이학박사 학위논문

Global Gradient Estimates for Elliptic and Parabolic Problems with Irregular Obstacles

(비정칙 장애물을 가진 타원형과 포물형 문제에 대한 대역적 그래디언트 가늠)

2014년 8월

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이 논문을 이학박사 학위논문으로 제출함

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Global Gradient Estimates for Elliptic and Parabolic Problems with Irregular Obstacles

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
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by

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Abstract

We consider nonhomogeneous elliptic and parabolic problems with irregular obstacles involving discontinuous nonlinearities over non-smooth domains in divergence form of p-Laplacian type. In this thesis, we establish the global Calderón-Zygmund estimate by proving that the gradient of the weak solution is as integrable as both the gradient of the obstacle and the nonhomogeneous term under the BMO smallness of the nonlinearity and sufficient flatness of the boundary in the Reifenberg sense.

Key words: Irregular obstacle, Calderón-Zygmund estimate, p-Laplacian, BMO space, Reifenberg domain
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Contents

Chapter 1

Introduction

The obstacle problem is a problem in the field of partial differential equations whose solution is limited by a given function, the so-called obstacle. This obstacle problem can be reformulated as a mathematical problem in the context of variational inequalities on various function spaces including Hölder spaces, Lebesgue spaces, Orlicz spaces.

In this thesis, we are interested in investigating how the regularity properties of the solution to the obstacle problem can be affected by those of the assigned obstacle and the nonhomogeneous term for a general elliptic and parabolic problem of the p-laplacian type. In particular, we prove that the spatial gradient of the solution is as integrable as both the spatial gradient of the obstacle and the nonhomogeneous term when the obstacle is allowed to be quite general, the associated nonlinearity is discontinuous, and the underlying domain is not necessarily given by graphs.

There have been many research activities on the regularity theory of obstacle problems. The Hölder regularity of a nonlinear elliptic and parabolic obstacle problem was studied by H. Choe, M. Fuchs, J. L. Lewis, G. M. Lieberman, P. Lindqvist, J. Mu, T. Norando, see [17, 18, 19, 25, 38, 42, 44, 46] and references therein. In [1] Bögelein, Duzaar and Mingione considered elliptic and parabolic variational problems involving divergence form of p-Laplacian type with irregular obstacles to establish the local Calderón-Zygmund theory for solutions, by proving that the (spatial) gradient of solutions are as integrable as the gradient of the obstacles. In [3], Bögelein and

Scheven established the self-improving property of integrability for the spatial gradient of solutions to parabolic variational inequalities satisfying an obstacle constraint and involving possibly degenerate respectively singular operators in divergence form. We mention also the paper [23] by Eleuteri and Habermann who considered a quasi-convex functional for a class of obstacle problems with nonstandard growths and established local Calderón–Zygmund type estimate when the associated integrand is continuous with respect to the x-variables. In [55], Scheven showed the existence of localizable solutions and Calderón–Zygmund estimates to parabolic obstacle problems. In [57], Scheven obtained gradient pointwise estimates for a nonlinear elliptic obstacle problem with nonhomogeneous measure data.

This work is a natural extension of the local Calderón-Zygmund theory in [1] to the global one. We want to answer as to what are minimal regularity requirements on the nonlinearity and what is the lowest level of geometric assumption on the boundary under which the gradient of the obstacle function and the nonhomogeneous term provide the gradient of a solution with the same regularity. Motivated the earlier work [14] where a local Calderón-Zygmund theory was obtained without an obstacle, we assume a smallness in bounded mean oscillation (BMO) on the nonlinearity with respect to the (spatial) variable. When it comes to a minimal geometric assumption we impose a Reifenberg flatness which turns out to be an appropriate one for nonlinear perturbation results, as in [13, 43, 51]. This is a sort of minimal regularity of the boundary guaranteeing the main results of the geometric analysis continue to hold true. In particular, C^1-smooth boundaries or Lipschitz continuous boundaries with a small Lipschitz constant belong to that category, but the class of Reifenberg flat domains extends beyond these common examples and contains domains with rough fractal boundaries such as the Van Koch snowflake, see [62].

In this thesis, we will present the following four papers.

- Chapter 2 : Calderon-Zygmund theory for nonlinear elliptic problems with irregular obstacles, with Sun-Sig Byun and Lihe Wang, Journal of Functional Analysis, Volume 263, Issue 10, Pages 3117-3143, 2012.
- Chapter 3 : Global weighted estimates for nonlinear elliptic obstacle

problems over Reifenberg domains, with Sun-Sig Byun and Dian K. Palagachev, Proceeding of American Mathematical Society, to appear.

- Chapter 4 : Nonlinear gradient estimates for parabolic problems with irregular obstacles, with Sun-Sig Byun, Nonlinear Analysis Series A: Theory, Methods and Applications, Volume 94, Pages 32-44, 2014.

- Chapter 5 : Nonlinear gradient estimates for parabolic obstacle problems in non-smooth domains, with Sun-Sig Byun, submitted.

We consider elliptic obstacle problems in Chaper 2 and 3. We let $p \in (1,\infty)$ to be a fixed real number and $\Omega \subset \mathbb{R}^n$ a bounded domain with $n \geq 2$. Given an obstacle $\psi \in W^{1,p}(\Omega)$ such that $\psi \leq 0$ on $\partial\Omega$, we define the convex admissible set

$$\mathcal{A} = \left\{ v \in W_0^{1,p}(\Omega) : \quad v \geq \psi \text{ a.e. in } \Omega \right\}. \tag{1.1}$$

We will deal with a function $u \colon \Omega \to \mathbb{R}$, belonging to $\mathcal{A}$, and such that

$$\int_\Omega \mathbf{a}(Du, x) \cdot D(v-u)\, dx \geq \int_\Omega |F|^{p-2} F \cdot D(v-u)\, dx \quad \text{for all } v \in \mathcal{A}, \tag{1.2}$$

where the nonhomogeneous term F is a vector valued function in $L^p(\Omega; \mathbb{R}^n)$. The aim of Chapter 2 is to find the minimal condition on the nonlinearity $\mathbf{a}(\xi, x)$ and geometric assumption on $\partial\Omega$ under which for each $q \in (1,\infty)$,

$$F \text{ and } D\psi \in L^{pq}(\Omega, \mathbb{R}^n) \Rightarrow Du \in L^{pq}(\Omega, \mathbb{R}^n).$$

In Chapter 3, we are going to derive a weighted version of the Calderón–Zygmund regularity estimate. Here a weight function is belonging to the Muckenhoupt class A_q (see the discussions in Chapter 3).

In Chapter 4 and 5, we study the following parabolic obstacle problems with time dependent obstacles. Let $p > \frac{2n}{n+2}$ be a fixed real number. We consider a function $u = u(x,t)$ lying in the convex admissible set

$$\mathcal{A} = \{ v \in C^0([0,T]; L^2(\Omega)) \cap L^p(0,T; W_0^{1,p}(\Omega)) : v(\cdot, 0) = 0, v \geq \psi \} \tag{1.3}$$

and satisfying the weak parabolic variational inequality

$$\int_0^T < v_t, v-u > dt + \int_{\Omega_T} a(Du,x,t)\cdot D(v-u)dxdt \\ \geq \int_{\Omega_T} |F|^{p-2}F\cdot D(v-u)dxdt, \quad (1.4)$$

for all testing functions $v \in \mathcal{A}'$, where

$$\mathcal{A}' = \{v \in \mathcal{A} : v_t \in L^{\frac{p}{p-1}}(0,T;W^{-1,\frac{p}{p-1}}(\Omega))\}, \quad (1.5)$$

and $< \cdot,\cdot >$ denotes the duality paring between $W^{-1,\frac{p}{p-1}}$ and $W_0^{1,p}$. The purpose of Chapter 4 is to establish the local natural Calderón-Zygmund theory for solutions to (1.4) under minimal regularity requirements on the nonlinearity. To be more precise, we want to find a reasonable answer as to what is the weakest condition on the nonlinearity a under which

$$|F|^p, |D\psi|^p, |\psi_t|^{\frac{p}{p-1}} \in L^q_{loc}(\Omega_T) \Rightarrow |Du|^p \in L^q_{loc}(\Omega_T), \text{ for every } q \in (1,\infty).$$

Finally, in Chapter 5, the result for the interior regularity in the Lebesgue space is extended to the global one in the Orlicz space which is a natural generalization of the Lebesgue space.

Chapter 2

Calderon-Zygmund theory for nonlinear elliptic problems with irregular obstacles

2.1 Preliminaries and results

Let Ω be bounded open domain in $\mathbb{R}^n$ with $n \geq 2$ and $1 < p < \infty$ be a fixed real number. We then consider the convex admissible set

$$\mathcal{A} = \{\phi \in W_0^{1,p}(\Omega) : \phi \geq \psi \quad \text{a.e. in } \Omega\} \tag{2.1}$$

with

$$\psi \in W^{1,p}(\Omega) \text{ and } \psi \leq 0 \text{ on } \partial\Omega. \tag{2.2}$$

Here, the condition of ψ on $\partial\Omega$ means that $\psi^+ \in W_0^{1,p}(\Omega)$. We are interested in functions $u \in \mathcal{A}$ satisfying the following variational inequality

$$\int_\Omega \mathbf{a}(Du, x) \cdot D(\phi - u)dx \geq \int_\Omega |F|^{p-2}F \cdot D(\phi - u)\ dx \tag{2.3}$$

for all $\phi \in \mathcal{A}$, where $F \in L^p(\Omega, \mathbb{R}^n)$.

We call such a function u to be a weak solution to the variational inequality (2.3). A given vector-valued function

$$\mathbf{a} = \mathbf{a}(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

is a Carathéodory function, namely, measurable in x and differentiable in ξ. Assume, moreover, the following boundedness and ellipticity conditions:

$$|\mathbf{a}(\xi,x)| + |\xi||D_\xi \mathbf{a}(\xi,x)| \le \Lambda|\xi|^{p-1} \tag{2.4}$$

and

$$D_\xi \mathbf{a}(\xi,x)\eta\cdot\eta \ge \mu|\xi|^{p-2}|\eta|^2 \tag{2.5}$$

for all $x,\ \xi,\ \eta \in \mathbb{R}^n$ and for some constants $0<\mu\le 1\le\Lambda$.

We point out that the primary structure conditions (2.4)-(2.5) imply the following monotonicity condition:

$$(\mathbf{a}(\xi,x)-\mathbf{a}(\eta,x))\cdot(\xi-\eta) \ge \gamma|\xi-\eta|^p \quad if \;\; p\ge 2, \tag{2.6}$$

$$(\mathbf{a}(\xi,x)-\mathbf{a}(\eta,x))\cdot(\xi-\eta) \ge \gamma|\xi-\eta|^2(|\xi|+|\eta|)^{p-2} \quad if \;\; 1<p<2. \tag{2.7}$$

Here γ is a positive constant depending only on μ, p, and n. Hereafter we employ the letter c to denote any constants that can be explicitly computed in terms of n, the geometric assumption on Ω, p, q, μ, and Λ, and so c might vary from line to line.

It is well known that there exists a unique weak solution $u \in \mathcal{A}$ to the variational inequality (2.3) from the theory of monotone operatoers, see [28]. We can also deduce the following estimates by taking $\phi = \psi^+$ in (2.3),

$$\|Du\|_{L^p(\Omega)} \le c(\|F\|_{L^p(\Omega)} + \|D\psi\|_{L^p(\Omega)}). \tag{2.8}$$

In order to measure the oscillation of $\frac{\mathbf{a}(\xi,x)}{|\xi|^{p-1}}$ in the variable x over the ball $\mathrm{B}_\rho(y)$, we define the function

$$\beta(\mathbf{a},B_\rho(y))(x) = \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \frac{|\mathbf{a}(\xi,x)-\overline{\mathbf{a}}_{B_\rho(y)}(\xi)|}{|\xi|^{p-1}}, \tag{2.9}$$

where $\overline{\mathbf{a}}_{B_\rho(y)}$ is the integral average of $\mathbf{a}(\xi,\cdot)$ over $B_\rho(y)$, as is defined by

$$\overline{\mathbf{a}}_{B_\rho(y)}(\xi) = \fint_{B_\rho(y)} \mathbf{a}(\xi,x)dx = \frac{1}{|B_\rho(y)|}\int_{B_\rho(y)} \mathbf{a}(\xi,x)dx. \tag{2.10}$$

Definition 2.1.1. $\mathbf{a}(\xi, x)$ *is* (δ, R)*-vanishing if we have*

$$\sup_{0<\rho\leq R}\sup_{y\in\mathbb{R}^n} \fint_{B_\rho(y)} |\beta(\mathbf{a}, B_\rho(y))(x)|dx \leq \delta.$$

The property of BMO(Bounded Mean Oscillation) space, see [59], implies that for any $1 \leq t < \infty$,

$$\sup_{0<\rho\leq R}\sup_{y\in\mathbb{R}^n} \fint_{B_\rho(y)} |\beta(\mathbf{a}, B_\rho(y))(x)|^t dx \leq c(t)\delta^t.$$

If $\mathbf{a}$ is (δ, R)-vanishing, we also find that

$$\sup_{0<\rho\leq R}\sup_{y\in\mathbb{R}^n} \fint_{B^+_\rho(y)} |\beta(\mathbf{a}, B^+_\rho(y))(x)|dx \leq 4\delta$$

from the following estimates :

$$\begin{aligned}
&\fint_{B^+_\rho(y)} |\beta(\mathbf{a}, B^+_\rho(y))(x)|dx \\
&\leq \fint_{B^+_\rho(y)} \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \frac{|\mathbf{a}(\xi,x) - \overline{\mathbf{a}}_{B_\rho(y)}(\xi)|}{|\xi|^{p-1}} dx + \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \frac{|\overline{\mathbf{a}}_{B^+_\rho}(\xi) - \overline{\mathbf{a}}_{B_\rho(y)}(\xi)|}{|\xi|^{p-1}} \\
&\leq 2\fint_{B^+_\rho(y)} |\beta(\mathbf{a}, B_\rho(y))(x)|dx \\
&\leq 2\frac{|B_\rho|}{|B^+_\rho|}\fint_{B_\rho(y)} |\beta(\mathbf{a}, B_\rho(y))(x)|dx,
\end{aligned}$$

where $B^+_\rho(y) = y + \{x \in B_\rho(0) : x_n > 0\}$.

Definition 2.1.2. Ω *is* (δ, R)*-Reifenberg flat if for every* $x \in \partial\Omega$ *and every* $\rho \in (0, R]$*, there exists a coordinate system* $\{y_1, ..., y_n\}$*, which can depend on* ρ *and* x *so that* $x = 0$ *in this coordinate system and that*

$$B_\rho(0) \cap \{y_n > \delta\rho\} \subset B_\rho(0) \cap \Omega \subset B_\rho(0) \cap \{y_n > -\delta\rho\}.$$

This geometric condition prescribes that under all scales the boundary can be trapped between two hyper planes, depending on the scale chosen. The domain can go beyond Lipschitz category, not necessarily given by graphs.

The following lemma shows that the obstacle problem under consideration has the invariance properties under scaling and normalization.

Lemma 2.1.1. *$u \in \mathcal{A}$ is the weak solution to the variational inequality (2.3). Assume that $\mathbf{a}$ is (δ, R)-vanishing and Ω is (δ, R)-Reifenberg flat. Fix $\lambda \geq 1$ and $0 < r < 1$. We define the rescaled maps*

$$\tilde{\mathbf{a}} = \frac{\mathbf{a}(\lambda\xi, rx)}{\lambda^{p-1}},\ \tilde{u}(x) = \frac{u(rx)}{\lambda r},\ \tilde{F}(x) = \frac{F(rx)}{\lambda},\ \tilde{\psi}(x) = \frac{\psi(rx)}{\lambda r}$$

and

$$\tilde{\Omega} = \{(1/r)x : x \in \Omega\}.$$

Then we have

1. *$\tilde{\mathbf{a}}$ satisfies (2.4) and (2.5) with the same constants $\mu,\ \Lambda$.*
2. *$\tilde{\mathbf{a}}$ is $(\delta, \frac{R}{r})$-vanishing.*
3. *$\tilde{\Omega}$ is $(\delta, \frac{R}{r})$-Reifenberg flat.*
4. *$\tilde{u} \in \tilde{\mathcal{A}} = \{\phi \in W_0^{1,p}(\tilde{\Omega}) : \phi \geq \tilde{\psi}$ a.e. in $\tilde{\Omega}\}$ is the weak solution to the following variational inequality*

$$\int_{\tilde{\Omega}} \tilde{\mathbf{a}}(D\tilde{u}, x) \cdot D(\phi - \tilde{u})dx \geq \int_{\tilde{\Omega}} |\tilde{F}|^{p-2}\tilde{F} \cdot D(\phi - \tilde{u})dx, \forall \phi \in \tilde{\mathcal{A}}.$$

From the above invariance properties, we can take any positive number for R. In this paper we use some artificial number for simplicity.

We then state the main result.

Theorem 2.1.1. *For any given $q \in (1, \infty)$, assume that $F \in L^{pq}(\Omega; \mathbb{R}^n)$ and $D\psi \in L^{pq}(\Omega; \mathbb{R}^n)$. Then there exists a constant $\delta = \delta(\mu, \Lambda, n, p, q) > 0$ such that if $\mathbf{a}(\xi, x)$ is (δ, R)-vanishing and Ω is (δ, R)-Reifenberg flat, then the weak solution u satisfies $Du \in L^{pq}(\Omega; \mathbb{R}^n)$ with the estimate*

$$\|Du\|_{L^{pq}(\Omega)} \leq c(\|D\psi\|_{L^{pq}(\Omega)} + \|F\|_{L^{pq}(\Omega)}),$$

where c is a positive constant depending on n, p, q, μ, Λ, and $|\Omega|$.

Remark 2.1.1. *As a consequence of the main result, we have Hölder regularity. More precisely, if we take q with $pq > n$, then it follows directly from Sobolev inequality that $u \in C^{0,1-\frac{n}{pq}}(\overline{\Omega})$.*

2.2 Analytic and geometric tools

We will prove the main theorem using the maximal function, some classical measure theory, a Vitali type covering lemma, and a comparison principle for the obstacle problems.

Definition 2.2.1. *The Hardy-Littlewood maximal function $\mathcal{M}f$ of a locally integrable function f is a function such that*

$$(\mathcal{M}f)(x) = \sup_{\rho>0} \fint_{B_\rho(x)} |f(y)|\ dy.$$

If f is not defined outside a bounded domain Ω,

$$\mathcal{M}_\Omega f = \mathcal{M}(f\chi_\Omega)$$

for the standard characteristic function χ on Ω.

Lemma 2.2.1. *[59] If $f \in L^t(\mathbb{R}^n)$ for $1 < t \leq \infty$, then $\mathcal{M}f \in L^t(\mathbb{R}^n)$ and for some $c = c(n,t) > 0$,*

$$\frac{1}{c}\|f\|_{L^t(\mathbb{R}^n)} \leq \|\mathcal{M}f\|_{L^t(\mathbb{R}^n)} \leq c\|f\|_{L^t(\mathbb{R}^n)}. \tag{2.11}$$

If $f \in L^1(\mathbb{R}^n)$, then for some $c = c(n) > 0$,

$$|\{x \in \mathbb{R}^n : (\mathcal{M}f)(x) > \lambda\}| \leq \frac{c}{\lambda}\int |f(x)|\ dx. \tag{2.12}$$

Lemma 2.2.2. *[12] Let C and D be measurable sets with $C \subset D \subset \Omega$. Assume that Ω is $(\delta,2)$-Reifenberg flat for some small $0 < \delta < \frac{1}{8}$. Assume further that there exists a small $\epsilon > 0$ such that*

$$|C| < \epsilon|B_1| \tag{2.13}$$

and that

$$\text{for } x \in \Omega \text{ and } r \in (0,1) \text{ with } |C \cap B_r(x)| \geq \epsilon|B_r(x)|,\ B_r(x)\cap\Omega \subset D. \tag{2.14}$$

Then we have

$$|C| \leq \epsilon_1|D|,$$

where $\epsilon_1 = \left(\frac{10}{1-4\delta}\right)^n \epsilon$.

Proof. We observe that for almost all $x \in C$, the function

$$f(r) = \frac{|C \cap B_r(x)|}{|B_r(x)|},\ r > 0,$$

is continuous with

$$\lim_{r \to 0} f(r) = 1 \text{ and } f(r) \le \frac{|C|}{|B_1(x)|} \overset{(2.13)}{<} \epsilon,\ \forall r \ge 1.$$

Then for a.e. $x \in C$, there exists a $r_x \in (0,1)$ such that

$$|C \cap B_{r_x}(x)| = \epsilon |B_{r_x}(x)| \text{ and } |C \cap B_r(x)| < \epsilon |B_r(x)|,\ \forall r > r_x. \tag{2.15}$$

Since $\{B_{r_x}(x); x \in C\}$ is covering of C, by Vitali's covering lemma there is a countable $\{x_i\}_{i=1}^{\infty}$ such that the balls $B_{r_i}(x_i)$ for $r_i = r_{x_i}$ are mutually disjoint and

$$C \subset \bigcup_i B_{5r_i}(x_i). \tag{2.16}$$

From (2.15) we find that

$$|C \cap B_{5r_i}(x_i)| < \epsilon |B_{5r_i}(x_i)| = 5^n \epsilon |B_{r_i}(x_i)|. \tag{2.17}$$

Now we claim that

$$|B_{r_i}(x_i)| \le \left(\frac{2}{1-4\delta}\right)^n |B_{r_i}(x_i) \cap \Omega|. \tag{2.18}$$

To do this, we first consider the case $\operatorname{dist}(x_i, \partial\Omega) \ge r_i$. Then

$$\frac{|B_{r_i}(x_i)|}{|B_{r_i}(x_i) \cap \Omega|} = 1$$

since $B_{r_i}(x_i) \subset \Omega$. If $\operatorname{dist}(x_i, \partial\Omega) < r_i$, then there is a $y_i \in \partial\Omega$ such that

$$\operatorname{dist}(x_i, \partial\Omega) = |x_i - y_i| < r_i \text{ and } B_{r_i}(x_i) \subset B_{2r_i}(y_i).$$

Since Ω is a $(\delta, 2)$-Reifenberg flat domain, there is a coordinate system with $y_i = 0$ such that

$$B_{2r_i}(y_i) \cap \{x_n > 2r_i\delta\} \subset B_{2r_i}(y_i) \cap \Omega \subset B_{2r_i}(y_i) \cap \{x_n > -2r_i\delta\}.$$

Then it follows that

$$B_{r_i}(x_i) \cap \{x_n > 2r_i\delta\} \subset B_{2r_i}(y_i) \cap \{x_n > 2r_i\delta\} \subset \Omega$$

and so

$$B_{r_i}(x_i) \cap \{x_n > 2r_i\delta\} \subset B_{r_i}(x_i) \cap \Omega.$$

Thus from the geometry and noting $\delta < \frac{1}{8}$, we see that

$$\frac{|B_{r_i}(x_i)|}{|B_{r_i}(x_i) \cap \Omega|} \leq \frac{|B_{r_i}(x_i)|}{|B_{r_i}(x_i) \cap \{x_n > 2r_i\delta\}|} \leq \left(\frac{2}{1-4\delta}\right)^n,$$

which shows (2.18). Finally from (2.17) and (2.18) we have

$$\begin{aligned} |C| &= \left|\bigcup_i (B_{5r_i}(x_i) \cap C)\right| \\ &\leq 5^n \epsilon \sum_i |B_{r_i}(x_i)| \\ &\leq \epsilon \left(\frac{10}{1-4\delta}\right)^n \sum_i |B_{r_i}(x_i) \cap \Omega|. \end{aligned}$$

Since the balls $B_{r_i}(x_i)$ are mutually disjoint and by (2.14) we get

$$\begin{aligned} |C| &\leq \epsilon \left(\frac{10}{1-4\delta}\right)^n \left|\bigcup_i (B_{r_i}(x_i) \cap \Omega)\right| \\ &\leq \epsilon \left(\frac{10}{1-4\delta}\right)^n |D|, \end{aligned}$$

which completes the proof. □

Lemma 2.2.3. *[15] Assume that f is a nonnegative and measurable function in $\mathbb{R}^n$. Assume further that f has a compact support in a bounded subset Ω of $\mathbb{R}^n$. Let $\theta > 0$ and $m > 1$ be constants. Then for $0 < t < \infty$, we have*

$$f \in L^t(\Omega) \Leftrightarrow S = \sum_{k \geq 1} m^{kt} |\{x \in \Omega : f(x) > \theta m^k\}| < \infty$$

and

$$\frac{1}{c} S \leq \|f\|^t_{L^t(\Omega)} \leq c(|\Omega| + S),$$

where $c > 0$ is a constant depending only on θ, m, and t.

Lemma 2.2.4. *Suppose that $\psi, v \in W^{1,p}(\Omega)$ satisfy*

$$\begin{cases} -div\ \mathbf{a}(D\psi, x) & \leq & -div\ \mathbf{a}(Dv, x) & in & \Omega, \\ \psi & \leq & v & on & \partial\Omega, \end{cases}$$

where (2.4) and (2.5) are assumed. Then there holds $\psi \leq v$ a.e. in Ω.

Proof. Let $\varphi \in W_0^{1,p}(\Omega)$ and $\varphi \geq 0$ a.e. in Ω. Then we have

$$\int_\Omega (\mathbf{a}(D\psi, x) - \mathbf{a}(Dv, x)) \cdot D\varphi \leq 0. \tag{2.19}$$

Since $\psi \leq v$ on $\partial\Omega$, we have $(\psi - v)^+ \in W_0^{1,p}(\Omega)$ and so we can take $\varphi = (\psi - v)^+$. Then it follows from (2.19) that

$$\int_\Omega (\mathbf{a}(D\psi, x) - \mathbf{a}(Dv, x)) \cdot D((\psi - v)^+)\ dx \leq 0,$$

which we rewrite as

$$\int_{\Omega\cap\{\psi>v\}} (\mathbf{a}(D\psi, x) - \mathbf{a}(Dv, x)) \cdot D(\psi - v)\ dx \leq 0. \tag{2.20}$$

If $p \geq 2$, from (2.6) and (2.20) we have

$$\begin{aligned} \int_\Omega |D((\psi - v)^+)|^p\ dx &= \int_{\Omega\cap\{\psi>v\}} |D(\psi - v)|^p\ dx \\ &\leq \frac{1}{\gamma} \int_{\Omega\cap\{\psi>v\}} (\mathbf{a}(D\psi, x) - \mathbf{a}(Dv, x)) \cdot D(\psi - v)\ dx \\ &\leq 0. \end{aligned}$$

Hence $\psi \leq v$ a.e. in Ω .

