박사 학위논문

On the Gross-Prasad Conjecture for some non-tempered case and a uniqueness theorem for the Extended Selberg class

(온화하지 않은 경우의 그로쓰-프라사드 예상과 확장된 셀버그 류에 있는 함수들의 유일성에 대하여)

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On the Gross-Prasad Conjecture for some non-tempered case and a uniqueness theorem for the Extended Selberg class

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
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Abstract

This thesis is composed of two parts. In the first part, we consider the Bessel period of two automorphic representations of $(U(3), U(2))$ involving a non-tempered one. For a pair of tempered representations of codimension 1 unitary groups, Gross and Prasad conjectured that the non-vanishing of their period would be equivalent to that of central critical L-value of their product L-function. Thereafter, Neal Harris has formualted their conjecture in a more refined way following Ichino-Ikeda work concerning orthogonal group. We investigate Neal Harris's conjecture for the non-tempered case. In the non-tempered case, the conjecture is false because the critical L-value may have a pole at $s = \frac{1}{2}$ and the local period may diverge. However, if we adopt the regularised local period, there is also an analogous formula and we suggest it for some specfic pair in the endoscopic A-packet of $(U(3), U(2))$.

In the second part, we study the *Selberg class*. The Selberg class is an axiomatically defined class of L-functions which are of arithmetic interests. We prove a uniqueness theorem for functions in the Extended Selberg Class which states that for every $c \neq 0$, the functions $L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^k}$ in the class having the positive degree are completely determined by $a(1)$ and $L^{-1}(c)$.

Key words: automorphic forms, unitary groups, theta correspondence, L-functions, period, non-tempered represenation,Extended Selberg class, Selberg class, zeros of L-functions
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국문초록

이 논문은 크게 두가지 주제로 구성되어 있다. 첫번째 주제는 비온화 표현이 들어간 $(U(3), U(2))$ 의 보형 형식 표현 쌍의 베셀 주기에 대한 것이다. 그로쓰와 프라사드는 상대적 차원이 1인 온화한 보형형식 표현 쌍이 있다면, 그들의 베쎌 주기함수가 0함수가 되는 것과 그들의 곱셈 L-함수의 $\frac{1}{2}$ 에서의 값이 0이 되는 것과 같은 운명을 가질것임을 추측하였다. 후에 이치노와 이케다는 직교군에 대해서 그 둘의 관련성을 정밀한 형태로 발전시켰는데, 닐 해리스는 유니터리 그룹에 대해서 이치노, 이케다가 한 것과 비슷한 예상을 만들었다. 우리는 닐 해리스의 예상을 온화한 표현쌍이 아닌 비온화 표현이 하나가 포함된 쌍에 대해서도 성립하는지 연구를 했는데, 이러한 쌍에 대해서는 닐 해리스의 예상이 성립하지 않음을 보였다. 뿐만 아니라 국소 주기 대신에 정규화된 국소기저를 도입하여 비온화 표현이 들어간 표현쌍에 대해서도 닐 해리스의 예상에 나오는 것과 같지는 않지만 비슷한 공식이 있음을 발견하였고, 그 공식을 제시하였다.

논문의 두번째 주제는 셀버그 류에 관한 것인데, 셀버그 류는 리만 제타함수나 디리끌레 L-함수 같이 산술적인 정보를 포함하고 있을 것이라고 생각되는 L-함수들의 모임을 공리적으로 정의한 것이다. 우리는 0이 아닌 임의의 복소수 c 에 대해서 셀버그 류에 있는 초기항이 같은 두 함수가 c 에 대한 역 이미지가 서로 같으면 두 L-함수는 서로 같은 함수가 됨을 보였다.

주요어휘 : 보형형식, 유니터리군, 쎄타 대응, L-함수, 주기, 비온화 표현, 확장된 셀버그 류, 셀버그 류, L-함수의 영점
학번 : 2007-20289

Contents

Introduction

The notion of L-functions is a central theme in number theory in that they encode many important arithmeric information. For example, the proof of two celebrated theorems in number theory, Dirichlet's Theorem on primes in arithmetic progression and the Prime Number Theorem all have to do with special L-values.

In the first part of this thesis, we briefly introduce some conjecture which relates special L-values and period. In 1992, Gross and Prasad gave a facinating conjecture(we call it GP conejcture) which connects the central critical L-value and period. The GP conjecture is reminiscent of a Gross-Zagier formula if we think period play a similar role of height pairing in the formula. Although Gross and Prasad conejctured it only for orthogonal group, a similar conjecture concerning unitary group exists and their conjecture have been refined by Ichino, Ikeda and Neal Harris.
We consider the refined GP conjecture for unitary group. In formulating the conjecture, there is some assumption such that a given pair of representations should be tempered. So, we investigated how it varies for non-tempered representations and tested it for $n = 2$ case taking π_3 a theta lifting from $U(1)$, which is the most well-known non-tempered representation.

In the second part, we study the Selberg class which axiomatically generalised the class of classical L-functions. Selberg introduced a class of meromorphic functions satisfying five axioms. We consider its uniqueness problem when the inverse image is specified. J.Steuding([38]) was a pioneer who first paved the way for this problem and then Li and Ki made a subtantial progress by removing more or less complicated assumstions in J.Steuding's theorem. Recently, we could remove the same functional equation condition appearing in Ki's theorem([22]) and verified that this theorem is optimal in the sense that the other conditions are indispensible in formulating the theorem.

Part I

The analogue of global Gross-Prasad conjecture for $(U(3), U(2))$ involving a non-tempered representation

Chapter 1

A preview of the first part

The Gross-Prasad conjecture is an outgrowth of the study of the restriction problem in the automorphic representation of classical groups and it has generated much interest in recent years. In this chapter, we first introduce its refined version, so-called the refined Gross-Prasad Conjecture formualted in [15] and then relate it to our result in [16].

Let E/F be a quadratic extension of number fields and $\mathbb{A}_F$, $\mathbb{A}_E$ are their adele rings respectively. Let $V_n \subset V_{n+1}$ be hermitian spaces of dimensions n and $n+1$ over E, respectively. Consider the unitary groups $U(V_n) \subset U(V_{n+1})$ defined over F. Write $G_i := U(V_i)$. Let π_n and π_{n+1} be irreducible tempered cuspidal automorphic representations of $G_n(\mathbb{A}_F)$ and $G_{n+1}(\mathbb{A}_F)$ respectively, and we fix isomorphisms $\pi_n \cong \otimes_v \pi_{n,v}$ and $\pi_{n+1} \cong \otimes_v \pi_{n+1,v}$. We suppose that $\mathrm{Hom}_{G_n(k_v)}(\pi_{n+1,v} \otimes \pi_{n,v}, \mathbb{C}) \neq 0$ for every place v of F.

We consider the following $G_n(\mathbb{A}_F) \times G_n(\mathbb{A}_F)$-invariant functional

$$\mathscr{P} : (V_{\pi_{n+1}} \boxtimes \bar{V}_{\pi_{n+1}}) \otimes (V_{\pi_n} \boxtimes \bar{V}_{\pi_n}) \to \mathbb{C}$$

defined by

$$\mathscr{P}(\phi_1, \phi_2; f_1, f_2) := \left(\int_{[G_n]} \phi_1(g) f_1(g) dg \right) \cdot \left(\int_{[G_n]} \overline{\phi_2(g) f_2(g)} dg \right) \tag{1.0.1}$$

for $\phi_i \in V_{\pi_{n+1}}, f_i \in V_{\pi_n}$ and $[G_n] = G_n(F) \quad G_n(\mathbb{A}_F)$. If $\phi_1 = \phi_2 = \phi$ and $f_1 = f_2 = f$, we simply write $\mathscr{P}(\phi, f) := \mathscr{P}(\phi_1, \phi_2; f_1, f_2)$ and we call $\mathscr{P}$ the global period.

On the other hand, there is another $G_n(\mathbb{A}_F) \times G_n(\mathbb{A}_F)$-invariant functional constructed from the local integral of matrix coefficients. To define matrix coefficients, for each place v of F, let F_v be its completion of F at v and denote $G_{i,v} := G_i(F_v)$. Fix the local pairings

$$\mathscr{B}_{\pi_{i,v}} : \pi_{i,v} \otimes \bar{\pi}_{i,v} \to \mathbb{C}$$

so that

$$\mathscr{B}_{\pi_i} = \prod_v \mathscr{B}_{\pi_{i,v}}$$

where $\mathscr{B}_{\pi_i}$ is the Petersson pairing

$$\mathscr{B}_{\pi_i}(f_1,f_2) \quad := \quad \int_{[G_i]} f_1(g_i)\overline{f_2(g_i)}dg_i$$

and the dg_i is Tamagawa measures on $G_i(\mathbb{A}_F)$. For each place v, we define a $G_{n,v} \times G_{n,v}$ invariant functional

$$\mathscr{P}_v^\natural : (\pi_{n+1,v} \boxtimes \bar{\pi}_{n+1,v}) \otimes (\pi_{n,v} \boxtimes \bar{\pi}_{n,v})$$

by $\mathscr{P}_v^\natural(\phi_{1,v}, \phi_{2,v};f_{1,v},f_{2,v}) :=$

$$\int_{G_{n,v}} \mathscr{B}_{\pi_{n+1,v}}(\pi_{n+1,v}(g_v)\phi_{1,v}, \phi_{2,v})\mathscr{B}_{\pi_{n,v}}(\pi_{n,v}(g_v)f_{1,v},f_{2,v})dg_v.$$

(Here, the dg_v are local Haar measures of $G_{n,v}$ such that $\prod_v dg_v = dg$.)

Write $\mathscr{P}_v^\natural(\phi_v, \phi_v;f_v,f_v) =: \mathscr{P}_v^\natural(\phi_v,f_v)$ and we set

$$\begin{aligned} \Delta_{G_i} &:= L(M_i^\vee(1), 0) \\ \Delta_{G_{i,v}} &:= L_v(M_i^\vee(1), 0) \end{aligned}$$

where $M_i^\vee(1)$ is the twisted dual of the motive M_i associated to G_i by Gross in [12]. It is known in [15, Prop. 2.1] that $\mathscr{P}_v^\natural$ converges absolutely if the $\pi_{i,v}$ is tempered. Furthermore, it is also known that for unramified data ϕ_v,f_v satisfying conditions $(1) - (7)$ in [15, p.6], we have

$$\mathscr{P}_v^\natural(\phi_v,f_v) = \Delta_{G_{n+1,v}} \frac{L_{E_v}(1/2, BC(\pi_{n,v}) \boxtimes BC(\pi_{n+1,v}))}{L_v(1, \pi_{n,v}, \mathrm{Ad})L_v(1, \pi_{n+1,v}, \mathrm{Ad})}$$

(Here, $BC(\pi_i)$ is the quadratic base-change of π_i to a representation of $GL_i(\mathbb{A}_E)$)

From this observation, we can normailze $\mathscr{P}_v^\natural$ as

$$\mathscr{P}_v := \Delta_{G_{n+1,v}}^{-1} \frac{L_v(1, \pi_{n,v}, \mathrm{Ad})L_v(1, \pi_{n+1,v}, \mathrm{Ad})}{L_{E_v}(1/2, BC(\pi_{n,v}) \boxtimes BC(\pi_{n+1,v}))} \mathscr{P}_v^\natural$$

and call this the local period.

Then

$$\prod_v \mathscr{P}_v : (V_{\pi_{n+1}} \boxtimes \bar{V}_{\pi_{n+1}}) \otimes (V_{\pi_n} \boxtimes \bar{V}_{\pi_n}) \to \mathbb{C}.$$

is also another $G_n(\mathbb{A}_F) \times G_n(\mathbb{A}_F)$-invariant functional.

The Refined Gross-Prasad Conjecture predicts that these two global $G_n(\mathbb{A}_F) \times G_n(\mathbb{A}_F)$-functionals $\mathscr{P}$ and $\prod_v \mathscr{P}_v$ differs by only a certain constant, that is the central L-value of the product L-function. The precise conjecture is as follows :

Conjecture 1.0.1 (Refined Gross-Prasad Conjecture for Unitary groups)**.**

$$\mathcal{P}(\phi,f) = \frac{\Delta_{G_{n+1}}}{2^{\beta}}\frac{L_E(1/2, BC(\pi_n)\boxtimes BC(\pi_{n+1}))}{L_F(1,\pi_n,\mathrm{Ad})L_F(1,\pi_{n+1},\mathrm{Ad})}\prod_v \mathcal{P}_v(\phi_v,f_v).$$

(Here ψ_i is the conjectural L-parameter for π_i and β is an integer such that $2^{\beta} = |S_{\psi_{n+1}}|\cdot|S_{\psi_n}|$ and $S_{\psi_i} := \mathrm{Cent}_{\widehat{G_i}}(\mathrm{Im}(\psi_i))$ is the associated component group.)

Beuzart Plessis has shown that the local period $\mathcal{P}_v$ is nonvanishing if and only if $\mathrm{Hom}_{G_n(k_v)}(\pi_{n+1,v}\otimes\pi_{n,v},\mathbb{C})\neq 0$. (Theorem 14.3.1 in [31]). Thus the above refined Gross-Prasad conjecture contains the following original Gross-Prasad conjecture,

Conjecture 1.0.2. *If $\mathrm{Hom}_{G_n(k_v)}(\pi_{n+1,v}\otimes\pi_{n,v},\mathbb{C})\neq 0$ for all all places v of F, then the global period $\mathcal{P}$ is nonvanishing if and only if $L_E(1/2, BC(\pi_n)\boxtimes BC(\pi_{n+1}))\neq 0$.*

In [15], N.Harris proved the above conjecture unconditionally for $n = 1$ using Waldspurger formula, and conditionally for $n = 2$ assuming π_3 is a Θ-lift of a representation on $U(2)$. Recently, Wei Zhang proved for general case using relative trace formula under some local conditions.[43, 44]

Our goal is to provide an analog of this conjecture for $n = 2$ and π_3 is a theta lift of $U(1)$. Note that in this case, π_3 is no longer tempered and so the above local periods may diverge. So we first regularize the local period using the function appearing in the doubling method. Once this is done, we can define a regularized local period and this enable us to establish the following formula which can be seen as an analogue of Refined Gross-Prasad conjecture.

Theorem 1.0.3. *Let F be a totally real field and E a totally imaginary quadratic extension of F such that all the finite places of F dividing 2 do not split in E. The unitary groups we are considering here are all associated to this extension. Let σ be an automorphic characters of $U(1)(\mathbb{A}_F)$ and $\pi_3 = \Theta(\bar{\sigma}), \pi_2 = \Theta(\bar{\mathbb{I}})$ be irreducible tempered cuspidal automorphic representations of $U(2)(\mathbb{A}_F)$ which comes from a theta lift of σ and trivial character $\mathbb{I}$, respectively. We assume that these two theta lifts are nonvanishing and cuspidal. Then for $f_3 = \otimes f_{3,v}\in\pi_3$ and $f_2 = \otimes f_{2,v}\in\pi_2$,*

$$\mathcal{P}(f_3,f_2) = c_{\pi_3,\pi_2,\gamma}\cdot\prod_v \mathcal{P}_v(f_{3,v},f_{2,v})$$

where

$$c_{\pi_3,\pi_2,\gamma} = -\frac{1}{2^3}\cdot\frac{L(3,\chi)}{L^2(1,\chi)}\cdot\frac{L_E(\frac{1}{2}, BC(\omega_{\pi_3}^{-1}\cdot\omega_{\pi_2}^{-1})\otimes\gamma)\cdot Res_{s=0}(L_E(s, BC(\pi_2)\otimes\gamma))}{L_E(\frac{3}{2}, BC(\omega_{\pi_3}^{-1})\otimes\gamma^3)}.$$

(here γ is a character of $\mathbb{A}_E^{\times}/E^{\times}$ such that $\gamma|_{\mathbb{A}_F^{\times}} = \chi_{E/F}$ and for $i = 1, 2$, ω_{π_i} is the central character of π_i. The regularized local periods $\mathcal{P}_v$'s are defined by

$$\mathcal{P}_v(f_{3,v},f_{2,v}) := c_v\cdot\lim_{s\to 0}\frac{\zeta_v(2s)}{L_v(s, BC(\pi_{2,v})\otimes\gamma_v)}\cdot\int_{U(2)_v}\mathcal{B}_{\pi_{3,v}}^{f_{3,v}}(g_v)\cdot\mathcal{B}_{\pi_{2,v}}^{f_{2,v}}(g_v)\cdot\Delta(g_v)^s dg_v.$$

(here, c_v is a constant for each v defined by

$$c_v := \frac{L_v^2(1, \chi_{E_v/F_v}) \cdot L_{E_v}(\frac{3}{2}, BC(\omega_{\pi_{3,v}}^{-1}) \otimes \gamma_v^3)}{L_v(3, \chi_{E_v/F_v}) \cdot L_{E_v}(\frac{1}{2}, BC(\omega_{\pi_{3,v}}^{-1} \cdot \omega_{\pi_{2,v}}^{-1}) \otimes \gamma_v)}$$

and $\mathcal{B}_{\pi_{i,v}}$'s are the fixed local pairings of $\theta(\bar{\sigma})_v$ s.t. $\mathcal{B}_{\pi_i} = \prod_v \mathcal{B}_{\pi_{i,v}}$, $\mathcal{B}_{\pi_{i,v}}^{f_{i,v}}(g_v) = \mathcal{B}_{\pi_{i,v}}(g_v \cdot f_{i,v}, f_{i,v})$ and $\Delta(g_v)$ is some function we will define in Section 3.)

Remark 1.0.4. *In ([17]), the author showed that the normailzed local period $\mathcal{P}_v$ in 1.0.3 is nonvanishing if and only if* $\mathrm{Hom}_{G_n(k_v)}(\pi_{n+1,v} \otimes \pi_{n,v}, \mathbb{C}) \neq 0$. *Thus we have the following corollary which can be seen as a non-tempered analogue of the original Gross-Prasad conjecture.*

Corollary 1.0.5. *Under the same condition as in Theorem 1.0.3, if* $\mathrm{Hom}_{G_n(k_v)}(\pi_{n+1,v} \otimes \pi_{n,v}, \mathbb{C}) \neq 0$ *for all places v of F, then the global period $\mathcal{P} \neq 0$ is equivalent to* $L_E(\frac{1}{2}, BC(\omega_{\pi_3}^{-1} \cdot \omega_{\pi_2}^{-1})) \neq 0$.

Remark 1.0.6. *The constant of propotionality between the above two global period can be rewritten as*

$$\frac{\Delta_{G_3}}{2^2} \lim_{s \to 0+} \frac{L_E(\frac{1}{2}, BC(\sigma) \otimes \gamma) \zeta_E(s) L_E(0, \gamma^2)}{\zeta_F(s) L^3(1, \chi_{E/F}) \zeta_F(2) L_E(\frac{3}{2}, BC(\sigma) \otimes \gamma^3)}$$

and the limit exists because both the denominator and numerator have simple pole at $s = 0$.
On the other hand, in the appendix, we shall see that

$$\frac{L_E(s, BC(\pi_2) \boxtimes BC(\pi_3))}{L_F(s + \frac{1}{2}, \pi_2, \mathrm{Ad}) L_F(s + \frac{1}{2}, \pi_3, \mathrm{Ad})}$$

would have double pole at $s = \frac{1}{2}$ in our case. Thus the refined Gross-Prasad does not hold even if we adopt the regularized local period instead of the original one. This shows that the conjecture cannot be extended to the nontempered case.

Remark 1.0.7. *In the $SO(n)$ version of the conjecture, Ichino was the first who considered the non-tempered case in [19], and recently, Yannan Qiu has brought his result into adelic setting including the former.[33]. Thus this article can be considered as an analogue of [33].*

The rest of the first part is organized as follows: in section 3, we introduce the theta correspondence for unitary groups, as well as the Weil representation. In section 3.2, we give several versions of the Rallis Inner Product Formula. With all these things put together, we prove Theorem 1.0.3 in chapter 4 under the assumption of a lemma which we prove in section 4.3. In the last chapter 5, we compare two L-values in Theorem 1.0.3 and Conjecture 1.0.1.

Chapter 2

Preliminaries

This chapter consists of preliminaries for the first part.

2.1 Unitary group

We give a brief introduction of the unitary group.
Let E/F be a quadratic extension of fields (local or global) and c is the nontrivial element in $Gal(E/F)$. Let V be a n- dimensional hermitian space over E equiped with a nondegenerate hermitian form

$$h : V \times V \to E$$

such that

$$h(\alpha v, \beta w) = \alpha \cdot c(\beta) h(v, w)$$

for $v, w \in V$, $\alpha, \beta \in E$ and $h(v, w) = c(h(w, v))$. The unitary group of V is a subgroup of $GL(V)$ which preserves the hermitian form h, that is,

$$U(V) = \{g \in GL(V) \mid h(gv, gw) = h(v, w) \text{ for all } v, w \in V\}.$$

We can easily check that this relation defines an algebraic group over F and identify $U(V)$ with its group of F-rational points $U(V)(F)$.

Two hermitian vector spaces are said equivalent if there is an isomorphism between them defined over F. But for two non-equivalent hermitian spaces V, V', their unitray groups can be isomorphic.
For two hermitian spaces V, V', one can construct the hermitian space $V \oplus V'$ in the obvious way.

If $E = \mathbb{C}, F = \mathbb{R}$ and $n = 1$, any hermitian form on $\mathbb{C}$ is equivalent to either

$$h_1(z_1, z_2) = z_1 \cdot \overline{z_2} \text{ or } h_2(z_1, z_2) = -z_1 \cdot \overline{z_2}.$$

We denote the corresponding hermitian spaces by V^+ and V^- respectively. Note that $U(V^+) \simeq U(V^-) = U_1$ the unit circle elements in $\mathbb{C}$. Any hermitian space over C of dimenstion of n is equal to $(V^+)^p \oplus (V^-)^q$ where p, q are nonnegative integers such that $p + q = n$. We denote by$U(p, q)$ the unitary group of the hermitian space of $(V^+)^p \oplus (V^-)^q$ and we say that it has signature (p, q). Then $U(p, q)$ and $U(q, p)$ are isomorphic and no two others are isomorphic. We also remark that that $U(n, 0)$ and $U(0, n)$ are compact unitary groups while others are non-compact Lie groups.

If E/F is a quadratic extension of p-adic fields, for n-dimensional hermitian space V over E, let $\mathrm{disc}V = (-1)^{\frac{(n-1)n}{2}} \cdot \det V$, so that $\mathrm{disc}V \in F^\times/Norm_{E/F}(E^\times)$. Let $\omega_{E/F}$ be a quadratic character of $F^\times$ corresponding to the nontrival Galois character of $Gal(E/F)$ by the local class field theory. We define $\epsilon(V) := \omega_{E/F}(\mathrm{disc}V)$ and call this the sign of V. It is a theorem of Landherr that there are exactly two types of hermitian spaces and all hermitian spaces are distinguished by their ϵ sign. Denote by V^ϵ for the hermitian space of sign ϵ. The two hermitian spaces of different signs are not equivalent.
We say that $v \in V$ is isotropic if $h(v, v) = 0$ and a subspace $W \subset V$ is isotropic if $h(w_1, w_2) = 0$ for all $w_1, w_2 \in W$. If a subspace $W \subset V$ has no isotropic vector, we say that W is anisotropic.

If $n = 1$, V can be identified with E and we can define its hermitian form $h(e_1, e_2) = ae_1\overline{e_2}$ for some $a \in E^\times$. We denote this hermitian space by $E(a)$ and we can easily check that $E(a) \simeq E(b)$ if and only if $\frac{b}{a} \neq N_{E/F}(E^\times)$ and thus 1-dimensional hermitian spaces are classifield by $E^\times/Norm_{E/F}(E^\times)$.

For $n = 2$, there are two types of hermitian spaces. Fixing some basis $\{e_1, e_2\}$ of E^2, we define the hermitian form of E^2 by $h(ae_1 + be_2, ce_1 + de_2) = \bar{a}d + \bar{b}c$ and the hermitian space with this form is called hyperbolic plane. The other type of 2-dimensionsal hermitian space is obtained by composing 1-dimensional hermitian spaces, that is, $E(a) \oplus E(b)$ where $-\frac{b}{a} \notin Norm_{E/F}(E^\times)$. This is anisotropic plane and all anisotropic planes are isomorphic.

For $n = 2m + 1$, there are exactly two types of hermitian spaces upto isomorphism. Those are $V^\pm \simeq mH \oplus W_1^\pm$ where H is a hyperbolic plane and $W_1^\pm = E(a)$ according to whether $a \in E^\times/Norm_{E/F}(E^\times)$ or not. Note that $U(V^+)$ and $U(V^-)$ are isomorphic and they are quasi-split. [1].

For $n = 2m$, we have $V^+ = mH$ and $V^- = (m-1)H \oplus W_2$ where W_2 is an anisotropic plane. In this case, $U(V^+)$ are not isomorphic to $U(V^-)$ and furthermore, $U(V^+)$ is quasi-split while $U(V^-)$ is not.

[1] In the context of algebraic groups, quasi-split means it has Borel subgroup defined over F.

