이학박사 학위논문

Heat kernel estimates for symmetric Markov processes in $C^{1,\eta}$ open sets

($C^{1,\eta}$ 열린집합에서 대칭 마르코브 과정의 열 핵 추정)

2015년 8월

서울대학교 대학원

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이 논문을 이학박사 학위논문으로 제출함

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Heat kernel estimates for symmetric Markov processes in $C^{1,\eta}$ open sets

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Abstract

In this thesis, we establish sharp two-sided heat kernel estimates for a large class of symmetric Markov processes in $C^{1,\eta}$ open sets for all $t > 0$. The processes are symmetric pure jump Markov processes with jumping kernel intensity

$$\kappa(x,y)\psi(|x-y|)^{-1}|x-y|^{-d-\alpha}$$

where $\alpha \in (0,2)$, ψ is an increasing function on $[0,\infty)$ with $\psi(r) = 1$ on $0 < r \leq 1$ and $c_1 e^{c_2 r^\beta} \leq \psi(r) \leq c_3 e^{c_4 r^\beta}$ on $r > 1$ for $\beta \in [0,\infty]$. A symmetric function $\kappa(x,y)$ is bounded by two positive constants and $|\kappa(x,y) - \kappa(x,x)| \leq c_5|x-y|^\rho$ for $|x-y| < 1$ and $\rho > \alpha/2$. As a corollary of our main result, we estimates sharp two-sided Green function for this process in $C^{1,\eta}$ open sets.

Key words: Dirihlet form, Markov process, heat kernel, transition density, Green function
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Contents

Chapter 1

Introduction

For a large class of second order elliptic differential operators $\mathcal{L}$ on $\mathbb{R}^d$, there is a diffusion process X in $\mathbb{R}^d$ associated with it so that $\mathcal{L}$ is the infinitesimal generator of X and vice versa. The fundamental solution $p(t, x, y)$ of $\mathcal{L}u = \partial_t u$, which is called the heat kernel of $\mathcal{L}$, describes the distribution of X, that is $\mathbb{P}(X \in A) = \int_A p(t, x, y)dy$. Thus obtaining sharp two-sided estimates for $p(t, x, y)$ is important in both analysis and probability theory and there are many interesting results for heat kernel estimates nowadays. It is well-known that the heat kernel $p(t, x, y)$ for uniformly elliptic differential operator in $\mathbb{R}^d$ enjoys the following Aronson's two-sided heat kernel estimate,

$$c_1 t^{-d/2} e^{-c_2|x-y|^2/t} \le p(t, x, y) \le c_3 t^{-d/2} e^{-c_4|x-y|^2/t}.$$

The infinitesimal generator of a discontinuous symmetric Markov process is no longer a differential operator but rather a non-local (or, integro-differential) operator. For example, the infinitesimal generator of symmetric α-stable process where $\alpha \in (0, 2)$ is the fractional Laplacian $-(-\Delta)^{\alpha/2}$ and it is defined as

$$\Delta^{\alpha/2} u(x) = \mathcal{A}(d, -\alpha) \lim_{\varepsilon \to 0} \int_{\{y \in \mathbb{R}^d : |x-y| > \varepsilon\}} (u(y) - u(x)) \frac{dy}{|x-y|^{d+\alpha}} \ ,$$

where Γ is the Gamma function and $\mathcal{A}(d, -\alpha) = \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma(\frac{d+\alpha}{2}) \Gamma(1 - \alpha/2)^{-1}$. Thus it is a pure jump process and the scaling property yields the following heat kernel estimates:

$$c_1\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)\le p(t,x,y)\le c_2\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right).$$

For any open set $D\subset\mathbb{R}^d$, let X^D be a subprocess of X killed upon leaving D, and $p_D(t,x,y)$ be a transition density of X^D. The transition density $p_D(t,x,y)$ describes the distribution of the process Y^D, in the sense that $\mathbb{P}(X^D\in A)=\int_A p_D(t,x,y)dy$ for $A\subset\mathbb{R}^d$. $p_D(t,x,y)$ is also called a Dirichlet heat kernel of the operator $\mathcal{L}$ on D, because it is a fundamental solution to the exterior Dirichlet problem $\mathcal{L}u=\partial_t u$ and $u=0$ on D^c. It is quite difficult to obtain two-sided estimates of $p_D(t,x,y)$ especially near the boundary, and it has been studied recently, see, [2], [7] –[9], [13] and [14].

In particular, the large time heat kernel estimates in unbounded open sets were only obtained very recently, since the results are changed differently depending on the geometry of the open set D. The research on the fractional Laplacian $-(-\Delta)^{\alpha/2}$ by Chen and Tokle in [17] gave us the foothold of the studies in unbounded open sets, e.g., the exterior $C^{1,1}$ open sets and the half space like open sets. The exterior $C^{1,1}$ open sets D are $C^{1,1}$ open sets satifying that D^c are compact sets, and the half space like open sets D are open sets satisfying that for any $b_1\le b_2$, $\mathbb{H}_{b_2}\subset D\subset\mathbb{H}_{b_1}$ where $\mathbb{H}_b=\{x=(x_1,\dots,x_d)\in\mathbb{R}^d: x_d>b\}$. Relativistic stable processes (i.e., $m-(m^{2/\alpha}-\Delta)^{\alpha/2}$) and mixed processes (i.e., $\Delta+\Delta^{\alpha/2}$) were studied in the exterior $C^{1,1}$ open sets and the half space like open sets, see [10]–[12].

In this thesis, we consider a large class of symmetric Markov processes (not necessarily Lévy processes) whose jumping kernels J are dominated by the kernel of the fractional Laplacian (see, Section 2.1 for the definition of the processes). We establish the two-sided estimates for Dirichlet heat kernels of the generators of such Markov processes in $C^{1,\eta}$ open sets and in exterior $C^{1,\eta}$ open sets D. When D is $\mathbb{R}^d$, such a problem has been discussed in [22, 28, 29] and [6]. These processes contain discontinous Markov processes which have been discussed in [7, 9], and far more.

The remainder of this thesis is organized as follows: In Chapter 2, we define the jumping intensity J and introduce the corresponding Markov processes. Also we give the main results. In Chapter 3, we solve the key estimate for exit distributions in Theorem 3.1.6. The proofs of two-sided Dirichlet heat kernel estimates in $C^{1,\eta}$ open sets when $t \le T$ are given in section 3.2–3.4, and in bounded connected $C^{1,\eta}$ open sets when $t \ge T$ is given in section 3.5. In Chapter 4, we give the proofs of the two-sided Dirichlet heat kernel in exterior $C^{1,\eta}$ open sets when $t \ge T$ and the Green function estimates are given as a Corollary.

Throughout this thesis we assume that $\beta \in [0, \infty]$, $\alpha \in (0, 2)$, and $d \in \{1, 2, 3, \dots\}$. We will use the symbol ":=," which is read as "is defined to be." For $a, b \in \mathbb{R}$, denote $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. For two nonnegative functions f and g, the notation $f \asymp g$ means that there are positive constants c_1 and c_2 such that $c_1 g(x) \le f(x) \le c_2 g(x)$ in the common domain of definition for f and g. For any Borel set $A \subset \mathbb{R}^d$, we will use $\mathrm{diam}(A)$ to denote its diameter and $|A|$ to denote its Lebesgue measure. The positive constants $C_1, C_2, C_*, C^* L_1, L_2, L_3, L_4, \gamma_1, \gamma_2$ will be fixed. In the statements of results and the proofs, the constants $c_i = c_i(a, b, c, \dots), i = 1, 2, 3, \dots$, denote generic constants depending on $a, b, c, \dots$ and there are given anew in each statement and each proof. The dependence of the constants on the dimension d, on α and on the positive constants $L_1, L_2, L_3, L_4, \gamma_1, \gamma_2$ will not be mentioned explicitly.

Chapter 2

Main results

2.1 Setting and preliminaries

Let ψ be an increasing function on $[0,\infty)$ where $\psi(r)=1$ on $0<r\le 1$, and

$$L_1 e^{\gamma_1 r^\beta} \le \psi(r) \le L_2 e^{\gamma_2 r^\beta} \qquad \text{on } \ 1<r<\infty \tag{2.1.1}$$

where $L_1, L_2, \gamma_1, \gamma_2$ are positive constants. Define

$$j(r) := r^{-d-\alpha}\psi(r)^{-1}, \qquad r>0. \tag{2.1.2}$$

Let $\kappa(x,y)$ be a positive symmetric function which is satisfying

$$L_3^{-1} \le \kappa(x,y) \le L_3, \qquad x,y\in\mathbb{R}^d, \tag{2.1.3}$$

and for $\rho>\alpha/2$,

$$|\kappa(x,y)-\kappa(x,x)|\mathbf{1}_{\{|x-y|<1\}} \le L_4|x-y|^\rho, \qquad x,y\in\mathbb{R}^d, \tag{2.1.4}$$

where L_3, L_4 are positive constants. Define a symmetric measurable function J on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\}$ as

$$J(x,y) := \kappa(x,y)j(|x-y|) = \begin{cases} \kappa(x,y)|x-y|^{-d-\alpha}\psi(|x-y|)^{-1} & \text{if } \beta \in [0,\infty), \\ \kappa(x,y)|x-y|^{-d-\alpha}\mathbf{1}_{\{|x-y|\le 1\}} & \text{if } \beta = \infty. \end{cases} \tag{2.1.5}$$

For $u \in L^2(\mathbb{R}^d, dx)$, $\mathcal{E}(u,u) := 2^{-1}\int_{\mathbb{R}^d\times\mathbb{R}^d}(u(x)-u(y))^2 J(x,y)dxdy$. We denote $C_c(\mathbb{R}^d)$ as the space of continuous functions with compact support in $\mathbb{R}^d$ and equipped with uniform topology, and define

$$\mathcal{D}(\mathcal{E}) := \{f \in C_c(\mathbb{R}^d) : \mathcal{E}(f,f) < \infty\}, \tag{2.1.6}$$

and $\mathcal{F} := \overline{\mathcal{D}(\mathcal{E})}^{\mathcal{E}_1}$ where $\mathcal{E}_1(u,u) := \mathcal{E}(u,u) + \int_{\mathbb{R}^d} u(x)^2 dx$. By [16, Proposition 2.2], $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d, dx)$ and there is a Hunt process Y in $\mathbb{R}^d$ associated with this Dirichlet form (see [19]).

We say **UJS** holds for $\widetilde{J}$ if for a.e. $x, y \in \mathbb{R}^d$,

$$\widetilde{J}(x,y) \le \frac{c}{r^d}\int_{B(x,r)} \widetilde{J}(z,y)dz \qquad \text{whenever } r \le |x-y|/2. \tag{\textbf{UJS}}$$

Since j is decreasing and $J(x,y) \asymp j(|x-y|)$, we have

$$\begin{aligned}\int_{B(x,r)} J(z,y)dz &\ge \int_{B(x,r)\cap\{|z-y|\le|x-y|\}} c_1 j(|z-y|)dz \\ &\ge c_2 r^d j(|x-y|) \ge c_3 r^d J(x,y) \qquad \text{for every } \ r \le |x-y|/2.\end{aligned}$$

Thus **UJS** holds for our J. Moreover, our conditions (2.1.1)–(2.1.3) and (2.1.5) imply that [6, (1.6)] holds with $\phi(r) = r^\alpha\psi(r)$. Thus the Hunt process Y associated with $(\mathcal{E}, \mathcal{F})$ belongs to a subclass of the processes considered in [6]. Therefore Y is conservative and it has a Hölder continuous transition density $p(t,x,y)$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ with respect to the Lebesgue measure.

The function J is called the jumping intensity kernel of Y, because it gives rise to a Lévy system for Y describing the jumps of the process Y. For any $x \in \mathbb{R}^d$, stopping time S (with respect to the filtration of Y), and nonnegative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s, y, y) = 0$, we have

$$\mathbb{E}_x \left[\sum_{s \le S} f(s, Y_{s-}, Y_s) \right] = \mathbb{E}_x \left[\int_0^S \left(\int_{\mathbb{R}^d} f(s, Y_s, y) J(Y_s, y) dy \right) ds \right] \qquad (2.1.7)$$

(e.g., see [16, Appendix A]).

For any positive constants a, b, T, define functions $F^1_{a,b,T}(t, r)$ on $(0, T] \times [0, \infty)$ as

$$F^1_{a,b,T}(t,r) := \begin{cases} t^{-d/\alpha} \wedge t r^{-d-\alpha} e^{-br^\beta} & \text{if } \beta \in [0,1], \\ t^{-d/\alpha} \wedge t r^{-d-\alpha} & \text{if } \beta \in (1,\infty],\ r < 1, \\ t \exp\left(-a\left(r\left(\log \frac{Tr}{t}\right)^{\frac{\beta-1}{\beta}} \wedge r^\beta\right)\right) & \text{if } \beta \in (1,\infty),\ r \ge 1, \\ (t/(Tr))^{ar} & \text{if } \beta = \infty,\ r \ge 1, \end{cases} \qquad (2.1.8)$$

and $F^2_{a,T}(t, r)$ on $[T, \infty) \times (0, \infty)$ as

$$F^2_{a,T}(t,r) := \begin{cases} t^{-d/\alpha} \wedge t r^{-d-\alpha} & \text{if } \beta = 0, \\ t^{-d/2} \exp\left(-a\left(r^\beta \wedge \frac{r^2}{t}\right)\right) & \text{if } \beta \in (0,1], \\ t^{-d/2} \exp\left(-a\left(r\left(1 + \log^+ \frac{Tr}{t}\right)^{(\beta-1)/\beta} \wedge \frac{r^2}{t}\right)\right) & \text{if } \beta \in (1,\infty), \\ t^{-d/2} \exp\left(-a\left(r\left(1 + \log^+ \frac{Tr}{t}\right) \wedge \frac{r^2}{t}\right)\right) & \text{if } \beta = \infty \end{cases} \qquad (2.1.9)$$

where $\log^+ x = \log x \cdot \mathbf{1}_{\{x \ge 1\}} + 0 \cdot \mathbf{1}_{\{x < 1\}}$.

Even though two-sided estimates for $p(t, x, y)$ are stated separately for the cases $0 < t \le 1$ and $t > 1$ in [16, Theorem 1.2] and [6, Theorems 1.2 and 1.4], the same proofs work for the cases $0 < t \le T$ and $t > T$ for some fixed constant $T > 0$. Also remark that in [6, Theorem 1.2(2.b) and Theorem 1.4(1.21)], the case $|x - y| \asymp t$ is missing. We state two-sided estimates for $p(t, x, y)$ for the case $0 < t \le T$ and $t > T$, and the correct form to include the case $|x - y| \asymp t$ when $t > T$ and $\beta \in (1.\infty]$ which we will use.

Theorem 2.1.1. *Suppose that Y is the symmetric pure jump Hunt process with the jumping intensity kernel J defined in* (2.1.5)*. Then the process Y has a continuous transition density function $p(t,x,y)$ on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. For each positive constant T, there are positive constants C_1 and $c\ge 1$ which depend on $\alpha,\beta,d,L_3,\psi,T$ such that for every $t\in(0,T]$*

$$(1)\qquad c^{-1}F^1_{C_1,\gamma_2,T}(t,|x-y|)\ \le\ p(t,x,y)\ \le\ c\,F^1_{C_1^{-1},\gamma_1,T}(t,|x-y|)$$

and for every $t\in[T,\infty)$

$$(2)\qquad c^{-1}F^2_{C_1,T}(t,|x-y|)\ \le\ p(t,x,y)\ \le\ c\,F^2_{C_1^{-1},T}(t,|x-y|).$$

Unlike those in [6, Theorem 1.2], the exponents γ_1 and γ_2 in Theorem 2.1.1(1) are explicit. The upper bound in Theorem 2.1.1(1) for $\beta\in[0,1]$ comes from [22, Theorem 2, Proposition 1]. The proof of the upper bound in Theorem 2.1.1(1) for $\beta\in[1,\infty]$, and Theorem 2.1.1(2) for $\beta\in[0,\infty]$ are the same as that of [16, Theorem 1.2] and [6, Theorems 1.2 and 1.4], as mentioned above. However the lower bounds in Theorem 2.1.1(1) and in Theorem 2.1.1(2) will be proved in Section 3.3 and in Section 4.2, respectively, as a special case of the preliminary lower bound on the heat kernel of the killed process.

2.2 Main theorems

The goal of this thesis is to establish the two-sided heat kernel estimates for Y in $C^{1,\eta}$ open set. An open set D in $\mathbb{R}^d$ (when $d\ge 2$) is said to be a $C^{1,\eta}$ open set with $\eta\in(0,1]$ if there exist a localization radius $R_0>0$ and a constant $\Lambda_0>0$ such that for every $Q\in\partial D$, there exists a $C^{1,\eta}$-function $\phi=\phi_Q:\mathbb{R}^{d-1}\to\mathbb{R}$ satisfying $\phi(0)=0$, $\nabla\phi(0)=(0,\dots,0)$, $\|\nabla\phi\|_\infty\le\Lambda_0$, $|\nabla\phi(x)-\nabla\phi(z)|\le\Lambda_0|x-z|^\eta$, and an orthonormal coordinate system CS_Q: $y=(y_1,\cdots,y_{d-1},y_d)=:(\widetilde{y},y_d)$ with its origin at Q such that $B(Q,R_0)\cap D=\{y=(\tilde{y},y_d)\in B(0,R_0)\text{ in }CS_Q:y_d>\phi(\widetilde{y})\}$. The pair (R_0,Λ_0) will be called the $C^{1,\eta}$ characteristics of the open set D. Note that a $C^{1,\eta}$ open set D with characteristics (R_0,Λ_0) can be unbounded and disconnected. By a $C^{1,\eta}$ open set in $\mathbb{R}$, we mean an open set that can be written as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is positive constant R_0.

Let Y^D be the subprocess of Y killed upon exiting D and $\tau_D := \inf\{t > 0 : Y_t \notin D\}$ be the first exit time from D. By the strong Markov property, $p_D(t,x,y) := p(t,x,y) - \mathbb{E}_x[p(t-\tau_D, Y_{\tau_D}, y); t > \tau_D]$ is the transition density of Y^D. By the continuity and estimate of p, $p_D(t,x,y)$ is symmetric and continuous(e.g., see the proof of Theorem 2.4 in [18]).

When $\beta \in (1,\infty]$, we need to make an assumption for D in order to obtain the lower bound of $p_D(t,x,y)$. We say that *the path distance in each connected component of D is comparable to the Euclidean distance with characteristic* λ_1 if for every x and y in the same component of D there is a rectifiable curve l in D which connects x to y such that the length of l is less than or equal to $\lambda_1|x-y|$. Clearly, such a property holds for all bounded $C^{1,\eta}$ open sets, $C^{1,\eta}$ open sets with compact complements, and connected open sets above graphs of $C^{1,\eta}$ functions.

We are now ready to state the first main result of this thesis. Let $\delta_D(x)$ be a distance between x and D^c, and let

$$\Psi(t,x) := \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right). \tag{2.2.1}$$

Recall that C_1 is the constant in Theorem 2.1.1 and the function $F^1_{a,b,T}(t,r)$ on $(0,T] \times [0,\infty)$ is defined as

$$F^1_{a,b,T}(t,r) = \begin{cases} t^{-d/\alpha} \wedge tr^{-d-\alpha} e^{-br^{\beta}} & \text{if } \beta \in [0,1], \\ t^{-d/\alpha} \wedge tr^{-d-\alpha} & \text{if } \beta \in (1,\infty],\ r < 1, \\ t \exp\left(-a\left(r\ \left(\log \frac{Tr}{t}\right)^{\frac{\beta-1}{\beta}} \wedge r^{\beta}\right)\right) & \text{if } \beta \in (1,\infty),\ r \geq 1, \\ (t/(Tr))^{ar} & \text{if } \beta = \infty,\ r \geq 1. \end{cases}$$

Theorem 2.2.1. *Suppose that Y is the symmetric pure jump Hunt process with the jumping intensity kernel J defined in* (2.1.5). *Suppose that $\eta \in (\alpha/2, 1]$, $T > 0$, and D is a $C^{1,\eta}$ open set in $\mathbb{R}^d$ with characteristics* (R_0, Λ_0).

Then the transition density $p_D(t,x,y)$ of Y^D has the following estimates.

(1) *There is a positive constant $c_1 = c_1(\alpha,\beta,\eta,R_0,\Lambda_0,T,d,\psi,L_3,L_4)$ such that for all $(t,x,y)\in(0,T]\times D\times D$ we have*

$$p_D(t,x,y) \le c_1\,\Psi(t,x)\Psi(t,y)\begin{cases} F^1_{C_1^{-1}\wedge\gamma_1,\gamma_1,T}(t,|x-y|/6) & \text{if } \beta\in[0,\infty),\\ F^1_{C_1^{-1},\gamma_1,T}(t,|x-y|/6) & \text{if } \beta=\infty.\end{cases}$$

(2) *There is a positive constant $c_2 = c_2(\alpha,\beta,\eta,R_0,\Lambda_0,T,d,\psi,L_3,L_4)$ such that for all $(t,x,y)\in(0,T]\times D\times D$ we have*

$$p_D(t,x,y) \ge c_2\,\Psi(t,x)\Psi(t,y) \cdot\begin{cases} t^{-d/\alpha}\wedge t\cdot\frac{e^{-\gamma_2|x-y|^\beta}}{|x-y|^{d+\alpha}} & \text{if } \beta\in[0,1],\\ t^{-d/\alpha}\wedge t\cdot\frac{1}{|x-y|^{d+\alpha}} & \text{if } \beta\in(1,\infty) \text{ and } |x-y|<1, \\ & \text{or } \beta=\infty \text{ and } |x-y|\le 4/5.\end{cases}$$

(3) *Suppose in addition that the path distance in each connected component of D is comparable to the Euclidean distance with characteristic λ_1. Then there are positive constants $c_i = c_i(\alpha,\beta,\eta,R_0,\Lambda_0,T,d,\psi,L_3,L_4,\lambda_1)$, $i=3,4$, such that if x,y are in the same component of D and $t\in(0,T]$, we have*

$$p_D(t,x,y) \ge c_3\,\Psi(t,x)\Psi(t,y) \cdot\begin{cases} F^1_{c_4,\gamma_2,T}(t,|x-y|) & \text{if } \beta\in(1,\infty) \text{ and } |x-y|\ge 1,\\ F^1_{c_4,\gamma_2,T}(t,5|x-y|/4) & \text{if } \beta=\infty \text{ and } |x-y|\ge 4/5.\end{cases}$$

(4) *If $\beta\in(1,\infty)$, there is a positive constant $c_5 = c_5(\alpha,\beta,\eta,R_0,\Lambda_0,T,d,\psi,L_3,L_4)$ such that for every x,y in the different components of D with $|x-y|\ge 1$ and $t\in(0,T]$ we have*

$$p_D(t,x,y) \ge c_5\,\Psi(t,x)\Psi(t,y)\frac{t\exp(-\gamma_2\,(5|x-y|/4)^\beta)}{|x-y|^{d+\alpha}}.$$

(5) *Suppose in addition that D is bounded and connected. Then there are positive constants $c_i = c_i(\alpha, \beta, \eta, R_0, \Lambda_0, T, d, \psi, L_3, L_4, \mathrm{diam}(D))$, $i = 6, 7$ such that for all $(t, x, y) \in [T, \infty) \times D \times D$ we have*

$$c_6\, e^{-t\,\lambda^D}\, \delta_D(x)^{\alpha/2}\, \delta_D(y)^{\alpha/2} \,\le\, p_D(t,x,y) \,\le\, c_7\, e^{-t\,\lambda^D}\, \delta_D(x)^{\alpha/2}\, \delta_D(y)^{\alpha/2},$$

where $-\lambda^D < 0$ is the largest eigenvalue of the generator of Y^D.

The cutoff values 5/4 in Theorem 2.2.1(3)–(4) (and 4/5 in Theorem 2.2.1(2)) are not essential. Further analysis reveals that for any $\varepsilon > 0$ we can choose $1+\varepsilon$ as the cutoff value. However, it seems that we cannot choose 1 as the cutoff value.

If D is a connected $C^{1,\eta}$ open set and the path distance in D is comparable to the Euclidean distance, then by Theorem 2.2.1(1)–(3) we can rewrite the two-sided estimates for $p_D(t,x,y)$.

Corollary 2.2.2. *Suppose that Y is the symmetric pure jump Hunt process with the jumping intensity kernel J defined in* (2.1.5)*. Suppose further that D is a connected $C^{1,\eta}$ open set with $\eta \in (\alpha/2, 1]$ and that the path distance in D is comparable to the Euclidean distance with characteristic λ_1. Then, for each $T > 0$ there exist $c_i = c_i(\alpha, \beta, \eta, R, \Lambda, T, d, \psi, L_3, L_4, \lambda_1) > 0$, $i = 1, 2$, such that for every $(t, x, y) \in (0, T] \times D \times D$ we have*

$$\begin{aligned} c_1^{-1}\, \Psi(t,x)\Psi(t,y) F^1_{c_2,\gamma_2,T}(t, 5|x-y|/4) &\le p_D(t,x,y) \\ &\le\, c_2\, \Psi(t,x)\Psi(t,y) F^1_{C_1^{-1}\wedge\gamma_1,\gamma_1,T}(t, |x-y|/6) \end{aligned}$$

where the constants C_1 is in Theorem 2.1.1 and γ_1, γ_2 are in (2.1.1)*.*

By Theorem 2.2.1, the large time Dirichlet heat kernel estimates are established only for the bounded and connected $C^{1,\eta}$ open sets since these estimates are different depending on the geometry of D for unbounded open sets (e.g., see [17] and [10, 11] for the symmetric α-stable processes and the relativistic stable processes, respectively).

Motivated by [17, 11], we establish the global sharp two-sided estimates on $p_D(t,x,y)$ in the exterior $C^{1,\eta}$ open set D, that is, D is $C^{1,\eta}$ open set satisfying D^c is compact. These open sets could be disconnected and consist of countably many bounded connected components, i.e., $D_0 \cup D_1 \cup \ldots \cup D_n = D$ where D_0 be an unbounded connected component and $D_1, \ldots, D_n$ be bounded connected components.

