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# Watson transformations and class numbers of ternary quadratic forms

(왓슨 변환과 삼변수 이차형식의 류수)

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# Watson transformations and class numbers of ternary quadratic forms

A dissertation
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by

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# Abstract

The class number of an integral quadratic form is defined by the number of inequivalent classes in its genus. Recently, Chan and Oh gave an explicit relation between the class number of a (positive definite integral) ternary quadratic form and the class number of its Watson transformation at an odd prime. In this thesis, we consider the case when the Watson transformation is taken at the prime 2 or 4 on arbitrary ternary quadratic forms and give an explicit relation under this situation. We finally give an effective inductive method on the computation of the class number of an arbitrary ternary quadratic form. As an example, we provide the class number formula for any Bell ternary quadratic form.

**Key words:** ternary quadratic forms, class number, Watson transformation
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# Contents

# Chapter 1

# Introduction

A quadratic homogeneous polynomial of three variables

$$f(x_1, x_2, x_3) = \sum_{1 \leqslant i,j \leqslant 3} a_{ij} x_i x_j \quad (a_{ij} = a_{ji} \in \mathbb{Q})$$

is called a ternary quadratic form. We say that $f$ is positive definite if the corresponding symmetric matrix $M_f = (a_{ij}) \in M_{3\times 3}(\mathbb{Z})$ is positive definite. We say that $f$ is integral if $a_{ij} \in \mathbb{Z}$ for any $i, j$. Throughout this thesis, we always assume that every ternary quadratic form is *positive definite and integral*, unless stated otherwise.

The *class number* of a ternary quadratic form $f$ is defined by the number of isometry classes in the genus of $f$. Computing the class number of a ternary quadratic form is an important problem in number theory and it is related with many other areas of mathematics. One of the most important connections is the correspondence between the class number of ternary quadratic forms and the type numbers of orders in quaternion algebras, which states that computing the class numbers of positive definite ternary quadratic forms over $\mathbb{Z}$ is equivalent to compute the type numbers of orders in definite quaternion algebras over $\mathbb{Q}$. Pizer [12] obtained an explicit formula for the type numbers of all Eichler orders by using the Selberg Trace Formula. A formula for an arbitrary order is obtained by Körner [7], but his formula requires the computation of the restricted embedding numbers of quadratic orders into quaternion orders, which can be obtained only for some special orders using results of [5, 10, 11, 12]. Recently, Chan and Oh [1] provided an effective method to compute the class number of a ternary quadratic form except some special cases by using, so called, Watson transformations which are first used by Watson in his doctoral thesis [14].

In this thesis, we provide a method of computing class numbers of arbitrary ternary quadratic forms. Our main tool is to use Watson transformations, which is quite similar to that of [1].

Unexplained notations and terminologies from the theory of quadratic spaces and lattices will follow those of O'Meara's book [8]. For convenience, a quadratic space is always a positive definite quadratic space over the field of rational numbers $\mathbb{Q}$, and the term "lattice" always refers to an integral $\mathbb{Z}$-lattice on a (not necessarily fixed) quadratic space over $\mathbb{Q}$. For a lattice $L$, the genus of $L$ is denoted by $\text{gen}(L)$, and $\text{gen}(L)/\sim$ is the set of isometry classes in $\text{gen}(L)$. It is well known that the latter set is finite and its cardinality is called the *class number of* $L$, denoted by $h(L)$. The isometry class containing $L$ in $\text{gen}(L)$ is denoted by $[L]$. The lattice $L$ is integral if its scale ideal $\mathfrak{s}(L)$ is inside $\mathbb{Z}$, and is "primitive" if its scale ideal $\mathfrak{s}(L)$ is exactly $\mathbb{Z}$. We write $L \cong A$ whenever $A$ is a Gram matrix of $L$, and the discriminant $dL$ is defined to be the determinant of $A$. A diagonal matrix with $a_1$,..., $a_n$ on the diagonal is denoted by $\langle a_1, ..., a_n \rangle$.

Let $L$ be a primitive ternary lattice, and let $k$ be a positive integer. We define

$$\Lambda_k(L) = \{x \in L : Q(x+z) \equiv Q(z) \pmod{k} \quad \text{for any } z \in L\}.$$

Then one may easily show that $\Lambda_k(L)$ is a sublattice of $L$. The primitive lattice obtained from $\Lambda_k(L)$ by a suitable scaling is defined by $\lambda_k(L)$. As one of class invariants, the *label* of a lattice was first defined in [1]. We define *the modified label* of a lattice $L$ consisting of the order of the orthogonal group of $L$, the norms of symmetries in the isometry group and some "additional information" depending on the structure of $L$ (for the exact definition, see Definition 4.2.1). In fact, this additional information was not necessary in [1], for they only consider $\lambda_p$ transformation for some odd prime $p$.

The basic strategy of the method of computing $h(L)$ provided in [1] is as follows:

(I) transform $L$, via a $finite$ sequence of Watson transformations, to a *stable* lattice $K$ (for the definition of a stable lattice, see Definition 3.1.3):

$$(*) \qquad L = L_0 \xrightarrow{\lambda_{k_1}} L_1 \xrightarrow{\lambda_{k_2}} \dots \xrightarrow{\lambda_{k_n}} L_n = K,$$

where each $k_i$ is either 4 or a prime divisor of $dL$;

(II) determine the labels of all the classes in $\text{gen}(K)$, and hence obtain $h(K)$;

(III) for each $i \leqslant n$, use the labels of the classes in $\text{gen}(L_i), \dots, \text{gen}(L_n)$ to determine the labels of all the classes in $\text{gen}(L_{i-1})$. In particular, $h(L_{i-1})$ is computed. Finally, compute the class number of $L = L_0$.

These three steps altogether provide an effective procedure to compute the class number $h(L)$. The initial inputs are the labels of the classes in $\text{gen}(K)$, which will be determined without any prior information. Note that this procedure will terminate in a finite number of iterations for any given lattice $L$. Chan and Oh [1] proved that this method works whenever each index $k_i$ of Watson transformations in $(*)$ is an odd prime.

In this thesis, we show that this method, in principle, works even when each index of Watson transformations in $(*)$ is 2 or 4 if we use the modified label explained above. There are some exceptional cases where Step (II) in the above procedure does not work. For these exceptional cases, we also provide the method on computing the labels by using some additional information.

In Chapter 2, we introduce some basic definitions and known results on quadratic forms. Most of them will also be found in [8].

In Chapter 3, we explain our method as a whole. In particular, the procedure on Step (I) is explained in this chapter.

As a main chapter, we compute the labels of lattices in Chapter 4. More precisely, for any lattice $K$, we compute the labels of all lattices $L$ such that $\lambda_{2e}(L) \simeq K$, where $e = 1$ if $L$ is odd, $e = 2$ otherwise. Since we add some more information to the definition of the label which is originally given in [1], we have to determine this additional information for each lattice $L$ by using the modified label of $K$. In fact, there are some exceptional cases that we can not determine the labels of lattices $L$ even if the label of $K$ is completely given. At the end of this chapter, we summarize all of these exceptional cases and describe the labels by using some more additional information such as the label of $\lambda_{2e}(K)$.

In Chapter 5, we consider stable lattices. We provide the class number formula for an arbitrary stable lattice together with the method on computing the labels of all lattices in the genus of the stable lattice. Hence Step (II) in the above procedure will be considered in this chapter.

In Chapter 6, as an application of our main result, we provide a closed formula for the class number of a *Bell ternary form* which is a lattice of the form $\langle 1, 2^m, 2^n \rangle$ for some positive integers $m$ and $n$ $(m \geqslant n)$.

In Chapter 7 which is an appendix, we list all of our results given in Chapter 4.

# Chapter 2

# Preliminaries

## 2.1 Quadratic spaces

Let $F$ be an abstract field of characteristic not 2, and $V$ be a finite dimensional vector space over $F$. We call $V$ a *quadratic space* over $F$ if the vector space $V$ is equipped with a symmetric bilinear form $B$, i. e. a mapping

$$B : V \times V \longrightarrow F$$

with the following properties:

$$B(x, y + z) = B(x, y) + B(x, z),$$

$$B(\alpha x, y) = \alpha B(x, y), \quad B(x, y) = B(y, x)$$

for all $x$, $y$, $z \in V$ and all $\alpha \in F$. We say that a quadratic space is binary, ternary, quaternary, quinary, ..., $n$-ary, according as its dimension is 2, 3, 4, 5,..., $n$. A *quadratic map* $Q$ of the bilinear form $B$ is defined as the map

$$Q : V \longrightarrow F$$

with $Q(x) = B(x, x)$ for all $x \in V$. Then the following identities hold:

$$Q(\alpha x) = \alpha^2 Q(x),$$

$$Q(x + y) = Q(x) + Q(y) + 2B(x, y)$$

for all $x$, $y \in V$ and all $\alpha \in F$. For a non-zero vector $x \in V$, we call $x$ *isotropic* if $Q(x) = 0$, we call it *anisotropic* if $Q(x) \neq 0$. For a non-zero quadratic space $V$, we call $V$ isotropic if it contains an isotropic vector, we call it anisotropic if it does not contain an isotropic vector.