If $1 < p < 2$, using Young's inequality for $\epsilon > 0$, (2.7) and (2.20) it follows that

$$\begin{aligned}
&\int_{\Omega} |D((\psi - v)^+)|^p \, dx = \int_{\Omega \cap \{\psi > v\}} |D\psi - Dv|^p \, dx \\
&= \int_{\Omega \cap \{\psi > v\}} (|D\psi| + |Dv|)^{\frac{p(2-p)}{2}} [(|D\psi| + |Dv|)^{\frac{p(p-2)}{2}} |D(\psi - v)|^p] \, dx \\
&\leq \epsilon \int_{\Omega \cap \{\psi > v\}} (|D\psi| + |Dv|)^p \, dx \\
&+ c(\epsilon) \int_{\Omega \cap \{\psi > v\}} (|D\psi| + |Dv|)^{p-2} |D(\psi - v)|^2 \, dx \\
&\leq c\epsilon \int_{\Omega \cap \{\psi > v\}} (|D\psi|^p + |Dv|^p) \, dx \\
&+ c(\epsilon) \int_{\Omega \cap \{\psi > v\}} (\mathbf{a}(D\psi, x) - \mathbf{a}(Dv, x)) \cdot (D\psi - Dv) \, dx \\
&\leq c\epsilon \int_{\Omega \cap \{\psi > v\}} (|D\psi|^p + |Dv|^p) \, dx.
\end{aligned}$$

By letting $\epsilon \to 0$, we have

$$\int_{\Omega} |D((\psi - v)^+)|^p \, dx \leq 0.$$

Therefore, $\psi \leq v$ a.e. in Ω. □

2.3 Gradient estimates for irregular obstacle problems

We start with interior comparison estimates. To do this, we assume that

$$B_6 \subset \Omega, \tag{2.21}$$

$$\sup_{0<\rho\leq 5} \fint_{B_\rho} \beta(\mathbf{a}; B_\rho) \, dx \leq \delta, \tag{2.22}$$

and

$$\fint_{B_5} |Du|^p dx \leq 1, \ \fint_{B_5} |F|^p dx \leq \delta^p, \ \fint_{B_5} |D\psi|^p \, dx \leq \delta^p. \tag{2.23}$$

Under these assumptions (2.21)-(2.23), we compare $u \in W_0^{1,p}(\Omega)$ to the unique weak solution $k \in W^{1,p}(B_5)$ of

$$\begin{cases} -div\ \mathbf{a}(Dk,x) = -div\ \mathbf{a}(D\psi,x) & \text{in} \quad B_5, \\ k = u & \text{on} \quad \partial B_5. \end{cases} \tag{2.24}$$

We then compare $k \in W^{1,p}(B_5)$ to the unique weak solution $w \in W^{1,p}(B_5)$ of

$$\begin{cases} -div\ \mathbf{a}(Dw,x) = 0 & \text{in} \quad B_5, \\ w = k & \text{on} \quad \partial B_5. \end{cases} \tag{2.25}$$

The limiting problem is

$$\begin{cases} -div\ \overline{\mathbf{a}}_{B_4}(Dv) = 0 & \text{in} \quad B_4, \\ v = w & \text{on} \quad \partial B_4. \end{cases} \tag{2.26}$$

The following is L^p estimate for (2.24). This estimate also can be applied for (2.25) and (2.26).

Lemma 2.3.1. *Let k be the weak solution of (2.24). Then we have*

$$⨍_{B_5} |Dk|^p\ dx \le c\left(⨍_{B_5} |Du|^p\ dx + ⨍_{B_5} |D\psi|^p\ dx\right). \tag{2.27}$$

Proof. We take $k - u \in W_0^{1,p}(B_5)$ as a test function in the weak formulation of (2.24). Then we have

$$⨍_{B_5} \mathbf{a}(Dk,x)Dk\ dx = ⨍_{B_5} \mathbf{a}(Dk,x)Du\ dx + ⨍_{B_5} \mathbf{a}(D\psi,x)D(k-u)\ dx. \tag{2.28}$$

In view of (2.4), (2.6) and (2.7), we estimate the left-hand side of (2.28) as follows:

$$⨍_{B_5} \mathbf{a}(Dk,x)Dk\ dx \ge \gamma ⨍_{B_5} |Dk|^p\ dx, \tag{2.29}$$

since $\mathbf{a}(0,x) = 0$.

Using (2.4) and Young's inequality with ϵ, we estimate the right-hand side of (2.28) as follows:

$$\begin{aligned} &\fint_{B_5} \mathbf{a}(Dk,x)Du\ dx + \fint_{B_5} \mathbf{a}(D\psi,x)D(k-u)\ dx \\ &\leq \Lambda \fint_{B_5} |Dk|^{p-1}|Du| + |D\psi|^{p-1}|Dk| + |D\psi|^{p-1}|Du|\ dx \\ &\leq c\epsilon \fint_{B_5} |Dk|^p\ dx + c(\epsilon)\left(\fint_{B_5} |Du|^p\ dx + \fint_{B_5} |D\psi|^p\ dx\right). \end{aligned} \tag{2.30}$$

We combine (2.28)-(2.30), and then take ϵ so small, in order to derive the conclusion (2.27). □

From a direct consequence of Lemma 2.3.1, we have L^p estimates for (2.25) and (2.26) as follows:

$$\fint_{B_5} |Dw|^p\ dx \leq c \fint_{B_5} |Dk|^p\ dx \tag{2.31}$$

and

$$\fint_{B_4} |Dv|^p\ dx \leq c \fint_{B_4} |Dw|^p\ dx. \tag{2.32}$$

Therefore, under the assumptions (2.21)-(2.26) we have

$$\fint_{B_4} |Dv|^p\ dx + \fint_{B_5} |Dw|^p\ dx + \fint_{B_5} |Dk|^p\ dx \leq c. \tag{2.33}$$

We have the following higher integrability result for (2.25)-(2.26).

Lemma 2.3.2. *Let w be the weak solution of the problem (2.25) with the assumptions (2.21)-(2.24). Then there exists a small positive constant $\epsilon_0 = \epsilon_0(n,p,\mu,\Lambda)$ such that $|Dw| \in L^{p+\epsilon_0}(B_4)$ with the uniform bound*

$$\fint_{B_4} |Dw|^{p+\epsilon_0}\ dx \leq c.$$

Proof. According to a well known improved regularity for the homogeneous problem (2.25), we find

$$\left(\fint_{B_4} |Dw|^{p+\epsilon_0}\ dx\right)^{\frac{1}{p+\epsilon_0}} \leq c\left(1 + \fint_{B_5} |Dw|^p\ dx\right) \tag{2.34}$$

for some small positive constant $\epsilon_0 = \epsilon_0(n, p, \mu, \Lambda)$ (see, [30]). Then the conclusion follows from (2.33) and (2.34). $\square$

The following Lipschitz regularity for the limiting problem (2.26) is crucial for the required $W^{1,pq}$ regularity for the obstacle problem under consideration.

Lemma 2.3.3. *Let v be the weak solution of the problem (2.26) with the assumptions (2.21)-(2.25). Then there is a positive constant $N_0 = N_0(n, p, \mu, \Lambda)$ such that*

$$\|Dv\|_{L^\infty(B_3)} \leq N_0.$$

Proof. Note that the nonlinearity for (2.26) is independent of x-variable, to see that

$$\|Dv\|^p_{L^\infty(B_3)} \leq c\left(\fint_{B_4} |Dv|^p \, dx + 1\right). \tag{2.35}$$

By this estimate (2.35) and (2.33), we complete the proof (see, [13],[22], [24]). $\square$

We are now ready to prove the following interior comparison estimate.

Lemma 2.3.4. *Let u be a weak solution to the variational inequality (2.3). Then for any $\epsilon > 0$, there is a small $\delta = \delta(\epsilon, \mu, \Lambda, n, p) > 0$ such that if the assumptions (2.21)-(2.23) hold, then there exists a weak solution $v \in W^{1,p}(B_4)$ of (2.26) such that*

$$\fint_{B_4} |D(u - v)|^p \leq \epsilon^p. \tag{2.36}$$

Proof. Let k be the weak solution of (2.24). Since $k = u \geq \psi$ a.e. on ∂B_5, it follows from Lemma 2.2.4 that $k \geq \psi$ a.e. in B_5. We next extend k to $\Omega \setminus B_5$ by u so that $k \in \mathcal{A}$ and $k - u = 0$ in $\Omega \setminus B_5$. Then the variational inequality (2.3) when $\phi = k$ implies that

$$\int_{B_5} \mathbf{a}(Du, x) \cdot D(k - u) \, dx \geq \int_{B_5} |F|^{p-2} F \cdot D(k - u) \, dx. \tag{2.37}$$

This inequality (2.37) and (2.28) imply that

$$\int_{B_5} (\mathbf{a}(Dk,x) - \mathbf{a}(Du,x)) \cdot D(k-u)\ dx \\ \leq \int_{B_5} (\mathbf{a}(D\psi,x) - |F|^{p-2}F) \cdot D(k-u)\ dx. \quad (2.38)$$

We first estimate the left-hand side of (2.38). If $p \geq 2$, it follows from (2.6) that

$$\gamma \fint_{B_5} |D(u-k)|^p\ dx \leq \fint_{B_5} (\mathbf{a}(Dk,x) - \mathbf{a}(Du,x)) \cdot D(k-u)\ dx. \quad (2.39)$$

If $1 < p < 2$, using Young's inequality with τ, (2.7) and (2.33), we estimate as follows:

$$\begin{aligned} &\fint_{B_5} |D(u-k)|^p\ dx \\ &= \fint_{B_5} (|Du| + |Dk|)^{\frac{p(2-p)}{2}} [(|Du| + |Dk|)^{\frac{p(p-2)}{2}} |D(u-k)|^p]\ dx \\ &\leq \tau \fint_{B_5} (|Du| + |Dk|)^p\ dx + c(\tau) \fint_{B_5} (|Du| + |Dk|)^{p-2} |D(u-k)|^2\ dx \\ &\leq c\tau + c(\tau) \frac{1}{\gamma} \fint_{B_5} (\mathbf{a}(Dk,x) - \mathbf{a}(Du,x)) \cdot D(k-u)\ dx, \end{aligned}$$

which implies that for any $\tau > 0$,

$$\gamma \fint_{B_5} |D(u-k)|^p\ dx \\ \leq c\tau + c(\tau) \fint_{B_5} (\mathbf{a}(Dk,x) - \mathbf{a}(Du,x)) \cdot D(k-u)\ dx. \quad (2.40)$$

Combining (2.39) and (2.40), we find that

$$\gamma \fint_{B_5} |D(u-k)|^p\ dx \\ \leq c\tau + c(\tau) \fint_{B_5} (\mathbf{a}(Dk,x) - \mathbf{a}(Du,x)) \cdot D(k-u)\ dx. \quad (2.41)$$

We next estimate the right-hand side of (2.38). Using (2.4), Young's inequality with $\sigma > 0$ and (2.23), we have

$$
\begin{aligned}
&\fint_{B_5} (\mathbf{a}(D\psi, x) - |F|^{p-2}F) \cdot D(k-u)\ dx \\
&\leq \Lambda \fint_{B_5} (|D\psi|^{p-1} + |F|^{p-1})|D(k-u)|\ dx \\
&\leq c\sigma \fint_{B_5} |D(u-k)|^p\ dx + c(\sigma) \fint_{B_5} (|D\psi|^p + |F|^p)\ dx \\
&\leq c\sigma \fint_{B_5} |D(u-k)|^p\ dx + c(\sigma)\delta^p.
\end{aligned} \tag{2.42}
$$

From (2.38), (2.41) and (2.42), we discover

$$\fint_{B_5} |D(u-k)|^p\ dx \leq c\tau + \sigma c(\tau) \fint_{B_5} |D(u-k)|^p\ dx + c(\tau)c(\sigma)\delta^p.$$

We then take τ, σ so small, respectively, in order to discover

$$\fint_{B_5} |D(u-k)|^p\ dx \leq c\delta^{\sigma_1}, \tag{2.43}$$

for some $\sigma_1 = \sigma_1(\mu, \Lambda, n, p) > 0$.

We now let w be the weak solution of the problem (2.25). Take a test function $\varphi = k - w \in W_0^{1,p}(B_5)$ for (2.24) and (2.25) to find

$$\fint_{B_5} (\mathbf{a}(Dk, x) - \mathbf{a}(Dw, x)) \cdot D(k-w)\ dx = \fint_{B_5} \mathbf{a}(D\psi, x) \cdot D(k-w)\ dx. \tag{2.44}$$

In the same way we have estimated (2.38), one can derive from (2.44) that

$$\fint_{B_5} |D(k-w)|^p\ dx \leq c\delta^{\sigma_2}, \tag{2.45}$$

for some positive number $\sigma_2 = \sigma_2(\mu, \Lambda, n, p)$.

We next consider the weak solution v of the problem (2.26). Take a test function $\varphi = w - v \in W_0^{1,p}(B_4)$ for (2.25) and(2.26), to find that

$$\fint_{B_4} (\mathbf{a}(Dw, x) - \overline{\mathbf{a}}_{B_4}(Dv)) \cdot (Dw - Dv)\ dx = 0,$$

which we write as follows:

$$\fint_{B_4} (\overline{\mathbf{a}}_{B_4}(Dw) - \overline{\mathbf{a}}_{B_4}(Dv)) \cdot (Dw - Dv)\ dx \\ = \fint_{B_4} (\overline{\mathbf{a}}_{B_4}(Dw) - \mathbf{a}(Dw, x)) \cdot (Dw - Dv)\ dx. \quad (2.46)$$

In view of (2.41), we estimate the left-hand side of (2.46) as follow:

$$\gamma \fint_{B_4} |D(w - v)|^p\ dx \\ \leq c\tau + c(\tau) \fint_{B_4} (\overline{\mathbf{a}}_{B_4}(Dw) - \overline{\mathbf{a}}_{B_4}(Dv)) \cdot D(w - v)\ dx. \quad (2.47)$$

Recalling (2.9) and using Lemma 2.3.2 and the smallness condition (2.22), we estimate the right-hand side of (2.46) as follows:

$$\begin{aligned}
&\fint_{B_4} (\overline{\mathbf{a}}_{B_4}(Dw) - \mathbf{a}(Dw, x)) \cdot (Dw - Dv) dx \\
&\leq \fint_{B_4} |\overline{\mathbf{a}}_{B_4}(Dw) - \mathbf{a}(Dw, x)||Dw - Dv| dx \\
&\leq \fint_{B_4} \beta(\overline{\mathbf{a}}_{B_4}, B_4)|Dw|^{p-1}|Dw - Dv| dx \\
&\leq \sigma \fint_{B_4} |Dw - Dv|^p dx + c(\sigma) \fint_{B_4} \beta^{\frac{p}{p-1}} |Dw|^p dx \\
&\leq \sigma \fint_{B_4} |Dw - Dv|^p dx + c(\sigma) \left(\fint_{B_4} \beta^{\frac{p(p+\epsilon_0)}{(p-1)\epsilon_0}} dx \right)^{\frac{\epsilon_0}{p+\epsilon_0}} \left(\fint_{B_4} |Dw|^{p+\epsilon_0} dx \right)^{\frac{p}{p+\epsilon_0}} \\
&\leq \sigma \fint_{B_4} |Dw - Dv|^p dx + c(\sigma) \delta^{\frac{p}{p-1}}.
\end{aligned}$$

That is, we find that,

$$\fint_{B_4} (\overline{\mathbf{a}}_{B_4}(Dw) - \mathbf{a}(Dw, x)) \cdot (Dw - Dv)\ dx \\ \leq \sigma \fint_{B_4} |Dw - Dv|^p\ dx + c(\sigma) \delta^{\frac{p}{p-1}}. \quad (2.48)$$

Then it follows from (2.46), (2.47) and (2.48) that for some universal constant $\sigma_3 = \sigma_3(\mu, \Lambda, n, p) > 0$

$$\fint_{B_4} |Dw - Dv|^p \, dx \leq c\delta^{\sigma_3}. \tag{2.49}$$

We now combine (2.43), (2.45) and (2.49), to derive that for some universal constant $\sigma_4 = \sigma_4(\mu, \Lambda, n, p) > 0$

$$\fint_{B_4} |Du - Dv|^p \, dx \leq c\delta^{\sigma_4},$$

from which we take $\delta > 0$ so small that have the conclusion (2.36). This completes the proof. □

We next extend the interior comparison estimate in Lemma 2.3.4 to find a boundary version. Here we use weak compactness method (Lemma 2.3.6) instead improved higher regularity (Lemma 2.3.2) for the interior case. To do this, we introduce the following notations:

$$\Omega_\rho = B_\rho \cap \Omega, \;\; B_\rho^+ = \{x \in B_\rho : x_n > 0\}$$

and

$$\partial_w \Omega_\rho = B_\rho \cap \partial\Omega, \quad T_\rho = \{x \in B_\rho : x_n = 0\}.$$

We now assume that

$$B_6^+ \subset \Omega_6 \subset B_6 \cap \{x_n > -12\delta\}, \tag{2.50}$$

$$\sup_{0<\rho\leq 6} \fint_{B_\rho^+} |\beta(\mathbf{a}, B_\rho^+)(x)| \, dx \;\leq \delta, \tag{2.51}$$

and

$$\fint_{\Omega_5} |Du|^p \, dx \leq 1, \; \fint_{\Omega_5} |F|^p \, dx \leq \delta^p, \; \fint_{\Omega_5} |D\psi|^p \, dx \leq \delta^p. \tag{2.52}$$

Under these assumptions (2.50)-(2.52) we consider the following problems:

$$\begin{cases} -div \; \mathbf{a}(Dk, x) & = & -div \; \mathbf{a}(D\psi, x) & \text{in} & \Omega_5, \\ k & = & u & \text{on} & \partial\Omega_5, \end{cases} \tag{2.53}$$

$$\begin{cases} -div\ \mathbf{a}(Dw,x) &= 0 \quad \text{in} \quad \Omega_5, \\ w &= k \quad \text{on} \quad \partial\Omega_5, \end{cases} \tag{2.54}$$

$$\begin{cases} -div\ \overline{\mathbf{a}}_{B_4^+}(Dh) &= 0 \quad \text{in} \quad \Omega_4, \\ h &= w \quad \text{on} \quad \partial\Omega_4, \end{cases} \tag{2.55}$$

and

$$\begin{cases} -div\ \overline{\mathbf{a}}_{B_4^+}(Dv) &= 0 \quad \text{in} \quad B_4^+, \\ v &= 0 \quad \text{on} \quad T_4. \end{cases} \tag{2.56}$$

We can now the following uniform boundedness in L^p for Dk, Dw and Dh in almost exactly the same as (2.33):

$$\fint_{\Omega_4} |Dh|^p\ dx + \fint_{\Omega_5} |Dw|^p\ dx + \fint_{\Omega_5} |Dk|^p\ dx \leq c. \tag{2.57}$$

Returning to the Reifenberg flatness conditions, see Definition 2.1.2, one can derive

$$|B_\rho(x_0)| \leq c(\delta,n)|\Omega_\rho(x_0)|,\ \forall x_0 \in \partial_w\Omega_\rho \text{ and } \forall \rho \in (0,6].$$

Thanks to this measure density condition, the Reifenberg domains are $W^{1,t}$-extension domains, $1 \leq t \leq \infty$ and the usual extension theorem, Sobolev inequality and Poincaré's inequality hold true on the Reifenberg domains, see [12, 43, 47, 48, 51] and the references therein. Moreover, this density condition guarantees a quantified higher integrability of the gradient of a weak solution of the homogeneous problem (2.54), see [30, 49, 50] and the references therein. Then using the L^p-uniform boundedness assumption (2.57) we observe that the homogeneous problem (2.54) has the following improved higher regularity with the uniform bound

$$\fint_{\Omega_4} |Dw|^{p+\sigma_*}\ dx \leq c, \tag{2.58}$$

where $\sigma_* = \sigma_*(n,p,\mu,\Lambda)$ is a positive small universal constant.

Lemma 2.3.5. *For any* $\epsilon > 0$*, there exists* $\delta = \delta(\epsilon,\mu,\Lambda,n,p) > 0$ *such that if the assumptions* (2.50)-(2.55) *holds, then*

$$\fint_{\Omega_4} |Dw - Dh|^p dx \leq \epsilon. \tag{2.59}$$

Proof. Like the way for the estimate (2.49) in the proof of Lemma 2.3.4, we can get that

$$\fint_{\Omega_4} |D(w-h)|^p \, dx \\ \leq c\tau + c(\tau) \fint_{\Omega_4} (\overline{\mathbf{a}}_{B_4^+}(Dw) - \mathbf{a}(Dw,x)) \cdot (Dw - Dh) \, dx. \quad (2.60)$$

Recalling (2.9) we estimate the right-hand side of (2.60):

$$\begin{aligned} &\fint_{\Omega_4} (\overline{\mathbf{a}}_{B_4^+}(Dw) - \mathbf{a}(Dw,x)) \cdot (Dw - Dh) \, dx \\ &\leq \fint_{\Omega_4} \beta(\overline{\mathbf{a}}_{B_4^+}, B_4^+) |Dw|^{p-1} |Dw - Dh| \, dx \\ &\leq \sigma \fint_{\Omega_4} |Dw - Dh|^p \, dx + c(\sigma) \fint_{\Omega_4} \beta(\overline{\mathbf{a}}_{B_4^+}, B_4^+)^{\frac{p}{p-1}} |Dw|^p \, dx. \quad (2.61) \end{aligned}$$

We then need the following calculations. From (2.4), (2.51) and (2.58),

$$\begin{aligned} &\fint_{\Omega_4} \beta(\overline{\mathbf{a}}_{B_4^+}, B_4^+)^{\frac{p}{p-1}} |Dw|^p \, dx \\ &\leq \left(\fint_{\Omega_4} \beta^{\frac{p(p+\sigma_*)}{(p-1)\sigma_*}} dx \right)^{\frac{\sigma_*}{p+\sigma_*}} \left(\fint_{\Omega_4} |Dw|^{p+\sigma_*} \, dx \right)^{\frac{p}{p+\sigma_*}} \\ &\leq c \left\{ \frac{1}{|B_4^+|} \left(\int_{B_4^+} \beta^{\frac{p(p+\sigma_*)}{(p-1)\sigma_*}} dx + \int_{\Omega_4 \setminus B_4^+} \beta^{\frac{p(p+\sigma_*)}{(p-1)\sigma_*}} dx \right) \right\}^{\frac{\sigma_*}{p+\sigma_*}} \\ &\leq c \left\{ \delta^{\frac{p}{p-1}} + (2\Lambda)^{\frac{p}{p-1}} \left(\frac{|\Omega_4 \setminus B_4^+|}{|B_4^+|} \right)^{\frac{\sigma_*}{p+\sigma_*}} \right\} \\ &\leq c\delta^{\sigma_{**}} \end{aligned} \quad (2.62)$$

for some $\sigma_{**} > 0$. Combining (2.60)-(2.62), we select a small $\delta > 0$ to conclude the estimate (2.59). □

We need the following Lipschitz regularity for a limiting problem (2.56).

Lemma 2.3.6. *[13, 24, 37] Let $v \in W^{1,p}(B_4^+)$ be a weak solution of (2.56). Then we have*

$$\|Dv\|^p_{L^\infty(B_3^+)} \leq c \left(\fint_{B_4^+} |Dv|^p \, dx + 1 \right). \quad (2.63)$$

In addition, if v_0 is the zero extension of v from B_4^+ to Ω_4, then we find

$$\|Dv_0\|^p_{L^\infty(\Omega_3)} = \|Dv\|^p_{L^\infty(B_3^+)} \leq c\left(\fint_{B_4^+} |Dv|^p \, dx + 1 \right). \tag{2.64}$$

Lemma 2.3.7. *For any $\epsilon > 0$, there exists a small $\delta = \delta(\epsilon) > 0$ such that if*

$$B_4^+ \subset \Omega_4 \subset B_4 \cap \{x_n > -12\delta\}$$

and $h \in W^{1,p}(\Omega_4)$ is a weak solution of

$$\begin{cases} -div\ \overline{\mathbf{a}}_{B_4^+}(Dh) = 0 & in \quad \Omega_4, \\ h = 0 & on \quad \partial_w \Omega_4, \end{cases}$$

with

$$\fint_{\Omega_4} |Dh|^p \, dx \leq 1,$$

then there exists a weak solution $v \in W^{1,p}(B_4^+)$ of (2.56) such that

$$\fint_{B_4^+} |Dv|^p \, dx \leq 1 \text{ and } \fint_{B_4^+} |h - v|^p \, dx \leq \epsilon^p.$$

Proof. We argue by contradiction. If not, there exist $\epsilon_0 > 0$, $\{h_k\}_{k=1}^\infty$ and $\{\Omega_4^k\}_{k=1}^\infty$ such that $h_k \in W^{1,p}(\Omega_4^k)$ is a weak solution of

$$\begin{cases} -div\ \overline{\mathbf{a}}_{B_4^+}(Dh_k) = 0 & \text{in} \quad \Omega_4^k, \\ h_k = 0 & \text{on} \quad \partial_w \Omega_4^k, \end{cases} \tag{2.65}$$

with

$$B_4^+ \subset \Omega_4^k \subset B_4 \cap \left\{ x_n > -\frac{12}{k} \right\}, \tag{2.66}$$

and

$$\fint_{\Omega_4^k} |Dh_k|^p \, dx \leq 1, \tag{2.67}$$

but for any weak solution $v \in W^{1,p}(B_4^+)$ of

$$\begin{cases} -div\ \overline{\mathbf{a}}_{B_4^+}(Dv) = 0 & \text{in} \quad B_4^+, \\ v = 0 & \text{on} \quad T_4, \end{cases} \tag{2.68}$$

with

$$\fint_{B_4^+} |Dv|^p \, dx \le 1, \tag{2.69}$$

we have

$$\int_{B_4^+} |h_k - v|^p \, dx > \epsilon_0^p. \tag{2.70}$$

We extend h_k by zero from Ω_4^k to B_4 and denote it by h_k also. Then by Poincaré's inequality and (2.67), we have $\|h_k\|_{W^{1,p}(B_4)} \le c$. That is, $\{h_k\}_{k=1}^{\infty}$ is uniformly bounded in $W^{1,p}(B_4^+)$. Therefore, there exists a subsequence, which we still denote by $\{h_k\}$, and $h_\infty \in W^{1,p}(B_4^+)$ such that

$$h_k \rightharpoonup h_\infty \text{ weakly in } W^{1,p}(B_4^+) \text{ and } h_k \to h_\infty \text{ strongly in } L^p(B_4^+). \tag{2.71}$$

Then we observe from (2.65), (2.66) and (2.71) that h_∞ is a weak solution of

$$\begin{cases} -div\ \overline{\mathbf{a}}_{B_4^+}(Dh_\infty) &= 0 \quad \text{in} \quad B_4^+, \\ h_\infty &= 0 \quad \text{on} \quad T_4, \end{cases} \tag{2.72}$$

see [13] for details. It follows from (2.66), (2.67) and weak lower semicontinuity property that

$$\fint_{B_4^+} |Dh_\infty|^p \, dx \le \liminf_{k\to\infty} \fint_{B_4^+} |Dh_k|^p \, dx \le \liminf_{k\to\infty} \frac{|\Omega_4^k|}{|B_4^+|} \fint_{\Omega_4^k} |Dh_k|^p \, dx \le 1.$$

We then reach a contradiction to (2.70) from (2.71). This completes the proof. □

Lemma 2.3.8. *Let u be the weak solution to the variational inequality (2.3). Then for any $\epsilon > 0$, there is a small $\delta = \delta(\epsilon) > 0$ such that if (2.50), (2.51) and (2.52) hold, then there exists a weak solution $v \in W^{1,p}(B_4^+)$ of (2.56) such that*

$$\|Dv_0\|_{L^\infty(\Omega_3)} \le N_2 \tag{2.73}$$

and

$$\fint_{\Omega_2} |D(u - v_0)|^p \le \epsilon^p, \tag{2.74}$$

where v_0 is the zero extension of v from B_4^+ to B_4 and N_2 is a positive constant depending only on n, p, μ, Λ.

Proof. Let $k \in W^{1,p}(\Omega_5)$ be the weak solution of (2.53), and then $w \in W^{1,p}(\Omega_5)$ the weak solution of (2.54), and then $h \in W^{1,p}(\Omega_4)$ be the weak solution of (2.55). Then we can derive in a similar way as in the proof of Lemma 2.3.4 that

$$\int_{\Omega_4} |Du - Dh|^p \, dx \le c\delta^{\sigma_5}, \tag{2.75}$$

where $\sigma_5 = \sigma_5(n, p, \mu, \Lambda)$ is a small positive constant.

From (2.57) and Lemma 2.3.7 we see that there is a weak solution $v \in W^{1,p}(B_4^+)$ of (2.56) such that

$$\fint_{B_4^+} |Dv|^p \, dx \le c \tag{2.76}$$

and

$$\int_{B_3^+} |h - v|^p \, dx \le c_* \epsilon^p, \tag{2.77}$$

where c_* is to be determined small in a universal way. We next let v_0 be the zero extension of v from B_4^+ to B_4. Then the Lipschitz bound (2.73) follows from Lemma 2.25 and (2.76).