Recall the functions $\Psi(t,x)$ defined in (2.2.1) satisfying

$$\Psi(1 \wedge t, x) = \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{1 \wedge \sqrt{t}}\right),$$

and $F^2_{a,T}(t,r)$ on $[T,\infty) \times (0,\infty)$ defined in (2.1.9) as

$$F^2_{a,T}(t,r) = \begin{cases} t^{-d/\alpha} \wedge tr^{-d-\alpha} & \text{if } \beta = 0, \\ t^{-d/2} \exp\left(-a\left(r^\beta \wedge \frac{r^2}{t}\right)\right) & \text{if } \beta \in (0,1], \\ t^{-d/2} \exp\left(-a\left(r\left(1+\log^+ \frac{Tr}{t}\right)^{(\beta-1)/\beta} \wedge \frac{r^2}{t}\right)\right) & \text{if } \beta \in (1,\infty), \\ t^{-d/2} \exp\left(-a\left(r\left(1+\log^+ \frac{Tr}{t}\right) \wedge \frac{r^2}{t}\right)\right) & \text{if } \beta = \infty \end{cases}$$

where $\log^+ x = \log x \cdot \mathbf{1}_{\{x \geq 1\}} + 0 \cdot \mathbf{1}_{\{x<1\}}$.

Theorem 2.2.3. *Suppose that Y is the symmetric pure jump Hunt process with the jumping intensity kernel J defined in* (2.1.5). *Let $d > 2 \cdot \mathbf{1}_{\{\beta \in (0,\infty]\}} + \alpha \cdot \mathbf{1}_{\{\beta=0\}}$ and $T > 0$. For any $\eta \in (\alpha/2, 1]$ and $R > 0$, let D be an exterior $C^{1,\eta}$ open set in $\mathbb{R}^d$ with $C^{1,\eta}$ characteristics (R_0, Λ_0) and $D^c \subset B(0,R)$. Then for any $t \geq T$ ($t > 0$ when $\beta = 0$, respectively) and $x, y \in D$, the transition density $p_D(t,x,y)$ of Y^D has the following estimates.*

(1) *For any $\beta \in [0,\infty]$, there are positive constants $c_i = c_i(\alpha, \beta, \eta, R_0, \Lambda_0, R, T, d, \psi, L_3, L_4)$ ($c_i = c_i(\alpha, \eta, R_0, \Lambda_0, R, d, \psi, L_3, L_4)$ when $\beta = 0$, respectively), $i = 1, 2$ such that*

$$p_D(t,x,y) \leq c_1 \Psi(1 \wedge t, x)\Psi(1 \wedge t, y) \cdot F^2_{c_2,T}(t, |x-y|).$$

(2) *Suppose that $\beta \in [0,1]$ or $\beta \in (1,\infty]$ with $|x-y| < 4/5$. Then there are positive constants $c_i = c_i(\alpha, \beta, \eta, R_0, \Lambda_0, R, T, d, \psi, L_3, L_4)$ ($c_i = c_i(\alpha, \eta, R_0, \Lambda_0, R, d, \psi, L_3, L_4)$ when $\beta = 0$, respectively), $i = 3, 4$ such*

that

$$p_D(t,x,y) \geq c_3 \Psi(1\wedge t, x)\Psi(1\wedge t, y)\cdot F^2_{c_4,T}(t,|x-y|).$$

(3) *Suppose that* $\beta \in (1,\infty]$ *with* $|x-y| \geq 4/5$ *and* x, y *are in a same component of* D.

(3.a) (Unbounded connected component) *There are positive constants* $c_i = c_i(\alpha,\beta,\eta,R_0,\Lambda_0,R,T,d,\psi,L_3,L_4)$, $i = 5,6$ *such that for* $x, y \in D_0$

$$p_D(t,x,y) \;\geq\; c_5 \Psi(1,x)\Psi(1,y)\cdot F^2_{c_6,T}(t,|x-y|).$$

(3.b) (Bounded connected component) *There is a positive constant* $c_7 = c_7(\alpha,\beta,\eta,R_0,\Lambda_0,R,T,d,\psi,L_3,L_4)$ *such that if* $x,y \in D_j$ *for some* $j = 1,\dots n$,

$$p_D(t,x,y) \;\geq\; c_7\, e^{-t\lambda_j}\, \delta_D(x)^{\alpha/2}\, \delta_D(y)^{\alpha/2}$$

where $-\lambda_j < 0$ *is the largest eigenvalue of the generator* Y^{D_j}, $j = 1\dots,n$.

(4) *Suppose that* $\beta \in (1,\infty)$ *with* $|x-y| \geq 4/5$ *and* x, y *are in different components of* D. *Then there are positive constants* $c_i = c_i(\alpha,\beta,\eta,R_0,\Lambda_0,R,T,d,\psi,L_3,L_4,\lambda_1,\dots,\lambda_n)$, $i = 8,9$ *such that*

$$p_D(t,x,y) \geq c_8 \Psi(1,x)\Psi(1,y)\cdot \frac{\exp\left(-c_9(|x-y|^\beta + t)\right)}{|x-y|^{d+\alpha}}$$

where $-\lambda_j < 0$ *is the largest eigenvalue of the generator* Y^{D_j}, $j = 1\dots,n$.

For a connected exterior $C^{1,\eta}$ open set, we can rewrite the sharp two-sided estimates on $p_D(t,x,y)$ for all $t>0$ in a simple form combining Theorem 2.2.1(1)–(3) and Theorem 2.2.3(1)–(3.a).

Corollary 2.2.4. *Suppose that* Y *is the symmetric pure jump Hunt process with the jumping intensity kernel* J *defined in* (2.1.5). *Let* $d > 2\cdot \mathbf{1}_{\{\beta\in(0,\infty]\}} +$

$\alpha \cdot \mathbf{1}_{\{\beta=0\}}$ and $T > 0$. For any $\eta \in (\alpha/2, 1]$ and $R > 0$, let D be an exterior $C^{1,\eta}$ open set in $\mathbb{R}^d$ with $C^{1,\eta}$ characteristics (R_0, Λ_0) and $D^c \subset B(0,R)$. Then there are positive constants $c_i = c_i(\alpha, \beta, \eta, R_0, \Lambda_0, R, T, d, \psi, L_3, L_4) > 1$, $i = 1, 2$ such that for every $(t, x, y) \in (0,\infty) \times D \times D$, we have

$$p_D(t,x,y) \le c_1 \Psi(1 \wedge t, x)\Psi(1 \wedge t, y) \cdot \begin{cases} F^1_{C_1^{-1} \wedge \gamma_1, \gamma_1, T}(t, |x-y|/6) & \text{if } t \in (0,T], \\ F^2_{c_2^{-1}, T}(t, |x-y|) & \text{if } t \in [T, \infty), \end{cases}$$

and in addition D is a connected, we have

$$p_D(t,x,y) \ge c_1^{-1} \Psi(1 \wedge t, x)\Psi(1 \wedge t, y) \cdot \begin{cases} F^1_{c_2, \gamma_2, T}(t, 5|x-y|/4) & \text{if } t \in (0,T], \\ F^2_{c_2, T}(t, |x-y|) & \text{if } t \in [T, \infty) \end{cases}$$

where the constants C_1 is in Theorem 2.1.1 and γ_1, γ_2 are in (2.1.1).

By integrating the heat kernel estimates in Corollary 2.2.4 with respect to $t \in (0,\infty)$, one gets the following sharp two-sided Green function estimates of Y^D in the connected exterior $C^{1,\eta}$ open sets.

Corollary 2.2.5. *Suppose that Y is the symmetric pure jump Hunt process with the jumping intensity kernel J defined in (2.1.5). Let $d > 2 \cdot \mathbf{1}_{\{\beta \in (0,\infty]\}} + \alpha \cdot \mathbf{1}_{\{\beta=0\}}$. For any $\eta \in (\alpha/2, 1]$ and $R > 0$, let D be a connected exterior $C^{1,\eta}$ open set in $\mathbb{R}^d$ with $C^{1,\eta}$ characteristics (R_0, Λ_0) and $D^c \subset B(0,R)$. Then there is a positive constant $c = c(\alpha, \beta, \eta, R_0, \Lambda_0, R, d, \psi, L_3, L_4) > 1$ such that for every $(x, y) \in D \times D$, we have*

$$\begin{aligned} &c^{-1} \left(\frac{1}{|x-y|^{d-\alpha}} + \frac{\mathbf{1}_{\{\beta \in (0,\infty]\}}}{|x-y|^{d-2}} \right) \left(1 \wedge \frac{\delta_D(x)}{|x-y| \wedge 1} \right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{|x-y| \wedge 1} \right)^{\alpha/2} \\ \le\ & G_D(x,y) \\ \le\ & c \left(\frac{1}{|x-y|^{d-\alpha}} + \frac{\mathbf{1}_{\{\beta \in (0,\infty]\}}}{|x-y|^{d-2}} \right) \left(1 \wedge \frac{\delta_D(x)}{|x-y| \wedge 1} \right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{|x-y| \wedge 1} \right)^{\alpha/2}. \end{aligned}$$

Chapter 3

Dirichlet heat kernel estimates in $C^{1,\eta}$ open sets

3.1 Estimates for exit distributions

In this section we give some key estimates for exit distributions. First, we introduce an inequality that is used several times in this thesis.

Lemma 3.1.1. *Suppose that $\beta \in [0,\infty)$. For any $r_0 > 0$, there exists a positive constant $c = c(\beta, \psi, r_0)$ such that*

$$j(r) \le c j(2r) \qquad \textit{for every} \quad r \in (0, r_0]. \tag{3.1.1}$$

Moreover, (3.1.1) *holds for $\beta = \infty$ if $r_0 \le 1/4$.*

Proof. The result follows immediately from $L_2^{-1} e^{-\gamma_2 r^\beta} r^{-d-\alpha} \le j(r) \le L_1^{-1} e^{-\gamma_1 r^\beta} r^{-d-\alpha}$. □

For $\varepsilon \in (0, 1/2)$, we define the operators $\mathcal{A}_\varepsilon$ and $\mathcal{A}$ by

$$\mathcal{A}_\varepsilon g(x) := \int_{\{y \in \mathbb{R}^d : |x-y| > \varepsilon\}} (g(y) - g(x)) J(x,y) dy \qquad \text{and} \quad \mathcal{A} g(x) := \lim_{\varepsilon \downarrow 0} \mathcal{A}_\varepsilon g(x)$$

whenever these exist pointwise. We use $C_c^2(\mathbb{R}^d)$ to denote the space of twice differentiable functions with compact support. For every $g \in C_c^2(\mathbb{R}^d)$ and

$r \in (\varepsilon/2, 1]$, we have

$$\begin{aligned}
\mathcal{A}_\varepsilon g(x) &= \left(\int_{\{y\in\mathbb{R}^d: r>|x-y|>\varepsilon\}} + \int_{\{y\in\mathbb{R}^d:|x-y|\geq r\}} \right) (g(y)-g(x))\kappa(x,y)j(|x-y|)dy \\
&= \kappa(x,x) \int_{\{y\in\mathbb{R}^d: r>|x-y|>\varepsilon\}} (g(y)-g(x)-(x-y)\cdot\nabla g(x))\, j(|x-y|)dy \\
&+ \int_{\{y\in\mathbb{R}^d: r>|x-y|>\varepsilon\}} (g(y)-g(x))(\kappa(x,y)-\kappa(x,x))j(|x-y|)dy \\
&+ \int_{\{y\in\mathbb{R}^d:|x-y|\geq r\}} (g(y)-g(x))\kappa(x,y)j(|x-y|)dy. \qquad (3.1.2)
\end{aligned}$$

By (2.1.2) and (2.1.4), we have that for $r<1$ and $g \in C_c^2(\mathbb{R}^d)$,

$$\begin{aligned}
&\mathbf{1}_{\{r>|x-y|>\varepsilon\}}(y)\, |(g(y)-g(x))(\kappa(x,y)-\kappa(x,x))j(|x-y|)| \\
&\leq L_4 \mathbf{1}_{\{r>|x-y|>\varepsilon\}}(y) \|\nabla g\|_\infty |x-y|^{-d-\alpha+\rho+1}.
\end{aligned}$$

By this and the assumption $\rho > \alpha/2$ which is greater than $\alpha - 1$, we see that $\mathcal{A}g$ is well defined in $\mathbb{R}^d$ and that $\mathcal{A}_\varepsilon g$ converges to $\mathcal{A}g$ locally uniformly in $\mathbb{R}^d$ as $\varepsilon \to 0$. Furthermore, for every $r \in (0,1]$ we have

$$\begin{aligned}
\mathcal{A}g(x) &= \kappa(x,x) \int_{\{y\in\mathbb{R}^d: r>|x-y|\}} (g(y)-g(x)-(x-y)\cdot\nabla g(x))\, j(|x-y|)dy \\
&+ \int_{\{\{y\in\mathbb{R}^d: r>|x-y|\}} (g(y)-g(x))(\kappa(x,y)-\kappa(x,x))j(|x-y|)dy \\
&+ \int_{\{\{y\in\mathbb{R}^d:|x-y|\geq r\}} (g(y)-g(x))\kappa(x,y)j(|x-y|)dy, \qquad (3.1.3)
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{A}g\|_\infty &\leq c_1 \int_{\mathbb{R}^d} \mathbf{1}_{\{1>|y|\}}(y)(|y|^{-d-\alpha+2} + |y|^{-d-\alpha+\rho+1}) + \mathbf{1}_{\{|y|\geq 1\}}(y)|y|^{-d-\alpha} dy \\
&< \infty.
\end{aligned}$$

Next, we solve the martingale-type problem for the operator $\mathcal{A}$ on $C_c^2(\mathbb{R}^d)$ and show that the Dynkin-type formula in terms of $\mathcal{A}$ is valid for every $f \in C_c^2(\mathbb{R}^d)$ (cf. [21, Section 6]).

Proposition 3.1.2. *For each $f \in C_c^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, there exists a $\mathbb{P}_x$-martingale M_t^f with respect to the filtration of Y such that $M_t^f = f(Y_t) - f(Y_0) - \int_0^t \mathcal{A}f(Y_s)ds$ $\mathbb{P}_x$-a.s. In particular, for every $f \in C_c^2(\mathbb{R}^d)$ and any bounded open subset U of $\mathbb{R}^d$ we have*

$$\mathbb{E}_x \int_0^{\tau_U} \mathcal{A}f(Y_t)dt = \mathbb{E}_x[f(Y_{\tau_U})] - f(x). \tag{3.1.4}$$

Proof. We fix $f \in C_c^2(\mathbb{R}^d)$ and assume that the support of f is a subset of $B(0, R_0/2)$. We use a strict version of Fukushima's decomposition [19, Theorem 5.2.5]. First, it is clear from (2.1.6) that $f \in \mathcal{F}$. The energy measure $\mu_{\langle f \rangle}$ of f has the density $\Gamma(f)(x) = \int_{\mathbb{R}^d}(f(x) - f(y))^2 J(x,y)dy$.

By Fubini's theorem and the dominated convergence theorem, we have for any $g \in C_c^2(\mathbb{R}^d)$

$$\begin{aligned}
\mathcal{E}(f,g) &= \frac{1}{2}\lim_{\varepsilon \downarrow 0} \int_{\{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d,\ |x-y|>\varepsilon\}} (g(y)-g(x))(f(y)-f(x))J(x,y)\,dx\,dy \\
&= -\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} g(x) \left(\int_{\{y\in\mathbb{R}^d:|x-y|>\varepsilon\}} (f(y)-f(x))J(x,y)dy \right) dx \\
&= -\int_{\mathbb{R}^d} g(x)\mathcal{A}f(x)\,dx.
\end{aligned}$$

Recall from [19] that S_0 is the collection of positive Radon measures of finite energy integrals and

$$S_{00} := \left\{ \mu \in S_0 : \mu(\mathbb{R}^d) < \infty,\ \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^\infty e^{-t} p(t,x,y)dt\mu(dy) < \infty \right\}.$$

Let $\nu := \nu_+ - \nu_-$ where $\nu_+(dx) := -\mathbf{1}_{\{\mathcal{A}f(x)<0\}}\mathcal{A}f(x)dx$ and $\nu_-(dx) := \mathbf{1}_{\{\mathcal{A}f(x)\geq 0\}}\mathcal{A}f(x)dx$, so that $\mathcal{E}(f,g) = \int_{\mathbb{R}^d} g(x)\nu(dx)$. Note that $\|\mathcal{A}f\|_\infty < \infty$ and that $|\mathcal{A}f(x)| \leq c_1|x|^{-d-\alpha}$ for $x \in B(0,R_0)^c$ and therefore $|\nu|(\mathbb{R}^d) < \infty$. Moreover since

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^\infty e^{-t}p(t,x,y)dt|\nu|(dy) \leq \|\mathcal{A}f\|_\infty \int_0^\infty e^{-t}dt < \infty,$$

$\nu_+,\ \nu_- \in S_{00}$. Similarly, since $\|\Gamma(f)\|_\infty < \infty$ and $|\Gamma(f)(x)| \leq c_2|x|^{-d-\alpha}$ for

$x \in B(0,R_0)^c$, we have $\mu_{\langle f\rangle}(\mathbb{R}^d)<\infty$ and so $\mu_{\langle f\rangle}\in S_{00}$.

Upon applying [19, Theorem 5.2.5], since $-\int_0^t \mathbf{1}_{\{\mathcal{A}f(X_s)<0\}}\mathcal{A}f(X_s)ds$ and $\int_0^t \mathbf{1}_{\{\mathcal{A}f(X_s)\geq 0\}}\mathcal{A}f(X_s)ds$ are positive continuous additive functionals in the strict sense with Revuz measures ν_+ and ν_-, respectively, we conclude that for every $x\in\mathbb{R}^d$

$$\begin{aligned} f(Y_t)-f(Y_0) &= M_t^f + \int_0^t \mathbf{1}_{\{\mathcal{A}f(x)\geq 0\}}\mathcal{A}f(Y_s)ds + \int_0^t \mathbf{1}_{\{\mathcal{A}f(x)<0\}}\mathcal{A}f(Y_s)ds \\ &= M_t^f + \int_0^t \mathcal{A}f(Y_s)ds, \end{aligned}$$

where M_t^f is a $\mathbb{P}_x$-martingale additive functional in the strict sense with Revuz measure $\mu_{\langle f\rangle}$. □

Using (3.1.4), we prove the following lemma, which is used several times in Section 3.2 and 3.4. The proof of the next result is well-known (e.g., see [24, Lemma 4.15].)

Lemma 3.1.3. *For every $a\in(0,1]$, there exists a positive constant $c = c(a,L_3,L_4)$ such that, for any $\beta\in[0,\infty]$, any $r\in(0,1]$, and any open sets U and D with $B(0,ar)\cap D\subset U\subset D$, we have*

$$\mathbb{P}_x\left(Y_{\tau_U}\in D\right) \leq c\, r^{-\alpha}\,\mathbb{E}_x[\tau_U], \qquad x\in D\cap B(0,ar/2).$$

Proof. For fixed $a\in(0,1]$, we take a sequence of radial functions $(\phi_m)_{m\geq 1}$ in $C_c^\infty(\mathbb{R}^d)$ such that $0\leq\phi_m\leq 1$, with

$$\phi_m(y)=\begin{cases} 0, & \text{if } |y|<a/2 \text{ or } |y|>m+2,\\ 1, & \text{if } a\leq |y|\leq m+1,\end{cases}$$

and

$$\sup_{m\geq 1}\left(\sum_{i=1}^d \left\|\frac{\partial}{\partial y_i}\phi_m\right\|_\infty + \sum_{i,j=1}^d \left\|\frac{\partial^2}{\partial y_i\partial y_j}\phi_m\right\|_\infty\right) < c_1 = c_1(a)<\infty.$$

For any $r \in (0,1]$, define $\phi_{m,r}(y) := \phi_m(y/r)$ so that $0 \le \phi_{m,r} \le 1$,

$$\phi_{m,r}(y) = \begin{cases} 0, & \text{if } |y| < ar/2 \text{ or } |y| > r(m+2), \\ 1, & \text{if } ar \le |y| \le r(m+1), \end{cases}$$

and

$$\sup_{m\ge 1} \sum_{i=1}^d \left\| \frac{\partial}{\partial y_i} \phi_{m,r} \right\|_\infty < c_1 r^{-1} \quad \text{and} \quad \sup_{m\ge 1} \sum_{i,j=1}^d \left\| \frac{\partial^2}{\partial y_i \partial y_j} \phi_{m,r} \right\|_\infty < c_1 r^{-2}. \tag{3.1.5}$$

Using (2.1.2)–(2.1.4), (3.1.5), and the assumption that $\rho > \alpha/2$, for every $x \in \mathbb{R}^d$, $r \in (0,1]$, and $m \ge 1$ we have

$$\begin{aligned} &\left| \kappa(x,x) \int_{\{y\in\mathbb{R}^d : |x-y|<r\}} (\phi_{m,r}(y) - \phi_{m,r}(x) - (x-y)\cdot\nabla\phi_{m,r}(x))\, j(|x-y|)dy \right| \\ &\quad + \int_{\{y\in\mathbb{R}^d : |x-y|<r\}} |\phi_{m,r}(y) - \phi_{m,r}(x)||\kappa(x,y) - \kappa(x,x)| j(|x-y|)dy \\ &\quad + \int_{\{y\in\mathbb{R}^d : |x-y|\ge r\}} |\phi_{m,r}(y) - \phi_{m,r}(x)||\kappa(x,y)| j(|x-y|)dy \\ &\le \frac{c_2}{r^2} \int_{\{|x-y|<r\}} |x-y|^{-d-\alpha+2} dy + \frac{c_2}{r} \int_{\{|x-y|<r\}} |x-y|^{-d-\alpha+1+\rho} dy \\ &\quad + c_2 \int_{\{|x-y|\ge r\}} |x-y|^{-d-\alpha} dy \ \le\ c_3(r^{-\alpha} + r^{-\alpha+\rho}) \le 2c_3 r^{-\alpha} \end{aligned} \tag{3.1.6}$$

for some $c_3 = c_3(a, L_3, L_4) > 0$. Since $\phi_{m,r}(x) = 0$ for any $x \in D \cap B(0, ar/2)$, by combining (3.1.3), (3.1.4), and (3.1.6), we find that

$$\begin{aligned} &\mathbb{P}_x\left(Y_{\tau_U} \in \{y \in D : ar \le |y| < (m+1)r\}\right) \\ &= \mathbb{E}_x\left[\phi_{m,r}\left(Y_{\tau_U}\right) : Y_{\tau_U} \in \{y \in D : ar \le |y| < (m+1)r\}\right] \\ &\le \mathbb{E}_x\left[\phi_{m,r}\left(Y_{\tau_U}\right)\right] = \mathbb{E}_x\left[\int_0^{\tau_U} \mathcal{A}\phi_{m,r}(Y_t)dt\right] \le \|\mathcal{A}\phi_{m,r}\|_\infty \mathbb{E}_x[\tau_U] \le 2c_3 r^{-\alpha}\mathbb{E}_x[\tau_U]. \end{aligned}$$

Therefore, since $B(0, ar) \cap D \subset U$, we obtain

$$\mathbb{P}_x\left(Y_{\tau_U} \in D\right) = \lim_{m\to\infty} \mathbb{P}_x\left(Y_{\tau_U} \in \{y \in D : ar \le |y| < (m+1)r\}\right) \le 2c_3 r^{-\alpha}\mathbb{E}_x[\tau_U].$$

□

For the remainder of this section we assume that $\eta \in (\alpha/2, 1]$ and that D is a $C^{1,\eta}$ open set with $C^{1,\eta}$ characteristics (R_0, Λ_0). Without loss of generality, we assume that $R_0 \le 1$ and $\Lambda_0 \ge 1$. For each fixed $Q \in \partial D$ and for every $r \le R_0$, we define

$$h_{Q,r}(y) := \delta_D(y)^{\alpha/2} \mathbf{1}_{D \cap B(Q,r)}(y). \tag{3.1.7}$$

The following two lemmas are used to obtain the key estimates for exit distribution. The next lemma and its proof are similar to [8, Lemma 2.3] and [23, Lemma 3.7] and their proofs. Recall that $\Delta^{\alpha/2}$ can be defined as

$$\Delta^{\alpha/2} u(x) = \mathcal{A}(d, -\alpha) \lim_{\varepsilon \to 0} \int_{\{y \in \mathbb{R}^d : |x-y| > \varepsilon\}} (u(y) - u(x)) \frac{dy}{|x-y|^{d+\alpha}},$$

where Γ is the Gamma function and $\mathcal{A}(d, -\alpha) = \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma(\frac{d+\alpha}{2}) \Gamma(1 - \frac{\alpha}{2})^{-1}$.

Lemma 3.1.4. *There exists a positive constant $c = c(\eta, R_0, \Lambda_0)$ independent of $Q \in \partial D$ such that $\Delta^{\alpha/2} h_{Q,R_0/2}$ is well defined in $D \cap B(Q, R_0/8)$ and*

$$|\Delta^{\alpha/2} h_{Q,R_0/2}(x)| \le c \quad \text{for all } x \in D \cap B(Q, R_0/8)\,.$$

Proof. Since the case of $d = 1$ is easier, we give the proof only for $d \ge 2$.

Let $h(\cdot) := h_{Q,R_0/2}(\cdot)$. Fix $x \in D \cap B(Q, R_0/8)$ and let $z_x \in \partial D$ such that $\delta_D(x) = |x - z_x|$. Let ϕ be a $C^{1,\eta}$ function and $CS = CS_{z_x}$ be an orthonormal coordinate system with z_x chosen so that $x = (\widetilde{0}, x_d)$, $B(0, R_0) \cap D = \{y = (\widetilde{y}, y_d) \text{ in } CS : y \in B(0, R_0), y_d > \phi(\widetilde{y})\}$, $\phi(\widetilde{0}) = 0$, $\nabla\phi(\widetilde{0}) = (0, \dots, 0)$, $\|\nabla\phi\|_\infty \le \Lambda_0$, and $|\nabla\phi(\widetilde{y}) - \nabla\phi_Q(\widetilde{z})| \le \Lambda_0 |\widetilde{y} - \widetilde{z}|^\eta$. Fix the function ϕ and the coordinate system CS.