Two subsets $V_1$ and $V_2$ of $V$ are called *orthogonal* if $B(x_1, x_2) = 0$ for all $x_1 \in V_1$ and all $x_2 \in V_2$. We say that $V$ has the *orthogonal splitting*

$$V = V_1 \perp \cdots \perp V_r$$

into subspaces $V_i (1 \leqslant i \leqslant r)$ if $V$ is the direct sum

$$V = V_1 \oplus \cdots \oplus V_r$$

with

$$B(V_i, V_j) = 0 \quad \text{for} \quad 1 \leqslant i \leqslant j \leqslant r.$$

For any fixed non-zero element $\alpha$ in $F$, we define

$$B^\alpha(x, y) = \alpha B(x, y)$$

for all $x$, $y \in V$. Then we let $V^\alpha$ denote the quadratic space $V$ equipped with the bilinear form $B^\alpha$ and called the quadratic space obtained from scaling $V$ by $\alpha$.

The quadratic space $V$ is said to *represent* an element $\alpha$ in $F$ if there is a vector $x$ in $V$ such that $Q(x) = \alpha$. Let $V$ and $W$ be quadratic spaces over $F$ and let $Q$ denote the quadratic map on each of them. Let $L_F(V, W)$ be the set of all linear transformations of $V$ into $W$. Then a linear transformation $\tau \in L_F(V, W)$ is called a *representation* of $V$ into $W$ with respect to the quadratic maps $Q$ on $V$ and $W$ if

$$Q(\tau x) = Q(x)$$

for all $x \in V$. If $\tau$ is injective, we call $\tau$ an *isometry* of $V$ into $W$. If there exists a bijective representation $\tau$ of $V$ onto $W$, we say that $V$ and $W$ are *isometric* and we write

$$V \cong W.$$

The set of all isometries of $V$ into $W$ is written

$$O(V, W).$$

If V=W, we write $O(V)$ instead of $O(V, V)$. Then the set $O(V)$ is a subgroup of the general linear group $GL_F(V)$ and called the *isometry group* of $V$ with respect to the quadratic map $Q$. For a fixed vector $x \in V$ with $Q(x) \neq 0$, we define a mapping $\tau_x : V \to V$ by

$$\tau_x(y) = y - \frac{2B(y, x)}{Q(x)} x.$$

Then we have $\tau_x \in O(V)$ and call $\tau_x$ the *symmetry* with respect to the vector $x$.

Let $\dim_F(V) = n$ and $x_1$,..., $x_n$ be a base of $V$. Then we may associate the $n \times n$ symmetric matrix $M$ with the above base by taking $M$ as

$$M := (B(x_i, x_j)).$$

We call $M$ the matrix of the quadratic space $V$ in the base $x_1$,..., $x_n$ and write $V \cong M$. The determinant of $M$ is called the *discriminant* of $V$ and denoted by $dV$. We note that the discriminant of $V$ is uniquely determined up to unit squares of $F$. We say that $V$ is regular if $dV \neq 0$.

Let $\mathbb{Q}$ be the rational number field and $\mathbb{Q}_p$ denote the field of $p$-adic completion of $\mathbb{Q}$ for any prime $p$. Suppose that $F = \mathbb{Q}_p$ for some prime $p$ and $V$ is a regular $n$-ary quadratic space over $F$. Then we may take a splitting

$$V \cong \langle \alpha_1 \rangle \perp \cdots \perp \langle \alpha_n \rangle$$

for some elements $\alpha_i$ $(1 \leqslant i \leqslant r)$ in $F$. We define the *Hasse symbol*

$$S_p(V) = \sum_{1 \leqslant i \leqslant j \leqslant n} \left( \frac{\alpha_i, \alpha_j}{p} \right),$$

where $\left(\frac{\cdot,\cdot}{\cdot}\right)$ is the Hilbert symbol.

**Theorem 2.1.1.** *Let $F = \mathbb{Q}_p$ for some prime $p$. Then two regular quadratic spaces $U$ and $V$ over $F$ are isometric if and only if*

$$\mathrm{dim}U = \mathrm{dim}V, \quad dU = dV, \quad S_pU = S_pV.$$

*Proof.* See 63:20 in [8].

## 2.2 Lattices on quadratic spaces

For any prime $p$, $\mathbb{Z}_p$ denotes the $p$-adic integer ring. Let $F$ denote $\mathbb{Q}$ or $\mathbb{Q}_p$ and $R$ denote $\mathbb{Z}$ or $\mathbb{Z}_p$, respectively, for a prime $p$. For a non-zero regular quadratic space $V$ over $F$, we consider a subset $L$ of $V$ which is a $R$-module under the laws induced by the vector space structure of $V$ over $F$. We call $L$ a *$R$-lattice* in $V$ if there is a base $x_1$,..., $x_n$ for $V$ such that

$$L \subseteq Rx_1 + \cdots + Rx_n.$$

We say that such $L$ is a $R$-lattice on $V$ if we have $FL = V$. In this case, the term 'on $V$' may be omitted for convenience. We say that a $R$-lattice $L$ is *integral* if $B(x, y) \in R$ for all $x, y \in L$. We define *scale* $\mathfrak{s}(L)$ of $L$ to be the ideal of $R$ generated by $B(x, y)$ for all $x, y \in L$, *norm* $\mathfrak{n}(L)$ of $L$ to be the ideal of $R$ generated by $Q(x)$ for all $x \in L$. For non-zero element $\alpha$ in $R$, we denote by $L^\alpha$ the $R$-lattice obtained from scaling $L$ by $\alpha$. If $L$ is a $R$-lattice on $V$, there is a base $x_1, ..., x_n$ for $V$ such that

$$L = Rx_1 + \cdots + Rx_n.$$

By the matrix of the lattice $L$ in the base $x_1, ..., x_n$ we mean the $n \times n$ matrix $M = (B(x_i, x_j))$, and we write

$$L \cong M.$$

Let $\mathfrak{u}$ denote the unit group in $R$. Then the canonical image of $\det(M)$ in $0 \cup (\dot{F}/\mathfrak{u}^2)$ is independent of the base chosen for $L$. It is called the *discriminant* of $L$, and written $dL$. We say a $R$-lattice $L$ of rank $n$ is a *modular lattice* if $(dL)R = \mathfrak{s}(L)^n$. In particular, a modular lattice $L$ is said to be *unimodular* if $\mathfrak{s}(L) = R$. Suppose that $L = L_1 \oplus \cdots \oplus L_r$ with

$$B(L_i, L_j) = 0 \quad \text{for } 1 \leqslant i < j \leqslant r,$$

for some sublattices $L_1, ..., L_r$ of $L$. Then we say that $L$ has the *orthogonal splitting*

$$L = L_1 \perp \cdots \perp L_r.$$

If $L$ is a $\mathbb{Z}_p$-lattice for some prime $p$, $L$ has an orthogonal splitting $L = L_1 \perp \cdots \perp L_r$ in which each $L_i$ $(1 \leqslant i \leqslant r)$ is modular and $\mathfrak{s}(L_r) \subsetneq \mathfrak{s}(L_{r-1}) \cdots \subsetneq \mathfrak{s}(L_1)$. We call this splitting a *Jordan decomposition* (see 91c in [8]).

Consider a $R$-lattice $L$ in $V$. Let $W$ be some other non-zero regular quadratic space over $F$ and $K$ be a $R$-lattice in $W$. We say that $K$ is *represented* by $L$ if there is a representation $\tau : FK \to FL$ such that $\tau K \subseteq L$. If there is an isometry $\tau \in O(W, V)$ such that $\tau K \subseteq L$, we say that there is an isometry of $K$ into $L$. We say that $K$ and $L$ are *isometric*, and write

$$K \cong L,$$

if there is an isometry $\tau : FK \to FL$ such that $\tau K = L$. We define the *isometry group* of $L$ to be

$$O(L) = \{\tau \in O(V) | \tau L = L\}.$$

Then $O(L)$ is a subgroup of $O(V)$. Consider $R$-lattices $K$, $L$ on $V$. We say that $K$ and $L$ are in the same *class* if $K \cong L$. This is an equivalence relation on the set of all lattices on $V$, and we use

$$\text{cls}(L)$$

to denote the class of $L$.