A direct computation shows that v_0 is a weak solution of

$$\begin{cases} -div\ \overline{\mathbf{a}}_{B_4^+}(Dv_0) = D_n g^n & \text{in} \quad \Omega_4, \\ v_0 = 0 & \text{on} \quad \partial_w \Omega_4, \end{cases} \tag{2.78}$$

where

$$g^n = \begin{cases} 0 & \text{if } x_n > 0, \\ \overline{\mathbf{a}}^n_{B_4^+}(Dv(x', 0)) & \text{if } x_n < 0. \end{cases} \tag{2.79}$$

Choose a cutoff function $\eta \in C_0^\infty(B_3)$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ on B_2 and $|D\eta| \le 2$. We test the problems (2.55) and (2.78) by $\varphi = \eta^p(h - v_0) \in W_0^{1,p}(\Omega_3)$, to discover

$$\fint_{\Omega_3} \left(\overline{\mathbf{a}}_{B_4^+}(Dh) - \overline{\mathbf{a}}_{B_4^+}(Dv_0) \right) \cdot D(\eta^p(h - v_0)) dx = \fint_{\Omega_3} g^n D_n \left(\eta^p (h - v_0) \right) dx,$$

from which we perform standard L^p estimate by making use of (2.4)-(2.7), to derive

$$\fint_{\Omega_2} |D(h - v_0)|^p \, dx \le c \left(\delta + \fint_{\Omega_3} \left(|h - v_0|^p + |g^n|^{\frac{p}{p-1}} \right) dx \right). \tag{2.80}$$

We then estimate the right-hand side of (2.80) as follows:

$$\begin{aligned}
\fint_{\Omega_3} |h - v_0|^p \, dx &\le \fint_{B_3^+} |h - v|^p \, dx + \frac{1}{|\Omega_3|} \int_{\Omega_3 \setminus B_3^+} |h|^p \, dx \\
&\le c_*^p \epsilon^p + \frac{1}{|\Omega_3|} \left(\int_{\Omega_3} |h|^{p^*} \, dx \right)^{\frac{p}{p^*}} \left| \Omega_3 \setminus B_3^+ \right|^{\frac{p^*-p}{p^*}} \\
&\le c_*^p \epsilon^p + c\delta^{\frac{p^*-p}{p^*}} \fint_{\Omega_3} |Dh|^p \, dx \\
&\le c_*^p \epsilon^p + c\delta^{\frac{p^*-p}{p^*}},
\end{aligned} \tag{2.81}$$

where $p^* = \frac{np}{n-p}$ for $p < n$, $p^* > p$ is arbitrary if $p \ge n$ and c_* is to be determined later. Here in the first line we have used (2.50) and the fact that $v_0 = 0$ in $\Omega_4 \setminus B_4^+$. In the second line we have used (2.77), (2.50) and Hölder's inequality. In the third line we have used (2.50) and Sobolev inequality, assuming $1 < p < n$, otherwise u is of class $C^{1-\frac{n}{p}}$ or BMO. In the last line we have used (2.57). In the last line we have used (2.57).

$$\begin{aligned}
\fint_{\Omega_3} |g^n|^{\frac{p}{p-1}} \, dx &\le \frac{1}{|\Omega_3|} \int_{\Omega_3 \setminus B_3^+} \left| \overline{\mathbf{a}}^n_{B_3^+}(Dv(x',0)) \right|^{\frac{p}{p-1}} dx \\
&\le c \frac{1}{|\Omega_3|} \int_{\Omega_3 \setminus B_3^+} |Dv(x',0)|^p \, dx \\
&\le c \frac{\left| \Omega_3 \setminus B_3^+ \right|}{\left| B_3^+ \right|} \\
&\le c\delta,
\end{aligned} \tag{2.82}$$

where we have used (2.79), (2.4), (2.73), and (2.50). Combining (2.80), (2.81) and (2.82), we deduce

$$\fint_{\Omega_2} |D(h - v_0)|^p \, dx \le c \left(c_* \epsilon + (c_* \epsilon + 1) \delta^{\sigma_6} \right) \tag{2.83}$$

for some positive constant $\sigma_6 = \sigma_6(n, p, \mu, \Lambda)$. But then (2.75) and (2.83) imply

$$\fint_{\Omega_2} |D(u - v_0)|^p \, dx \le c \left(c_* \epsilon + (c_* \epsilon + 1) \delta^{\sigma_7} \right) \tag{2.84}$$

for some positive constant $\sigma_7 = \sigma_7(n, p, \mu, \Lambda)$. Finally, taking c_* small enough, and then δ, in order to arrive at the conclusion

$$\fint_{\Omega_2} |D(u - v_0)|^p \, dx \le \epsilon^p.$$

□

Lemma 2.3.9. *Given a vector-valued function $F \in L^p(\Omega, \mathbb{R}^n)$, let $u \in W_0^{1,p}(\Omega)$ be the weak solution of the variational problem (2.3). Then, there exists a universal constant $N = N(\mu, \Lambda, n, p) > 1$ such that for each $0 < \epsilon < 1$ one can select a small $\delta = \delta(\epsilon) > 0$ such that if $\mathbf{a}$ is $(\delta, 48)$-vanishing, Ω is $(\delta, 48)$-Reifenberg flat, and $B_r(y)$ with $y \in \Omega$ and $r \in (0, 1)$ satisfies*

$$|\{x \in \Omega : \mathcal{M}(|Du|^p) > N^p\} \cap B_r(y)| \ge \epsilon \, |B_r(y)| \tag{2.85}$$

for such a small δ, then we have

$$\begin{aligned} &B_r(y) \cap \Omega = \Omega_r(y) \\ &\subset \{x \in \Omega : \mathcal{M}(|Du|^p) > 1\} \cup \{x \in \Omega : \mathcal{M}(|F|^p) > \delta^p\} \\ &\qquad\qquad \cup \{x \in \Omega : \mathcal{M}(|D\psi|^p) > \delta^p\}. \end{aligned} \tag{2.86}$$

Proof. We argue by contradiction. If $B_r(y)$ satisfies (2.85)and the claim (2.86) is false, then there exists a point $y_1 \in \Omega_r(y) = B_r(y) \cap \Omega$ such that for every $\rho > 0$,

$$\begin{aligned} &\frac{1}{|B_\rho(y_1)|} \int_{\Omega_\rho(y_1)} |Du|^p \, dx \le 1, \\ &\qquad \frac{1}{|B_\rho(y_1)|} \int_{\Omega_\rho(y_1)} |F|^p \, dx \le \delta^p, \ \frac{1}{|B_\rho(y_1)|} \int_{\Omega_\rho(y_1)} |D\psi|^p \, dx \le \delta^p. \end{aligned} \tag{2.87}$$

We first consider the interior case that $B_{6r}(y) \subset \Omega$. Since $B_{5r}(y) \subset \Omega_{6r}(y_1)$, it follows from (2.87) that

$$\begin{aligned} \fint_{B_{5r}(y)} |Du|^p \, dx &\le \frac{1}{|B_{5r}(y)|} \int_{\Omega_{6r}(y_1)} |Du|^p \, dx \\ &\le \frac{|B_{6r}(y)|}{|B_{5r}(y)|} \frac{1}{|B_{6r}(y_1)|} \int_{\Omega_{6r}(y_1)} |Du|^p \, dx \le \left(\frac{6}{5}\right)^n < 2^n. \end{aligned} \tag{2.88}$$

Likewise, we find

$$\fint_{\Omega_{5r}(y)} |F|^p \, dx \le 2^n \delta^p, \; \fint_{\Omega_{5r}(y)} |D\psi|^p \, dx \le 2^n \delta^p. \tag{2.89}$$

Without loss of generality we assume $y = 0$. We then consider the re-scale maps

$$\tilde{\mathbf{a}}(\xi, x) = \frac{\mathbf{a}\left(2^{\frac{n}{p}}\xi, rx\right)}{2^{\frac{n(p-1)}{p}}}, \; \tilde{\Omega} = \left\{\frac{1}{r}x : x \in \Omega\right\}. \tag{2.90}$$

and

$$\tilde{u}(x) = \frac{u(rx)}{2^{\frac{n}{p}} r}, \; \tilde{F} = \frac{F(rx)}{2^{\frac{n}{p}}}, \; \tilde{\psi}(x) = \frac{\psi(rx)}{2^{\frac{n}{p}} r} \tag{2.91}$$

with $x \in B_6 \subset \tilde{\Omega}$ and $\xi \in \mathbb{R}^n$. Because of Lemma 2.1.1 and (2.88)-(2.91), we are in the setting of Lemma 2.3.4. This lemma and Lemma 2.3.3 imply, after scaling back, that there exists $v \in W^{1,p}(B_{4r})$ such that

$$\|Dv\|_{L^\infty(B_{3r})} \le 2^{\frac{n}{p}} N_0 =: \overline{N}_0 \tag{2.92}$$

for some positive constant $N_0 = N_0(\mu, \Lambda, n, p)$, and

$$\fint_{B_{4r}} |D(u-v)|^p \, dx \le 2^n \epsilon_* \epsilon^p, \tag{2.93}$$

where ϵ_* is to be determined in a universal way as below. Now we let

$$N_1 = \max\{2\overline{N}_0, 2^{\frac{n}{p}}\},$$

then we claim that

$$\{x \in B_r : \mathcal{M}(|Du|^p) > N_1^p\} \subset \left\{x \in B_r : \mathcal{M}_{B_{3r}}(|D(u-v)|^p) > \overline{N}_0^p\right\}. \tag{2.94}$$

To show this, take $x_1 \in \{x \in B_r : \mathcal{M}_{B_{3r}}(|D(u-v)|^p) \le \overline{N}_0^p\}$. For $0 < \rho < 2r$, since $B_\rho(x_1) \subset B_{3r}$,

$$\begin{aligned} \fint_{B_\rho(x_1)} |Du|^p dx &\le 2^{p-1} \fint_{B_\rho(x_1)} (|D(u-v)|^p + |Dv|^p) dx \\ &\le 2^p \overline{N}_0^p \le N_1^p. \end{aligned}$$

For $\rho \geq 2r$, since $B_\rho(x_1) \subset B_{2\rho}(y_1)$,

$$\frac{1}{|B_\rho|}\int_{\Omega_\rho(x_1)} |Du|^p dx \leq \frac{|B_{2\rho}|}{|B_\rho|}\frac{1}{|B_{2\rho}|}\int_{\Omega_{2\rho}(y_1)} |Du|^p dx \leq 2^n \leq N_1^p.$$

Hence we get (2.94).

By (2.94), (2.12) in Lemma 2.2.1 and (2.93), we conclude that

$$\begin{aligned}
&\frac{1}{|B_r|}\left|\{x \in B_r : \mathcal{M}(|Du|^p) > N_1^p\}\right| \\
&\leq \frac{1}{|B_r|}\left|\left\{x \in B_r : \mathcal{M}_{B_{3r}}(|D(u-v)|^p) > \overline{N}_0^p\right\}\right| \\
&\leq c\fint_{B_{3r}} |D(u-v)|^p \, dx \\
&\leq (c\epsilon_*)\epsilon < \epsilon,
\end{aligned}$$

from the choice of a sufficiently small ϵ_*. Then we arrive at a contradiction to (2.85).

We next consider the boundary case when $B_{6r}(y) \not\subset \Omega$. In this case, there is a boundary point $y_0 \in \partial\Omega \cap B_{6r}(y)$. From the Reifenberg flatness condition and small BMO condition, we assume that there exists a new coordinate system, modulo reorientation of the axes and translation, depending on y_0 and r, whose variables we denote by $z = (z_1, ..., z_n)$ such that in this new coordinate system the origin is y_0 and

$$B_{48r} \cap \{z_n > 48r\delta\} \ \subset \ \Omega_{48r} \ \subset \ B_{48r} \cap \{z_n > -48r\delta\}. \tag{2.95}$$

We translate this coordinate system to the z_n-direction by $48r\delta$, to have a coordinate system, still say $z = (z_1, ..., z_n)$, such that

$$B_{48r}^+ \ \subset \ \Omega_{48r} \ \subset \ \{z \in B_\rho : z_n > -96r\delta\} \tag{2.96}$$

and

$$\sup_{0<\rho\leq 48} \fint_{B_\rho^+} \left|\beta(\mathbf{a}, B_\rho^+)(z)\right| dz \ \leq \delta. \tag{2.97}$$

Since $\delta \in (0, \frac{1}{8})$, we have $|y| \leq 12r$ and $|y_1| \leq 13r$, which imply $\Omega_r(y) \subset \Omega_{13r}$ and $\Omega_{40r} \subset \Omega_{53r}(y_1)$. Then it follows from (2.93) and (2.87) that

$$\fint_{\Omega_{40r}} |Du|^p \, dz \leq 2\left(\frac{53}{40}\right)^n \fint_{\Omega_{53r}(y_1)} |Du|^p \, dz \leq 2\left(\frac{53}{40}\right)^n < 2^{n+1} \tag{2.98}$$

and

$$\fint_{\Omega_{40r}} |F|^p \, dz \le 2^{n+1}\delta^p, \quad \fint_{\Omega_{40r}} |D\psi|^p \, dz \le 2^{n+1}\delta^p. \tag{2.99}$$

As for the interior case, we apply Lemma 2.1.1 by taking $\rho = 8r$ and $\lambda = 2^{\frac{n+1}{p}}$, and then use (2.96)-(2.99), to observe that we are in the hypotheses of Lemma 2.3.8, which yields that there exists an function $v_0 \in W^{1,p}(\Omega_{32r})$ with the properties

$$\fint_{\Omega_{16r}} |D(u - v_0)|^p \, dz \le \epsilon_{**}\epsilon^p\lambda^p$$

for ϵ_{**} as selected below, and

$$\|Dv_0\|_{L^\infty(\Omega_{24r})} \le N_2\lambda =: \overline{N}_2,$$

where N_2 is a universal constant depending on μ, Λ, n and p. Setting

$$N_3 = \max\{2\overline{N}_2, 10^{\frac{n}{p}}\},$$

we conclude, as in the interior case, that

$$\{z \in \Omega_{13r} : \mathcal{M}(|Du|^p) > N_3^p\} \subset \{z \in \Omega_{13r} : \mathcal{M}_{\Omega_{16r}}(|D(u-v)|^p) > \overline{N}_2^p\}$$

and so

$$\frac{1}{|B_r|}\left|\{z \in \Omega_{13r} : \mathcal{M}(|Du|^p) > N_3^p\}\right| \le c\epsilon_{**}\epsilon.$$

From $\Omega_r(y) \subset \Omega_{13r}$,

$$\frac{1}{|B_r(y)|}\left|\{x \in \Omega_r(y) : \mathcal{M}(|Du|^p) > N_3^p\}\right| \le c\epsilon_{**}\epsilon.$$

Then if $c\epsilon_{**} < 1$, we reach a contradiction. Now we set $N = \max\{N_1, N_3\}$ to complete the proof. □

2.4 Global Calderón-Zygmund theory for obstacle problems

In this section, we prove Theorem 2.1.1. This proof is based on the Vitali type covering lemma and the Hardy-Littlewood maximal function.

Proof. Fix $N > 1$, $\epsilon \in (0,1)$ and the corresponding $\delta \in \left(0, \frac{1}{8}\right)$ given by Lemma 2.3.9. Our strategy is to derive

$$\|Du\|_{L^{pq}(\Omega,\mathbb{R}^n)} \leq c \tag{2.100}$$

under the assumptions

$$\|F\|_{L^{pq}(\Omega,\mathbb{R}^n)} \leq \delta,\ \|D\psi\|_{L^{pq}(\Omega,\mathbb{R}^n)} \leq \delta. \tag{2.101}$$

Then a direct computation with Lemma 2.20 and (2.101) shows

$$\sum_{k=1}^{\infty} N^{pqk}|\{x \in \Omega : \mathcal{M}(|F|^p) > \delta^p N^{pk}\}| \leq c\frac{1}{\delta^{pq}}\|\mathcal{M}(|F|)\|^{pq}_{L^{pq}(\Omega)} \leq c \tag{2.102}$$

and

$$\sum_{k=1}^{\infty} N^{pqk}|\{x \in \Omega : \mathcal{M}(|D\psi|^p) > \delta^p N^{pk}\}| \leq c\frac{1}{\delta^{pq}}\|\mathcal{M}(|D\psi|)\|^{pq}_{L^{pq}(\Omega)} \leq c. \tag{2.103}$$

We now set

$$C = \{x \in \Omega : \mathcal{M}(|Du|^p) > N^p\}$$

and

$$D = \{x \in \Omega : \mathcal{M}(|Du|^p) > 1\} \cup \{x \in \Omega : \mathcal{M}(|F|^p) > \delta^p\} \\ \cup \{x \in \Omega : \mathcal{M}(|D\psi|^p) > \delta^p\}.$$

Then it follows from Lemma 2.2.1, standard L^p estimate (2.8) and (2.101) that

$$|C| \leq c(n,p)\|Du\|^p_{L^p(\Omega)} \leq c\left(\|F\|^p_{L^p(\Omega)} + \|D\psi\|^p_{L^p(\Omega)}\right) \leq c\delta^p < \epsilon|B_1| \tag{2.104}$$

from a choice of δ corresponding to ϵ. Then it is clear from (2.104) and Lemma 2.3.9 that we are under the hypotheses of Lemma 2.2.2. Consequently, we get

$$\begin{aligned} &|\{x \in \Omega : \mathcal{M}(|Du|^p) > N^p\}| \\ &\quad \leq \epsilon_1|\{x \in \Omega : \mathcal{M}(|Du|^p) > 1\}| + \epsilon_1|\{x \in \Omega : \mathcal{M}(|F|^p) > \delta^p\}| \\ &\qquad + \epsilon_1|\{x \in \Omega : \mathcal{M}(|D\psi|^p) > \delta^p\}|. \end{aligned} \tag{2.105}$$

Set $u_1 = \frac{u}{N}$, $F_1 = \frac{F}{N}$ and $\psi_1 = \frac{\psi}{N}$. Then u_1 is the weak solution for the variational inequality (2.3) and we obtan

$$\{x \in \Omega : \mathcal{M}(|Du_1|^p) > N^p\} = \{x \in \Omega : \mathcal{M}(|Du|^p) > N^{2p}\} \leq |C| \leq \epsilon|B_1|.$$

One can derive the same estimate (2.105) replaced by the normalized functions u_1, F_1 and ψ_1. We then iterate the estimate (2.105) for $k \geq 2$, to find

$$\begin{aligned} &|\{x \in \Omega : \mathcal{M}(|Du|^p) > N^{pk}\}| \\ &\leq \epsilon_1^k|\{x \in \Omega : \mathcal{M}(|Du|^p) > 1\}| + \sum_{i=1}^{k} \epsilon_1^i|\{x \in \Omega : \mathcal{M}(|F|^p) > \delta^p N^{p(k-i)}\}| \\ &+ \sum_{i=1}^{k} \epsilon_1^i|\{x \in \Omega : \mathcal{M}(|D\psi|^p) > \delta^p N^{p(k-i)}\}| \end{aligned} \tag{2.106}$$

Then in view of Lemma 2.2.3, (2.101), (2.102)-(2.103) and (2.106), we compute as follows:

$$\begin{aligned} &\|\mathcal{M}(|Du|^p)\|_{L^q(\Omega)}^q \\ &\leq c\left(|\Omega| + \sum_{k=1}^{\infty} N^{pqk}|\{x \in \Omega : \mathcal{M}(|Du|^p) > N^{pk}\}|\right) \\ &\leq c\left(1 + \sum_{i=1}^{\infty}(N^{pq}\epsilon_1)^i \sum_{k=i}^{\infty} N^{pq(k-i)}|\{x \in \Omega : \mathcal{M}(|F|^p) > \delta^p N^{p(k-i)}\}|\right) \\ &+c\sum_{i=1}^{\infty}(N^{pq}\epsilon_1)^i \sum_{k=i}^{\infty} N^{pq(k-i)}|\{x \in \Omega : \mathcal{M}(|D\psi|^p) > \delta^p N^{p(k-i)}\}| \\ &\leq c\left(1 + \sum_{k=1}^{\infty}(N^{pq}\epsilon_1)^k\right). \end{aligned}$$

Taking ϵ so small, in order to get

$$N^{pq}\epsilon_1 = N^{pq}\left(\frac{10}{1-4\delta}\right)^n \epsilon \leq N^{pq}20^n\epsilon < 1,$$

we conclude that $\|\mathcal{M}(|Du|^p)\|_{L^q(\Omega)} \leq c$. But then, by (2.12) in Lemma 2.2.1, we arrive at the claim (2.100).

Now we need to drop the a priori assumptions (2.101). To do this, we consider the normalized functions

$$\widetilde{u} = \frac{u}{\frac{1}{\delta}\left(\|F\|_{L^{pq}(\Omega)} + \|D\psi\|_{L^{pq}(\Omega)}\right)},$$

$$\widetilde{F} = \frac{F}{\frac{1}{\delta}\left(\|F\|_{L^{pq}(\Omega)} + \|D\psi\|_{L^{pq}(\Omega)}\right)},$$

and

$$\widetilde{\psi} = \frac{\psi}{\frac{1}{\delta}\left(\|F\|_{L^{pq}(\Omega)} + \|D\psi\|_{L^{pq}(\Omega)}\right)}.$$

Clearly, we have

$$\|\widetilde{F}\|_{L^{pq}(\Omega,\mathbb{R}^n)} \leq \delta,\ \|D\widetilde{\psi}\|_{L^{pq}(\Omega,\mathbb{R}^n)} \leq \delta.$$

As a consequence, we conclude

$$\|D\widetilde{u}\|_{L^{pq}(\Omega,\mathbb{R}^n)} \leq c,$$

from which we finally obtain the required estimate

$$\|Du\|_{L^{pq}(\Omega,\mathbb{R}^n)} \leq c\left(\|F\|_{L^{pq}(\Omega,\mathbb{R}^n)} + \|D\psi\|_{L^{pq}(\Omega,\mathbb{R}^n)}\right).$$

□

Remark 2.4.1. *In this work, we think it is valuable to prove a similar global estimate in the case of parabolic problems, and for linear problems where measurable dependence is considered with respect to one variable. In parabolic problems, it might be proved by using the maximal function free technique in order to surmount the lack of scaling of degenerate parabolic problems.*

Chapter 3

Global weighted estimates for nonlinear elliptic obstacle problems over Reifenberg domains

3.1 Hypotheses, Preliminaries and Main Results

We let $p \in (1, \infty)$ to be a fixed real number and $\Omega \subset \mathbb{R}^n$ a bounded domain with $n \geq 2$. Given a vector field $\mathbf{a}(\xi, x) \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, we will suppose it defines a C^1-Carathéodory map, that is, $\mathbf{a}(\xi, x)$ is differentiable with respect to ξ for almost all (a.a.) $x \in \mathbb{R}^n$ and is measurable in x for all $\xi \in \mathbb{R}^n$. Moreover, we will assume that there exist constants $0 < \mu \leq 1 \leq \Lambda$ such that $\mathbf{a}(\xi, x)$ satisfies the following growth and ellipticity conditions

$$|\mathbf{a}(\xi, x)| + |\xi||D_\xi \mathbf{a}(\xi, x)| \leq \Lambda |\xi|^{p-1} \tag{3.1}$$

and

$$D_\xi \mathbf{a}(\xi, x)\eta \cdot \eta \geq \mu |\xi|^{p-2}|\eta|^2 \tag{3.2}$$

for a.a. $x \in \mathbb{R}^n$ and for all $\xi, \eta \in \mathbb{R}^n$.

Given an obstacle $\psi \in W^{1,p}(\Omega)$ such that $\psi \leq 0$ on $\partial\Omega$, we define the convex admissible set

$$\mathcal{A} = \left\{ v \in W_0^{1,p}(\Omega) \colon \quad v \geq \psi \text{ a.e. in } \Omega \right\}. \tag{3.3}$$

We will deal with a function $u \colon \Omega \to \mathbb{R}$, belonging to $\mathcal{A}$, and such that

$$\int_\Omega \mathbf{a}(Du, x) \cdot D(v - u)\, dx \geq \int_\Omega |F|^{p-2} F \cdot D(v-u)\, dx \quad \text{for all } v \in \mathcal{A}, \tag{3.4}$$

where the nonhomogeneous term F is a vector valued function in $L^p(\Omega; \mathbb{R}^n)$. According to the common terminology, the function u will be called a solution of the *variational inequality* (3.4).

It is well known that, under the structure conditions (3.1) and (3.2), there exists a unique solution $u \in \mathcal{A}$ of (3.4). Moreover,

$$\|Du\|_{L^p(\Omega;\mathbb{R}^n)} \leq c \left(\|F\|_{L^p(\Omega;\mathbb{R}^n)} + \|D\psi\|_{L^p(\Omega;\mathbb{R}^n)} \right) \tag{3.5}$$

with constant c depending only on n, p, Λ and μ.

The main result we are going to derive here is a weighted version of the Calderón–Zygmund regularity estimate. Precisely, assuming $F \in L^{pq}_w(\Omega; \mathbb{R}^n)$ and $D\psi \in L^{pq}_w(\Omega; \mathbb{R}^n)$, we are interested in a bound of the type

$$\|Du\|_{L^{pq}_w(\Omega;\mathbb{R}^n)} \leq c \left(\|F\|_{L^{pq}_w(\Omega;\mathbb{R}^n)} + \|D\psi\|_{L^{pq}_w(\Omega;\mathbb{R}^n)} \right) \tag{3.6}$$

holding for each $q \in (1, \infty)$, which in turn implies $Du \in L^{pq}_w(\Omega; \mathbb{R}^n)$. Here $w = w(x)$ is a weight function belonging to the Muckenhoupt class A_q (see the discussions in the next section) and c is a constant depending on n, p, Λ and μ as before, and on Ω, q and w as well.

In what follows, given a point $y \in \mathbb{R}^n$ and a number $\rho > 0$, we set $B_\rho(y) = \{x \in \mathbb{R}^n : |x - y| < \rho\}$ for the open ball centered at y and of radius ρ.

The main geometric assumption on the boundary of the underlying domain Ω is its δ-*Reifenberg flatness* which is introduced in Chapter 2.

Definition 3.1.1. *We say that Ω is (δ, R)-Reifenberg flat if there exist positive constants δ and R such that for each $x \in \partial\Omega$ and each $\rho \in (0, R]$ there* *is a coordinate system* $\{y_1, \cdots, y_n\}$*, which may depend on* x *and* ρ*, with the origin at* x *and such that*

$$B_\rho(x) \cap \{y\colon\ y_n > \delta\rho\} \subset B_\rho(x) \cap \Omega \subset B_\rho(x) \cap \{y\colon\ y_n > -\delta\rho\}.$$

Indeed, the definition is significant for small values of δ, and R could be any constant as it follows by the scaling invariance property. The flatness of the boundary in Reifenberg sense means that it is well approximated by hyperplanes at every point and at each scale. It was Reifenberg who first defined that concept in his studies [54] on Plateau problems, proving that such a domain is locally a topological disc for small enough δ. The δ-Reifenberg flatness exhibits a sort of a minimal geometric condition on the boundary ensuring validity of some natural properties of geometric analysis and partial differential equations such as $W^{1,p}$-extension, nontangential accessibility property, measure density condition, the Poincaré inequality and so on. We refer the reader to [21, 34, 36, 62] and the references therein for further details.

It is worth noting that the C^1-smooth domains are Reifenberg flat with vanishing δ when $R \searrow 0^+$. More generally, Reifenberg flat is any domain with boundary which is locally a graph of Lipschitz continuous function with small Lipschitz constant. Actually, the class of Reifenberg flat domains goes beyond these common examples and contains domains with rough fractal boundaries. For instance, the von Koch snowflake is a Reifenberg flat when the angle of the spike with respect to the horizontal is small enough.