For the half space in CS, $H^+ = \{y = (\widetilde{y}, y_d) \text{ in } CS : y_d > 0\}$, define a function $h_x(y) := \delta_{H^+}(y)^{\alpha/2}$. Since $\Delta^{\alpha/2} h_x(y) = 0$ for any $y \in H^+$ (see, [20, (6.6)]), it is enough to show that $\Delta^{\alpha/2}(h - h_x)(x)$ is well defined and that there exists a constant $c_1 = c_1(\eta, R_0, \Lambda_0) > 0$ independent of $x \in D \cap B(Q, R_0/8)$

and $Q \in \partial D$ such that

$$\int_{D\cup H^+} \frac{|h(y)-h_x(y)|}{|x-y|^{d+\alpha}}dy \le c_1 < \infty. \tag{3.1.8}$$

Define a function $\widehat{\phi} : B(\widetilde{0}, R_0) \subset \mathbb{R}^{d-1} \to \mathbb{R}$ as $\widehat{\phi}(\widetilde{y}) := 2\Lambda_0|\widetilde{y}|^{\eta+1}$. By the fact that $\nabla\phi(0) = 0$ and the mean value theorem, $-\widehat{\phi}(\widetilde{y}) \le \phi(\widetilde{y}) \le \widehat{\phi}(\widetilde{y})$ for any $y \in D \cap B(x, R_0/8)$. Let $A := \{y \in (D \cup H^+) \cap B(x, R_0/8) : -\widehat{\phi}(\widetilde{y}) \le y_d \le \widehat{\phi}(\widetilde{y})\}$ and $E := \{y \in B(x, R_0/8) : y_d > \widehat{\phi}(\widetilde{y})\}$. We prove (3.1.8) by showing that $I + II + III \le c_1$, where

$$I := \int_{B(x,R_0/8)^c} \frac{h(y)+h_x(y)}{|x-y|^{d+\alpha}}dy,$$
$$II := \int_A \frac{h(y)+h_x(y)}{|x-y|^{d+\alpha}}dy, \quad \text{and} \qquad III := \int_E \frac{|h(y)-h_x(y)|}{|x-y|^{d+\alpha}}dy.$$

Since $h = 0$ on $B(Q, R_0/2)^c$ and $\delta_{H^+}(y) = y_d \le |y-z| + z_d \le 2|y-z|$ for $0 < z_d < R_0/8$ and $y \in B(z, R_0/8)^c \cap H^+$, we have

$$\begin{aligned}
I &\le \left(\frac{R_0}{2}\right)^{\alpha/2} \int_{B\left(x,\frac{R_0}{8}\right)^c} \frac{1}{|x-y|^{d+\alpha}}dy + \sup_{\{z\in\mathbb{R}^d:0<z_d<R_0/8\}} \int_{B\left(z,\frac{R_0}{8}\right)^c\cap H^+} \frac{\delta_{H^+}(y)^{\alpha/2}}{|y-z|^{d+\alpha}}dy \\
&\le \left(\frac{R_0}{2}\right)^{\alpha/2} \int_{B\left(0,\frac{R_0}{8}\right)^c} \frac{1}{|y|^{d+\alpha}}dy + \int_{B\left(0,\frac{R_0}{8}\right)^c} \frac{2^{\alpha/2}}{|y|^{d+\alpha/2}}dy < \infty.
\end{aligned}$$

For any $y \in A$, we have $h(y) + h_x(y) \le c_2|\widetilde{y}|^{(1+\eta)\alpha/2}$ and $m_{d-1}(\{y : |\widetilde{y}| = r, -\widehat{\phi}(\widetilde{y}) \le y_d \le \widehat{\phi}(\widetilde{y})\}) \le c_3 r^{d+\eta-1}$ for $r \le R_0/8$, where $m_{d-1}(dy)$ is the surface measure. Since $\alpha/2 < \eta$, we have

$$\begin{aligned}
II &\le c_2 \int_0^{R_0/8} \int_{|\widetilde{y}|=r} \mathbf{1}_A(y)|\widetilde{y}|^{(1+\eta)\alpha/2}|\widetilde{y}|^{-d-\alpha} m_{d-1}(dy)dr \\
&\le c_4 \int_0^{R_0/8} r^{-\alpha/2+\eta-1+\eta\alpha/2}dr < \infty.
\end{aligned}$$

Next we estimate III. When $0 < y_d = \delta_{H^+}(y) \le \delta_D(y)$, we have $\delta_D(y) -$

$y_d \le 4\Lambda_0|\widetilde{y}|^{1+\eta}$ and

$$h(y) - h_x(y) \le (y_d + 4\Lambda_0|\widetilde{y}|^{1+\eta})^{\alpha/2} - y_d^{\alpha/2} \le 2\alpha\Lambda_0|\widetilde{y}|^{1+\eta} y_d^{\frac{\alpha}{2}-1}.$$

When $y_d = \delta_{H^+}(y) > \delta_D(y)$, we have $\delta_D(y) \ge y_d - 2\Lambda_0|\widetilde{y}|^{\eta+1}$ and

$$h_x(y) - h(y) \le y_d^{\alpha/2} - (y_d - 2\Lambda_0|\widetilde{y}|^{\eta+1})^{\alpha/2} \le \alpha\Lambda_0|\widetilde{y}|^{1+\eta}(y_d - 2\Lambda_0|\widetilde{y}|^{\eta+1})^{\frac{\alpha}{2}-1}.$$

In both cases, we use the fact that $s \to s^{\alpha/2}$ for $\alpha \in (0,2)$ is concave. Since $|y_d|^{\frac{\alpha}{2}-1} \le (y_d - 2\Lambda_0|\widetilde{y}|^{\eta+1})^{\frac{\alpha}{2}-1}$, we only consider the second case. Since $E \subset \{|\widetilde{y}| < R_0/4, \widehat{\phi}(\widetilde{y}) < y_d < \widehat{\phi}(\widetilde{y}) + R_0/4\}$, by the change of variable $s = y_d - \widehat{\phi}(r)$, we have

$$\begin{aligned} III &\le c_5 \int_E \frac{|\widetilde{y}|^{1+\eta}(y_d - 2\Lambda_0|\widetilde{y}|^{\eta+1})^{\frac{\alpha}{2}-1}}{(|\widetilde{y}| + |x_d - y_d|)^{d+\alpha}} dy \\ &\le c_6 \int_0^{R_0/4} \int_{\widehat{\phi}(r)}^{\widehat{\phi}(r)+R_0/4} \frac{(y_d - \widehat{\phi}(r))^{\frac{\alpha}{2}-1}}{(r + |x_d - y_d|)^{\alpha+1-\eta}} dy_d dr \\ &= c_6 \int_0^{R_0/4} \int_0^{R_0/4} \frac{s^{\frac{\alpha}{2}-1}}{(r + |x_d - (s + \widehat{\phi}(r))|)^{\alpha+1-\eta}} ds dr. \end{aligned}$$

Therefore by [27, Lemma 4.4], which is a consequence of the rearrangement inequality, we obtain

$$III \le 2c_6 \int_0^{R_0/2} \left(\int_0^u t^{\frac{\alpha}{2}-1} dt \right) u^{-\alpha-1+\eta} du = \frac{4c_6}{\alpha} \int_0^{R_0/2} u^{-\frac{\alpha}{2}-1+\eta} du < \infty.$$

□

Recall that $h_{Q,r}(y)$ is defined in (3.1.7) for each $Q \in \partial D$ and $r \le R_0$.

Lemma 3.1.5. *For $k > 0$, let* $B_k := \left\{ y \in D \cap B\left(Q, \frac{r}{8}\right) : \delta_{D \cap B\left(Q, \frac{r}{8}\right)}(y) \ge 2^{-k} \right\}$. *Then, for every $|z| < 2^{-k}$,*

$$\widehat{\mathcal{A}}_z h_{Q,r/2}(w) := \lim_{\varepsilon \to 0} \int_{|(w-z)-y|>\varepsilon} \left(h_{Q,r/2}(y) - h_{Q,r/2}(w-z) \right) J(w, z+y) dy \tag{3.1.9}$$

is well defined in B_k. Moreover, there exists $C_* = C_*(\eta, R_0, \Lambda_0, \rho) > 0$ independent of Q, k, and $r \le R_0$ such that

$$|\widehat{\mathcal{A}}_z h_{Q,r/2}(w)| \le C_* r^{-\alpha/2} \quad \text{for all } w \in B_k,\ |z| < 2^{-k}.$$

Proof. For $x \in D \cap B(Q, r/8)$, let

$$I = I(x) := \int_{\mathbb{R}^d} \left| h_{Q,r/2}(y) - h_{Q,r/2}(x) \right| \frac{|x-y|^\rho \wedge 1}{|x-y|^{d+\alpha}} dy \qquad \text{and}$$
$$II_\varepsilon = II_\varepsilon(x) := \int_{|x-y|>\varepsilon} \left(h_{Q,r/2}(y) - h_{Q,r/2}(x) \right) \frac{dy}{|x-y|^{d+\alpha}}.$$

For $r \le R_0$, let $x^r = r^{-1}x$, $Q^r = r^{-1}Q$, and $D^r = r^{-1}D$. Then D^r are $C^{1,\eta}$ open sets with the same $C^{1,\eta}$ characteristics $(1, \Lambda_0)$ for all $r \le R_0$, and

$$II_\varepsilon = r^{-\alpha/2} \int_{\{|v-x^r|>\varepsilon r^{-1}\}} \left(\delta_{D^r}(v)^{\alpha/2} \mathbf{1}_{D^r \cap B(Q^r,1/2)}(v) - \delta_{D^r}(x^r)^{\alpha/2} \right) \frac{dv}{|x^r - v|^{d+\alpha}}.$$

By Lemma 3.1.4, $\lim_{\varepsilon\to 0} II_\varepsilon$ exists and satisfies $|\lim_{\varepsilon\to 0} II_\varepsilon| \le c_1 r^{-\alpha/2}$.

Similarly we obtain

$$I = r^{-\alpha/2} \int_{\mathbb{R}^d} \left| \delta_{D^r}(v)^{\alpha/2} \mathbf{1}_{D^r \cap B(Q^r,1/2)}(v) - \delta_{D^r}(x^r)^{\alpha/2} \right| \frac{r^\rho |x^r - v|^\rho \wedge 1}{|x^r - v|^{d+\alpha}} dv.$$

For $r \le 1$, since $|\delta_{D^r}(v)^{\alpha/2} - \delta_{D^r}(x^r)^{\alpha/2}| \le |\delta_{D^r}(v) - \delta_{D^r}(x^r)|^{\alpha/2} \le |v - x^r|^{\alpha/2}$,

$$\begin{aligned} I &\le \int_{D^r \cap B(Q^r,1/2)} \frac{|x^r - v|^{\rho+\alpha/2}}{|x^r - v|^{d+\alpha}} dv + c_2 \int_{(D^r \cap B(Q^r,1/2))^c} \frac{1}{|x^r - v|^{d+\alpha}} dv \\ &\le c_3 \int_{\mathbb{R}^d} \frac{|u|^{\rho+\alpha/2} \wedge 1}{|u|^{d+\alpha}} du < \infty. \end{aligned}$$

The last inequality holds since $\rho > \alpha/2$.

From (2.1.2) and (2.1.5), we observe that

$$\begin{aligned}&\int_{\{|(w-z)-y|>\varepsilon\}} \big(h_{Q,r/2}(y)-h_{Q,r/2}(w-z)\big)\ J(w,z+y)\,dy\\ =&\int_{\{|(w-z)-y|>\varepsilon\}} \big(h_{Q,r/2}(y)-h_{Q,r/2}(w-z)\big)\ \frac{\kappa(w,z+y)}{|w-z-y|^{d+\alpha}\psi(|w-z-y|)}dy\\ =&\int_{\{|(w-z)-y|>\varepsilon\}} (h_{Q,r/2}(y)-h_{Q,r/2}(w-z))\frac{(\kappa(w,z+y)-\kappa(w,w))+\kappa(w,w)}{|w-z-y|^{d+\alpha}\psi(|w-z-y|)}dy\end{aligned}$$

and

$$\begin{aligned}&\left|\int_{\{y:|(w-z)-y|>\varepsilon\}} \big(h_{Q,r/2}(y)-h_{Q,r/2}(w-z)\big)\ J(w,z+y)\,dy\right|\\ &\le c_4 I(w-z)+c_4 II_\varepsilon(w-z).\end{aligned}$$

Therefore $\widehat{\mathcal{A}}_z h_{Q,r/2}(w)$ exists on B_k and we have $|\widehat{\mathcal{A}}_z h_{Q,r/2}(w)|\le c_5 r^{-\alpha/2}$ for every $w\in B_k$ and $|z|<2^{-k}$. □

Using Lemma 3.1.5, we prove the following theorem which plays a critical role in estimating the exit distribution. The proof of the next theorem is modeled after the proof of [25, Lemma 4.5]. In the next theorem for $x\in D$, we use z_x to denote a point on ∂D such that $|z_x-x|=\delta_D(x)$. Recall that D be a $C^{1,\eta}$ open set with $C^{1,\eta}$ characteristics (R_0,Λ_0), and there are the coordinate system CS_{z_x} and the $C^{1,\eta}$ function $\phi:=\phi_{z_x}$.

Theorem 3.1.6. *There are constants $b_1=b_1(\eta,R_0,\Lambda_0,\rho)\in(0,1/10)$ and $c_1=c_1(\eta,R_0,\Lambda_0)>1$ such that for any $r\le b_1(R_0\wedge 1)/2$ and $x\in D$ with $\delta_D(x)<r$, we have*

$$\mathbb{E}_x\left[\tau_{D\cap B(z_x,r)}\right]\le c_1\, r^{\alpha/2}\delta_D(x)^{\alpha/2}, \tag{3.1.10}$$

and for any $r\le(R_0\wedge 1)/4$, $\lambda\ge 4$ and $x\in D$ with $\delta_D(x)<\lambda^{-1}r/2$, we have

$$\mathbb{P}_x\left(Y_{\tau_{D\cap B(z_x,\lambda^{-1}r)}}\in\left\{2\Lambda_0|\widetilde{y}|<y_d,\lambda^{-1}r<|y|<r \text{ in } CS_{z_x}\right\}\right)\ge c_1^{-1}\frac{\delta_D(x)^{\alpha/2}}{r^{\alpha/2}}. \tag{3.1.11}$$

Proof. Without loss of generality, we assume that $z_x = 0$. For any $0 < a < b$, let $A(a,b) := B(0,b)\backslash B(0,a)$ and for $r \le (R_0 \wedge 1)/2$, let $h(y) := h_{0,r/2}(y)$ (see (3.1.7)). Suppose that f be a nonnegative smooth radial function such that $f(y) = 0$ for $|y| > 1$ and $\int_{\mathbb{R}^d} f(y)dy = 1$. For $k \ge 1$, define $f_k(y) := 2^{kd} f(2^k y)$ and $h^{(k)}(z) := (f_k * h)(z) \in C_c^\infty(\mathbb{R}^d)$, and let $B_k := \{y \in D \cap B(0,r/8) : \delta_{D\cap B(0,r/8)}(y) \ge 2^{-k}\}$.

By Lemma 3.1.5, $\widehat{\mathcal{A}}_z h(w)$ exists for $w \in B_k$ and $z \in B(0, 2^{-k})$, with $-C_* r^{-\alpha/2} \le \widehat{\mathcal{A}}_z h(w) \le C_* r^{-\alpha/2}$, where $\widehat{\mathcal{A}}_z h(w)$ is defined in (3.1.9) and C_* is the constant in Lemma 3.1.5. By letting $\varepsilon \to 0$ and using the dominated convergence theorem, it follows that $\mathcal{A}h^{(k)}$ is well defined everywhere and for large k and $|z| < 2^{-k}$, we have for $w \in B_k$,

$$|\mathcal{A}h^{(k)}(w)| = \left| \int_{|z|<2^{-k}} f_k(z) \widehat{\mathcal{A}}_z h(w)\, dz \right| \le C_* r^{-\alpha/2} \int_{|z|<2^{-k}} f_k(z)\, dz \;\le\; C_* r^{-\alpha/2}.$$

Applying (3.1.4) to $U_\lambda^k := D \cap B(0, \lambda^{-1} r) \cap B_k$ with $\lambda \ge 8$ and $h^{(k)}$ and using the above inequality, we have for $w \in U_\lambda^k$,

$$\begin{aligned} \mathbb{E}_x \left[h^{(k)}\big(Y_{\tau_{U_\lambda^k}}\big) \right] - C_* r^{-\alpha/2} \mathbb{E}_x \left[\tau_{U_\lambda^k} \right] &\le h^{(k)}(x) \\ &\le \mathbb{E}_x \left[h^{(k)}\big(Y_{\tau_{U_\lambda^k}}\big) \right] + C_* r^{-\alpha/2} \mathbb{E}_x \left[\tau_{U_\lambda^k} \right]. \end{aligned}$$

Since $h^{(k)} \in C_c^\infty(\mathbb{R}^d)$, by letting $k \to \infty$, we obtain that for all $\lambda \ge 8$ and $x \in D \cap B(0, \lambda^{-1} r)$,

$$\delta_D(x)^{\alpha/2} \ge \mathbb{E}_x \left[h\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}} \right) \right] - C_* r^{-\alpha/2} \mathbb{E}_x \left[\tau_{D\cap B(0,\lambda^{-1}r)} \right] \quad \text{and} \tag{3.1.12}$$

$$\delta_D(x)^{\alpha/2} \le \mathbb{E}_x \left[h\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}} \right) \right] + C_* r^{-\alpha/2} \mathbb{E}_x \left[\tau_{D\cap B(0,\lambda^{-1}r)} \right]. \tag{3.1.13}$$

For any $z \in D \cap B(0, \lambda^{-1} r)$ and $y \in D \cap (B(0, 2^{-1} r) \backslash B(0, \lambda^{-1} r))$, $2|y| \le r \le 1/2$ implies $j(|y - z|) \ge j(|y| + |z|) \ge j(2|y|) \ge c_1 j(|y|)$. From (2.1.7), we obtain that

$$\mathbb{E}_x\left[h\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}}\right)\right] = \mathbb{E}_x\int_{D\cap A(\lambda^{-1}r,2^{-1}r)}\int_0^{\tau_{D\cap B(0,\lambda^{-1}r)}} J(Y_t,y)dt\delta_D(y)^{\alpha/2}dy$$
$$\geq c_2\,\mathbb{E}_x\left[\tau_{D\cap B(0,\lambda^{-1}r)}\right]\int_{D\cap A(\lambda^{-1}r,2^{-1}r)} j(|y|)\delta_D(y)^{\alpha/2}dy. \qquad (3.1.14)$$

Similarly with $V := \{2\Lambda_0|\widetilde{y}| < y_d\}$, we have

$$\mathbb{P}_x\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}} \in V\cap A(\lambda^{-1}r,2^{-1}r)\right)$$
$$\geq c_3\,\mathbb{E}_x\left[\tau_{D\cap B(0,\lambda^{-1}r)}\right]\int_{V\cap A(\lambda^{-1}r,2^{-1}r)} j(|y|)dy. \qquad (3.1.15)$$

Note that for any $a>0$, $\lambda\geq 8$,

$$\int_{V\cap A(\lambda^{-1}r,2^{-1}r)} |y|^{-d-a}dy \geq \frac{c_4}{a}r^{-a}\left(\lambda^a-2^a\right) \geq \frac{c_4}{2a}r^{-a}\left(\lambda^a-1\right) \qquad (3.1.16)$$

and for any $y\in B(0,R_0)\cap D$ with $2\Lambda_0|\widetilde{y}|<y_d$,

$$\delta_D(y) \geq (1+\Lambda_0)^{-1}\left(y_d-\phi(\widetilde{y})\right) \geq (2\Lambda_0)^{-1}(y_d-\Lambda_0|\widetilde{y}|)$$
$$> (4\Lambda_0)^{-1}y_d \geq ((2\Lambda_0)^{-2}+1)^{-1/2}(4\Lambda_0)^{-1}|y|.$$

Using these two inequalities, (2.1.2) and by changing to polar coordinates with $|y|=s$, we obtain

$$\int_{D\cap A(\lambda^{-1}r,2^{-1}r)} j(|y|)\delta_D(y)^{\alpha/2}dy \geq \int_{V\cap A(\lambda^{-1}r,2^{-1}r)} j(|y|)\delta_D(y)^{\alpha/2}dy$$
$$\geq c_5\int_{V\cap A(\lambda^{-1}r,2^{-1}r)} |y|^{-d-\alpha}|y|^{\alpha/2}dy \geq c_6 r^{-\alpha/2}\left(\lambda^{\alpha/2}-1\right). \qquad (3.1.17)$$

Then, combining (3.1.14) and (3.1.17) yield

$$\mathbb{E}_x\left[h\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}}\right)\right] \geq c_7 r^{-\alpha/2}\left(\lambda^{\alpha/2}-1\right)\mathbb{E}_x\left[\tau_{D\cap B(0,\lambda^{-1}r)}\right] \qquad (3.1.18)$$

and (3.1.15) and (3.1.16) with (2.1.2) yield

$$\mathbb{P}_x\left(Y_{\tau_{D\cap B(0,\lambda^{-1}R_0)}} \in V \cap A(\lambda^{-1}r, 2^{-1}r)\right) \geq c_7 r^{-\alpha}\left(\lambda^{\alpha}-1\right)\mathbb{E}_x\left[\tau_{D\cap B(0,\lambda^{-1}r)}\right]. \tag{3.1.19}$$

Hence, by (3.1.12) and (3.1.18), we find that for every $\lambda \geq \lambda_0 := (2 + 2C_*/c_7)^{2/\alpha} \vee (10)$ and $\delta_D(x) < \lambda^{-1}r$ we have

$$\begin{aligned}\delta_D(x)^{\alpha/2} &\geq \left(c_7\lambda^{\alpha/2} - (c_7 + C_*)\right) r^{-\alpha/2}\mathbb{E}_x[\tau_{D\cap B(0,\lambda^{-1}r)}] \\ &\geq \frac{c_7}{2}(\lambda^{-1}r)^{-\alpha/2}\mathbb{E}_x[\tau_{D\cap B(0,\lambda^{-1}r)}].\end{aligned} \tag{3.1.20}$$

Thus we have proved (3.1.10) with $b_1 = \lambda_0^{-1}$ and $r = (R_0 \wedge 1)/2$.

On the other hand, since h is zero on D^c and bounded above by $(r/2)^{\alpha/2}$,

$$\mathbb{E}_x\left[h\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}}\right)\right] \leq (r/2)^{\alpha/2}\mathbb{P}_x\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}} \in D\right).$$

Thus using (3.1.13) and this, and then using Lemma 3.1.3 and (3.1.19), we find that for every $\lambda \geq 8$ and $\delta_D(x) < \lambda^{-1}r/2$ we obtain

$$\begin{aligned}\delta_D(x)^{\alpha/2} &\leq (r/2)^{\alpha/2}\mathbb{P}_x\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}} \in D\right) + C_* r^{-\alpha/2}\mathbb{E}_x\left[\tau_{D\cap B(0,\lambda^{-1}r)}\right] \\ &\leq r^{-\alpha/2}\left(c_8\lambda^{\alpha} + C_*\right)\mathbb{E}_x\left[\tau_{D\cap B(0,\lambda^{-1}r)}\right] \\ &\leq r^{\alpha/2}\frac{c_8\lambda^{\alpha} + C_*}{(\lambda^{\alpha}-1)c_7}\mathbb{P}_x\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}} \in V \cap A(\lambda^{-1}r, 2^{-1}r)\right) \\ &\leq r^{\alpha/2}\frac{(c_8 + C_*)\lambda^{\alpha}}{(\lambda^{\alpha}-(\lambda/2)^{\alpha})c_7}\mathbb{P}_x\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}} \in V \cap A(\lambda^{-1}r, 2^{-1}r)\right) \\ &= r^{\alpha/2}\frac{c_8 + C_*}{(1-2^{-\alpha})c_7}\mathbb{P}_x\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}} \in V \cap A(\lambda^{-1}r, 2^{-1}r)\right).\end{aligned}$$

Thus we have proved (3.1.11). □

3.2 Upper bound estimates

In this section, we derive the upper bound estimate for $p_D(t,x,y)$ as stated in Theorem 2.2.1. We first introduce a lemma that appears in [13]. The proof of the next lemma is identical to that of [13, Lemma 3.1] and we give the proof for the completeness.