Let $L$ be a $\mathbb{Z}$-lattice of rank $n$ and $M$ be the corresponding matrix. Then we may write

$$L = \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n$$

for some base $x_1, ..., x_n$ of the quadratic space $\mathbb{Q}L$. we say that $L$ is *positive definite* (or *indefinite*) if $M$ is positive definite (or indefinite, respectively). We say that $L$ is *even* if $Q(x) \in 2\mathbb{Z}$ for all $x \in L$, and say that $L$ is *odd* otherwise. We define

$$L_p := \mathbb{Z}_p x_1 + \cdots + \mathbb{Z}_p x_n$$

for any prime $p$. Then $L_p$ is a $\mathbb{Z}_p$-lattice. We define the *genus* $\text{gen}(L)$ of the lattice $L$ on $V$ to be the set of all lattices $K$ on $V$ with the following property: for each prime $p$, there exists an isometry $\sigma_p \in O(V_p)$ such that $K_p = \sigma_p L_p$. We define the *class number* $h(L)$ as the number of non-isometric classes in $\text{gen}(L)$. It is well known that the class number of a $\mathbb{Z}$-lattice is always finite (see 103:4 in [8]).

# Chapter 3

# Watson transformations

## 3.1 Properties of Watson transformations

Let $k$ be a positive integer. For any integral $\mathbb{Z}$-lattice $L$, we define

$$\Lambda_k(L) = \{x \in L : Q(x+y) \equiv Q(y) \mod k \text{ for all } y \in L\},$$

and for any prime number $p$, we define

$$\Lambda_k(L_p) = \{x \in L_p : Q(x+y) \equiv Q(y) \mod k \text{ for all } y \in L_p\}.$$

Then it is clear that they are sublattices of $L$ and $L_p$ respectively, and $\Lambda_k(L)_p = \Lambda_k(L_p)$ for every prime number $p$. Moreover, $\Lambda_k(L)_p = L_p$ if $p$ does not divide $k$. For more properties of the operators $\Lambda_k$, see [3] and [2]. We denote by $\lambda_k(L)$ the primitive lattice obtained by scaling the quadratic map on $\Lambda_k(L)$ suitably. The mappings $\lambda_k$ are called Watson transformations.

Now we describe $\Lambda_k(L)$ when $m$ is a prime or 4, although only the case when $m$ is 2 or 4 will be used in this paper. For any prime number $p$, we write $L_p = M_p \perp U_p$, where $M_p$ is the leading Jordan component and $\mathfrak{s}(U_p) \subseteq p\mathfrak{s}(M_p)$.

**Lemma 3.1.1.** *Suppose that $M_p$ is even unimodular and $\mathfrak{n}(U_p) \subseteq 2p\mathbb{Z}_p$. Then*

$$\Lambda_{ep}(L)_p = pM_p \perp U_p,$$

*where $e = 2$ if $p = 2$, and $1$ otherwise. In particular,*

(a) *if* $\mathrm{ord}_p(dL) \geqslant 2$, *then* $\mathrm{ord}(d\lambda_{ep}(L)) < \mathrm{ord}_p(dL)$;

(b) *if $m$ is an odd squarefree positive integer and* $\mathrm{ord}_p(dL) \leqslant 1$ *for all* $p \mid m$, *then* $\lambda_m^2(L) = L$.

*Proof.* See [1].

**Lemma 3.1.2.** *Suppose that $L$ is a ternary integral lattice with* $\mathfrak{n}(L_2) = \mathbb{Z}_2$.

(a) *If* $\mathrm{rank}(M_2) = 2$ *and* $\mathfrak{s}(U_2) \subseteq 2\mathbb{Z}_2$, *then*

$$\lambda_2(L)_2 \cong \begin{cases} M_2 \perp U_2^{\frac{1}{2}} & \text{if } dM \equiv 1 \bmod 8; \\ M_2^3 \perp U_2^{\frac{1}{2}} & \text{if } dM \equiv 5 \bmod 8; \\ \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp U_2^{\frac{1}{2}} & \text{if } dM \equiv 3 \bmod 8; \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp U_2^{\frac{1}{2}} & \text{if } dM \equiv 7 \bmod 8. \end{cases}$$

(b) *If* $\mathrm{rank}(M_2) = 1$ *and* $\mathfrak{s}(U_2) \subseteq 4\mathbb{Z}_2$, *then* $\lambda_2(L)_2 \cong M_2 \perp U_2^{\frac{1}{4}}$.

(c) *If* $\mathrm{rank}(M_2) = 1$ *and* $\mathfrak{s}(U_2) = 2\mathbb{Z}_2$, *then* $\lambda_2(L)_2 \cong M_2^2 \perp U_2^{\frac{1}{2}}$.

*In particular, if* $\mathrm{ord}_2(dL) \geqslant 2$, *then* $\mathrm{ord}_2(d\lambda_2(L)) < \mathrm{ord}_2(dL)$; *and if* $\mathrm{ord}_2(dL) = 1$, *then* $\lambda_2(L)_2$ *is unimodular.*

*Proof.* See [1].

**Definition 3.1.3.** A primitive ternary lattice $K$ is called *stable* if $\mathrm{ord}_p(dK) \leqslant 1$ for all primes $p$, and $\mathrm{ord}_2(dK) = 1$ if and only if $K$ is even.

**Corollary 3.1.4.** *A primitive ternary lattice $L$ can be transformed, via a finite sequence of Watson transformations at the primes dividing $dL$ or at $4$, to a stable lattice.*

*Proof.* See [1].

**Lemma 3.1.5.** *Let $L$ be an integral lattice on a quadratic space $V$ and $k$ be a fixed prime number or 4. Then*

(1) $\sigma \circ \Lambda_k(L) = \Lambda_k \circ \sigma(L)$ *for every* $\sigma \in O(V)$ *in particular, the group* $O(L)$ *is a subgroup of* $O(\Lambda_k(L))$.

(2) $\lambda_k$ *induces a surjective function from* $\mathrm{gen}(L)/\sim$ *onto* $\mathrm{gen}(\lambda_k(L))/\sim$.

*Proof.* See [1]. The case when $k = 4$ may be shown by a similar way.

## 3.2 Definition of $\gamma_k^L(K)$

Let $L$ be an integral lattice and $k$ be a prime number or 4. For any lattice $K$ in $\mathrm{gen}(\lambda_k(L))$, we define

$$\gamma_k^L(K) := \{M \in \mathrm{gen}(L) \mid \lambda_k(M) = K\} \quad \text{and}$$

$$\gamma_k^L(K)/\sim := \{[M] \in \mathrm{gen}(L)/\sim \mid \lambda_k(M) = K\}.$$

Clearly,

$$h(L) = \sum_{[K]\in \mathrm{gen}(\lambda_k(L))} |\gamma_k^L(K)/\sim|.$$

**Proposition 3.2.1.** *Let $k$ be any prime number or 4. For any $K \in \mathrm{gen}(\Lambda_k(L))$,*

$$|\gamma_k^L(K)| = \frac{\mathfrak{w}(L)}{\mathfrak{w}(\lambda_k(L))},$$

*where $\mathfrak{w}(L)$ and $\mathfrak{w}(\lambda_k(L))$ are the mass of* $\mathrm{gen}(L)$ *and* $\mathrm{gen}(\lambda_k(L))$ *respectively.*

*Proof.* See [1]. The case when $k = 4$ follows a similar procedure.