A remarkable feature of the Reifenberg flat domains, which is an immediate consequence of the definition, is a two-sided variant of the so-called *(A)-condition of Ladyzhenskaya and Ural'tseva* (cf. [33]). Namely, the Lebesgue measure of $B_\rho(x) \cap \Omega$ is comparable to the measure of the ball $B_\rho(x)$ itself for any $x \in \overline{\Omega}$ and any $\rho \in (0, \operatorname{diam}\Omega)$. In other words, for each δ there exists a constant $A_\Omega(\delta) \in (0,1)$ such that

$$A_\Omega(\delta)\rho^n \leq |B_\rho(x) \cap \Omega| \leq \big(1 - A_\Omega(\delta)\big)\rho^n \quad \forall x \in \overline{\Omega},\ \forall \rho \in (0, \operatorname{diam}\Omega). \tag{3.7}$$

The lower bound here excludes interior cusps at each point of $\partial\Omega$ and this guarantees the validity of the Sobolev embedding theorem in the spaces $W^{1,p}(\Omega)$. The upper bound in (3.7) instead ensures that no exterior cusps exist on $\partial\Omega$ and this serves to get fine regularity properties of solutions to nonlinear PDEs such as better integrability of the gradient based of reverse Hölder inequalities ([47, 48]), regularity in Morrey and Hölder spaces of solutions to semilinear problems ([8]) and essential boundedness of the weak solutions to a very general class of quasilinear elliptic equations ([9]). We refer to [21, 29, 62] for an exhaustive discussion on the properties and regularity of the Reifenberg domains.

To introduce the main assumption regarding the principal part $\mathbf{a}(\xi, x)$ of the nonlinear differential operator considered we define

$$\Theta(\mathbf{a}, B_\rho(y))(x) = \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|\mathbf{a}(\xi, x) - \overline{\mathbf{a}}_{B_\rho(y)}(\xi)|}{|\xi|^{p-1}}, \tag{3.8}$$

where $\overline{\mathbf{a}}_{B_\rho(y)}(\xi)$ stands for the integral average

$$\fint_{B_\rho(y)} \mathbf{a}(\xi, z)\, dz = \frac{1}{|B_\rho(y)|} \int_{B_\rho(y)} \mathbf{a}(\xi, z)\, dz$$

of $\mathbf{a}(\xi, \cdot)$ over $B_\rho(y)$.

Definition 3.1.2. *We say that $\mathbf{a}(\xi, x)$ is (δ, R)-vanishing if*

$$\sup_{0<\rho\leq R} \sup_{y\in\mathbb{R}^n} \fint_{B_\rho(y)} \left|\Theta(\mathbf{a}, B_\rho(y))(x)\right| dx \leq \delta.$$

It is worth noting that the (δ, R)-vanishing of $\mathbf{a}(\xi, x)$ means that the function $x \mapsto \frac{\mathbf{a}(\xi,x)}{|\xi|^{p-1}}$ has a small mean oscillation around its integral average, uniformly in ξ. This allows, of course, discontinuities of $\mathbf{a}(\xi, x)$ with respect to x, measured in terms of *smallness* of the BMO-seminorm. That is a quite general condition to impose on the behaviour of $\mathbf{a}(\xi, x)$ in x which is minimal in some sense, and which is surely satisfied if $x \mapsto \frac{\mathbf{a}(\xi,x)}{|\xi|^{p-1}}$ is VMO or continuous with respect to x, uniformly in ξ (see [6, 12, 43] and the references therein).

For what concerns the constant R in Definitions 3.1.1 and 3.1.2, it could be any positive number due to the scaling invariance property of the problem (3.4), while δ remains the same under such a scaling. Further on, the constant δ will be sufficiently small to be selected in a universal way so that it will be independent of the nonhomogeneous term F and the obstacle ψ.

For the purposes of this Chapter, F and $D\psi$ will be taken to lie in an appropriate weighted Lebesgue space. For the sake of completeness, let us recall the definition of the *Muckenhoupt classes* A_q of weights with $1 < q < \infty$. A weight w is a positive, locally integrable function on $\mathbb{R}^n$. Given $q \in (1, \infty)$, the class A_q is defined as the collection of all weights w for which

$$[w]_q = \sup_{y\in\mathbb{R}^n} \sup_{r>0} \left(\fint_{B_r(y)} w(x)\ dx \right) \left(\fint_{B_r(y)} w(x)^{\frac{-1}{q-1}}\ dx \right)^{q-1} < \infty. \tag{3.9}$$

A typical example of a weight in A_q is given by the function $w_\sigma(x) = |x|^\sigma$ when $-n < \sigma < n(q-1)$.

Now, the weighted Lebesgue space $L^q_w(\Omega)$ related to A_q consists of all measurable functions $f\colon \Omega \to \mathbb{R}^n$ such that

$$\|f\|_{L^q_w(\Omega)} = \left(\int_\Omega |f(x)|^q w(x)\ dx \right)^{\frac{1}{q}} < \infty.$$

We set further $w(E)$ for the weighted Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$, given by

$$w(E) = \int_E w(x)\ dx. \tag{3.10}$$

In the sequel, we will use the following relationship between the Lebesgue and the weighted measures.

Lemma 3.1.1. (see [43]) *Let E a measurable subset of Ω and $w \in A_q$ for some $1 < q < \infty$.*

Then there exist positive constants ν and α, depending only on $[w]_q$ and n, such that

$$\frac{1}{\alpha}\left(\frac{|E\cap B_r(y)|}{|B_r(y)|}\right)^q \le \frac{w(E\cap B_r(y))}{w(B_r(y))} \le \alpha \left(\frac{|E\cap B_r(y)|}{|B_r(y)|}\right)^\nu.$$

The main result of the paper is the next theorem.

Theorem 3.1.1. *Let $1 < p < \infty$ and w be a weight in A_q for some $q \in (1, \infty)$. Suppose that $|F|^p \in L^q_w(\Omega)$ and $|D\psi|^p \in L^q_w(\Omega)$.*

There exists a positive constant $\delta = \delta(n, p, q, \Lambda, \mu, [w]_q)$ *such that if* $\mathbf{a}(\xi, x)$ *is* (δ, R)*-vanishing and* Ω *is* (δ, R)*-Reifenberg flat, then the gradient* Du *of the weak solution to the variational inequality* (3.4) *satisfies* $|Du|^p \in L^q_w(\Omega)$ *and we have the estimate*

$$\int_\Omega |Du|^{pq} w(x)\, dx \le c \left(\int_\Omega |F|^{pq} w(x)\, dx + \int_\Omega |D\psi|^{pq} w(x)\, dx \right) \tag{3.11}$$

with a constant c *depending only on is* $n, p, q, \Lambda, \mu, [w]_q$ *and* Ω.

Let us point out the reader attention to the fact that in the special, unweighted case ($w = 1$) Theorem 3.1.1 gives rise to a regularity result already proved in the earlier paper [6]. The technique employed in proving (3.11) is based on local comparison estimates, maximal function and Vitali covering lemma, and is more or less analogous to that in [6], adapted to the settings of the weighted spaces here considered. Indeed, that is possible thanks to Lemma 3.1.1 which implies the associated weight measure is comparable to the Lebesgue one. In that sense, Theorem 3.1.1 is a natural extension of the work in [6] to the framework of weighted Lebesgue spaces.

To proceed further with our second result, which is a particular outgrowth of Theorem 3.1.1, we need to recall the definition of the Morrey classes. Namely, given $p \in (1, \infty)$ and $\gamma \in (0, n)$, the *Morrey space* $L^{p,\gamma}(\Omega)$ is the collection of all functions $f \in L^p(\Omega)$ for which

$$\|f\|_{L^{p,\gamma}(\Omega)} = \sup_{\substack{x_0 \in \Omega \\ r \in (0, \operatorname{diam} \Omega)}} \left(\frac{1}{r^\gamma} \int_{B_r(x_0) \cap \Omega} |f(x)|^p\, dx \right)^{1/p} < \infty.$$

That quantity defines a norm which makes $L^{p,\gamma}(\Omega)$ a Banach space. The limit cases $\gamma = 0$ and $\gamma = n$ give rise to $L^p(\Omega)$ and $L^\infty(\Omega)$, respectively.

Our second results yields Sobolev–Morrey regularity of the weak solution to the variational inequality (3.4) and follows from Theorem 3.1.1 with a particular choice of the weight.

Theorem 3.1.2. *Given* $p, q \in (1, \infty)$ *and* $\gamma \in (0, n)$, *assume that* $|F|^p \in L^{q,\gamma}(\Omega)$ *and* $|D\psi|^p \in L^{q,\gamma}(\Omega)$.

There exists a positive constant $\delta = \delta(n, p, q, \gamma, \Lambda, \mu)$ *such that if* $\mathbf{a}(\xi, x)$ *is* (δ, R)*-vanishing and* Ω *is* (δ, R)*-Reifenberg flat, then the gradient* Du *of* *the weak solution to the variational inequality* (3.4) *belongs to the Morrey space* $L^{pq,\gamma}(\Omega;\mathbb{R}^n)$ *and satisfies the estimate*

$$\|Du\|_{L^{pq,\gamma}(\Omega;\mathbb{R}^n)} \le c\left(\|F\|_{L^{pq,\gamma}(\Omega;\mathbb{R}^n)} + \|D\psi\|_{L^{pq,\gamma}(\Omega;\mathbb{R}^n)}\right)$$

with a constant c *depending only on* $n, p, q, \gamma, \Lambda, \mu$ *and* Ω.

An important consequence of Theorem 3.1.2, based on the known properties of functions having gradients in Morrey spaces (cf. [16]) and the (A)-condition (3.7) of $\partial\Omega$, is the next Corollary which asserts better integrability and even Hölder continuity of the weak solution to the variational inequality (3.4) for appropriate values of p, q and γ.

Corollary 3.1.1. *Under the hypotheses of Theorem 3.1.2, let* $u \in W_0^{1,p}(\Omega)$ *be a weak solution to the variational inequality* (3.4).

Then

1. $u \in L^{\frac{npq}{n-pq},\frac{n\gamma}{n-pq}}(\Omega) \subset L^{pq,\gamma+pq}(\Omega)$ *if* $pq+\gamma<n$;
2. $u \in L^{\tilde{p},\tilde{\gamma}}(\Omega)$ *for any* $\tilde{p}<\infty$ *and any* $\tilde{\gamma}<n$, *if* $pq+\gamma=n$;
3. $u \in C^{0,1-\frac{n-\gamma}{pq}}(\overline{\Omega})$ *if* $pq+\gamma>n$.

Let us point out that, without essential difficulties, the result of Theorem 3.1.2 could be extended to the case of variational inequalities in the settings of the generalized Morrey spaces (cf. [60]).

3.2 Proofs of the Main Results

We start this section with reviewing some standard properties of the maximal function and basic facts from the measure theory with respect to the Muckenhoupt weights.

Our approach in proving Theorem 3.1.1 is based on the Hardy–Littlewood maximal function operator. Recall that the maximal function $\mathcal{M}h$ of a locally integrable function h is given by

$$(\mathcal{M}h)(x) = \sup_{\rho>0} \fint_{B_\rho(x)} |h(y)|\, dy.$$

Indeed, if h is defined only on a bounded domain U we assume tacitly that it is extended as zero outside U and then apply the maximal operator.

It follows from [43, 45, 61] that if a weight w belongs to the Muckenhoupt class A_q for some $q \in (1, \infty)$, then there exists a constant $c = c(n, q, [w]_q) > 0$ such that

$$\frac{1}{c}\|h\|_{L^q_w(\mathbb{R}^n)} \leq \|\mathcal{M}h\|_{L^q_w(\mathbb{R}^n)} \leq c\|h\|_{L^q_w(\mathbb{R}^n)} \quad \text{for each } h \in L^q_w(\mathbb{R}^n). \tag{3.12}$$

In the particular case when $w(x) \equiv 1$, we have

$$\big|\{x \in \mathbb{R}^n \colon (\mathcal{M}h)(x) > \lambda\}\big| \leq \frac{c}{\lambda}\int |h(x)|\, dx \quad \text{for every } \lambda > 0 \tag{3.13}$$

with a constant $c = c(n)$.

We will use the following technical lemma the proof of which can be found in [61, 43].

Lemma 3.2.1. *Assume that h is a nonnegative measurable function on a bounded subset U of $\mathbb{R}^n$. Let $\theta > 0$ and $m > 1$ be constants and $w \in A_q$ with $0 < q < \infty$.*

Then

$$h \in L^q_w(U) \iff S = \sum_{k\geq 1} m^{kq} w\big(\{x \in U \colon h(x) > \theta m^k\}\big) < \infty$$

and

$$\frac{1}{c}S \leq \|h\|^q_{L^q_w(U)} \leq c(w(U) + S),$$

where $c > 0$ is a constant depending only on θ, m and q.

The following Vitali type covering Lemma will be useful in the sequel. We refer the reader to [10, Lemma 3.3] or [43, Lemma 3.8] for the corresponding proof.

Lemma 3.2.2. *Assume that Ω is a bounded domain satisfying the (δ, R)-Reifenberg flatness condition with $0 < \delta < \frac{1}{8}$, and $w \in A_q$ for some $q \in (1, \infty)$. Let C and D be measurable sets with $C \subset D \subset \Omega$. Assume further that there exists a small constant $\epsilon > 0$ such that*

$$w(C) < \epsilon w(B_1(y)) \quad \forall y \in \Omega \tag{3.14}$$

and, for every $y \in \Omega$ *and for every* $r \in (0,1)$ *one has*

$$B_r(y) \cap \Omega \subset D \quad \textit{whenever} \quad w(C \cap B_r(y)) \geq \epsilon w(B_r(y)). \tag{3.15}$$

Then

$$w(C) \leq c^* \epsilon w(D)$$

with a positive constant c^* *depending only on* n, q *and* $[w]_q$.

The next lemma is the main ingredient of the principal result in [6] which treats unweighted variational inequalities.

Lemma 3.2.3. (see Lemma 2.3.9) *Assume that* $|F| \in L^p(\Omega)$ *and* $|D\psi| \in L^p(\Omega)$ *with* $\psi \leq 0$ *a.e. on* $\partial\Omega$. *Suppose that* $u \in W_0^{1,p}(\Omega)$ *is a weak solution of the variational inequality* (3.4).

Then there exists a universal constant $N = N(\mu, \Lambda, n, p) > 1$ *so that for every fixed* $0 < \epsilon < 1$ *one can find a small enough* $\delta = \delta(\epsilon, \mu, \Lambda, n, p) > 0$ *with the property that if* $\mathbf{a}(\xi, x)$ *is* $(\delta, 48)$*-vanishing,* Ω *is* $(\delta, 48)$*-Reifenberg flat, and* $B_r(y)$, *with* $y \in \Omega$ *and* $r \in (0,1)$, *satisfies*

$$\left|\left\{x \in \Omega \cap B_r(y) \colon \mathcal{M}(|Du|^p) > N^p\right\}\right| \geq \epsilon \left|B_r(y)\right|, \tag{3.16}$$

then we have

$$\begin{aligned} B_r(y) \cap \Omega \subset & \left\{x \in \Omega \colon \mathcal{M}(|Du|^p) > 1\right\} \\ & \cup \left\{x \in \Omega \colon \mathcal{M}(|F|^p) > \delta^p\right\} \cup \left\{x \in \Omega \colon \mathcal{M}(|D\psi|^p) > \delta^p\right\}. \end{aligned} \tag{3.17}$$

The weighted counterpart of Lemma 3.14 follows, which relies on Lemma 3.1.1.

Lemma 3.2.4. *Assume that* $w \in A_q$ *for some* $q \in (1, \infty)$, $|F| \in L^p_w(\Omega)$ *and* $|D\psi| \in L^p_w(\Omega)$ *with* $\psi \leq 0$ *a.e. on* $\partial\Omega$. *Suppose that* $u \in W_0^{1,p}(\Omega)$ *is the weak solution of the variational inequality* (3.4).

Then there exists a universal constant $N = N(\mu, \Lambda, n, p) > 1$ *such that for every fixed* $0 < \epsilon < 1$ *one can find small* $\delta = \delta(\epsilon, \mu, \Lambda, n, p, [w]_q) > 0$ *such that if* $\mathbf{a}(\xi, x)$ *is* $(\delta, 42)$*-vanishing,* Ω *is* $(\delta, 48)$*-Reifenberg flat, and* $B_r(y)$ *with* $y \in \Omega$ *and* $r \in (0,1)$ *satisfies*

$$w\big(\left\{x \in \Omega \cap B_r(y) \colon \mathcal{M}(|Du|^p) > N^p\right\}\big) \geq \epsilon w\big(B_r(y)\big), \tag{3.18}$$

then we have

$$B_r(y)\cap\Omega\subset\left\{x\in\Omega\colon \mathcal{M}(|Du|^p)>1\right\} \cup\left\{x\in\Omega\colon \mathcal{M}(|F|^p)>\delta^p\right\}\cup\left\{x\in\Omega\colon \mathcal{M}(|D\psi|^p)>\delta^p\right\}. \tag{3.19}$$

Proof. By Lemmas 3.1.1 and 3.2.4 we have

$$\begin{aligned}&\left|\left\{x\in\Omega\colon \mathcal{M}(|Du|^p)>N^p\right\}\cap B_r(y)\right|\\ &\quad\geq\left(\frac{1}{\alpha}\frac{w\left(\{x\in\Omega\colon \mathcal{M}(|Du|^p)>N^p\}\cap B_r(y)\right)}{w(B_r(y))}\right)^{\frac{1}{\nu}}|B_r(y)|\\ &\quad\geq\left(\frac{\epsilon}{\alpha}\right)^{1/\nu}|B_r(y)|\end{aligned}$$

according to (3.18). We apply now Lemma 3.2.4 with ϵ replaced by $\left(\frac{\epsilon}{\alpha}\right)^{\frac{1}{\nu}}$ and select $\delta=\delta(\epsilon,\mu,\Lambda,n,p,[w]_q)$ in order to get (3.19). □

Proof of Theorem 3.1.1.. We assert first of all that there exists a universal constant c depending on n,p,q,μ,Λ,Ω and $[w]_q$ such that

$$\|Du\|_{L^{pq}_w(\Omega)}\leq c \tag{3.20}$$

if the nonhomogeneous term F and the obstacle ψ satisfy

$$\|F\|_{L^{pq}_w(\Omega;\mathbb{R}^n)}+\|D\psi\|_{L^{pq}_w(\Omega)}\leq\delta. \tag{3.21}$$

In fact, the Hölder inequality implies

$$\|F\|^p_{L^p(\Omega;\mathbb{R}^n)}\leq\int_\Omega|F|^p w^{\frac{1}{q}}w^{-\frac{1}{q}}\,dx\leq\left(\int_\Omega|F|^{pq}w\,dx\right)^{\frac{1}{q}}\underbrace{\left(\int_\Omega w^{\frac{-1}{q-1}}\,dx\right)^{\frac{q-1}{q}}}_{I}. \tag{3.22}$$

Since Ω is bounded, we have $\Omega\subset B_{\frac{d}{2}}(x_0)$ for some $x_0\in\Omega$ where d stands for the diameter of Ω. Then, employing (3.9) and (3.10), we estimate I as follows

$$
\begin{aligned}
\left(\int_\Omega w^{\frac{-1}{q-1}}\,dx\right)^{q-1} &\le \left(\int_{B_{\frac{d}{2}}(x_0)} w^{\frac{-1}{q-1}}\,dx\right)^{q-1}\\
&\le \frac{\left(\fint_{B_{\frac{d}{2}}(x_0)} w\,dx\right)\left(\fint_{B_{\frac{d}{2}}(x_0)} w^{\frac{-1}{q-1}}\,dx\right)^{q-1} |B_{\frac{d}{2}}(x_0)|^{q-1}}{\fint_{B_{\frac{d}{2}}(x_0)} w\,dx}\\
&\le \frac{\left|B_{\frac{d}{2}}(x_0)\right|^q}{w\left(B_{\frac{d}{2}}(x_0)\right)}[w]_q\\
&\le \frac{d^{nq}|B_1|^q}{w(\Omega)}[w]_q.
\end{aligned}
$$

This estimate, combined with (3.21) and (3.22), gives

$$
\|F\|^p_{L^p(\Omega;\mathbb{R}^n)} \le \frac{d^n|B_1|}{w(\Omega)^{\frac{1}{q}}}[w]_q^{\frac{1}{q}}\delta^p.
$$

In the same manner we get

$$
\|D\psi\|^p_{L^p(\Omega)} \le \frac{d^n|B_1|}{w(\Omega)^{\frac{1}{q}}}[w]_q^{\frac{1}{q}}\delta^p,
$$

whence

$$
\|F\|^p_{L^p(\Omega;\mathbb{R}^n)} + \|D\psi\|^p_{L^p(\Omega)} \le c\delta^p. \tag{3.23}
$$

with a suitable constant $c = c(n, q, [w]_q, \Omega)$.

Now take N and ϵ and select the corresponding $\delta > 0$ as given by Lemma 3.2.4. Set further

$$
C = \{x \in \Omega \colon \mathcal{M}(|Du|^p) > N^p\}
$$

and

$$
D = \{x \colon \mathcal{M}(|Du|^p) > 1\} \cup \{x \colon \mathcal{M}(|F|^p) > \delta^p\} \cup \{x \colon \mathcal{M}(|D\psi|^p) > \delta^p\}.
$$

The estimates (3.13) and (3.5) yield that for any $y \in \Omega$ and for some constant $c = c(\mu, \Lambda, n, p, q, \Omega, [w]_q)$ one has

$$\begin{aligned}|C| &\leq c \int_\Omega |Du|^p \, dx \\ &\leq c \left(\|F\|^p_{L^p(\Omega;\mathbb{R}^n)} + \|D\psi\|^p_{L^p(\Omega)} \right) \\ &\leq c\delta^p \\ &\leq \left(\frac{\epsilon}{\alpha}\right)^{\frac{1}{\nu}} |B_1(y)|,\end{aligned}$$

where δ is additionally taken small enough, if necessary, in order to ensure the last inequality. We apply then Lemma 3.1.1 in order to get

$$w\,(C) \leq \alpha \left(\frac{|C|}{|B_1(y)|} \right)^\nu w(B_1(y)) \leq \epsilon w\,(B_1(y))\,. \tag{3.24}$$

At this point the hypotheses of Lemma 3.13 hold because of (3.24) and Lemma 3.15, and as consequence we have

$$\begin{aligned}w(\{x \in \Omega\colon \mathcal{M}(|Du|^p) > N^p\}) \leq \epsilon_1 w(\{x \in \Omega : \mathcal{M}(|Du|^p) > 1\}) \\ + \epsilon_1 \big[w(\{x \in \Omega\colon \mathcal{M}(|F|^p) > \delta^p\}) \\ + w(\{x \in \Omega\colon \mathcal{M}(|D\psi|^p) > \delta^p\}) \big]\end{aligned}$$

with $\epsilon_1 = c^*\epsilon$ and c^* depending only on n, q and $[w]_q$.

Running induction in k, yields the following power decay estimate

$$\begin{aligned}w(\{x \in \Omega\colon \mathcal{M}(|Du|^p) > N^{pk}\}) \leq \epsilon_1^k w(\{x \in \Omega\colon \mathcal{M}(|Du|^p) > 1\}) \\ + \sum_{i=1}^k \epsilon_1^i w(\{x \in \Omega\colon \mathcal{M}(|F|^p) > \delta^p N^{p(k-i)}\}) \\ + \sum_{i=1}^k \epsilon_1^i w(\{x \in \Omega\colon \mathcal{M}(|D\psi|^p) > \delta^p N^{p(k-i)}\}).\end{aligned}$$

Then, using that estimate, we compute as follows

$$\begin{aligned}
\sum_{k=1}^{\infty} & N^{pqk} w(\{x\in\Omega\colon \mathcal{M}(|Du|^p) > N^{pk}\}) \\
&\le \sum_{k=1}^{\infty}(N^{pq}\epsilon_1)^k w(\{x\in\Omega\colon \mathcal{M}(|Du|^p) > 1\}) \\
&\quad + \sum_{i=1}^{\infty}(N^{pq}\epsilon_1)^i \underbrace{\sum_{k=i}^{\infty} N^{pq(k-i)} w(\{x\in\Omega\colon \mathcal{M}(|F|^p) > \delta^p N^{p(k-i)}\})}_{S_1} \\
&\quad + \sum_{i=1}^{\infty}(N^{pq}\epsilon_1)^i \underbrace{\sum_{k=i}^{\infty} N^{pq(k-i)} w(\{x\in\Omega\colon \mathcal{M}(|D\psi|^p) > \delta^p N^{p(k-i)}\})}_{S_2} \\
&\le \sum_{k=1}^{\infty}(N^{pq}\epsilon_1)^k w(\Omega) + \sum_{i=1}^{\infty}(N^{pq}\epsilon_1)^i \left[S_1+S_2\right].
\end{aligned}$$

In view of Lemma 3.12, (3.12) and (3.21), we have

$$S_1 \le c\frac{1}{\delta}\|F\|_{L^{pq}_w(\Omega;\mathbb{R}^n)} \le c$$

and

$$S_2 \le c\frac{1}{\delta}\|D\psi\|_{L^{pq}_w(\Omega)} \le c,$$

for some constant $c = c(n,p,q,\mu,\Lambda,[w]_q)$.

This way, we conclude that there exists a constant $c = c(n,p,q,\mu,\Lambda,\Omega,[w]_q)$ such that

$$\sum_{k=1}^{\infty} N^{pqk} w(\{x\in\Omega\colon \mathcal{M}(|Du|^p) > N^{pk}\}) \le c\sum_{k=1}^{\infty}(N^{pq}\epsilon_1)^k \le c,$$

after choosing ϵ so small that $N^{pq}\epsilon_1 < 1$.

According to Lemma 3.2.4 we find a corresponding $\delta > 0$ which depends on $n,p,q,\mu,\Lambda,[w]_q$ and Ω, and the claim (3.20) follows from (3.12).

To proceed further, we consider the the normalized functions

$$u_\lambda = \frac{u}{\lambda}, \quad F_\lambda = \frac{F}{\lambda}, \quad \psi_\lambda = \frac{\psi}{\lambda},$$

where $\lambda = \frac{1}{\delta}\left(\|F\|_{L^{pq}_w(\Omega;\mathbb{R}^n)} + \|D\psi\|_{L^{pq}_w(\Omega)}\right)$. It follows that

$$\|F_\lambda\|_{L^{pq}_w(\Omega;\mathbb{R}^n)} + \|D\psi_\lambda\|_{L^{pq}_w(\Omega)} \le \delta,$$

which implies that for some constant $c = c(n, p, q, \mu, \Lambda, \Omega, [w]_q)$, one has

$$\|Du_\lambda\|_{L^{pq}_w(\Omega)} \le c.$$

Indeed, the last bound leads to the desired estimate

$$\|Du\|_{L^{pq}_w(\Omega)} \le c\left(\|F\|_{L^{pq}_w(\Omega;\mathbb{R}^n)} + \|D\psi\|_{L^{pq}_w(\Omega)}\right)$$

and this completes the proof of Theorem 3.1.1. □

Proof of Theorem 3.1.2. Without loss of generality we may assume that the nonhomogeneous term F and the gradient of the obstacle $D\psi$ are taken to be zero outside Ω, so that

$$\|F\|_{L^{pq,\gamma}(\mathbb{R}^n;\mathbb{R}^n)} = \|F\|_{L^{pq,\gamma}(\Omega;\mathbb{R}^n)} \quad \text{and} \quad \|D\psi\|_{L^{pq,\gamma}(\mathbb{R}^n;\mathbb{R}^n)} \le c\|D\psi\|_{L^{pq,\gamma}(\Omega;\mathbb{R}^n)}.$$

Let $x_0 \in \Omega$ and $r > 0$ be arbitrary. We set $\chi_{B_r(x_0)}$ for the characteristic function of the ball $B_r(x_0)$ and $\mathcal{M}\chi_{B_r(x_0)}(x)$ for its Hardy–Littlewood maximal function.