Now we consider the case E/F are quadratic number field extensions. Let v be a place of F and we assume v splits in E, that is, $v = w_1 w_2$ in E where w_1, w_2 are places of E. Then $E_v \simeq E_{w_1} \times E_{w_2} \simeq F_{w_1} \times F_{w_2}$ and so V_v, the the scalar extension of V to E_v, is $V \otimes_E E_v \simeq V \otimes (E_{w_1} \oplus E_{w_2}) = V_{w_1} \oplus V_{w_2}$ and c acts on V_{w_i} by swithching two components. Thus for $i = 1, 2$, there are two hermitian forms $h_i : V_{w_i} \times V_{w_i} \to F_{w_i}$ such that the extended hermitian form $h_v = h \otimes E_v$ can be written

$$h_v(x, y) = h_1(x_1, y_1) \cdot h_2(x_2, y_2)$$

where $x = x_1 + x_2$ and $y = y_1 + y_2$. (here x_i, y_i are vectors in V_{w_i}.) Thus we can view $U(V_v)$ as the subgroup of $GL(V_{w_1}) \times GL(V_{w_2})$ and we have an isomorphism

$$U(V_v) \simeq GL(V_w)$$

under the projection map

$$(g_1, g_2) \to g_1.$$

2.2 Automorphic L-function

Let F be a fixed number field, F_v denote the completion of F with respect to a place v. If v is finite place, let o_v denote the ring of integer of F_v and q_v the order of corresponding the residue field. We shall write $\mathbb{A} = \mathbb{A}_{\mathbb{F}}$ for the adele ring of F. Though automorphic L-function can be defined for arbitrary reductive algebraic group defined over F, we confine ourselves to general linear group GL_n and denote it by G. Then $G(\mathbb{A})$ is the restricted tensor product, over all primes v, of the groups $G(F_v) = GL_n(F_v)$. In other words, $G(\mathbb{A})$ is the topological direct limit of the groups

$$G_s = \prod_{v \in S} G(F_v) \cdot \prod_{v \notin S} G(o_v)$$

in which S ranges over all finite sets of places of F containing the set S_∞ of archimedean places.
One is interested in the set $\prod(G(\mathbb{A}))$, the equivalent classes of irreducible, admissible representations of $G(\mathbb{A})$. (here, the admissible representation of $G(\mathbb{A})$ is the one whose restriction to the maximal compact subgroup

$$K = \prod_{v:\text{complex}} U(n, \mathbb{C}) \times \prod_{v:\text{real}} O(n, \mathbb{R}) \times \prod_{v:\text{finite}} GL_n(o_v)$$

contain each irreducible representation of $K(\mathbb{A})$ with only finite multiplicity.) Similarly, one has the set $\prod(G(F_v))$ of equivalence classes of irreducible admissible representations of $G(F_v)$. It is known that any $\pi \in \prod(G(\mathbb{A}))$ can be decomposed into a restricted tensor product

$$\otimes_v \pi_v, \qquad \pi_v \in \prod(G(F_v)),$$

of irreducible, admissible representations of the local groups.
The *unramified principal series* is a particularly simple subset to describe. Suppose that v is finite place. For Borel subgroup

$$B(F_v) = \{b = \begin{pmatrix} b_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_n \end{pmatrix}\} \subseteq G(F_v)$$

of $G(F_v)$, and for any n-tuple $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$,

$$b \to \chi_z(b) = |b_1|^{z_1} \cdots |b_n|^{z_n}$$

gives a quasi-character on $B(F_v)$. Let $\tilde{\pi}_{v,z}$ be the representation of $G(F_v)$ obtained by inducing χ_z from $B(F_v)$ to $G(F_v)$. That means, $\tilde{\pi}_{v,z}$ acts on the space of locally constant functions ϕ on $G(F_v)$ such that

$$\phi(bx) = \chi_z(b) \cdot (\prod_{i=0}^{n-1} |b_i|_v^{\frac{n-1}{2}-i}) \cdot \phi(x), \qquad b \in B(F_v)\ , x \in G(F_v)$$

and that

$$\tilde{\pi}_{v,z}(\phi)(x) = \phi(xy)$$

for any such ϕ. We shall assume that

$$Re(z_1) \geq Re(z_2) \geq \cdots \geq Re(z_n).$$

It is known that $\tilde{\pi}_{v,z}$ has a unique irreducible quotient $\pi_{v,z}$. The representations $\{\pi_{v,z}\}$ obtained in this way is called unramified principal representation and they are precisely the representations in $\prod(G(F_v))$ whose restrictions to $G(o_v)$ contain the trivial representation. If π_v is any representation in $\prod(G(F_v))$ which is equivalent to some $\pi_{v,z}$, it makes sense to define a semisimple conjugacy class which is equivalent to

$$\sigma(\pi_v) = \begin{pmatrix} q_1^{-z_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & q_n^{-z_n} \end{pmatrix} \in GL_n(\mathbb{C})$$

and $\sigma(\pi_v)$ does not depend on the equivalence class of π_v.
Suppose that $\pi = \otimes_v \pi_v$ is a representation in $\prod(G(\mathbb{A}))$. Since π is admissible, almost all the local constituents pi_v belong to the unramified principal series. Thus π gives rise to a family $\{\sigma(\pi_v) : v \notin S\}$ of semisimple conjugacy classes in $GL_n(\mathbb{C})$ where S is some finite set containing S_∞. Since semisimple conjugacy class is determined by its characteristic polynomial, we define the local L-functions

$$L_v(s, \pi) = L(s, \pi_v) = \det(1 - \sigma(\pi_v)q_v^{-s})^{-1}, \qquad s \in \mathbb{C}, v \notin S.$$

Then the global L-function, which is developed by Jacquet, Shalika and Piatetskii-Shapiro, is then given as a formal product

$$L_S(s,\pi) = \prod_{v \notin S} L_v(s,\pi). \tag{2.2.1}$$

Using the doubling method of Piatetski-Shapiro and Rallis, we can also define local L-function at $v \notin S$. But it will take us to afar from our course, we will not discuss here and refer the reader to [41].

If the global L-function is to have interesting arithmetic properties, one needs to assume that π is L^2-automorphic. We shall briefly review the notion of an automorphic representation.

Let $\mathscr{Z} = \mathscr{Z}(\mathfrak{g})$ denote the center of the universal envelopong algebra $\mathscr{U}(\mathfrak{g})$ of the complexified Lie algebra $\mathfrak{g}$ of $G_\infty = GL_n(\mathbb{C})^{r_1} \times GL_n(\mathbb{R})^{r_2}$ where r_1 and r_2 are the number of complex and real embeddings of F.

Definition 1. *A smooth function $\varphi : G(\mathbb{A}) \to \mathbb{C}$ is called a smooth automorphic form if it satisfies:*
(i) $\varphi(\gamma z g) = \varphi(g)$ for all $\gamma \in G(F)$ and $z \in Z(\mathbb{A})$;
(ii) there is a compact open subgroup $L \subset G_{finite} = \prod_{v<\infty} GL_n(o_v)$ such that $\varphi(gl) = \varphi(g)$ for all $l \in L$;
(iii) there exist an ideal $\mathfrak{J} \subset \mathscr{Z}$ of finite co-dimension such that $\mathfrak{J}\varphi = 0$;
(iv) there exist a positive integer r such that for all differential operators $X \in \mathscr{U}(\mathfrak{g})$

$$X\varphi(g) \leq C_X \|g\|^r.$$

Note that the group $G(F)$ embeds diagonally as a discrete subgroup of

$$G(\mathbb{A})^1 = \{g \in G(\mathbb{A}) : |\det g| = 1\}.$$

Since smooth automorphic forms are invariant under the right regular representation of $Z(\mathbb{A})$ and $G(F)$, we can regard it as a function on $G(F)\backslash G(\mathbb{A})^1$.

If a smooth automorphic forms φ satisfies

$$\int_{G(F)\backslash G(\mathbb{A})^1} |\varphi(g)|^2 dg < \infty,$$

we write $\varphi \in L^2(G(F)\backslash G(\mathbb{A})^1)$.
The space of cusp forms on $G(\mathbb{A})^1$ consists of the functions $\phi \in L^2(G(F)\backslash G(\mathbb{A})^1)$ such that

$$\int_{N_P(F)\backslash N_P(\mathbb{A})} \phi(nx) dn = 0$$

for almost all $x \in G(\mathbb{A})^1$, and for the unipotent radical N_P of any proper, standard parabolic group. (here, standard parabolic subgroups are subgroups of the form

$$P(\mathbb{A}) = \{p = \begin{pmatrix} p_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_n \end{pmatrix} : p_k \in GL_{n_k}\},$$

where $(n_1, n_2, \cdots, n_k)$ is a partition of n). The space of cusp forms is a closed, right $G(\mathbb{A})^1$-invariant subspace of $L^2(G(F)\backslash G(\mathbb{A})^1)$, which is known to decompose into a discrete direct sum of irreducible representations of $G(\mathbb{A})^1$. A representation $\pi \in \prod(G(\mathbb{A}))$ is said to be cuspidal if its restriction to $G(\mathbb{A})^1$ is equivalent to an irreducible constituent of the space of cusp forms. Now suppose that $p \in P(\mathbb{A})$ is as above, with P a given standard parabolic subgroup, and that for each i, $1 \leq i \leq r$, π_i is a cuspidal automorphic representation of $GL_{n_i}(\mathbb{A})$. Then $p \to \pi_1(p_1) \otimes \cdots \otimes \pi_r(p_r)$ is a representation of $P(\mathbb{A})$, which we can induce to $G(\mathbb{A})$. The automorphic representations of $G(\mathbb{A})$ are the irreducible constituents of induced representation of this form and we denote the subset of automorphic representations in $\prod(G(\mathbb{A}))$ by $\prod_{\text{aut}}(G)$.

Theorem. (Godement, Jacquet, Langlands) *If $\pi \in \prod_{aut}(G)$, then the product in* (2.2.1) *converges in some right half plane. It extends to a meromorphic function on $\mathbb{C}$. When $n = 1$ and π is trivial, its only singularity is a simple pole at $s = 1$. Otherwise, $L(s, \pi)$ is entire. In both cases, $L(s, \pi)$ satisfies the following functional equation $L(s, \pi) = w_\pi L(1 - s, \tilde{\pi})$ where $\tilde{\pi}$ denotes the representation "contragradient" to π and ω_π is of absolute value of 1.*

There is another way to define automorphic L-function using the local Langlands conjecture. Though this relies on some 'big' conjecture, it is very powerful in that it enables us to define local L-function at every places at once while we have discussed the local L-function only for $v \notin S$ in the previous argument. Thus we shall give a brief survey of the construction of the automorphic L-function using the local Langlands conjecture.

Let E/F be a quadratic extension of number fields, G be a reductive algebraic group defined over F and $\pi = \otimes_v \pi_v$ is an automorphic representation of $G(F)$. Given two datum (π, γ), where γ is a smooth hohomorphism $\gamma :^L G \to GL_m(\mathbb{C})$, we want to define $L(s, \pi, \gamma)$, where s is a complex variable. Since our main concerns in this paper are the general linear group and unitary group, we give an ad-hoc definition of $^L G$ for both them.

If $G = GL(n)$, we define $^L G = GL_n(\mathbb{C})$ and if $G = U(n)$, we define $^L G = GL_n(\mathbb{C}) \rtimes W_F$, where W_F is the Weil group and acts on $GL_n(\mathbb{C})$ through the projection map $W_F \to Gal(E/F)$, and the nontrivial element c of $Gal(E/F)$ acts on $GL_n(\mathbb{C})$ as follows;

$$c \cdot g := J^t g^{-1} J^{-1}$$

where

$$J := \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ (-1)^{n+1} & & & \end{pmatrix}$$

We also give a definition of local L-groups ${}^L G_v$. Let v be a place of F. When v splits in E, ${}^L G_v = GL_n(\mathbb{C})$ and if E_v is a field, then ${}^L G_v = GL_n(\mathbb{C}) \rtimes Gal(\bar{F}_v/F_v)$ where $Gal(\bar{F}_v/F_v)$ acts on $GL_n(\mathbb{C})$ analogous to the global case.

Our goal is to define $L(s, \pi, \gamma)$ and to do it, the notion of local L-function $L_v(s, \pi_v, \gamma)$ should be proceeded. The local L-function can be defined using the local Langlands correspondence and since it is of the central issue in the Langland program, we give just a glimpse look here.

The local Langlands conjecture relates equivalence classes of irreducible admissible representations of G_v to equivalence classes of L-parameters of F_v. The L-parameter is a continuous semisimple homomorphism of the Weil-Deligne group[2] WD_{F_v} to ${}^L G_v$ under some restriction on the image of $SL_2(\mathbb{C})$. For the precise definition of this, we refer the reader [1]. We say that two L-parameters are equivalent to each other when they are conjugate via an element of ${}^L G_v^0$, the identity component of ${}^L G_v^0$.
Denote the set of equivalent classes of L-parameters of F by $\Phi(G_v)$ and the equivalence classes of the admissible representations of G_v by $\Pi(G_v)$. Given an L-parameter $\phi : WD_{F_v} \to^L G_v$ and $\gamma :^L G_v \to GL(V)$, we can associate L-function

$$L(s, \gamma \circ \phi) := \det(1 - Frob_v q^{-s} | V^{I_v})$$

where $Frob_v$ is a geometric Frobenius element in W_{F_v}, $q :=$ the cardinality of the residue field of F, I_v the inertia group of W_{F_v} and V^{I_v} is the invariant subspace of V under the action of I_v. This can be seen as a generalization of the Artin L-function developed by Deligne and Langlands.

Then we can state the local Langlands conjecture as follows;

Conjecture. *There is a 'natural' finite-to-one map between* $\Pi(G_v) \to \Phi(G_v)$.

The hypthetical preimage of an L-parameter ϕ is called the L-packet whose L-parameter is ϕ. The terminology 'natural' is perhaps the most important condition in formulating this conjecture. Roughly, it forces the map to preserve L and ϵ-factors.

[2]The Weil-Deligne group is just the Weil group W_{F_v} for archimedian place and $WD_{F_v} = W_{F_v} \rtimes SL_2(\mathbb{C})$ for non-archimedean place.

This conjecture was completely proven for $G = GL(n)$ by Harris-Taylor [14] and independently by Henniart [18] at the almost same time. Recently, this conjecture for other classical groups was also proved by James Arthur (orthogonal, symplectic group) and Chung Pang Mok (quasi-split unitary group) under the assumption of the stabilization of the trace formula. So assuming this conjecture, we can give an alternative definition of the local L-function of $GL(n)$ and $U(n)$.

For $G_v = GL_n(F_v)$ or $U(n)(F_v)$, suppose we are given π_v an admissible representation of G_v and $\gamma :^L G_v \to GL(V)$ for some finite dimensional complex vector space V. Then the local L-function is defined by

$$L_v(s, \pi_v, \gamma) := L_v(s, \gamma \circ \phi)$$

where ϕ is the corresponding L-parameter of π_v via the local Langlands correspondence and the RHS is the Artin L-function we defined ahead.
With this definition of the local L-function at hands, we can define the global automorphic L-function by an Euler product;

$$L(s, \pi, \gamma) := \prod_v L_v(s, \pi_v, \gamma).$$

It is known that this global L-function is well-defined for sufficiently large $s \gg 0$ and has meromorphic continuation to whole complex plane.
Especially when $G = GL(n)$, ${}^L G_v = GL_n(\mathbb{C})$ and so we can take $\gamma :^L G_v \to GL_n(\mathbb{C})$ as the tautological representation of $GL_n(\mathbb{C})$. We write such γ as St and call $L(s, \pi, St)$ the standard L-function and briefly denote by $L(s, \pi)$. This is the same L-function with the one given by Jacquet, Shalika and Piatetskii-Shapiro in the previous discussion.
For $G = GL_n \times GL_m$, there is another canonical choice for γ given by the tensor product. Since ${}^L G_v = GL_n(\mathbb{C}) \times GL_m(\mathbb{C})$, we can take γ as the tensor product representation of ${}^L G_v$ on $V \otimes W$ where V, W are n, m-dimensional vector spaces over $\mathbb{C}$ and $GL_n(\mathbb{C}), GL_m(\mathbb{C})$ act on V, W respectively in a tautological way.
Using this canonical tensor product homomorphism $\gamma_{t.p}$, for two automorphic representations π_n, π_m of $GL(n), GL(m)$ respectively, we define

$$L(s, \pi_n \times \pi_m) := L(s, \pi_n \times \pi_m, \gamma_{t,p})$$

and call this the *Rankin-Selberg* L-function.

When $G = U(n)$, it is cumbersome to deal with the L-parameter $\phi : WD_{F_v} \to^L G_v$ becuase ${}^L G_v$ involves the $Gal(\bar{F}_v/F_v)$ action. However, there is the following useful proposition.

Proposition 2.2.1. *Restriction to W_{F_v} defines a bijection between the set of L-parameters for $U(n)$ and the set of equivalent classes of Frobenius semisimple, conjugate-self-dual representations $\phi : WD_{E_v} \to GL_n(\mathbb{C})$ of sign $(-1)^{n-1}$.*

For the precise definition of the terms in the above statement, we refer the reader to [5].

This proposition says that the L-parameters of F_v for unitary groups are essentially L-parameter of E_v for general linear group with 'some' property. So using this proposition, we can transfer the standard L-function of unitary group to that of general linear group.

The Base change

There is a natural transfer of automorphic representaion of $U(n)(F_v)$ to that of $GL(n)(F_v)$ which corresponds to the restriction functor of the Galois group $Gal(\bar{F}/F)$ to $Gal(\bar{E}/E)$. This functor is called the base change and to state our main theorem in the first part, we require this concept. However, rather than giving a full account of this, we introduce it only to the extent that meets our purpose.

Let π_v be an irreducible admissible representation of $U(n)(F_v)$ and ϕ be the corresponding L-parameter of π_v. Then we can obtain another L-parameter of $GL_n(F_v)$ by restricting the domain WD_{F_v} of ϕ to WD_{E_v}. By the definition of L-parameter, ϕ composed with the projection maps from ${}^LG_v$ to $Gal(E_v/F_v)$ should commute with the projection map from WD_{F_v} to $Gal(E_v/F_v)$. Thus if we restrict the domain of ϕ to WD_{E_v}, we obtain $\varphi_\phi : WD_{E_v} \to GL_n(\mathbb{C})$, the L-parameter of $GL_n(F_v)$. By the local Langlands conjecture of $GL_n(F_v)$, there is the irreducible admissible representation $BC(\pi_v)$ of $GL_n(F_v)$ whose L-parameter is φ_ϕ. Thus we associated $BC(\pi_v)$, the representation of $GL_n(F_v)$, to π_v and it is called the quadratic base change of unitary group. Since the L-parameter of $BC(\pi_v)$ is φ_ϕ, we see that L-function of $BC(\pi)$ should be

$$L(s, BC(\pi_v)) = L(s, St \circ \varphi_\phi)$$

where the LHS is the Artin L-function.

Tempered representation

In this subsection we introduce the terminology of the tempered representation. To define it we first define the admissibility of a representation.

Definition 2.2.1. *Let G be a reductive group defined over F and K a maximal compact subgroup of G. A continuous repreentation (π, V) of $G(F)$ on a complex Hilbert space is called admissible if π restricted to K is unitary and each irreducible unitary representationof K occurs in it with finite multiplicity.*

Given an admissible representation π of $G(F)$, we can define the matrix coefficient, that is a function of $G(F)$ defined by

$$g \to B(\pi(g)\phi_1, \phi_2)$$

where $\phi_1, \phi_2 \in \pi$.

The temperedness can be descried using this matrix coefficients.

Definition 2.2.2. *For an admissible representation π, if all matrix coefficients are in $L^{2+\epsilon}(G(F))$ for any $\epsilon > 0$, we say that π is tempered.*

Thus the temperedness is a local condition by its nature, but people abuse it in the global situation.

Definition 2.2.3. *Let F be a number field and π an automorphic representation of an algebraic group $G(\mathbb{A}_F)$. Fix a group decomposition $G(\mathbb{A}_F) = \prod_v G(F_v)$, measure decompostion $dg = \prod_v dg_v$ and tensor decomposition $\pi = \otimes \pi_v$. Then with this decompostion, we say that π is tempered when all local components π_v's are tempered.*

Remark 2.2.4. *The global temperedness depends on the choice of the decomposition.*

Chapter 3

The Theta correspondence for Unitary groups

We review the Weil Representation and Θ-correspondence. Most of this section are excerpts from [15].

3.1 The Weil Representation for Unitary Groups

In this subsection, we introduce the Weil representation. Since the constructiuons of global and local Weil representation are similar, we will treat both of them simultaneously. For an algebraic group G, if the same statement can be applied to both the local and global cases, we will not use the distinguished notation $G(F_v)$ and $G(\mathbb{A}_F)$, but just refer them to G.
Let $(V, \langle , \rangle_V)$ and $(W, \langle , \rangle_W)$ be two hermitian and skew-hermitian spaces of dimension m, n respectively. Denote $G := U(V)$ and $H := U(W)$ and we regard them as an algebraic group over F.

Define the symplectic space

$$\mathbb{W} := \mathrm{Res}_{E/F} V \otimes_E W$$

with the symplectic form

$$\langle v \otimes w, v' \otimes w' \rangle_{\mathbb{W}} := \frac{1}{2} \mathrm{tr}_{E/F} \left(\langle v, v' \rangle_V \otimes \langle w, w' \rangle_W \right).$$

We also consider the associated symplectic group $Sp(\mathbb{W})$ preserving $\langle \cdot, \cdot \rangle_{\mathbb{W}}$ and the metaplectic group $\widetilde{Sp}(\mathbb{W})$ satisfying the following short exact sequence :

$$1 \to \mathbb{C}^\times \to \widetilde{Sp}(\mathbb{W}) \to Sp(\mathbb{W}) \to 1.$$

Let $\mathbb{X}$ be a Lagrangian subspace of $\mathbb{W}$ and we fix an additive character $\psi : \mathbb{A}_F/F \to \mathbb{C}^\times$ (globally) or $\psi : F_v \to \mathbb{C}^\times$ (locally). Then we have a Schrödinger model of the Weil Representation ω_ψ of $\widetilde{Sp}(\mathbb{W})$ on $\mathscr{S}(\mathbb{X})$, where $\mathscr{S}$ is the Schwartz-Bruhat function space.

Throughout the rest of the paper, let $\chi_{E/F}$ be the quadratic character of $\mathbb{A}_F^\times/F^\times$ or $F_v^\times$ associated to E/F by the global and local class field theory. (For split place v, we define $\chi_{E/F}$ the trivial character.) And we also fix some unitary character γ of $\mathbb{A}_E^\times/E^\times$ or $E_v^\times$ whose restriction to $A_F^\times$ or $F_v^\times$ is $\chi_{E/F}$.

If we set

$$\begin{aligned} \gamma_V &:= \gamma^m \\ \gamma_W &:= \gamma^n, \end{aligned}$$

then (γ_V, γ_W) gives a splitting homomorphism

$$\imath_{\gamma_V,\gamma_W} : G \times H \to \widetilde{Sp}(\mathbb{W})$$

and so by composing this to ω_ψ, we have a Weil representation of $G \times H$ on $\mathbb{S}(\mathbb{X})$.

When the choice of ψ and (γ_V, γ_W) is fixed as above, we simply write

$$\omega_{W,V} := \omega_\psi \circ \imath_{\gamma_V,\gamma_W}.$$

Remark 3.1.1. *For $n = 1$, the image of $H = U(1)$ in $\widetilde{Sp}(\mathbb{W})$ coincides with the image of the center of G, so we can regard the Weil representation of $G \times H$ as the representation of G.*

The Local Θ-Correspondence

In this subsection, we deal with only the local case and so we suppress v from the notation. (Note that if v is non-split, E is the quadratic extension of F and in the split case, $E = F \oplus F$.) As in previous subsection, for non-split v, we denote $\chi_{E/F}$ the quadratic character associated to E/F by local class field theory and for the split case, $\chi_{E/F}$ is trivial.

Howe Duality

Suppose that (G, G') is a dual reductive pair of unitary groups in a symplectic group $Sp(\mathbb{W})$. (Recall that a dual reductive pair (G, G') in $Sp(\mathbb{W})$ is a pair of reductive subgroups of $Sp(\mathbb{W})$ which are mutual centralizers, i.e. $Z_{Sp(\mathbb{W})}(G) = G'$ and $Z_{Sp(\mathbb{W})}(G') = G$.)

After fixing the characters ψ and γ as in subsection 2.1, we obtain a Weil representation $(\omega_{\psi,\gamma}, \mathcal{S})$ of $G \times G'$. For an irreducible admissible representation π of G, the maximal π-isotypic quotient of ω, say $\mathcal{S}(\pi)$, is of the form

$$\mathcal{S}(\pi) \cong \pi \otimes \Theta(\pi).$$

The *Howe Duality Principle* says that if $\Theta(\pi)$ is nonzero, then

1. $\Theta(\pi)$ is a finite-length admissible representation of G'.
2. $\Theta(\pi)$ has the unique maximal semisimple quotient $\theta(\pi)$ and it is irreducible.
3. The correspondence $\pi \mapsto \theta(\pi)$ gives a bijection between the irreducible admissible representations of G and G' that occur as the maximal semisimple quotients of $\mathcal{S}$.

The third is called the local Θ-correspondence. The Howe duality is now known to hold for all places. (see [3])

The Explicit Local Weil representation for $GL(3)(F_v)$

The local Weil representation of unitary groups is explicitly described in [13]. In particular, if v splits, $U(3)(F_v) = \{(A, B) \in M_3(F_v) \mid AB = I\}$ and so by sending (x, x^{-1}) to x, it is identified to $GL(3)(F_v)$. We record here the explicit local Weil representation of $GL(3)(F_v)$ for later use.
Let $X = F_v^3$ be a 3-dimensional vector space over F_v with a fixed basis. Then there is a Weil-representation ω of $GL(3)(F_v)$ realized on $\mathcal{S}(F_v^3)$, which is uniquely determined by the following formula:

$$\omega(g)f(x) = \gamma(\det(g))|\det(g)|^{\frac{1}{2}} f(g^t x), \qquad x \in F_v^3 \tag{3.1.1}$$

Since $E_v = F_v \times F_v$ and γ, we defined in [3.1], is trivial on F_v, we can write $\gamma = (\gamma_1, \gamma_1^{-1})$ for some unitary character γ_1 of F_v. Using the above isomorphism of $U(3)$ and $GL(3)$, we can write $\gamma(\det(g)) = \gamma_1^2(\det(g))$. We will use this formula in Section. 5.

The Global Θ-Correspondence

The global Θ-correspondence is realized using Θ-series. To do this, we first define the theta kernel as follows. For any $\varphi \in \mathcal{S}(\mathbb{X}(\mathbb{A}_F))$, let

$$\theta(g, h, \varphi) := \sum_{\lambda \in \mathbb{X}(F)} \omega_{W,\gamma_W,V,\gamma_V,\psi}(g, h)(\varphi)(\lambda).$$

Note that this is slowly increasing function. Thus if f is some cusp form on $G(\mathbb{A}_F)$, it is rapidly decreasing and so we can define

$$\theta(f,\varphi)(h) := \int_{[G]} \theta(g,h,\varphi)\overline{f(g)}\, dg \tag{3.1.2}$$

where dg is the Tamagawa measure.