Using Minkowski-Siegel mass formula, we have

$$\frac{\mathfrak{w}(L)}{\mathfrak{w}(\lambda_k(L))} = \left(\frac{dL}{d(\lambda_k(L))}\right)^2 \frac{\alpha_p(\lambda_k(L_p), \lambda_k(L)_p)}{\alpha_p(L_p, L_p)}$$

where $\alpha_p(,)$ are the local densities and $p = k$ if $k$ is odd prime, $p = 2$ otherwise. These local densities can be computed by Theorem 5.6.3 of [13]. The values of $\frac{\mathfrak{w}(L)}{\mathfrak{w}(\lambda_{2e}(L))}$ are displayed in Table 3.1, where $e = 2$ if $L_2$ is even, $e = 1$ otherwise. In this table, $L_2$ is arranged by the Jordan decomposition and $\epsilon_i \in \mathbb{Z}_2^\times$ $(i = 1, 2, 3)$. The letter $T$ denotes an even unimodular lattice of rank 2 and the values $e_{ij}$ are equal to 1 if $\epsilon_i \equiv \epsilon_j \pmod 4$, and 0 otherwise. For any integral lattice $K$, $\chi(K_2) = 1$ if $K_2$ is isotropic, and $-1$ otherwise, and the quantity $\tau$ is defined as follows;

$$\tau = \begin{cases} 1 & \text{if } \epsilon_1\epsilon_2 \equiv 7 \bmod 8, \\ -1 & \text{if } \epsilon_1\epsilon_2 \equiv 3 \bmod 8. \end{cases}$$

Table 3.1: Values of $\frac{\mathfrak{w}(L)}{\mathfrak{w}(\lambda_{2e}(L))}$

| $L_2$ | $m$ | $\frac{\mathfrak{w}(L)}{\mathfrak{w}(\lambda_{2e}(L))}$ | $L_2$ | $m, \alpha, \beta$ | $\frac{\mathfrak{w}(L)}{\mathfrak{w}(\lambda_{2e}(L))}$ |
|---|---|---|---|---|---|
| $T \perp \langle 2^m\epsilon\rangle$ | $m = 0,1,2$ | 1 | $\langle \epsilon_1, 4\epsilon_2, 2^m\epsilon_3\rangle$ | $m = 2$ | $2^{1-\epsilon_{23}} \frac{3}{2+\chi(L_2)}$ |
| | $m = 3$ | $2 + \chi(T)$ | | $m = 3$ | 2 |
| | $m \geqslant 4$ | 4 | | $m \geqslant 4$ | $2^{1+\epsilon_{12}}$ |
| $\langle\epsilon\rangle \perp 2^m T$ | $m = 0, 1, 2$ | 1 | $\langle \epsilon_1, 2^\alpha\epsilon_2, 2^\beta\epsilon_3\rangle$ | $\alpha = 3,\ \beta = 3$ | $\frac{2}{1+\epsilon_{23}}$ |
| | $m = 3$ | $2 + \chi(T)$ | | $\alpha = 4,\ \beta = 4$ | 2 |
| | $m \geqslant 4$ | 4 | | $\alpha \geqslant 5,\ \beta \geqslant 5$ | 4 |
| $\langle \epsilon_1, \epsilon_2, 2^m\epsilon_3\rangle$ | $m = 1$ | $\frac{3}{2+\tau}$ | | $\alpha = 3,\ \beta = 4$ | 1 |
| $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$ | $m = 2$ | 3 | | $\alpha = 4,\ \beta = 5$ | 2 |
| | $m \geqslant 3$ | $2(2-\tau)$ | | $\alpha \geqslant 5,\ \beta = \alpha + 1$ | 4 |
| $\langle \epsilon_1, \epsilon_2, 2^m\epsilon_3\rangle$ | $m = 1$ | $\frac{3}{2+\chi(L_2)}$ | | $\alpha = 3{,}4,\ \beta = \alpha + 2$ | 2 |
| $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | $m = 2{,}3$ | 1 | | $\alpha \geqslant 5,\ \beta = \alpha + 2$ | 4 |
| | $m \geqslant 4$ | 2 | | $\alpha = 3,\ \beta \geqslant 6$ | 2 |
| $\langle \epsilon_1, 2\epsilon_2, 2^m\epsilon_3\rangle$ | $m = 1{,}3{,}4$ | 1 | | $\alpha = 4,\ \beta \geqslant 7$ | 2 |
| | $m = 2,\ m \geqslant 5$ | 2 | | $\alpha \geqslant 5,\ \beta \geqslant \alpha + 3$ | 4 |

# Chapter 4

# Labels of classes

## 4.1 Isometry groups

In this section, we will classify ternary quadratic forms into several types by considering their isometry groups in order to define the ‘label’ of an isometry class later. Throughout this section, $K$ is a primitive ternary lattice and we let $S(K)$ be the set of symmetries of $K$. Given a non-zero vector $x$ in the quadratic space underlying $K$, $\tau_x$ denotes the associated symmetry of the space.

By a result of Minkowski [9], $|O(K)|$ is a divisor of 48. More precisely, it is one of the numbers $2, 4, 8, 12, 16, 24, 48$. Now we describe $K$ in each case.

**Lemma 4.1.1.** *Let $K$ be a primitive ternary lattice whose isometry group is of order 48. Then $K \cong \mathbf{I}$, $\mathbf{A}$, or $\mathbf{J}$, where*

$$\mathbf{I} \cong \langle 1,1,1\rangle, \quad \mathbf{A} \cong \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \quad \textit{and} \quad \mathbf{J} \cong \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$

*The isometry groups of the above three lattices are isomorphic and they are generated by $-I$ and symmetries. If $\{x_1, x_2, x_3\}$ is the basis which yields any one of the above Gram matrices, then*

$$S(\mathbf{I}) = \{\tau_{x_1}, \tau_{x_2}, \tau_{x_3}\} \cup \{\tau_{x_i \pm x_j} \colon 1 \leqslant i < j \leqslant 3\},$$

$$S(\mathbf{A}) = \{\tau_{x_1}, \tau_{x_2}, \tau_{x_3}, \tau_{x_1-x_2}\tau_{x_2-x_3}\tau_{x_1-x_2+x_3}\} \cup \{\tau_{x_1-x_3}, \tau_{x_1+x_3}, \tau_{x_1-2x_2+x_3}\},$$

*and*

$$S(\mathbf{J}) = \{\tau_{x_i+x_j} \colon 1 \leqslant i < j \leqslant 3\}$$
$$\cup\{\tau_{x_1-x_2}, \tau_{x_1-x_3}, \tau_{x_2-x_3}, \tau_{x_1+2x_2+x_3}, \tau_{x_1+x_2+2x_3}, \tau_{2x_1+x_2+x_3}\}.$$

*Proof.* See [1].

**Lemma 4.1.2.** *Let $K$ be a primitive ternary lattice whose isometry group is of order 12. Then $K$ is isometric to a lattice of the form*

$$K_{12}(a,b) := \begin{pmatrix} 2a & -a & -a \\ -a & 2a & 0 \\ -a & 0 & b \end{pmatrix},$$

*where $a$, $b$ are relatively prime positive integers.*
*Conversely, for relatively prime positive integers $a$ and $b$, we have $|O(K_{12}(a,b))| = 12$ unless $(a,b) = (1,1)$, $(1,2)$ or $(4.3)$. The isometry group $O(K_{12}(a,b))$ is generated by $-I$ and symmetries, and excluding the above three cases, we have*

$$S(K_{12}(a,b)) = \{\tau_{x_1}, \tau_{x_2}, \tau_{x_1+x_2}\},$$

*where $\{x_1, x_2, x_3\}$ is the basis which yields the Gram matrix $K_{12}(a,b)$.*

*Proof.* See [1]. Note that $K_{12}(1,1) \cong \mathbf{I}$, $K_{12}(1,2) \cong \mathbf{A}$ and $K_{12}(4,3) \cong \mathbf{J}$.

**Lemma 4.1.3.** *Let $K$ be a primitive ternary lattice. Then the isometry group of $K$ is of order 24 if and only if it is isometric to a lattice of the form*

$$K_{24}(a,b) := \begin{pmatrix} 2a & a \\ a & 2a \end{pmatrix} \perp \langle b \rangle,$$

*where $a$, $b$ are relatively prime positive integers. The isometry group $O(K_{24}(a,b))$ is generated by $-I$ and symmetries, and we have*

$$S(K_{24}(a,b)) = \{\tau_{x_1}, \tau_{x_2}, \tau_{x_1-x_2}\} \cup \{\tau_{x_1-2x_2}, \tau_{2x_1-x_2}, \tau_{x_1+x_2}\} \cup \{\tau_{x_3}\},$$

*where* $\{x_1, x_2, x_3\}$ *is the basis which yields the Gram matrix* $K_{24}(a, b)$*.*

*Proof.* See [1].

**Lemma 4.1.4.** *Let* $K$ *be a primitive ternary lattice whose isometry group is of order* 16*. Then* $K$ *has only one of the forms*

$$K_{16,\mathrm{I}}(a,b) := \langle a, a, b\rangle \quad \text{and} \quad K_{16,\mathrm{II}}(a,b) := \begin{pmatrix} 2a & 0 & -a \\ 0 & 2a & -a \\ -a & -a & b \end{pmatrix}$$

*up to isometry, where a,b are relatively prime positive integers.*
*Conversely, for relatively prime positive integers* $a$ *and* $b$*, the isometry groups of* $K_{16,\mathrm{I}}(a,b)$ *and* $K_{16,\mathrm{II}}(a,b)$ *are of order* 16*, except for* $K_{16,\mathrm{I}}(1,1)$*,*$K_{16,\mathrm{II}}(1,2)$*, and* $K_{16,\mathrm{II}}(2,3)$*. The isometry groups* $O(K_{16,\mathrm{I}}(a,b))$ *and* $O(K_{16,\mathrm{II}}(a,b))$ *are generated by* $-I$ *and symmetries, and we have*

$$S(K_{16,\mathrm{I}}(a,b)) = \{\tau_{x_1}, \tau_{x_2}\} \cup \{\tau_{x_1+x_2}, \tau_{x_1-x_2}\} \cup \{\tau_{x_3}\}$$

*and*

$$S(K_{16,\mathrm{II}}(a,b)) = \{\tau_{x_1}, \tau_{x_2}\} \cup \{\tau_{x_1+x_2}, \tau_{x_1-x_2}\} \cup \{\tau_{x_1+x_2+2x_3}\}.$$

*Proof.* See [1]. Note that $K_{16,\mathrm{I}}(1,1) \cong \mathbf{I}$, $K_{16,\mathrm{II}}(1,2) \cong \mathbf{A}$, and $K_{16,\mathrm{II}}(2,3) \cong \mathbf{J}$.