For an arbitrary exponent $\sigma \in (0,1)$, it is a classical fact (see e.g. Proposition 2 in [20]) that

$$\mathcal{M}\Big(\left(\mathcal{M}\chi_{B_r(x_0)}(x)\right)^\sigma\Big) \le c\left(\mathcal{M}\chi_{B_r(x_0)}(x)\right)^\sigma \quad \text{for a.a. } x \in \mathbb{R}^n.$$

In other words, $\left(\mathcal{M}\chi_{B_r(x_0)}(x)\right)^\sigma$ belongs to the Muckenhoupt class A_1 and therefore $\left(\mathcal{M}\chi_{B_r(x_0)}(x)\right)^\sigma \in A_q$ for each $q \in (1, \infty)$ with

$$\left[\left(\mathcal{M}\chi_{B_r(x_0)}(x)\right)^\sigma\right]_q = c(n, q, \sigma).$$

Fix now an arbitrary $\sigma \in \big(\gamma/n, 1\big)$ and apply Theorem 3.1.1. It follows that there exist constants $\delta > 0$ and c, depending on $n, p, q, \Lambda, \mu, \sigma$ and Ω, such that if $\mathbf{a}(\xi, x)$ is (δ, R)-vanishing and Ω is (δ, R)-Reifenberg flat, the estimate (3.11) yields

$$\begin{aligned} \int_{B_r(x_0)\cap\Omega} |Du(x)|^{pq}\,dx &= \int_\Omega |Du(x)|^{pq}\left(\chi_{B_r(x_0)}(x)\right)^\sigma\,dx \\ &\le \int_\Omega |Du(x)|^{pq}\left(\mathcal{M}\chi_{B_r(x_0)}(x)\right)^\sigma\,dx \\ &\le c\int_\Omega |K(x)|^{pq}\left(\mathcal{M}\chi_{B_r(x_0)}(x)\right)^\sigma\,dx \\ &= c\int_{\mathbb{R}^n} |K(x)|^{pq}\left(\mathcal{M}\chi_{B_r(x_0)}(x)\right)^\sigma\,dx \end{aligned} \tag{3.25}$$

where we have set

$$K(x) = |F(x)| + |D\psi(x)|$$

for the sake of simplicity.

With the aid of the dyadic decomposition

$$\mathbb{R}^n = B_{2r}(x_0)\bigcup\Big(\bigcup_{k=1}^{\infty} B_{2^{k+1}r}(x_0)\setminus B_{2^k r}(x_0)\Big)$$

the last integral above decomposes into

$$\int_{\mathbb{R}^n} |K(x)|^{pq}\left(\mathcal{M}\chi_{B_r(x_0)}(x)\right)^\sigma\,dx = I_0(r,x_0) + \sum_{k=1}^{\infty} I_k(r,x_0), \tag{3.26}$$

with

$$\begin{aligned} I_0(r,x_0) &= \int_{B_{2r}(x_0)} |K(x)|^{pq}\left(\mathcal{M}\chi_{B_r(x_0)}(x)\right)^\sigma\,dx, \\ I_k(r,x_0) &= \int_{B_{2^{k+1}r}(x_0)\setminus B_{2^k r}(x_0)} |K(x)|^{pq}\left(\mathcal{M}\chi_{B_r(x_0)}(x)\right)^\sigma\,dx. \end{aligned}$$

We use the inequality $\mathcal{M}\chi_{B_r(x_0)}(x) \le 1$ a.e. $\mathbb{R}^n$ in order to estimate $I_0(r,x_0)$. Namely,

$$\begin{aligned} I_0(r,x_0) &\le \int_{B_{2r}(x_0)} |K(x)|^{pq}\,dx \le c(n)r^\gamma \|K\|^{pq}_{L^{pq,\gamma}(\Omega)} \\ &\le c(n)r^\gamma\left(\|F\|^{pq}_{L^{pq,\gamma}(\Omega;\mathbb{R}^n)} + \|D\psi\|^{pq}_{L^{pq,\gamma}(\Omega;\mathbb{R}^n)}\right). \end{aligned} \tag{3.27}$$

Later on, it is clear that

$$\frac{1}{|B_\rho(x)|}\int_{B_\rho(x)} \left|\chi_{B_r(x_0)}(y)\right|\ dy \leq \frac{|B_r(x_0)|}{|B_\rho(x)|} = \frac{r^n}{\rho^n} \tag{3.28}$$

for each $x \in B_{2^{k+1}r}(x_0) \setminus B_{2^k r}(x_0)$ and each $\rho > 0$. This way, the term on the left-hand side above is positive only for values $\rho > 2^k r - r$, and the simple inequality $2^k - 1 \geq 2^{k-1}$ which holds for all $k \geq 1$, reduces (3.28) to

$$\frac{1}{|B_\rho(x)|}\int_{B_\rho(x)} \left|\chi_{B_r(x_0)}(y)\right|\ dy \leq \frac{r^n}{2^{n(k-1)} r^n} = \frac{1}{2^{n(k-1)}}.$$

Take now the supremum in $\rho > 0$ in order to get

$$\left(\mathcal{M}\chi_{B_r(x_0)}(x)\right)^\sigma \leq \frac{1}{2^{\sigma n(k-1)}}.$$

With this bound at hand, we estimate $I_k(r, x_0)$ from (3.26) as follows

$$\begin{aligned} I_k(r,x_0) &\leq \frac{1}{2^{\sigma n(k-1)}} \int_{B_{2^{k+1}r}(x_0)\setminus B_{2^k r}(x_0)} |K(x)|^{pq}\ dx \\ &\leq \frac{1}{2^{\sigma n(k-1)}} \int_{B_{2^{k+1}r}(x_0)} |K(x)|^{pq}\ dx \\ &\leq \frac{(2^{k+1}r)^\gamma}{2^{\sigma n(k-1)}} \frac{1}{(2^{k+1}r)^\gamma} \int_{B_{2^{k+1}r}(x_0)} |K(x)|^{pq}\ dx \\ &\leq 2^{\gamma+\sigma n}(2^{\gamma-\sigma n})^k r^\gamma \, \|K\|^{pq}_{L^{pq,\gamma}(\Omega)} \\ &\leq 2^{\gamma+\sigma n}(2^{\gamma-\sigma n})^k r^\gamma \left(\|F\|^{pq}_{L^{pq,\gamma}(\Omega;\mathbb{R}^n)} + \|D\psi\|^{pq}_{L^{pq,\gamma}(\Omega;\mathbb{R}^n)}\right). \end{aligned} \tag{3.29}$$

A substitution of (3.27) and (3.29) into (3.25) yields

$$\begin{aligned} \int_{B_r(x_0)\cap\Omega} |Du(x)|^{pq}\ dx &\leq c r^\gamma \left(\sum_{k=0}^{\infty} (2^{\gamma-\sigma n})^k\right)\left(\|F\|^{pq}_{L^{pq,\gamma}(\Omega;\mathbb{R}^n)} + \|D\psi\|^{pq}_{L^{pq,\gamma}(\Omega;\mathbb{R}^n)}\right) \\ &= c r^\gamma \left(\|F\|^{pq}_{L^{pq,\gamma}(\Omega;\mathbb{R}^n)} + \|D\psi\|^{pq}_{L^{pq,\gamma}(\Omega;\mathbb{R}^n)}\right) \end{aligned}$$

thanks to our choice $\sigma \in \left(\gamma/n, 1\right)$ which ensures the convergence of the series above.

To complete the proof of Theorem 3.1.2, it remains to divide the both sides above by r^{γ} and to take the supremum with respect to $x_0 \in \Omega$ and $r > 0$ in order to get $|Du| \in L^{pq,\gamma}(\Omega)$ with the desired estimate

$$\|Du\|_{L^{pq,\gamma}(\Omega;\mathbb{R}^n)} \leq c \left(\|F\|_{L^{pq,\gamma}(\Omega;\mathbb{R}^n)} + \|D\psi\|_{L^{pq,\gamma}(\Omega;\mathbb{R}^n)} \right).$$

□

The *Proof of Corollary 3.1.1* is an immediate of the known pointwise properties of functions with gradients in Morrey spaces (cf. [16]) and the (A)-condition (3.7). We left the details to the reader. □

Chapter 4

Nonlinear gradient estimates for parabolic problems with irregular obstacles

4.1 Results

Let p be a fixed number with $\frac{2n}{n+2} < p < \infty$ and assume that a given Carathéodory function,

$$a = a(\xi, x, t) = a(\xi, z) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n,$$

satisfies the following basic structural conditions:

$$\begin{cases} |a(\xi, x, t)| + |\xi||D_\xi a(\xi, x, t)| \leq \Lambda |\xi|^{p-1}, \\ D_\xi a(\xi, x, t)\eta \cdot \eta \geq \mu |\xi|^{p-2}|\eta|^2, \end{cases} \tag{4.1}$$

for every $\xi, \eta \in \mathbb{R}^n$, for almost every $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, and for some constants $0 < \mu \leq 1 \leq \Lambda$. We clearly point out that the conditions (4.1) imply the following standard monotonicity conditions:

$$\begin{cases} (a(\xi, x, t) - a(\eta, x, t)) \cdot (\xi - \eta) \geq \gamma |\xi - \eta|^p, \text{ if } p \geq 2, \\ (a(\xi, x, t) - a(\eta, x, t)) \cdot (\xi - \eta) \geq \gamma |\xi - \eta|^2 (|\xi| + |\eta|)^{p-2}, \text{ if } 1 < p < 2, \end{cases} \tag{4.2}$$

where γ is a positive constant depending only on n, μ and p.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and write $\Omega_T = \Omega \times (0,T)$ for some constant $T > 0$. We then consider a function ψ as the time dependent obstacle with

$$\psi \in L^p(0,T;W^{1,p}(\Omega)),\ \psi_t \in L^{\frac{p}{p-1}}(\Omega_T) \text{ and } \psi \leq 0 \text{ a.e. on } \partial\Omega \times (0,T), \quad (4.3)$$

and a measurable function F as the inhomogeneity with

$$F \in L^p(\Omega_T, \mathbb{R}^n). \quad (4.4)$$

For the sake of simplicity, we take zero initial value, as we assume

$$u(\cdot,0) = 0 \text{ and } 0 \geq \psi(\cdot,0). \quad (4.5)$$

A Solution under consideration is a function $u = u(x,t)$ lying in the convex admissible set

$$\mathcal{A} = \{v \in C^0([0,T];L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega)) : v(\cdot,0) = 0, v \geq \psi\} \quad (4.6)$$

and satisfying the weak parabolic variational inequality

$$\int_0^T < v_t, v-u > \ dt + \int_{\Omega_T} a(Du,x,t) \cdot D(v-u)\ dxdt \geq \int_{\Omega_T} |F|^{p-2}F \cdot D(v-u)\ dxdt, \quad (4.7)$$

for all testing functions $v \in \mathcal{A}'$, where

$$\mathcal{A}' = \{v \in \mathcal{A} : v_t \in L^{\frac{p}{p-1}}(0,T;W^{-1,\frac{p}{p-1}}(\Omega))\}, \quad (4.8)$$

and $< \cdot,\cdot >$ denotes the duality paring between $W^{-1,\frac{p}{p-1}}$ and $W_0^{1,p}$.

It is well known that with the basic conditions (4.1) on $a = a(\xi,x,t)$, there is a unique solution u to the parabolic variational inequality (4.7) with standard L^p-estimate, provided (4.3), (4.4) and (4.5) hold true, see [1, 3, 32, 58] and references therein. More precisely, we have the following:

Lemma 4.1.1. *There is a unique solution $u \in \mathcal{A}$ to the variational inequality (4.7) and we have the following estimate:*

$$\sup_{t\in(a,a+T)} \int_{\Omega} |u(\cdot,t)|^2 \, dx + \int_{\Omega_T} |Du|^p \, dxdt \\ \leq c \left[\int_{\Omega_T} \left(|F|^p + |D\psi|^p + |\psi_t|^{\frac{p}{p-1}} \right) \, dxdt \right],$$

for some positive constant $c = c(n, p, \mu, \Lambda)$.

The purpose of this chapter is to establish the local natural Calderón-Zygmund theory for solutions to (4.7) under minimal regularity requirements on the nonlinearity. To be more precise, we want to find a reasonable answer as to what is the weakest condition on the nonlinearity in addition to (4.1) under which for every $q \in (1, \infty)$,

$$|F|^p, |D\psi|^p, |\psi_t|^{\frac{p}{p-1}} \in L^q_{loc}(\Omega_T) \Rightarrow |Du|^p \in L^q_{loc}(\Omega_T). \tag{4.9}$$

According to the very interesting work in [3], (4.9) holds true when $q \in (1, 1+\sigma_0)$ for some small $\sigma_0(n, p, \mu, \Lambda)$ without any extra condition to (4.1)-(4.2). However, in order that (4.9) holds true for every $q \in (1, \infty)$, the basic structural conditions (4.1) are not enough, even for the stationary case, see [6, 47].

To state the main assumption on the nonlinearity, we introduce some notation. Given a point $(y, s) \in \mathbb{R}^n \times \mathbb{R}$ and positive numbers ρ, θ, the parabolic cylinder under consideration is

$$Q_{(\rho,\theta)}(y, s) = B_\rho(y) \times (s - \theta, s + \theta)$$

where $B_\rho(y) = \{x \in \mathbb{R}^n : |x - y| < \rho\}$. If the center is $(0, 0)$, or, it is clear in the context, we do not specify the center. In the special case $\theta = \rho^2$, we simply write

$$Q_\rho(y, s) = Q_{(\rho,\rho^2)}(y, s).$$

In order to measure the deviation of $a(\xi, x, t)$ in a fixed parabolic cylinder $Q_{(\rho,\theta)}(y, s)$ from being the function $\overline{a}_{B_\rho(y)}(\xi, t)$, which is the integral average of $a(\xi, \cdot, t)$ over the ball $B_\rho(y)$, we define a function on $Q_{(\rho,\theta)}(y, s)$ by

$$\beta \left[a, Q_{(\rho,\theta)}(y, s)\right](x, t) = \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \frac{|a(\xi, x, t) - \overline{a}_{B_\rho(y)}(\xi, t)|}{|\xi|^{p-1}}, \tag{4.10}$$

where

$$\overline{a}_{B_\rho(y)}(\xi,t) = \fint_{B_\rho(y)} a(\xi,x,t)\ dx = \frac{1}{|B_\rho(y)|}\int_{B_\rho(y)} a(\xi,x,t)\ dx. \tag{4.11}$$

The main assumption is that $\frac{a(\xi,x,t)}{|\xi|^{p-1}}$ has small BMO semi-norms in the x-variable, but there is no regularity assumption in the t-variable, uniformly in $\xi \in \mathbb{R}^n \setminus \{0\}$. More precisely, we have the following definition.

Definition 4.1.1. $a = a(\xi,x,t)$ *is weakly* (δ,R)*-vanishing if we have*

$$\sup_{0<\rho\leq R}\ \sup_{0<\theta\leq R^2}\ \sup_{(y,s)\in\mathbb{R}^{n+1}} \fint_{Q_{\rho,\theta}(y,s)} \beta\left[a, Q_{(\rho,\theta)}(y,s)\right](x,t)\ dxdt \leq \delta.$$

We now state the main result of the paper.

Theorem 4.1.1. *Let* $\frac{2n}{n+2} < p < \infty$ *and let* $1 < q < \infty$. *Suppose that*

$$|\psi_t|^{\frac{p}{p-1}}, |D\psi|^p, |F|^p \in L^q_{loc}(\Omega_T).$$

Then there exists a constant $\delta = \delta(\mu,\Lambda,n,p,q) > 0$ *such that if* $a = a(\xi,x,t)$ *is weakly* (δ,R_0)*-vanishing for some* R_0, *then* $|Du|^p \in L^q_{loc}(\Omega_T)$ *with the estimate*

$$\begin{aligned}\left(\fint_{Q_R(y_0,s_0)} |Du|^{pq} dxdt\right)^{\frac{1}{d}} &\leq c\left[1+\left(\fint_{Q_{2R}(y_0,s_0)} |Du|^p dxdt\right)^q\right] \\ &+ c\fint_{Q_{2R}(y_0,s_0)} \left(|\psi_t|^{\frac{pq}{p-1}} + |D\psi|^{pq} + |F|^{pq}\right) dxdt, \end{aligned}\tag{4.12}$$

for any $Q_{2R}(y_0,s_0) \Subset \Omega_T$, $R \leq R_0$, *and for some constant* $c = c(n,p,q,\mu,\Lambda,R)$, *where*

$$d = \begin{cases} \frac{p}{2} & \text{if } p \geq 2, \\ \frac{2p}{p(n+2)-2n} & \text{if } p < 2. \end{cases} \tag{4.13}$$

4.2 Nonlinear comparison estimates based on local approximation

In this section we find a local estimate of solutions to the parabolic variational inequality by comparison with solutions to the reference problem of the type

$$\overline{a}_{B_R}(Dv,t) = 0 \text{ in } Q_R,$$

naturally enjoying interior Lipschitz regularity with respect to x-variable, uniformly t-variable. Throughout this section we employ c to denote any universal constants that can be explicitly computed in terms of known quantities such as μ, Λ, n, p.

To start with, we consider a solution u to the variational inequality (4.7) and assume that

$$Du \in L^{\infty}_{loc}(\Omega_T) \text{ and } u_t \in L^{\frac{p}{p-1}}(a, a+T; W^{-1,\frac{p}{p-1}}(\Omega)). \tag{4.14}$$

This assumption (4.15) can be ensured in Section 4.4 by an approximation scheme based on standard mollification in space.

We then localize our interest into Q_{16} by assuming

$$Q_{16} \Subset \Omega_T, \tag{4.15}$$

$$\sup_{0<\rho\leq 8} \sup_{0<\theta\leq 8^2} \fint_{Q_{(\rho,\theta)}} \beta\left[a, Q_{(\rho,\theta)}\right] dxdt \leq \delta, \tag{4.16}$$

and

$$\fint_{Q_8} |Du|^p dxdt + \frac{1}{\delta^p} \fint_{Q_8} \Psi^p dxdt \leq 1, \tag{4.17}$$

where

$$\Psi = |\psi_t|^{\frac{1}{p-1}} + |D\psi| + |F| \tag{4.18}$$

and δ is to be determined later in a universal way, being dependent on n, μ, Λ, p, q.

Then we let

$$w \in L^p(-8^2, 8^2; W^{1,p}(B_8)) \cap W^{1,\frac{p}{p-1}}(-8^2, 8^2; W^{-1,\frac{p}{p-1}}(B_8))$$

be the weak solution to

$$\begin{cases} w_t - div\ a(Dw, x, t) &= \psi_t - div\ a(D\psi, x, t) \quad \text{in} \quad Q_8, \\ w &= u(\geq \psi) \quad \text{on} \quad \partial_p Q_8, \end{cases} \tag{4.19}$$

where

$$\partial_p Q_R = \partial B_R \times (-R^2, R^2) \ \cup\ B_R \times \{t = -R^2\}$$

is the parabolic boundary of Q_R. From standard L^p-estimate and (4.1), we find that

$$\begin{aligned}\fint_{Q_8} |Dw|^p dxdt &\leq c\left[\fint_{Q_8}\left(|Du|^p + |\psi_t|^{\frac{p}{p-1}} + |a(D\psi,x,t)|^{\frac{p}{p-1}} + |F|^p\right) dxdt\right] \\ &\leq c\left[\fint_{Q_8}\left(|Du|^p + |\psi_t|^{\frac{p}{p-1}} + |D\psi|^p + |F|^p\right) dxdt\right].\end{aligned}$$

We then employ (4.17) and (4.18) to find

$$\fint_{Q_8} |Dw|^p dxdt \leq c\,(1+\delta^p+\delta^p) \leq c. \tag{4.20}$$

Lemma 4.2.1. *There exists a constant*

$$\sigma_1 = \sigma_1(\mu,\Lambda,n,p)$$

such that

$$\fint_{Q_8} |D(u-w)|^p dxdt \leq c\delta^{\sigma_1}.$$

Proof. We recall the obstacle constraint $u \geq \psi$ a.e. on Ω_T and use the initial-boundary condition of (4.19), to see that $w \geq \psi$ a.e. on $\partial_p Q_8$. We then apply a comparison principle to the problem (4.19), see Lemma 2.8 in [1], to discover that

$$w \geq \psi \text{ a.e. on } Q_8.$$

We next extend w by u from $Q_8 = B_8 \times (-8^2, 8^2)$ to $\Omega \times (a, 8^2)$. Then by a proper localization in time, we assume that w is an admissible function for the variational inequality (4.7) up to the time 8^2. In this respect, we discover that

$$\int_{-8^2}^{8^2} < w_t, w-u > dt + \int_{Q_8} a(Du,x,t)\cdot D(w-u)dxdt \geq \int_{Q_8} |F|^{p-2}F\cdot D(w-u)dxdt. \tag{4.21}$$

We test the problem (4.19) by $w-u$ to find that

$$\int_{-8^2}^{8^2} < w_t, w-u > dt + \int_{Q_8} a(Dw,x,t)\cdot D(w-u)dxdt = \int_{Q_8} \psi_t(w-u)dxdt + \int_{Q_8} a(D\psi,x,t)\cdot D(w-u)dxdt. \tag{4.22}$$

Subtracting the above inequality (4.21) from the weak formulation identity (4.22), we have

$$\begin{aligned}&\fint_{Q_8}(a(Dw,x,t)-a(Du,x,t))\cdot D(w-u)dxdt\\&\leq \fint_{Q_8}\psi_t(w-u)dxdt+\fint_{Q_8}\big(a(D\psi,x,t)-|F|^{p-2}F\big)\cdot D(w-u)dxdt. \quad (4.23)\end{aligned}$$

In the case that $p \geq 2$, from (4.2) we have

$$\fint_{Q_8}|D(u-w)|^p dxdt \leq \frac{1}{\gamma}\fint_{Q_8}(a(Dw,x,t)-a(Du,x,t))\cdot D(w-u)dxdt$$

In the case that $p < 2$, it follows from Young's inequality with $\tau_1 > 0$ and (4.2) that

$$\begin{aligned}&\fint_{Q_8}|D(u-w)|^p dxdt\\&=\fint_{Q_8}(|Du|+|Dw|)^{\frac{p(2-p)}{2}}[(|Du|+|Dw|)^{\frac{p(p-2)}{2}}|D(u-w)|^p]dxdt\\&\leq \tau\fint_{Q_8}(|Du|+|Dw|)^p dxdt + c(\tau_1)\fint_{Q_8}(|Du|+|Dw|)^{p-2}|D(u-w)|^2 dxdt\\&\leq c\tau_1 + c(\tau_1)\frac{1}{\gamma}\fint_{Q_8}(a(Dw,x,t)-a(Du,x,t))\cdot D(w-u)dxdt,\end{aligned}$$

where we have used (4.17) and (4.20) in the last line. Therefore, in either case, we have

$$\begin{aligned}&\fint_{Q_8}|D(u-w)|^p dxdt\\&\qquad\leq c\tau_1 + c(\tau_1)\frac{1}{\gamma}\fint_{Q_8}(a(Dw,x,t)-a(Du,x,t))\cdot D(w-u)dxdt. \quad (4.24)\end{aligned}$$

Using Young's inequality with $\tau_2 > 0$ and Poincaré's inequality, we have

$$\begin{aligned}&\fint_{Q_8}|\psi_t(w-u)|dxdt\\&\qquad\leq c(\tau_2)\fint_{Q_8}|\psi_t|^{\frac{p}{p-1}}dxdt + \tau_2\fint_{Q_8}|w-u|^p dxdt\\&\qquad\leq c(\tau_2)\fint_{Q_8}|\psi_t|^{\frac{p}{p-1}}dxdt + c\tau_2\fint_{Q_8}|Dw-Du|^p dxdt. \quad (4.25)\end{aligned}$$

We use (4.1) and Young's inequality with $\tau_3 > 0$, to derive that

$$\begin{aligned}
&\fint_{Q_8} |a(D\psi, x, t) - |F|^{p-2}F||D(w-u)|dxdt \\
&\qquad \leq c \fint_{Q_8} \left(|D\psi|^{p-1} + |F|^{p-1}\right)|Dw - Du|dxdt \\
&\qquad \leq c(\tau_3) \fint_{Q_8} \left(|D\psi|^{p} + |F|^{p}\right) dxdt + \tau_3 \fint_{Q_8} |Dw - Du|^p dxdt. \quad (4.26)
\end{aligned}$$

We then combine (4.23), (4.24), (4.25) and (4.26), to discover that

$$\begin{aligned}
&\fint_{Q_8} |D(u-w)|^p dxdt \\
&\leq c\tau_1 + c(\tau_1)\frac{1}{\gamma} \fint_{Q_8} (a(Dw, x, t) - a(Du, x, t)) \cdot D(w-u) dxdt \\
&\leq c\tau_1 + c(\tau_1)c(\tau_2) \fint_{Q_8} |\psi_t|^{\frac{p}{p-1}} dxdt + c(\tau_1)\tau_2 \fint_{Q_8} |D(w-u)|^p dxdt \\
&+ c(\tau_1)c(\tau_3) \fint_{Q_8} \left(|D\psi|^p + |F|^p\right) dxdt + c(\tau_1)\tau_3 \fint_{Q_8} |Dw - Du|^p dxdt.
\end{aligned}$$

Finally, we take τ_1, τ_2 and τ_3, sufficiently small, in order to derive the conclusion of the lemma. This completes the proof. □

We next let

$$h \in L^p(-8^2, 8^2; W^{1,p}(B_8)) \cap W^{1,\frac{p}{p-1}}(-8^2, 8^2; W^{-1,\frac{p}{p-1}}(B_8))$$

be the weak solution to

$$\begin{cases} h_t - div\ a(Dh, x, t) = 0 & \text{in} \quad Q_8, \\ h = w & \text{on} \quad \partial_p Q_8. \end{cases} \quad (4.27)$$

Then, again by standard L^p-estimate and (4.20), we have

$$\fint_{Q_8} |Dh|^p dxdt \leq c. \quad (4.28)$$

Lemma 4.2.2. *There is a positive constant $\sigma_2 = \sigma_2(n, p, \mu, \Lambda)$ such that*

$$|Dh| \in L^{p+\sigma_2}(Q_4)$$

with the uniform bound

$$\fint_{Q_4} |Dh|^{p+\sigma_2} dxdt \leq c.$$

Proof. The proof follows from a higher integrability result for (4.27) and (4.28). □

Finally, we let

$$v \in L^p(-4^2, 4^2; W^{1,p}(B_4)) \cap W^{1,\frac{p}{p-1}}(-4^2, 4^2; W^{-1,\frac{p}{p-1}}(B_4))$$

be the weak solution to

$$\begin{cases} v_t - div\ \overline{a}_{B_4}(Dv, t) & = & 0 & \text{in} & Q_4, \\ v & = & h & \text{on} & \partial_p Q_4. \end{cases} \tag{4.29}$$

Then it follows again from standard L^p-estimate and (4.28) that

$$\fint_{Q_4} |Dv|^p dxdt \leq c. \tag{4.30}$$

Lemma 4.2.3. *There is a positive constant*

$$n_1 = n_1(n, p, \mu, \Lambda) \geq 1$$

such that

$$\sup_{Q_2} |Dv| \leq n_1. \tag{4.31}$$

Proof. According to Lipschitz regularity for (4.29),

$$\sup_{Q_2} |Dv|^p \leq c \left(\fint_{Q_4} |Dv|^p dxdt + 1 \right).$$

Then the conclusion follows from (4.30). □

We now prove the following comparison estimate.