Then the Θ-lift of a cuspidal representation of G as follows:

Definition 3.1.2. *For a cuspidal automorphic representation π of $G(\mathbb{A}_F)$,*

$$\Theta_{V,W,\gamma_W,\gamma_V,\psi}(\pi) = \{\theta(f,\varphi) : f \in \pi, \varphi \in \mathscr{S}(\mathbb{X}(\mathbb{A}_F))\}$$

is called the Θ-lift of π with data $(\gamma_W, \gamma_V, \psi)$.

The Howe Duality Principle implies the following. ([9], proposition 1.2)

Proposition 3.1.3. *If $\Theta(\pi)$ is a cuspidal representation of $U(V)(\mathbb{A})$, then it is irreducible and is isomorphic to the restricted tensor product $\otimes_v \theta(\pi_v)$.*

Remark 3.1.4. *Since we integrated $\overline{f}$ (instead of f) against the theta series, π and $\Theta(\pi)$ have the same central characters.*

Remark 3.1.5. *In the theory of theta lift, there are two main issues, that is, the cuspidality and non-vanishing of the theta lift. The cuspidality issue was treated by Rallis in terms of so-called tower property.[35] So to make our Theorem (1.0.3) not vacuous, we record the criterion in [3.2] which ensures the non-vanishing of two theta lifts π_3 and π_2.*

3.2 The Rallis Inner Product Formula

The Rallis inner product formula enables us to express the Petersson inner product of the global theta lift with respect to the source information. Since we will need three different version of Rallis inner product formulas, we record them for lifts from $U(1)$ to $U(3)$, $U(1)$ to $U(1)$ and $U(1)$ to $U(2)$. To give a uniform description, we introduce some related notions.

Global and Local zeta-integral

Let V be a hermitian space over E of dimension m, and W be a skew-hermitian space of dimension n. Let V^- be the same space as V, but with hermitian form $-\langle\cdot,\cdot\rangle_V$. Note that $U(V) = U(V^-)$. Let τ be a irreducible cupspidal automorphic representation of $U(V)$.
Denote $G := U(V) = U(V^-), H := U(W)$, $G^\diamond := U(V \oplus V^-)$ and $i : G \times G \to G^\diamond$

be the inclusion map $U(V) \times U(V^-) \hookrightarrow U(V \oplus V^-)$. Let v be a finite place of F and $\mathcal{O}_v$ the ring of integer of F_v and denote by ϖ a generator of its maximal ideal. We fix a maximal compact subgroup $K = \prod_v K_v$ of G such that $K_v := G(\mathcal{O}_v)$ for finite places and $K_v := G(F_v) \cap U(2m)$ for archimedean places. Let P be a Siegel-parabolic subgroup of $G^\diamond$ stabilizing $V^\triangle := \{(x,x) \in V \oplus V^-\}$ with Levi-component $GL(V^\triangle)$ and $\tilde{K}$ a maximal compact subgroup of $G^\diamond$ such that $i(K \times K) \hookrightarrow \tilde{K}$ and $G^\diamond = P\tilde{K}$. Let $I(s, \gamma_W) := \mathrm{Ind}_{P(\mathbb{A}_F)}^{G^\diamond(\mathbb{A}_F)}(\gamma_W \circ \det) \cdot |\det|^s$ be the degenerate principal series representation induced from the character γ_W of $\mathbb{A}_E^\times$ and $|\det|^s$. (Here, we took γ_W as the one we defined in [3.1] and the determinants are taken with respect to $GL(V^\Delta)$ which is isomorphic to the Levi of P.)

Then for $\Phi_s \in I(\gamma_W, s)$, we define the Eisenstein series

$$E(\Phi_s, \tilde{g}) := \sum_{x \in P(F)\backslash G^\diamond(F)} \Phi_s(x\tilde{g})$$

for $\tilde{g} \in G^\diamond$. Then for $f_1, f_2 \in \tau$, we can define

Definition 3.2.1. *The Piatetski-Shapiro-Rallis zeta integral is defined as follows:*

$$Z(s, f_1, f_2, \Phi_s, \gamma_W) := \int_{[G\times G]} f_1(g_1)\overline{f_2(g_2)}E(\Phi_s, \imath(g_1, g_2))\gamma_W^{-1}(\det_{U(V^-)} g_2)dg_1dg_2.$$

This integral converges only for $\mathrm{Re}(s) \gg 0$. However, once the convergence is ensured, it can be factored into the product of the local-zeta integrals. So we define the local zeta-integrals. Assume that $\Phi_s = \otimes_v \Phi_{s,v}$ and $f_i = \otimes_v f_{i,v}$. Then for each place v, the local zeta-integral is defined by

$$Z_v(s, f_{1,v}, f_{2,v}, \Phi_{s,v}) := \int_{U(V)_v} \Phi_{s,v}(i(g_v, 1))\langle \pi_v(g_v) f_{1,v}, f_{2,v}\rangle_{\pi_v} dg_v$$

We note that the integral defining the Z_v converges for $\mathrm{Re}(s)$ sufficiently large. However, Z_v can be extended to all of $\mathbb{C}$ by meromorphic continuation. For large s, there is a factorization theorem of the zeta integral. (See [32] for more detail)

Theorem 3.2.2. *For* $\mathrm{Re}(s) \gg 0$,

$$Z(s, f_1, f_2, \Phi_s, \gamma_W) = \prod_v Z_v(s, f_{1,v}, f_{2,v}, \Phi_{s,v})$$

The local-zeta integral has a simple form for unramified places. Take S to be a sufficiently large finite set of places of F such that for all $v \notin S$, the relevant data is unramified, and the local vectors $f_{i,v}$ are normalized spherical vectors so that $\langle f_{1,v}, f_{2,v}\rangle_{\pi_v} = 1$. Recall that $m = \dim_E V$, $n = \dim_E W$ and set

$$d_m(s, \gamma_W) := \prod_{r=0}^{m-1} L(2s + m - r, \chi_{E/F}^{n+r}).$$

It is known that for $v \notin S$, Z_v has the following simple form,

$$Z_v(s, f_{1,v}, f_{2,v}, \Phi_{s,v}) = \frac{L_v(s+1/2, \pi \otimes \gamma_W)}{d_{m,v}(s, \gamma_W)} \tag{3.2.1}$$

and so we can normalize them defining $Z_v^\#$ by

$$Z_v^\#(s, f_{1,v}, f_{2,v}, \Phi_{s,v}) = \frac{d_{m,v}(s, \gamma_W)}{L_v(s+1/2, \pi \otimes \gamma_W)} \cdot Z_v(s, f_{1,v}, f_{2,v}, \Phi_{s,v})$$

Thus, we can rewrite Theorem 3.2.2 as follows:
For $f_1, f_2 \in \tau$, we have

$$Z(s, f_1, f_2, \Phi_s, \gamma_W) = \frac{L(s+1/2, \pi \otimes \gamma_W)}{d_m(s, \gamma_W)} \cdot \prod_v Z_v^\#(s, f_{1,v}, f_{2,v}, \Phi_{s,v}) \tag{3.2.2}$$

The Siegel-Weil section

The Rallis Inner Product Formula relates the Petersson inner product of the global theta lifts to the global zeta-integral for a special section $\Phi_s \in I(s, \gamma_W)$, so called Siegel-Weil section. In this section, we give the definition of the Siegel-Weil section introducing the doubled Weil representation.

The setting for the doubled Weil representation is as follows.

We have

$$\mathbb{W} := \mathrm{Res}_{E/F}\, 2V \otimes_E W$$

where $2V := V \oplus V^-$. We also denote

$$V^\nabla := \{(v, -v) : v \in V\} \subset V \oplus V^-.$$

Since $V^\nabla \otimes W$ is a Lagrangian subspace of $\mathbb{W}$ over F, with some fixed choice of characters ψ and γ, we have a Schrödinger model of the Weil representation $\tilde{\omega}$ of $G^\diamond \times H$ realized on $\mathcal{S}((V^\nabla \otimes W))$.

Now, fix polarizations

$$\begin{aligned} V &= X^+ \oplus Y^+ \\ V^- &= X^- \oplus Y^- \end{aligned}$$

and denote

$$\begin{aligned} X &:= X^+ \oplus X^- \\ Y &:= Y^+ \oplus Y^-. \end{aligned}$$

Then

$$2V = X \oplus Y$$

and so we have another Lagrangian $X \otimes W$ of $\mathbb{W}$.
If we set

$$\begin{array}{rcl}\mathbb{X} & := & X \otimes W \\ \mathbb{X}^{+} & := & X^{+} \otimes W \\ \mathbb{X}^{-} & := & X^{-} \otimes W,\end{array}$$

then there is a $U(V)(\mathbb{A}_F) \times U(V^{-})(\mathbb{A}_F)$-intertwining map

$$\rho_{m,n} : \mathscr{S}(\mathbb{X}^{+}(\mathbb{A}_F)) \otimes \mathscr{S}(\mathbb{X}^{-}(\mathbb{A}_F)) \to \mathscr{S}(\mathbb{X}(\mathbb{A}_F)) \to \mathscr{S}((V^{\nabla} \otimes W)(\mathbb{A}_F))$$

where the first map is the obvious one, and the second map is given by the Fourier transform. Furthermore, it satisfies $\rho_{m,n}(\varphi_1 \otimes \bar{\varphi_2})(0) =< \varphi_1, \varphi_2 >$ and so $(\tilde{\omega}(i(g,1)) \cdot \rho_{m,n}(\varphi_1 \otimes \bar{\varphi_2}))(0) =< \omega_{W,V}(g) \cdot \varphi_1, \varphi_2 >$.([27, p.182]) Let $s_m = \frac{n-m}{2}$. By the explicit formula for $\tilde{\omega}$ described in [25], there is an intertwining map $[\ ] : \mathscr{S}(V^{\nabla} \otimes W) \to I(s_m, \gamma_W)$ given by $\Phi \to f_{\Phi}^{s_m}(\tilde{g}) = \tilde{\omega}(\tilde{g})\Phi(0)$. We can also extend $f_{\Phi}^{s_m}$ to $f_{\Phi}^{s} \in I(s, \gamma_W)$ for all $s \in \mathbb{C}$ by defining $f_{\Phi}^{s} := f_{\Phi}^{s_m} \cdot |\det|^{s-s_m}$ and call this the Siegel-Weil section in $I(s, \gamma_W)$. (Here the determinant map was taken as in 3.2.) Then we can define the function Δ_m of G as $\Delta_m(g) := |\det(i(g,1))|$ and using Δ_m, we can write the Siegel-Weil section as

$$f^{s}_{[\rho_{m,n}(\varphi_1 \otimes \bar{\varphi_2})]}(g) = \langle \omega_{W,V}(g) \cdot \varphi_1, \varphi_2 \rangle \cdot \Delta_m(g)^{s-s_m}. \tag{3.2.3}$$

Note that $\Delta_m(g)$ is $K \times K$ invariant and $\Delta_m(1) = 1$. (For $k_1, k_2 \in K$, $(k_1 g k_2, 1) = (k_1, k_1) \cdot (g, 1) \cdot (k_2, k_1^{-1})$ and $(k_1, k_1) \in P$, $(k_2, k_1^{-1}) \in \tilde{K}$.) Using the similar argument of Prop.6.4 in [32], Yamana[[42], Lemma A.4.] computed $\Delta_m(g_v)$ explicitly for split place v of F. We record his computation for the non-archimedean split places not dividing 2.

Let v be a finite place of F which splits in E and not divide 2. Let $\mathscr{O}_v$ be the ring of integer of F_v and ϖ a generator of its maximal ideal. Since v splits, $U(m)(F_v) \simeq GL(m)(F_v)$ and by Cartan decomposition, $GL(m)(F_v) = K_m D_m^{+} K_m$ where $K_m = GL(m)(\mathscr{O}_v)$ and $D_m^{+} = \mathrm{diag}[\varpi^{a_1}, \cdots, \varpi^{a_m}]$. Then,

$$\Delta_m(g_v) = |\varpi|^{\sum_{i=1}^{m} |a_i|} \tag{3.2.4}$$

Remark 3.2.3. *Since $|a+b| \neq |a| + |b|$, we cannot expect $\Delta_m(g_v l_v) \neq \Delta_m(g_v)\Delta_m(l_v)$ for central diagonal matrix $l_v = \mathrm{diag}[\varpi^{c}, \cdots, \varpi^{c}] \in GL(m)(F_v)$.*

Now, we are ready to state the three versions of Rallis Inner Product formula. The first one is as follows;

Lifting from $U(1)$ to $U(3)$

Here, $\dim V = 1$, $\dim W = 3$ and τ is a irreducible automorphic representation of $U(1)(\mathbb{A}_F)$. Suppose that $f_i = \otimes_v f_{i,v} \in \tau$, $\varphi_1 = \otimes_v \varphi_{1,v} \in \mathscr{S}(\mathbb{X}^+(\mathbb{A}_F))$ and $\varphi_2 = \otimes_v \varphi_{2,v} \in \mathscr{S}(\mathbb{X}^-(\mathbb{A}_F))$. Let $\Phi_{s,v} \in I(s, \gamma^3)$ is a holomorphic Siegel-Weil section given by $[\rho_{1,3}(\varphi_1 \otimes \bar{\varphi_2})]$. Then,

Theorem 3.2.4.

$$\langle \theta(\bar{f_1}, \varphi_1), \theta(\bar{f_2}, \varphi_2) \rangle_{\Theta(\bar{\tau})} = \frac{L_E(\frac{3}{2}, BC(\tau) \otimes \gamma^3)}{L(3, \chi_{E/F})} \prod_v Z_v^\sharp(1, f_{1,v}, f_{2,v}, \Phi_{1,v})$$

where

$$Z_v^\sharp := \frac{L_v(3, \chi_{E_v/F_v})}{L_{E_v}(\frac{3}{2}, BC(\tau_v) \otimes \gamma_v^3)} \cdot Z_v$$

Proof. This follows immediately from Theorem 2.1 in [27] and (3.2.2) the normalization of the local-zeta integral. □

The next following two versions of Rallis Inner product formula come from Lemma 10.1 in [41]:

Lifting from $U(1)$ to $U(1)$

Here, $\dim V = \dim W = 1$ and τ is a irreducible automorphic representation of $U(1)(\mathbb{A}_F)$. Suppose that $f_i = \otimes_v f_{i,v} \in \tau$, $\varphi_1 = \otimes_v \varphi_{1,v} \in \mathscr{S}(\mathbb{X}^+(\mathbb{A}_F))$ and $\varphi_2 = \otimes_v \varphi_{2,v} \in \mathscr{S}(\mathbb{X}^-(\mathbb{A}_F))$. Let $\Phi_{s,v} \in I(s, \gamma)$ is a holomorphic Siegel-Weil section given by $[\rho_{1,1}(\varphi_1 \otimes \bar{\varphi_2})]$. By [41, Theorem 4.1] and 3.2.2 , we have

Theorem 3.2.5.

$$\langle \theta(\bar{f_1}, \varphi_1), \theta(\bar{f_2}, \varphi_2) \rangle_{\Theta(\bar{\tau})} = \frac{1}{2} \cdot \frac{L_E(\frac{1}{2}, BC(\tau) \otimes \gamma)}{L(1, \chi_{E/F})} \prod_v Z_v^\sharp(0, f_{1,v}, f_{2,v}, \Phi_{0,v})$$

where

$$Z_v^\sharp = \frac{L_v(1, \chi_{E_v/F_v})}{L_{E_v}(\frac{1}{2}, BC(\tau_v) \otimes \gamma_v)} \cdot Z_v$$

Lifting from $U(2)$ to $U(1)$

Here, $\dim V = 2$, $\dim W = 1$ and τ is a irreducible automorphic representation of $U(2)(\mathbb{A}_F)$. Suppose that $f_i = \otimes_v f_{i,v} \in \tau$, $\varphi_1 = \otimes_v \varphi_{1,v} \in \mathscr{S}(\mathbb{X}^+(\mathbb{A}_F))$ and $\varphi_2 = \otimes_v \varphi_{2,v} \in \mathscr{S}(\mathbb{X}^-(\mathbb{A}_F))$. Let $\Phi_{s,v} \in I(s, \gamma)$ be a holomorphic Siegel-Weil section given by $[\rho_{2,1}(\varphi_1 \otimes \bar{\varphi_2})]$. Then,

Theorem 3.2.6.
$$\langle\theta(\bar{f}_1,\varphi_1),\theta(\bar{f}_2,\varphi_2)\rangle_{\Theta(\bar{\tau})} = \frac{-Res_{s=0}(L_E(s,BC(\tau)\otimes\gamma))}{L(1,\chi_{E/F})}\prod_v Z_v^{\sharp}(-\frac{1}{2},f_{1,v},f_{2,v},\Phi_{s,v})$$
where
$$Z_v^{\sharp}(-\frac{1}{2},f_{1,v},f_{2,v},\Phi_{s,v}) = \lim_{s\to 0}\frac{L_v(2s+1,\chi_{E_v/F_v})\cdot\zeta_v(2s)}{L_{E_v}(s,BC(\tau_v)\otimes\gamma_v)}\cdot Z_v(s-\frac{1}{2},f_{1,v},f_{2,v},\Phi_{s-\frac{1}{2},v})$$

Proof. By Lemma 10.1 (2) in [41] and (3.2.2),
$$\langle\theta(\bar{f}_1,\varphi_1),\theta(\bar{f}_2,\varphi_2)\rangle_{\Theta(\bar{\tau})} = \frac{1}{2}\cdot\lim_{s\to 0}\frac{L_E(s,BC(\tau)\otimes\gamma)}{L(2s+1,\chi_{E/F})\zeta_F(2s)}\prod_v Z_v^{\sharp}(-\frac{1}{2},f_{1,v},f_{2,v},\Phi_{s-\frac{1}{2},v}).$$
By Theorem 9.1 and Lemma 10.2 in [41], if $\theta(\bar{\tau})$ doesn't vanish, $L_E(s,BC(\tau)\otimes\gamma)$ has a simple pole at $s=0$. Note that $\zeta_F(s)$ is the completed Dedekind zeta function of F and it has a simple pole at $s=0$. Since $Res_{s=0}\zeta_F(s) = -1$ and $L(1,\chi_{E/F})$ is nonzero, we get
$$\lim_{s\to 0}\frac{L_E(s,BC(\tau)\otimes\gamma)}{L(2s+1,\chi_{E/F})\zeta_F(2s)} = \frac{-Res_{s=0}(L_E(s,BC(\tau)\otimes\gamma))}{L(1,\chi_{E/F})}.$$
For each v, $d_2(s-\frac{1}{2},\gamma_W)\cdot\Phi_{s-\frac{1}{2},v}(g)$ is not holomorphic but good section (see, [41]), so by Theorem 5.2 in[41], the quotient of $L_v(2s+1,\chi_{E_v/F_v})\cdot\zeta_v(2s)\cdot Z_v(s-\frac{1}{2},f_{1,v},f_{2,v},\Phi_{s-\frac{1}{2},v})$ by $L_{E_v}(s,BC(\tau_v)\otimes\gamma_v)$ is holomorphic.
Thus each $Z^{\sharp}_{v,s=-\frac{1}{2}}(s,f_{1,v},f_{2,v},\Phi_{s,v})$ exists and it proves theorem when $\theta(\bar{\tau})$ is nonvanishing. When $\theta(\bar{\tau})$ is zero, then $L_E(s,BC(\tau)\otimes\gamma)$ is holomorphic by Lemma 10.2 in [41], and so $Res_{s=0}(L_E(s,BC(\tau)\otimes\gamma)$ is zero. So the theorem also holds in this case. □

The local-to-global criterion for the non-vanishing of the theta lifts

Since we will assume π_3 and π_2 are non-vanishing, we descrive the non-vanishing criterion of the theta lifts π_3,π_2 as well as from $U(1)$ to $U(1)$.

Theta lift from $U(1)$ to $U(3)$

Let τ be a character of $U(1)$. By the [Lemma 5.3 , [27]], the Euler product $L_E(s,BC(\tau)\otimes\gamma^3)$ absolutely converges and nonzero at $s=\frac{3}{2}$. Then by (3.2.4), we see that $\pi_3 = \Theta(\bar{\tau})$ does not vanish when the local zeta integral $Z_v(1,\cdot)\in Hom(I(1,\gamma_v^3)\otimes\tau_v^\vee\otimes\tau_v)$ is nonzero for all the places v.

Theta lift from $U(1)$ to $U(2)$

Let τ be a character of $U(1)$. Then by [Theorem 5.10, [15]], the theta lift $\pi_3 = \Theta(\bar{\tau})$ does not vanish when $L_E(1,BC(\tau)\otimes\gamma^2)\neq 0$ and local theta lift $\theta_v(\bar{\tau}_v)\neq 0$ for all the places v.

Theta lift form $U(1)$ to $U(1)$

Let V (resp, W) be a hermitian (resp, skew-hermitian) space of dimension 1 over E. Let τ be a character of $U(V)(\mathbb{A}_F)$. Then by (3.2) and [Theorem 6.1, [13]], the theta lift $\Theta(\bar{\tau})$ is non-vanishing if and only if $L_E(\frac{1}{2}, BC(\tau) \otimes \gamma) \neq 0$ and for all v, $\epsilon_v(\frac{1}{2}, \tau_v \otimes \gamma_v, \psi_v) = \epsilon_V \cdot \epsilon_W$. (Here, $\epsilon_V(s, \cdot), \epsilon_W(s, \cdot)$ are the local root number and ϵ_v is the sign of V_{E_v},W_{E_v} respectively.)

Chapter 4

Proof of Theorem 1.0.2

In this chapter, we give the proof of Theorem 1.0.3. We first remind the reader of our setting.

4.1 The Setup

F is a totally real number field and E a totally imaginary quadratic extension of F. We consider the following seesaw diagram:

$$\begin{tikzcd}[arrows=dash]
U(V \oplus L) \arrow[d] \arrow[dr] & U(W) \times U(W) \arrow[d] \arrow[dl] \\
U(V) \times U(L) & U(W)
\end{tikzcd} \tag{4.1.1}$$

(Here, V is a 2-dimensional hermitian space over E/F and W is a 1-dimensional skew-hermitian space over E/F and L is a hermitian line over E/F.
Using the seesaw duality, we can relate the period integral in Theorem to the triple product integral over $U(W)$.

We first fix the following:

- $\pi_2 = \otimes \pi_{2,v}$ is an irreducible, cuspidal, tempered, automorphic representation of $U(V)(\mathbb{A}_F)$.

- $\sigma = \otimes \sigma_v$ is an automorphic character of $U(W)(\mathbb{A}_F)$.

- $\mu := w_{\pi_2}^{-1} \cdot \sigma$ is an automorphic character of $U(L)(\mathbb{A}_F)$, where ω_{π_2} is the central character of π_2 and $\mu = \otimes \mu_v$ where $\mu_v = w_{\pi_{2,v}}^{-1} \cdot \sigma_v$.

- $(\omega_{V\oplus L,W}, \psi)$ is a Weil representation of $\widetilde{Sp}(\mathbb{W})(\mathbb{A}_F)$. (See Chapter 3 for notation.)

We also fix local pairings $\mathcal{B}_{\pi_{2,v}}, \mathcal{B}_{\sigma_v}, \mathcal{B}_{\mu_v}$ such that $\prod_v \mathcal{B}_{\pi_{2,v}}, \prod_v \mathcal{B}_{\sigma_v}, \prod_v \mathcal{B}_{\mu_v}$ give the respective Petersson inner products on the global representation and $\mathcal{B}_{\mu_v}(\mu_v, \mu_v) = \mathcal{B}_{\sigma_v}(\sigma_v, \sigma_v)$ for all places v. (Since $\mathcal{B}_\sigma(\sigma, \sigma) = \mathcal{B}_\mu(\mu, \mu) = \mathrm{Vol}([U(1)])$, these choices are possible stand with no conflict)

We take $\gamma_L, \gamma_W = \gamma$ and $\gamma_V = \gamma^2$, where γ is a unitary character of $\mathbb{A}_E^\times/E^\times$ such that $\gamma|_{\mathbb{A}_F^\times} = \chi_{E/F}$and fix additive character $\psi : \mathbb{A}_F \to \mathbb{C}$. After fixing these splitting data $(\gamma_V, \gamma_L, \gamma_W, \psi)$, we can define the relevent theta lifts and denote them $\Theta(\bar{\pi}_2) := \Theta_{W,V\gamma_W,\gamma_V,\psi}(\bar{\pi}_2)$ on $U(W)(\mathbb{A}_F)$, $\Theta(\bar{\sigma}) := \Theta_{W,V\oplus L,\gamma_W\gamma_V,\gamma_L,\psi}(\bar{\sigma})$ on $U(V \oplus L)(\mathbb{A}_F)$, and $\Theta(\bar{\mu}) := \Theta_{W,L\gamma_W,\gamma_L,\psi}(\bar{\mu})$ on $U(W)(\mathbb{A}_F)$. We assume that all Θ-lifts we consider here are non-vanishing and cuspidal.

4.2 Proof of Theorem 1.0.3

In the course of the proof, we will regard μ and σ as automorphic forms in the 1-dimension representations of μ and σ and take$f_\mu = \mu$, and$f_\sigma = \sigma$. Since $\omega_{W,V\oplus L} = \omega_{W,V} \otimes \omega_{W,L}$, we prove the theorem assuming $\varphi = \varphi_1 \otimes \varphi_2$ for $\varphi_1 \in \omega_{W,V}$ and $\varphi_2 \in \omega_{W,L}$.