Lemma 3.2.1. *Suppose that U_1, U_3, E are open subsets of $\mathbb{R}^d$, with $U_1, U_3 \subset E$ and $\mathrm{dist}(U_1, U_3) > 0$. Let $U_2 := E \backslash (U_1 \cup U_3)$. If $x \in U_1$ and $y \in U_3$, then for every $t > 0$ we have*

$$\begin{aligned} p_E(t,x,y) &\leq \mathbb{P}_x\left(Y_{\tau_{U_1}} \in U_2\right) \cdot \sup_{s<t, z\in U_2} p_E(s,z,y) \\ &\quad + \int_0^t \mathbb{P}_x\left(\tau_{U_1} > s\right) \mathbb{P}_y\left(\tau_E > t-s\right) ds \cdot \sup_{u\in U_1, z\in U_3} J(u,z) \qquad (3.2.1) \\ &\leq \mathbb{P}_x\left(Y_{\tau_{U_1}} \in U_2\right) \cdot \sup_{s<t, z\in U_2} p(s,z,y) \\ &\qquad + \left(t \wedge \mathbb{E}_x[\tau_{U_1}]\right) \cdot \sup_{u\in U_1, z\in U_3} J(u,z). \qquad (3.2.2) \end{aligned}$$

Proof. Using the strong Markov property of Y, we have that

$$\begin{aligned} p_E(t,x,y) &= \mathbb{E}_x\left[\, p_E(t-\tau_{U_1}, Y_{\tau_{U_1}}, y) : \tau_{U_1} < t \,\right] \\ &= \mathbb{E}_x\left[\, p_E(t-\tau_{U_1}, Y_{\tau_{U_1}}, y) : \tau_{U_1} < t \,, Y_{\tau_{U_1}} \in U_2 \,\right] \\ &\quad + \mathbb{E}_x\left[\, p_E(t-\tau_{U_1}, Y_{\tau_{U_1}}, y) : \tau_{U_1} < t \,, Y_{\tau_{U_1}} \in U_3 \,\right] =: I + II. \end{aligned}$$

Clearly,

$$I \leq \mathbb{P}_x\left(Y_{\tau_{U_1}} \in U_2\right) \left(\sup_{s<t, z\in U_2} p_E(s,z,y)\right) \leq \mathbb{P}_x\left(Y_{\tau_{U_1}} \in U_2\right) \left(\sup_{s<t, z\in U_2} p(s,z,y)\right)$$

On the other hand, by (2.1.7) and the symmetry,

$$\begin{aligned} II &\le \int_0^t \left(\int_{U_1} p_{U_1}(s,x,u) \left(\int_{U_3} J(u,z) p_E(t-s,z,y) dz \right) du \right) ds \\ &\le \left(\sup_{u\in U_1, z\in U_3} J(u,z) \right) \int_0^t \mathbb{P}_x(\tau_{U_1} > s) \left(\int_{U_3} p_E(t-s,z,y) dz \right) ds \\ &\le \left(\sup_{u\in U_1, z\in U_3} J(u,z) \right) \int_0^t \mathbb{P}_x(\tau_{U_1} > s) \mathbb{P}_y(\tau_E > t-s) ds. \end{aligned}$$

Finally

$$\int_0^t \mathbb{P}_x(\tau_{U_1} > s) \mathbb{P}_y(\tau_E > t-s) ds \le \int_0^t \mathbb{P}_x(\tau_{U_1} > s) ds \le t \wedge \mathbb{E}_x[\tau_{U_1}].$$

This completes the proof of the lemma. □

For the remainder of this section we assume that $T > 0$, $\eta \in (\alpha/2, 1]$, and D is a $C^{1,\eta}$ open set with characteristics (R_0, Λ_0). Without loss of generality, we assume that $\Lambda_0 > 1$ and $R_0 < 10^{-1}$. Recall that b_1 is the constant in Theorem 3.1.6. We let

$$a_T = a_{T,R_0} := 2^{-1} b_1 R_0 T^{-1/\alpha} < (200)^{-1} T^{-1/\alpha},$$

and for $x \in D$ recall that z_x be the point on ∂D such that $|z_x - x| = \delta_D(x)$.

We first obtain the upper bound for the survival probability. Recall that Ψ is defined in (2.2.1).

Lemma 3.2.2. *There exists a positive constant $c = c(\beta, R_0, \Lambda_0, T, \eta, \rho)$ such that for any $(t,x) \in (0,T] \times D$ we have $\mathbb{P}_x(\tau_D > t) \le c\Psi(t,x)$.*

Proof. By the definition of Ψ, we need to prove the lemma only for $\delta_D(x) \le a_T t^{1/\alpha}/8$. Let $U := D \cap B(z_x, a_T t^{1/\alpha})$. Since $\mathbb{P}_x(\tau_D > t) \le \mathbb{P}_x(\tau_U > t) + \mathbb{P}_x(Y_{\tau_U} \in D)$, by Chebyshev's inequality, Lemma 3.1.3, and (3.1.10) we have $\mathbb{P}_x(\tau_D > t) \le t^{-1}\mathbb{E}_x[\tau_U] + c_1(a_T t^{1/\alpha})^{-\alpha}\mathbb{E}_x[\tau_U] \le c_2 \delta_D(x)^{\alpha/2}/\sqrt{t} \le c_3 \Psi(t,x)$. □

Next, we use (3.2.2) to obtain the intermediate upper bound in which one boundary decay appears.

Proposition 3.2.3. *For any $a \le a_T$ and $\beta \in [0,\infty]$, there exists a positive constant $c = c(\beta, R_0, \Lambda_0, T, \eta, \rho, a)$ such that for every $(t,x,y) \in (0,T]\times D\times D$ with $|x-y| \ge 12at^{1/\alpha}\mathbf{1}_{\beta\in[0,1]} + 2\cdot\mathbf{1}_{\beta\in(1,\infty)} + 2(1+2at^{1/\alpha})\cdot\mathbf{1}_{\beta=\infty}$ we have*

$$p_D(t,x,y) \le c\Psi(t,x)\cdot\begin{cases} F^1_{C_1^{-1}\wedge\gamma_1,\gamma_1,T}(t,|x-y|/3) & \text{if } \beta\in[0,\infty),\\ F^1_{C_1^{-1},\gamma_1,T}(t,|x-y|/2) & \text{if } \beta=\infty,\end{cases}$$

where C_1 is the constant in Theorem 2.1.1 and γ_1 is the constant in (2.1.1).

Proof. By virtue of Theorem 2.1.1 and the fact that $r \to F^1_{a,\gamma,T}(t,r)$ is decreasing, the theorem holds for $\delta_D(x) \ge at^{1/\alpha}/2$.

We now fix $(t,x,y) \in (0,T]\times D\times D$ with $\delta_D(x) < at^{1/\alpha}/2$ and $|x-y| \ge 12at^{1/\alpha}\mathbf{1}_{\beta\in[0,1]}+2\cdot\mathbf{1}_{\beta\in(1,\infty)}+2(1+2at^{1/\alpha})\cdot\mathbf{1}_{\beta=\infty}$, and we define $r_t := at^{1/\alpha}$. Let $U_1 := B(z_x, r_t)\cap D$, $U_3 := \{z\in D : |z-x| > |x-y|/2\}$, and $U_2 := D\backslash(U_1\cup U_3)$. Then, $x\in U_1$ and $y\in U_3$. For $z\in U_2$, $|x-y|/2 \le |x-y|-|x-z| \le |z-y|$. Thus, by virtue of Theorem 2.1.1, we have

$$\begin{aligned}\sup_{s<t, z\in U_2} p(s,z,y) &\le c_0 \sup_{s<t,|z-y|>|x-y|/2} F^1_{C_1^{-1},\gamma_1,T}(s,|z-y|)\\ &\le c_1\left(1\vee(6a)^{-d-\alpha}\right) F^1_{C_1^{-1},\gamma_1,T}(t,|x-y|/2).\end{aligned}$$

In fact, if $\beta\in(1,\infty]$, we have $|z-y| \ge |x-y|/2 > 1$ and so $F^1_{C_1^{-1},\gamma_1,T}(s,|z-y|)$ is increasing in s. Also, if $\beta\in[0,1]$, we have $|z-y|\ge|x-y|/2\ge 6at^{1/\alpha}$ and $sr^{-\alpha-d}e^{-\gamma r^\beta}$ is increasing in s. Thus, combining these observations with the fact $r\to F^1_{C_1^{-1},\gamma_1,T}(t,r)$ is decreasing, the second inequality above holds.

Moreover, from Lemma 3.1.3 and (3.1.10) in Theorem 3.1.6 we obtain

$$\mathbb{P}_x\left(Y_{\tau_{U_1}}\in U_2\right) \le \mathbb{P}_x\left(Y_{\tau_{U_1}}\in D\right) \le c_2 t^{-1}\mathbb{E}_x[\tau_{U_1}] \le c_3\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}. \qquad (3.2.3)$$

Hence, the first part of (3.2.2) in Lemma 3.2.1 is bounded as follows:

$$\mathbb{P}_x\left(Y_{\tau_{U_1}}\in U_2\right)\left(\sup_{s<t,z\in U_2} p(s,z,y)\right) \le c_4\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}F^1_{C_1^{-1},\gamma_1,T}(t,|x-y|/2). \qquad (3.2.4)$$

If $\beta \in [0, \infty)$, since $|x-y| \geq 12at^{1/\alpha}$ we have for $u \in U_1$ and $z \in U_3$ that

$$\begin{aligned} |u-z| &\geq |z-x| - |x-z_x| - |u-z_x| \\ &> |x-y|/2 - 2at^{1/\alpha} \geq |x-y|/3. \end{aligned} \tag{3.2.5}$$

Then, from (2.1.1)–(2.1.3), (2.1.5) and (3.1.10) we obtain

$$\begin{aligned} \mathbb{E}_x[\tau_{U_1}]\Big(\sup_{u\in U_1, z\in U_3} J(u,z)\Big) &\leq\ c_5\sqrt{t}\delta_D(x)^{\alpha/2}\frac{e^{-\gamma_1(|x-y|/3)^\beta}}{|x-y|^{d+\alpha}} \\ &\leq\ c_6\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}F^1_{\gamma_1,\gamma_1,T}(t,|x-y|/3). \end{aligned} \tag{3.2.6}$$

If $\beta = \infty$, since $|u-z| \geq |x-y|/2 - 2at^{1/\alpha} \geq 1$ we have $J(u,z) = 0$ on $U_1 \times U_3$. Hence, by applying (3.2.4) and (4.4.6) to (3.2.2) for the case $\beta \in [0, \infty)$ and applying (3.2.4) to (3.2.2) for the case $\beta = \infty$, we reach the conclusion. □

For notational convenience, we denote by X the process Y in the case $\beta = 0$, and we let $J^X(x,y) := \kappa(x,y)|x-y|^{-d-\alpha}$ be its jumping kernel. By Meyer's construction (e.g., see [16, 4.1]), when $\beta \in (0, \infty]$ the process Y can be constructed from X by removing jumps of size greater than 1 with suitable rate. Let $p^X_D(t,x,y)$ be the transition density function of X on D. For $\beta \in (0, \infty]$, we define

$$\mathcal{J}(x) := \int_{\mathbb{R}^d} \kappa(x,y)|x-y|^{-(d+\alpha)}\left(1 - \psi(|x-y|)^{-1}\right)dy,$$

where $\psi(|x-y|)$ is defined in (2.1.1). Then, $\|\mathcal{J}\|_\infty \leq c_1 \int_{|z|\geq 1} |z|^{-(d+\alpha)}dz < \infty$. By [1, Lemma 3.6] we have that for any $(t,x,y) \in (0,T] \times D \times D$

$$p_D(t,x,y)\ \leq\ e^{T\|\mathcal{J}\|_\infty}p^X_D(t,x,y)\ . \tag{3.2.7}$$

The equation (3.2.7) and the upper bound of $p^X_D(t,x,y)$, which is given next, imply the upper bound of $p_D(t,x,y)$ for $|x-y| < M$ for some $M > 0$.

Proposition 3.2.4. *There exists a positive constant* $c = c(R_0, \Lambda_0, T, \eta, \rho)$

such that for any $(t,x,y) \in (0,T] \times D \times D$ we have

$$p_D^X(t,x,y) \le c\Psi(t,x)\Psi(t,y)\left(t^{-d/\alpha} \wedge t|x-y|^{-\alpha-d}\right).$$

Proof. The semigroup property, Theorem 2.1.1 (for $\beta = 0$), and Lemma 3.2.2 yield

$$\begin{aligned} p_D^X(t/2,x,y) &\le \left(\sup_{z,w\in D} p_D^X(t/4,z,w)\right)\int_D p_D^X(t/4,x,z)dz \\ &\le c_1 t^{-d/\alpha}\mathbb{P}_x(\tau_D > t/4) \le c_2 t^{-d/\alpha}\Psi(t,x). \end{aligned}$$

By Proposition 3.2.3 and Theorem 2.1.1 (for $\beta = 0$), we obtain

$$p_D^X(t/2,x,y) \le c_3\Psi(t,x)\cdot\left(t^{-d/\alpha} \wedge t|x-y|^{-\alpha-d}\right) \le c_4\Psi(t,x)p^X(t/2,x,y).$$

Combining this with Theorem 2.1.1 (for $\beta = 0$), we conclude that

$$\begin{aligned} p_D^X(t,x,y) &= \int_D p_D^X(t/2,x,z)\cdot p_D^X(t/2,z,y)dz \\ &\le c_3^2\Psi(t,x)\Psi(t,y)\int_{\mathbb{R}^d} p^X(t/2,x,z)p^X(t/2,z,y)dz \\ &= c_4^2\Psi(t,x)\Psi(t,y)p^X(t,x,y) \le c_5\Psi(t,x)\Psi(t,y)\left(t^{-d/\alpha} \wedge t|x-y|^{-\alpha-d}\right). \end{aligned}$$

□

Combining Propositions 3.2.3 and 3.2.4, we have the following proposition.

Proposition 3.2.5. *There exists a positive constant $c = c(\beta, R_0, \Lambda_0, T, \eta, \rho)$ such that for every $(t,x,y) \in (0,T] \times D \times D$ we have*

$$p_D(t,x,y) \le c\ \Psi(t,x)\cdot\begin{cases} F^1_{C_1^{-1}\wedge\gamma_1,\gamma_1,T}(t,|x-y|/3) & \text{if } \beta \in [0,\infty), \\ F^1_{C_1^{-1},\gamma_1,T}(t,|x-y|/2) & \text{if } \beta = \infty, \end{cases}$$

where C_1 is the constant in Theorem 2.1.1 and γ_1 is the constant in (2.1.1).

Next we provide the upper bound estimates for $p_D(t,x,y)$ in the case

$\beta \in (0, \infty]$.

Proof of Theorem 2.2.1(1). Let $r_t := a_T t^{1/\alpha}$. By Proposition 3.2.5 and the symmetry of $p_D(t,x,y)$, we may assume that $\delta_D(x) \vee \delta_D(y) < r_t$.

If $\beta = \infty$ and $6 < |x-y| \le 6C_1$, by (3.2.7) and Proposition 3.2.4, we have

$$p_D(t,x,y) \le c_1 \Psi(t,x)\Psi(t,y)(t/T) \le c_1 \Psi(t,x)\Psi(t,y)(t/T)^{C_1|x-y|/6}.$$

If $\beta \in [0,\infty)$ and $|x-y| \le 6C_1$ or $\beta = \infty$ and $|x-y| \le 6$, by (3.2.7) and Proposition 3.2.4, we have

$$p_D(t,x,y) \le e^{T\|\mathcal{J}\|_\infty} p_D^X(t,x,y) \le c_2 \Psi(t,x)\Psi(t,y)\left(t^{-d/\alpha} \wedge t|x-y|^{-\alpha-d}\right).$$

Thus, the theorem holds for $|x-y| \le 6C_1$.

For the remainder of the proof, we assume that $\delta_D(x) \vee \delta_D(y) < r_t$ and $|x-y| > 6C_1$ For any x with $\delta_D(x) < r_t$, let $z_x \in \partial D$ such that $\delta_D(x) = |z_x - x|$. Let $U_1 := B(z_x, r_t) \cap D$, $U_3 := \{z \in D : |z-x| > |x-y|/2\}$, and $U_2 := D \backslash (U_1 \cup U_3)$. Note that $x \in U_1$ and $y \in U_3$ and $|x-y|/2 \le |z-y|$ for $z \in U_2$. By Proposition 3.2.5 we have

$$\begin{aligned}
\sup_{s<t, z\in U_2} p_D(s,z,y) \le& \sup_{s<t, z\in U_2} c_4 \frac{\delta_D(y)^{\alpha/2}}{\sqrt{s}} \cdot F^1_{C_1^{-1}\wedge\gamma_1,\gamma_1,T}(s, |z-y|/3) \cdot \mathbf{1}_{\beta\in[0,\infty)} \\
&+ \sup_{s<t, z\in U_2} c_4 \frac{\delta_D(y)^{\alpha/2}}{\sqrt{s}} \cdot F^1_{C_1,\gamma_1,T}(s, |z-y|/2) \cdot \mathbf{1}_{\beta=\infty} \\
\le&\ c_5 \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \cdot F^1_{C_1^{-1}\wedge\gamma_1,\gamma_1,T}(t, |x-y|/6) \cdot \mathbf{1}_{\beta\in[0,\infty)} \\
&+ c_5 \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \cdot F^1_{C_1^{-1},\gamma_1,T}(t, |x-y|/4) \cdot \mathbf{1}_{\beta=\infty}. \quad (3.2.8)
\end{aligned}$$

The last inequality is clear for $\beta \in [0,\infty)$ by definition of $F^1_{a,\gamma,T}$, and for $\beta = \infty$ by the fact that $s \to s^{-1/2}(s/Tr)^{ar}$ is increasing if $ar \ge 1$. Hence, from (3.2.3) and (3.2.8) we obtain

$$\begin{aligned}
&\mathbb{P}_x\left(Y_{\tau_{U_1}} \in U_2\right)\left(\sup_{s<t, z\in U_2} p_D(s,z,y)\right) \\
&\le c_6 \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \cdot \begin{cases} F^1_{C_1^{-1}\wedge\gamma_1,\gamma_1,T}(t,|x-y|/6) & \text{if } \beta\in[0,\infty), \\ F^1_{C_1^{-1},\gamma_1,T}(t,|x-y|/6) & \text{if } \beta=\infty. \end{cases} \qquad (3.2.9)
\end{aligned}$$

Also, from Lemma 3.2.2 we have

$$\begin{aligned}
\int_0^t \mathbb{P}_x(\tau_{U_1}>s)\mathbb{P}_y(\tau_D>t-s)ds &\le \int_0^t \mathbb{P}_x(\tau_D>s)\mathbb{P}_y(\tau_D>t-s)ds \\
&\le c_7\, \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}\int_0^t s^{-1/2}(t-s)^{-1/2}ds \\
&\le c_8\, t\, \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}. \qquad (3.2.10)
\end{aligned}$$

For $\beta\in[0,\infty)$, $|u-z|\ge|x-y|/3$ for $(u,z)\in U_1\times U_3$ as in (3.2.5). Combining (2.1.1)–(2.1.3) and (3.2.10), we obtain

$$\begin{aligned}
&\int_0^t \mathbb{P}_x\left(\tau_{U_1}>s\right)\mathbb{P}_y\left(\tau_D>t-s\right)ds\cdot\left(\sup_{u\in U_1, z\in U_3} J(u,z)\right) \\
&\le c_9 \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} t \frac{e^{-\gamma_1(|x-y|/3)^\beta}}{|x-y|^{d+\alpha}} \\
&\le c_9 \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} F^1_{\gamma_1,\gamma_1,T}(t,|x-y|/3). \qquad (3.2.11)
\end{aligned}$$

If $\beta=\infty$, since $|u-z|>1$, $J(u,z)=0$ on $U_1\times U_3$. Therefore, by applying (3.2.9) and (3.2.11) in (3.2.1) of Lemma 3.2.1 for $\beta\in[0,\infty)$ and applying (3.2.9) for $\beta=\infty$, we prove the theorem for $|x-y|>6C_1$ and $\delta_D(x)\vee\delta_D(y)<r_t$. □

3.3 Inteiror lower bound estimates

In this section, we discuss a preliminary lower bound for $p_D(t,x,y)$.

Note that conditions in [5] are even weaker than conditions in [6]. Thus Y satisfies conditions imposed in [5]. Using [5, Theorem 1.4 and Lemma 2.5], the proof of the next lemma is the same as that of [9, Lemma 3.1]. Thus, we omit the proof.

Lemma 3.3.1. *Let T, a, and b be positive constants. For any $\beta \in [0, \infty]$, there exists a constant $c = c(a, b, \beta, T) > 0$ such that for all $\lambda \in (0, T]$ we have*

$$\inf_{\substack{y \in \mathbb{R}^d \\ |y-z| \le b\lambda^{1/\alpha}}} \mathbb{P}_y \left(\tau_{B(z, 2b\lambda^{1/\alpha})} > a\lambda \right) \ge c.$$

Next, we give some preliminary lower bound estimates for $p_D(t, x, y)$ on $\delta_D(x) \wedge \delta_D(y) \wedge T \ge t^{1/\alpha}$, which are used to derive the sharp two-sided estimates for $p_D(t, x, y)$. We first consider D an arbitrary nonempty open set, and we use the convention that $\delta_D(\cdot) \equiv \infty$ when $D = \mathbb{R}^d$. This convention allows us to derive the lower bound of $p(t, x, y)$ simultaneously.

Using [5, Theorem 1.4] and Lemma 3.3.1, the proof of the next lemma is the same as that of [9, Proposition 3.2]. Thus, we omit the proof.

Proposition 3.3.2. *Let D be an arbitrary open set and let a and T be positive constants. Suppose that $(t, x, y) \in (0, T] \times D \times D$, with $\delta_D(x) \ge at^{1/\alpha} \ge 2|x - y|$. Then, for any $\beta \in [0, \infty]$, there exists a positive constant $c = c(a, \beta, T)$ such that $p_D(t, x, y) \ge c\, t^{-d/\alpha}$.*

Proposition 3.3.3. *Let D be an arbitrary open set and let a and T be positive constants. Suppose that $(t, x, y) \in (0, T] \times D \times D$, with $\delta_D(x) \wedge \delta_D(y) \ge at^{1/\alpha}$ and $at^{1/\alpha} \le 2|x-y|$. Then, for any $\beta \in [0, \infty]$, there exists a positive constant $c = c(a, \beta, T)$ such that $p_D(t, x, y) \ge ctj(|x - y|)$.*

Proof. By Lemma 3.3.1, starting at $z \in B(y,\, 4^{-1}at^{1/\alpha})$, with probability at least $c_1 = c_1(a, \beta, T) > 0$ the process Y does not move more than $6^{-1}at^{1/\alpha}$ by time t. Thus, by the strong Markov property

$$\mathbb{P}_x \left(Y_t^D \in B\big(y,\, 2^{-1}at^{1/\alpha}\big) \right) \ge c_1 \mathbb{P}_x \left(Y^D \text{ hits the ball } B(y,\, 4^{-1}at^{1/\alpha}) \text{ by time } t \right).$$

Now using this and the Lévy system in (2.1.7), we obtain

$$\begin{aligned}
&\mathbb{P}_x\left(Y_t^D \in B\left(y,\, 2^{-1}at^{1/\alpha}\right)\right)\\
\ge\, & c_1\mathbb{P}_x\left(Y^D_{t\wedge\tau_{B\left(x,6^{-1}at^{1/\alpha}\right)}} \in B\left(y, 4^{-1}at^{1/\alpha}\right), t\wedge\tau_{B\left(x,6^{-1}at^{1/\alpha}\right)} \text{ is a jumping time}\right)\\
=\, & c_1\mathbb{E}_x\left[\int_0^{t\wedge\tau_{B(x,6^{-1}at^{1/\alpha})}}\int_{B(y,\,4^{-1}at^{1/\alpha})} J(Y_s,u)\,du\,ds\right]. \qquad (3.3.1)
\end{aligned}$$

Lemma 3.3.1 also implies that for all $t\in(0,T]$,

$$\mathbb{E}_x\left[t\wedge\tau_{B(x,6^{-1}at^{1/\alpha})}\right] \;\ge\; t\,\mathbb{P}_x\left(\tau_{B(x,6^{-1}at^{1/\alpha})}\ge t\right) \;\ge\; c_2\,t. \qquad (3.3.2)$$

We fix the point w on the line connecting $|x-y|$ (i.e., $|x-y| = |x-w|+|w-y|$) such that $|w-y| = 7\cdot 2^{-5}at^{1/\alpha}$, which is possible because $\delta_D(y)\ge at^{1/\alpha}$. Then, $B(w,2^{-5}at^{1/\alpha})\subset B(y,\,4^{-1}at^{1/\alpha})$. Moreover, for every $(z,u)\in B(x,6^{-1}at^{1/\alpha})\times B(w,2^{-5}at^{1/\alpha})$ we have

$$|z-u| < 6^{-1}at^{1/\alpha}+2^{-5}at^{1/\alpha}+|x-w| = |x-y|+(6^{-1}+2^{-5}-7\cdot 2^{-5})at^{1/\alpha} < |x-y|.$$

Thus, $B(w,2^{-5}at^{1/\alpha})\subset B(y,\,4^{-1}at^{1/\alpha})\cap\{u:|u-z|<|x-y|\}$. Combining this result with (2.1.3) and (3.3.2), we obtain

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^{t\wedge\tau_{B(x,6^{-1}at^{1/\alpha})}}\int_{B(y,\,4^{-1}at^{1/\alpha})} J(Y_s,u)duds\right]\\
\ge&\mathbb{E}_x\left[\int_0^{t\wedge\tau_{B(x,6^{-1}at^{1/\alpha})}}\int_{B(w,2^{-5}at^{1/\alpha})} J(Y_s,u)\mathbf{1}_{\{|Y_s-u|<|x-y|\}}duds\right]\\
\ge& c_3\,\mathbb{E}_x\left[t\wedge\tau_{B(x,6^{-1}at^{1/\alpha})}\right]\left|B(w,2^{-5}at^{1/\alpha})\right| j(|x-y|)\\
>& c_4\,t^{1+d/\alpha}j(|x-y|). \qquad (3.3.3)
\end{aligned}$$

Then, using the semigroup property along with (3.3.3) and Proposition 3.3.2, the proposition follows from the proof of [9, Proposition 3.4]. □

Combining Propositions 3.3.2 and 3.3.3 with the definition of j, we obtain a lower bound for $p_D(t,x,y)$ that yields the preliminary lower bound for $p_D(t,x,y)$ and $p(t,x,y)$ for the case $\beta\in[0,1]$ and the case $\beta\in(1,\infty]$ with

$|x-y|<1$.

Proposition 3.3.4. *Let D be an arbitrary open set and let a and T be positive constants. Suppose that $(t,x,y)\in(0,T]\times D\times D$, with $\delta_D(x)\wedge\delta_D(y)\geq at^{1/\alpha}$. Then, for any $\beta\in[0,\infty]$, there exists a positive constant $c=c(a,\beta,T)$ such that*

$$p_D(t,x,y)\ \geq\ c\left(t^{-d/\alpha}\wedge tj(|x-y|)\right).$$

We next consider cases $\beta\in(1,\infty]$ with $|x-y|\geq 1$. We will closely follow the proofs of [4, Theorem 3.6] and [6, Theorem 5.5].

For the remainder of this section, we assume that D is an open set with the following property: there exist $\lambda_1\in[1,\infty)$ and $\lambda_2\in(0,1]$ such that for every $r\leq 1$ and x,y in the same component of D with $\delta_D(x)\wedge\delta_D(y)\geq r$ there exists in D a length parameterized rectifiable curve l connecting x to y with the length $|l|$ of l less than or equal to $\lambda_1|x-y|$ and $\delta_D(l(u))\geq\lambda_2 r$ for $u\in[0,|l|]$.

Under this assumption, we prove the preliminary lower bound of $p_D(t,x,y)$ on $|x-y|\geq 1$ separately for the case $\beta\in(1,\infty)$ and the case $\beta=\infty$.