Lemma 4.2.4. *Let u be a solution to the variational inequality (4.7) with the regularity (4.14). Then for any $\epsilon > 0$, there is a small $\delta = \delta(\epsilon, \mu, \Lambda, n, p) > 0$ such that if the assumptions (4.15), (4.16), (4.17) and (4.18) hold, then there exists a weak solution*

$$v \in L^p(-4^2, 4^2; W^{1,p}(B_4)) \cap W^{1,\frac{p}{p-1}}(-4^2, 4^2; W^{-1,\frac{p}{p-1}}(B_4))$$

to (4.29) satisfying (4.31) such that

$$\fint_{Q_4} |D(u-v)|^p dxdt \leq \epsilon^p. \tag{4.32}$$

Proof. Taking a test function $w - h$ in (4.19) and (4.27), we obtain

$$\int_{-8^2}^{8^2} < (w-h)_t, w-h > dt + \int_{Q_8} (a(Dw,x,t) - a(Dh,x,t)) \cdot D(w-h) dxdt$$
$$= \int_{Q_8} \psi_t (w-h) dxdt + \int_{Q_8} a(D\psi, x, t) \cdot D(w-h) dxdt.$$

We know that

$$\int_{-8^2}^{8^2} < (w-h)_t, w-h > dt = \frac{1}{2} \int_{B_8} |w(\cdot, 8^2) - h(\cdot, 8^2)|^2 dx \geq 0, \tag{4.33}$$

which implies that

$$\begin{aligned}
&\fint_{Q_8} (a(Dw,x,t) - a(Dh,x,t)) \cdot D(w-h) dxdt \\
&\leq \fint_{Q_8} \psi_t (w-h) dxdt + \fint_{Q_8} a(D\psi, x, t) \cdot D(w-h) dxdt \\
&\leq c\tau_4 \fint_{Q_8} |D(w-h)|^p dxdt + c(\tau_4) \fint_{Q_8} |\psi_t|^{\frac{p}{p-1}} + |D\psi|^p dxdt \\
&\leq c\tau_4 \fint_{Q_8} |D(w-h)|^p dxdt + c(\tau_4) \delta^p
\end{aligned}$$

for any $\tau_4 > 0$, where we have used (4.1), Young's inequality with $\tau_4 > 0$ and Poincaré's inequality. Then similarly in the proof of Lemma 4.2.1, we find that

$$\fint_{Q_8} |D(w-h)|^p dxdt \leq c\delta^{\sigma_3}, \tag{4.34}$$

for some positive constant $\sigma_3 = \sigma_3(\mu, \Lambda, n, p)$.

We now take $h - v$ as a test function in both (4.27) and (4.29), to find

$$\int_{-4^2}^{4^2} < (h-v)_t, h-v > dt$$
$$= \int_{Q_4} (\overline{a}_{B_4}(Dv,t) - a(Dh,x,t)) \cdot (Dh - Dv) dxdt. \quad (4.35)$$

But then, since the left hand side of (4.35) is nonnegative from the same reason for (4.33), we have

$$\fint_{Q_4} (a(Dh,x,t) - \overline{a}_{B_4}(Dv,t)) \cdot (Dh - Dv) dxdt \leq 0,$$

which can be written as follows. which is as follows

$$\fint_{Q_4} (\overline{a}_{B_4}(Dh,t) - \overline{a}_{B_4}(Dv,t)) \cdot (Dh - Dv) dxdt \quad (4.36)$$
$$\leq \fint_{Q_4} (\overline{a}_{B_4}(Dh,t) - a(Dh,x,t)) \cdot (Dh - Dv) dxdt.$$

Again, we see that for each $\tau_5 > 0$,

$$\gamma \fint_{Q_4} |D(h-v)|^p dxdt$$
$$\leq c\tau_5 + c(\tau_5) \fint_{Q_4} (\overline{a}_{B_4}(Dh,t) - \overline{a}_{B_4}(Dv,t)) \cdot D(h-v) dxdt. \quad (4.37)$$

We now recall the definition of β (4.10), use Young's inequality with $\tau_6 > 0$ and Hölder's inequality, applying Lemma 4.2.2, and finally using the assumption (4.16), to estimate the right-hand side of (4.37) as follows:

$$\begin{aligned}
&\fint_{Q_4} (\overline{a}_{B_4}(Dh,t) - a(Dh,x,t)) \cdot (Dh - Dv) dxdt \\
&\qquad \leq \fint_{Q_4} |\overline{a}_{B_4}(Dh,t) - a(Dh,x,t)||Dh - Dv| dxdt \\
&\qquad \leq \fint_{Q_4} \beta(a, Q_4)|Dh|^{p-1}|Dh - Dv| dxdt \\
&\qquad \leq \tau_6 \fint_{Q_4} |Dh - Dv|^p dxdt + c(\tau_6) \fint_{Q_4} \beta^{\frac{p}{p-1}} |Dh|^p dxdt \\
&\qquad \leq \tau_6 \fint_{Q_4} |Dh - Dv|^p dxdt \\
&\qquad\quad + c(\tau_6) \left(\fint_{Q_4} \beta^{\frac{p(p+\sigma_2)}{(p-1)\sigma_2}} dxdt \right)^{\frac{\sigma_2}{p+\sigma_2}} \left(\fint_{Q_4} |Dh|^{p+\sigma_2} dxdt \right)^{\frac{p}{p+\sigma_2}} \\
&\qquad \leq \tau_6 \fint_{Q_4} |Dh - Dv|^p dxdt + c(\tau_6) \delta^{\frac{p}{p-1}}.
\end{aligned}$$

Consequently, (4.36) and (4.37) imply

$$\begin{aligned}
&\fint_{Q_4} |D(h - v)|^p \, dxdt \\
&\qquad \leq c\tau_5 + c(\tau_5)\tau_6 \fint_{Q_4} |Dh - Dv|^p \, dxdt + c(\tau_5)c(\tau_6)\delta^{\frac{p}{p-1}}. \qquad (4.38)
\end{aligned}$$

Take τ_5 and τ_6 so small, in order to derive that for some constant $\sigma_4 = \sigma_4(\mu, \Lambda, n, p) > 0$,

$$\fint_{Q_4} |Dh - Dv|^p dxdt \leq c\delta^{\sigma_4}. \qquad (4.39)$$

We now combine Lemma 4.2.1, (4.34) and (4.39), to find

$$\fint_{Q_4} |Du - Dv|^p dxdt \leq c \left(\delta^{\sigma_1} + \delta^{\sigma_3} + \delta^{\sigma_4} \right) \leq \epsilon^p, \qquad (4.40)$$

by taking $\delta = \delta(\epsilon, \mu, \Lambda, n, p) > 0$ so that the last inequality in (4.40) holds true. This completes the proof. □

4.3 The a priori estimates

In this section we essentially establish the required estimate (4.12) as an a priori estimate under the a priori assumption (4.14). To do this, we fix any parabolic cylinder

$$Q_{2R}(y_0, s_0) \Subset \Omega_T. \tag{4.41}$$

With the exponent d in (4.13) and the function Ψ in (4.18), we write

$$\lambda_0^{\frac{p}{d}} = \fint_{Q_{2R}(y_0,s_0)} |Du|^p dxdt + \frac{1}{\delta^p} \fint_{Q_{2R}(y_0,s_0)} \Psi^p dxdt + 1, \tag{4.42}$$

where $\delta > 0$ is to be determined later in a universal way.

4.3.1 Covering Argument

We next fix two numbers ρ_1, ρ_2 with

$$R \le \rho_1 < \rho_2 \le 2R, \tag{4.43}$$

and so we have

$$Q_R(y_0, s_0) \subset Q_{\rho_1}(y_0, s_0) \subset Q_{\rho_2}(y_0, s_0) \subset Q_{2R}(y_0, s_0) \tag{4.44}$$

and

$$Q_{\rho_2-\rho_1}(y, s) \subset Q_{\rho_2}(y_0, s_0),\ \forall (y, s) \in Q_{\rho_1}(y_0, s_0). \tag{4.45}$$

For the sake of simplicity, we write

$$d_1 = \begin{cases} p-2 & \text{if} \quad p \ge 2, \\ \frac{n(2-p)}{2} & \text{if} \quad p < 2 \end{cases} \tag{4.46}$$

and use the following notation for the intrinsic parabolic cylinders:

$$Q_r^\lambda(y, s) = \begin{cases} Q_{(r,\lambda^{2-p}r^2)}(y, s) & \text{if} \quad p \ge 2, \\ Q_{\left(\lambda^{\frac{p-2}{2}}r, r^2\right)}(y, s) & \text{if} \quad p < 2, \end{cases} \tag{4.47}$$

where $\lambda > 1$, $r > 0$ and $(y, s) \in \Omega_T$.

Let us take a point $(y,s) \in Q_{\rho_1}(y_0,s_0)$ and a number $r \in \left[\frac{\rho_2-\rho_1}{2^5}, \rho_2-\rho_1\right)$. Then we see from (4.43) to (4.47) that $Q_r^\lambda(y,s) \subset Q_{\rho_2}(y_0,s_0) \subset Q_{2R}(y_0,s_0)$. Thus we calculate as follows:

$$\begin{aligned}
&\fint_{Q_r^\lambda(y,s)} |Du|^p dxdt + \frac{1}{\delta^p}\fint_{Q_r^\lambda(y,s)} \Psi^p dxdt \\
&\leq \frac{|Q_{2R}(y_0,s_0)|}{|Q_r^\lambda(y,s)|}\left(\fint_{Q_{2R}(y_0,s_0)} |Du|^p dxdt + \frac{1}{\delta^p}\fint_{Q_{2R}(y_0,s_0))} \Psi^p dxdt\right) \\
&\leq \left(\frac{2^6 R}{\rho_2-\rho_1}\right)^{n+2} \lambda^{d_1}\ \lambda_0^{\frac{p}{d}} \\
&< \lambda^p,
\end{aligned} \tag{4.48}$$

provided that

$$\lambda > \left(\frac{2^6 R}{\rho_2-\rho_1}\right)^{\frac{n+2}{p-d_1}} \lambda_0^{\frac{p}{d(p-d_1)}} = \left(\frac{2^6 R}{\rho_2-\rho_1}\right)^{\frac{(n+2)d}{p}} \lambda_0 = B\lambda_0, \tag{4.49}$$

where we have used (4.13) and (4.46). Combining (4.48) and (4.49), we infer that

$$\fint_{Q_r^\lambda(y,s)} |Du|^p dxdt + \frac{1}{\delta^p}\fint_{Q_r^\lambda(y,s)} \Psi^p dxdt < \lambda^p, \tag{4.50}$$

for every $\lambda > B\lambda_0$, for every $(y,s) \in Q_{\rho_1}(y_0,s_0)$ and for every $r \in \left[\frac{\rho_2-\rho_1}{2^5}, \rho_2-\rho_1\right)$.

Next, for λ, as in (4.49), we consider the following upper-level set:

$$E(\lambda,\rho_1) = \{(y,s) \in Q_{\rho_1}(y_0,s_0) : |Du(y,s)| > \lambda\}. \tag{4.51}$$

Then for any point $(y,s) \in E(\lambda,\rho_1)$, we have

$$\lim_{r\to 0}\left(\fint_{Q_r^\lambda(y,s)} |Du|^p dxdt + \frac{1}{\delta^p}\fint_{Q_r^\lambda(y,s)} \Psi^p dxdt\right) \geq |Du(y,s)|^p > \lambda^p. \tag{4.52}$$

Therefore, we conclude from (4.49)-(4.52) that for any $(y,s) \in E(\lambda,\rho_1)$, there exists $r_{(y,s)} \in \left(0, \frac{\rho_2-\rho_1}{2^5}\right)$, depending on (y,s), such that

$$\fint_{Q^\lambda_{r_{(y,s)}}(y,s)} |Du|^p dxdt + \frac{1}{\delta^p}\fint_{Q^\lambda_{r_{(y,s)}}(y,s)} \Psi^p dxdt = \lambda^p \tag{4.53}$$

and for all $r \in (r_{(x,y)}, \rho_2 - \rho_1)$,

$$\fint_{Q_r^\lambda(y,s)} |Du|^p dxdt + \frac{1}{\delta^p} \fint_{Q_r^\lambda(y,s)} \Psi^p dxdt < \lambda^p. \tag{4.54}$$

Applying Vitali covering lemma, we see from (4.53)-(4.54) that there exists a countable collection of disjoint parabolic cylinders,

$$\left\{ Q_{r_i}^\lambda(y_i, s_i) : (y_i, s_i) \in E(\lambda, \rho_1),\ 0 < r_i < \frac{\rho_2 - \rho_1}{2^5} \right\}_{i \geq 1} \tag{4.55}$$

such that

$$\fint_{Q_{r_i}^\lambda(y_i,s_i)} |Du|^p dxdt + \frac{1}{\delta^p} \fint_{Q_{r_i}^\lambda(y_i,s_i)} \Psi^p dxdt = \lambda^p, \tag{4.56}$$

$$E(\lambda, \rho_1) \subset \bigcup_{i=1}^{\infty} Q_{5r_i}^\lambda(y_i, s_i) \cup \text{ negligible set} \tag{4.57}$$

and

$$\fint_{Q_r^\lambda(y,s)} |Du|^p dxdt + \frac{1}{\delta^p} \fint_{Q_r^\lambda(y,s)} \Psi^p dxdt < \lambda^p,\ \forall r \in (r_i, \rho_2 - \rho_1). \tag{4.58}$$

For simplicity, we write

$$Q_i^j = Q_{2^j r_i}^\lambda(y_i, s_i),\ j = 0, 1, 2, 3, 4, 5. \tag{4.59}$$

We now recall (4.43) and see from (4.55) and (4.57)-(4.59) that

$$E(\lambda, \rho_1) \subset \bigcup_{i=1}^{\infty} Q_i^3 \cup \text{ negligible set} \tag{4.60}$$

and

$$\fint_{Q_i^j} |Du|^p dxdt + \frac{1}{\delta^p} \fint_{Q_i^j} \Psi^p dxdt < \lambda^p,\ j = 1, 2, 3, 4, 5. \tag{4.61}$$

Moreover one can discover from (4.56) and (4.59) that

$$|Q_i^0| \leq \frac{2}{\lambda^p} \left(\int_{\{(y,s) \in Q_i^0 : |Du|^p > \frac{\lambda^p}{4}\}} |Du|^p dxdt + \frac{1}{\delta^p} \int_{\{(y,s) \in Q_i^0 : \Psi^p > \frac{\delta^p \lambda^p}{4}\}} \Psi^p dxdt \right). \tag{4.62}$$

4.3.2 Comparison estimates

We next apply a scaling argument and employ the comparison estimates which have already been obtained on uniformed and standard parabolic cylinders in Section 4.2, to find counterparts on the intrinsic parabolic cylinders which have already been constructed from the Vitali covering argument. We first recall (4.18) and (4.61) to find that

$$\fint_{Q_i^5} |Du|^p dxdt + \frac{1}{\delta^p} \fint_{Q_i^5} \Psi^p dxdt < \lambda^p. \tag{4.63}$$

We consider the following rescaled functions:

$$\begin{cases} \tilde{a}_i(\xi, x, t) = \frac{a(\lambda\xi, Y_i, S_i)}{\lambda^{p-1}}, & \tilde{F}_i(x,t) = \frac{F(Y_i,S_i)}{\lambda}, \\ \tilde{u}_i(x,t) = \frac{u(Y_i,S_i)}{R_i}, & \tilde{\psi}_i(x,t) = \frac{\psi(Y_i,S_i)}{R_i}, \end{cases}$$

where if $p \geq 2$,

$$\begin{cases} Y_i = y_i + (2^2 r_i)x, \\ S_i = s_i + \lambda^{2-p}(2^2 r_i)^2 t, \\ R_i = \lambda 2^2 r_i, \end{cases}$$

and on the other hand, if $\frac{2n}{n+2} < p < 2$,

$$\begin{cases} Y_i = y_i + \lambda^{\frac{p-2}{2}}(2^2 r_i)x, \\ S_i = s_i + (2^2 r_i)^2 t, \\ R_i = \lambda^{\frac{p}{2}}(2^2 r_i). \end{cases}$$

Then one can see that $\tilde{a}_i(\xi, x, t)$ satisfies (4.1) with the same constants μ, Λ, and it is weakly $\left(\delta, \frac{R_0}{(2^2 r_i)}\right)$-vanishing. The condition (4.16) follows by taking R_0 with $R_0 \geq R$, since $r_i < \frac{\rho_2 - \rho_1}{2^5}$. On the other hand, the assumption (4.17) follows from (4.63). Thus we are in the hypotheses of Lemma 4.2.4, which concludes that there exists a function

$$\tilde{v}_i \in L^p(-4^2, 4^2; W^{1,p}(B_4)) \cap W^{1,\frac{p}{p-1}}(-4^2, 4^2; W^{-1,\frac{p}{p-1}}(B_4))$$

such that

$$\fint_{Q_4} |D(\tilde{u}_i - \tilde{v}_i)|^p \, dxdt \leq \epsilon^p \text{ and } \sup_{Q_2} |D\tilde{v}_i| \leq n_1,$$

where $n_1 = n_1(\mu, \Lambda, n, p) \geq 1$ is a universal constant.

By rescaling back, we discover that there exists a function v_i defined on Q_i^4 such that

$$\fint_{Q_i^4} |D(u - v_i)|^p dxdt \leq \epsilon^p \lambda^p \text{ and } \sup_{Q_i^3} |Dv_i| \leq n_1 \lambda. \tag{4.64}$$

4.3.3 The a priori estimates

We next integrate the estimates (4.64) for the upperlevel sets for the gradient of the weak solution to the obstacle problem (4.7).

We start with two standard identities: For any measurable, nonnegative function f in a bounded measurable set U in $\mathbb{R}^n \times \mathbb{R}$ and for any $\beta > 0$,

$$\int_U f^\beta dxdt = \int_0^\infty \beta \lambda^{\beta-1} |\{(x,t) \in U : f > \lambda\}| d\lambda. \tag{4.65}$$

If $\beta > \alpha > 0$, then

$$\int_U f^\beta dxdt = (\beta - \alpha) \int_0^\infty \lambda^{\beta-\alpha-1} \left(\int_{\{(x,t)\in U : f > \lambda\}} g^\alpha dxdt \right) d\lambda. \tag{4.66}$$

The weak (1,1)-type estimate is

$$|\{(x,t) \in U : g > \lambda\}| \leq \frac{1}{\lambda} \int_U g dxdt. \tag{4.67}$$

The next lemma is a standard tool which can be found from Lemma 4.3 in [27].

Lemma 4.3.1. *Let $\phi(\rho)$ be a bounded nonnegative function in the interval $[R, 2R]$. Suppose that for $R \leq \rho_1 < \rho_2 \leq 2R$, we have*

$$\phi(\rho_1) \leq \theta f(\rho_2) + \frac{A_1}{(\rho_2 - \rho_1)^\alpha} + A_2$$

with $A_1, A_2 \geq 0$, $\alpha > 0$ and $0 < \theta < 1$. Then

$$f(R) \leq c \left(\frac{A_1}{R^\alpha} + A_2 \right),$$

for some positive constant $c = c(\alpha, \theta)$.

We are now in a position to derive the gradient estimates (4.12). For λ in (4.49), Q_i^3 in (4.59) and n_1 in (4.64), we calculate that

$$\begin{aligned}
&|\{(y,s)\in Q_i^3 : |Du| > 2n_1\lambda\}| \\
&\quad \le |\{(y,s)\in Q_i^3 : |D(u-v_i)| > n_1\lambda\}| + |\{(y,s)\in Q_i^3 : |Dv_i| > n_1\lambda\}| \\
&\quad \overset{(4.64),(4.67)}{\le} \frac{1}{n_1^p\lambda^p}\int_{Q_i^4} |D(u-v_i)|^p dxdt \\
&\quad \overset{(4.59),(4.64)}{\le} \frac{1}{n_1^p}\epsilon^p 2^{4(n+2)}|Q_i^0| \\
&\quad \overset{(4.62)}{\le} \epsilon^p \frac{c_1}{\lambda^p}\left(\int_{\{(y,s)\in Q_i^0:|Du|^p>\frac{\lambda^p}{4}\}} |Du|^p dxdt + \frac{1}{\delta^p}\int_{\{(y,s)\in Q_i^0:\Psi^p>\frac{\delta^p\lambda^p}{4}\}} \Psi^p dxdt\right),
\end{aligned} \tag{4.68}$$

for some universal constant $c_1 = c_1(\mu,\Lambda,n,p)$.

We next recall that the collection $\{Q_i^0\}$ is disjoint, $Q_i^0 \subset Q_{\rho_2}(y_0,s_0)$, and that

$$\bigcup_{i=1}^{\infty} Q_i^3 \supset E(\rho_1,\lambda) \supset E(\rho_1, 2n_1\lambda). \tag{4.69}$$

For λ_0 in (4.42) and B in (4.49), we calculate that

$$\begin{aligned}
&\int_{B\lambda_0}^{\infty} \lambda^{pq-1}|E(\rho_1,2n_1\lambda)|\, d\lambda \\
&\quad \overset{(4.69)}{\le} \int_{B\lambda_0}^{\infty} \lambda^{pq-1}\sum_{i\ge 1}|\{(y,s)\in Q_i^3 : |Du| > 2n_1\lambda\}|\, d\lambda \\
&\quad \overset{(4.68)}{\le} c_1\epsilon^p \int_{B\lambda_0}^{\infty} \lambda^{pq-p-1}\left(\int_{\left\{(y,s):Q_{\rho_2}(y_0,s_0):|Du|^p>\frac{\lambda^p}{4}\right\}} |Du|^p dxdt\right) d\lambda \\
&\qquad + \frac{c_1}{\delta^p}\epsilon^p \int_{B\lambda_0}^{\infty} \lambda^{pq-p-1}\left(\int_{\left\{(y,s)\in Q_{\rho_2}(y_0,s_0):\Psi^p>\frac{\delta^p\lambda^p}{4}\right\}} |\Psi|^p dxdt\right) d\lambda \\
&\quad \overset{(4.66)}{\le} c_2\epsilon^p\left(\int_{Q_{\rho_2}(y_0,s_0)} |Du|^{pq} dxdt + \frac{1}{\delta^{pq}}\int_{Q_{\rho_2}(y_0,s_0)} \Psi^{pq} dxdt\right),
\end{aligned} \tag{4.70}$$

for some constant $c_2 = c_2(\mu, \lambda, n, p, q)$. Consequently, we have

$$\begin{aligned}
&\int_{Q_{\rho_1}(y_0,s_0)} |Du|^{pq} dxdt \\
&\overset{(4.65)}{=} pq(2n_1)^{pq} \left(\int_0^{B\lambda_0} \lambda^{pq-1} |E(\rho_1, 2n_1\lambda)| d\lambda + \int_{B\lambda_0}^{\infty} \lambda^{pq-1} |E(\rho_1, 2n_1\lambda)| d\lambda \right) \\
&\overset{(4.70)}{\leq} (2n_1 B)^{pq} \lambda_0^{pq} |Q_{\rho_1}(y_0, s_0)| \\
&\quad + pq(2n_1)^{pq} c_2 \epsilon^p \left(\int_{Q_{\rho_2}(y_0,s_0)} |Du|^{pq} dxdt + \frac{1}{\delta^{pq}} \int_{Q_{\rho_2}(y_0,s_0)} \Psi^{pq} dxdt \right).
\end{aligned}$$

Thus it follows from (4.43), (4.49) and (4.70) that

$$\begin{aligned}
\int_{Q_{\rho_1}(y_0,s_0)} |Du|^{pq} dxdt \;\leq\; & c_3 \epsilon^p \int_{Q_{\rho_2}(y_0,s_0)} |Du|^{pq} dxdt \\
& + c_3 \epsilon^p \frac{1}{\delta^{pq}} \int_{Q_{2R}(y_0,s_0)} \Psi^{pq} dxdt \\
& + c_4 \left(\frac{R}{\rho_2 - \rho_1} \right)^{(n+2)d} |Q_{2R}(y_0, s_0)| \lambda_0^{pq}, \quad (4.71)
\end{aligned}$$

for some constants $c_3 = c_3(\mu, \Lambda, n, p, q)$ and $c_4 = c_4(\mu, \Lambda, n, p, q)$.

We now choose $\epsilon > 0$ small enough such that

$$0 < c_3 \epsilon^p \leq \frac{1}{2},$$

and thereby there exists a positive number $\delta = \delta(\mu, \Lambda, n, p, q)$ from Lemma 4.2.4. As a consequence, (4.71) can be written as

$$\begin{aligned}
\int_{Q_{\rho_1}(y_0,s_0)} |Du|^{pq} dxdt \;\leq\; & \frac{1}{2} \int_{Q_{\rho_2}(y_0,s_0)} |Du|^{pq} dxdt \\
& \frac{c_4 |Q_{2R}(y_0, s_0)| \lambda_0^{pq} R^{(n+2)d}}{(\rho_2 - \rho_1)^{(n+2)d}} + c_5 \int_{Q_{2R}(y_0,s_0)} \Psi^{pq} dxdt, \quad (4.72)
\end{aligned}$$

for some constants $c_5 = c_5(\mu, \Lambda, n, p, q)$. Applying Lemma 4.3.1 and recalling (4.42), we now easily deduce the required estimates (4.12).

4.4 Proof of Theorem 4.1.1

We have established the estimates (4.12) in Theorem 4.1.1 under the a priori assumption (4.14). In this section we want to complete the Theorem 4.1.1 by removing this assumption (4.14) by an proper approximation argument. In particular, an approximation procedure along the same vein has been systematically and thoroughly discussed in the recent papers [1, 3] in a different situation, respectively, we sketch its proof for the sake of completeness.