Step 1. First, we consider another the global period

$$\mathcal{P}' : V_{\Theta(\bar{\sigma})} \otimes V_{\pi_2} \otimes V_\mu \to \mathbb{C}$$

defined by

$$\mathcal{P}'(f_{\Theta(\bar{\sigma})}, f_{\pi_2}, f_\mu) := \left| \int_{[U(V)\times U(L)]} f_{\Theta(\bar{\sigma})}(i(g,l)) f_{\pi_2}(g) f_\mu(l) dg dl \right|^2 .$$

(Here, i is the natural embedding $i : U(V) \times U(L) \hookrightarrow U(V \oplus L)$.)
By making a change of variables $g \to gl$, we see that

$$\int_{[U(V)\times U(L)]} f_{\Theta(\bar{\sigma})}(i(g,l)) f_{\pi_2}(g) f_\mu(l) dg dl = \int_{[U(V)\times U(L)]} f_{\Theta(\bar{\sigma})}(i(gl,l)) f_{\pi_2}(gl) f_\mu(l) dg dl.$$

By Remark 3.1.4, the central character of $\Theta(\bar{\sigma})$ is $\omega_\sigma^{-1} = \sigma^{-1}$. So, after observing that (l,l) is in the center of $U(V \oplus L)$ and l is in the center of $U(V)$, we have

$$\begin{aligned}
&\int_{[U(V)\times U(L)]} f_{\Theta(\bar{\sigma})}(i(gl,l)) f_{\pi_2}(gl) \mu(l) dg dl \\
&= \int_{[U(V)\times U(L)]} \omega_{\Theta(\bar{\sigma})}(l) \omega_{\pi_2}(l) \mu(l) f_{\Theta(\bar{\sigma})}|_{U(V)}(g) f_{\pi_2}(g) dg dl \\
&= \int_{[U(V)\times U(L)]} f_{\Theta(\bar{\sigma})}|_{U(V)}(g) f_{\pi_2}(g) dg dl \\
&= \mathrm{Vol}([U(L)]) \int_{[U(V)]} f_{\Theta(\bar{\sigma})}|_{U(V)}(g) f_{\pi_2}(g) dg \\
&= 2 \int_{[U(V)]} f_{\Theta(\bar{\sigma})}|_{U(V)}(g) f_{\pi_2}(g) dg. \quad \text{(note that } \mathrm{Vol}([U(1)]) = 2)
\end{aligned}$$

Thus, we get $\mathcal{P}(f_{\Theta(\bar{\sigma})}, f_{\pi_2}) = \frac{1}{4}\mathcal{P}'(f_{\Theta(\bar{\sigma})}, f_{\pi_2}, f_{\mu})$.

Step 2. By the global seesaw duality, we see that

$$\int_{[U(V)\times U(L)]} \theta(\bar{\sigma},\varphi)(i(g,l)) f_{\pi_2}(g)\mu(l)dgdl = \int_{[U(W)]} \theta(\bar{f_{\pi_2}},\varphi_1)(h)\theta(\bar{\mu},\varphi_2)(h)\sigma(h)dh$$

(The order change of integration is justified by the rapidly decreasing property of cusp forms and the moderate growth of the theta series.)
Since $\Theta(\bar{\pi_2})$ and $\Theta(\bar{\mu})$ have central characters $\omega_{\pi_2}^{-1}$ and μ^{-1} respectively, we see that

$$\mathcal{P}'(\theta(\bar{f_{\pi_2}},\varphi_1)),\theta(\bar{f_{\mu}},\varphi_2)), f_{\sigma}) = |\theta(\bar{f_{\pi_2}},\varphi_1)(1)\theta(\bar{\mu},\varphi_2)(1)\sigma(1)|^2 \cdot \mathrm{Vol}([U(W)])^2.$$

For $\tau = \pi_2$ or μ and $i = 1, 2$,

$$\mathcal{B}_{\Theta(\bar{\tau})}(\theta(\bar{f_{\tau}},\varphi_i),\theta(\bar{f_{\tau}},\varphi_i)) = |\theta(\bar{f_{\tau}},\varphi_i)(1)|^2 \cdot \mathrm{Vol}([U(W)]) \text{ and } \sigma(1) = 1.$$

Thus we can write $\mathcal{P}'(\theta(\bar{f_{\pi_2}},\varphi_1)),\theta(\bar{f_{\mu}},\varphi_2)), f_{\sigma}) =$

$$\mathcal{B}_{\Theta(\bar{\pi_2})}(\theta(\bar{f_{\pi_2}},\varphi_1),\theta(\bar{f_{\pi}},\varphi_1)) \cdot \mathcal{B}_{\Theta(\bar{\mu})}(\theta(\bar{f_{\mu}},\varphi_2),\theta(\bar{f_{\mu}},\varphi_2)).$$

By theorem 3.2.5 and 3.2.6, we see that $\mathcal{P}'(\theta(\bar{f_{\pi_2}},\varphi_1)),\theta(\bar{f_{\mu}},\varphi_2)), f_{\sigma}) =$

$$-\frac{1}{2}\cdot\frac{L_E(\frac{1}{2}, BC(\mu)\otimes\gamma)}{L(1,\chi_{E/F})}\cdot\frac{Res_{s=0}(L_E(s, BC(\pi_2)\otimes\gamma))}{L(1,\chi_{E/F})}\cdot\prod_v Z_v^{\#}(f_{\mu_v}, f_{\pi_{2,v}},\varphi_{1,v},\varphi_{2,v})$$

where $Z_v^{\#}(f_{\pi_{2,v}}, f_{\mu_v},\varphi_{1,v},\varphi_{2,v}) = Z^{\#}_{v,s=-\frac{1}{2}}(s, f_{\pi_{2,v}}, f_{\pi_{2,v}},\Phi_{s,v}) \cdot Z_v^{\#}(0, f_{\mu_v}, f_{\mu_v},\Phi_{0,v})$
and
$\Phi_{s,v} = [\rho_{2,1}(\varphi_1\otimes\bar{\varphi_1})] \in I(s,\gamma)$, $\Phi_{0,v} = [\rho_{1,1}(\varphi_2\otimes\bar{\varphi_2})] \in I(0,\gamma)$.
(Note that $Z_v^{\#}(f_{\pi_{2,v}}, f_{\mu_v},\varphi_{1,v},\varphi_{2,v}) = 1$ for unramified data)

Step 3. Recall the abbreviations for various matrix coefficients made in Theorem 1.0.3.

$$\mathcal{B}^{\varphi_{1,v}}_{\omega_{W,V}}(g_v) := \mathcal{B}_{\omega_{W,V}}(\omega_{W,V}(g_v)\cdot\varphi_{1,v},\varphi_{1,v}),\ \mathcal{B}^{\varphi_{2,v}}_{\omega_{W,L}}(l_v) := \mathcal{B}_{\omega_{W,L}}(\omega_{W,L}(l_v)\cdot\varphi_{2,v},\varphi_{2,v})$$

$\mathcal{B}^{\varphi_v}_{\omega_{W,V\oplus L}}(g_v,l_v) := \mathcal{B}_{\omega_{W,V\oplus L}}(\omega_{W,V\oplus L}(i(g_v,1),l_v)\varphi_v,\varphi_v)$ and

$\mathcal{B}^{f_{\pi_{2,v}}}_{\pi_{2,v}}(g_v) := \mathcal{B}_{\pi_{2,v}}(g_v\cdot f_{\pi_{2,v}}, f_{\pi_{2,v}})$, $\mathcal{B}^{f_{\tau_v}}_{\tau_v}(l_v) := \mathcal{B}_{\tau_v}(l_v\cdot f_{\tau_v}, f_{\tau_v})$ for $\tau = \sigma$ or μ.

If we unfold $Z^{\#}_{v,s=-\frac{1}{2}}(s, f_{\pi_{2,v}}, f_{\pi_{2,v}},\Phi_{s,v})$ in $Z_v^{\#}(f_{\pi_{2,v}}, f_{\mu_v},\varphi_{1,v},\varphi_{2,v})$, we can write

$$Z_v^{\#}(f_{\pi_{2,v}}, f_{\mu_v},\varphi_{1,v},\varphi_{2,v}) = \lim_{\Re(s)\to 0+}\frac{L_v(2s+1,\chi_{E_v/F_v})\cdot\zeta_v(2s)}{L_{E_v}(s, BC(\pi_{2,v})\otimes\gamma_v)}.$$

$$\int_{U(V)_v} Z_v^\sharp(0,f_{\mu_v},f_{\mu_v},\Phi_{0,v})\mathcal{B}^{\varphi_{1,v}}_{\omega_{W,V}}(g_v)\mathcal{B}^{f_{\pi_{2,v}}}_{\pi_{2,v}}(g_v)\Delta_2(g_v)^s dg_v$$

$$= \frac{L_v^2(1,\chi_{E_v/F_v})}{L_{E_v}(\frac{1}{2},BC(\mu_v)\otimes\gamma_v)}\cdot \lim_{\Re(s)\to 0+}\frac{\zeta_v(2s)}{L_{E_v}(s,BC(\pi_{2,v})\otimes\gamma_v)}\cdot I_v(s,\varphi_{1,v},\varphi_{2,v},f_{\pi_{2,v}},f_{\mu_v})$$

where

$$I_v(s,\varphi_{1,v},\varphi_{2,v},f_{\pi_{2,v}},f_{\mu_v}) :=$$
$$\int_{U(V)_v}\left(\int_{U(L)_v}\mathcal{B}^{\varphi_{2,v}}_{\omega_{W,L}}(l_v)\cdot\mathcal{B}^{f_{\mu_v}}_{\mu_v}(l_v)dl_v\right)\cdot\mathcal{B}^{\varphi_{1,v}}_{\omega_{W,V\oplus L}}(g_v)\cdot\mathcal{B}^{f_{\pi_{2,v}}}_{\pi_{2,v}}(g_v)\cdot\Delta_2(g_v)^s dg_v.$$

Set $J(s,g_v,l_v,\varphi_{1,v},\varphi_{2,v},f_{\pi_v},f_{\mu_v}) := \mathcal{B}^{\varphi_{2,v}}_{\omega_{W,L}}(l_v)\cdot\mathcal{B}^{f_{\mu_v}}_{\mu_v}(l_v)\cdot\mathcal{B}^{\varphi_{1,v}}_{\omega_{W,V\oplus L}}(g_v)\cdot\mathcal{B}^{f_{\pi_{2,v}}}_{\pi_{2,v}}(g_v)\cdot\Delta_2(g_v)^s$. Then we can write I_v as a double integral,

$$I_v(s,\varphi_{1,v},\varphi_{2,v},f_{\pi_{2,v}},f_{\mu_v}) = \int_{U(V)_v\times U(L)_v} J(s,g_v,l_v,\varphi_{1,v},\varphi_{2,v},f_{\pi_{2,v}},f_{\mu_v})dg_v dl_v.$$

Since π_2 is tempered, by Lemma 7.2 in [41], $Z_v(s,f_{\pi_{2,v}},f_{\pi_{2,v}},[\rho(\varphi_{1,v}\otimes\bar{\varphi}_{1,v})])$ absolutely converge for $\Re(s) > -\frac{1}{2}$ and so $Z_v(0,f_{\mu_v},f_{\mu_v},[\rho(\varphi_{2,v}\otimes\bar{\varphi}_{2,v})])$ does. For $\Re(s) > 0$, $I_v(s)$ is just the product of $Z_v(s,f_{\pi_{2,v}},f_{\pi_{2,v}},[\rho(\varphi_{1,v}\otimes\bar{\varphi}_{1,v})])$ and $Z_v(0,f_{\mu_v},f_{\mu_v},[\rho(\varphi_{2,v}\otimes\bar{\varphi}_{2,v})])$, the above doubled integral for $I_v(s)$ absolutely converges for $\Re(s) > 0$.

Step 4. By making a change of variables $g_v \to g_v l_v$,

$$\begin{aligned}
I_v(s,\varphi_{1,v},\varphi_{2,v},f_{\pi_{2,v}},f_{\mu_v}) &= \int_{U(V)_v\times U(L)_v} J(s,g_v l_v,l_v,\varphi_{1,v},\varphi_{2,v},f_{\pi_{2,v}},f_{\mu_v})dg_v dl_v\\
&= \int_{U(V)_v\times U(L)_v}\mathcal{B}^{\varphi_{1,v}}_{\omega_{W,V}}(g_v l_v)\cdot\mathcal{B}^{\varphi_{2,v}}_{\omega_{W,L}}(l_v)\cdot\mathcal{B}^{f_{\pi_{2,v}}}_{\pi_{2,v}}(g_v l_v)\cdot\mathcal{B}^{f_{\mu_v}}_{\mu_v}(l_v)\cdot\Delta_2(g_v l_v)^s dg_v dl_v\\
&= \int_{U(V)_v\times U(L)_v}\mathcal{B}^{\varphi_v}_{\omega_{W,V\oplus L}}(g_v,l_v)\cdot\mathcal{B}^{f_{\pi_{2,v}}}_{\pi_{2,v}}(g_v)\cdot\omega_{\pi_{2,v}}(l_v)\cdot\mathcal{B}^{f_{\mu_v}}_{\mu_v}(l_v)\cdot\Delta_2(g_v l_v)^s dg_v dl_v\\
&= \int_{U(V)_v\times U(L)_v}\mathcal{B}^{\varphi_v}_{\omega_{W,V\oplus L}}(g_v,l_v)\cdot\mathcal{B}^{f_{\pi_{2,v}}}_{\pi_{2,v}}(g_v)\cdot\omega_{\sigma_v}(l_v)\cdot\mathcal{B}^{f_{\mu_v}}_{\mu_v}(1_v)\cdot\Delta_2(g_v l_v)^s dg_v dl_v\\
&= \int_{U(V)_v\times U(L)_v}\mathcal{B}^{\varphi_v}_{\omega_{W,V\oplus L}}(g_v,l_v)\cdot\mathcal{B}^{f_{\pi_{2,v}}}_{\pi_{2,v}}(g_v)\cdot\mathcal{B}^{f_{\sigma_v}}_{\sigma_v}(l_v)\cdot\Delta_2(g_v l_v)^s dg_v dl_v
\end{aligned}$$

(The last equality follows from $\mathcal{B}_{\sigma_v}(f_{\sigma_v},f_{\sigma_v}) = \mathcal{B}_{\mu_v}(f_{\mu_v},f_{\mu_v})$).

Step 5. Write $d_{\pi_{2,v},\gamma_v}(s) = \frac{\zeta_v(2s)}{L_{E_v}(s,BC(\pi_{2,v})\otimes\gamma_v)}$. By the lemma **??** in the next section, we see that

$$\lim_{\Re(s)\to 0+} d_{\pi_{2,v},\gamma_v}(s)\cdot\int_{U(V)_v\times U(L)_v}\mathcal{B}^{\varphi_v}_{\omega_{W,V\oplus L}}(g_v,l_v)\cdot\mathcal{B}^{f_{\pi_{2,v}}}_{\pi_{2,v}}(g_v)\cdot\mathcal{B}^{f_{\sigma_v}}_{\sigma_v}(l_v)\cdot\Delta_2(g_v l_v)^s dg_v dl_v$$

$$=$$

$$\lim_{\Re(s)\to 0+} d_{\pi_{2,v},\gamma_v}(s)\cdot\int_{U(V)_v\times U(L)_v}\mathcal{B}^{\varphi_v}_{\omega_{W,V\oplus L}}(g_v,l_v)\cdot\mathcal{B}^{f_{\pi_{2,v}}}_{\pi_{2,v}}(g_v)\cdot\mathcal{B}^{f_{\sigma_v}}_{\sigma_v}(l_v)\cdot\Delta_2(g_v)^s dg_v dl_v$$

$$=$$

$$\lim_{\Re(s)\to 0+} d_{\pi_{2,v},\gamma_v}(s)\cdot\int_{U(V)_v} Z_v(1,f_{\sigma_v},f_{\sigma_v},[\rho(g_v\cdot\varphi_v\otimes\bar{\varphi}_v)])\cdot\mathscr{B}_{\pi_{2,v}}^{f_{\pi_{2,v}}}(g_v)\cdot\Delta_2(g_v)^s dg_v.$$

We normalize $Z_v(1,f_{\sigma_v},f_{\sigma_v},[\rho(\varphi_v\otimes\bar{\varphi}_v)])$ by

$$Z_v^\sharp(1,f_{\sigma_v},f_{\sigma_v},[\rho(\varphi_v\otimes\bar{\varphi}_v)]) := \frac{L_v(3,\chi_{E_v/F_v})}{L_{E_v}(3/2,BC(\sigma_v)\otimes\gamma_v^3)}\cdot Z_v(1,f_{\sigma_v},f_{\sigma_v},[\rho(\varphi_v\otimes\bar{\varphi}_v)]).$$

We define the local inner product $\mathscr{B}_{\theta(\bar{\sigma}_v)}$ on $\theta_v(\bar{\sigma}_v)$ as follows:

$$\mathscr{B}_{\theta(\bar{\sigma}_v)}(\theta_v(\bar{f}_{\sigma_v},\varphi_v),\theta_v(\bar{f}_{\sigma_v},\varphi_v)) := \begin{cases} \frac{L_E(3/2,BC(\sigma)\otimes\gamma^3)}{L(3,\chi_{E/F})}\cdot Z_v^\sharp(1,f_{\sigma_v},f_{\sigma_v},[\rho(\varphi_v\otimes\bar{\varphi}_v)]) \\ \text{for some place } v \\ \\ Z_v^\sharp(1,f_{\sigma_v},f_{\sigma_v},[\rho(\varphi_v\otimes\bar{\varphi}_v)]) \\ \text{for the remaining places} \end{cases}$$

Then we see that

$$\mathscr{B}_{\Theta(\bar{\sigma})}(\theta(\bar{f}_\sigma,\varphi),\theta(\bar{f}_\sigma,\varphi)) = \prod_v \mathscr{B}_{\theta(\bar{\sigma}_v)}(\theta_v(\bar{f}_{\sigma_v},\varphi_v),\theta_v(\bar{f}_{\sigma_v},\varphi_v))$$

and $\mathscr{B}_{\theta(\bar{\sigma}_v)}(\theta_v(\bar{f}_{\sigma_v},\varphi_v),\theta_v(\bar{f}_{\sigma_v},\varphi_v)) = 1$ for unramified data (f_{σ_v},φ_v).
(Note that the $'$small$'$ local theta-lift is the maximal semisimple quotient of the $'$big$'$ theta-lift, and so we should check whether these pairings are well-defined. But since we are assuming $\Theta(\bar{\sigma})$ is cuspidal, it is semisimple and so $\mathscr{B}_{\Theta(\bar{\sigma})}(\theta(\bar{f}_\sigma,\varphi),\theta(\bar{f}_\sigma,\varphi))$ factors as a map $\sigma_v\otimes\bar{\sigma}_v\otimes\varpi_{\omega_{W,V\oplus L}}\otimes\bar{\varpi}_{\omega_{W,V\oplus L}}\to\Theta(\bar{\sigma})\otimes\overline{\Theta(\bar{\sigma})}$. Thus theorem (3.2.4) shows that $\mathscr{B}_{\Theta(\bar{\sigma}_v)}$ descends to $\mathscr{B}_{\theta(\bar{\sigma}_v)}$.)

Step 6. With the things we developed so far, we see that

$$\mathscr{P}(f_{\Theta(\bar{\sigma})},f_{\pi_2}) = \frac{1}{4}\mathscr{P}'(f_{\Theta(\bar{\sigma})},f_{\pi_2},f_\mu) = \frac{1}{4}\mathscr{P}'(\theta(\bar{f}_{\pi_2},\varphi_1)),\theta(\bar{f}_\mu,\varphi_2)),f_\sigma)$$

$$= -\frac{1}{2^3}\cdot\frac{L_E(\frac{1}{2},BC(\mu)\otimes\gamma)}{L(1,\chi_{E/F})}\cdot\frac{Res_{s=0}(L_E(s,BC(\pi_2)\otimes\gamma))}{L(1,\chi_{E/F})}\cdot\prod_v Z_v^\sharp(f_{\mu_v},f_{\pi_{2,v}},\varphi_{1,v},\varphi_{2,v})$$

$$= -\frac{1}{2^3}\cdot\frac{L_E(\frac{1}{2},BC(\mu)\otimes\gamma)}{L(1,\chi_{E/F})}\cdot\frac{Res_{s=0}(L_E(s,BC(\pi_2)\otimes\gamma))}{L(1,\chi_{E/F})}\frac{L(3,\chi_{E/F})}{L_E(3/2,BC(\sigma)\otimes\gamma^3)}\cdot$$
$$\prod_v \mathscr{P}_v(\theta_v(\bar{f}_{\sigma_v},\varphi_v),f_{\pi_{2,v}})$$

This proves the theorem.

Remark 4.2.1. *Since $Z_v^\sharp(0,f_{\mu_v},f_{\mu_v},\Phi_{0,v})\cdot Z^\sharp_{v,s=-\frac{1}{2}}(s,f_{\pi_{2,v}},f_{\pi_{2,v}},\Phi_{-\frac{1}{2},v}) = 1$ for unramified vectors, our local periods $\mathscr{P}_v^\sharp$'s are also 1 at infinitely many places and so the above product is indeed a finite product.*

4.3 Proof of Lemma 3.3.1

In this section, we prove the lemma upon which we developed *Step* 5 in the proof of 1.0.3. We retain the same notations as in the previous section and since everything occurs in local case, we suppress v from the notation. We remind the reader that π_2 is given by the theta lift of the trivial character $\mathbb{I}$ of $U(1)$.

Lemma 4.3.1. *Let t be the order of $\frac{\zeta(2s)}{L_E(s,BC(\pi_2)\otimes\gamma)}$ at $s=0$. Then,*

$$\lim_{\Re(s)\to 0+} s^t\cdot\int_{U(V)\times U(L)} \mathcal{B}^{\varphi}_{\omega_{W,V\oplus L}}(g,l)\cdot\mathcal{B}^{f_{\pi_2}}_{\pi_2}(g)\cdot\mathcal{B}^{f_\sigma}_{\sigma}(l)\cdot(\Delta_2(gl)^s-\Delta_2(g)^s)dgdl=0 \tag{4.3.1}$$

Proof. When E is quadratic field extension of F, $U(L)$ is the centralizer of $U(V)$ and compact and so it is included in every maximal compact subgroup of $U(V)$. Then $\Delta_2(gl)^s-\Delta_2(g)^s=0$ and so the lemma is immediate in this case. So we assume $E=F\times F$ and by our hypothesis, all archimedean places do not split, and so we consider only p-adic case.
Since $E=F\times F, U(n)\simeq GL_n(F)$ and by Cartan decomposition, $GL_1(F)=\bigcup_{l\in\mathbb{Z}}\varpi^lK_1$, $GL_2(F)=\bigcup_{n\in\mathbb{Z},m\in\mathbb{N}}K_2\begin{pmatrix}\varpi^{n+m} & \\ & \varpi^n\end{pmatrix}K_2$. (here, $\mathcal{O}$ is the ring of integer of F and ϖ is a uniformizer of $\mathcal{O}$ and $K_i=GL_i(\mathcal{O})$.)
Since the theta lift preserves the central character, $\omega_{\pi_2}(\varpi)=1$ and let $\alpha=\sigma(\varpi)$. For $i=1,2$ and diagonal matrix $m\in GL_i(F)$, let $\mu_i(m):=\frac{Vol(K_imK_i)}{Vol(K_i)^2}$. Since $GL_1(F)$ is abelian, $\mu_1(m)=1$ and by the Lemma 2.1 in ([34]), $\mu_2(\text{diag}(a,b))=C\cdot|\frac{b}{a}|$ for some constant $C\in\mathbb{R}_{>0}$.
Then the measure decomposition formula turns 4.3.1 to show

$$\lim_{\Re(s)\to 0+} s^t\cdot\sum_{n,l\in\mathbb{Z},m\geq 0}\alpha^l\cdot|\varpi|^{-m}\cdot(|\varpi|^{s(|n+m+l|+|n+l|)}-|\varpi|^{s(|n+m|+|n|)})\cdot I(s,\varphi,f_{\pi_2},m,n,l)=0$$

where $I(s,\varphi,f_{\pi_2},m,n,l)=$

$$\int_{K_1\times K_2\times K_2}\mathcal{B}^{\varphi}_{\omega_{W,V\oplus L}}(k_2\,\text{diag}(\varpi^{n+m},\varpi^n)k_2',\varpi^lk_1)\cdot\mathcal{B}^{f_{\pi_2}}_{\pi_2}(k_2\,\text{diag}(\varpi^m,1)k_2')dk_1dk_2dk_2'.$$

Since φ and f_{π_2} are $K\times K$-finite functions, we are sufficient to show

$$\lim_{\Re(s)\to 0+} s^t\cdot\Big(\sum_{n,l\in\mathbb{Z},m\geq 0}\alpha^l\cdot|\varpi|^{-m}\cdot(|\varpi|^{s(|n+m+l|+|n+l|)}-|\varpi|^{s(|n+m|+|n|)})\cdot c_{n,m,l}\cdot d_m\Big)=0$$

where $c_{n,m,l}=\mathcal{B}^{\varphi}_{\omega_{W,V\oplus L}}(\text{diag}(\varpi^{n+m},\varpi^n),\varpi^l)$ and $d_m=\mathcal{B}^{f_{\pi_2}}_{\pi_2}(\text{diag}(\varpi^m,1))$.
Now we invoke the asymptotic fomulas of $c_{n,m,l}$ and d_m. Recall (3.1.1) in Section 2.2

and write $c = \gamma_1^2(\varpi)$. (Note that $|c| = 1$.) Since φ is locally constant and has compact support, there is $l_1 \in \mathbb{N}$ such that for $X, Y \in F^3$, if $|X - Y| \le |\varpi|^{l_1} \cdot \mathrm{Sup}\{|X| \mid X \in supp(\varphi) \subset F^3\}$, then $\varphi(X) = \varphi(Y)$. Thus

$$c_{n,m,l} = \begin{cases} c^{2n+m+l} \cdot |\varpi|^{n+\frac{m+l}{2}} \cdot \int_{F^3} \varphi(\varpi^{n+m}x_1, \varpi^n x_2, 0) \cdot \varphi(x_1,x_2,x_3)dX, & \text{if } l \ge l_1 \\ c^{2n+m+l} \cdot |\varpi|^{n+\frac{m-l}{2}} \cdot \int_{F^3} \varphi(\varpi^{n+m}x_1, \varpi^n x_2, x_3) \cdot \varphi(x_1,x_2,0)dX, & \text{if } l \le -l_1. \end{cases}$$

Write $a_{n,m} = \int_{F^3} \varphi(\varpi^{n+m}x_1, \varpi^n x_2, 0) \cdot \varphi(x_1,x_2,x_3)dX$,
$b_{n,m} = \int_{F^3} \varphi(\varpi^{n+m}x_1, \varpi^n x_2, x_3) \cdot \varphi(x_1,x_2,0)dX$.
Then $a_{n,m} = \begin{cases} a^1_{n,m}, & \text{if } n \ge l_1 \\ |\varpi|^{-n} \cdot a^2_{n,m}, & \text{if } n \le -l_1. \end{cases}$ where
$a^1_{n,m} = \int_{F^3} \varphi(\varpi^{n+m}x_1, 0, 0) \cdot \varphi(x_1,x_2,x_3)dX$,
$a^2_{n,m} = \int_{F^3} \varphi(\varpi^{n+m}x_1, x_2, 0) \cdot \varphi(x_1,0,x_3)dX$
and $b_{n,m} = \begin{cases} b^1_{n,m}, & \text{if } n \ge l_1 \\ |\varpi|^{-n} \cdot b^2_{n,m}, & \text{if } n \le -l_1. \end{cases}$ where
$b^1_{n,m} = \int_{F^3} \varphi(\varpi^{n+m}x_1, 0, x_3) \cdot \varphi(x_1,x_2,0)dX$,
$b^2_{n,m} = \int_{F^3} \varphi(\varpi^{n+m}x_1, x_2, x_3) \cdot \varphi(x_1,0,0)dX$.
Again $a^i_{n,m} = \begin{cases} k^i_1 & \text{if } n+m \ge l_1 \\ |\varpi|^{-(n+m)} \cdot k^i_2 & \text{if } n+m \le -l_1 \end{cases}$ and
$b^i_{n,m} = \begin{cases} k^i_3 & \text{if } n+m \ge l_1 \\ |\varpi|^{-(n+m)} \cdot k^i_4 & \text{if } n+m \le -l_1 \end{cases}$ for some constants $\{k^i_1, k^i_2, k^i_3, k^i_4\}_{i=1,2}$.