Proposition 3.3.5. *Suppose that $T>0$, $a\in(0,4^{-1}T^{-1/\alpha}]$, and $\beta\in(1,\infty)$. Then, there exist constants $c_i=c_i(a,\beta,T,\lambda_1,\lambda_2)>0$, $i=1,2$ such that for any x,y in the same component of D with $\delta_D(x)\wedge\delta_D(y)\geq at^{1/\alpha}$, $|x-y|\geq 1$, and $t\leq T$ we have*

$$p_D(t,x,y)\geq c_1 t\exp\left(-c_2\left(|x-y|\left(\log\frac{T|x-y|}{t}\right)^{\frac{\beta-1}{\beta}}\wedge(|x-y|)^{\beta}\right)\right).$$

Proof. We fix $T>0$ and $a\in(0,4^{-1}T^{-1/\alpha}]$, and we let $R_1:=|x-y|$. If either $1\leq R_1\leq 2$ or $R_1(\log(TR_1/t))^{(\beta-1)/\beta}\geq(R_1)^{\beta}$, the proposition holds by virtue of Proposition 3.3.4. Thus, for the remainder of this proof we assume that $R_1>2$ and $R_1(\log(TR_1/t))^{(\beta-1)/\beta}<(R_1)^{\beta}$, which is equivalent to $R_1\exp\{-(R_1)^{\beta}\}<t/T$.

Let $k \geq 2$ be a positive integer such that

$$1 < R_1 \left(\log \frac{TR_1}{t}\right)^{-1/\beta} \leq k < R_1 \left(\log \frac{TR_1}{t}\right)^{-1/\beta} + 1 < 2R_1 \left(\log \frac{TR_1}{t}\right)^{-1/\beta}. \tag{3.3.4}$$

By our assumption for D, there is a length parameterized curve $l \subset D$ connecting x and y such that the total length $|l|$ of l is less than or equal to $\lambda_1 R_1$ and $\delta_D(l(u)) \geq \lambda_2 a t^{1/\alpha}$ for every $u \in [0, |l|]$. We define $r_t := (2^{-1}\lambda_2 a t^{1/\alpha}) \wedge ((6\lambda_1)^{-1}(\log(TR_1/t))^{1/\beta})$. Then, by (4.2.1) and the assumption $((\log(TR_1/t))^{1/\beta}) \vee 2 < R_1$ we have

$$\left(\frac{\lambda_2}{2} a T^{1/\alpha} \left(\frac{t}{TR_1}\right)^{1/\alpha}\right) \wedge \left(\frac{(2\log 2)^{1/\beta}}{6\lambda_1} \left(\frac{t}{TR_1}\right)^{1/\beta}\right) \leq r_t \leq \frac{1}{6\lambda_1}\left(\log \frac{TR_1}{t}\right)^{1/\beta} < \frac{R_1}{3\lambda_1 k}. \tag{3.3.5}$$

We define $x_i := l(i|l|/k)$ and $B_i := B(x_i, r_t)$, with $i = 0, \ldots, k$. Then, $\delta_D(y_i) \geq 2^{-1}\lambda_2 a t^{1/\alpha} > 2^{-1}\lambda_2 a (t/k)^{1/\alpha}$ for every $y_i \in B_i$. Note that from (3.3.5) we obtain

$$|y_i - y_{i+1}| \leq |x_i - x_{i+1}| + 2r_t \leq \left(\lambda_1 + \frac{2}{3\lambda_1}\right)\frac{R_1}{k}. \tag{3.3.6}$$

Thus, using Proposition 3.3.4 along with (4.2.1) and (4.2.4) we obtain

$$\begin{aligned} p_D(t/k, y_i, y_{i+1}) &\geq c_1 \left((t/k)^{-d/\alpha} \wedge \frac{t}{k} j(|y_i - y_{i+1}|)\right) \\ &\geq c_2 \left(1 \wedge \left(\frac{t}{k}(R_1/k)^{-d-\alpha} e^{-c_3 (R_1/k)^\beta}\right)\right) \geq c_4 \frac{t}{TR_1}\left(\frac{k}{R_1}\right)^{d+\alpha-1} e^{-c_3 (R_1/k)^\beta} \\ &\geq c_4 \frac{t}{TR_1}\left(\log \frac{TR_1}{t}\right)^{-\frac{d+\alpha-1}{\beta}} \left(\frac{t}{TR_1}\right)^{c_3} \geq c_4 \left(\frac{t}{TR_1}\right)^{c_5}. \end{aligned} \tag{3.3.7}$$

Since the lower bound of r_t in (3.3.5) yields $r_t \geq c_6 (t/(TR_1))^{(\alpha\wedge\beta)^{-1}}$, by using

(4.2.5) and the semigroup property we conclude that

$$\begin{aligned} p_D(t,x,y) &\geq \int_{B_1} \cdots \int_{B_{k-1}} p_D(t/k,x,y_1)\cdots p_D(t/k,y_{k-1},y)dy_1\cdots dy_{k-1} \\ &\geq c_7 \exp\{-c_8 k \log(TR_1/t)\} \\ &\geq c_7 \exp\left\{-c_8\left(R_1 \log\left(\frac{TR_1}{t}\right)^{-1/\beta} + 1\right)\log\frac{TR_1}{t}\right\} \\ &\geq c_7 \exp\left\{-c_9\left(R_1 \log\left(\frac{TR_1}{t}\right)^{1-1/\beta}\right)\right\}. \end{aligned}$$

□

Proposition 3.3.6. *Suppose that $T > 0$, $a \in (0, 4^{-1}T^{-1/\alpha}]$, and $\beta = \infty$. Then, there exist positive constants $c_i = c_i(a, T, \lambda_1, \lambda_2)$, $i = 1, 2$, such that for any x, y in the same component of D with $\delta_D(x) \wedge \delta_D(y) \geq at^{1/\alpha}$, $|x-y| \geq 1$, and $t \leq T$, we have*

$$p_D(t,x,y) \;\geq\; c_1 \left(\frac{t}{T|x-y|}\right)^{c_2|x-y|}.$$

Proof. We fix $T > 0$ and $a \in (0, 4^{-1}T^{-1/\alpha}]$, and we let $R_1 := |x-y| \geq 1$. By our assumption of D, there is a length parameterized curve $l \subset D$ connecting x and y such that the total length $|l|$ of l is less than or equal to $\lambda_1 R_1$ and $\delta_D(l(u)) \geq \lambda_2 a t^{1/\alpha}$ for every $u \in [0, |l|]$. We define k as the integer satisfying $(4 \leq) 4\lambda_1 R_1 \leq k < 4\lambda_1 R_1 + 1 < 5\lambda_1 R_1$ and $r_t := 2^{-1}\lambda_2 a t^{1/\alpha} \leq 8^{-1}$. Let $x_i := l(i|l|/k)$ and $B_i := B(x_i, r_t)$, with $i = 0, 1, 2, \ldots, k$. Then, $\delta_D(x_i) > 2r_t$ and $B_i = B(x_i, r_t) \subset B(x_i, 2r_t) \subset D$, with $i = 0, 1, 2, \ldots, k$.

Since $4\lambda_1 R_1 \leq k$, for each $y_i \in B_i$ we have

$$\begin{aligned} |y_i - y_{i+1}| &\leq |y_i - x_i| + |x_i - x_{i+1}| + |x_{i+1} - y_{i+1}| \\ &\leq 8^{-1} + |l|/k + 8^{-1} < \lambda_1 R_1/(4\lambda_1 R_1) + 4^{-1} \leq 2^{-1}. \quad (3.3.8) \end{aligned}$$

Moreover, $\delta_D(y_i) \geq \delta_D(x_i) - |y_i - x_i| > r_t > r_{t/k}$.

Thus, by Proposition 3.3.4 and (3.3.8), there are constants $c_i = c_i(a, T, \lambda_2) >$

0, $i = 1, 2$, such that for $(y_i, y_{i+1}) \in B_i \times B_{i+1}$,

$$p_D(t/k, y_i, y_{i+1}) \geq c_1 \left((t/k)^{-d/\alpha} \wedge \frac{t/k}{|y_i - y_{i+1}|^{d+\alpha}} \right) \geq c_2\, t/(Tk). \quad (3.3.9)$$

Observe that $4\lambda_1 R_1 \leq k < 2(k-1) < 8\lambda_1 R_1$ and $r_t \geq T^{1/\alpha} r_{t/(Tk)}$. Thus, from (3.3.9) we obtain

$$\begin{aligned} p_D(t,x,y) &\geq \int_{B_1} \dots \int_{B_{k-1}} p_D(t/k, x, y_1) \dots p_D(t/k, y_{k-1}, y) dy_{k-1} \dots dy_1 \\ &\geq (c_2 t (Tk)^{-1})^k \Pi_{i=1}^{k-1} |B_i| \geq c_3 (c_4 t (Tk)^{-1})^{c_5 k} \\ &\geq c_6 (c_7 t (TR_1)^{-1})^{c_8 R_1} \geq c_9 (t (TR_1)^{-1})^{c_{10} R_1}. \end{aligned}$$

□

Proof of the lower bound in Theorem 2.1.1(1) . The proof for the two cases $\beta \in [0,1]$ and $\beta \in (1, \infty]$ with $|x - y| < 1$ follow from Proposition 3.3.4 with $D = \mathbb{R}^d$. The proof for the remaining cases follows from Propositions 3.3.5 and 3.3.6 with $D = \mathbb{R}^d$. □

3.4 Lower bound estimates

We proved the preliminary lower bound estimates in Section 3.3. In this section, combining these results with the key estimate in (3.1.11), we give the full lower bound estimate for $p_D(t, x, y)$ with the boundary decay terms. We first introduce the next lemma which is introduce in [10, Lemma 3.3] and give you the proof for the completeness.

Lemma 3.4.1. *Suppose that* $U_1, U_2 \subset E$ *are open subsets of* $\mathbb{R}^d$ *with* $\mathrm{dist}(U_1, U_2) > 0$*. If* $x \in U_1$ *and* $y \in U_2$*, then for all* $t > 0$ *we have*

$$p_E(t, x, y) \geq t\ \mathbb{P}_x(\tau_{U_1} > t)\ \mathbb{P}_y(\tau_{U_2} > t) \inf_{(u,w) \in U_1 \times U_2} J(u, w).$$

Proof. Using the strong Markov property, we have

$$p_E(t,x,y) \geq \mathbb{E}_x \left[p_E\left(t-\tau_{U_1}, Y_{\tau_{U_1}}, y\right) : \tau_{U_1} < t, Y_{\tau_{U_1}} \in U_2 \right].$$

By (2.1.7), we have that

$$\begin{aligned} p_E(t,x,y) &\geq \int_0^t \left(\int_{U_1} p_{U_1}(s,x,u) \left(\int_{U_2} J(u,z) p_E(t-s,z,y) dz \right) du \right) ds \\ &\geq \inf_{u \in U_1, z \in U_2} J(u,z) \int_0^t \int_{U_2} p_E(t-s,z,y) \mathbb{P}_x(\tau_{U_1} > s) dz\, ds \end{aligned}$$

and

$$\begin{aligned} &\int_0^t \int_{U_2} p_E(t-s,x,y) \mathbb{P}_x(\tau_{U_1} > s)\, dz\, ds \\ &\geq \mathbb{P}_x(\tau_{U_1} > t) \int_0^t \mathbb{P}_y(\tau_{U_2} > t-s)\, ds \geq t\, \mathbb{P}_x(\tau_{U_1} > t)\, \mathbb{P}_y(\tau_{U_2} > t). \end{aligned}$$

□

For the remainder of this section we assume that $T > 0$, $\eta \in (\alpha/2, 1]$, and D is a $C^{1,\eta}$ open set with characteristics (R_0, Λ_0). Without loss of the generality, we assume that $\Lambda_0 > 4$ and $R < 10^{-1}$. We let

$$\widehat{a}_T = a_{T,R} := 2^{-5} R T^{-1/\alpha} < 2^{-5} 10^{-1} T^{-1/\alpha},$$

and for $x \in D$ recall that z_x be the point on ∂D such that $|z_x - x| = \delta_D(x)$.

The next two lemmas are crucial to obtain the lower bound on the survival probability where x is near the boundary of D.

Lemma 3.4.2. *For any $a \leq \widehat{a}_T$, there exists a positive constant $c = c$ $(a, \beta, R_0, \Lambda_0, T, \eta, \rho)$ such that for every $t < T$ and $x \in D$ with $\delta_D(x) < at^{1/\alpha}$ we have*

$$\mathbb{P}_x(\tau_{B(z_x, 10at^{1/\alpha}) \cap D} > t/3) \geq c\ \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}.$$

Proof. Without loss of generality, we assume that $z_x = 0$. Consider a coordinate system $CS := CS_0$ such that $B(0, R_0) \cap D = \{y = (\widetilde{y}, y_d) \in$

$B(0,R_0)$ in $CS : y_d > \phi(\widetilde{y})\}$, where ϕ is a $C^{1,\eta}$ function such that $\phi(0)=0$, $\nabla\phi(0)=(0,\dots,0)$, $\|\nabla\phi\|_\infty \le \Lambda_0$, and $|\nabla\phi(\widetilde{y})-\nabla\phi(\widetilde{w})| \le \Lambda_0|\widetilde{y}-\widetilde{w}|^\eta$.

Let $\psi(\widetilde{y}) = 2\Lambda_0|\widetilde{y}|$ and $V := \{y=(\widetilde{y},y_d)\in B(0,R_0)$ in $CS : y_d > \psi(\widetilde{y})\}$. Then, since $\psi(\widetilde{y}) \ge 2\Lambda_0|\widetilde{y}|^{\eta+1}$, the mean value theorem yields $\{y=(\widetilde{y},y_d)\in B(0,R_0)$ in $CS : y_d > \psi(\widetilde{y})\} \subset B(0,R_0)\cap D$.

Let $U_1 := B(0,2at^{1/\alpha})\cap D$, $U_2 := B(0,10at^{1/\alpha})\cap D$, and

$$W := \{y=(\widetilde{y},y_d)\in B(0,8at^{1/\alpha})\backslash B(0,2at^{1/\alpha}) \text{ in } CS : y_d > \psi(\widetilde{y})\}.$$

Since $\Lambda|\widetilde{w}| = \psi(\widetilde{w})/2 < w_d/2$ for $w\in W$, we have

$$\begin{aligned}\delta_D(w) &> (1+\Lambda)^{-1}(w_d-\phi(\widetilde{w}))\\ &> (1+\Lambda)^{-1}(w_d-\Lambda|\widetilde{w}|) > 2^{-1}(1+\Lambda)^{-1}w_d. \end{aligned}\tag{3.4.1}$$

Moreover, since $|\widetilde{w}| \le (2\Lambda)^{-1}|w| \le \Lambda^{-1}4at^{1/\alpha} \le at^{1/\alpha}$ for $w\in W$, we have

$$w_d^2 = |w|^2-|\widetilde{w}|^2 \ge (2at^{1/\alpha})^2-(at^{1/\alpha})^2 = 3(at^{1/\alpha})^2 \text{ for } w\in W. \tag{3.4.2}$$

Combining (3.4.1) and (3.4.2), we obtain $\delta_D(w) > (1+\Lambda)^{-1}at^{1/\alpha}$. Thus, $B(w,r_1at^{1/\alpha})\subset U_2$ for $w\in W$, where $r_1 := (1+\Lambda)^{-1}$. Hence, by virtue of the strong Markov property, Lemma 3.3.1, and (3.1.11), we have

$$\begin{aligned}&\mathbb{P}_x(\tau_{U_2}>t/3) \ge \mathbb{P}_x(\tau_{U_2}>t/3, Y_{\tau_{U_1}}\in W) = \mathbb{E}_x[\mathbb{P}_{Y_{\tau_{U_1}}}(\tau_{U_2}>t/3) : Y_{\tau_{U_1}}\in W]\\ &\ge \mathbb{E}_x[\mathbb{P}_{Y_{\tau_{U_1}}}(\tau_{B(Y_{\tau_{U_1}},r_1at^{1/\alpha})}>t/3) : Y_{\tau_{U_1}}\in W]\\ &\ge \left(\inf_{z\in\mathbb{R}^d}\mathbb{P}_z(\tau_{B(z,r_1at^{1/\alpha})}>t/3)\right)\mathbb{P}_x(Y_{\tau_{U_1}}\in W) \ge c_1\,\mathbb{P}_x(Y_{\tau_{U_1}}\in W) \ge c_2\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}.\end{aligned}$$

□

We introduce the following definition for the subsequent lemma.

Definition 3.4.3. *Let $0<\kappa\le 1/2$. We say that an open set D is κ-fat if there is $R_1>0$ such that for all $x\in\overline{D}$ and all $r\in(0,R_1]$ there is a ball $B(A_r(x),\kappa r)\subset D\cap B(x,r)$. The pair (R_1,κ) are called the characteristics of the κ-fat open set D.*

It is clear that a $C^{1,\eta}$ open set D with characteristics (R_0,Λ_0) is always a

κ-fat set whose characteristics (R_1, κ) depend only on R_0, Λ_0, and d. Hereinafter, without loss of generality, we assume that $R_0 \le R_1$ (by choosing R_0 smaller if necessary) and that $A_r(x)$ is always the point $A_r(x) \in D$ in Definition 3.4.3 for D. Recall that Ψ is defined in (2.2.1).

Lemma 3.4.4. *For any $\beta \in [0, \infty]$, there exists a positive constant $c = c(\beta, R_0, \Lambda_0, T, \eta, \rho) > 0$ such that, for every $t < T$ and $x \in D$, we can find x_1 with $\delta_D(x_1) \ge 2^{-1}\kappa \widehat{a}_T t^{1/\alpha}$ and $|x_1 - x| \le 6\widehat{a}_T t^{1/\alpha}$ such that*

$$\int_{B(x_1,(\kappa/4)\widehat{a}_T t^{1/\alpha})} p_D(t/3, x, z)dz \ge c\Psi(t, x).$$

Proof. For any $\delta_D(x) < 2^{-1}\kappa\widehat{a}_T t^{1/\alpha}$, let $x_1 = A_{6\widehat{a}_T t^{1/\alpha}}(z_x)$. Let $B_{x_1} := B(x_1, (\kappa/4)\widehat{a}_T t^{1/\alpha})$ and $B_{z_x} := B(z_x, 5\kappa\widehat{a}_T t^{1/\alpha}) \cap D$ so that $B_{x_1} \cap B_{z_x} = \emptyset$. By Lemmas 4.3.2, 4.3.3, and 3.3.1,

$$\begin{aligned}
&\int_{B_{x_1}} p_D(t/3, x, z)dz \\
\ge& \frac{t}{3} \int_{B_{x_1}} \mathbb{P}_x(\tau_{B_{z_x}} > t/3)\, \mathbb{P}_z(\tau_{B_{x_1}} > t/3) \cdot \inf_{(u,w) \in B_{z_x} \times B_{x_1}} J(u, w)dz \\
\ge& \frac{t}{3}\, \mathbb{P}_x(\tau_{B_{z_x}} > t/3) \cdot c_1 \int_{B_{x_1}} dz \cdot \frac{c_2}{(t^{1/\alpha})^{d+\alpha}} = c_3 \mathbb{P}_x(\tau_{B_{z_x}} > t/3) \ge c_4\, \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}.
\end{aligned}$$

For $\delta_D(x) \ge 2^{-1}\kappa\widehat{a}_T t^{1/\alpha}$, let $x_1 = x$ and $B_{x_1} := B(x_1, (\kappa/4)\widehat{a}_T t^{1/\alpha})$. By Lemma 3.3.1, there exists a constant $c_5 = c_5(\alpha, \beta, R_0, T, d, L_3) > 0$ such that

$$\int_{B_{x_1}} p_D(t/3, x, z)dz \ge \int_{B_{x_1}} p_{B_{x_1}}(t/3, x, z)dz = \mathbb{P}_x(\tau_{B_{x_1}} > t/3) > c_5.$$

This proves the lemma. □

We are now ready to give the proof of the lower bound estimates for $p_D(t, x, y)$. Recall our assumption that $\eta \in (\alpha/2, 1]$ and D is a $C^{1,\eta}$ open set. For the cases $\beta \in (1, \infty)$ with $|x - y| \ge 1$ and $\beta = \infty$ with $|x - y| > 4/5$, we assume in addition that the path distance in each connected component of D is comparable to the Euclidean distance with characteristic λ_1. Note that combining this assumption with $C^{1,\eta}$ assumption entails that D satisfies the assumption made before Proposition 3.3.5.

Proof of Theorem 2.2.1(2) and (3). By Lemma 3.4.4, for any $x, y \in D$, there exists $x_1, y_1 \in D$ such that $\delta_D(x_1) \wedge \delta_D(y_1) \geq 2^{-1}\kappa\widehat{a}_T t^{1/\alpha}$ and $|x_1 - x| \vee |y_1 - y| \leq 6\widehat{a}_T t^{1/\alpha}$, and there exists a constant $c_1 = c_1(\eta, \rho, \beta, R_0, \Lambda_0, T) > 0$ independent of x, y such that

$$\int_{B_{x_1}} p_D(t/3, x, z)dz \int_{B_{y_1}} p_D(t/3, y, z)dz \geq c_1\Psi(t, x)\Psi(t, y), \tag{3.4.3}$$

where $B_{x_1} := B(x_1, (\kappa/4)\widehat{a}_T t^{1/\alpha})$ and $B_{y_1} := B(y_1, (\kappa/4)\widehat{a}_T t^{1/\alpha})$. Thus, by the semigroup property we have

$$\begin{aligned} p_D(t, x, y) &= \int_D \int_D p_D(t/3, x, u)p_D(t/3, u, w)p_D(t/3, w, y)dudw \\ &\geq \int_{B_{x_1}} p_D(t/3, x, u)du \int_{B_{y_1}} p_D(t/3, y, w)dw \left(\inf_{(u,w)\in B_{x_1}\times B_{y_1}} p_D(t/3, u, w) \right) \\ &\geq c_1\Psi(t, x)\Psi(t, y) \inf_{(u,w)\in B_{x_1}\times B_{y_1}} p_D(t/3, u, w). \end{aligned} \tag{3.4.4}$$

We now carefully calculate the lower bounds of $p_D(t/3, u, w)$ on $B_{x_1} \times B_{y_1}$. Since $|x - x_1| \vee |y - y_1| \leq 6\widehat{a}_T t^{1/\alpha}$, for $u \in B_{x_1}$ and $w \in B_{y_1}$ we have

$$\begin{aligned} &|x - y| - 20^{-1} \leq |x - y| - (12 + (\kappa/2))\widehat{a}_T t^{1/\alpha} \\ &\quad \leq |u - w| \leq |x - y| + (12 + (\kappa/2))\widehat{a}_T t^{1/\alpha} \leq |x - y| + 20^{-1} \end{aligned} \tag{3.4.5}$$

and $\delta_D(u) \wedge \delta_D(w) \geq (\kappa/4)\widehat{a}_T t^{1/\alpha}$.

If $\beta \in [0, 1]$, then by considering the cases $|x - y| \leq 15\widehat{a}_T t^{1/\alpha}$ and $|x - y| > 15\widehat{a}_T t^{1/\alpha}$ separately using Proposition 3.3.4 and (3.4.5) we obtain

$$\begin{aligned} p_D(t/3, u, w) &\geq c_2 \left(t^{-d/\alpha} \wedge t|u - w|^{-d-\alpha} e^{-\gamma_2|u-w|^\beta} \right) \\ &\geq c_3 \left(t^{-d/\alpha} \wedge t|x - y|^{-d-\alpha} e^{-\gamma_2|x-y|^\beta} \right). \end{aligned}$$

If $\beta \in (1, \infty]$ and $|x - y| \leq 4/5$, then (3.4.5) yields $|u - w| \leq |x - y| + 20^{-1} < 1$. Thus, by considering the cases $|x - y| < 15\widehat{a}_T t^{1/\alpha}$ and

$|x-y| > 15\widehat{a}_T t^{1/\alpha}$ separately using Proposition 3.3.4 and (3.4.5), we have $p_D(t/3,u,w) \geq c_4 \left(t^{-d/\alpha} \wedge (t|x-y|^{-d-\alpha})\right)$.

If $\beta \in (1,\infty]$ and $4/5 \leq |x-y|$, then (3.4.5) yields $|u-w| \asymp |x-y|$.

We now consider $p_D(t/3,u,w)$ in each of the remaining cases.

(1) If $\beta \in (1,\infty)$ and $4/5 \leq |x-y| < 2$, then $|u-w| \asymp 1$. Thus, by Proposition 3.3.4, we have $p_D(t/3,u,w) \geq c_5 t$.