We first extend $\psi^+ = \max\{\psi, 0\} \in \mathcal{A}'$ to $\mathbb{R}^n \times (0,T)$ by zero, again denoted by the same ψ^+, such that

$$\psi^+ \in L^p(0,T;W^{1,p}(\mathbb{R}^n)),\ \psi_t^+ \in L^{\frac{p}{p-1}}\left(\mathbb{R}^n \times (0,T)\right).$$

We next recall that Ω is a bounded Lipschitz domain, to see from a standard extend theorem that

$$\begin{cases} \psi \in L^{pq}(0,T;W^{1,pq}(\mathbb{R}^n)),\ \psi_t \in L^{\frac{pq}{p-1}}\left(\mathbb{R}^n \times (0,T)\right), \\ \psi \le \psi^+ \text{ a.e. on } \mathbb{R}^n \times (0,T). \end{cases}$$

Then by a standard interpolation argument, we see that

$$\psi,\ \psi^+ \in L^\infty(0,T;L^2(\Omega)).$$

We also extend F to $\mathbb{R}^n \times (a, a+T)$ by zero, such that

$$F \in L^{pq}\left(\mathbb{R}^n \times (0,T), \mathbb{R}^n\right).$$

By a proper approximation using the mollification techniques, one can take

$$\psi_k(\cdot,t),\ \partial_t[\psi_k(\cdot,t)],\ \psi_k^+(\cdot,t),\ \partial_t[\psi_k^+(\cdot,t)] \in C^\infty(\mathbb{R}^n),$$

$$F_k(\cdot,t) \in C^\infty(\mathbb{R}^n,\mathbb{R}^n) \text{ and } a_k(\cdot,\cdot,t) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$$

for almost every $t \in (0,T)$, such that

$$\begin{cases} |a_k(\xi,x,t)| + |D_x a_k(\xi,x,t)| \le c\Lambda|\xi|^{p-1}, \\ |D_\xi a_k(\xi,x,t)| \le \Lambda|\xi|^{p-1},\ D_\xi a_k(\xi,x,t)\eta\cdot\eta \ge \mu|\xi|^{p-2}|\eta|^2, \end{cases}$$

for all $\xi,\eta \in \mathbb{R}^n$, for almost every $(x,t) \in \mathbb{R}^{n+1}$, and for some constant $c = c(p)$,

$$|\partial_t\psi_k(\cdot)| + \left|\operatorname{div}\left(|F_k(\cdot)|^{p-2}F_k(\cdot)\right)\right| + |\operatorname{div}\left[a_k(D\psi_k(\cdot),\cdot)\right]| \in L^{n+2}(\Omega_T),$$

and

$$\begin{cases} \psi_k^+ \to \psi^+ \text{ in } L^p(0,T;W^{1,p}(\Omega)) \cap L^\infty\left(0,T;L^2(\Omega)\right), \\ \psi_k \to \psi \text{ in } L^{pq}(0,T;W^{1,pq}(\Omega) \cap L^\infty\left(0,T;L^2(\Omega)\right), \\ \partial_t[\psi_k^+] \to \partial_t\psi^+ \text{ in } L^{\frac{p}{p-1}}\left(\Omega_T\right), \\ \partial_t\psi_k \to \partial_t\psi \text{ in } L^{\frac{pq}{p-1}}\left(\Omega_T\right), \\ F_k \to F \text{ in } L^{pq}(\Omega_T), \\ a_k(\xi,x,t) \to a(\xi,x,t) \text{ for all } \xi \in \mathbb{R}^n \text{ and for a.e. } (x,t) \in \mathbb{R}^n \times \mathbb{R}. \end{cases} \tag{4.73}$$

Then, according to existence and regularity theory for nonlinear parabolic obstacle problems, there exists a solution

$$u_k \in \psi_k^+ + L^p\left(0,T;W_0^{1,p}(\Omega)\right) \cap W^{1,\frac{p}{p-1}}\left(0,T;W^{-1,\frac{p}{p-1}}(\Omega)\right)$$

to the (strong) variational inequality

$$\int_0^T < \partial_t u_k, v-u_k > dt + \int_{\Omega_T} a_k(Du_k,x,t)D(v-u_k)dxdt \geq \int_{\Omega_T} |F_k|^{p-2}F_k D(v-u_k)dxdt,$$

for every $v \in \psi_k^+ + L^p\left(0,T;W_0^{1,p}(\Omega)\right) \cap W^{1,\frac{p}{p-1}}\left(0,T;W^{-1,\frac{p}{p-1}}(\Omega)\right)$ with $v(\cdot,a)=0$ and $v \geq \psi_k$, such that we have the regularity

$$Du_k \in L^\infty_{loc}(\Omega_T,\mathbb{R}^n). \tag{4.74}$$

A careful analysis carried out in [1, 3] claims that there exists a solution $\overline{u}$ to the original problem (4.7) such that, up to a non-relabeled subsequence,

$$Du_k \to D\overline{u} \text{ strongly in } L^p(\Omega_T,\mathbb{R}^n). \tag{4.75}$$

On the other hand, with the Lipschitz regularity (4.74) and in the same spirit as used in the previous section, we obtain that

$$\begin{aligned}\left(\fint_{Q_R(y_0,s_0)} |Du_k|^{pq} dxdt\right)^{\frac{1}{d}} &\leq c\left[1+\left(\fint_{Q_{2R}(y_0,s_0)} |Du_k|^p dxdt\right)^q\right]\\ &+c\fint_{Q_{2R}(y_0,s_0)} \left(|\partial_t \psi_k|^{\frac{pq}{p-1}} + |D\psi_k|^{pq} + |F_k|^{pq}\right) dxdt\\ &\leq c\left[1+\left(\fint_{Q_{2R}(y_0,s_0)} |D\overline{u}|^p dxdt\right)^q\right]\\ &+c\fint_{Q_{2R}(y_0,s_0)} \left(|\partial_t \psi|^{\frac{pq}{p-1}} + |D\psi|^{pq} + |F|^{pq}\right) dxdt \leq c,\end{aligned}$$

where we have used (4.73) and (4.75). Therefore, applying subsequently Fatou's lemma via a standard weak convergence method, we finally derive the estimate required one (4.12) from the uniqueness of the original problem (4.7). This completes the proof of Theorem 4.1.1.

Chapter 5

Nonlinear gradient estimates for parabolic obstacle problems in non-smooth domains

5.1 Assumptions and Results

Let Ω is a bounded domain of $\mathbb{R}^n$ with $n \geq 2$, and we denote by $\Omega_T = \Omega \times (0, T)$, $0 < T < \infty$, the cylindrical domain. Then given a fixed real number $p \in \left(\frac{2n}{n+2}, \infty\right)$, we consider a Carathéodory nonlinearity

$$a = a(\xi, x, t) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

satisfying the following basic structural conditions: for every $\xi, \eta \in \mathbb{R}^n$, for almost every $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ and for some constants $0 < \mu \leq 1 \leq \Lambda$,

$$|a(\xi, x, t)| + |\xi||D_\xi a(\xi, x, t)| \leq \Lambda |\xi|^{p-1} \text{ and } D_\xi a(\xi, x, t)\eta \cdot \eta \geq \mu |\xi|^{p-2}|\eta|^2. \tag{5.1}$$

On the other hand, (5.1) implies the following monotonicity condition:

$$\begin{cases} (a(\xi, x, t) - a(\eta, x, t)) \cdot (\xi - \eta) \geq \gamma |\xi - \eta|^p, \text{ if } p \geq 2, \\ (a(\xi, x, t) - a(\eta, x, t)) \cdot (\xi - \eta) \geq \gamma |\xi - \eta|^2 (|\xi| + |\eta|)^{p-2}, \text{ if } 1 < p < 2, \end{cases} \tag{5.2}$$

where γ is a positive constant depending only on n, μ and p.

The nonhomogeneous term is a function $F \in L^p(\Omega_T, \mathbb{R}^n)$, and the obstacle is a function ψ satisfying

$$\begin{cases} \psi \in L^p(0,T;W^{1,p}(\Omega)),\ \psi_t \in L^{\frac{p}{p-1}}(\Omega_T), \\ \psi \le 0 \text{ a.e. on } \partial\Omega \times (0,T),\ \text{and } \psi(\cdot,0) \le 0 \text{ a.e. in } \Omega. \end{cases} \tag{5.3}$$

Let $\mathcal{A}$ be the admissible set of all $v \in C^0([0,T];L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ such that

$$v(\cdot,0) = 0 \text{ a.e. in } \Omega \text{ and } v \ge \psi \text{ a.e. in } \Omega_T. \tag{5.4}$$

We then call a function $u \in \mathcal{A}$ to be a solution of the variational inequality if there holds

$$\int_0^T < v_t, v-u > \ dt + \int_{\Omega_T} a(Du,x,t) \cdot D(v-u)\ dxdt \ge \int_{\Omega_T} |F|^{p-2}F \cdot D(v-u)\ dxdt, \tag{5.5}$$

for all $v \in \mathcal{A}' = \{v \in \mathcal{A} : v_t \in L^{\frac{p}{p-1}}(0,T;W^{-1,\frac{p}{p-1}}(\Omega))\}$. Here, $< \cdot, \cdot >$ denotes the duality pairing between $W^{-1,\frac{p}{p-1}}$ and $W_0^{1,p}$.

The variational inequality (2.5) is formulated as a weak form, not involving u_t, since for a parabolic variational inequality we do not know that u_t lies in $L^{\frac{p}{p-1}}(0,T;W^{-1,\frac{p}{p-1}}(\Omega))$. Thus a weak solution may not be admissible as a test function. This problem is not resolved by Steklov averages because of the obstacle constraint depending on time. Many papers on nonlinear problems with time dependent obstacle imposed monotonicity type assumptions on a obstacle function with respect to time. Recently, in [1], without any monotonicity conditions on the obstacle, the existence of the weak solution for (2.5) has been showed when the obstacle function ψ is weakly differentiable with respect to time variable and satisfies $\psi_t \in L^{\frac{p}{p-1}}(\Omega_T)$.

Lemma 5.1.1. *[1] There is a unique solution $u \in \mathcal{A}$ of the variational inequality (5.5) and we have the estimate*

$$\sup_{t\in(0,T)} \int_\Omega |u(\cdot,t)|^2 dx + \int_{\Omega_T} |Du|^p dxdt \le c \left[\int_{\Omega_T} \left(|F|^p + |D\psi|^p + |\psi_t|^{\frac{p}{p-1}} \right) dxdt \right],$$

for some positive constant $c = c(\mu, \Lambda, n, p)$.

In the very interesting paper [1] Böegelein, Duzaar and Mingione proved the interior L^q, $p \leq q < \infty$, estimates for the solutions u of this obstacle problems, essentially proving that the gradient Du of the solution u is as locally integrable as the one $D\psi$ of the assigned obstacle as well as the nonhomogeneous term F. The main purpose in this paper is two-fold. One is to find a global version of the results in [1], considering a non-smooth domain with δ-Reifenberg flatness which may be beyond the Lipschitz category with a small Lipschitz constant. The other is to extend the functional frame from Lebesgue spaces to Orlicz spaces. An Orlicz space is a type of a function space which generalizes the L^q spaces. To this end, we first give a brief discussion on Orlicz spaces.

A real valued function ϕ defined on $[0,\infty)$ is called a Young function if it is convex, nondecreasing and having the following properties:

$$\phi(0) = 0;\ \lim_{z\to+\infty} \phi(z) = +\infty;\ \lim_{z\to 0+} \frac{\phi(z)}{z} = 0;\ \lim_{z\to+\infty} \frac{\phi(z)}{z} = +\infty.$$

Here the young function ϕ is assumed to satisfy the so called Δ_2 and ∇_2 conditions.

Definition 5.1.1. *A Young function ϕ is said to satisfy the Δ_2 condition, written as $\phi \in \triangle_2$, if there is a constant $\kappa_1 > 0$ such that*

$$\phi(2z) \leq \kappa_1 \phi(z) \text{ for all } z \geq 0.$$

Moreover, ϕ is said to satisfy the ∇_2 condition, written as $\phi \in \nabla_2$, if there is a constant $\kappa_2 > 1$ such that

$$\phi(z) \leq \frac{1}{2\kappa_2}\phi(\kappa_2 z) \text{ for all } z \geq 0.$$

Δ_2 and ∇_2 conditions are a type of doubling conditions, ensuring that ϕ grows neither too fast nor too slowly. For instance, $\exp(t^2)$ is ruled out by Δ_2, so is $t\ln(1+t)$ by ∇_2. It is worthwhile to point out that they are unavoidable in the regularity theory we are considering here, as follows from the earlier works [4, 7, 63, 64, 65] and the references therein. For simplicity we denote by $\phi \in \Delta_2 \cap \nabla_2$ to mean that ϕ satisfies both Δ_2 and ∇_2 conditions. On the other hand $\phi \in \Delta_2 \cap \nabla_2$ implies that there exist constants $A > 0$, $B > 0$ and $1 < \alpha_2 \leq \alpha_1$ such that for any $0 < z_2 \leq 1 \leq z_1 < \infty$,

$$\phi(z_1 z) \leq A z_1^{\alpha_1} \phi(z) \text{ and } \phi(z_2 z) \leq B z_2^{\alpha_2} \phi(z) \text{ for all } z \geq 0. \tag{5.6}$$

For a bounded domain $U \subset \mathbb{R}^n \times \mathbb{R}$ and a Young function $\phi \in \Delta_2 \cap \nabla_2$, the Orlicz space $L^\phi(U)$ consists of all measurable functions $f : U \to \mathbb{R}$ for which the Luxemburg norm

$$\|f\|_{L^\phi(U)} := \inf \left\{ z > 0 : \int_U \phi\left(\frac{|f|}{z}\right) dxdt \leq 1 \right\}$$

is finite. We then observe that $L^{\alpha_1}(U) \subset L^\phi(U) \subset L^{\alpha_2}(U) \subset L^1(U)$ with the constants α_1, α_2 in (5.6). Moreover there is a positive constant c such that for $f \in L^\phi(U)$,

$$\frac{1}{c} \min_{\alpha=\{\alpha_1,\alpha_2\}} \left(\int_U \phi(|f|)dx \right)^{\frac{1}{\alpha}} \leq \|f\|_{L^\phi(U)} \leq c \max_{\alpha=\{\alpha_1,\alpha_2\}} \left(\int_U \phi(|f|)dx \right)^{\frac{1}{\alpha}}.$$

For typical examples of a Young function $\phi \in \Delta_2 \cap \nabla_2$, $\phi(z) = z^q$ or $\phi(z) = z^q \log(1+z)$, $q > 1$. As a consequence, the Lebesgue space L^q, $1 < q < \infty$, is a special case of the Orlicz spaces we are considering here. We refer the reader to [31, 41, 52, 53] for more details on the Orlicz spaces.

We will use the following lemma.

Lemma 5.1.2. *For a bounded domain $U \subset \mathbb{R}^n \times \mathbb{R}$ and a Young function $\phi \in \Delta_2 \cap \nabla_2$, let $g \in L^\phi(U)$. Then there exists a constant $m_0 = m_0(\phi) > 1$ such that for any $\nu > 0$ and $m \geq m_0$, there holds*

$$\sum_{k=0}^{\infty} \frac{\phi(\nu m^k)}{\nu m^k} \int_{\{(x,t)\in U : |g| > \nu m^k\}} |g| \; dxdt \leq c \int_U \phi(|g|) \; dxdt, \tag{5.7}$$

where $c = c(\phi) > 0$.

Proof. We start from the following estimate, which can be found, for example from Lemma 2.8 in [64].

$$\underbrace{\int_0^\infty \frac{1}{\lambda} \left(\int_{\{(x,t)\in U : |g| > \lambda\}} |g| \; dxdt \right) d[\phi(\lambda)]}_{I} \leq c \int_U \phi(|g|) \; dxdt, \tag{5.8}$$

where $c = c(\phi)$ is a positive constant. We now split I as follows.

$$I = I_0 + \sum_{k=1}^{\infty} I_k \tag{5.9}$$

for

$$I_0 = \int_0^{\nu} \frac{1}{\lambda} \left(\int_{\{(x,t)\in U:|g|>\lambda\}} |g| \; dxdt \right) d[\phi(\lambda)]$$

and

$$I_k = \int_{\nu m^{k-1}}^{\nu m^k} \frac{1}{\lambda} \left(\int_{\{(x,t)\in U:|g|>\lambda\}} |g| \; dxdt \right) d[\phi(\lambda)].$$

It is easy to check that

$$I_0 \geq \frac{\phi(\nu)}{\nu} \int_{\{(x,t)\in U:|g|>\nu\}} |g| \; dxdt.$$

On the other hand we observe from (2.6) that

$$\begin{aligned} I_k &\geq \frac{\phi(\nu m^k) - \phi(\nu m^{k-1})}{\nu m^k} \int_{\{(x,t)\in U:|g|>\nu m^k\}} |g| \; dxdt \\ &\geq \frac{(1 - Bm^{-\alpha_2})\phi(\nu m^k)}{\nu m^k} \int_{\{(x,t)\in U:|g|>\nu m^k\}} |g| \; dxdt. \end{aligned}$$

Therefore

$$c_* \sum_{k=0}^{\infty} \frac{\phi(\nu m^k)}{\nu m^k} \int_{\{(x,t)\in U:|g|>\nu m^k\}} |g| \; dxdt \leq I, \tag{5.10}$$

for the constant $c_* = \min\{1, 1 - Bm^{-\alpha_2}\}$. Choosing a large m_0 such that $1 - Bm_0^{-\alpha_2} > 0$ and recalling (5.8), (5.9) and (5.10), we derive the required estimate (5.7) for $m \geq m_0$. □

We next introduce the main assumptions on the nonlinearity $a = a(\xi, x, t)$ and the boundary $\partial\Omega$ of the domain Ω. Here we consider symmetric parabolic cylinders of the type

$$Q_{\rho,\theta}(y, s) = B_\rho(y) \times (s - \theta, s + \theta)$$

where $(y, s) \in \mathbb{R}^n \times \mathbb{R}$, $B_\rho(y) = \{x \in \mathbb{R}^n : |x - y| < \rho\}$ and $\rho, \ \theta > 0$. When dealing with the standard case that $\theta = \rho^2$, we write $Q_\rho(y, s) = Q_{\rho,\rho^2}(y, s)$.

In order to measure the oscillation of $a(\xi, x, t)$ in the variable x over the ball $B_\rho(y)$ we consider the function

$$\beta(x,t) = \beta(a, Q_{\rho,\theta}(y,s))(x,t) = \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|a(\xi,x,t) - \overline{a}_{B_\rho(y)}(\xi,t)|}{|\xi|^{p-1}}, \tag{5.11}$$

where $\overline{a}_{B_\rho(y)}(\xi, t)$ is the integral average $a(\xi, \cdot, t)$ over $B_\rho(y)$, i.e.,

$$\overline{a}_{B_\rho(y)}(\xi,t) = \fint_{B_\rho(y)} a(\xi,x,t)dx = \frac{1}{|B_\rho(y)|}\int_{B_\rho(y)} a(\xi,x,t)dx. \tag{5.12}$$

Remark 5.1.1. *In view of (5.15), (5.11) and (5.12), we see that*

$$\beta \leq 2\Lambda.$$

Definition 5.1.2. *We say that $a(\xi, x, t)$ is weakly (δ, R)-vanishing if we have*

$$\sup_{\substack{0<\rho\leq R \\ 0<\theta\leq R^2}} \sup_{(y,s)\in\mathbb{R}^{n+1}} \fint_{Q_{\rho,\theta}(y,s)} \beta(x,t)\ dxdt \leq \delta.$$

Roughly speaking, the nonlinearity $a(\xi, x, t)$ has a small BMO semi-norm for the spatial variable x while it is allowed to be merely measurable in the time variable, uniformly in ξ. This kind of regularity assumption is weaker than any other one reported in this literature.

On the other hand, we deal with the Reifenberg flatness of the boundary which has been widely studied in the area of geometric analysis as a minimal geometric assumption coming from various minimizing problems, see [26, 29, 35, 54, 62] and the references therein. In particular, thanks to the scaling invariant property and the measure density condition δ-Reifenberg flat domains have, they are naturally used in the study of optimal boundary regularity theory for elliptic and parabolic problems, see [4, 6, 7, 11].

Definition 5.1.3. *Ω is (δ, R)-Reifenberg flat if for every $x \in \partial\Omega$ and every $\rho \in (0, R]$, there exists a coordinate system $\{y_1, \cdots, y_n\}$, which can depend on ρ and x so that $x = 0$ in this coordinate system and that*

$$B_\rho(0) \cap \{y_n > \delta\rho\} \subset B_\rho(0) \cap \Omega \subset B_\rho(0) \cap \{y_n > -\delta\rho\}.$$

Here we intend to prove for the variational inequality (5.5) that

$$|\partial_t\psi|^{\frac{p}{p-1}},\ |D\psi|^p,\ |F|^p \in L^\phi(\Omega_T) \Rightarrow |Du|^p \in L^\phi(\Omega_T) \qquad \forall\ \phi \in \Delta_2 \cap \nabla_2,$$

under the small BMO assumption on the nonlinearity in x variable and the δ-Reifenberg flatness condition on the boundary. More precisely, we have the following:

Theorem 5.1.1. *Let $\frac{2n}{n+2} < p < \infty$ and $\phi \in \Delta_2 \cap \nabla_2$. We suppose that*

$$|\partial_t\psi|^{\frac{p}{p-1}},\ |D\psi|^p,\ |F|^p \in L^\phi(\Omega_T).$$

Let $0 < R < 1$. Then there exists a small constant $\delta = \delta(\mu, \Lambda, n, p, \phi) > 0$ such that if $a(\xi, x, t)$ is weakly (δ, R)-vanishing and Ω is (δ, R)-Reifenberg flat, then for the solution u of (2.5), $|Du|^p \in L^\phi(\Omega_T)$ and we have the estimate

$$\int_{\Omega_T} \phi(|Du|^p)\ dxdt \le c\left\{\phi\left[\left(\fint_{\Omega_T} \Psi^p\ dxdt + 1\right)^d\right] + \int_{\Omega_T} \phi(\Psi^p)\ dxdt\right\} \tag{5.13}$$

for some positive constant $c = c(\mu, \Lambda, n, p, \phi, R, |\Omega_T|)$, where

$$\Psi = |\partial_t\psi|^{\frac{1}{p-1}} + |D\psi| + |F| \ \text{and}\ d = \begin{cases} \frac{p}{2} & if\ \ p \ge 2, \\ \frac{2p}{p(n+2)-2n} & if\ \ p < 2. \end{cases} \tag{5.14}$$

The previous theorem not only extends to the constrained case of the earlier results in [7] which hold in the unconstrained case, but also provides a global version in the setting of Orlicz spaces of the interior results in [1]. We refer to [6] for the elliptic obstacle problem in the stationary case.

Hereafter we denote by c to mean any universal constant which can be explicitly computed in terms of known quantities such as μ, Λ, n, p, ϕ. Thus the exact value c may change in different occurrences. We also point out that our weak solutions $u \in C^0([0, T]; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$ of (5.5) will be assumed to be defined on $\Omega \times \mathbb{R}$ from the following reasons. The solution u and the variational inequality (5.5) can be extended forward when $t \ge T$ by taking $F = 0$ and extending ψ by the even extension, $\psi(x, t) = \psi(x, 2T - t)$, so that all properties in question are preserved. For backward extension one can use the zero extension of u.

5.2 L^p estimates by approximation

In this section we find a localized estimate of the weak solution of the obstacle problem (5.5) by comparison with solutions of fixed parabolic operators. Our smallness assumption in BMO space on $a(\xi, x, t)$ for the spatial variables and the δ-Reifenberg flatness condition on $\partial\Omega$ will be mainly used in this process as well as the desired Lipschitz regularity for the fixed operator on the flat boundary.

We only consider the estimates near the boundary. For the interior estimates, we refer to the recent paper [5]. To find desired error estimates near the boundary, we recall the notation for the standard parabolic cube $Q_\rho = Q_{\rho,\rho^2}(0,0)$ and add some more geometric notations.

1. $\Omega_\rho = B_\rho \cap \Omega,\ K_\rho = \Omega_\rho \times (-\rho^2, \rho^2)$.
2. $Q_\rho^+ = Q_\rho \cap \{(x,t) = (x', x_n) : x_n > 0\}, T_\rho = B_\rho \cap \{x = (x', x_n) : x_n = 0\}$.
3. $\partial_p K_\rho = \Omega_\rho \times \{t = -\rho^2\} \cup \partial\Omega_\rho \times [-\rho^2, \rho^2]$.

With the same notations and settings, we assume

$$Q_8^+ \subset K_8 \subset Q_8 \cap \{(x,t) : x_n > -16\delta\}. \tag{5.15}$$

We write

$$\Psi(x,t) = \left(|\psi_t|^{\frac{1}{p-1}} + |D\psi| + |F|\right)(x,t),\ (x,t) \in K_8. \tag{5.16}$$

Then we further assume

$$\fint_{K_8} |Du|^p \, dxdt + \frac{1}{\delta^p} \fint_{K_8} \Psi^p \, dxdt \le 1 \tag{5.17}$$

and

$$\fint_{Q_8^+} \beta[a, Q_8^+] \, dxdt \le \delta, \tag{5.18}$$

where δ is to be determined later depending on $n,\ \mu,\ \Lambda,\ p$.

To find comparison estimates, we need the weak solution w of

$$\begin{cases} w_t - div\ a(Dw, x, t) = \psi_t - div\ a(D\psi, x, t) & \text{in} \quad K_8, \\ w = u & \text{on} \quad \partial_p K_8. \end{cases} \tag{5.19}$$

In this problem, it is necessary that the lateral boundary data has a time derivative in $L^{\frac{p}{p-1}}\left(-8^2, 8^2; W^{-1,\frac{p}{p-1}}(\Omega_8)\right)$. However, for the weak solution u of (2.5), it is not a priori clear whether u_t lies in $L^{\frac{p}{p-1}}\left(-8^2, 8^2; W^{-1,\frac{p}{p-1}}(\Omega_8)\right)$ or not. The authors in [1] solved this problem by considering such approximating solutions of regularized obstacle problems. We do not follow the approximation procedure used in [1], instead, we adopt the concept of the localizable solution which was introduced by C. Scheven in [56]. This localizable solution has an extension property and satisfies the variational inequality on any parabolic cylinder $U_I = U \times I \subset \Omega_T$ where $U = \widetilde{U} \cap \Omega$ with a Lipschitz regular domain $\widetilde{U} \subset \mathbb{R}^n$. Moreover, C. Scheven has showed that if the bounded domain Ω admits a $W^{1,p}$-extension property, then the weak solution u treated in our work is a localizable solution. However, a δ-Reifenberg domain has a $W^{1,p}$-extension property, see [26], which implies that this u is a localizable solution. More precisely, in the problem (5.19), one can replace u by $\widetilde{u}$ so that $\widetilde{u}|_{\partial_p K_8} = u|_{\partial_p K_8}$ in the usual weak sense and $\widetilde{u}_t \in L^{\frac{p}{p-1}}\left(-8^2, 8^2; W^{-1,\frac{p}{p-1}}(\Omega_8)\right)$, see [56] for a further discussion on localizable solutions in the literature. As a consequence, one can consider a unique weak solution of (5.19).

With this w, we let h be the weak solution of

$$\begin{cases} h_t - div\ a(Dh, x, t) &= 0 \quad \text{in} \quad K_8, \\ h &= w \quad \text{on} \quad \partial_p K_8. \end{cases} \tag{5.20}$$

We also consider a weak solution of

$$\begin{cases} v_t - div\ \overline{a}_{Q_4^+}(Dv, t) &= 0 \quad \text{in} \quad Q_4^+, \\ v &= 0 \quad \text{on} \quad T_4 \times [-16, 16]. \end{cases} \tag{5.21}$$

We now return to (5.19) and observe that $w = u \geq \psi$ a.e. on $\partial_p K_8$. Then according to a well known comparison principle, see [1], we discover that

$$w \geq \psi \text{ a.e on } K_8.$$

We then test (5.5) with w and (5.19) with $w-u$, respectively to find that

$$\begin{aligned}&\fint_{K_8}(a(Dw,x,t)-a(Du,x,t))\cdot D(w-u)dxdt\\&\qquad\le\fint_{K_8}\psi_t(w-u)+(a(D\psi,x,t)-|F|^{p-2}F)\cdot D(w-u)dxdt.\end{aligned}\tag{5.22}$$

If $p>2$, we can find from (5.1), (5.2), and Young's inequality,

$$\fint_{K_8}|D(w-u)|^p dxdt\le c\left(\fint_{K_8}|\psi_t|^{\frac{p}{p-1}}+|D\psi|^p+|F|^p dxdt\right)\tag{5.23}$$

If $p<2$, for any $\tau_1>0$,

$$\begin{aligned}|D(w-u)|^p &= (|Dw|+|Du|)^{\frac{p(2-p)}{2}}(|Dw|+|Du|)^{\frac{p(p-2)}{2}}|Dw-Du|^p\\&\le \tau_1(|Dw|+|Du|)^p+c(\tau_1)(|Dw|+|Du|)^{p-2}|Dw-Du|^2\\&\le \tau_1(|Dw|+|Du|)^{p-2}(2|Du|+|Dw-Du|)^2\\&\quad +c(\tau_1)(|Dw|+|Du|)^{p-2}|Dw-Du|^2\\&\le 8\tau_1|Du|^p+(2\tau_1+c(\tau_1))(|Dw|+|Du|)^{p-2}|Dw-Du|^2.\end{aligned}$$

Here, we used $p-2<0$ in the last inequality. Using this estimate, we have from (5.1), (5.2), and (5.22),

$$\begin{aligned}\fint_{K_8}|D(w-u)|^p dxdt &\le 8\tau_1\fint_{K_8}|Du|^p dxdt+\tau_2 c(\tau_1)\fint_{K_8}|D(w-u)|^p dxdt\\&\qquad + c(\tau_1,\tau_2)\fint_{K_8}|\psi_t|^{\frac{p}{p-1}}+|D\psi|^p+|F|^p dxdt,\end{aligned}\tag{5.24}$$

for any $\tau_2>0$. We then arrive from (5.16), (5.17), (5.23), and (5.24),

$$\fint_{K_8}|D(u-w)|^p\ dxdt\le c\delta^{\sigma_1}\tag{5.25}$$

for some σ_1 depending on $\mu,\ \Lambda,\ n$ and p, and so

$$\fint_{K_8}|Dw|^p dxdt\le 2^{p-1}\left(\fint_{K_8}|Du|^p dxdt+\fint_{K_8}|D(u-w)|^p dxdt\right)\le c.\tag{5.26}$$

Then the standard L^p estimates for (5.20) and the estimate (5.26) yield

$$\fint_{K_8} |Dh|^p dxdt \leq c. \tag{5.27}$$

Likewise, testing (5.19) and (5.20) with $h - w$ and using (5.15)-(5.18), we discover that

$$\fint_{K_8} |D(w-h)|^p \; dxdt \leq c\delta^{\sigma_2} \tag{5.28}$$

for some positive constant $\sigma_2 = \sigma(\mu, \Lambda, n, p)$.