Note that in codimension 0, 1 case, the theta lift sends a tempered representation to a tempered one. Thus we know that π_2 is tempered and by [Prop.8.1, [5]], we see that it is the irreducible unitary induced representation $B(\gamma_1^2, \gamma_1^{-2})$ of $GL(2)(F)$. (here, since $\gamma = (\gamma_1, \gamma_1^{-1})$, if we regard γ as a character of $F^\times$ using the isomorphism of $U(1)$ and $GL(1)$, $\gamma(x) = \gamma_1^2(x)$.) Then by ([34], Lemma 3.9), if we take l_1 large enough, we assume that for $m \ge l_1$, $d_m = |\varpi|^{\frac{m}{2}} \cdot (c_1 \cdot c^m + c_2 \cdot c^{-m})$ where c_1, c_2 are constants.

If π is an unramified representation of $U(W_n)$ and $\theta(\pi)$ is the theta lift of π to $U(V_{n+1})$, then $BC(\theta(\pi)) \simeq BC(\pi)\gamma^{-1} \boxplus \gamma^n$ by (8.1.2) in [40]. Recall that $GU_{2,0}(\mathbb{A}_F) \simeq (D^\times \times E^\times)/\Delta F^\times$ where D is the quaternion division algebra over F and $GU_{1,1}(\mathbb{A}_F) \simeq (GL_2(F) \times E^\times)/\Delta F^\times$. Since $GL_2(F)$ and $D^\times$ have the strong multiplicity one theorem and global theta lift is the product of local theta lifts, the unramified computations of the local theta lifts completely determine the global theta lift from $U(1)$ to $U(2)$ not at the level of individual represenations but of L-parameters. Thus since π_2 is the theta lift of the trivial representation, we have the L-parameter relation $BC(\pi_2) = BC(\mathbb{I})\gamma^{-1} \boxplus \gamma$ for all places and so $L_E(s, BC(\pi_2) \otimes \gamma) = (\frac{1}{1-q^{-s}})^2 \cdot \frac{1}{1-\gamma_1^2(\varpi)q^{-s}} \cdot \frac{1}{1-\gamma_1^{-2}(\varpi)q^{-s}}$. (Recall $\gamma = (\gamma_1, \gamma_1^{-1})$ for some unitary character γ_1 of

F.) Thus if $\gamma_1^2(\varpi) = 1$, $L_E(s, BC(\pi_2) \otimes \gamma)$ has a quadruple pole at $s = 0$ and if $\gamma_1^2(\varpi) \neq 1$, then it has double pole at $s = 0$.) So in any cases, we have $t \geq 1$.

Now, we introduce two notation that we will use in this argument :

• If two meromorphic functions f_1,f_2 differ by a constant multiplication, we write $f_1 \approx f_2$.

• For two meromorphic functions f_1,f_2 and $m \in \mathbb{N}$, if $\lim_{\Re(s)\to 0+} s^m \cdot (f_1(s) - f_2(s)) = 0$, we write $f_1 \overset{m}{\sim} f_2$ and if $f_1 \overset{0}{\sim} f_2$, we simply write $f_1 \sim f_2$.

Since the integral in the Lemma absolutely converges on $\Re(s) > 0$, to prove it, it suffices to show that each component of the integral

$$\sum_{n\in\mathbb{Z},m\geq 0} c^{2n+m}|\varpi|^{n-\frac{m}{2}} d_m a_{n,m} \cdot \Big(\sum_{l\geq l_1} c^l \alpha^l |\varpi|^{\frac{l}{2}} (|\varpi|^{s(|n+m+l|+|n+l|)} - |\varpi|^{s(|n+m|+|n|)} \Big) \tag{4.3.2}$$

$$\sum_{n\in\mathbb{Z},m\geq 0} |\varpi|^{-m} d_m \cdot \Big(\sum_{-l_1<l<l_1} \alpha^l (|\varpi|^{s(|n+m+l|+|n+l|)} - |\varpi|^{s(|n+m|+|n|)}) \cdot c_{n,m,l} \Big) \tag{4.3.3}$$

$$\sum_{n\in\mathbb{Z},m\geq 0} c^{2n+m}|\varpi|^{n-\frac{m}{2}} d_m a_{n,m} \cdot \Big(\sum_{l<-l_1} c^l \alpha^l |\varpi|^{\frac{l}{2}} (|\varpi|^{s(|n+m+l|+|n+l|)} - |\varpi|^{s(|n+m|+|n|)} \Big) \tag{4.3.4}$$

are all $\overset{1}{\sim} 0$.

We will first show (4.3.2)$\overset{1}{\sim} 0$. To do this, we write

$$r_{l,m,n}(s) = c^l \alpha^l |\varpi|^{\frac{l}{2}} (|\varpi|^{s(|n+m+l|+|n+l|)} - |\varpi|^{s(|n+m|+|n|)})$$

and decompose (4.3.2) into three component.

$$\sum_{n\in\mathbb{Z},m\geq 0} c^{2n+m}|\varpi|^{n-\frac{m}{2}} d_m a_{n,m} \cdot \Big(\sum_{l\geq l_1, l<-(n+m)} r_{l,m,n}(s) \Big)$$
$$+ \sum_{n\in\mathbb{Z},m\geq 0} c^{2n+m}|\varpi|^{n-\frac{m}{2}} d_m a_{n,m} \cdot \Big(\sum_{l\geq l_1, -(n+m)\leq l<-n} r_{l,m,n}(s) \Big)$$
$$+ \sum_{n\in\mathbb{Z},m\geq 0} c^{2n+m}|\varpi|^{n-\frac{m}{2}} d_m a_{n,m} \cdot \Big(\sum_{l\geq l_1, l\geq -n} r_{l,m,n}(s) \Big)$$

and show each component is $\overset{1}{\sim} 0$.

For fixed $m \in \mathbb{N}$ and small $\Re(s) > 0$,

$$\sum_{n\in\mathbb{Z}} c^{2n+m}|\varpi|^{n-\frac{m}{2}} d_m a_{n,m} \cdot \sum_{l\geq l_1, l<-(n+m)} r_{l,m,n}(s) =$$

$$\sum_{n\leq -(m+l_1+1)} c^{2n+m} \cdot |\varpi|^{n(1-2s)-m(\frac{1}{2}+s)} d_m a_{n,m} \cdot \left(f_1(s) - f_2^{m,n}(s)\right)$$

where

$$f_1(s) = \frac{(c\alpha|\varpi|^{\frac{1}{2}-2s})^{l_1}}{1-c\alpha|\varpi|^{\frac{1}{2}-2s}} - \frac{(c\alpha|\varpi|^{\frac{1}{2}})^{l_1}}{1-c\alpha|\varpi|^{\frac{1}{2}}} \text{ and}$$

$$f_2^{m,n}(s) = \frac{(c\alpha|\varpi|^{\frac{1}{2}-2s})^{-(n+m)}}{1-c\alpha|\varpi|^{\frac{1}{2}-2s}} - \frac{(c\alpha|\varpi|^{\frac{1}{2}})^{-(n+m)}}{1-c\alpha|\varpi|^{\frac{1}{2}}}.$$

Note that $f_1, f_2^{m,n} \sim 0$.

Since

$$\sum_{m\geq 0}\sum_{n\leq -(m+l_1+1)} c^{2n+m} \cdot |\varpi|^{n(1-2s)-m(\frac{1}{2}+s)} d_m a_{n,m}$$

$$\approx \sum_{m\geq 0} d_m (c|\varpi|^{-s-\frac{3}{2}})^m \Big(\sum_{n\leq -(m+l_1+1)} (c^2|\varpi|^{-1-2s})^n\Big)$$

$$\approx (c^{-2}|\varpi|^{1+2s})^{l_1+1} \sum_{m\geq 0} \frac{d_m (c^{-1}|\varpi|^{s-\frac{1}{2}})^m}{1-c^{-2}|\varpi|^{1+2s}}$$

$$= \frac{(c^{-2}|\varpi|^{1+2s})^{l_1+1}}{1-c^{-2}|\varpi|^{1+2s}} \cdot \Big(\Big(\sum_{m=0}^{l_1-1} d_m (c^{-1}|\varpi|^{s-\frac{1}{2}})^m\Big) + c_1 \cdot \frac{|\varpi|^{sl_1}}{1-|\varpi|^s} + c_2 \cdot \frac{(c^{-2}|\varpi|^s)^{l_1}}{1-c^{-2}|\varpi|^s}\Big)$$

and so $(\sum_{m\geq 0}\sum_{n\leq -(m+l_1+1)} c^{2n+m} \cdot |\varpi|^{n(1-2s)-m(\frac{1}{2}+s)} d_m a_{n,m}) \cdot f_1(s) \overset{1}{\sim} 0$.

Furthermore,

$$\sum_{m\geq 0}\sum_{n\leq -(m+l_1+1)} c^{2n+m} \cdot |\varpi|^{n(1-2s)-m(\frac{1}{2}+s)} d_m a_{n,m} \cdot \Big(\frac{(c\alpha|\varpi|^{\frac{1}{2}-2s})^{-(n+m)}}{1-c\alpha|\varpi|^{\frac{1}{2}-2s}} - \frac{(c\alpha|\varpi|^{\frac{1}{2}})^{-(n+m)}}{1-c\alpha|\varpi|^{\frac{1}{2}}}\Big) \approx$$

$$\sum_{m\geq 0} d_m \alpha^{-m} \Big(\sum_{n\leq -(m+l_1+1)} |\varpi|^{(s-2)m} \cdot (c|\varpi|^{-\frac{3}{2}}\alpha^{-1})^n + |\varpi|^{(-s-2)m} \cdot (c|\varpi|^{-\frac{3}{2}-2s}\alpha^{-1})^n\Big) =$$

$$\Big(\frac{(c^{-1}|\varpi|^{\frac{3}{2}}\alpha)^{l_1+1}}{1-c^{-1}|\varpi|^{\frac{3}{2}}\alpha} - \frac{(c^{-1}|\varpi|^{\frac{3}{2}}\alpha)^{l_1+1}}{1-c^{-1}|\varpi|^{\frac{3}{2}+2s}\alpha}\Big) \cdot \sum_{m\geq 0} d_m (c^{-1}|\varpi|^{(s-\frac{1}{2})})^m \approx$$

$$\Big(\frac{(c^{-1}|\varpi|^{\frac{3}{2}}\alpha)^{l_1+1}}{1-c^{-1}|\varpi|^{\frac{3}{2}}\alpha} - \frac{(c^{-1}|\varpi|^{\frac{3}{2}}\alpha)^{l_1+1}}{1-c^{-1}|\varpi|^{\frac{3}{2}+2s}\alpha}\Big) \cdot \Big(c_1 \cdot \frac{|\varpi|^{sl_1}}{1-|\varpi|^s} + c_2 \cdot \frac{(c^{-2}|\varpi|^s)^{l_1}}{1-c^{-2}|\varpi|^s}\Big) \overset{1}{\sim} 0.$$

Thus we see that

$$\sum_{n\in\mathbb{Z},m\geq 0} c^{2n+m}|\varpi|^{n-\frac{m}{2}} d_m a_{n,m}\cdot(\sum_{l\geq l_1, l<-(n+m)} r_{l,m,n}(s)) \overset{1}{\sim} 0.$$

Next we will show

$$\sum_{m\in\mathbb{N}} c^m d_m |\varpi|^{-\frac{m}{2}} \sum_{n\in\mathbb{Z}} c^{2n}|\varpi|^n a_{n,m}\cdot \sum_{l\geq l_1, -(n+m)\leq l<-n} (c\alpha|\varpi|^{\frac{1}{2}})^l\cdot(|\varpi|^{sm}-|\varpi|^{s(|n+m|-n)}) \overset{1}{\sim} 0.$$

Let

$$p_{n,m}(s) = c^{2n}|\varpi|^n a_{n,m}\cdot \sum_{l\geq l_1, -(n+m)\leq l<-n} (c\alpha|\varpi|^{\frac{1}{2}})^l\cdot(|\varpi|^{sm}-|\varpi|^{s(|n+m|-n)}).$$

Then

$$\sum_{n\in\mathbb{Z}} p_{n,m}(s) =$$

$$\sum_{n<\min\{-l_1,-m\}} c^{2n}|\varpi|^n a_{n,m}\cdot(|\varpi|^{sm}-|\varpi|^{(-2n-m)s})\cdot\frac{(c\alpha|\varpi|^{\frac{1}{2}})^{\max\{l_1,-(n+m)\}}-(c\alpha|\varpi|^{\frac{1}{2}})^{-n}}{1-c\alpha|\varpi|^{\frac{1}{2}}}$$

and so to show $\sum_{m\in\mathbb{N}} c^m d_m|\varpi|^{-\frac{m}{2}}\sum_{n\in\mathbb{Z}} p_{n,m}(s) \overset{1}{\sim} 0$, it is suffcient to check

$$\sum_{0\leq m<l_1} c^m d_m|\varpi|^{-\frac{m}{2}}\cdot(\sum_{-l_1-m<n<-l_1} p_{n,m}(s)) \overset{1}{\sim} 0 \tag{4.3.5}$$

$$\sum_{0\leq m<l_1} c^m d_m|\varpi|^{-\frac{m}{2}}\cdot(\sum_{n\leq -l_1-m} p_{n,m}(s)) \overset{1}{\sim} 0 \tag{4.3.6}$$

$$\sum_{m\geq l_1} c^m d_m|\varpi|^{-\frac{m}{2}}\cdot(\sum_{-l_1-m<n\leq -m} p_{n,m}(s)) \overset{1}{\sim} 0 \tag{4.3.7}$$

$$\sum_{m\geq l_1} c^m d_m|\varpi|^{-\frac{m}{2}}\cdot(\sum_{n\leq -l_1-m} p_{n,m}(s)) \overset{1}{\sim} 0. \tag{4.3.8}$$

For each $0\leq m<l_1, -l_1-m\leq n<-l_1$,

$$c^m d_m|\varpi|^{-\frac{m}{2}} p_{n,m}(s) \overset{1}{\sim} 0$$

and so (4.3.5) easily follows.
For each $m\in\mathbb{N}$,

$$\sum_{n\leq -l_1-m} p_{n,m}(s) \approx (c^{-2}|\varpi|^s)^m\cdot g_1(s) - (c^{-1}\alpha|\varpi|^{\frac{1}{2}+s})^m\cdot g_2(s)$$

where

$$g_1(s) = \frac{(c^{-1}\alpha|\varpi|^{\frac{3}{2}})^{l_1}}{1-c^{-2}|\varpi|} - \frac{(c^{-1}\alpha|\varpi|^{\frac{3}{2}+2s})^{l_1}}{1-c^{-2}|\varpi|^{1+2s}},\ g_2(s) = \frac{(c^{-1}\alpha|\varpi|^{\frac{3}{2}})^{l_1}}{1-c^{-1}\alpha|\varpi|^{\frac{3}{2}}} - \frac{(c^{-1}\alpha|\varpi|^{\frac{3}{2}+2s})^{l_1}}{1-c^{-1}\alpha|\varpi|^{\frac{3}{2}+2s}}$$

and so (4.3.6) and (4.3.8) follow from this.

For each $-l_1 < k < 0$, note that

$$\sum_{m\geq l_1} c^m d_m |\varpi|^{-\frac{m}{2}} p_{k-m,m}(s) \approx$$

$$(1-|\varpi|^{-2ks})\cdot \sum_{m\geq l_1}(c_1|\varpi|^{sm}+c_2(c^{-2}|\varpi|^s)^{2m})\cdot\left((c\alpha|\varpi|^{\frac{1}{2}})^{l_1}-(c\alpha|\varpi|^{\frac{1}{2}})^{m-k}\right)\sim 0$$

and so we have (4.3.7).

To prove

$$\sum_{m\in\mathbb{N}} c^m d_m|\varpi|^{-\frac{m}{2}}\sum_{n\in\mathbb{Z}} c^{2n}|\varpi|^n a_{n,m}\sum_{l\geq l_1, l\geq -n} r_{l,m,n}(s) \overset{1}{\sim} 0,$$

we first decompose $\sum_{n\in\mathbb{Z}} c^{2n}|\varpi|^n a_{n,m}\sum_{l\geq l_1,l\geq -n} r_{l,m,n}(s)$ for fixed m into three components

$$\sum_{n\geq 0} c^{2n}|\varpi|^n a_{n,m}\cdot|\varpi|^{s(2n+m)}\left(\frac{(c\alpha|\varpi|^{\frac{1}{2}+2s})^{l_1}}{1-c\alpha|\varpi|^{\frac{1}{2}+2s}}-\frac{(c\alpha|\varpi|^{\frac{1}{2}})^{l_1}}{1-c\alpha|\varpi|^{\frac{1}{2}}}\right)$$

$+$

$$\sum_{-m\leq n<0} c^{2n}|\varpi|^n a_{n,m}\left(|\varpi|^{s(2n+m)}\cdot\frac{(c\alpha|\varpi|^{\frac{1}{2}+2s})^{\max\{l_1,-n\}}}{1-c\alpha|\varpi|^{\frac{1}{2}+2s}}-|\varpi|^{sm}\cdot\frac{(c\alpha|\varpi|^{\frac{1}{2}})^{\max\{l_1,-n\}}}{1-c\alpha|\varpi|^{\frac{1}{2}}}\right)$$

$+$

$$\sum_{n<-m} c^{2n}|\varpi|^n a_{n,m}\left(|\varpi|^{s(2n+m)}\cdot\frac{(c\alpha|\varpi|^{\frac{1}{2}+2s})^{\max\{l_1,-n\}}}{1-c\alpha|\varpi|^{\frac{1}{2}+2s}}-|\varpi|^{-s(2n+m)}\cdot\frac{(c\alpha|\varpi|^{\frac{1}{2}})^{\max\{l_1,-n\}}}{1-c\alpha|\varpi|^{\frac{1}{2}}}\right).$$

Using the asymptotic formulae of d_m and $a_{n,m}$, one can easily see that

$$\sum_{m\in\mathbb{N}} c^m d_m|\varpi|^{-\frac{m}{2}}\sum_{n\geq 0} c^{2n}|\varpi|^n a_{n,m}\cdot|\varpi|^{s(2n+m)}\left(\frac{(c\alpha|\varpi|^{\frac{1}{2}+2s})^{l_1}}{1-c\alpha|\varpi|^{\frac{1}{2}+2s}}-\frac{(c\alpha|\varpi|^{\frac{1}{2}})^{l_1}}{1-c\alpha|\varpi|^{\frac{1}{2}}}\right)\overset{1}{\sim} 0.$$

Write $p^1_{n,m}(s) =$

$$c^m d_m|\varpi|^{-\frac{m}{2}}c^{2n}|\varpi|^n a_{n,m}\left(|\varpi|^{s(2n+m)}\cdot\frac{(c\alpha|\varpi|^{\frac{1}{2}+2s})^{\max\{l_1,-n\}}}{1-c\alpha|\varpi|^{\frac{1}{2}+2s}}-|\varpi|^{sm}\cdot\frac{(c\alpha|\varpi|^{\frac{1}{2}})^{\max\{l_1,-n\}}}{1-c\alpha|\varpi|^{\frac{1}{2}}}\right)$$

and note that $p^1_{n,m}(s)\sim 0$. The second sum is decomposed into

$$\sum_{0\leq m<l_1}\sum_{-m\leq n<0} p^1_{n,m}(s)+\sum_{l_1\leq m}\sum_{-l_1\leq n<0} p^1_{n,m}(s)+\sum_{l_1\leq m}\sum_{-m\leq n<-l_1} p^1_{n,m}(s)$$

and since $\sum_{0\le m<l_1}\sum_{-m\le n<0} p^1_{n,m}(s)$ is a finite sum, it is $\overset{1}{\sim}$ 0. For each $-l_1 \le n < 0$, one can easily check $\sum_{l_1\le m} p^1_{n,m}(s) \overset{1}{\sim} 0$ and so $\sum_{l_1\le m}\sum_{-l_1\le n<0} p^1_{n,m}(s) \overset{1}{\sim} 0$.
If $n < -l_1$,

$$|\varpi|^{s(2n+m)} \cdot \frac{(c\alpha|\varpi|^{\frac{1}{2}+2s})^{\max\{l_1,-n\}}}{1-c\alpha|\varpi|^{\frac{1}{2}+2s}} - |\varpi|^{sm} \cdot \frac{(c\alpha|\varpi|^{\frac{1}{2}})^{\max\{l_1,-n\}}}{1-c\alpha|\varpi|^{\frac{1}{2}}} = 0$$

and so $\sum_{l_1\le m}\sum_{-m\le n<-l_1} p^1_{n,m}(s) = 0$. Thus the second sum $\sum_{m\in\mathbb{N}}\sum_{-m<n\le 0} p^1_{n,m}(s) = 0$.

To show the third sum is $\overset{1}{\sim}$ 0, write $p^2_{n,m}(s) =$

$$c^{m+2n}d_m|\varpi|^{n-\frac{m}{2}}a_{n,m}\Big(|\varpi|^{s(2n+m)}\cdot\frac{(c\alpha|\varpi|^{\frac{1}{2}+2s})^{\max\{l_1,-n\}}}{1-c\alpha|\varpi|^{\frac{1}{2}+2s}} - |\varpi|^{-s(2n+m)}\cdot\frac{(c\alpha|\varpi|^{\frac{1}{2}})^{\max\{l_1,-n\}}}{1-c\alpha|\varpi|^{\frac{1}{2}}}\Big).$$

We decompose

$$\sum_{m\in\mathbb{N}}\sum_{n<-m} p^2_{n,m}(s) = \sum_{m\in\mathbb{N}}\sum_{-m-l_1<n<-m} p^2_{n,m}(s) + \sum_{m\in\mathbb{N}}\sum_{n\le -m-l_1} p^2_{n,m}(s).$$

Write $k = m+n$ and for each $-l_1 < k < 0$,

$$\sum_{m\in\mathbb{N}} p^2_{k-m,m}(s) \approx \sum_{m\ge l_1} p^2_{k-m,m}(s) = c^k(c_1(c\alpha|\varpi|^s)^m + c_2(c^{-1}\alpha|\varpi|^s)^m)\cdot g_k(s) \overset{1}{\sim} 0$$

where

$$g_k(s) = \frac{(c\alpha|\varpi|^{\frac{1}{2}})^{-k}}{1-c\alpha|\varpi|^{\frac{1}{2}+2s}} - \frac{(c\alpha|\varpi|^{\frac{1}{2}+s})^{-k}}{1-c\alpha|\varpi|^{\frac{1}{2}}}.$$

Thus $\sum_{m\in\mathbb{N}}\sum_{-m-l_1<n<-m} p^2_{n,m}(s) = 0$.

Next, for each $m \in \mathbb{N}$, some calculation shows that

$$\sum_{n\le -m-l_1} p^2_{n,m}(s) = c^m d_m|\varpi|^{-\frac{m}{2}}\cdot k_2^2\cdot(c^{-1}\alpha|\varpi|^{\frac{1}{2}+s})^m\cdot g(s) \text{ where}$$

$$g(s) = \frac{(c^{-1}\alpha|\varpi|^{\frac{3}{2}})^{l_1}}{(1-c\alpha|\varpi|^{\frac{1}{2}+2s})(1-c^{-1}\alpha|\varpi|^{\frac{3}{2}})} - \frac{(c^{-1}\alpha|\varpi|^{\frac{3}{2}+2s})^{l_1}}{(1-c\alpha|\varpi|^{\frac{1}{2}})(1-c^{-1}\alpha|\varpi|^{\frac{3}{2}+2s})}$$

and so $\sum_{m\in\mathbb{N}}\sum_{n\le -m-l_1} p^2_{n,m}(s) \sim 0$. Thus we have showed (4.3.2)$\overset{1}{\sim}$ 0.

Now, we will show (4.3.3) $\overset{1}{\sim}$ 0. To do this, for each $-l_1 < l < l_1$, we decompose

$$\sum_{n\in\mathbb{Z},m\ge 0} |\varpi|^{-m}d_m\cdot\alpha^l(|\varpi|^{s(|n+m+l|+|n+l|)} - |\varpi|^{s(|n+m|+|n|)})\cdot c_{n,m,l}$$

into three summands $\sum_{m\in\mathbb{N},n\geq l_1} + \sum_{m\in\mathbb{N},-l_1<n<l_1} + \sum_{m\in\mathbb{N},n\leq -l_1}$ and show that each is $\overset{1}{\sim} 0$.