(2) If $\beta = \infty$ and $4/5 \leq |x-y| < 2$, then by Propositions 3.3.4 and 3.3.6 we have

$$p_D(t/3,u,w) \geq c_6 \frac{4t}{5T|x-y|} \geq c_6 \left(\frac{4t}{5T|x-y|}\right)^{5|x-y|/4}.$$

(3) If $\beta \in (1,\infty)$ and $2 \leq |x-y|$, then $1 < |u-w|$. Since

$$1 < T|u-w|/t \leq T(|x-y| + 20^{-1})/t \leq (T|x-y|/t)^2, \qquad (3.4.6)$$

and from Proposition 3.3.6 and (3.4.5), we obtain

$$\begin{aligned} p_D(t/3,u,w) &\geq c_7 t \exp\left\{-c_8\left(|u-w|\left(\log\frac{T|u-w|}{t}\right)^{\frac{\beta-1}{\beta}} \wedge |u-w|^\beta\right)\right\} \\ &\geq c_7 t \exp\left\{-c_9\left(|x-y|\left(\log\frac{T|x-y|}{t}\right)^{\frac{\beta-1}{\beta}} \wedge |x-y|^\beta\right)\right\}. \end{aligned}$$

(4) If $\beta = \infty$ and $2 \leq |x-y|$, then $1 < |u-w|$. Similarly, by (3.4.6), Proposition 3.3.6 and (3.4.5), we have

$$\begin{aligned} p_D(t/3,u,w) &\geq c_{11}\left(\frac{t}{T|u-w|}\right)^{c_{10}|u-w|} \geq c_{11}\left(\frac{t}{T|x-y|}\right)^{2c_{12}|x-y|} \\ &\geq c_{11}\left(\frac{4t}{5T|x-y|}\right)^{2c_{12}|x-y|}. \end{aligned}$$

Hence, combining (3.4.4) with the above observations on the lower bound of $p_D(t/3,u,w)$, we have proved Theorem 2.2.1(2) and (3). □

Proof of Theorem 2.2.1(4). Let $D(x)$ and $D(y)$ be connected components containing x and y, respectively. By definition of a $C^{1,\eta}$ open set, the distance between x and y is at least R. Using Lemma 3.4.4, we find that $x_1 \in D(x)$ and $y_1 \in D(y)$. We then define B_{x_1} and B_{y_1} in the same way as when beginning the proof of Theorem 2.2.1(2) and (3) so that (3.4.3) holds and for any $u \in B_{x_1}$ and $w \in B_{y_1}$ we have $3R/4 \leq 3|x-y|/4 \leq |u-w| \leq 5|x-y|/4$. By Proposition 3.3.4, for every $u \in B_{x_1}$ and $w \in B_{y_1}$ we have

$$p_D(t/3, u, w) \geq c_1 \frac{t}{|u-w|^{d+\alpha}} e^{-\gamma_2 |u-w|^\beta} \geq c_2 \frac{t}{|x-y|^{d+\alpha}} e^{-\gamma_2 (5|x-y|/4)^\beta}.$$

Therefore,

$$\begin{aligned} p_D(t,x,y) &\geq \int_{B_{x_1}} \int_{B_{y_1}} p_D(t/3,x,w) p_D(t/3,u,w) p_D(t/3,w,y) dw dv \\ &\geq \int_{B_{x_1}} p_D(t/3,x,u) du \int_{B_{y_1}} p_D(t/3,y,w) dw \cdot \inf_{(u,w)\in B_{x_1}\times B_{y_1}} p_D(t/3,u,w) \\ &\geq c_3 \Psi(t,x) \Psi(t,y) \cdot \frac{t}{|x-y|^{d+\alpha}} e^{-\gamma_2 (5|x-y|/4)^\beta}. \end{aligned}$$

□

3.5 Large time heat kernel estimates

Proof of Theorem 2.2.1(5). Let D be a bounded $C^{1,\eta}$ open set in $\mathbb{R}^d$. The path distance condition is satisfied in bounded $C^{1,\eta}$ open set D with λ_1 depending only on R_0, Λ_0 and $\mathrm{diam}(D)$. Thus by Theorem 2.2.1 (1)–(3), it suffices to prove for $T = 3$.

By the monotonicity of the domain of $p_D(t,x,y)$ and Theorem 2.1.1,

$$\int_{D\times D} p_D(t,x,y)^2 dx dy \leq \int_D p\left(2t,x,x\right) dx \leq c_1 \left(t^{-d/\alpha} \vee 1\right) |D| < \infty,$$

and it implies that the semigroup $\{P_t^D, t > 0\}$ of Y is a Hilbert-Schmidt operatior in $L^2(D, dx)$ and hence compact. So P_t^D has discrete spectrun

$\{e^{-\lambda_k t}, \, k \geq 1\}$ arranged in decreasing order and repeated according to their multiplicity. Let $\{\phi_k, \; k \geq 1\}$ be the corresponding eigenfucnctions with unit L^2- norm, which forms an orthonormal basis for $L^2(D, dx)$. Clearly,

$$\int_D (1 \wedge \delta_D(x))\phi_1(x)dx \leq |D|^{1/2}\|\phi_1\|_{L^2(D)} \leq |D|^{1/2}. \tag{3.5.1}$$

Note that the eigenfunction expansion $p_D(t,x,y) = \sum_{k=1}^{\infty} e^{-\lambda_k t}\phi_k(x)\phi_k(y)$ and $\|f\|_2^2 = \sum_{k=1}^{\infty} e^{-\lambda_k t}\left(\int_D f(x)\phi_k(x)dx\right)^2$. Using these facts, and since λ_k is increasing, we have that

$$\begin{aligned}
&\int_{D\times D} (1 \wedge \delta_D(x))p_D(t,x,y)(1 \wedge \delta_D(y))dxdy \\
=&\sum_{k=1}^{\infty} e^{-\lambda_k t}\left(\int_D (1 \wedge \delta_D(x))\phi_k(x)dx\right)^2 & (3.5.2)\\
\leq& e^{-\lambda_1 t}\int_D (1 \wedge \delta_D(x))^2 dx \leq e^{-\lambda_1 t}|D|. & (3.5.3)
\end{aligned}$$

On the other hand, by Theorem 2.2.1(1) and (3.5.1), for any $x \in D$,

$$\begin{aligned}
&\phi_1(x) = e^{\lambda_1}\int_D p_D(1,x,y)\phi_1(y)dy \\
\leq& \, c_2\, e^{\lambda_1}(1 \wedge \delta_D(x))\int_D (1 \wedge \delta_D(y))\phi_1(y)dy \leq \; c_2\, e^{\lambda_1}|D|^{1/2}(1 \wedge \delta_D(x)). \quad (3.5.4)
\end{aligned}$$

Also, it follows from (3.5.2) and (3.5.4) that for every $t > 0$,

$$\begin{aligned}
&\int_{D\times D} (1 \wedge \delta_D(x))p_D(t,x,y)(1 \wedge \delta_D(y))dxdy \geq e^{-\lambda_1 t}\left(\int_D (1 \wedge \delta_D(x))\phi_1(x)dx\right)^2 \\
&\geq e^{-\lambda_1 t}\left(\int_D (c_2 e^{\lambda_1}|D|^{1/2})^{-1}\phi_1(x)^2 dx\right)^2 = c_2^{-2}|D|^{-1}e^{-\lambda_1(t+2)}. \quad (3.5.5)
\end{aligned}$$

For $t \geq 3$ and $x, y \in D$, we have that

$$p_D(t,x,y) = \int_{D\times D} p_D(1,x,z)p_D(t-2,z,w)p_D(1,z,y)dzdw. \tag{3.5.6}$$

By (3.5.6), Thoeorem 2.2.1(1) and (3.5.3)

$$\begin{aligned}
&p_D(t,x,y)\\
&\le c_3\,(1\wedge\delta_D(x))(1\wedge\delta_D(y))\int_{D\times D}(1\wedge\delta_D(z))p_D(t-2,z,w)(1\wedge\delta_D(w))dzdw\\
&\le c_3|D|e^{-\lambda_1(t-2)}(1\wedge\delta_D(x))(1\wedge\delta_D(y))\le\ c_4\,e^{-\lambda_1 t}(1\wedge\delta_D(x))(1\wedge\delta_D(y)).
\end{aligned}$$

By (3.5.6) and (3.5.5), and since D is connected, applying Thoeorem 2.2.1(2)–(3),

$$\begin{aligned}
&p_D(t,x,y)\\
&\ge c_5\,F^1_{c_6,\gamma_2,1}(1,2\,\mathrm{diam}(D))^2(1\wedge\delta_D(x))(1\wedge\delta_D(y))\\
&\qquad\qquad\cdot\int_{D\times D}(1\wedge\delta_D(z))p_D(t-2,z,w)(1\wedge\delta_D(w))dzdw\\
&\ge c_7F^1_{c_6,\gamma_2,1}(1,2\,\mathrm{diam}(D))^2(1\wedge\delta_D(x))(1\wedge\delta_D(y))|D|^{-1}e^{-\lambda_1 t}\\
&=c_8\,e^{-\lambda_1 t}(1\wedge\delta_D(x))(1\wedge\delta_D(y)).
\end{aligned}$$

The theorem is now proved. □

Chapter 4

Large time heat kernel estimates in exterior $C^{1,\eta}$ open sets

4.1 Upper bound estimates

We first give elementary lemmas which are used several times to estimates the upper and lower bound on $p_D(t,x,y)$ where $t \geq T$ ($t > 0$ when $\beta = 0$, respectively). Recall the function $F^1(t,r)$ defined on $[0,T] \times (0,\infty)$ as

$$F^1_{a,b,T}(t,r) := \begin{cases} t^{-d/\alpha} \wedge tr^{-d-\alpha} e^{-br^\beta} & \text{if } \beta \in [0,1], \\ t^{-d/\alpha} \wedge tr^{-d-\alpha} & \text{if } \beta \in (1,\infty],\ r < 1, \\ t \exp\left(-a\left(r\,\left(\log \frac{Tr}{t}\right)^{\frac{\beta-1}{\beta}} \wedge r^\beta\right)\right) & \text{if } \beta \in (1,\infty),\ r \geq 1, \\ (t/(Tr))^{ar} & \text{if } \beta = \infty,\ r \geq 1, \end{cases}$$

and the function $F^2(t,r)$ defined on $[T,\infty)\times(0,\infty)$ as

$$F^2_{a,T}(t,r) := \begin{cases} t^{-d/\alpha} \wedge t r^{-d-\alpha} & \text{if } \beta = 0, \\ t^{-d/2} \exp\left(-a\left(r^\beta \wedge \frac{r^2}{t}\right)\right) & \text{if } \beta \in (0,1], \\ t^{-d/2} \exp\left(-a\left(r\left(1+\log^+ \frac{Tr}{t}\right)^{(\beta-1)/\beta} \wedge \frac{r^2}{t}\right)\right) & \text{if } \beta \in (1,\infty), \\ t^{-d/2} \exp\left(-a\left(r\left(1+\log^+ \frac{Tr}{t}\right) \wedge \frac{r^2}{t}\right)\right) & \text{if } \beta = \infty, \end{cases}$$

where $\log^+ x = \log x \cdot \mathbf{1}_{\{x\geq 1\}} + 0 \cdot \mathbf{1}_{\{x<1\}}$.

Lemma 4.1.1. *Let $t_0 > 0$ and $a,b,c \geq 1$ be fixed constants. For any $\beta \in (0,\infty]$, suppose that N_1, N_2 be positive constants satisfying $N_2 \geq N_1 \cdot (ab \vee c^{2/\beta})$. Then there exist positive constants $c_i = c_i(t_0), i = 1,2$ such that for every $r > 0$, we have that*

$$(1) \qquad F^1_{b^{-1},c^{-1},t_0}(t_0, N_1^{-1} r) \leq c_1 F^1_{a,c,t_0}(t_0, N_2^{-1} r) \qquad \textit{and}$$
$$(2) \qquad F^1_{a^{-1},c^{-1},t_0}(t_0, N_2 r) \leq c_2 F^1_{b,c,t_0}(t_0, N_1 r).$$

Proof. When $\beta \in (0,1]$, since $N_2 \geq N_1 c^{2/\beta}$, we have (1) and (2).

When $\beta \in (1,\infty]$, since $t_0^{-d/\alpha} \wedge t_0 r^{-d/\alpha} \asymp 1$ for any $r < 1$, we only consider the case $1 \leq N_2^{-1} r (\leq N_1^{-1} r)$ to prove (1) and $1 \leq N_1 r (\leq N_2 r)$ to prove (2). In these cases, since $\log x$ is increasing in x and $N_2 \geq N_1 ab$, we have (1) and (2). □

Lemma 4.1.2. *Let T, a and b be positive constants. (1) If $b \geq 1$, there exists a positive constant $c = c(b)$ such that for every $t \in [T,\infty)$ and $r > 0$, we have that*

$$F^2_{a,T}(t, b^{-1} r) \leq F^2_{ab^{-2},T}(t,r).$$

(2) In addition, for $a,b \geq 1$ and $\beta \in (0,\infty]$, suppose that N be a positive constant satisfying $N \geq (ab)^{1/(\beta\wedge 1)}$. Then for every $t \in [T,\infty)$ and $r > 0$, we have that

$$F^2_{b^{-1},T}(t,r) \leq F^2_{a,T}(t, N^{-1} r).$$

Proof. Since $b \geq 1$, it is easy to prove (1) when $\beta \in [0,1]$. Also, since

$$b\left(1+\log^+\frac{Tb^{-1}r}{t}\right) \geq (1+\log b)\cdot\left(1+\log^+\frac{Tb^{-1}r}{t}\right) \geq \left(1+\log^+\frac{Tr}{t}\right),$$

for any $b \geq 1$, we have (1) when $\beta \in (1,\infty]$.

On the other hand, since $N \geq (ab)^{1/\beta}(\geq 1)$, we have that

$$\begin{aligned} b^{-1}\left(r^\beta \wedge \frac{r^2}{t}\right) &\geq b^{-1}N^\beta\left((N^{-1}r)^\beta \wedge \frac{(N^{-1}r)^2}{t}\right) \\ &\geq a\left((N^{-1}r)^\beta \wedge \frac{(N^{-1}r)^2}{t}\right). \end{aligned} \tag{4.1.1}$$

Also, since $r \to 1+\log^+ r$ is non-decreasing and $N \geq ab\,(\geq 1)$, we have that

$$\begin{aligned} &b^{-1}\left(r\left(1+\log^+\frac{Tr}{t}\right)^{(\beta-1)/\beta} \wedge \frac{r^2}{t}\right) \\ &\geq b^{-1}N\left(N^{-1}r\left(1+\log^+\frac{N^{-1}Tr}{t}\right)^{(\beta-1)/\beta} \wedge \frac{(N^{-1}r)^2}{t}\right) \\ &\geq a\left(N^{-1}r\left(1+\log^+\frac{N^{-1}Tr}{t}\right)^{(\beta-1)/\beta} \wedge \frac{(N^{-1}r)^2}{t}\right). \end{aligned} \tag{4.1.2}$$

Hence, by (4.1.1) for $\beta \in (0,1]$ and by (4.1.2) for $\beta \in (1,\infty]$, we have (2). □

We now prove the upper bound estimates in Theorem 2.2.3(1).

Proof of Theorem 2.2.3(1) When $\beta = 0$, by Theorem 2.2.1(1), we may assume that $t \geq T$. Wthout loss of the generality, we may assume that $T = 3$. By the semigroup property and Theorem 2.2.1(1), we have that for $t-2 \geq 1$ and $x, y \in D$,

$$\begin{aligned} p_D(t,x,y) &= \int_D\int_D p_D(1,x,z)p_D(t-2,z,w)p_D(1,w,y)dzdw \\ &\leq c_1\,(1\wedge\delta_D(x))^{\alpha/2}\,(1\wedge\delta_D(y))^{\alpha/2}\,f_1(t,x,y). \end{aligned} \tag{4.1.3}$$

Here $f_1(t,x,y)$ is defined by

$$f_1(t,x,y) = \int_{\mathbb{R}^d\times\mathbb{R}^d} F^1_{C_2^{-1},\gamma^{-1},1}(1,|x-z|/6)\, p(t-2,z,w) \\ F^1_{C_2^{-1},\gamma^{-1},1}(1,|y-w|/6)dzdw \qquad (4.1.4)$$

where $C_2 := C_1 \vee \gamma_1^{-1}$ and $\gamma := \gamma_1^{-1} \vee \gamma_2$ for the constancts C_1 in Theorem 2.1.1 and γ_1, γ_2 in (2.1.1).

Let $A_1 := \max\{C_1^{2/(\beta\wedge 1)}, 6\gamma^{2/\beta}, 6C_1C_2\}$ ($A_1 = 6$ when $\beta = 0$, respectively). Then by Theorem 2.1.1(2), there exists constants $c_i = c_i(\beta) > 0$, $i = 2,3$ such that

$$\begin{aligned} p(t-2,z,w) &\le c_2\, F^2_{C_1,1}(t-2,|z-w|) \\ &\le c_2\, F^2_{C_1,1}(t-2,A_1^{-1}|z-w|) \le c_3\, p(t-2,A_1^{-1}z,A_1^{-1}w). \end{aligned}$$

For the second inequaltiy, when $\beta \in (0,\infty]$, we use (2) in Lemma 4.1.2 with $N = A_1$, $a = b = C_1$ and the fact $A_1 \ge C_1^{2/(\beta\wedge 1)}$. When $\beta = 0$, the second inequality holds since $A_1 \ge 1$.

Also, by Theorem 2.1.1(1), there exist constants $c_i = c_i(\beta) > 0, i = 4,5$ such that

$$\begin{aligned} F^1_{C_2^{-1},\gamma^{-1},1}(1,|x-z|/6) &\le c_4\, F^1_{C_1,\gamma,1}(1,A_1^{-1}|x-z|) \le c_5\, p(1,A_1^{-1}x,A_1^{-1}z), \\ F^1_{C_2^{-1},\gamma^{-1},1}(1,|y-w|/6) &\le c_4\, F^1_{C_1,\gamma,1}(1,A_1^{-1}|y-w|) \le c_5\, p(1,A_1^{-1}y,A_1^{-1}w). \end{aligned}$$

For the first inequalties above, when $\beta \in (0,\infty]$, we use (1) in Lemma 4.1.1 along with $a = C_1$, $b = C_2$, $c = \gamma$, $N_1 = 6$ and $N_2 = A_1$ and the fact $A_1 \ge 6(C_1C_2 \vee \gamma^{2/\beta})$. When $\beta = 0$, the first inequalities hold since $A_1 = 6$.

Applying the above observations to (4.1.4) and by the change of variable $\hat{z} = A_1^{-1}z$, $\hat{w} = A_1^{-1}w$, the semigroup property and Theorem 2.1.1(2), we conclude that

$$\begin{aligned} f_1(t,x,y) &\le c_6 \int_{\mathbb{R}^d\times\mathbb{R}^d} p(1, A_1^{-1}x, \hat{z})\, p(t-2, \hat{z}, \hat{w}) p(1, A_1^{-1}y, \hat{w}) d\hat{z} d\hat{w} \\ &= c_6\, p(t, A_1^{-1}x, A_1^{-1}y) \le c_7 F^2_{C_1^{-1},T}(t, A_1^{-1}|x-y|) \\ &\le c_8 F^2_{C_1^{-1}A_1^{-2},T}(t, |x-y|). \end{aligned} \tag{4.1.5}$$

We have applied (1) in Lemma 4.1.2 with $a = C_1^{-1}$ and $b = A_1$ for the last inequality. Applying (4.1.5) to (4.1.3), we have proved the upper bound estimates in Theorem 2.2.3(1).

4.2 Interior lower bound estimates

The goal of this section is to the establish interior lower bound estimate on the heat kernel $p_{\overline{B}^c_R}(t,x,y)$ for $t \ge T$ ($t > 0$ when $\beta = 0$, respectively) where $B_R = B(x_0, R)$ for some $R > 0$ and $x_0 \in \mathbb{R}^d$. We will combine ideas from [10] and section 3.3.

First, we introduce a Lemma which will be used in the proof of Lemma 4.2.2 and Proposition 4.2.3. Define $\varphi(r) := r^2 \cdot \mathbf{1}_{\{\beta\in(0,\infty]\}} + r^\alpha \cdot \mathbf{1}_{\{\beta=0\}}$, then the inverse function $\varphi^{-1}(t) = t^{1/2} \cdot \mathbf{1}_{\{\beta\in(0,\infty]\}} + t^{1/\alpha} \cdot \mathbf{1}_{\{\beta=0\}}$.

Lemma 4.2.1. *For any $a > 0$ and $\beta \in [0,\infty]$, there exists a positive constant $c = c(a,\beta,T)$ ($c = c(a)$ when $\beta = 0$, respectively) such that for all $t \in [T,\infty)$ ($t > 0$ when $\beta = 0$, respectively), we have*

$$\inf_{y\in\mathbb{R}^d} \mathbb{P}_y\left(\tau_{B(y,a\varphi^{-1}(t))} > t\right) \ge c.$$

Proof. When $\beta = 0$, using [16, Theorem 4.12 and Proposition 4.9], the proof is almost identical to that of [9, Lemma 3.1]. When $\beta \in (0,\infty]$, using [6, Theorem 4.8], the proof is the same as that of [10, Lemma 3.2]. So we omit the proof detail. □

Lemma 4.2.2. *Let D be an arbitrary open set. Suppose that $a > 0$ and $\beta \in [0,\infty]$. Then there exists a positive constant $c = c(a,\beta,T)$ ($c = c(a)$*

when $\beta = 0$, respectively) such that for all $t \in [T, \infty)$ ($t > 0$ when $\beta = 0$, respectively) and $x, y \in D$ with $\delta_D(x) \wedge \delta_D(y) \geq a\varphi^{-1}(t)$ and $|x - y| \geq 2^{-1}a\varphi^{-1}(t)$, we have

$$\mathbb{P}_x\left(Y_t^D \in B\big(y,\, 2^{-1}a\varphi^{-1}(t)\big)\right) \geq c\,t \cdot \varphi^{-d}(t) j(|x-y|).$$

Proof. Using Lemma 4.2.1, the strong Markov property and Lévy system (2.1.7), the proof of the lemma is similar to that of Proposition 4.2.5. So we omit the proof detail. □

For the remainder of this section, we assume that D is a domain with the following property which is introduced in section 3.3: there exist $\lambda_1 \in [1, \infty)$ and $\lambda_2 \in (0, 1]$ such that for every $r \leq 1$ and x, y in the same component of D with $\delta_D(x) \wedge \delta_D(y) \geq r$, there exists in D a length parameterized rectifiable curve l connecting x to y with the length $|l|$ of l is less than or equal to $\lambda_1|x - y|$ and $\delta_D(l(u)) \geq \lambda_2 r$ for $u \in (0, |l|]$. Clearly, such a property holds for all $C^{1,\eta}$ domains with compact complements, and domains above graphs of $C^{1,\eta}$ functions.

The following Propositions are motivated by [10].

Proposition 4.2.3. *For any $a > 0$ and $\beta \in [0, \infty]$, there exists a positive constant $c = c(a, \beta, T, \lambda_1, \lambda_2)$ ($c = c(a, \lambda_1, \lambda_2)$ when $\beta = 0$, respectively) such that for all $t \in [T, \infty)$ ($t > 0$ when $\beta = 0$, respectively) and $x, y \in D$ with $\delta_D(x) \wedge \delta_D(y) \geq a\varphi^{-1}(t) \geq 2|x - y|$, we have $p_D(t, x, y) \geq c/\varphi^{-d}(t)$.*

Proof. By the same proof as that of [10, Proposition 3.4], we deduce the proposition using the parabolic Harnack inequality(see [16, Theorem 4.12] for $\beta = 0$ and [6, Theorem 4.11] for $\beta \in (0, \infty]$) and Lemma 4.2.2. □

Proposition 4.2.4. *For any $a > 0$ and $\beta \in [0, \infty]$, there exists a positive constant $c = c(a, \beta, T, \lambda_1, \lambda_2)$ ($c = c(a, \lambda_1, \lambda_2)$ when $\beta = 0$, respectively) such that for all $t \in [T, \infty)$ ($t > 0$ when $\beta = 0$, respectively) and $x, y \in D$ with $\delta_D(x) \wedge \delta_D(y) \geq a\varphi^{-1}(t)$ and $|x - y| \geq 2^{-1}a\varphi^{-1}(t)$, we have $p_D(t, x, y) \geq c\,t j(|x - y|)$.*

Proof. By the same proof as that of [10, Proposition 3.5], we deduce the proposition using the semigroup property, Lemma 4.2.2 and Proposition 4.2.3. □

Also, since the proof of the following proposition is almost identical to that of [10, Proposition 3.6] using Proposition 4.2.3, we skip the proof.

Proposition 4.2.5. *Let $\beta \in (1, \infty]$ and a and C^* be positive constants. Then there exist positive constants $c_i = c_i(a, \beta, C^*, \lambda_1, \lambda_2)$, $i = 1, 2$ such that for every $t \in (0, \infty)$ and $x, y \in D$ with $\delta_D(x) \wedge \delta_D(y) \geq a\sqrt{t}$, we have*

$$p_D(t, x, y) \;\geq\; c_1\, t^{-d/2} \exp\left(-c_2 \frac{|x-y|^2}{t}\right) \;\text{ when }\; C^*|x-y| \leq t \leq |x-y|^2.$$

Now, we estimates the interior lower bound for $p_D(t, x, y)$ where $\beta \in (1, \infty]$ and $T \leq t \leq C^*T|x-y|$ for any positive constant $C^* < 1$. The following Proposition 4.2.6 and Proposition 4.2.7 are counterparts of Propition 3.3.5 and Proposition 3.3.6, respectively (See, also [6, Theorem 5.5]) and [4, Theorem 3.6], respectively).

Proposition 4.2.6. *Let $a > 0$, $\beta \in (1, \infty)$ and $C^* \in (0, 1)$. Then there exist positive constants $c_i = c_i(a, \beta, T, C^*, \lambda_1, \lambda_2)$, $i = 1, 2$ such that for every $t \in [T, \infty)$ and $x, y \in D$ with $\delta_D(x) \wedge \delta_D(y) \geq a\sqrt{t}$, we have*

$$p_D(t, x, y) \;\geq\; c_1 \exp\left(-c_2 |x-y| \left(1 + \log \frac{T|x-y|}{t}\right)^{(\beta-1)/\beta}\right)$$

when $C^*T|x-y| \geq t$.