Next, we consider primitive ternary lattices whose isometry groups are of order 8. Let $K$ be such a $\mathbb{Z}$-lattice. Then, $O(K)$ has precisely three symmetries and their corresponding primitive vectors are orthogonal each other by [1]. So let $x$, $y$, $z$ be primitive vectors in $K$ such that $S(K) = \{\tau_x, \tau_y, \tau_z\}$.

**Definition 4.1.5.** For a primitive vector $x$ in $K$ with $\tau_x \in O(K)$, if $Q(x)$ divides $B(x, y)$ for any $y \in K$, then $\tau_x$ is said to be of type 1, and type 2 otherwise.

First, without loss of generality, we assume that $\tau_x$ is of type 1. Then $x$ splits $K$ and two vectors $y$, $z$ are contained in the orthogonal component of $x$ in $K$. If one of $\tau_y$ and $\tau_z$ is of type 1, the Gram matrix of K is represented as

$\langle a, b, c\rangle$ for some pairwise relatively prime positive integers $a$, $b$, $c$. Otherwise it is represented as

$$\langle a\rangle \perp \begin{pmatrix} 2b & b \\ b & c \end{pmatrix}$$

for some relatively prime positive integers $a$, $b$, $c$. Next, assume that each symmetry in $O(K)$ is of type 2. Then we can take two vectors $u$, $v$ in $K$ such that $\{x, u, v\}$ is a basis of $K$ with the corresponding Gram matrix

$$\begin{pmatrix} 2a & a & 0 \\ a & b & c \\ 0 & c & d \end{pmatrix}.$$

We denote the orthogonal component of $x$ in $K$ by $x^{\perp}$ and let $\widetilde{K} := \mathbb{Z}x \perp x^{\perp}$. Then, $\widetilde{K} = \mathbb{Z}x + \mathbb{Z}(x - 2u) + \mathbb{Z}v$ and for any $w \in x^{\perp}$, $B(\tau_y(w), x) = B(\tau_y(w), \tau_y(x)) = B(w, x) = 0$. This implies that $\tau_y(w) \in x^{\perp}$ for any $w \in x^{\perp}$ and so $\tau_y(x^{\perp}) = x^{\perp}$. Therefore $\tau_y$ is also an isometry of $\widetilde{K}$ and we can represent $\widetilde{K}$ as

$$\widetilde{K} = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z' \cong \langle 2a, b', c'\rangle \quad \text{or} \quad \langle 2a\rangle \perp \begin{pmatrix} 2b' & b' \\ b' & c' \end{pmatrix}$$

for some positive integers $b'$ and $c'$ where $Q(y) = b'$ or $2b'$ respectively. It is clear that $[\widetilde{K} : 2K] = 4$ and there exists a basis $\{x_1, x_2, x_3\}$ of $\widetilde{K}$ such that $\mathbb{Z}x_1 + \mathbb{Z}(2x_2) + \mathbb{Z}(2x_3) = 2K$. Then, all candidates of a lattice $2K$ are the following lattices ;

$$\begin{array}{ll} \mathbb{Z}(2x) + \mathbb{Z}(2y) + \mathbb{Z}z', & \mathbb{Z}(2x) + \mathbb{Z}(y) + \mathbb{Z}(2z'), \\ \mathbb{Z}(2x) + \mathbb{Z}(y + z') + \mathbb{Z}(2z'), & \mathbb{Z}x + \mathbb{Z}(2y) + \mathbb{Z}(2z'), \\ \mathbb{Z}(x + y) + \mathbb{Z}(2y) + \mathbb{Z}(2z'), & \mathbb{Z}(x + z') + \mathbb{Z}(2y) + \mathbb{Z}(2z'), \\ \mathbb{Z}(x + y + z') + \mathbb{Z}(2y) + \mathbb{Z}2z'. & \end{array}$$

Suppose that $\widetilde{K} \cong \langle 2a, b', c'\rangle$. Considering their Gram matrices, we may conclude that only the last one is possible and then

$$K \cong \begin{pmatrix} b' & 0 & \frac{b'}{2} \\ 0 & c' & \frac{c'}{2} \\ \frac{b'}{2} & \frac{c'}{2} & \frac{2a+b'+c'}{4} \end{pmatrix}.$$

Next, suppose that $\widetilde{K} \cong \langle 2a \rangle \perp \begin{pmatrix} 2b' & b' \\ b' & c' \end{pmatrix}$. Then, only the fifth lattice in the above list is possible since the last two lattices don't have $\tau_y$ as their isometry and we have

$$K \cong \begin{pmatrix} 2b' & b' & b' \\ b' & \frac{a+b'}{2} & \frac{b'}{2} \\ b' & \frac{b'}{2} & c' \end{pmatrix}.$$

Therefore we obtain the following result.

**Lemma 4.1.6.** *Let $K$ be a primitive ternary lattice whose isometry group is of order 8. Then $K$ is isometric to only one of the following forms;*

$$K_{8,\mathrm{I}}(a,b,c) := \langle a,b,c \rangle, \quad K_{8,\mathrm{II}}(a,b,c) := \langle a \rangle \perp \begin{pmatrix} 2b & b \\ b & c \end{pmatrix},$$

$$K_{8,\mathrm{III}}(a,b,c) := \begin{pmatrix} 2a & 0 & a \\ 0 & 2b & b \\ a & b & c \end{pmatrix}, \quad K_{8,\mathrm{IV}}(a,b,c) := \begin{pmatrix} 4a & 2a & 2a \\ 2a & a+b & a \\ 2a & a & c \end{pmatrix}.$$

*Proof.* We have only to show that $K$ is not isometric to two of the above forms simultaneously. The sets of all symmetries of above forms are $\{\tau_x, \tau_y, \tau_z\}$, $\{\tau_x, \tau_y, \tau_{y-2z}\}$, $\{\tau_x, \tau_y, \tau_{x+y-2z}\}$, $\{\tau_x, \tau_{x-2y}, \tau_{x-2z}\}$, respectively. For any lattice $L$, let denote the product of all primitive vectors associated to each symmetry of $L$ by $N(L)$. Then we obtain

$$\begin{aligned} \mathrm{ord}_2(d(K_{8,\mathrm{I}}(a,b,c))) &= \mathrm{ord}_2(N(K_{8,\mathrm{I}}(a,b,c))), \\ \mathrm{ord}_2(d(K_{8,\mathrm{II}}(a,b,c))) &= 4\mathrm{ord}_2(N(K_{8,\mathrm{II}}(a,b,c))), \\ \mathrm{ord}_2(d(K_{8,\mathrm{III}}(a,b,c))) &= 4\mathrm{ord}_2(N(K_{8,\mathrm{III}}(a,b,c))), \\ \mathrm{ord}_2(d(K_{8,\mathrm{IV}}(a,b,c))) &= 16\mathrm{ord}_2(N(K_{8,\mathrm{IV}}(a,b,c))). \end{aligned}$$

Hence we have distinguished all pairs of structures except one pair, $K_{8,\mathrm{II}}(a,b,c)$

and $K_{8,\mathrm{III}}(a,b,c)$. Since all symmetries of $K_{8,\mathrm{III}}(a,b,c)$ are of type 2, we are done. □

Finally, we discuss about the primitive ternary lattices whose isometry groups are of order 4. Let $K$ be a such lattice. Then it is clear that there exists only one symmetry in $O(K)$. Therefore we can obtain the following result.