Write $f_{n,m,l}(s) = |\varpi|^{-m}d_m \cdot \alpha^l(|\varpi|^{s(|n+m+l|+|n+l|)} - |\varpi|^{s(|n+m|+|n|)}) \cdot c_{n,m,l}$ and note that for each fixed $n, m, l,\ f_{n,m,l} \sim 0$.
For each $-l_1 < l < l_1$, we see that
$\sum_{m\in\mathbb{N},n\geq l_1} f_{n,m,l}(s) \approx (\sum_{n\geq l_1}(c^2|\varpi|^{1+2s})^n)\cdot(\sum_{m\in\mathbb{N}} c_1\cdot(c^2|\varpi|^s)^m + c_2|\varpi|^{sm})\cdot(|\varpi|^{2ls}-1) \overset{1}{\sim} 0.$

For all $-l_1 < n, l < l_1$, there exists $N_1 \in \mathbb{N}$ such that $N_1 > 2l_1$ and if $m \geq N_1$, then $c_{n,m,l} = (c|\varpi|^{\frac{1}{2}})^m \cdot f_{n,l}$ for some constants $f_{n,l}$. Thus

$$\sum_{m\geq 0,-l_1<n<l_1} f_{n,m,l}(s) = \sum_{0\leq m<N_1,-l_1<n<l_1} f_{n,m,l}(s) +$$

$$\sum_{m\geq N_1,-l_1<n<l_1} c_1(c^2|\varpi|^s)^m + c_2|\varpi|^{sm}) \cdot (|\varpi|^{s(n+l+|n+l|)} - |\varpi|^{s(n+|n|)})\alpha^l \cdot f_{n,l}$$

and so

$$\sum_{m\geq 0,-l_1<n<l_1} f_{n,m,l}(s) \overset{1}{\sim} 0.$$

Next we decompose

$$\sum_{m\geq 0,n\leq -l_1} f_{n,m,l}$$

into four summands

$$\sum_{n\leq -l_1,m+n\geq \max\{-l,0\}} + \sum_{n\leq -l_1,-l\leq m+n<0} + \sum_{n\leq -l_1,0\leq m+n<-l} + \sum_{n\leq -l_1,m+n<\min\{-l,0\}}.$$

The first sum is zero. The second sum is $\sum_{-l\leq k<0}\sum_{m\geq k+l_1} f_{k-m,m,l}$ and for each $-l \leq k < 0$, there exists $N_2 \in \mathbb{N}$ such that $N_2 \geq l_1$ and if $m \geq N_2$, then $c_{k-m,m,l} \approx |\varpi|^{\frac{m}{2}} \cdot c^{-m}$. Thus

$$\sum_{-l\leq k<0}\sum_{m\geq k+l_1} f_{k-m,m,l} \approx$$

$$(\sum_{k+l_1\leq m<N_2} f_{k-m,m,l}) + \Big((1 - |\varpi|^{-2ks}) \cdot \sum_{m\geq N_2}(c_1|\varpi^s|^m + c_2|c^{-2}\varpi^s|^m)\Big) \overset{1}{\sim} 0.$$

Similarly, we can show the third sum $\overset{1}{\sim} 0$.
The fourth sum is decomposed into

$$\sum_{n\leq -l_1,-l_1<m+n<\min\{-l,0\}} f_{n,m,l} + \sum_{n\leq -l_1,m+n\leq -l_1} f_{n,m,l}$$

and as we have done in the above, it is easy to see

$$\sum_{n\leq -l_1,-l_1<m+n<\min\{-l,0\}} f_{n,m,l} \overset{1}{\sim} 0.$$

Note

$$\sum_{n\le -l_1, m+n\le -l_1} f_{n,m,l} = \sum_{0\le m<l_1, n\le -l_1, m+n\le -l_1} f_{n,m,l} + \sum_{m\ge l_1, n\le -l_1, m+n\le -l_1} f_{n,m,l}.$$

For each $0 \le m < l_1$,

$$\sum_{n\le -l_1-m} f_{n,m,l} \approx d_m (c|\varpi|^{-(\frac{3}{2}+s)})^m \cdot (|\varpi|^{-2ls} - 1) \cdot \sum_{n\le -l_1-m} (c^2|\varpi|^{-(1+2s)})^n \overset{1}{\sim} 0.$$

On the other hand,

$$\begin{aligned} &\sum_{m\ge l_1} \sum_{n\le -l_1-m} f_{n,m,l} \\ &\approx (|\varpi|^{-2ls} - 1)(c_1(c^2|\varpi|^{-(1+s)})^m + c_2|\varpi|^{-(1+s)m}) \cdot \sum_{n\le -l_1-m} (c^2|\varpi|^{-(1+2s)})^n \\ &= (c^{-2}|\varpi|^{1+2s})^{l_1} \cdot (|\varpi|^{-2ls} - 1) \cdot \Big(\sum_{m\ge l_1} c_1 \cdot |\varpi|^{sm} + c_2 \cdot (c^{-2}|\varpi|^{s})^m \Big) \overset{1}{\sim} 0. \end{aligned}$$

Thus we see that the fourth sum $\sum_{n\le -l_1, m+n<\min\{-l,0\}} f_{n,m,l}$ is also $\overset{1}{\sim} 0$ and we showed (4.3.3) $\overset{1}{\sim} 0$.

Last, we will show (4.3.4) $\overset{1}{\sim} 0$. To do this, write $\sum_{l\le -l_1} c^l \alpha^l |\varpi|^{-\frac{l}{2}} (|\varpi|^{s(|n+m+l|+|n+l|)} - |\varpi|^{s(|n+m|+|n|)}$ as

$$\sum_{l\ge l_1} (c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^l \cdot (|\varpi|^{s(|n+m-l|+|n-l|)} - |\varpi|^{s(|n+m|+|n|)}$$

and decompose it into three summands

$$\sum_{l\ge l_1, l>(n+m)} \Big((c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+2s})^l \cdot |\varpi|^{-s(2n+m)} - (c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^l \cdot |\varpi|^{s(|n+m|+|n|)} \Big)$$

$+$

$$\sum_{l\ge l_1, n<l\le n+m} (c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^l \cdot (|\varpi|^{sm} - |\varpi|^{s(|n+m|+|n|)})$$

$+$

$$\sum_{l\ge l_1, l\le n} \Big((c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}-2s})^l \cdot |\varpi|^{s(2n+m)} - (c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^l \cdot |\varpi|^{s(2n+m)} \Big).$$

We write $M_{n,m} = \max\{l_1, m+n+1\}$. Then for fixed $m, n \in \mathbb{N}$ and small $\Re(s) > 0$,

$$\sum_{l\ge l_1, l>(n+m)} \Big((c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+2s})^l \cdot |\varpi|^{-s(2n+m)} - (c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^l \cdot |\varpi|^{s(|n+m|+|n|)} \Big) =$$

$$\frac{|\varpi|^{-s(2n+m)} (c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+2s})^{M_{n,m}}}{1 - c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+2s}} - \frac{|\varpi|^{s(|n+m|+|n|)} (c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^{M_{n,m}}}{1 - c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}}}.$$

Denote

$$c^{2n+m}|\varpi|^{n-\frac{m}{2}}d_m a_{n,m}\cdot\Big(\frac{|\varpi|^{-s(2n+m)}(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+2s})^{M_{n,m}}}{1-c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+2s}}-\frac{|\varpi|^{s(|n+m|+|n|)}(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^{M_{n,m}}}{1-c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}}}\Big)$$

by $g_{n,m}(s)$ and note $g_{n,m}(s)\sim 0$. We shall show $\sum_{m\geq 0,n\in\mathbb{N}} g_{n,m}(s) \overset{1}{\sim} 0$.

Decompose $\sum_{m\geq 0,n\in\mathbb{N}} g_{n,m}(s)$ into

$$\sum_{m\geq 0,n\geq 0} g_{n,m}(s)+\sum_{m\geq -n,n<0} g_{n,m}(s)+\sum_{m<-n,n<0} g_{n,m}(s)$$

and the first sum decomposes again into

$$\sum_{0\leq m\leq l_1-1}\ \sum_{0\leq n\leq l_1-m-1} g_{n,m}(s)+\sum_{0\leq m\leq l_1-1}\ \sum_{l_1-m\leq n} g_{n,m}(s)+\sum_{l_1\leq m}\sum_{0\leq n} g_{n,m}(s).$$

Since the first term in the above is a finite sum, $\sum_{0\leq m\leq l_1-1}\sum_{0\leq n\leq l_1-m-1} g_{n,m}(s)\sim 0$. For each $0\leq m\leq l_1-1$, $\sum_{l_1-m\leq n} g_{n,m}(s)\sim\sum_{l_1\leq n} g_{n,m}(s)\approx d_m(\alpha^{-1}|\varpi|^{s})^m\cdot\sum_{n\geq l_1} g_n^1(s)$ where

$$g_n^1(s)=\frac{(c\alpha^{-1}|\varpi|^{\frac{3}{2}})^n(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+2s})}{1-c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+2s}}-\frac{(c\alpha^{-1}|\varpi|^{\frac{3}{2}+2s})^n(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+2s})}{1-c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}}}$$

and note that $\sum_{n\geq l_1} g_n^1(s)\sim 0$. Thus the second term $\sum_{0\leq m\leq l_1-1}\sum_{l_1-m\leq n} g_{n,m}(s)\sim 0$.

The third term $\sum_{l_1\leq m}\sum_{0\leq n} g_{n,m}(s)$ is $\sum_{l_1\leq m}\sum_{0\leq n<l_1} g_{n,m}(s)+\sum_{l_1\leq m}\sum_{l_1\leq n} g_{n,m}(s)$. For each $0\leq n<l_1$, $\sum_{l_1\leq m} g_{n,m}=\big(\sum_{l_1\leq m} c_1(c\alpha^{-1}|\varpi|^{\frac{1}{2}+s})^m+c_2(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+s})^m\big)\cdot g_n^2(s)$ where

$$g_n^2(s)=\Big(\frac{(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^n(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+2s})}{1-c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+2s}}-\frac{(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+2s})^n(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})}{1-c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}}}\Big).$$

Thus $\sum_{l_1\leq m}\sum_{0\leq n<l_1} g_{n,m}(s)\sim 0$ and

$$\sum_{l_1\leq m}\sum_{l_1\leq n} g_{n,m}(s)\approx\Big(\sum_{n\geq l_1} g_n^1(s)\Big)\cdot\Big(\sum_{l_1\leq m} c_1(c\alpha^{-1}|\varpi|^{\frac{1}{2}+s})^m+c_2(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+s})^m\Big)\sim 0.$$

Since the above three components of $\sum_{m\geq 0,n\geq 0} g_{n,m}(s)$ are all ~ 0,

$$\sum_{m\geq 0,n\geq 0} g_{n,m}(s)\sim 0.$$

Next we divide $\sum_{m\geq -n, n<0} g_{n,m}(s) = \sum_{m\geq -n, -l_1<n<0} g_{n,m}(s) + \sum_{m\geq -n, n\leq -l_1} g_{n,m}(s)$. For each $-l_1 < n < 0$,

$$\sum_{m\geq -n+l_1} g_{n,m}(s) = \Big(\frac{c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+2s}}{1-c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+2s}} - \frac{c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}}}{1-c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}}}\Big)k_n \cdot (c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^n \cdot f_n(s)$$

where $k_n = \int_{F^3} \varphi(0, \varpi^n x_2, 0)\cdot \varphi(x_1, x_2, x_3)dX$ and

$$f_n^1(s) = \frac{c_1(\alpha^{-1}|\varpi|^{\frac{1}{2}+s})^{-n+l_1}}{1-\alpha^{-1}|\varpi|^{\frac{1}{2}+s}} + \frac{c_2(c^{-2}\alpha^{-1}|\varpi|^{\frac{1}{2}+s})^{-n+l_1}}{1-c^{-2}\alpha^{-1}|\varpi|^{\frac{1}{2}+s}}.$$

Thus

$$\sum_{m\geq -n, -l_1<n<0} g_{n,m}(s) = \sum_{-n\leq m<-n+l_1, -l_1<n<0} g_{n,m}(s) + \sum_{m\geq -n+l_1, -l_1<n<0} g_{n,m}(s) \sim 0.$$

Next, we divide

$$\sum_{m<-n, n<0} g_{n,m}(s) = \sum_{n<-l_1, 0\leq m<-n-l_1} g_{n,m}(s) + \sum_{n<0, -n-l_1\leq m<-n} g_{n,m}(s).$$

Again,

$$\sum_{n<-l_1, 0\leq m<-n-l_1} g_{n,m}(s) \sim \sum_{n\leq -2l_1, 0\leq m<l_1} g_{n,m}(s) + \sum_{n\leq -2l_1, l_1\leq m<-n-l_1} g_{n,m}$$

and for each $0\leq m<l_1$, $\sum_{n\leq -2l_1} g_{n,m} =$

$$d_m(c|\varpi|^{-\frac{3}{2}-s})^m \cdot \Big(\frac{(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+2s})^{l_1}}{1-c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+2s}} - \frac{(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^{l_1}}{1-c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}}}\Big)\cdot \sum_{n\leq -2l_1} (c^2|\varpi|^{-1-2s})^n \sim 0.$$

Note that $\sum_{n\leq -2l_1, l_1\leq m<-n-l_1} g_{n,m} =$

$$\Big(\frac{(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+2s})^{l_1}}{1-c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+2s}} - \frac{(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^{l_1}}{1-c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}}}\Big)\cdot \sum_{n\leq -2l_1} f_n^2(s)\cdot (c^2|\varpi|^{-1-2s})^n$$

where

$$f_n^2(s) = c_1 \cdot \frac{(c^2|\varpi|^{-1-s})^{l_1} - (c^2|\varpi|^{-1-s})^{-n-l_1}}{1-c^2|\varpi|^{-1-s}} + c_2\cdot \frac{(|\varpi|^{-1-s})^{l_1} - (|\varpi|^{-1-s})^{-n-l_1}}{1-|\varpi|^{-1-s}}.$$

Thus $\sum_{n\le -2l_1, l_1\le m<-n-l_1} g_{n,m}(s) \overset{1}{\sim} 0$ and so $\sum_{n<-l_1, 0\le m<-n-l_1} g_{n,m}(s) \overset{1}{\sim} 0$.

To show $\sum_{n<0,-n-l_1\le m<-n} g_{n,m}(s) \overset{1}{\sim} 0$, let $k = n+m$ and for each $-l_1 \le k < 0$, we will check $\sum_{n<0} g_{n,k-n}(s) \overset{1}{\sim} 0$.

$$\sum_{n<0} g_{n,k-n}(s) \sim \sum_{n<-2l_1} g_{n,k-n}(s) \approx (c_1 \cdot \sum_{n<-2l_1} |\varpi|^{(-\frac{1}{2}-s)n} + c_2 \cdot \sum_{n<-2l_1} (c^2|\varpi|^{(-\frac{1}{2}-s)})^n)$$

where

$$f_n^3(s) = \frac{(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+2s})^{l_1}}{1-c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+2s}} - \frac{(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^{l_1}}{1-c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}}}.$$

Thus $\sum_{n<0,-n-l_1\le m<-n} g_{n,m}(s) \overset{1}{\sim} 0$ and so we checked

$$\sum_{l\ge l_1, l>(n+m)} \big((c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}+2s})^l \cdot |\varpi|^{-s(2n+m)} - (c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^l \cdot |\varpi|^{s(|n+m|+|n|)}\big) \overset{1}{\sim} 0.$$

Next we turn to show

$$\sum_{n\in\mathbb{Z}, m\ge 0} c^{2n+m}|\varpi|^{n-\frac{m}{2}} d_m a_{n,m} \cdot \Big(\sum_{l\ge l_1, n<l\le n+m} (c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^l \cdot (|\varpi|^{sm} - |\varpi|^{s(|n+m|+|n|)})\Big) \sim 0.$$

It equals $\sum_{m\ge 0} c^m d_m |\varpi|^{-\frac{m}{2}} \sum_{n\ge 0} f_{n,m}(s)$ where

$$f_{n,m}(s) = c^{2n}|\varpi|^n a_{n,m} \cdot \Big(\sum_{l\ge l_1, n<l\le n+m} (c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^l \cdot (|\varpi|^{sm} - |\varpi|^{s(2n+m)})\Big).$$

Then

$$\sum_{n\ge 0} f_{n,m}(s) = \sum_{0\le n\le l_1-1} f_{n,m}(s) + \sum_{l_1\le n} f_{n,m}(s)$$

and

$$\sum_{m\ge 0} c^m d_m |\varpi|^{-\frac{m}{2}} (\sum_{0\le n\le l_1-1} f_{n,m}(s)) \sim \sum_{m\ge 2l_1} c^m d_m |\varpi|^{-\frac{m}{2}} (\sum_{0\le n\le l_1-1} f_{n,m}(s)) =$$

$$\sum_{0\le n\le l_1-1} c^{2n}|\varpi|^n(1-|\varpi|^{2ns}) \sum_{m\ge 2l_1} c^m d_m |\varpi|^{(s-\frac{1}{2})m} a_{n,m} \cdot \frac{(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^{l_1} - (c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^{n+m+1}}{1-c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}}}.$$

For each $0 \le n \le l_1 - 1$,

$$\sum_{m\ge 2l_1} c^m d_m |\varpi|^{(s-\frac{1}{2})m} a_{n,m} \cdot \frac{(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^{l_1} - (c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^{n+m+1}}{1-c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}}} \overset{1}{\sim} 0$$

and so

$$\sum_{m\geq 0} c^m d_m |\varpi|^{-\frac{m}{2}} \Big(\sum_{0\leq n\leq l_1-1} f_{n,m}(s) \Big) \overset{1}{\sim} 0.$$

For each $m \in \mathbb{N}$,

$$\sum_{n\geq l_1} f_{n,m} \approx |\varpi|^{sm}\big(1-(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^m\big)\cdot\Big(\sum_{n\geq l_1}(c\alpha^{-1}|\varpi|^{\frac{3}{2}})^n-(c\alpha^{-1}|\varpi|^{\frac{3}{2}+2s})^n\Big).$$

Thus $\sum_{m\geq 0} c^m d_m|\varpi|^{-\frac{m}{2}}\sum_{n\geq l_1} f_{n,m} \overset{1}{\sim} 0$ and so we showed

$$\sum_{n\in\mathbb{Z},m\geq 0} c^{2n+m}|\varpi|^{n-\frac{m}{2}} d_m a_{n,m}\cdot\Big(\sum_{l\geq l_1, n<l\leq n+m}(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^l\cdot(|\varpi|^{sm}-|\varpi|^{s(|n+m|+|n|)})\Big) \overset{1}{\sim} 0.$$

Finally, we investigate the last sum

$$\sum_{n\in\mathbb{Z},m\geq 0} c^{2n+m}|\varpi|^{(1+2s)n+(s-\frac{1}{2})m} d_m a_{n,m}\cdot\Big(\sum_{l_1\leq l\leq n}(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}-2s})^l-(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^l\Big). \tag{4.3.9}$$

It equals

$$k_1^1\cdot\sum_{m\geq 0} d_m(c|\varpi|^{s-\frac{1}{2}})^m\Big(\sum_{n\geq l_1}(c^2|\varpi|^{1+2s})^n\cdot g_n(s)\Big) \text{ where}$$

$g_n(s) =$

$$\frac{(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}-2s})^{l_1}-(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}-2s})^{n+1}}{1-c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}-2s}}-\frac{(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^{l_1}-(c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}})^{n+1}}{1-c^{-1}\alpha^{-1}|\varpi|^{\frac{1}{2}}}.$$

Thus $\sum_{n\geq l_1}(c^2|\varpi|^{1+2s})^n\cdot g_n(s) \sim 0$ and $\sum_{m\geq 0} d_m(c|\varpi|^{s-\frac{1}{2}})^m \overset{2}{\approx} 0$, and so we see that 4.3.9 $\overset{1}{\sim}$ 0.
We have checked (4.3.4)$\overset{1}{\sim}$ 0.

Putting all these things together, we verified our claim (4.3.1).

□

Chapter 5

The comparison of two L-values

In this chapter, for $\pi_3 = \Theta(\bar{\sigma})$ and $\pi_2 = \Theta(\bar{\mathbb{I}})$, we shall show that the L-function in the Refined Gross-Prasad conjecture 1.0.1

$$\frac{L_E(s, BC(\pi_2) \boxtimes BC(\pi_3))}{L_F(s+\frac{1}{2}, \pi_2, \mathrm{Ad})L_F(s+\frac{1}{2}, \pi_3, \mathrm{Ad})}$$

would have double pole at $s = \frac{1}{2}$ and relate this to our L-value

$$\lim_{s \to 0+} \frac{L_E(\frac{1}{2}, BC(\sigma) \otimes \gamma)\zeta_E(s)L_E(0, \gamma^2)}{\zeta_F(s)L^3(1, \chi_{E/F})\zeta_F(2)L_E(\frac{3}{2}, BC(\sigma) \otimes \gamma^3)} \text{ appearing in Remark 1.0.6.}$$

Since $BC(\theta_v(\bar{\sigma}_v))$ is non-tempered for almost all places, by Theorem 5.1.1 in [7], there exists a set of finite places S such that for $v \notin S$, $BC(\theta_v(\bar{\sigma_v}))$ has L-parameter $\gamma \cdot |\cdot|_{E_v}^{\frac{1}{2}} \oplus BC(\bar{\sigma}_v) \cdot \gamma^{-2} \oplus \gamma \cdot |\cdot|_{E_v}^{-\frac{1}{2}}$ as a representation of $WD(E_v)$. We also know that the L-parameter of $BC(\theta_v(\bar{\mathbb{I}}_v))$ is $BC(\bar{\mathbb{I}}_v)\gamma^{-1} \boxplus \gamma$.

On the other hand, by results in [6], if π is an automorphic represenation of $U(n)(\mathbb{A}_F)$, then

$$L_F(s, \pi, Ad) = L_F(s, BC(\pi), \mathrm{Asai}^{(-1)^n}).$$

(Here, we view $BC(\pi)$ as a representation of $GL(n)(F)$ via restriction of scalar $Res_{E/F}$ and RHS L-function is 'Asai' (if n is even) or 'twisted Asai' (if n is odd) L-function.)

Then for $v \notin S$, we can easily check the following

(1) $L_{F_v}(s, \theta_v(\bar{\sigma_v}), Ad) = L_{F_v}(s+1, \chi_{E_v/F_v}) \cdot L_{F_v}(s-1, \chi_{E_v/F_v}) \cdot L_{F_v}(s, \chi_{E_v/F_v})^3 \cdot$

$$L_{E_v}(s+\frac{1}{2}, BC(\bar{\sigma_v})\gamma^{-3}) \cdot L_{E_v}(s-\frac{1}{2}, BC(\bar{\sigma_v})\gamma^{-3}),$$

(2) $L_{F_v}(s, \theta_v(\bar{\mathbb{I}}_v), Ad) = L_{F_v}(s, \chi_{E_v/F_v})^2 \cdot L_{E_v}(s, \gamma^2)$,

(3) $L_{E_v}(s, BC(\theta_v(\bar{\sigma_v})) \boxtimes BC(\theta_v(\bar{\mathbb{I}}_v))) = \zeta_{E_v}(s+\frac{1}{2}) \cdot \zeta_{E_v}(s-\frac{1}{2})\cdot$

$$L_{E_v}(s, BC(\bar{\sigma_v}) \cdot \gamma^{-1}) \cdot L_{E_v}(s, BC(\bar{\sigma_v}) \cdot \gamma^{-3}) \cdot L_{E_v}(s+\frac{1}{2}, \gamma^2) \cdot L_{E_v}(s-\frac{1}{2}, \gamma^2).$$

Thus for $s \notin S$,

$$\frac{L_{E_v}(s, BC(\theta_v(\bar{\mathbb{I}}_v) \boxtimes BC(\theta_v(\bar{\sigma_v})))}{L_{F_v}(s+\frac{1}{2}, \theta_v(\bar{\mathbb{I}}_v), \mathrm{Ad}) L_{F_v}(s+\frac{1}{2}, \theta_v(\bar{\sigma_v}), \mathrm{Ad})} =$$

$$\frac{L_E(s, BC(\bar{\sigma_v}) \otimes \gamma^{-1}) \cdot \zeta_{E_v}(s+\frac{1}{2}) \cdot \zeta_{E_v}(s-\frac{1}{2}) \cdot L_{E_v}(s-\frac{1}{2}, \gamma^2)}{L_{E_v}(s+1, BC(\bar{\sigma_v}) \cdot \gamma^{-3}) \cdot L_{F_v}(s-\frac{1}{2}, \chi_{E_v/F_v}) \cdot L_{F_v}(s+\frac{1}{2}, \chi_{E_v/F_v})^5 \cdot L_{F_v}(s+\frac{3}{2}, \chi_{E_v/F_v})}$$

and so the partial L-function

$$\frac{L_E^S(s, BC(\theta(\bar{\mathbb{I}}) \boxtimes BC(\theta(\bar{\sigma}))}{L_F^S(s+\frac{1}{2}, \theta(\bar{\mathbb{I}}), \mathrm{Ad}) \cdot L_F^S(s+\frac{1}{2}, \theta(\bar{\sigma}), \mathrm{Ad})}$$

has at most double pole at $s = \frac{1}{2}$ and so does the complete L-function because all local L-factors are holomorphic and nonzero there.
By Lemma 3.5 in [6], we see that

$$L_{E_v}(\frac{1}{2}, BC(\bar{\sigma_v}) \otimes \gamma^{-1}) = L_{E_v}(\frac{1}{2}, BC(\sigma_v) \otimes \gamma) \text{ and}$$

$$L_{E_v}(s+1, BC(\bar{\sigma_v}) \cdot \gamma^{-3}) = L_{E_v}(s+1, BC(\sigma_v) \cdot \gamma^3).$$

Thus our partial L-value in Remark 1.0.6 can be written as the limit of

$$\frac{L^S(s, \chi_{E/F}) \cdot L^S(s+1, \chi_{E/F})^2 \cdot L^S(s+2, \chi_{E/F})}{\zeta^S(s) \cdot \zeta_E^S(s+1) \cdot \zeta^S(s+2)} \times$$

$$\frac{L_E^S(s+\frac{1}{2}, BC(\theta(\bar{\mathbb{I}}) \boxtimes BC(\theta(\bar{\sigma}))}{L_F^S(s+1, \theta(\bar{\mathbb{I}}), \mathrm{Ad}) \cdot L_F^S(s+1, \theta(\bar{\sigma}), \mathrm{Ad})}$$

as s goes to 0.