Proof. We let $r := |x-y|$ and fix $C^* \in (0, 1)$. Note that $r \geq (C^*)^{-1}t/T > t/T \geq 1$ and $r\exp(-r^\beta) \leq \exp(-1)(< 1)$ for $\beta > 1$. So we only consider the case $Tr\exp(-r^\beta) < t\ (\leq C^*Tr)$ which is equivalent to $r\left(\log(Tr/t)\right)^{-1/\beta} > 1$. Let $k \geq 2$ be a positive integer such that

$$r\left(\log \frac{Tr}{t}\right)^{-1/\beta} \leq k < r\left(\log \frac{Tr}{t}\right)^{-1/\beta} + 1 < 2r\left(\log \frac{Tr}{t}\right)^{-1/\beta}. \quad (4.2.1)$$

Then we have that

$$\frac{t}{k} \le \frac{t}{r}\left(\log \frac{Tr}{t}\right)^{1/\beta} \le T \cdot \sup_{s \ge (C^*)^{-1}} s^{-1}(\log s)^{1/\beta} =: t_0 < \infty \tag{4.2.2}$$

By our assumption on D, there is a length parameterized curve $l \subset D$ connecting x and y such that the total length $|l|$ of l is less than or equal to $\lambda_1 r$ and $\delta_D(l(u)) \ge \lambda_2 a\sqrt{t}$ for every $u \in [0,|l|]$. We define $r_t := (2^{-1}\lambda_2 a\sqrt{t}) \wedge ((6\lambda_1)^{-1}(\log(Tr/t))^{1/\beta})$. Then by (4.2.1) and the assumption $\log((C^*)^{-1}) < \log(Tr/t)$, we have that

$$\begin{aligned} 0 < r_0 &:= \left(\frac{\lambda_2 a\sqrt{T}}{2}\right) \wedge \left(\frac{(\log((C^*)^{-1}))^{1/\beta}}{6\lambda_1}\right) \\ &\le r_t \le \frac{1}{6\lambda_1}\left(\log \frac{Tr}{t}\right)^{1/\beta} < \frac{r}{3\lambda_1 k}. \end{aligned} \tag{4.2.3}$$

Define $x_i := l(i|l|/k)$ and $B_i := B(x_i, r_t)$ for $i = 0,1,2,\dots,k$ then $\delta_D(x_i) \ge \lambda_2 a\sqrt{t} > r_t$ and $B_i \subset D$. For every $y_i \in B_i$, we have that $\delta_D(y_i) \ge 2^{-1}\lambda_2 a\sqrt{t} > 2^{-1}\lambda_2 a\sqrt{t/k}$ and

$$|y_i - y_{i+1}| \le |x_i - x_{i+1}| + 2r_t \le \left(\lambda_1 + \frac{2}{3\lambda_1}\right)\frac{r}{k}. \tag{4.2.4}$$

Thus by Proposition 4.2.3 and 4.2.4 along with the definition of j, (4.2.1), (4.2.2) and (4.2.4), there exist constants $c_i > 0, i = 1,\dots,5$ such that

$$\begin{aligned} p_D(t/k, y_i, y_{i+1}) &\ge c_1\left(\left(\frac{t}{k}\right)^{-d/2} \wedge \frac{t}{k}\cdot j(|y_i - y_{i+1}|)\right) \\ &\ge c_2\left(1 \wedge \left(\frac{t}{k}\frac{e^{-c_3(r/k)^\beta}}{(r/k)^{d+\alpha}}\right)\right) \ge c_4\frac{t}{Tr}\left(\frac{k}{r}\right)^{d+\alpha-1} e^{-c_3(r/k)^\beta} \\ &\ge c_4\frac{t}{Tr}\left(\log\frac{Tr}{t}\right)^{-\frac{d+\alpha-1}{\beta}}\left(\frac{t}{Tr}\right)^{c_3} \ge c_4\left(\frac{t}{Tr}\right)^{c_5}. \end{aligned} \tag{4.2.5}$$

Therefore, by the semigroup property, (4.2.3) and (4.2.5), we conclude

$$
\begin{aligned}
p_D(t,x,y) &\geq \int_{B_1}\cdots\int_{B_{k-1}} p_D(t/k,x,y_1)\cdots p_D(t/k,y_{k-1},y)dy_1\cdots dy_{k-1} \\
&\geq \left(c_4\left(\frac{t}{Tr}\right)^{c_5}\right)^k \Pi_{i=1}^{k-1}|B_i| \;\geq\; \left(\frac{c_6 t}{Tr}\right)^{c_5 k} \geq c_7\exp\left(-c_5 k\left(\log\frac{Tr}{c_8 t}\right)\right) \\
&\geq c_7\exp\left(-c_9 r\left(\log\frac{Tr}{t}\right)^{1-1/\beta}\right) \geq c_7\exp\left(-c_9 r\left(1+\log\frac{Tr}{t}\right)^{1-1/\beta}\right).
\end{aligned}
$$

□

Proposition 4.2.7. *Let $a>0$, $\beta=\infty$ and $C^*\in(1/2,1)$. Then there exist positive constants $c_i=c_i(a,T,C^*,\lambda_1,\lambda_2)$, $i=1,2$ such that for every $t\in[T,\infty)$ and $x,y\in D$ with $\delta_D(x)\wedge\delta_D(y)\geq a\sqrt{t}$, we have*

$$
p_D(t,x,y) \;\geq\; c_1\exp\left(-c_2|x-y|\left(1+\log\frac{T|x-y|}{t}\right)\right),
$$

when $C^*T|x-y|\geq t$.

Proof. Let $r:=|x-y|$ and fix $C^*\in(1/2,1)$. Since $T\leq t\leq C^*Tr$, we note that $1\leq C^*r$. By our assumption on D, there is a length parameterized curve $l\subset D$ connecting x and y such that the total length $|l|$ of l is less than or equal to $\lambda_1 r$ and $\delta_D(l(u))\geq\lambda_2 a\sqrt{t}$ for every $u\in[0,|l|]$. Let $k\geq 2$ be a positive integer satisfying

$$
1<8\lambda_1C^*r\leq k<8\lambda_1C^*r+1\leq(8\lambda_1+1)C^*r. \tag{4.2.6}
$$

Define $r_t:=(\lambda_2 a\sqrt{t}/2)\wedge 8^{-1}$, $x_i:=l(i|l|/k)$ and $B_i:=B(x_i,r_t)$ for $i=0,1,\dots,k$. Then $\delta_D(x_i)>2r_t$ and $B_i\subset B(x_i,2r_t)\subset D$. For every $y_i\in B_i$, since $t/k<t/(8\lambda_1C^*r)\leq T/(8\lambda_1)$, we have $\delta_D(y_i)>r_t>c_1\sqrt{t/k}$ for some constant $c_1=c_1(a,T,\lambda_1,\lambda_2)>0$. Also, for each $y_i\in B_i$,

$$
\begin{aligned}
|y_i-y_{i+1}| &\leq |y_i-x_i|+|x_i-x_{i+1}|+|x_{i+1}-y_{i+1}| \\
&\leq 8^{-1}+|l|/k+8^{-1}<\lambda_1 r/(8\lambda_1C^*r)+4^{-1}\leq 2^{-1}.
\end{aligned} \tag{4.2.7}
$$

By Proposition 4.2.3 and 4.2.4 along with the definition of j, (4.2.7) and the fact that $t/k < T/(8\lambda_1)$, there are constants $c_i = c_i(a, T, \lambda_1) > 0$, $i = 2, \dots, 4$, such that for $(y_i, y_{i+1}) \in B_i \times B_{i+1}$,

$$\begin{aligned} p_D(t/k, y_i, y_{i+1}) &\geq c_2 \left((t/k)^{-d/\alpha} \wedge \frac{t/k}{|y_i - y_{i+1}|^{d+\alpha}} \right) \\ &\geq c_3 \, (1 \wedge t/k) \geq c_4 \, t/(Tk). \end{aligned} \tag{4.2.8}$$

Thus, by the semigroup property combining the fact $r_t \geq r_T \wedge 8^{-1}$, (4.2.6) and (4.2.8), we obtain that

$$\begin{aligned} p_D(t, x, y) &\geq \int_{B_1} \cdots \int_{B_{k-1}} p_D(t/k, x, y_1) \dots p_D(t/k, y_{k-1}, y) dy_{k-1} \dots dy_1 \\ &\geq \left(\frac{c_4 \, t}{Tk} \right)^k \Pi_{i=1}^{k-1} |B_i| \geq \left(\frac{c_5 \, t}{Tk} \right)^k \geq c_6 \left(\frac{c_7 \, t}{Tr} \right)^k \\ &\geq c_6 \exp \left(-c_8 r \log \frac{Tr}{c_7 t} \right) \geq \exp \left(-c_9 r \left(1 + \log \frac{Tr}{t} \right) \right). \end{aligned}$$

□

Recall that $B_R = B(x_0, R)$. Note that a exterior ball $\overline{B}_R^c$ is a domain in which the path distance is comparable to the Euclidean distance with characteristics (λ_1, λ_2) independent of x_0 and R. Hence, the previous propositions yield the following Theorem.

Theorem 4.2.8. *Let a and T be positive constants. Then for any $\beta \in [0, \infty]$, there exists positive constants $c_i = c_i(a, \beta, T)$ ($c = c(a)$ when $\beta = 0$, respectively) , $i = 1, 2$, such that for every $R > 0$, $t \in [T, \infty)$ ($t > 0$ when $\beta = 0$, respectively) and $x, y \in \overline{B}_R^c$ with $\delta_{\overline{B}_R^c}(x) \wedge \delta_{\overline{B}_R^c}(y) \geq a\varphi^{-1}(t)$, we have*

$$p_{\overline{B}_R^c}(t, x, y) \;\geq\; c_1 F^2_{c_2, T}(t, |x - y|)$$

where $F^2_{c_2,T}(t, r)$ is defined in (2.1.8).

Proof. Let $r := |x - y|$. For any $\beta \geq 0$, if $\varphi(r) < t$, by Proposition 4.2.3, we have the conclusion.

Suppose $t \le \varphi(r)$. When $\beta \in [0,1]$, we have the conclusion by Proposition 4.2.4 and Proposition 4.2.5. When $\beta \in (1,\infty)$, using Proposition 4.2.5 and Proposition 4.2.6 , and when $\beta = \infty$, using Proposition 4.2.5 and Proposition 4.2.7, we have the conclusion. □

Proof of the lower bound in Theorem 2.1.1(2). For $D = \mathbb{R}^d$, using the same proof of Theorem 4.2.8, we obtain the conclusion. □

4.3 Lower bound estimates

In this section, we assume that the dimension $d > 2 \cdot \mathbf{1}_{\{\beta\in(0,\infty]\}} + \alpha \cdot \mathbf{1}_{\{\beta=0\}}$. To establish the lower bound estimates in Theorem 2.2.3(2)–(4), we first consider the lower bound estimates on $p_{\overline{B}_R^c}(t,x,y)$ for $t \ge T$ ($t > 0$ when $\beta = 0$, respectively) where B_R is a ball of radius $R > 0$ centered at x_0. Since all following estimates are independent of x_0, we may assume that $x_0 = 0$.

We define the Green function $G(x,y)$ of Y in $\mathbb{R}^d$ as $G(x,y) := \int_0^\infty p(t,x,y)dt$ for every $x, y \in \mathbb{R}^d$. Then by the fact that $\int_0^\infty (t^{-d/\alpha} \wedge tr^{-d-\alpha})dt \asymp r^{\alpha-d}$ for $d > \alpha$ when $\beta = 0$ and by [6, Theorem 6.1] when $\beta \in (0,\infty]$, we have that

$$G(x,y) \asymp \left(|x-y|^{\alpha-d} + |x-y|^{2-d} \cdot \mathbf{1}_{\{\beta\in(0,\infty]\}}\right). \tag{4.3.1}$$

For any Borel set $A \subset \mathbb{R}^d$, recall the first exit time of A as $\tau_A = \inf\{t > 0 : Y_t \notin A\}$ and define the first hitting time of A as $T_A := \inf\{t > 0 : Y_t \in A\}$. The next lemma provide us the beginning point for the lower bound estimates which proof is almost identical to that of [11, Lemma 4.1] using (4.3.1), so we omit the proof.

Lemma 4.3.1. *There is a constant $C_3 > 1$ such that for any $R > 0$ and $|x| \ge 2R$,*

$$\begin{aligned} &C_3^{-1} \frac{R^d}{R^\alpha + R^2} \left(|x|^{\alpha-d} + |x|^{2-d} \cdot \mathbf{1}_{\{\beta\in(0,\infty]\}}\right) \\ &\quad \le \mathbb{P}_x(T_{\overline{B}_R} < \infty) \le C_3 \frac{R^d}{R^\alpha + R^2} \left(|x|^{\alpha-d} + |x|^{2-d} \cdot \mathbf{1}_{\{\beta\in(0,\infty]\}}\right). \end{aligned}$$

The following ideas of obtaining the lower bound estimates on $p_{\overline{B}_R^c}(t,x,y)$ are motivated by that of Section 5 in [11] and for the sake of completeness, we give proofs detail. For the simplicity of the notation, hereafter for any $y \in \mathbb{R}^d\backslash\{0\}$ and $r>0$, we define $H(y,r) := \{z \in B(y,r) : z\cdot y \geq 0\}$. Recall that $\varphi(r) = r^2\cdot\mathbf{1}_{\{\beta\in(0,\infty]\}}+r^\alpha\cdot\mathbf{1}_{\{\beta=0\}}$ and $\varphi^{-1}(t) = t^{1/2}\cdot\mathbf{1}_{\{\beta\in(0,\infty]\}}+t^{1/\alpha}\cdot\mathbf{1}_{\{\beta=0\}}$.

Lemma 4.3.2. *For any $T>0$ and $\beta\in[0,\infty]$, there exists constants $\varepsilon = \varepsilon(\beta,T)>0$ and $M_1 = M_1(\beta,T)\geq 3$ ($\varepsilon>0$ and $M_1\geq 3$ when $\beta=0$, respectively) such that the following holds: for any $R>0$, $t\in[T,\infty)$ ($t>0$ when $\beta=0$, respectively) and x,y satisfying $|x|>M_1R$, $|y|>R$ and $y\in B(x,9\varphi^{-1}(t))$, we have*

$$\mathbb{P}_x\left(Y_t^{\overline{B}_R^c}\in H(y,\varphi^{-1}(t)/2)\right)\geq\varepsilon.$$

Proof. Applying Theorem 2.1.1(2) (Applying Theorem 2.1.1(1) and (2) when $\beta=0$, respectively) and by the change of variable with $v=z/\varphi^{-1}(t)$, for any $t\geq T$ ($t>0$ when $\beta=0$, respectively), there are constants $c_i = c_i(\beta,T)>0$ ($c_i>0$ when $\beta=0$, respectively), $i=1,\cdots,3$ such that

$$\begin{aligned}
&\mathbb{P}_x\left(Y_t\in H(y,\varphi^{-1}(t)/2)\right)\geq\inf_{w\in B(y,9\varphi^{-1}(t))}\mathbb{P}_w\left(Y_t\in H(y,\varphi^{-1}(t)/2)\right)\\
&\quad\geq\ c_1\inf_{w\in B(y,9\varphi^{-1}(t))}\int_{H(y,\varphi^{-1}(t)/2)}F^2_{C_1,T}(t,|w-z|)\cdot\mathbf{1}_{\{\beta\in(0,\infty]\}}\,dz\\
&\qquad\quad+c_1\inf_{w\in B(y,9\varphi^{-1}(t))}\int_{H(y,\varphi^{-1}(t)/2)}\left(t^{-d/\alpha}\wedge t|w-z|^{-d-\alpha}\right)\cdot\mathbf{1}_{\{\beta=0\}}\,dz\\
&\quad\geq\ c_2\inf_{w\in B(y,9\varphi^{-1}(t))}\int_{H(y,\varphi^{-1}(t)/2)}\frac{1}{\varphi^{-d}(t)}\exp\left(-C_1\frac{|w-z|^2}{t}\right)\cdot\mathbf{1}_{\{\beta\in(0,\infty]\}}\,dz\\
&\qquad\quad+c_2\inf_{w\in B(y,9\varphi^{-1}(t))}\int_{H(y,\varphi^{-1}(t)/2)}\frac{1}{\varphi^{-d}(t)}\mathbf{1}_{\{\beta=0\}}\,dz\\
&\quad=\ c_3\inf_{w_0\in B(y_0,9)}\int_{H(y_0,1/2)}\exp\left(-C_1|w_0-v|^2\right)\cdot\mathbf{1}_{\{\beta\in(0,\infty]\}}+\mathbf{1}_{\{\beta=0\}}\,dv\\
&\quad\geq\ 2^{-1}c_3|B(0,1/2)|\left(e^{-C_1 10^2}\cdot\mathbf{1}_{\{\beta\in(0,\infty]\}}+\mathbf{1}_{\{\beta=0\}}\right)
\end{aligned}$$

where $y_0 := y/\varphi^{-1}(t)$ and $w_0 := w/\varphi^{-1}(t)$. When $\beta=0$, since $|w-z|\leq 10t^{1/\alpha}$, the third inequality holds. Hence, there is $\varepsilon\in(0,1/4)$ so that for any

$t \geq T$ ($t > 0$ when $\beta = 0$, respectively), $x \in \mathbb{R}^d$ and $y \in B(x, 9\varphi^{-1}(t))$, we have

$$\varepsilon < \frac{1}{2}\mathbb{P}_x\left(Y_t \in H(y, \varphi^{-1}(t)/2)\right). \tag{4.3.2}$$

For $d > 2 \cdot \mathbf{1}_{\{\beta\in(0,\infty]\}} + \alpha \cdot \mathbf{1}_{\{\beta=0\}}$ and the constant $C_3 > 1$ in Lemma 4.3.1, we may choose $M_1 \geq 3$ so that $C_3(M_1^{2-d} + M_1^{\alpha-d} \cdot \mathbf{1}_{\{\beta\in(0,\infty]\}}) \leq \varepsilon$. For any x with $|x| > M_1 R$, by Lemma 4.3.1, we have that

$$\begin{aligned}
\mathbb{P}_x\left(\tau_{\overline{B}_R^c} \leq t\right) = \mathbb{P}_x\left(T_{\overline{B}_R} < \infty\right) &\leq C_3 \frac{R^d}{R^\alpha + R^2}(|x|^{2-d} + |x|^{\alpha-d} \cdot \mathbf{1}_{\{\beta\in(0,\infty]\}}) \\
&\leq C_3\left(\frac{R^2}{R^\alpha + R^2}M_1^{2-d} + \frac{R^\alpha}{R^\alpha + R^2}M_1^{\alpha-d} \cdot \mathbf{1}_{\{\beta\in(0,\infty]\}}\right) \\
&\leq C_3(M_1^{2-d} + M_1^{\alpha-d} \cdot \mathbf{1}_{\{\beta\in(0,\infty]\}}) \leq \varepsilon. \qquad (4.3.3)
\end{aligned}$$

Hence, combining (4.3.2) and (4.3.3), we obtain that

$$\begin{aligned}
&\mathbb{P}_x\left(Y_t^{\overline{B}_R^c} \in H(y, \varphi^{-1}(t)/2)\right) \\
=&\mathbb{P}_x\left(\tau_{\overline{B}_R^c} > t\right) - \mathbb{P}_x\left(Y_t^{\overline{B}_R^c} \notin H(y, \varphi^{-1}(t)/2); \tau_{\overline{B}_R^c} > t\right) \\
\geq&\mathbb{P}_x\left(\tau_{\overline{B}_R^c} > t\right) - \mathbb{P}_x\left(Y_t \notin H(y, \varphi^{-1}(t)/2)\right) \geq (1-\varepsilon) - (1-2\varepsilon) = \varepsilon.
\end{aligned}$$

□

Lemma 4.3.3. *Let $T > 0$, $\beta \in [0, \infty]$, and $M_1 = M_1(\beta, T/8) \geq 3$ ($M_1 \geq 3$ when $\beta = 0$, respectively) be the constant in Lemma 4.3.2. Then there exists a positive constant $c = c(\beta, T) > 0$ ($c > 0$ when $\beta = 0$, respectively) such that for any $R > 0$, $t \in [T, \infty)$ ($t > 0$ when $\beta = 0$, respectively) and x, y satisfying $|x| > M_1 R$, $|y| > M_1 R$ and $|x - y| \leq \varphi^{-1}(t)/6$, we have that $p_{\overline{B}_R^c}(t, x, y) \geq c/\varphi^{-d}(t)$.*

Proof. Without loss of generality we may assume that $|y| \geq |x|$. If $\delta_{\overline{B}_R^c}(y) > \varphi^{-1}(t)/2$, then $\delta_{\overline{B}_R^c}(x) \geq \delta_{\overline{B}_R^c}(y) - |x - y| \geq \varphi^{-1}(t)/3$, and hence the lemma follows immediately from Proposition 4.2.3.

Now we assume that $\delta_{\overline{B}_R^c}(y) \leq \varphi^{-1}(t)/2$. By the semigroup property and the parabolic Harnack inequality(see [6, Theorem 4.11]) , we have

$$\begin{aligned} p_{\overline{B}_R^c}(t,x,y) &\geq \int_{H(y,\varphi^{-1}(t/2))} p_{\overline{B}_R^c}(t/2,x,z)p_{\overline{B}_R^c}(t/2,z,y)dz \\ &\geq c_1\,\mathbb{P}_x\left(Y_{t/2}^{\overline{B}_R^c}\in H(y,\varphi^{-1}(t/2))\right)p_{\overline{B}_R^c}\left(t/2-\varphi(2\delta_{\overline{B}_R^c}(y))/4,y,y\right). \end{aligned} \tag{4.3.4}$$

Note that $t\geq s:=t/2-\varphi(2\delta_{\overline{B}_R^c}(y))/4\geq t/4\geq T/4$ ($s\geq t/4>0$ when $\beta=0$, respectively). So by the semigroup property, the Cauchy-Schwarz inequality and Lemma 4.3.2, we obtain that

$$\begin{aligned} p_{\overline{B}_R^c}(s,y,y)\geq&\int_{H(y,\varphi^{-1}(s)/2)}\left(p_{\overline{B}_R^c}(s/2,y,z)\right)^2dz \\ \geq&\frac{2}{|B(y,\varphi^{-1}(s)/2)|}\mathbb{P}_y\left(Y_{s/2}^{\overline{B}_R^c}\in H(y,\varphi^{-1}(s)/2)\right)^2 \\ \geq&c_2/\varphi^{-d}(s)\geq c_2/\varphi^{-d}(t). \end{aligned} \tag{4.3.5}$$

Applying Lemma 4.3.2 again and (4.3.5) to (4.3.4), we have that $p_{\overline{B}_R^c}(t,x,y)\geq c_3/\varphi^{-d}(t)$. □

Proposition 4.3.4. *Let $T>0$, $\beta\in[0,\infty]$, and $M_1=M_1(\beta,T/16)\geq 3$ ($M_1\geq 3$ when $\beta=0$, respectively) be the constant in Lemma 4.3.2. Then there exist positive constants $c=c(\beta,T)$ and $C_4=C_4(\beta,T)$ (c, $C_4>0$ when $\beta=0$, respectively) such that for any $R>0$, $t\in[T,\infty)$ ($t>0$ when $\beta=0$, respectively) and x,y satisfying $|x|>M_1R$, $|y|>M_1R$, we have that $p_{\overline{B}_R^c}(t,x,y)\geq cF^2_{C_4,T}(t,|x-y|)$, where $F^2_{a,T}(t,r)$ is defined in* (2.1.8).

Proof. By Lemma 4.3.3, we only need to prove the proposition for $|x-y|>\varphi^{-1}(t)/6$.

If $t\leq 2\varphi(60R)$, then $\delta_{\overline{B}_R^c}(x)\wedge\delta_{\overline{B}_R^c}(y)\geq(M_1-1)R\geq 2R\geq(30)^{-1}\varphi^{-1}(t/2)$. In this case the Proposition holds by Theorem 4.2.8. So we only consider the following case: $t\geq T\wedge 2\varphi(60R)$ ($t\geq 2\varphi(60R)$ when $\beta=0$, respectively) and $|x-y|>\varphi^{-1}(t)/6$. Without loss of generality, we may assume that $|y|\geq|x-y|/2$. Let $x_1:=x+20^{-1}\varphi^{-1}(t/2)x/|x|$ then we have $B(x_1,20^{-1}\varphi^{-1}(t/2))\subset\overline{B}^c_{|x|}\subset\overline{B}^c_R$.

For every $z \in B(x_1, 20^{-1}\varphi^{-1}(t/2))$, we obtain

$$|x - z| \leq \frac{1}{20}\varphi^{-1}(t/2) + |x_1 - z| \leq \frac{1}{10}\varphi^{-1}(t/2) \leq \frac{1}{6}\varphi^{-1}(t/2). \qquad (4.3.6)$$

Since $|y| \geq |x-y|/2$ and $R \leq 60^{-1}\varphi^{-1}(t/2)$, we have

$$\begin{aligned} \delta_{\overline{B}_R^c}(y) = |y| - R &\geq \frac{1}{2}|x-y| - \frac{1}{60}\varphi^{-1}(t/2) \\ &> \frac{1}{12}\varphi^{-1}(t) - \frac{1}{60}\varphi^{-1}(t/2) \geq \frac{1}{15}\varphi^{-1}(t/2). \end{aligned} \qquad (4.3.7)$$

For $z \in B(x_1, 60^{-1}\varphi^{-1}(t/2))$, we have

$$\begin{aligned} &\delta_{\overline{B}_R^c}(z) = |z| - R \geq |x_1| - |x_1 - z| - \frac{1}{60}\varphi^{-1}(t/2) \\ &\geq |x| + \frac{1}{20}\varphi^{-1}(t/2) - \frac{1}{60}\varphi^{-1}(t/2) - \frac{1}{60}\varphi^{-1}(t/2) \geq \frac{1}{60}\varphi^{-1}(t/2) \end{aligned} \qquad (4.3.8)$$

and

$$|z - y| \leq |z - x| + |x - y| \leq \frac{1}{15}\varphi^{-1}(t/2) + |x - y| \leq 2|x - y|.$$

By the semigroup property, Lemma 4.3.3 with (4.3.6), Theorem 4.2.8 with (4.3.7) and (4.3.8) and the fact $r \to F^2_{a,T}(t,r)$ is decreasing, there exist constants $c_i = c_i(\beta, T) > 0$ ($c_i > 0$ when $\beta = 0$, respectively), $i = 1, \ldots, 4$ such that

$$\begin{aligned} p_{\overline{B}_R^c}(t,x,y) &= \int_{\overline{B}_R^c} p_{\overline{B}_R^c}(t/2,x,z) p_{\overline{B}_R^c}(t/2,z,y) dz \\ &\geq \int_{B(x_1,\varphi^{-1}(t/2)/60)} p_{\overline{B}_R^c}(t/2,x,z) p_{\overline{B}_R^c}(t/2,z,y) dz \\ &\geq c_1 \int_{B(x_1,\varphi^{-1}(t/2)/60)} 1/(\varphi^{-d}(t/2)) F^2_{c_2,T/2}(t/2, |z-y|) dz \\ &\geq c_3\, F^2_{2c_2,T}(t, 2|x-y|) \geq\, c_4\, F^2_{2^3 c_2,T}(t, |x-y|). \end{aligned}$$

The last inequality holds by (1) in Lemma 4.1.2 with $a = 2c_2$ and $b = 2$ and we have proved the proposition. □

The following elementary lemma is used to prove the lower bound estimates on $p_D(t,x,y)$ where $t \in [T,\infty)$ ($t > 0$ when $\beta = 0$, respectively). Recall the function $F^1_{a,b,T}(t,r)$ which is defined in (2.1.8).

