This will contradict the fact

that T is maximal, proving the theorem. From Lemma 4.4.2 and Corollary 4.5.6, the functions $g(\cdot, t)$ satisfy conditions (4.1.5), for all $0 \leq t < T$, with constant c independent of t. Hence, it will be enough to establish the uniform $C_s^{2+\gamma}$ regularity of g, on $0 \leq t \leq T$, up to the interface, whose proof follows the same line of argument as in [19].

$\square$

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국문초록

본 학위논문은 알파 지수 가우스 곡률을 속력으로 갖는 n+1차원 유클리드 공간에 있는 n차원 순볼록 초곡면의 변형을 연구한다. 알파의 범위가 $\frac{1}{n} < \alpha \leq 1$일 때 초기 초곡면이 순볼록이고 매끄러우면 순볼록인 매끄러운 해들이 존재하고 한 점으로 수렴한다. 또한, 척도변환된 초곡면의 점근적 행동에 대하여 토론한다. 즉, 척도 변환된 다양체는 순볼록 매끄러운 다양체로 수렴한다. 더욱이, 극한값이 어떤 방정식을 만족하는 부분수열이 존재한다. $\frac{1}{2} < \alpha \leq 1$의 범위에 있는 알파에 대하여 알파 가우스 곡률에 의해 주어진 속력을 갖는 3차원 유클리드 공간에 있는 볼록 곡면에 대하여, n차원의 경우에서와는 다른 어떤 추정값을 이용하여 초기 곡면이 매끄럽고 순볼록일 때 매끄러운 해들이 존재한다는 것을 보인다. 게다가, 만약 초기 곡면이 단지 볼록인 경우에 축소 시간 전에 $C^{1,1}$ 측도값을 갖는 viscosity 해가 존재하며, 볼록 곡면의 평탄면이 한동안 지속된다. 또한 비퇴화의 초기 조건과 어떤 정칙성 아래에서 곡면의 평탄한 부분과 순볼록 부분 사이의 접촉면이 평탄면이 사라지는 시간 전까지 매끈함을 유지한다는 것을 보인다.

주요어휘: 가우스 곡률 흐름, 알파 가우스 곡률 흐름의 정칙성, 자유 경계 문제, 비선형 포물형 편미분 방정식
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