Part II

A uniqueness theorem for functions in the extended Selberg class

Chapter 6

A preview of the second part

Most L-functions used in number theory share some common analytic properties such as meromorphic continuation and functional equation and Euler product. Furthermore, they are also expected to satisfy certain Riemann hypothesis type conjectures. This observations push to make an axiomatic definition of a class which contains all these L-functions. In [36] A. Selberg introduced a class of meromorphic functions $L(s)$, now called the *Selberg class* and denoted by $\mathscr{S}$, satisfying the following five axioms:
(1) (*Dirichlet series*) $L(s)$ has a Dirichlet series representation

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \qquad (s = \sigma + it)$$

which is absolutely convergent for $\sigma > 1$.
(2) (*Analytic continuation*) There is a nonnegative integer m such that $(s-1)^m L(s)$ is an entire function of finite order.
(3) (*Functional equation*) $L(s)$ satisfies a functional equation of the form

$$\Phi(s) = \omega \overline{\Phi(1-\bar{s})},$$

where

$$\Phi(s) = Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + \mu_j) L(s)$$

with $Q > 0$, $\lambda_j > 0$, $\operatorname{Re} \mu_j \geqslant 0$ and $|\omega| = 1$.
(4) (*Ramanujan hypothesis*) For every $\epsilon > 0$, we have $a(n) \ll_\epsilon n^\epsilon$.
(5) (*Euler product*) For all sufficiently large σ,

$$L(s) = \prod_p \exp\Big(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}} \Big),$$

where the product is over all primes p and $b(p^k) \ll p^{k\theta}$ for some fixed $\theta < \frac{1}{2}$.

It is sometimes convenient to consider the wider class of functions that satisfy axioms (1)–(3) above and do not vanish identically. J. Kaczorowski and A. Perelli [23] call this set the *extended Selberg class* and denote it by $\mathscr{S}^\sharp$. For $L(s)$ in either class, we define the *degree* of $L(s)$ to be

$$\lambda = 2\sum_j \lambda_j. \tag{6.0.1}$$

Given a Dirichlet series $L(s)$ and a complex number c, we let $L^{-1}(c)$ denote the preimage of c under L, that is, $L^{-1}(c) = \{s \in \mathbb{C} : L(s) = c\}$. It is relatively straightforward to show that if two Dirichlet series $L_1(s)$ and $L_2(s)$ satisfy axioms (1)–(3), have constant coefficients $a_1(1) = 1$ and $a_2(1) = 1$, and take a value c at exactly the same points with the same multiplicities, then $L_1(s) \equiv L_2(s)$ (see J. Steuding [38, p. 152]). However, if we drop the requirement that all multiplicities match, this becomes a more difficult problem. Let $N_L^c(T)$ denote the number of zeros of $L(s) - c$ in the rectangle $0 \leqslant \operatorname{Re} s \leqslant 1$, $|t| \leqslant T$ counting multiplicities, and let $\widetilde{N}_L^c(T)$ denote the number of distinct zeros in this rectangle. J. Steuding [38, p. 152] proved the following theorem.

Theorem A. *Suppose that two Dirichlet series $L_1(s)$ and $L_2(s)$ satisfy axioms (1)–(4), share the same functional equation, and have leading coefficients $a_1(1) = a_2(1) = 1$. Suppose also that $L_1^{-1}(c_j) = L_2^{-1}(c_j)$ for two distinct complex numbers c_1 and c_2, and that for either $j = 1$ or 2 we have*

$$\liminf_{T\to\infty} \frac{\widetilde{N}_{L_j}^{c_1}(T) + \widetilde{N}_{L_j}^{c_2}(T)}{N_{L_j}^{c_1}(T) + N_{L_j}^{c_2}(T)} > \frac{1}{2}, \tag{6.0.2}$$

Then $L_1(s) \equiv L_2(s)$.

The condition (6.0.2) is quite difficult to verify and, as of this writing, is not known to hold for any L-function of degree greater than 1. Thus, B. Q. Li [27] made a substantial improvement by removing it.

Theorem B. *Suppose that two Dirichlet series $L_1(s)$ and $L_2(s)$ satisfy axioms (1)–(4), share the same functional equation, and have leading coefficients $a_1(1) = a_2(1) = 1$. If $L_1^{-1}(c_j) = L_2^{-1}(c_j)$ for two distinct complex numbers c_1 and c_2, then $L_1(s) \equiv L_2(s)$.*

Recently, Ki [22] made a further improvement by showing that if $L_1(s)$ and $L_2(s)$ have positive degree, one can dispense with axiom (4) and, more importantly, one only needs to assume $L_1^{-1}(c) = L_2^{-1}(c)$ for a *single* nonzero value of c.

Theorem C. *Suppose that two Dirichlet series $L_1(s)$ and $L_2(s)$ are in the extended Selberg class, that is, they satisfy axioms (1)–(3), have positive degree, have leading coefficients $a_1(1) = a_2(1) = 1$, and share the same functional equation. If $L_1^{-1}(c) = L_2^{-1}(c)$ for a nonzero complex number c, then $L_1(s) \equiv L_2(s)$.*

Our purpose is to show that one can even dispense with the condition that $L_1(s)$ and $L_2(s)$ satisfy the same functional equation and that $a_1(1) = a_2(1)$.

Main Theorem. *([11]) Suppose that two Dirichlet series $L_1(s)$ and $L_2(s)$ are in the extended Selberg class, have positive degree. If $L_1^{-1}(c) = L_2^{-1}(c)$ for a nonzero complex number c, then $L_1(s) \equiv L_2(s)$.*

To show that the conclusion of Main Theorem (and Theorem C) need not hold if the L-functions have degree zero, H. Ki [22] notes that

$$L_1(s) = 1 + \frac{\sqrt{6}}{2^s} + \frac{2}{4^s}, \qquad L_2(s) = 1 + \frac{3\sqrt{6}}{2^s} + \frac{18}{4^s} + \frac{6\sqrt{6}}{8^s} + \frac{4}{16^s}$$

satisfy all the other conditions of Theorem 1 and that

$$L_2(s) - 1 = \frac{4^s}{2}\left(L_1(s) - 1\right)^3.$$

Thus, $L_1^{-1}(1) = L_2^{-1}(1)$, but $L_1(s) \not\equiv L_2(s)$. To see that the case $c = 0$ must be excluded, one can take $L_1(s) = L(s)$ and $L_2(s) = L(s)^2$ for any nontrivial $L(s) \in \mathscr{S}^\sharp$.

The question naturally arises as to what additional conditions must be imposed in order for Theorem 1 to remain valid when $c = 0$. We say that a nontrivial function $L(s)$ in the extended Selberg class $S^\sharp$ is *primitive* if $L(s) \equiv L_1(s)L_2(s)$ for some two functions $L_1(s), L_2(s) \in S^\sharp$, then $L_1(s) = \text{const}$ or $L_2(s) = \text{const}$. In [10] it was shown that the main theorem with $c = 0$ is true for degree 1 functions $L_1(s), L_2(s)$ in the Selberg class. We conjecture that main theorem with $c = 0$ holds for two functions $L_1(s), L_2(s)$ in the extended Selberg class $\mathscr{S}^\sharp$ provided that $L_1(s)$ and $L_2(s)$ are primitive. Indeed, there is a fundamental conjecture in this direction as follow.

Conjecture. *No two distinct primitive function in the Selberg class share any nontrivial complex zeros.*

It is widely believed that it should be true, but it seems to be in a very remote future.

This rest of the part is organized as follows. In chapter 2, we give three prominent examples of Selberg class, Dirichlet L-functions, Artin-L-functions and automorphic L-functions and survey some conjectures on Selberg class. In chapter 3, we first prove 3 lemmas and then prove our main theorem.

Chapter 7

Examples and related conjectures

In this chapter, we give three examples of Selberg class. Since Jerzy Kaczorowski made a neat exposition on them, we his treatise in [21].

7.1 Examples of Selberg class

Riemann zeta function and Dirichlet L-functions

For $\sigma > 1$, the Riemann zeta function is defined by the absolutely convergent Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and its analytic continuation elsewhere. The only singularity is the simple pole at $s = 1$ whose residue is 1. For $\sigma > 1$, It is well known that it has the Euler product

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \tag{7.1.1}$$

where p runs over primes. Furthermore, using the transformation formula of elliptic theta series and Mellin inversion formula of it, the functional equation

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)$$

where $\Gamma(s)$ is the Gamma function.

By (7.1.1), $\zeta(s)$ has no zero in the half plane $\sigma > 1$ and by functional equation and since the Gamma function that $\Gamma(s)$ has simple poles at $s = 0, -1, -2, \ldots$ and no zeros, it has trivial zeros at even negative integers. It is known that infinitely many non-trivial zeros are in critical strip $0 < \sigma < 1$ and Riemann hypothesis predicts that all lie in critical line $s = \frac{1}{2}$.

Riemann zeta function can be generalized to Dirichlet L-function. For an integer $k > 0$, a group homomorphism $\chi : (\mathbb{Z}/k\mathbb{Z})^\times \to S^1$ can be extended to on integers relatively prime to k and then whole interger if we define $\chi(n) = 0$ for $(n,k) \neq 1$. These characters are called Dirichlet character (mod k) and

$$\chi_0(n) = \begin{cases} 1 & \text{if } (n,k) = 1 \\ 0 & \text{otherwise} \end{cases}$$

is called principal character (mod k). For $k'|k$, there is a natural map $(\mathbb{Z}/k\mathbb{Z})^\times \to (\mathbb{Z}/k'\mathbb{Z})^\times$ and if a Dirichlet character χ (mod k) does not factor through χ' (mod k') for a proper divisor $k'|k$, then we call χ primitive Dirichlet character. Note that all Dirichlet characters (mod 1) are trivial and is called trivial character. (i.e. $\chi(n) = 1$ for all nonzero integer $n \in \mathbb{Z}$ and $\chi(0) = 0$)

Dirichlet L-function associated with a Dirichlet character χ is defined for $\sigma > 1$ by the absolute convergence series

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

and by analytic continuation elsewhere. It has Euler product

$$L(s, \chi) = \prod_p (1 - \frac{\chi(p)}{p^s})^{-1}$$

and in the case of primitive chracter χ (mod k), it has functional equation

$$\Phi(s, \chi) = \omega_\chi \Phi(1 - s, \bar{\chi})$$

where

$$\Phi(s, \chi) = (\frac{k}{\pi})^{\frac{s}{2}} \cdot \Gamma(\frac{s + a(\chi)}{2}) \cdot L(s, \chi),$$

$$a(\chi) = \begin{cases} 0 & \text{if} \quad \chi(-1) = 1 \\ 1 & \text{otherwise,} \end{cases}$$

and

$$\omega_\chi = \frac{\tau(\chi)}{i^{a(\chi)}\sqrt{k}},$$

and $\tau(\chi)$ denotes the corresponding Gaussian sum. We have $|\omega_\chi| = 1$. As in the Riemann zeta function, $L(s, \chi)$ does not vanish for $\sigma > 1$ and for $k > 1$ and primitive Dirichlet chracter χ (mod k), trivial zeros occurs at points

$$s = -2n - a(\chi), \quad n \geq 0.$$

According to Generalized Riemann Hypothesis, all non-trivial zeros lie on the critical line.

Artin L-function

Let K/k be a normal extension of algebraic number fields with the Galois group G and rings of integers $\mathcal{O}_K$ and $\mathcal{O}_k$ respectively. Denote by ρ a finite dimensional representation of G in a vector space V. Moreover, let χ denote its character.

Given a prime $\mathfrak{p}$ of k, we choose a prime $\mathfrak{P}$ of K lying above $\mathfrak{p}$. Let $D_{\mathfrak{P}} := \{\sigma \in G : \mathfrak{P}^\sigma = \mathfrak{P}\}$ and $I_{\mathfrak{P}} := \{\sigma \in G : \sigma(a) = a (\mathrm{mod} \mathfrak{P})$ for every $a \in \mathcal{O}_{\mathcal{K}}\}$.denote the decomposition group and inertia group respectively. The quotient group $D_{\mathfrak{P}}/I_{\mathfrak{P}}$ is canonically isomorphic to the Galois group of $\mathcal{O}_k/\mathfrak{p} \subset \mathcal{O}_K/\mathfrak{P}$. Let $\sigma_{\mathfrak{P}} = [\frac{K/k}{\mathfrak{P}}]$he corresponding denote the corresponding Frobenius substitution. We write

$$V^{I_{\mathfrak{P}}} := \{v \in V : \rho(\sigma)(v) = v \text{ for every } \sigma \in I_{\mathfrak{P}}\}.$$

We define the local Artin's L-function corresponding to a finite prime $\mathfrak{P}$ of k by the formula

$$L_{\mathfrak{P}}(s, K/k, \rho) = \frac{1}{\det\left(I - N(\mathfrak{p})^{-s}\rho(\sigma_{\mathfrak{P}})\right)}, \tag{7.1.2}$$

where I denotes the unit matrix of dimension $\dim V^{I_{\mathfrak{P}}}$ and s denotes a complex number with positive real part. One checks without difficulty that the RHS of (2.2) does not depend on the particular choice of $\mathfrak{P}$ above $\mathfrak{p}$ and that it is the same for all equivalent representations. Therefore, $L_{\mathfrak{p}}(s, K/k, \rho)$ depends only on χ and we can write $L_{\mathfrak{p}}(s, K/k, \chi)$ instead of $L_{\mathfrak{p}}(s, K/k, \rho)$.

Let us fix a rational prime p and consider the product $L_p(s, K/k, \chi)$ of all local Artin's L-functoins taken over all finite primes $\mathfrak{p}$ of k lying above p,

$$L_p(s, K/k, \chi) = \prod_{\mathfrak{p}|p} L_{\mathfrak{p}}(s, K/k, \chi).$$

Suppose for simplicity that p is unramified in K/Q. Then $V^{I_{\mathfrak{p}}} = V$ for every $\mathfrak{p}|p$. Moreover, we can assume without loss of generality that $\rho(\sigma_{\mathfrak{P}})$ is represented by a diagonal matrix

$$\begin{pmatrix} \epsilon_1 & & 0 \\ & \ddots & \\ 0 & & \epsilon_n \end{pmatrix}, \ (n = \dim V).$$

Since G is a finite group, ϵ_j's are roots of unity. Therefore,

$$L_{\mathfrak{p}}(s, K/k, \chi) = \prod_{j=1}^{n} (1 - \epsilon_j N(\mathfrak{p})^{-s})^{-1}.$$

If $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_n$ are all primes of k lying above p, then $N(\mathfrak{P}_j) = p^{f_j}$ for every $j = 1, \cdots, t$ and $\sum_{j=1}^{t} f_j = [k : Q]$. Hence

$$L_p(s, K/k, \chi) = \prod_{j=1}^{n[k:Q]} (1 - \zeta_{j,p} p^{-s})^{-1} \tag{7.1.3}$$

for certain roots of unity $\zeta_{j,p}$. So $L_p(s, K/k, \chi)$ is the inverse of a polynomial of degree $[k:\mathbb{Q}]\mathrm{dim}V$ at p^{-s}. Roots of polynomial in question are roots of unity. For the infinite number of ramifying primes p, we have a similar statement but in these cases degress of the involved polynomials are smaller than $[k:\mathbb{Q}]\mathrm{dim}V$.

The global *Artin's L-function* $L(s, K/k, \rho)$ is defined as the product of all local factors : $L(s, K/k, \rho) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(s, K/k, \chi)$. The product converges for $Re(s) > 1$ and hence $L(s, K/k, \chi)$ is holomorphic in this half plane.

Expanding local factors in (2.3), one can write $L(s, K/k, \chi)$ for $Re(s) > 1$ as an absolutely convergent Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

say. Absolute values of coefficients are bounded by appropriate divisor function and therefore the following *Ramanujan condition* holds

$$a(n) \ll n^{\epsilon}$$

for every positie ϵ.

Theorem. (Artin) *We have*
1. For two characters χ_1 and χ_2 of G we have

$$L(s, K/k, \chi_1 + \chi_2) = L(s, K/k, \chi_1)L(s, K/k, \chi_2).$$

2. If H is a subgroup of G and E denotes the corresponding field then for every character χ of H

$$L(s, K/k, \chi) = L(s, K/k, Ind_H^{G(\chi)}),$$

where $Ind_H^G(\chi)$ denotes the induced chracter of G.
3. If H is a normal subgroup of G then every character χ of the quotient group G/H defines in a canonical way a character χ' of G and

$$L(s, E/k, \chi) = L(s, K/k, \chi')$$

4. (Artin's reciprocity law) *If K/k is abelian, then for every character χ of G there exists an ideal $\mathfrak{f} \in \mathcal{O}_k$ and a character χ^* of the ideal class group $H_{\mathfrak{f}}{}^*$ such that*

$$L(s, K/k, \chi) = L_k(s, \chi^*)$$

where $L_k(s, \chi^*)$ denotes the Hecke L-function of k associated with χ^*.

The first property reduces study of Artin L-functions to the case of irreducible representations. The last property provides analytic continuation of all abelian Artin L-functions. Using 3, we can define Artin's L-functions to every (virtual) character of

$Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Let us take $H = 1$ in 2. Then the induced representtion is just the regular representation of G, and the induced chracter is $\sum_{\chi} (\dim \chi)\, \chi$, where the sum is over all irreducible chracters of G. Since $L(s, K/k, 1)$ is the Dedekind zeta function of K, as a consequnce we obtain

$$\zeta_K(s) = \prod_{\chi} L(s, K/k, \chi)^{\dim \chi}.$$

Conjecture. (Artin's conjecture) *Every $L(s, K/k, \chi)$, where χ is the character of an irreducible representation admits meromorphic continuation to the whole complex plane. It is entire if $\chi \neq 1$ and has a simple pole at $s = 1$ otherwise.*

The most successful approach to this conjecture uses Theorem 2 and a theorem of Brauer that every character of a finite group is a linear combination with integer coefficients of characters induced by characters of degree 1. Hence by theorem 2, we can write

$$L(s, K/k, \chi) = \prod_{j=1}^{J} L^{n_j}(s, K/E_j, \chi_j) \tag{7.1.4}$$

for certain intermediate fields $k \subset E_j \subset K$, degree one characters χ_j of groups $G_j = Gal(K/E_j)$ and certain integers n_j. Every character of degree one factors through G_j^{ab}, the quotient of G_j by its commutative subgroup. Hence using theorem Theorem 2, we see that the corresponding factors on the RHS of (2.4) coincide with Artin L-functions of certain abelian extensions F_j/E_j with $F_j \subset K$ and hence, by the Artin's reciprocity law, they are Hecke L-functions.

Therefore we see that every Artin's L-function can be written as a quotient of products of Hecke L-functions associated with finite order Hecke characters of certain intermediate fields $k \subset E_j \subset K$. In particular, it admits meromorphic continuation to the whole complex plane and satisfies a functional equation with multiple gamma factors.

Let us consider the problem of functional equation with more care. Let v be a real infinite prime of k and let w be an infinite prime of K lying above v. Let σ_w denote the generator of the inertia group $G(w) = \{\sigma \in G : \sigma w = w\}$. Note that G(w) is cyclic of order at most 2 and hence σ_w exists. The matrix $\rho(\sigma_w)$ has at most two eigenvalues +1 or -1. Accordingly, V splits into the direct sum of two subspaces $V = V_v^+ \oplus V_v^-$.

For complex s, we write

$$g(s) = \pi^{-\frac{s}{2}} \cdot \gamma(\frac{s}{2}).$$

Then for every infinite prime v of k, let

$$\gamma_v(s) = \begin{cases} g(s)^{\dim V} g(s+1)^{\dim V} & \text{if } v \text{ is complex },\\ g(s)^{\dim V_v^+ g(s+1)^{\dim V_v^-}} & \text{if } v \text{ is real.} \end{cases} \tag{7.1.5}$$

We define the *gamma factor* of χ as follows:

$$\gamma_\chi = \prod_v \gamma_v(s),$$

where the products is taken over all infinite primes of k. In order to define the gamma factor of $L(s, K/k, \chi)$ and write the functional equation we have to introduce Artin's conductor of χ. We proceed locally. Let $\mathfrak{p}$ be a prime of k and let $\mathfrak{P}$ be a prime of K lying above $\mathfrak{p}$. We denote by $G_i (i \geq 0)$ the corresponding ramification groups. Write

$$n(\chi, \mathfrak{p}) = \sum_{i=0}^{\infty} \frac{G_i}{G_0} codim V^{G_i}.$$

Artin proved that this is an integer. We have $n(\chi, \mathfrak{p}) = 0$ for unramified $\mathfrak{p}$. Hence the following product

$$f(\chi, K/k) = \prod_{\mathfrak{p}} \mathfrak{p}^{n(\chi, \mathfrak{p})}$$

is well defined and represents an ideal of k, called the *Artin conductor*.

Theorem. *The completed Artin L-function*

$$\Lambda(s, K/k, \chi) = A(\chi)^{\frac{s}{2}} \gamma_\chi(s) L(s, K/k, \chi),$$

where

$$A(\chi) = |D_k|^{dimV} N_{k/\mathbb{Q}}(f(\chi, K/k))$$

and D_k denotes the absolute discriminant of k, satisfies the following functional equation

$$\Lambda(1-s, K/k, \chi) = W(\chi)\Lambda(s, K/k, \bar{\chi}),$$

for some constant $W(\chi)$ of absolute value 1 (the Artin root number*)*

As we have already seen every Artin's L-function can be expressed as a product of Hecke L-functions. If

$$L(s, K/k, \chi) = \prod_{j=1}^{J} L_{E_j}^{n_j}(s, \chi_j),$$

where E_j's are intermediate fields ($k \subset E_j \subset K$), χ_j's are Hecke characters of finite order and n_j's are integers, then

$$W(\chi) = \prod_{j=1}^{J} w(\chi_j)^{n_j}.$$

So that the Artin root number is expressed in terms of root numbers of Hecke L-functions and therefore in terms of generalized Gaussian sums It follows in particular that $W(\chi)$ is always an algebraic number.

Remark 7.1.1. *Theorem 2.2 shows that the L-functions attached to cuspidal automorphic satisfy the axioms of Selberg class except for* Ramanujan hypothesis. *It is believed that the functions in the Selberg class would be automorphic L-functions.*

7.2 Several conjectures on Selberg class

We collect several basic properties of Selberg class. As with the Riemann zeta function, an element L of $\mathscr{S}$ has trivial zeroes that arise from the poles of the Gamma factor $\Gamma(s)$. The other zeroes are referred to as the *non-trivial zeroes* of L. All these will be located in some strip $\{s \in \mathbb{C} \quad | \quad 1 - A \leq Re(s) \leq A\}$ for some $0 \leq A \leq 1$. Selberg showed that $N_L^0(T)$, the number of non-trivial zeroes of L in $\{s \in \mathbb{C} \quad | \quad |\mathrm{Im}\, s| \leq T\}$ counting multiplicity,

$$N_L^0(T) = \chi_L \frac{T \log T}{\pi} + c_L T + O(\log T).$$

where c_L is a constant and χ_L is the degree we defined in (6.0.1).

If L_1 and L_2 are in the Selberg class, then so is their product and

$$\chi_{L_1 L_2} = \chi_{L_1} + \chi_{L_2}.$$

Every function $L \neq 1$ of S can be written as a product of primitive functions. Selberg's conjectures 1,2, described below, imply that the factorization into primitive functions is unique.

Conjecture 1. *For all L in $\mathscr{S}$, there is an integer n_L such that $\sum_{p \leq x} \frac{|a_p|^2}{p} = n_L \log\log x + O(1)$ and $n_L = 1$ whenever L is primitive.*

Conjecture 2. *For two distinctive primitive $L(s), L'(s) \in \mathscr{S}$,*

$$\sum_{p \leq x} \frac{a_p a_p'}{p} = O(1)$$

Conjecture 3. (Riemann hypothesis for $\mathscr{S}$)
For all L in $\mathscr{S}$, the non-trivial zeros of L all lie on the line $Re(s) = \frac{1}{2}$

Main Theorem. *([2], [30]) We asume Selberg's conjecture 1, 2. Then factoriazation into primitie functions in $\mathscr{S}$ is unique up to the order of factors.*

Proof. Let $P_1, P_2, \cdots, P_m, Q_1, Q_2, \cdots, Q_n$ be in pairs different primitive elements of the Selberg class such that

$$P_1^{e_1}(s) \cdots P_m^{e_m}(s) = Q_1^{f_1}(s) \cdots Q_n^{f_n}(s)$$

for certain positive integers $e_1, e_2, \cdots, e_m$ and $f_1, f_2, \cdots, f_n$. Comparing p-th coefficients of both sides we have

$$e_1 a_{P_1}(p) + \cdots + e_m a_{P_m}(p) = f_1 a_{Q_1}(p) + \cdots + f_n a_{Q_n}(p).$$

Multiplying $\frac{\overline{a_{P_1}(p)}}{p}$ and summing over primes $p \leq x$, we obtain

$$e_1 \sum_{p\leq x} \frac{|a_{P_1}(p)|^2}{p} + \sum_{j=2}^{m} e_j \sum_{p\leq x} \frac{a_{P_j}(p)\overline{a_{P_1}(p)}}{p} = \sum_{j=1}^{n} f_j \sum_{p\leq x} \frac{a_{Q_j}(p)\overline{a_{P_1}(p)}}{p}.$$

By the Selberg's conjecture 1,2 , LHS is $e_1 \log x \log x + O(1)$ whereas RHS is $O(1)$, a contradiction.

If $F = P_1^{e_1} \cdots P_m^{e_m}$ is a factorization into powers of distinct primitive functions, then Selberg's conjecture 1,2 implies that

$$\sum_{p\leq x} \frac{|a_F(p)|^2}{p} = n_F \log x \log x + O(1)$$

, where n_F is an integer given by the formula

$$n_F = \sum_{i=1}^{m} {e_i}^2.$$

Hence, under Selberg's conjecture 1, 2, F is primitive if and only if $n_F = 1$.

□

There is also another important conjecture so called *General Converse Conjecture* and we briefly introduce it.