Lemma 4.3.5. *Let K, R, b and t_0 be fixed positive constants and $\beta \in [0,\infty]$. Suppose that $x, x_1 \in \mathbb{R}^d$ satisfy $|x - x_1| = K^2R$. Then there exists a positive constant $c = c(K,R,b,t_0,\beta)$ such that for any $a > 0$ and $z \in \mathbb{R}^d$, we have $F^1_{a,b,t_0}(t_0, 5|x-z|/4) \geq c\, F^1_{a,b,t_0}(t_0, 2|x_1 - z|)$.*

Proof. Let $r := |x-z|$ and $r_1 := |x_1-z|$. For any $z \in B(x,KR) \cup B(x_1,KR)$, we have that $r \leq (K+1)KR$. So $F^1_{a,b,t_0}(t_0, 5r/4)$ is bounded below and the lemma holds.

Suppose that $z \notin B(x,KR) \cup B(x_1,KR)$. When $r \leq 4K^2R \vee 4/5$, then $F^1_{a,b,t_0}(t_0,5r/4)$ is bounded below and hence the lemma holds. Let $r > 4K^2R \vee 4/5$. By the triangle inequality, we have that $3r/4 < r - K^2R \leq r_1 \leq r + K^2R < 5r/4$ and hence $1 \leq 5r/4 \leq 5r_1/3 \leq 2r_1$. In this case, since $r \to F^1_{a,b,t_0}(t_0,r)$ is non-increasing, the lemma holds. □

Now, we are ready to prove the lower bound estimates on $p_D(t,x,y)$. For the remainder of this thesis, we assume that $\eta \in (\alpha/2, 1]$ and D is an exterior $C^{1,\eta}$ open set in $\mathbb{R}^d$ with $C^{1,\eta}$ characteristics (R_0, Λ_0) and $D^c \subset B(0,R)$ for some $R > 0$. Such an open set D can be disconnected. When $\beta \in (1,\infty]$ and $|x-y| \geq 4/5$, we will consider the following two cases that x, y are in the same component and in different components in D, separately.

Proof of Theorem 2.2.3(2)–(3) Due to Theorem 2.2.1(5) and the domain monotonicity of $p_D(t,x,y)$, the Theorem holds when x, y are in the same bounded connected component of D. So we only need to prove Theorem 2.2.3(2)–(3.a).

When $\beta = 0$, by Theorem 2.2.1(1), we may assume that $t \geq T$. Without loss of generality, we may assume that $T = 3$. For x and y in D, let $v \in \mathbb{R}^d$ be any unit vector satisfying $x \cdot v \geq 0$ and $y \cdot v \geq 0$. Let $M_2 := M_1(\beta, 3(16)^{-1}) (\geq 3)$, where M_1 is the constant in Lemma 4.3.2. Define

$$x_1 := x + M_2^2 R v \quad \text{and} \quad y_1 := y + M_2^2 R v.$$

By the semigroup property and Theorem 2.2.1(2)-(3), we have that for every $t-2 \geq 1$ and $x, y \in D$,

$$\begin{aligned} p_D(t,x,y) &= \int_D \int_D p_D(1,x,z) p_D(t-2,z,w) p_D(1,w,y) dz dw \\ &\geq c_1 (1 \wedge \delta_D(x))^{\alpha/2} (1 \wedge \delta_D(y))^{\alpha/2} f_2(t,x,y). \end{aligned} \tag{4.3.9}$$

Here $f_2(t,x,y)$ is defined by

$$\begin{aligned} f_2(t,x,y) &= \int_{B(0,M_2R)^c \times B(0,M_2R)^c} (1 \wedge \delta_D(z))^{\alpha/2} F^1_{C_2,\gamma,1}(1, 5|x-z|/4) \\ &\cdot\, p_D(t-2,z,w)(1 \wedge \delta_D(w))^{\alpha/2} F^1_{C_2,\gamma,1}(1,5|y-w|/4) dz dw, \end{aligned} \tag{4.3.10}$$

for some constant $C_5 > 0$ and $\gamma = \gamma_1^{-1} \vee \gamma_2$ where γ_1, γ_2 in (2.1.1).

Let $A_2 := \max\{(C_1C_4)^{1/(\beta\wedge 1)}, 2\gamma^{2/\beta}, 2C_1C_5\} (\geq 2)$ ($A_2 = 2$ when $\beta = 0$, respectively) where C_1 is the constant in Theorem 2.1.1 and C_4 is the constant in Proposition 4.3.4. By Lemma 4.3.5 and Theorem 2.1.1(1), there exists $c_i = c_i(\beta) > 0, i = 2, \dots, 4$ such that

$$\begin{aligned} F^1_{C_5,\gamma,1}(1,5|x-z|/4) &\geq c_2 F^1_{C_5,\gamma,1}(1,2|x_1-z|) \geq c_3 F^1_{C_1^{-1},\gamma^{-1},1}(1,A_2|x_1-z|) \\ &\geq c_4 p(1, A_2x_1, A_2z), \\ F^1_{C_5,\gamma,1}(1,5|y-w|/4) &\geq c_2 F^1_{C_5,\gamma,1}(1,2|y_1-w|) \geq c_3 F^1_{C_1^{-1},\gamma^{-1},1}(1,A_2|y_1-w|) \\ &\geq c_4 p(1, A_2y_1, A_2w). \end{aligned} \tag{4.3.11}$$

When $\beta \in (0, \infty]$, the second inequalities hold by (2) in Lemma 4.1.1 along with $t_0 = 1$, $a = C_1$, $b = C_5$, $c = \gamma$, $N_1 = 2$ and $N_2 = A_2$ and the fact $A_2 \geq 2(C_1C_5 \vee \gamma^{2/\beta})$. When $\beta = 0$, the second inequalities hold since $A_2 = 2$.

For $z, w \in B(0, M_2R)^c$ and $t - 2 \in [1, \infty)$, by Proposition 4.3.4 and Theorem 2.1.1(2), we have that

$$\begin{aligned} &p_D(t-2,z,w) \geq p_{\overline{B}_R^c}(t-2,z,w) \geq c_5 F^2_{C_4,1}(t-2,|z-w|) \\ &\geq c_6 F^2_{C_1^{-1},1}(t-2, A_2|z-w|) \geq c_7 p(t-2, A_2z, A_2w). \end{aligned} \tag{4.3.12}$$

For the third inequality above, we use (2) in Lemma 4.1.2 along with $T = 1$, $a = C_4$, $b = C_1$ and $N = A_2$ and the fact $A_2 \geq (C_1C_4)^{1/(\beta\wedge 1)}$ when $\beta \in (0, \infty]$. When $\beta = 0$, the third inequality holds since $A_2 \geq 1$.

For $z \in B(0, M_2R)^c$, $\delta_D(z) \geq \delta_{\overline{B}_R^c}(z) = |z| - R \geq M_2R - R$. So applying (4.3.11) and (4.3.12) to (4.3.10) and by the change of variables $\hat{z} = A_2z$, $\hat{w} = A_2w$ and semigroup property, we have that

$$\begin{aligned} f_2(t, x, y) &\geq c_8 \int_{B(0,M_2R)^c\times B(0,M_2R)^c} p(1, A_2x_1, A_2z) \\ &\qquad\qquad \cdot p(t-2, A_2z, A_2w)p(1, A_2y_1, A_2w)dzdw \\ &\geq c_9 \int_{B(0,A_2M_2R)^c\times B(0,A_2M_2R)^c} p_{B(0,A_2M_2R)^c}(1, A_2x_1, \hat{z}) \\ &\qquad\qquad \cdot p_{B(0,A_2M_2R)^c}(t-2, \hat{z}, \hat{w})p_{B(0,A_2M_2R)^c}(1, A_2y_1, \hat{w})d\hat{z}d\hat{w} \\ &= c_9\, p_{B(0,A_2M_2R)^c}(t, A_2x_1, A_2y_1). \end{aligned} \tag{4.3.13}$$

Since $A_2|x_1| \wedge A_2|y_1| \geq M_2(A_2M_2R)$, by Proposition 4.3.4 and (1) in Lemma 4.1.2 with $a = C_4$ and $b = A_2$, we have that

$$\begin{aligned} p_{B(0,A_2M_2R)^c}(t, A_2x_1, A_2y_1) &\geq c_{10}\, F^2_{C_4,T}(t, A_2|x_1 - y_1|) \\ &= c_{10}F^2_{C_4,T}(t, A_2|x - y|) \geq c_{11}F^2_{A_2^2C_4,T}(t, |x - y|). \end{aligned} \tag{4.3.14}$$

Combining (4.3.9) with (4.3.13) and (4.3.14), we have proved the lower bound estimates in Theorem 2.2.3(2)–(3.a). □

For the remainder of this section, we assume that $T > 0$, $\beta \in (1, \infty)$ and $(t, x, y) \in [T, \infty) \times D \times D$ where $|x - y| \geq 4/5$ and x, y are in different components of D.

Recall the definition of κ-fat in Definition 3.4.3, and note an exterior $C^{1,\eta}$ open set is κ-fat. Hereinafter, we assume that $A_r(x)$ is such the point in D.

Lemma 4.3.6. *Suppose that $D_b \subset B(0, R)$ be a bounded connected component of D. Then there exists a positive constant $c = c(\beta, \eta, r_0, \Lambda_0, T)$ such that for every $t \geq T$ and $x \in D_b$, we can find a ball $B \subset D_b$ such that*

$$\int_B p_{D_b}(2^{-1}t - 3^{-1}T, x, z)dz \geq c\, e^{-t\lambda^{D_b}}\delta_{D_b}(x)^{\alpha/2}$$

where $-\lambda^{D_b} < 0$ *be the largest eigenvalue of the generator of* Y^{D_b}.

Proof. For any $x \in D_b$, let $z_x \in \overline{D_b}$ be the point so that $|z_x - x| = \delta_{D_b}(x)$. Let $x_1 := A_{r_0}(z_x)$ and $B := B(x_1, \kappa r_0)$. For any $z \in B$, we have that $\delta_{D_b}(z) \geq \kappa r_0$. Hence since $2^{-1}t - 3^{-1}T \geq 6^{-1}T$, by Theorem 2.2.1(5) along with the bounded connected component D_b, there exist constants $c_i = c_i(\beta, \eta, r_0, \Lambda_0, T) > 0$, $i = 1, \dots, 3$ such that for any $x \in D_b$

$$\int_B p_{D_b}(2^{-1}t - 3^{-1}T, x, z)dz \geq c_1 e^{-t\lambda^{D_b}} \int_B \delta_{D_b}(x)^{\alpha/2}\delta_{D_b}(z)^{\alpha/2}dz$$
$$\geq c_2 e^{-t\lambda^{D_b}} \delta_{D_b}(x)^{\alpha/2} \int_B dz \geq c_3 e^{-t\lambda^{D_b}} \delta_{D_b}(x)^{\alpha/2}.$$

□

Now, we are ready to prove the lower bound estimates on $p_D(t, x, y)$ for any $\beta \in (1, \infty)$ and $(t, x, y) \in [T, \infty) \times D \times D$ where $|x - y| \geq 4/5$ and x, y are in different components of D.

Proof of Theorem 2.2.3(4) Let $D(x)$ and $D(y)$ be connected components containing x and y, respectively with $D(x) \cap D(y) \neq \emptyset$. Without loss of generality, we may assume that $D(x)$ is a bounded connected component and $T = 3$.

By the semigroup property and the domain monotonicity of $p_D(t, x, y)$, we first observe that

$$p_D(t, x, y) \geq \int_{D(x)} \int_{D(y)} p_{D(x)}(2^{-1}t - 1, x, z) \cdot p_D(2, z, w) p_{D(y)}(2^{-1}t - 1, y, w) dw dz. \tag{4.3.15}$$

For bounded connected component D_j of D and the largest eigenvalue $-\lambda_j < 0$ of the generator Y^{D_j}, define $\overline{\lambda} := \max\{\lambda_j : j = 1, \dots, n\}$. By Lemma 4.3.6, there exist a ball $B_x \subset D(x)$ and a constant $c_1 = c_1(\beta, \eta, r_0, \Lambda_0) > 0$ such that

$$\int_{B_x} p_{D(x)}(2^{-1}t - 1, x, z)dz \geq c_1 e^{-t\overline{\lambda}} \delta_D(x)^{\alpha/2}. \tag{4.3.16}$$

Similarly, if $D(y)$ is a bounded connected component, $\int_{B_y} p_{D(y)}(2^{-1}t - 1, y, w)dw \geq c_2 e^{-t\overline{\lambda}} \delta_D(y)^{\alpha/2}$ for some a ball $B_y \subset D(y)$ and a constant $c_2 > 0$. For any $(z, w) \in B_x \times B_y$, note that $r_0 \leq |z-w| \leq 2R$ and $\delta_D(z) \wedge \delta_D(w) \geq c_3$. So by Theorem 2.2.1(2) and (4), we have that $\inf_{(z,w)\in B_x\times B_y} p_D(2, z, w) \geq c_4$. Hence, we have the conclusion when $D(x)$ and $D(y)$ are bounded connected components of D.

When $D(y)$ is an unbounded connected component, let $y_1 := y + 2Ry/|y|$ and $B_{y_1} := B(y_1, 2^{-1}R) \subset D(y)$. For any $w \in B_{y_1}$, we have that $\delta_{D(y)}(w) \geq R/2$ and $|y - w| \leq |y - y_1| + |y_1 - w| \leq 5R/2$. Hence for $2^{-1}t - 1 \geq 1/2$, by Theorem 2.2.3(2)–(3.a) and the fact $t/2 - 1 \asymp t$, there exist constants $c_i = c_i(\beta, \eta, r_0, \Lambda_0, R) > 0$, $i = 5, \ldots, 8$ such that

$$\begin{aligned}
&\int_{B_{y_1}} p_{D(y)}(2^{-1}t - 1, y, w)dw \\
&\geq c_5 \int_{B_{y_1}} (1 \wedge \delta_{D(y)}(y))^{\alpha/2}(1 \wedge \delta_{D(y)}(w))^{\alpha/2} t^{-d/2} \exp(-c_6|y - w|^2/t)dw \\
&\geq c_7\, (1 \wedge \delta_D(y))^{\alpha/2} t^{-d/2} \int_{B_{y_1}} dw = c_8(1 \wedge \delta_D(y))^{\alpha/2} t^{-d/2}. \qquad (4.3.17)
\end{aligned}$$

For any $(z, w) \in B_x \times B_{y_1}$, we have that $\delta_D(z) \wedge \delta_D(w) \geq c_9$ and

$$|z - w| \leq |z - x| + |x - y| + |y - w| \leq 2R + |x - y| + 5R/2 \leq c_{10}|x - y|.$$

The last inequality holds since $|x - y| \geq 4/5$. So by Theorem 2.2.1(2) and (4), there are constants $c_i = c_i(\beta, \eta, r_0, \Lambda_0, R) > 0$, $i = 11, \ldots, 14$ such that

$$\begin{aligned}
\inf_{(z,w)\in B_x\times B_{y_1}} p_D(2, z, w) &\geq c_{11} \left(\frac{\exp(-c_{12}|z - w|^{\beta})}{|z - w|^{d+\alpha}} \wedge 1 \right) \\
&\geq c_{13} \frac{\exp(-c_{14}|x - y|^{\beta})}{|x - y|^{d+\alpha}}. \qquad (4.3.18)
\end{aligned}$$

Combining (4.3.16), (4.3.17) and (4.3.18) with (4.3.15), we have the conclusion when $D(x)$ is a bounded connected component and $D(y)$ is an unbounded connected component of D □

Remark 4.3.7. *Let D be an exterior $C^{1,\eta}$ open set in $\mathbb{R}^d$ with $C^{1,\eta}$ characteristics (r_0, Λ_0) and $D^c \subset B(0,R)$. Then the number of bounded connected components of D is finite, say $D_1, \dots, D_n$. According to the proof of Theorem 2.2.3(4), there exists a constant $c > 0$ such that if $x, y \in D$ are in different bounded connected components of D*

$$p_D(t,x,y) \geq c\, \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2} \exp\left(-t \sum_{j=1}^{n} \lambda_j \Big(\mathbf{1}_{D_j}(x) + \mathbf{1}_{D_j}(y)\Big)\right)$$

where $-\lambda_j < 0$ is the largest eigenvalue of the generator Y^{D_j}, $j = 1, \cdots, n$.

4.4 Green function estimates

In this section, we present a proof of Corollary 2.2.5. We recall that $G_D(x,y) = \int_0^\infty p_D(t,x,y)dt$. When $\beta = 0$, the proof of Corollary 2.2.5 is similar to that of [17, Corollary 1.5], we only consider the case $\beta \in (0, \infty]$.

Proof of Corollary 2.2.5 By Corollary 2.2.4, there exist constants $c_i > 1$, $i = 1, 2$ such that

$$\begin{aligned} G_D(x,y) \leq c_1 \int_0^1 \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2} F^1_{c_2^{-1}, \gamma^{-1}, 30}(t, |x-y|/6) dt \\ + c_1 \left(1 \wedge \delta_D(x)\right)^{\alpha/2} \left(1 \wedge \delta_D(y)\right)^{\alpha/2} \int_1^\infty F^2_{c_2^{-1}, 1}(t, |x-y|) dt, \\ G_D(x,y) \geq c_1^{-1} \mathbf{1}_{\{|x-y| < 4/5\}} \int_0^1 \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2} \\ \cdot F^1_{c_2, \gamma, 30}(t, 5|x-y|/4) dt \\ + c_1^{-1} \mathbf{1}_{\{|x-y| \geq 4/5\}} \left(1 \wedge \delta_D(x)\right)^{\alpha/2} \left(1 \wedge \delta_D(y)\right)^{\alpha/2} \int_1^\infty F^2_{c_2, 1}(t, |x-y|) dt \end{aligned}$$

where $\gamma := \gamma_1^{-1} \vee \gamma_2$ for the constants γ_1, γ_2 in (2.1.1).

Without loss of generality, we may assume that $c_2 = 1$ and we define I_1,

I_2 and II by

$$\begin{aligned} I_1 &:= \int_0^1 \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2} \left(t^{-d/\alpha} \wedge t|x-y|^{-\alpha-d}\right) dt \\ I_2 &:= \int_0^1 \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2} F^1_{1,\gamma^{-1},30}(t, |x-y|/6) dt \\ II &:= \int_1^\infty F^2_{1,1}(t, |x-y|) dt. \end{aligned}$$

For any $a, b > 0$, if $b|x-y| < 1$, we have that $F^1_{1,a,30}(t, b|x-y|) \asymp t^{-d/\alpha} \wedge t|x-y|^{-\alpha-d}$. So when $|x-y| < 4/5$, it suffices to show that

$$\begin{aligned} I_1 &\asymp \left(\frac{1}{|x-y|^{d-\alpha}} + \frac{1}{|x-y|^{d-2}}\right) \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right)^{\alpha/2}, \\ II &\le c_3 \le c_4 \left(\frac{1}{|x-y|^{d-\alpha}} + \frac{1}{|x-y|^{d-2}}\right). \end{aligned} \tag{4.4.1}$$

When $|x-y| \ge 4/5$, we will show that

$$I_2 \le c_5 \left(\frac{1}{|x-y|^{d-\alpha}} + \frac{1}{|x-y|^{d-2}}\right) (1 \wedge \delta_D(x))^{\alpha/2} (1 \wedge \delta_D(y))^{\alpha/2}, \tag{4.4.2}$$

$$II \asymp \frac{1}{|x-y|^{d-2}} \asymp \left(\frac{1}{|x-y|^{d-\alpha}} + \frac{1}{|x-y|^{d-2}}\right). \tag{4.4.3}$$

Let $r := |x-y|$. Suppose that $r < 4/5$. By [[7], (4.3), (4.4) and (4.6)], we have

$$\begin{aligned} I_1 &\asymp \frac{1}{r^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x)}{r}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{r}\right)^{\alpha/2} \\ &\asymp \left(\frac{1}{r^{d-\alpha}} + \frac{1}{r^{d-2}}\right) \left(1 \wedge \frac{\delta_D(x)}{r}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{r}\right)^{\alpha/2}. \end{aligned} \tag{4.4.4}$$

Note that for every $s \in [0, \infty]$,

$$\int_s^\infty t^{-d/2} e^{-r^2/t} dt = r^{2-d} \int_0^{r^2/s} u^{d/2-2} e^{-u} du. \tag{4.4.5}$$

For $r < 1$ and $1 < t$, we have $F^2_{1,1}(t,r) = t^{-d/2}e^{-r^2/t}$ and

$$II = r^{2-d}\int_0^{r^2} u^{d/2-2}e^{-u}du \asymp r^{2-d}\int_0^{r^2} u^{d/2-2}du = \frac{2}{d-2}. \tag{4.4.6}$$

Hence we obtain (4.4.1) by (4.4.4) and (4.4.6).

Suppose that $r \geq 4/5$. Note that for $0 < t \leq 1$, we have

$$\begin{aligned} F^1_{1,\gamma^{-1},30}(t, r/6) &= \begin{cases} t^{-d/\alpha} \wedge t(r/6)^{-d-\alpha}e^{-\gamma^{-1}(r/6)^{\beta}} & \text{if } \beta \in (0,1] \\ t\exp(-((r/6)(\log(5r/t))^{(\beta-1)/\beta} \wedge (r/6)^{\beta})) & \text{if } \beta \in (1,\infty) \\ (t/(5r))^{r/6} & \text{if } \beta = \infty \end{cases} \\ &\leq \begin{cases} t(r/6)^{-d-\alpha} & \text{if } \beta \in (0,1] \\ te^{-c_6 r} & \text{if } \beta \in (1,\infty) \\ t^{2/15}e^{-c_6 r} & \text{if } \beta = \infty \end{cases} \leq \; c_7 t^{2/15} r^{-d-\alpha} \end{aligned}$$

for some constant $c_i = c_i(\beta) > 0$, $i = 6, 7$. Thus by the change of variable $u = r^{\alpha}/t$, there exist constants $c_i > 0, i = 8, 9$ such that

$$\begin{aligned} I_2 &\leq c_7 r^{-d-\alpha}\int_0^1 t^{\frac{2}{15}}\left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2}\left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2} dt \\ &= c_7 r^{-d+\frac{2}{15}\alpha}\int_{r^{\alpha}}^{\infty} u^{-\frac{2}{15}-2}\left(1 \wedge \frac{\sqrt{u}\delta_D(x)^{\alpha/2}}{r^{\alpha/2}}\right)\left(1 \wedge \frac{\sqrt{u}\delta_D(y)^{\alpha/2}}{r^{\alpha/2}}\right) du \\ &= c_7 r^{-d+\frac{2}{15}\alpha}\int_{r^{\alpha}}^{\infty} u^{-\frac{2}{15}-1}\left(\frac{1}{\sqrt{u}} \wedge \frac{\delta_D(x)^{\alpha/2}}{r^{\alpha/2}}\right)\left(\frac{1}{\sqrt{u}} \wedge \frac{\delta_D(y)^{\alpha/2}}{r^{\alpha/2}}\right) du \\ &\leq c_8 r^{-d+\frac{2}{15}\alpha}\int_{r^{\alpha}}^{\infty} u^{-\frac{2}{15}-1}du\left(1 \wedge \frac{\delta_D(x)}{r}\right)^{\alpha/2}\left(1 \wedge \frac{\delta_D(y)}{r}\right)^{\alpha/2} \\ &= \frac{15}{2}c_8 r^{-d}\left(1 \wedge \frac{\delta_D(x)}{r}\right)^{\alpha/2}\left(1 \wedge \frac{\delta_D(y)}{r}\right)^{\alpha/2} \\ &\leq c_9 r^{2-d}\left(1 \wedge \delta_D(x)\right)^{\alpha/2}\left(1 \wedge \delta_D(y)\right)^{\alpha/2} \end{aligned}$$

and it yields (4.4.2). For (4.4.3), because of (4.4.6), we may assume that

$r \geq 1$. By (4.4.5), we have that

$$\begin{aligned} II &\geq \int_1^\infty t^{-d/2} e^{-r^2/t} dt \geq r^{2-d} \int_0^1 u^{d/2-2} e^{-u} du \geq c_{10} r^{2-d} \\ II &\leq \int_1^{r^{2-(\beta\wedge 1)}} t^{-d/2} e^{-r^{(\beta\wedge 1)}} dt + \int_{r^{2-(\beta\wedge 1)}}^\infty t^{-d/2} e^{-r^2/t} dt \\ &\leq c_{11} e^{-r^{(\beta\wedge 1)}} + r^{2-d} \int_0^{r^{(\beta\wedge 1)}} u^{d/2-2} e^{-u} du \leq c_{12} r^{2-d}. \end{aligned}$$

This implies $II \asymp r^{2-d}$ and hence (4.4.3) holds. So we have proved the Corollary. □

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국문초록

이 논문에서는 전체 시간 $t > 0$에 대하여, $C^{1,\eta}$ 열린집합에서 정의된 마르코브 과정의 큰 집합에 대한 날카로운 양측 열 핵의 추정을 계산한다. 이 과정은 점프 핵 강도가

$$\kappa(x,y)\psi(|x-y|)^{-1}|x-y|^{-d-\alpha}$$

인 순수 점프 마르코브 과정이다. 이때 $\alpha \in (0,2)$, 그리고 $[0,\infty)$에서 정의된 증가함수 ψ는 $0 < r \le 1$일 경우 $\psi(r) = 1$이며, $r > 1$일 경우 임의의 $\beta \in [0,\infty]$에 대하여 $c_1 e^{c_2 r^\beta} \le \psi(r) \le c_3 e^{c_4 r^\beta}$이다. 대칭함수 $\kappa(x,y)$는 위, 아래로 유계이며, 임의의 $|x-y| < 1$와 $\rho > \alpha/2$에 대하여 $|\kappa(x,y) - \kappa(x,x)| \le c_5 |x-y|^\rho$를 만족한다. 주요 결과로부터 열린집합에서 이 과정의 양측 그린 함수를 추정한다.

주요어휘: 디리슐렛 형식, 마르코브 과정, 열 핵, 전이확률밀도, 그린 함수
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