**Lemma 4.1.7.** *Let $K$ be a primitive ternary lattice whose isometry group is of order 4. Then, $K$ is isometric to only one of the following forms;*

$$K_{4,\mathrm{I}}(a,b,c,d) := \langle a\rangle \perp \begin{pmatrix} b & c \\ c & d \end{pmatrix}, \quad K_{4,\mathrm{II}}(a,b,c,d) := \begin{pmatrix} 2a & a & 0 \\ a & b & c \\ 0 & c & d \end{pmatrix}.$$

*Proof.* The first form occurs when the symmetry is of type 1, and otherwise the second form occurs. □

## 4.2 Definition of labels

Let $K$ be a ternary lattice. Every symmetry $\sigma$ in $O(K)$ is of the form $\tau_x$, where $x$ is a primitive vector of $K$. We define $Q_K(\sigma)$ to be $Q(x)$.

**Definition 4.2.1.** Let $K$ be a ternary lattice, and $\sigma_1$, ..., $\sigma_t$ be all symmetries of $K$. If $|O(K)|$=2, 12, 24 or 48, the *label* of $K$ is defined as

$$\mathrm{label}(K) := [\![|O(K)|; Q_K(\sigma_1), ..., Q_K(\sigma_t)]\!].$$

If $|O(K)| = 4$, 8 or 16, the *label* of $K$ is defined as

$$\mathrm{label}(K) := [\![|O(K)|, \text{'the type of } K\text{'}; Q_K(\sigma_1), ..., Q_K(\sigma_t)]\!],$$

where the type of $K$ is a Roman numeral which denotes the type of the form described in the above chapter. If $|O(K)| = 24$, or $|O(K)| = 8$ and $K$ is of type II, we add a dot on top of the norm of the symmetry of type 1.

For example, if $K \cong K_{8,\mathrm{II}}(a,b,c)$, we denote the label of $K$ by

$$\mathrm{label}(K) := [\![8, \mathrm{II}; \dot{a}, 2b, -2b+4c]\!].$$

In the cases dealt in [1], we do not need the information of 'the type' of $K$ and 'a dot' on a type 1 symmetry. But we need this information in our cases.

Throughout this section, we always assume that $L$ is a primitive ternary $\mathbb{Z}$-lattice such that $\mathrm{ord}_2(dL) \geqslant 2$, or $\mathrm{ord}_2(dL) = 1$ and $L$ is odd, and $K := \lambda_{2e}(L)$, where $e = 1$ if $L$ is odd, $e = 2$ otherwise. Hence $L$ is not stable over $\mathbb{Z}_2$. The above assumption is the general setting of Step (II) in our method provided in Chapter 1. Let $\mathrm{label}(\gamma_{2e}^L(K))$ denote the *multiset* of the labels of all classes in $\gamma_{2e}^L(K)/\sim$. In order to prove the possibility of Step (II), we claim as follows.

*Claim* : *One may determine the multiset* $\mathrm{label}(\gamma_{2e}^L(K))$ *by the label and discriminant of* $K$*, and the structure of* $L_2$.

If the above claim is true in all possible cases, then we may execute Step (II) successfully by the surjective map in Lemma 3.1.5 (2). Although there are some exceptional cases in which the above claim does not hold, it can be enough to recover to achieve our purpose. All exceptional cases are contained in the case when $|O(K)| = 4$. Now we introduce two typical exceptional cases of the above claim. First, let

$$L := \langle 1 \rangle \perp \begin{pmatrix} 6 & 8 \\ 8 & 40 \end{pmatrix}, \quad L' := \begin{pmatrix} 4 & 2 & 0 \\ 2 & 7 & 4 \\ 0 & 4 & 10 \end{pmatrix},$$

$$K := \langle 2 \rangle \perp \begin{pmatrix} 3 & 4 \\ 4 & 20 \end{pmatrix}, \quad \text{and} \quad K' := \langle 2 \rangle \perp \begin{pmatrix} 5 & 4 \\ 4 & 12 \end{pmatrix}.$$

Then we have $\lambda_2(L) \cong K$ and $\lambda_2(L') \cong K'$. Furthermore, we may verify that $\mathrm{label}(K) = \mathrm{label}(K') = [\![4, \mathrm{I}; 2]\!]$, $dK = dK'$ and $L_2 \cong L'_2$. But

$$\mathrm{label}(\gamma_2^L(K)) = \left\{[\![4, \mathrm{I}; 1]\!]\right\} \neq \left\{[\![4, \mathrm{II}; 4]\!]\right\} = \mathrm{label}(\gamma_2^{L'}(K'))$$

since $|\gamma_2^L(K)| = |\gamma_2^{L'}(K')| = 1$ by Table 3.1. Therefore we can not determine the multiset label$(\gamma_2^L(K))$ by the given information in this case. In such a case, we may determine label$(\gamma_2^L(K))$ by using label$(\lambda_2(K))$. See Subcase(4.7.1.19-2) below for details.

Next, we put

$$L := \langle 4\rangle \perp \begin{pmatrix} 4 & 1 \\ 1 & 6 \end{pmatrix}, \quad L' := \begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 7 \\ 0 & 7 & 24 \end{pmatrix},$$

$$K := \langle 1\rangle \perp \begin{pmatrix} 4 & 1 \\ 1 & 6 \end{pmatrix}, \quad \text{and} \quad K' := \langle 1\rangle \perp \begin{pmatrix} 3 & 4 \\ 4 & 13 \end{pmatrix}.$$

Then we have $\lambda_2(L) \cong K$ and $\lambda_2(L') \cong K'$. Furthermore, we may verify that label$(K) =$ label$(K') = [\![4, \mathrm{I}; 1]\!]$, $dK = dK'$ and $L_2 \cong L'_2$. But

$$\text{label}(\gamma_4^L(K)) = \left\{[\![4, \mathrm{I}; 4]\!]\right\} \neq \left\{[\![4, \mathrm{II}; 4]\!]\right\} = \text{label}(\gamma_4^{L'}(K')).$$

Therefore we also can not determine the multiset label$(\gamma_2^L(K))$ by the given information in this case. Let denote $x$ be the primitive vector in $K$ corresponding to the symmetry in $O(K)$. In such a case, we have to check that the orthogonal complement of $x$ in $K$ is either even or odd. The method to obtain this information is given in Section 2 of Chapter 5. This is an example of Subcase(4.7.1.1) below.

From now on, we will check the above claim for every possible case.

## 4.3 The case when $|O(K)| = 24$

Assume that $|O(K)| = 24$. Then $K \cong K_{24}(a, b)$ for some relatively prime positive integers $a$, $b$ by Lemma 4.1.3. We define the notations

$$\mathbb{H} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbb{A} := \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then it is clear that $K_2 \cong 2^m\mathbb{A} \perp \langle\epsilon\rangle$ or $\mathbb{A} \perp \langle 2^m\epsilon\rangle$ for some nonnegative integer $m$ and some unit $\epsilon$ in $\mathbb{Z}_2$. Since $\lambda_{2e}(L_2) \cong K_2$, we may reduce the number of possible local structures of $L_2$.

**Lemma 4.3.1.** *Let $L$ be a primitive ternary lattice which is not stable over $\mathbb{Z}_2$ and let $K := \lambda_{2e}(L)$. Suppose that $K \cong K_{24}(a,b)$ for some relatively prime positive integers $a$ and $b$. Then $L_2$ is isomorphic to one of the following forms;*

$$\langle \epsilon_1, 4\epsilon_2, 4\epsilon_3\rangle, \quad \mathbb{A} \perp \langle 2^m\epsilon\rangle \ (m \geqslant 2), \quad \langle \epsilon\rangle \perp 2^m\mathbb{A} \ (m \geqslant 1),$$

$$\mathbb{H} \perp \langle 8\epsilon\rangle, \quad \langle \epsilon\rangle \perp 8\mathbb{H}, \quad \langle 1, 3, 2^m\epsilon\rangle \ (m \geqslant 1),$$

*where $m$ is a positive integer and $\epsilon, \epsilon_i$ $(i = 1, 2, 3)$ are unit elemtnts in $\mathbb{Z}_2$.*

*Proof.* By taking the Watson transformation, we may easily verify that $L_2$ is isometric to one of the following forms;

$$\begin{array}{lll} \langle \epsilon_1, 4\epsilon_2, 4\epsilon_3\rangle, & \mathbb{A} \perp \langle 2^m\epsilon\rangle \ (m \geqslant 2), & \langle \epsilon\rangle \perp 2^m\mathbb{A} \ (m \geqslant 1), \\ \mathbb{H} \perp \langle 8\epsilon\rangle, & \langle \epsilon\rangle \perp 8\mathbb{H}, & \langle 1, 3, 2^m\epsilon\rangle \ (m \geqslant 1), \\ \langle 1, 7, 4\epsilon\rangle, & \langle \epsilon_1, \epsilon_2, 2\epsilon_3\rangle \ ( \ \epsilon_1 \equiv \epsilon_2 \pmod 4)). & \end{array}$$