For $d \geq 0$, let

$$S_d := \{F \in S : d_F = d\},$$
$$S_d^\sharp := \{F \in S^\sharp : d_F = d\}.$$

Then the General converse conjecture says

Conjecture. 1. (Degree conjecture) *For $d \notin \mathbb{N} \cup \{0\}$, $S_d^\sharp = S_d = \emptyset$.*

2. *For $d \in \mathbb{N} \cup \{0\}$, if $F \in S_d$, then F is an automorphic L-function.*

Main Theorem. *Let $Q > 0$ and for j=1,2, $\cdots, r$, $\lambda_j > 0$, $\mu_j \in \mathbb{C}, Re(\mu_j) \geq 0$ and $w \in \mathbb{C}, |w| = 1$ be arbitrary. Moreover, put*

$$\gamma(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j).$$

Then the functional equation

$$\gamma(s)F(s) = w\overline{\gamma(1-\bar{s})F(\bar{s})}$$

has uncountably many independent solutions in the set of Generalized Dirichlet series

$$\sum_{n=1}^{\infty} a(n)e^{-\theta_n s} \quad \textit{where } \theta_n > 0.$$

Corollary 7.2.1. *General Converse conjecture fails in case of the Generalized Dirichlet series.*

For now, the general converse conjecture is known to be true for $0 \leq d < 2$ and unknown for $d \geq 2$. In particular, for $d = 1$, if $F \in S_1$, then F should be $L(s + i\theta, \chi)$ where χ is primitive and $\theta \in \mathbb{R}$.

Chapter 8

Proof of the Main Theorem

8.1 Lemmas

In this section we state some basic facts and prove the lemmas required for the proof of the main theorem. We begin with some observations about functions with positive degree in the extended Selberg class $S^\sharp$.

When $L(s)$ has positive degree, the functional equation in axiom (3) may be written

$$L(s) = \chi(s)\overline{L(1-\overline{s})}, \tag{8.1.1}$$

where

$$\chi(s) = \omega Q^{1-2s} \prod_{j=1}^{K} \frac{\Gamma\left(\lambda_j(1-s)+\overline{\mu_j}\right)}{\Gamma\left(\lambda_j s+\mu_j\right)}, \tag{8.1.2}$$

$|\omega| = 1$, $Q, \lambda_j > 0$, and $\operatorname{Re} \mu_j \geq 0$.

The Gamma function $\Gamma(s)$ has simple poles at $s = 0, -1, -2, \ldots$, and no zeros. Thus, for $1 \leq j \leq K$, $\chi(s)$ has simple poles at the points

$$s = 1 + \frac{m+\overline{\mu_j}}{\lambda_j} \qquad (m = 0, 1, 2, \ldots),$$

and zeros at the points

$$s = -\frac{m+\mu_j}{\lambda_j} \qquad (m = 0, 1, 2, \ldots).$$

By (8.1.1) these zeros are also zeros of $L(s)$, with the possible exception of $s = 0$, which occurs if one or more of the $\mu_j = 0$. For if $L(s)$ has a pole at $s = 1$, it could cancel this zero. In any case, we call these zeros of $L(s)$ "trivial" zeros. They all have real part less than or equal to 0, and may have multiplicity greater than one. We shall denote them by $\rho_1, \rho_2, \rho_3, \ldots$, where $\operatorname{Re} \rho_1 \geq \operatorname{Re} \rho_2 \geq \operatorname{Re} \rho_3 \geq \cdots$, and where each zero is listed as many times as its multiplicity.

The following observations will be useful.

(i) $|\mathrm{Im}\,\rho_n| \le B_0$ for $n = 1, 2, 3, \ldots$, where $B_0 = \max_{1\le j\le K} |\mathrm{Im}\,\mu_j|/\lambda_j$;

(ii) $D_0 = \min_{\rho_m \ne \rho_n} |\rho_m - \rho_n|$ exists and $D_0 > 0$;

(iii) $\displaystyle\sum_{-U<\mathrm{Re}\,\rho_n\le 0} 1 = \Big(\sum_{j\le K}\lambda_j\Big)U + O(1) = \lambda U/2 + O(1)$, as $U \to \infty$;

(iv) there is a number $A_0 > 0$ such that $L(s)$ has only trivial zeros in $\sigma \le -A_0$ and these are the same, counting multiplicities, as the zeros of $\chi(s)$ in this half-plane.

To see why the last assertion is true, note that we may write

$$L(s) = \frac{a(k)}{k^s}(1 + o(1)) \quad \text{as} \quad \sigma \to \infty,$$

where $a(k) \ne 0$ and $a(l) = 0$ for $l < k$. Hence, If $A_0 > 0$ is sufficiently large, $L(s) \ne 0$ for $\sigma > A_0$. It follows from this and (8.1.1) that $L(s)$ only has trivial zeros in $\sigma \le -A_0$. We have already seen that these are also zeros of $\chi(s)$ with the same multiplicities. Note that the constants A_0, B_0, C_0, and D_0 depend at most on K and the μ_j and λ_j.

For an arbitrary meromorphic function $F(s)$, let ρ denote a generic one of its zeros, and for $\sigma_1, \sigma_2, T > 0$ define

$$N_F(\sigma_1, \sigma_2) = \sum_{\sigma_1<\mathrm{Re}\,\rho\le\sigma_2} 1$$

$$N_F(\sigma_1, \sigma_2; t) = \sum_{\substack{\sigma_1<\mathrm{Re}\,\rho\le\sigma_2\\ |\mathrm{Im}\,\rho|\le t}} 1.$$

Then with A_0 and B_0 as above, we clearly have

$$N_L(-U, -A_0) = N_L(-U, -A_0; B_0) = \lambda U/2 + O(1). \tag{8.1.3}$$

We now proceed to our lemmas.

Lemma 8.1.1. *Suppose that $L(s)$ is in the extended Selberg class and has positive degree. For any fixed complex number $c \ne 0$, there exist positive constants A_1, B_1, and C_1 depending at most on K and the μ_j and λ_j, such that*

(a) $N_{L-c}(-U, -A_1) = N_{L-c}(-U, -A_1; B_1) = \lambda U/2 + O(1)$, as $U \to \infty$;

(b) each zero of $L(s) - c$ in $\sigma \le -A_1$ is within $|\rho_n|^{-C_1 \log|\rho_n|}$ of a trivial zero ρ_n of $L(s)$;

(c) all the zeros of $L(s) - c$ in $\sigma \le -A_1$ are simple.

Proof. Let $A_0 > 0$ be as in (iv) above, so that the only zeros of $L(s)$ in $\sigma \le -A_0$ are trivial zeros of $L(s)$, and let $\rho_n = \beta_n + i\gamma_n$ be one of these. Then ρ_n is a zero of at least one of the factors

$$\frac{\Gamma\left(\lambda_j(1-s) + \overline{\mu_j}\right)}{\Gamma\left(\lambda_j s + \mu_j\right)}$$

in (8.1.2). Using the identity

$$\Gamma(s) = \frac{\pi}{\Gamma(1-s)\sin \pi s},$$

we rewrite this factor as

$$\Gamma\left(\lambda_j(1-s) + \overline{\mu_j}\right)\ \Gamma\left(1 - (\lambda_j s + \mu_j)\right)\ \frac{\sin(\pi(\lambda_j s + \mu_j))}{\pi}. \tag{8.1.4}$$

As in (ii) above, let D_0 denote the minimum distance between any two distinct trivial zeros, and let $\mathscr{C}_n = \{s : |s - \rho_n| = d\}$, where d is fixed with $0 < d < D_0/2$. Then unless ρ_m and ρ_n coincide (so that ρ_n is a multiple zero), ρ_m is not inside or on $\mathscr{C}_n$. Now $\sin(\pi(\lambda_j s + \mu_j)) \gg d$ on $\mathscr{C}_n$ so, using the estimate

$$\Gamma(s) = (2\pi)^{1/2} e^{(s-1/2)\log s - s}\left(1 + O\left(|s|^{-1}\right)\right) \qquad (|\arg s| < \pi), \tag{8.1.5}$$

we find that for $s \in \mathscr{C}_n$ with n is sufficiently large, (8.1.4) is

$$\begin{aligned} &\gg d\ e^{\left(\lambda_j(1-\beta_n)-1/2\right)\log(\lambda_j|\beta_n|/e)}\ e^{-\lambda_j\beta_n\log(\lambda_j|\beta_n|/e)} \\ &\gg d\ e^{|\rho_n|\lambda_j\log(\lambda_j|\rho_n|)}. \end{aligned}$$

Here the implied constant depends on μ_j and λ_j. Thus, there is a constant $c_1 > 0$ depending at most on K, the λ_j, and the μ_j, such that for n sufficiently large and $s \in \mathscr{C}_n$,

$$|\chi(s)| \geq c_1 d^K Q^{2|\rho_n|+1} e^{|\rho_n|\sum_{j\leq K}\lambda_j\log(|\rho_n|\lambda_j)}.$$

Next, for some $k \geq 1$, we have $L(s) = a(k)k^{-s}(1 + o(1))$ as $\sigma \to \infty$. Thus, by (8.1.1) there is a constant $A_1 > 0$ such that

$$|L(s)| = \frac{|a(k)|}{|k^{1-s}|}(1 + o(1))\ |\chi(s)| \geq (2k)^{\sigma}|\chi(s)|$$

for $\sigma \leq -A_1$. We may assume that $A_1 > A_0$. Then if n is sufficiently large,

$$|L(s)| \geq (2k)^{-|\rho_n|-d} c_1 d^K Q^{2|\rho_n|+1} e^{|\rho_n|\sum_{j\leq K}\lambda_j\log(|\rho_n|\lambda_j)} \tag{8.1.6}$$

for $s \in \mathscr{C}_n$. Now assume that n is so large that the right-hand side is $> 2|c|$. We observe that $L(s) - c$ has no zeros in the intersection of the sets $\{s : |s - \rho_k| \geq d\}$ with $k \geq n$, and applying Rouche's theorem to $L(s)$ and the function $f(s) = -c$, we find that $L(s)$ and $L(s) - c$ have the same number of zeros inside $\mathscr{C}_n$ counting multiplicity. We see that (a) follows from this and (8.1.3) on increasing the size of A_1, if necessary.

To prove (b) we suppose that $L(s_n) = c$ with s_n inside the open disc bounded by $\mathscr{C}_n$, and set $d_0 = |s_n - \rho_n|$. Thus $d_0 < d$. By (8.1.6)

$$|c| \geq (2k)^{-|\rho_n|-d} c_1 d_0^K Q^{2|\rho_n|+1} e^{|\rho_n|\sum_{j\leq K}\lambda_j\log(|\rho_n|\lambda_j)}.$$

Hence

$$d_0 \le \left(\frac{2|c|(2k)^{|\rho_n|+d}}{c_1}\right)^{1/K} Q^{-\frac{1}{K}(2|\rho_n|+1)} e^{-\frac{1}{K}|\rho_n|\sum_{j=1}^{K}\lambda_j \log(|\rho_n|\lambda_j)} \le |\rho_n|^{-C_1 \log|\rho_n|}, \tag{8.1.7}$$

where C_1 depends on k, c, K, the λ_j, and the μ_j. This proves (b).

Next we prove (c). By (8.1.1), if σ is negative and $|\sigma|$ is sufficiently large, then

$$\frac{L'}{L}(s) = \frac{\chi'}{\chi}(s) + O(1).$$

From this, (8.1.2), (8.1.4), and (8.1.5) we deduce that

$$\frac{L'}{L}(s) = -2\sum_{j=1}^{K}\lambda_j \log(\lambda_j|s|) + \pi\sum_{j=1}^{K}\lambda_j \cot(\pi(\lambda_j s + \mu_j)) + O(1).$$

Now, if s_n is a zero of $L(s) - c$ inside $\mathscr{C}_n$ as in the proof of (b), then $s_n = \rho_n + d_0 e^{i\theta_0}$ for some real number θ_0. If s_n were a zero of multiplicity $m > 1$, the left-hand side of this equation would equal 0, and we would have

$$0 = \frac{m}{d_0 e^{i\theta_0}} + O(\log|\rho_n|).$$

The estimate (8.1.7) shows that this is impossible if n is sufficiently large. Again making A_1 larger if necessary, we obtain (c). This completes the proof of the lemma. □

Lemma 8.1.2. *Let $L_1(s)$ and $L_2(s)$ be two Dirichlet series in the extended Selberg class with positive degrees. Let their functional equations be*

$$L_l(s) = \chi_l(s)\overline{L_l(1-\bar{s})} \qquad (l = 1, 2),$$

with $\chi_l(s)$ as in (8.1.2). *If $L_1^{-1}(c) = L_2^{-1}(c)$ for some nonzero complex number c, then the degree of $L_1(s)$ equals the degree of $L_2(s)$. Moreover,*

$$\chi_2(s) = \chi_1(s)R(s)e^{as},$$

where a is a complex number and $R(s)$ is a rational function.

Proof. That the degrees of $L_1(s)$ and $L_2(s)$ are equal follows immediately from Lemma 8.1.1 (a).

By (iv) above, there is a constant $A_2 > 0$ such that $L_1(s)$ and $\chi_1(s)$ have the same zeros with the same multiplicities in $\sigma < -A_2$, as do $L_2(s)$ and $\chi_2(s)$. Suppose, for the moment, that we can show that $L_1(s)$ and $L_2(s)$ have the same zeros with the same multiplicities for $\sigma < -A_2$. Then by (8.1.1) we would have

$$\frac{\chi_1(s)}{\chi_2(s)} \sim \frac{L_1(s)}{L_2(s)}$$

as $\sigma \to -\infty$. Now, writing

$$\chi_l(s) = \omega_l Q_l^{1-2s} \prod_{j=1}^{K_l} \frac{\Gamma\left(\lambda_{lj}(1-s) + \overline{\mu_{lj}}\right)}{\Gamma\left(\lambda_{lj}s + \mu_{lj}\right)}, \tag{8.1.8}$$

we see that

$$\frac{\chi_1(s)}{\chi_2(s)} = \frac{\omega_1}{\omega_2}\Big(\frac{Q_1}{Q_2}\Big)^{1-2s} \prod_{j=1}^{K_1} \frac{\Gamma\left(\lambda_{1j}(1-s) + \overline{\mu_{1j}}\right)}{\Gamma\left(\lambda_{1j}s + \mu_{1j}\right)} \prod_{j=1}^{K_2} \frac{\Gamma\left(\lambda_{2j}s + \mu_{2j}\right)}{\Gamma\left(\lambda_{2j}(1-s) + \overline{\mu_{2j}}\right)}.$$

In particular, the poles of $\prod_{j\le K_1} \Gamma\left(\lambda_{1j}s + \mu_{1j}\right)$ and $\prod_{j\le K_2} \Gamma\left(\lambda_{2j}s + \mu_{2j}\right)$ in $\sigma < -A_2$ must exactly match, and the poles of $\prod_{j\le K_1} \Gamma\left(\lambda_{1j}(1-s) + \overline{\mu_{1j}}\right)$ and $\prod_{j\le K_2} \Gamma\left(\lambda_{2j}(1-s) + \overline{\mu_{2j}}\right)$ must match in $\sigma > A_2$. It follows that $\chi_1/\chi_2(s)$ is meromorphic with only finitely many zeros and poles. It must therefore be of the form $R(s)e^{as}$ for some rational function $R(s)$ and complex constant a. This would prove the second assertion of the lemma, so it remains to show that $L_1(s)$ and $L_2(s)$ have the same zeros with the same multiplicities in $\sigma < -A_2$.

The zeros of each in this region are of the form

$$-\frac{n + \mu_{lj}}{\lambda_{lj}} \qquad (j = 1, 2, \ldots, K_l;\ \lambda_{lj} \neq 0) \quad (l = 1, 2)$$

for all sufficiently large positive integers n. Thus, there is an absolute constant $D > 0$ such that the distance between any two of these zeros that are distinct is $> D$. Assume now that the zeros of $L_1(s)$ and $L_2(s)$, counting multiplicities, are not identical in $\sigma < -A_2$. Then there is a sequence of complex numbers $\{\varrho_n = b_n + ig_n\}_{n=1}^{\infty}$ with $-A_2 > b_1 > b_2 > \cdots$ and $b_n \to -\infty$, such that either:
(1) $L_1(\varrho_n) = 0$ but $L_2(\varrho_n) \neq 0$, or
(2) $L_1(\varrho_n) = L_2(\varrho_n) = 0$, but the multiplicities are different.
We consider case (1) first. By Lemma 8.1.1 (b), $L_1(s)-c$ has a zero s_n within $D/4$ (say) of ϱ_n for all n sufficiently large. Since $L_1(s) - c$ and $L_2(s) - c$ have exactly the same zeros, s_n is also a zero of $L_2(s)-c$. Therefore $L_2(s)$ must have a zero within $D/4$ of ϱ_n. However, its closest zero is at least a distance D away from s_n, a contradiction. Now consider case (2). Let ϱ_n be a common zero of $L_1(s)$ and $L_2(s)$ with multiplicities m_1 and m_2, respectively, with $m_1 \neq m_2$. By Lemma 8.1.1 (b) and (c), $L_1(s) - c$ has m_1 simple zeros within $D/4$ of ϱ_n, and $L_2(s) - c$ has m_2 such zeros. But $L_1(s) - c$ and $L_1(s) - c$ have the same zeros, a contradiction. Thus, $L_1(s)$ and $L_2(s)$ have the same zeros with the same multiplicities in $\sigma < -A_2$. This completes the proof of Lemma 8.1.2.

□

Lemma 8.1.3. *Let $L_1(s)$ and $L_2(s)$ be two different Dirichlet series in the extended Selberg class with positive degree. If $L_1^{-1}(c) = L_2^{-1}(c)$ for some nonzero complex* *number c, then there exist constants $A_3 > 0$ and $B_3 > 0$ such that*

$$N_{L_2-L_1}(-U, -A_3; B_3) = \lambda U/2 + O(1)$$

as $U \to \infty$.

Proof. By (8.1.1) and Lemma 8.1.2, we have

$$L_2(s) - L_1(s) = \chi_1(s)\big(e^{as}R(s)\overline{L_2(1-\overline{s})} - \overline{L_1(1-\overline{s})}\big) = \chi_1(s)F(s), \qquad (8.1.9)$$

say. By (8.1.3) there are constants $A_0 > 0$ and $B_0 > 0$ such that $\chi_1(s)$ has $\lambda U/2 + O(1)$ zeros for $-U < \sigma \leq -A_0$ and $|\mathrm{Im}\, s| \leq B_0$. Thus, it suffices to prove that there exist positive constants A_3 and B_3 such that $F(s)$ has no zeros in $\sigma \leq -A_3$ and $|\mathrm{Im}\, s| \leq B_3$.

For $L_1(s)$ and $L_2(s)$, let k_1 and k_2 be such that $a_1(k_1) \neq 0$, $a_1(l) = 0$ for $l < k_1$ and $a_2(k_2) \neq 0$, $a_2(l) = 0$ for $l < k_2$. We will prove Lemma 3 only when $k_1 = k_2$ because the proof for the case $k_1 \neq k_2$ is similar. So assume that $k = k_1 = k_2$. We consider a number of cases.

case 1. $|R(s)| \to \infty$ or $|R(s)| \to 0$ as $|s| \to \infty$.

Then

$$|e^{as}R(s)| \to \infty \quad \text{or} \quad 0$$

as $\sigma \longrightarrow -\infty$ with $|\mathrm{Im}\, s| \leq B_3$. Hence,

$$F(s) = \begin{cases} e^{as}R(s)\overline{a_2(k)}k^{s-1}(1+o(1)) & \text{if } |e^{as}R(s)| \to \infty, \\ -\overline{a_1(k)}k^{s-1}(1+o(1)) & \text{if } |e^{as}R(s)| \to 0. \end{cases}$$

In either case $F(s)$ does not vanish for σ negative with $|\sigma|$ sufficiently large and $|\mathrm{Im}\, s| \leq B_3$.

case 2. $R(s) \to r$ as $|s| \to \infty$, where r is a nonzero complex number.

Then either

$$|e^{as}R(s)| \to \infty \quad \text{or} \quad 0 \quad \text{or} \quad r$$

as $\sigma \longrightarrow -\infty$ with $|\mathrm{Im}\, s| \leq B_3$. The first two cases are handled exactly as in case 1. For the third case, we observe that a must be pure imaginary, say $a = i\theta$ for some real number θ. That is,

$$e^{as}R(s) = e^{i\theta s}R(s).$$

Thus, we have

$$\chi_2(s) = e^{i\theta s}R(s)\chi_1(s). \qquad (8.1.10)$$

We next show that $\theta = 0$. Suppose that $\theta \neq 0$. Without loss of generality, we can assume that $\theta > 0$. From (8.1.5) it follows that for any fixed complex numbers v_1, v_2 we have

$$\frac{\Gamma(s+v_1)}{\Gamma(s+v_2)} = s^{v_1-v_2}(1 + O(|s|^{-1})) \qquad (\,|\arg(s+v_i)| < \pi,\ i = 1, 2\,).$$

We use this, (8.1.8), and (8.1.10) with $s = -it,\ t > 0$. Taking absolute values of both sides in (8.1.10), we deduce that

$$Q_2 \prod_{j=1}^{K_2} (t\lambda_{2j})^{\lambda_{2j}} = (r + O(t^{-1}))Q_1 e^{\theta t} \prod_{j=1}^{K_1} (t\lambda_{1j})^{\lambda_{1j}} \qquad (t \to \infty).$$

By Lemma 8.1.2

$$\sum_{j=1}^{K_1} \lambda_{1j} = \sum_{j=1}^{K_2} \lambda_{2j} = \lambda/2,$$

so we see that

$$e^{\theta t} = (r + O(t^{-1}))Q_2 Q_1^{-1} \prod_{j=1}^{K_2} (\lambda_{2j})^{\lambda_{2j}} \prod_{j=1}^{K_1} (\lambda_{1j})^{-\lambda_{1j}} \qquad (t \to \infty).$$

This is clearly impossible, so $\theta = 0$.

We now have that $\theta = 0$, so

$$\chi_2(s) = R(s)\chi_1(s)$$

and $R(s) \to r \neq 0$ as $\sigma \to -\infty$ with $|\mathrm{Im}\, s| \le B_3$.
subcase a. If $r \neq \overline{a_1(k)}/\overline{a_2(k)}$, then we have

$$(e^{as} R(s)\overline{L_2(1-\overline{s})} - \overline{L_1(1-\overline{s})} = \frac{r\overline{a_2(k)} - \overline{a_1(k)}}{k^{1-s}}(1 + o(1)) \qquad (\sigma \to -\infty).$$

Thus, there are no zeros of $F(s)$ in $\sigma \le -A_3$, $|\mathrm{Im}\, s| \le B_3$ if A_3 is sufficiently large.
subcase b. Next suppose that $r = \overline{a_1(k)}/\overline{a_2(k)}$ and $R(s) = r$. If $L_1(s) \not\equiv L_2(s)$, there is a least integer $N > 0$ such that

$$L_1(s) - L_2(s) = (a_1(N) - a_2(N))N^{-s}(1 + o(1)) \qquad (\sigma \to \infty),$$

where $a_1(N) \neq a_2(N)$. Thus,

$$F(s) = e^{as} R(s)\overline{L_2(1-\overline{s})} - \overline{L_1(1-\overline{s})} = \overline{L_2(1-\overline{s})} - \overline{L_1(1-\overline{s})} = \frac{\overline{a_{2N}} - \overline{a_{1N}}}{N^{1-s}}(1 + o(1))$$

as $\sigma \to -\infty$. Again, there is an $A_3 > 0$ such that $F(s)$ is nonzero when $\sigma \le -A_3$ and $|\mathrm{Im}\, s| \le B_3$.
subcase c. Finally, assume that $r = \overline{a_1(k)}/\overline{a_2(k)}$ and $R(s) \neq r$. Then there is a nonzero complex number b and a positive integer m, such that

$$R(s) = r + bs^{-m}(1 + o(1)) \qquad (|s| \to \infty).$$

Furthermore, $L_1(s) = a_1(k)k^{-s} + O((k+1)^{-\sigma})$ and $L_2(s) = a_2(k)k^{-s} + O((k+1)^{-\sigma})$ as $\sigma \to \infty$. Thus, for any fixed $B_3 > 0$ we have

$$F(s) = e^{as} R(s)\overline{L_2(1-\overline{s})} - \overline{L_1(1-\overline{s})} = \frac{b\overline{a_2(k)}}{k^{1-s}} \left(\frac{1}{s^m} + O\left(\left(\frac{k+1}{k} \right)^{\sigma} \right) \right) (1 + o(1))$$

for $\sigma \to -\infty$, $|\mathrm{Im}\, s| < B_3$. Thus, in this case also, there is an $A_3 > 0$ such that $F(s)$ is nonzero in $\sigma \leq -A_3$ and $|\mathrm{Im}\, s| \leq B_3$.

This completes the proof of the lemma. □

8.2 Proof of the Main Theorem

Observe that for $j = 1, 2$, $L_j(s)$ and $L_j(s) - c$ cannot have any zeros in common. Moreover, any common zero of $L_1(s) - c$ and $L_2(s) - c$ is a zero of $L_2(s) - L_1(s)$. Also, by (8.1.9) the zeros of $\chi_1(s)$ are zeros of $L_2(s) - L_1(s)$ in $\sigma < -A_3$ if $A_3 > 0$ is sufficiently large. Moroever, for A_3 large enough these are also zeros of $L_1(s)$ and $L_2(s)$. Let $B_3 = \max\{B_0, B_1\}$, where B_0 and B_1 are as in observation (1) and Lemma 8.1.2 (a). Then it is easy to see that

$$N_{L_2 - L_1}(-U, -A_3; B_3) \geqslant N_{L_1}(-U, -A_3; B_3) + N^*_{L_1 - c}(-U, -A_3; B_3) + O(1),$$

where $N^*_{L_1 - c}(-U, -A_3; B_3)$ is the number of distinct zeros of $L_1(s) - c$ in the region $-U < \sigma < -A_3$, $|\mathrm{Im}\, s| \leq B_3$. By Lemma 8.1.3 and (8.1.3) we now find that

$$N^*_{L_1 - c}(-U, -A_3; B_3) = O(1).$$

On the other hand, by Lemma 8.1.1 (a) and (c),

$$N^*_{L_1 - c}(-U, -A_3) \to \infty, \qquad T \to \infty.$$

It follows that $L_1(s) \equiv L_2(s)$. This completes the proof of the main theorem.

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