The following higher integrability result for the homogeneous problem (5.20) is a consequence of the fact that the δ-Reifenberg flatness implies the uniform capacity density condition, see Theorem 2.2 of [2].

Lemma 5.2.1. *Under the assumptions as in (5.15)-(5.20), one can find a small constant $\kappa = \kappa(\mu, \Lambda, n, p) > 0$ such that*

$$\fint_{K_4} |Dh|^{p+\kappa} \; dxdt \leq c. \tag{5.29}$$

Proof. From a self-improving property of (5.20) there exists a positive constant $\kappa = \kappa(\mu, \Lambda, n, p)$ such that

$$\fint_{K_4} |Dh|^{p+\kappa} \; dxdt \leq c \left(\fint_{K_8} |Dh|^p \; dxdt \right)^{\frac{\kappa+d}{d}} + c.$$

Then the conclusion follows from the uniform L^p boundedness (5.27). □

We find a uniform Lipschitz regularity for the limiting (5.21).

Lemma 5.2.2. *[39, 40] Under the assumption as in (5.15)-(5.18), we further suppose that*

$$\fint_{Q_4^+} |Dv|^p \; dxdt \leq c.$$

Then we have

$$\sup_{Q_3^+} |Dv|^p \leq c \left(\fint_{Q_4^+} |Dv|^p \; dxdt + 1 \right) \leq c. \tag{5.30}$$

As a consequence, if we extend v, denoted again v, by zero from Q_4^+ to Q_4, we obtain

$$\sup_{Q_3} |Dv| = \sup_{Q_3^+} |Dv| \le c. \tag{5.31}$$

We will use the following approximation lemma.

Lemma 5.2.3. *[7] For any $\epsilon > 0$, there is a small $\delta = \delta(\epsilon) > 0$ such that if*

$$Q_4^+ \subset K_4 \subset Q_4 \cap \{(x,t) : x_n > -16\delta\}$$

and if k is a weak solution of

$$\begin{cases} k_t - div\ \overline{a}_{Q_4^+}(Dk,t) &= 0 \quad in \quad K_4, \\ k &= 0 \quad on \quad \partial_w K_4 \end{cases}$$

with

$$\fint_{K_4} |Dk|^p\ dxdt \le 1,$$

then there exists a weak solution v of (5.21) *such that*

$$\fint_{Q_4^+} |Dv|^p\ dxdt \le 1 \text{ and } \fint_{K_2} |D(k-v)|^p\ dxdt \le \epsilon^p,$$

where v is extended by zero from Q_4^+ to Q_4.

We now proceed to find a localized comparison estimate near the flat boundary.

Lemma 5.2.4. *For any $\epsilon > 0$, there is a small $\delta = \delta(\epsilon, \mu, \Lambda, n, p) > 0$ such that if (5.15)-(5.18) hold true, then there exists a function v satisfying*

$$\|Dv\|_{L^\infty(K_3)} \le n_1 \text{ and } \fint_{K_2} |D(u-v)|^p\ dxdt \le \epsilon^p$$

for some universal constant $n_1 = n_1(\mu, \Lambda, n, p) \ge 1$.

Proof. We first recall the uniform estimates (5.25) and (5.28). Let k be the weak solution of

$$\begin{cases} k_t - div\ \overline{a}_{B_4^+}(Dk,t) &= 0 \quad \text{in} \quad K_4, \\ k &= h \quad \text{on} \quad \partial_p K_4. \end{cases} \tag{5.32}$$

As usual, we take $k-h$ as a test function for (5.20) and (5.32), respectively, to find that

$$\int_{-16}^{16} < (k-h)_t, k-h > \ dt$$
$$= \int_{K_4} (\overline{a}_{B_4^+}(Dk,t) - a(Dh,x,t)) \cdot D(h-k) \ dxdt. \quad (5.33)$$

We observe

$$\int_{-16}^{16} < (k-h)_t, k-h > \ dt = \frac{1}{2}\int_{\Omega_4} |k(\cdot,16) - h(\cdot,16)|^2 \ dx \geq 0$$

to see from (5.33) that

$$\fint_{K_4} (\overline{a}_{B_4^+}(Dh,t) - \overline{a}_{B_4^+}(Dk,t)) \cdot D(h-k) \ dxdt$$
$$\leq \underbrace{\fint_{K_4} (\overline{a}_{B_4^+}(Dh,t) - a(Dh,x,t)) \cdot D(h-k) \ dxdt}_{I}. \quad (5.34)$$

From the monotonicity condition (2.2) we induce that for any $\tau_1 > 0$,

$$\fint_{K_4} |D(h-k)|^p \ dxdt$$
$$\leq c\tau_1 + c(\tau_1)\fint_{K_4} \left(\overline{a}_{B_4^+}(Dh,t) - \overline{a}_{B_4^+}(Dk,t)\right) \cdot D(h-k) \ dxdt. \quad (5.35)$$

We now return to I in (5.34). Recall (5.11), the definition of β, and use Young's inequality with $\tau_2 > 0$ to discover that

$$\begin{aligned} I &\leq \fint_{K_4} \beta(a, Q_4^+)|Dh|^{p-1}|D(h-k)| \ dxdt \\ &\leq c(\tau_2)\fint_{K_4} \beta^{\frac{p}{p-1}}|Dh|^p \ dxdt + \tau_2\fint_{K_4} |D(h-k)|^p \ dxdt. \end{aligned}$$

By Lemma 5.2.1, the higher integrability result for (5.20), Hölder's inequality, Remark 5.1.1, (5.15) and (5.18), we find that

$$\begin{aligned}
&\fint_{K_4} \beta(a, Q_4^+)^{\frac{p}{p-1}} |Dh|^p \, dxdt \\
&\leq \left(\fint_{Q_4^+} \beta^{\frac{p(p+\kappa)}{(p-1)\kappa}} \, dxdt + \frac{1}{|Q_4^+|} \int_{K_4 \setminus Q_4^+} \beta^{\frac{p(p+\kappa)}{(p-1)\kappa}} \, dxdt \right)^{\frac{\kappa}{p+\kappa}} \left(\int_{K_4} |Dh|^{p+\kappa} \, dxdt \right)^{\frac{p}{p+\kappa}} \\
&\leq c \left(\fint_{Q_4^+} \beta^{\frac{p(p+\kappa)}{(p-1)\kappa}} \, dxdt + \frac{|K_4 \setminus Q_4^+|}{|Q_4^+|} (2\Lambda)^{\frac{p(p+\kappa)}{(p-1)\kappa}} \right)^{\frac{\kappa}{p+\kappa}} \\
&\leq c \delta^{\frac{\kappa}{p+\kappa}}.
\end{aligned}$$

Consequently, we have

$$I \leq c(\tau_2) \delta^{\frac{\kappa}{p+\kappa}} + \tau_2 \fint_{K_4} |D(h-k)|^p \, dxdt. \tag{5.36}$$

Combining (5.34)-(5.36), we discover that

$$\fint_{K_4} |D(h-k)|^p \, dxdt \leq c\tau_1 + c(\tau_1, \tau_2) \delta^{\frac{\kappa}{p+\kappa}} + \tau_2 c(\tau_1) \fint_{K_4} |D(h-k)|^p \, dxdt.$$

We then take first τ_1 and then τ_2 so small in order to have that for some $\sigma_3 = \sigma_3(\mu, \Lambda, n, p) > 0$,

$$\fint_{K_4} |D(h-k)|^p \, dxdt \leq c \delta^{\sigma_3}. \tag{5.37}$$

This estimate (5.37) and (5.27) imply

$$\fint_{K_4} |Dk|^p \, dxdt \leq c.$$

Therefore, we can apply Lemma 5.2.2 and Lemma 5.2.3 to see that one can find a small constant $\delta > 0$ and a weak solution v of (5.21) such that

$$\|Dv\|_{L^\infty(K_3)} \leq n_1 \quad \text{and} \quad \fint_{K_2} |D(k-v)|^p \, dxdt \leq \frac{1}{2} \epsilon^p, \tag{5.38}$$

for some constant $n_1 = n_1(\mu, \Lambda, n, p) \geq 1$. Next we recall (5.25) and (5.28) to further select a smaller $\delta > 0$ so that

$$\fint_{K_2} |D(u-k)|^p \, dxdt \leq \frac{1}{2} \epsilon^p. \tag{5.39}$$

The conclusion follows from (5.38) and (5.39). This completes the proof. □

5.3 Global gradient estimates in Orlicz spaces

In this section we obtain a global gradient estimate for the problem (5.5) under a small BMO assumption on the $\frac{a(\xi,x,t)}{|\xi|^{p-1}}$ with respect to x-variables and a sufficient flatness condition on $\partial\Omega$, addressed earlier in Section 5.1. To this end, we first set

$$E(|Du|,\lambda)=\{(y,s)\in\Omega_T : |Du|>\lambda\},\ E(\Psi,\lambda)=\{(y,s)\in\Omega_T : \Psi>\lambda\} \quad (5.40)$$

for $\lambda > 0$ and

$$\lambda_0^{\frac{p}{d}} = \fint_{\Omega_T} |Du|^p \, dxdt + \frac{1}{\delta^p}\fint_{\Omega_T} \Psi^p \, dxdt + 1, \quad (5.41)$$

where u is the weak solution to (5.5), $\Psi = |\partial_t\psi|^{\frac{1}{p-1}} + |D\psi| + |F|$, the exponent d is defined as in (5.14) and $0 < \delta < \frac{1}{8}$ will be chosen later in a universal way that it depends only on $\mu,\ \Lambda,\ n,\ p,\ \phi$. As usual, we consider intrinsic parabolic cylinders

$$Q_r^\lambda(y,s) = \begin{cases} Q_{(r,r^2\lambda^{2-p})}(y,s) & \text{if} \quad p\geq 2, \\ Q_{\left(r\lambda^{\frac{p-2}{2}},r^2\right)}(y,s) & \text{if} \quad p<2 \end{cases} \quad \text{and } K_r^\lambda(y,s) = Q_r^\lambda(y,s)\cap\Omega_T,$$

where $\lambda \geq 1$, $r > 0$ and $(y,s)\in\Omega_T$.

Hereafter we fix any $R > 0$. Assume that $\partial\Omega$ is (δ, R)-Reifenberg flat and a is weakly (δ, R)-vanishing. Then if $r > \frac{R}{32}$, we have

$$\begin{aligned}
&\fint_{K_r^\lambda(y,s)} |Du|^p \, dxdt + \frac{1}{\delta^p}\fint_{K_r^\lambda(y,s)} \Psi^p \, dxdt \\
&\leq \frac{|Q_r^\lambda(y,s)|}{|K_r^\lambda(y,s)|}\frac{|\Omega_T|}{|Q_r^\lambda(y,s)|}\left(\fint_{\Omega_T} |Du|^p \, dxdt + \frac{1}{\delta^p}\fint_{\Omega_T} \Psi^p \, dxdt\right) \\
&< 2\left(\frac{2}{1-2\delta}\right)^n \frac{|\Omega_T|}{|Q_1|}\left(\frac{32}{R}\right)^{n+2} \lambda^{p-\frac{p}{d}}\lambda_0^{\frac{p}{d}},\ \text{from the } \delta\text{-flatness condition.} \\
&\leq 2\left(\frac{8}{3}\right)^n \frac{|\Omega_T|}{|Q_1|}\left(\frac{32}{R}\right)^{n+2} \lambda^{p-\frac{p}{d}}\lambda_0^{\frac{p}{d}} \\
&\leq \lambda^p,
\end{aligned} \quad (5.42)$$

by taking $\lambda \geq \lambda_1$ for

$$\lambda_1 = \left(2 \left(\frac{8}{3} \right)^n \frac{|\Omega_T|}{|Q_1|} \left(\frac{32}{R} \right)^{n+2} + 1 \right)^{\frac{d}{p}} \lambda_0. \tag{5.43}$$

In addition, note that for any $(y, s) \in E(|Du|, \lambda)$,

$$\lim_{r \to 0} \left(\fint_{K_r^\lambda(y,s)} |Du|^p \, dxdt + \frac{1}{\delta^p} \fint_{K_r^\lambda(y,s)} \Psi^p \, dxdt \right) \geq |Du(y,s)|^p > \lambda^p. \tag{5.44}$$

Using (5.40)-(5.44) and the Vitali covering Lemma, we discover that for any $\lambda \geq \lambda_1$, one can select a collection of disjoint intrinsic parabolic cylinders

$$\left\{ K_{r_i}^\lambda(y_i, s_i) : (y_i, s_i) \in E(|Du|, \lambda), \ 0 < r_i \leq \frac{R}{32} \right\}_{i \geq 1} \tag{5.45}$$

such that

$$\fint_{K_{r_i}^\lambda(y_i,s_i)} |Du|^p \, dxdt + \frac{1}{\delta^p} \fint_{K_{r_i}^\lambda(y_i,s_i)} \Psi^p \, dxdt = \lambda^p, \tag{5.46}$$

$$\fint_{K_r^\lambda(y_i,s_i)} |Du|^p \, dxdt + \frac{1}{\delta^p} \fint_{K_r^\lambda(y_i,s_i)} \Psi^p \, dxdt < \lambda^p \text{ for all } r > r_i \tag{5.47}$$

and

$$E(|Du|, \lambda) \subset \bigcup_{i=1}^{\infty} K_{2^3 r_i}^\lambda(y_i, s_i) \cup \text{ negligible set.} \tag{5.48}$$

We denote

$$Q_i^j = Q_{2^j r_i}^\lambda(y_i, s_i) \text{ and } K_i^j = K_{2^j r_i}^\lambda(y_i, s_i) \tag{5.49}$$

for $j = 0, 1, 2, 3, 4, 5$. Then we know from (5.46) that

$$\fint_{K_i^j} |Du|^p \, dxdt + \frac{1}{\delta^p} \fint_{K_i^j} \Psi^p \, dxdt < \lambda^p, \ j = 1, 2, 3, 4, 5. \tag{5.50}$$

It can be easily seen from (5.47) that

$$|K_i^0| \leq \frac{2}{\lambda^p} \left(\int_{K_i^0 \cap E\left(|Du|, \frac{\lambda}{4^{\frac{1}{p}}}\right)} |Du|^p \, dxdt + \frac{1}{\delta^p} \int_{K_i^0 \cap E\left(\Psi, \frac{\delta\lambda}{4^{\frac{1}{p}}}\right)} \Psi^p \, dxdt \right). \tag{5.51}$$

We next proceed the comparison estimates on the intrinsic parabolic cylinders obtained by the above Vitali covering argument. Define the rescaled functions defined on K_8 as follows:

$$\begin{cases} \tilde{a}_i(\xi,x,t) = \frac{a(\lambda\xi,Y_i,S_i)}{\lambda^{p-1}}, & \tilde{F}_i(x,t) = \frac{F(Y_i,S_i)}{\lambda}, \\ \tilde{u}_i(x,t) = \frac{u(Y_i,S_i)}{R_i}, & \tilde{\psi}_i(x,t) = \frac{\psi(Y_i,S_i)}{R_i}, \end{cases}$$

where if $p \geq 2$,

$$\begin{cases} Y_i = y_i + (2^2 r_i)x, \\ S_i = s_i + \lambda^{2-p}(2^2 r_i)^2 t, \\ R_i = \lambda 2^2 r_i, \end{cases}$$

and on the other hand, if $\frac{2n}{n+2} < p < 2$,

$$\begin{cases} Y_i = y_i + \lambda^{\frac{p-2}{2}}(2^2 r_i)x, \\ S_i = s_i + (2^2 r_i)^2 t, \\ R_i = \lambda^{\frac{p}{2}}(2^2 r_i). \end{cases}$$

We recall (5.50) for $j = 5$ and note $0 < r_i \leq \frac{R}{32}$ to observe that we are under the hypotheses (5.15)-(5.18). We then apply Lemma 5.2.4 to find a function $\tilde{v}_i$ defined in K_4 satisfying

$$\|D\tilde{v}_i\|_{L^\infty(K_2)} \leq n_1 \text{ and } \fint_{K_2} |D(\tilde{u} - \tilde{v}_i)|^p \, dxdt \leq \epsilon^p.$$

Taking rescaling back, we have a function v_i defined in K_i^4 such that

$$\|Dv_i\|_{L^\infty(K_i^3)} \leq n_1\lambda \text{ and } \fint_{K_i^3} |D(u - v_i)|^p \, dxdt \leq \epsilon^p \lambda^p. \tag{5.52}$$

We now ready to prove the main result, Theorem 5.1.1.

Proof. Let $N = \max\{n_1, m_0\}$ where m_0 is the constant as in Lemma 5.1.2. To derive the desired regularity estimate (5.13), we set for $k = 1, 2, 3, \cdots$,

$$A_k = E\left(|Du|, (2N)^k 4^{\frac{k}{p}} \lambda_1\right) \setminus E\left(|Du|, (2N)^{k+1} 4^{\frac{k+1}{p}} \lambda_1\right). \tag{5.53}$$

We have

$$\begin{aligned}
\int_{\Omega_T} \phi(|Du|^p)\, dxdt &= \int_{\Omega_T \setminus E\left(|Du|,(2N)4^{\frac{1}{p}}\lambda_1\right)} \phi(|Du|^p) dxdt \\
&\quad + \sum_{k=1}^{\infty} \int_{A_k} \frac{\phi(|Du|^p)}{|Du|^p} |Du|^p dxdt \\
&\le \phi\left(4(2N)^p \lambda_1^p\right) |\Omega_T| \\
&\quad + \underbrace{\sum_{k=1}^{\infty} \frac{\phi\left((2N)^{p(k+1)} 4^{k+1} \lambda_1^p\right)}{(2N)^{pk} 4^k \lambda_1^p} \int_{E\left(|Du|,(2N)^k 4^{\frac{k}{p}}\lambda_1\right)} |Du|^p dxdt}_{I} \\
&\overset{(5.6),(5.41),(5.43)}{\le} c_1 \phi\left(\lambda_0^p\right) + I,
\end{aligned} \tag{5.54}$$

for some positive constant $c_1 = c_1(\mu, \Lambda, n, p, \phi, R, |\Omega_T|)$. We now recall (5.48)-(5.49) to find that for every $\lambda \ge \lambda_1$,

$$\int_{E(|Du|,2N\lambda)} |Du|^p dxdt \le \sum_{i=1}^{\infty} \int_{K_i^3 \cap E(|Du|,2N\lambda)} |Du|^p dxdt. \tag{5.55}$$

If $(y,s) \in K_i^3 \cap E\left(|Du|, 2N\lambda\right)$, then we observe from (5.52)

$$\begin{aligned}
|Du(y,s)| &\le |D(u-v_i)(y,s)| + |Dv_i(y,s)| \\
&\le |D(u-v_i)(y,s)| + N\lambda \\
&\le |D(u-v_i)(y,s)| + \frac{1}{2}|Du(y,s)|,
\end{aligned}$$

which leads us to conclude that for every $(y,s) \in K_i^3 \cap E\left(|Du|, 2N\lambda\right)$,

$$|Du| \le 2|D(u-v_i)|. \tag{5.56}$$

We then calculate as follows.

$$
\begin{aligned}
&\int_{E(|Du|,2N\lambda)} |Du|^p \, dxdt \\
&\overset{(5.55),(5.56)}{\leq} 2^p \sum_{i=1}^{\infty} \int_{K_i^3} |D(u-v_i)|^p \, dxdt \\
&\overset{(5.52)}{\leq} 2^p \epsilon^p \lambda^p \sum_{i=1}^{\infty} |K_i^3| \\
&\leq 2^{p+3n+6} \epsilon^p \lambda^p \sum_{i=1}^{\infty} \frac{|Q_i^0|}{|K_i^0|} |K_i^0| \\
&\leq 2^{p+3n+7} \left(\frac{8}{3}\right)^n \epsilon^p \lambda^p \sum_{i=1}^{\infty} |K_i^0| \\
&\overset{(5.51)}{\leq} c_2 \epsilon^p \sum_{i=1}^{\infty} \left(\int_{K_i^0 \cap E\left(|Du|, \frac{\lambda}{4^{\frac{1}{p}}}\right)} |Du|^p \, dxdt + \frac{1}{\delta^p} \int_{K_i^0 \cap E\left(\Psi, \frac{\delta\lambda}{4^{\frac{1}{p}}}\right)} \Psi^p \, dxdt \right) \\
&\overset{(5.45)}{\leq} c_2 \epsilon^p \int_{E\left(|Du|, \frac{\lambda}{4^{\frac{1}{p}}}\right)} |Du|^p \, dxdt + \frac{c_2 \epsilon^p}{\delta^p} \int_{E\left(\Psi, \frac{\delta\lambda}{4^{\frac{1}{p}}}\right)} \Psi^p \, dxdt,
\end{aligned}
$$

for every $\lambda \geq \lambda_1$ and for some positive constant $c_2 = c_2(n,p)$.

That is, for any $\lambda \geq \lambda_1$,

$$
\begin{aligned}
&\int_{E(|Du|,2N\lambda)} |Du|^p \, dxdt \\
&\leq c_2 \epsilon^p \int_{E\left(|Du|, \frac{\lambda}{4^{\frac{1}{p}}}\right)} |Du|^p \, dxdt + \frac{c_2 \epsilon^p}{\delta^p} \int_{E\left(\Psi, \frac{\delta\lambda}{4^{\frac{1}{p}}}\right)} \Psi^p \, dxdt.
\end{aligned}
\tag{5.57}
$$

From $(2N)^{k-1} 4^{\frac{k}{p}} \lambda_1 \geq \lambda_1$ for $k \geq 1$ we iterate (2.37) to discover that for any $k \geq 1$,

$$
\begin{aligned}
\int_{E\left(|Du|, (2N)^k 4^{\frac{k}{p}} \lambda_1\right)} |Du|^p \, dxdt &\leq (c_2 \epsilon^p)^k \int_{E(|Du|,\lambda_1)} |Du|^p \, dxdt \\
&\quad + \sum_{i=1}^{k} \frac{(c_2 \epsilon^p)^i}{\delta^p} \int_{E\left(\Psi, \delta (2N)^{k-i} 4^{\frac{k-i}{p}} \lambda_1\right)} \Psi^p dxdt.
\end{aligned}
\tag{5.58}
$$

We now return to (5.54) and estimate I as follows:

$$\begin{aligned} I \quad & \overset{(5.58)}{\leq} \quad \sum_{k=1}^{\infty} \frac{\phi((2N)^{p(k+1)}4^{k+1}\lambda_1^p)}{(2N)^{pk}4^k\lambda_1^p}(c_2\epsilon^p)^k \int_{\Omega_T} |Du|^p \, dxdt \\ & \quad + \sum_{k=1}^{\infty} \frac{\phi((2N)^{p(k+1)}4^{k+1}\lambda_1^p)}{(2N)^{pk}4^k\lambda_1^p} \sum_{i=1}^{k} \frac{(c_2\epsilon^p)^i}{\delta^p} \int_{E\left(\Psi,\delta(2N)^{k-i}4^{\frac{k-i}{p}}\lambda_1\right)} \Psi^p dxdt \\ & \overset{(5.6),(5.41),(5.43)}{\leq} c_3\lambda_0^{\frac{p(1-d)}{d}} \phi(\lambda_0^p) \sum_{k=1}^{\infty} (c_4\epsilon^p)^k \\ & \quad + c_5 \sum_{i=1}^{\infty} (c_6\epsilon^p)^i \sum_{j=0}^{\infty} \frac{\phi(\delta^p\lambda_1^p[4(2N)^p]^j)}{\delta^p\lambda_1^p[4(2N)^p]^j} \int_{E\left(\Psi,\delta\lambda_1[4^{\frac{1}{p}}(2N)]^j\right)} \Psi^p \, dxdt, \end{aligned}$$

for some constants $c_4, \ c_6$ depending only on $\mu, \ \Lambda, \ n, \ p, \ \phi$ and for some positive constants $c_3 = c_3(\mu, \Lambda, n, p, \phi, R, |\Omega_T|)$ and $c_5 = c_5(\mu, \Lambda, n, p, \phi, \delta)$. Applying Lemma 5.1.2 to $\nu = \delta^p\lambda_1^p$ and $m = 4(2N)^p$ we find that

$$I \leq c_3\lambda_0^{\frac{p(1-d)}{d}} \phi(\lambda_0^p) \sum_{k=1}^{\infty} (c_4\epsilon^p)^k + c_7 \sum_{i=1}^{\infty} (c_6\epsilon^p)^i \int_{\Omega_T} \phi(\Psi^p) \, dxdt,$$

for some positive constant $c_7 = c_7(\mu, \Lambda, n, p, \phi, \delta)$. We then note that $\lambda_0^{\frac{p(1-d)}{d}} \leq 1$ and take a sufficiently small ϵ in order to conclude that

$$I \leq c_8 \left(\phi(\lambda_0^p) + \int_{\Omega_T} \phi(\Psi^p) \, dxdt \right) \tag{5.59}$$

for some constant c_8 depending only on $\mu, \ \Lambda, \ n, \ p, \ \phi, \ R, \ |\Omega_T|$.

Combining (5.54) and (5.59) and using Lemma 5.1.1, we finally derive that

$$\begin{aligned} \int_{\Omega_T} \phi(|Du|^p) \, dxdt \quad & \leq \quad c \left\{ \phi(\lambda_0^p) + \int_{\Omega_T} \phi(\Psi^p) \, dxdt \right\} \\ & \leq \quad c \left\{ \phi\left(\left[\fint_{\Omega_T} \Psi^p \, dxdt + 1 \right]^d \right) + \int_{\Omega_T} \phi(\Psi^p) \, dxdt \right\}. \end{aligned}$$

□

Specially, if we put $\phi(t) = t^q$, $1 < q < \infty$, then we deduce the following Calderón-Zygmund type estimate in the setting of Lebesgue space.

Corollary 5.3.1. *Let* $\frac{2n}{n+2} < p < \infty$ *and* $1 < q < \infty$*. Assume that*

$$|\partial_t \psi|^{\frac{p}{p-1}},\ |D\psi|^p,\ |F|^p \in L^q(\Omega_T).$$

Let $0 < R < 1$*. Then there exists a small constant* $\delta = \delta(\mu, \Lambda, n, p, q) > 0$ *such that if* $a(\xi, x, t)$ *is weakly* (δ, R)*-vanishing and* Ω *is* (δ, R)*-Reifenberg flat for such* δ*, then we have* $|Du|^p \in L^q(\Omega_T)$ *with the estimate*

$$\left(\int_{\Omega_T} |Du|^{pq} dxdt\right)^{\frac{1}{q}} \leq c\left[1 + \left\{\int_{\Omega_T}\left(|\partial_t \psi|^{\frac{pq}{p-1}} + |D\psi|^{pq} + |F|^{pq}\right) dxdt\right\}^{\frac{1}{q}}\right]^d$$

for some positive constant $c = c(\mu, \Lambda, n, p, q, R, |\Omega_T|)$*, where* d *is the scaling deficit in* (5.14).

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국문초록

우리는 비정칙 장애물을 가진, p-라플라시안 형태의 불연속 비선형 계수함수를 포함하는 비제차 타원형 및 포물형 문제를 매끄럽지 않은 경계를 가진 영역에서 다룬다. 이 논문의 목적은 비선형 계수 함수의 BMO semi-norm이 충분히 작을 때, 라이펜버그 센스로 편평한 경계를 가진 영역 하에서 약해의 그래디언트가 장애물 함수의 그래디언트의 적분 가능성과 비제차 항의 적분 가능성 만큼의 정칙성을 가진다는 것을 보임으로써, 대역적 칼데론-지그먼드 가늠을 이끌어내는 것이다.

주요어휘: 비정칙 장애물, 칼데론-지그먼드 가늠, p-라플라시안, BMO 공간, 라이펜버그 영역
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