By changing basis, we may show that the last two forms are included in the fom $\langle 1, 3, 2^m\epsilon\rangle$, and so we proved the assertion. $\square$

Let $M$ be a primitive $\mathbb{Z}$-lattice and let $N$ be a sublattice of $M$ which is primitive. If there is a basis $\{x_1, x_2, x_3\}$ for $M$ such that $\{x_1, 2^s x_2, 2^t x_3\}$ forms a basis of $N$ for some nonnegative integers $s$ and $t$, we write

$$M \underset{(1,2^s,2^t)}{\supset} N.$$

Suppose that the leading Jordan component of $L_2$ is odd. Then, there is a basis $\{x_1, x_2, x_3\}$ for $L$ such that $\{x_1, x_2, 2x_3\}$ forms a basis of $\Lambda_2(L)$, and then

$$L \underset{(1,1,2)}{\supset} \Lambda_2(L) \underset{(1,2,2)}{\supset} 2L.$$

Let $\Lambda_2(L)^s = \lambda_2(L)$, where $s = \frac{1}{2}$ or $\frac{1}{4}$. Then for any $L' \in \gamma_2^L(K)$,

$$K = \lambda_2(L') = \Lambda_2(L')^s \underset{(1,2,2)}{\supset} (L')^{4s}. \tag{4.3.1}$$

By the assumption, $K \cong K_{24}(a,b)$ and let $\{x,\ y,\ z\}$ be the corresponding basis of $K$. Then all candidates of the lattice $(L')^{4s}$ are the following seven lattices;

$$\begin{aligned}
&L_{\mathrm{I}} := \mathbb{Z}(2x) + \mathbb{Z}(2y) + \mathbb{Z}z, && L_{\mathrm{II}} := \mathbb{Z}(2x) + \mathbb{Z}y + \mathbb{Z}(2z),\\
&L_{\mathrm{III}} := \mathbb{Z}x + \mathbb{Z}(2y) + \mathbb{Z}(2z), && L_{\mathrm{IV}} := \mathbb{Z}(x+y) + \mathbb{Z}(2y) + \mathbb{Z}(2z),\\
&L_{\mathrm{V}} := \mathbb{Z}(2x) + \mathbb{Z}(y+z) + \mathbb{Z}(2z), && L_{\mathrm{VI}} := \mathbb{Z}(x+z) + \mathbb{Z}(2y) + \mathbb{Z}(2z),\\
&L_{\mathrm{VII}} := \mathbb{Z}(x+y+z) + \mathbb{Z}(2y) + \mathbb{Z}(2z).
\end{aligned} \tag{4.3.2}$$

We may easily show that $L_{\mathrm{II}}$, $L_{\mathrm{III}}$ and $L_{\mathrm{IV}}$ are isometric to each other and so are $L_{\mathrm{V}}$, $L_{\mathrm{VI}}$ and $L_{\mathrm{VII}}$. The Gram matrices of them are as follows;

$$L_{\mathrm{I}} \cong \begin{pmatrix} 8a & 4a \\ 4a & 8a \end{pmatrix} \perp \langle b \rangle, \quad L_{\mathrm{II}} = \tau_{x-y}(L_{\mathrm{III}}) = \tau_{x-2y}(L_{\mathrm{IV}}) \cong \langle 2a, 6a, 4b \rangle,$$

$$L_{\mathrm{V}} = \tau_{x-y}(L_{\mathrm{VI}}) = \tau_{x-2y}(L_{\mathrm{VII}}) \cong \begin{pmatrix} 4b & 2b & 2b \\ 2b & 6a+b & b \\ 2b & b & 2a+b \end{pmatrix}.$$

Next, suppose that the leading Jordan component of $L_2$ is even. Then by a similar argument as above, we obtain

$$L \underset{(1,2,2)}{\supset} \Lambda_4(L) \underset{(1,1,2)}{\supset} 2L,$$

and hence, for any $L' \in \gamma_4^L(K)$,

$$K = \lambda_4(L') = \Lambda_4(L')^{\frac{1}{4}} \underset{(1,1,2)}{\supset} L' \tag{4.3.3}$$

since $L$ is not stable over $\mathbb{Z}_2$ . In this case, we obtain all candidates of the lattice $L'$ as following seven lattices;

$$\begin{aligned}
&L'_{\mathrm{I}} := \mathbb{Z}(2x) + \mathbb{Z}y + \mathbb{Z}z, && L'_{\mathrm{II}} := \mathbb{Z}x + \mathbb{Z}(2y) + \mathbb{Z}z,\\
&L'_{\mathrm{III}} := \mathbb{Z}(x+y) + \mathbb{Z}(2y) + \mathbb{Z}z, && L'_{\mathrm{IV}} := \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}(2z),\\
&L'_{\mathrm{V}} := \mathbb{Z}(x+z) + \mathbb{Z}y + \mathbb{Z}(2z), && L'_{\mathrm{VI}} := \mathbb{Z}x + \mathbb{Z}(y+z) + \mathbb{Z}(2z),\\
&L'_{\mathrm{VII}} := \mathbb{Z}(x+z) + \mathbb{Z}(y+z) + \mathbb{Z}(2z).
\end{aligned} \tag{4.3.4}$$

Furthermore we may obtain

$$L'_{\mathrm{I}} = \tau_{x-y}(L'_{\mathrm{II}}) = \tau_{x-2y}(L'_{\mathrm{III}}) \cong \langle 2a, 6a, b\rangle, \quad L'_{\mathrm{IV}} \cong \begin{pmatrix} 2a & a \\ a & 2a \end{pmatrix} \perp \langle 4b\rangle,$$

$$L'_{\mathrm{V}} = \tau_{x-y}(L'_{\mathrm{VI}}) = \tau_{x}(L'_{\mathrm{VII}}) \cong \begin{pmatrix} 2a & 0 & a \\ 0 & 4b & 2b \\ a & 2b & 2a+b \end{pmatrix}.$$

Now, we will show the above claim in this case.

**Theorem 4.3.2.** *Let $L$ be a primitive ternary lattice which is not stable over $\mathbb{Z}_2$. Let $K := \lambda_{2e}(L)$ and suppose that $|O(K)| = 24$. Then the multiset* $\mathrm{label}(\gamma_{2e}^L(K))$ *is completely determined by the label of $K$ and the structure of $L_2$.*

*Proof.* We note that $\mathrm{label}(K) = [\![24; 2a, 2a, 2a, 6a, 6a, 6a, b]\!]$ for some relatively prime positive integers $a$ and $b$. Hence it is possible to determine the values $a$ and $b$ from the label of $K$ and hence it decides the class $[K]$ itself. By Lemma 4.3.1, we only have to check the assertion for the reduced structures for $L_2$.

**Case(4.3.2.1)** $L_2 \cong \mathbb{A} \perp \langle 2^m\epsilon\rangle$ $(m \geqslant 2)$.
**Subcase(4.3.2.1-1)** $m = 2$. In this case, we obtain that $|\gamma_4^L(K)| = 1$ from Table 3.1, and $K_2 = \lambda_4(L_2) \cong \mathbb{A} \perp \langle \epsilon\rangle$. Since $K = K_{24}(a, b)$, two integers $a$ and $b$ are odd. By the relation (4.3.3), we obtain

$$K = \lambda_4(L) \underset{(1,1,2)}{\supset} L. \tag{4.3.5}$$

Now we have to find all elements of $\gamma_4^L(K)$ among the seven candidates $L'_{\mathrm{I}}$,..., $L'_{\mathrm{VII}}$ in (4.3.4) by considering their local structures. In this case, only possible candidate is $L'_{\mathrm{IV}}$, for $L_2$ is an even unimodular lattice. By Lemma 4.1.3,

$$\mathrm{label}(\gamma_4^L(K)) = \left\{[\![24; 2a, 2a, 2a, 6a, 6a, 6a, \dot{4b}]\!]\right\}.$$

**Subcase(4.3.2.1-2)** $m = 3$. In this case, we have $|\gamma_4^L(K)| = 1$ and $K_2 = \lambda_4(L_2) \cong \mathbb{A} \perp \langle 2\epsilon\rangle$. Since $K = K_{24}(a, b)$, the integer $a$ is odd and $b \equiv 2$

(mod 4), and we get the same relation (4.3.5) as Subcase(4.3.2.1-1). The lattice $(L'_{\mathrm{I}})_2$ is not isometric to $L_2$ since it is an odd lattice. We may also obtain that $(L'_{\mathrm{IV}})_2 \cong \mathbb{H} \perp \langle 8\epsilon \rangle$ and it is not isometric to $L_2$. Hence we obtain the same result as Subcase(4.3.2.1-1).

**Subcase(4.3.2.1-3)** $m \geqslant 4$. In this case, we have $|\gamma_4^L(K)| = 4$ and $K_2 = \lambda_4(L_2) \cong \mathbb{A} \perp \langle 2^{m-2}\epsilon \rangle$. Then the integer $a$ is odd and $\mathrm{ord}_2(b) \geqslant 2$, and we get the relation (4.3.5) again. Computing each local structure, we may show that $(L'_{\mathrm{I}})_2$, $(L'_{\mathrm{II}})_2$ and $(L'_{\mathrm{III}})_2$ are not isometric to $L_2$, and the others are all possible. Since the number of the possible cases coincides with the order of $\gamma_4^L(K)$, all elements in $\gamma_4^L(K)$ are founded. Since $\lambda_4(L'_{\mathrm{V}}) = K$, the order $|O(L'_{\mathrm{V}})|$ divides $|O(K)|$. Then we may easily verify that the isometry group of $L'_{\mathrm{V}}$ is of order 8 and the type of its lavel is III by observing the symmetries, structure of $L'_{\mathrm{V}}$ and Lemma 4.1.6. Therefore

$$\mathrm{label}(\gamma_4^L(K)) = \left\{ [\![8, \mathrm{III}; 2a, 6a, 4b]\!], [\![24; 2a, 2a, 2a, 6a, 6a, 6a, \dot{4b}]\!] \right\}.$$

**Case(4.3.2.2)** $L_2 \cong \langle \epsilon \rangle \perp 2^m\mathbb{A}$ $(m \geqslant 1)$.

**Subcase(4.3.2.2-1)** $m = 1$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle 2\epsilon \rangle \perp \mathbb{A}$. Then the integer $a$ is odd and $b \equiv 2 \pmod 4$. By (4.3.1), we obtain the relation

$$K = \lambda_2(L) \underset{(1,2,2)}{\supset} L^2. \tag{4.3.6}$$

Since $(L_{\mathrm{II}})_2$ has no even modular component and $(L_{\mathrm{V}})_2$ has a 2-modular component of rank 2, they are not isometric to $(L_2)^2$. Now there is an only possible candidate $L_{\mathrm{I}}$ and thus, $L = (L_{\mathrm{I}})^{\frac{1}{2}}$. Therefore

$$\mathrm{label}(\gamma_2^L(K)) = \left\{ [\![24; 4a, 4a, 4a, 12a, 12a, 12a, \frac{\dot{b}}{2}]\!] \right\}$$

by Lemma 4.1.3.

**Subcase(4.3.2.2-2)** $m = 2$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon \rangle \perp \mathbb{A}$. Then the integers $a$, $b$ are odd and we obtain the relation

$$K = \lambda_2(L) \underset{(1,2,2)}{\supset} L \tag{4.3.7}$$

by (4.3.1). Clearly $(L_{\mathrm{II}})_2$ is not isometric to $L_2$. Suppose that $(L_{\mathrm{V}})_2$ is isometric to $L_2$. Since $2a+b$ is odd, we may write $(L_{\mathrm{V}})_2 \cong \langle 2a+b\rangle \perp K'$ for some binary $\mathbb{Z}_2$-lattice $K'$. Since $L_2 \cong \langle \epsilon \rangle \perp 4\mathbb{A}$, we have $d((K')_2) = d(4\mathbb{A}) = 48$. But $d((K')_2) = 48a^2b(2a+b) \neq 48$ and this is a contradiction. Therefore only possible one is $L_{\mathrm{I}}$ and we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \left\{[\![24; 8a, 8a, 8a, 24a, 24a, 24a, \dot{b}]\!]\right\}.$$

**Subcase(4.3.2.2-3)** $m = 3$. In this case, we may obtain the same result as Subcase(4.3.2.2-2) by the same procedure.
**Subcase(4.3.2.2-4)** $m \geqslant 4$. In this case, we have $|\gamma_2^L(K)| = 4$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon \rangle \perp 2^{m-2}\mathbb{A}$. Then $\mathrm{ord}_2(a) \geqslant 2$ and $b$ is odd, and we get the relation (4.3.7). It is clear that $(L_{\mathrm{II}})_2$, $(L_{\mathrm{III}})_2$ and $(L_{\mathrm{IV}})_2$ are not isometric to $L_2$, and the other candidates are all possible by their local structures. In addition, we may verify that the order of $O(L_{\mathrm{V}})$ is 8 and the type of the label for $L_{\mathrm{V}}$ is IV. Therefore

$$\mathrm{label}(\gamma_2^L(K)) = \left\{[\![8, \mathrm{IV}; 8a, 24a, 4b]\!], [\![24; 8a, 8a, 8a, 24a, 24a, 24a, \dot{b}]\!]\right\}.$$

**Case(4.3.2.3)** $L_2 \cong \langle 1, 3, 2^m\epsilon \rangle$ $(m \geqslant 1)$.
**Subcase(4.3.2.3-1)** $m = 1$. In this case, we have $|\gamma_2^L(K)| = 3$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon \rangle \perp \mathbb{A}$. Then the integers $a$ and $b$ are odd and we get the relation (4.3.6). Since $\mathfrak{s}((L_{\mathrm{I}})_2) = \mathfrak{s}((L_{\mathrm{V}})_2) = \mathbb{Z}_2$, the four candidates are excluded. The rest of three lattices are possible to be isometric to $L_2$. Therefore

$$\mathrm{label}(\gamma_2^L(K)) = \left\{[\![8, \mathrm{I}; a, 3a, 2b]\!]\right\}.$$

**Subcase(4.3.2.3-2)** $m = 2$. In this case, we obtain the same result as Subcase(4.3.2.3-1) in a similar way.
**Subcase(4.3.2.3-3)** $m \geqslant 3$. In this case, we have $|\gamma_2^L(K)| = 6$ and $K_2 = \lambda_2(L_2) \cong \mathbb{A} \perp 2^{m-1}\langle \epsilon \rangle$. Then the integer $a$ is odd and $\mathrm{ord}_2(b) \geqslant 2$, and we get the relation (4.3.6). Among the candidates, $(L_{\mathrm{I}})^{\frac{1}{2}}$ is the only impossible case and hence

$$\mathrm{label}(\gamma_2^L(K)) = \left\{[\![8, \mathrm{I}; a, 3a, 2b]\!], [\![8, \mathrm{IV}; 4a, 12a, 2b]\!]\right\}.$$

**Case(4.3.2.4)** $L_2 \cong \langle \epsilon_1, 4\epsilon_2, 4\epsilon_3 \rangle$. In this case, we have

$$|\gamma_2^L(K)| = 2^{1-\epsilon_{23}} \frac{3}{2+\chi(L_2)} \quad \text{and} \quad K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, \epsilon_2, \epsilon_3 \rangle.$$

Then the integers $a$ and $b$ are odd and we get the relation (4.3.7). Since $\lambda_2(L_2) \cong \langle b \rangle \perp \mathbb{A}$, this is anisotropic and so is $L_2$. Therefore $|\gamma_2^L(K)| = 3$. Then we may easily show that $\gamma_2^L(K) = \{L_{\mathrm{V}}, L_{\mathrm{VI}}, L_{\mathrm{VII}}\}$ and hence

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![8, \mathrm{IV}; 8a, 24a, 4b]\!]\big\}.$$

**Case(4.3.2.5)** $L_2 \cong \langle \epsilon \rangle \perp 8\mathbb{H}$. In this case, we have $|\gamma_2^L(K)| = 3$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon \rangle \perp 2\mathbb{H}$. Then we have $a \equiv 2 \pmod 4$ and the integer $b$ is odd, and we get the relation (4.3.7). By a simple calculation, we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![8, \mathrm{IV}; 8a, 24a, 4b]\!]\big\}.$$

**Case(4.3.2.6)** $L_2 \cong \mathbb{H} \perp \langle 8\epsilon \rangle$. In this case, we have $|\gamma_4^L(K)| = 3$ and $K_2 = \lambda_4(L_2) \cong \mathbb{H} \perp \langle 2\epsilon \rangle$. Then the integer $a$ is odd and $b \equiv 2 \pmod 4$, and we get the relation (4.3.5). By a simple calculation, we obtain that $\gamma_4^L(K) = \{L'_{\mathrm{V}}, L'_{\mathrm{VI}}, L'_{\mathrm{VII}}\}$ and hence

$$\mathrm{label}(\gamma_4^L(K)) = \big\{[\![8, \mathrm{III}; 2a, 6a, 4b]\!]\big\}.$$

□

## 4.4 The case when $|O(K)| = 16$

Assume that $|O(K)| = 16$. Then $K \cong K_{16,\mathrm{I}}(a,b)$ or $K_{16,\mathrm{II}}(a,b)$. for some relatively prime positive integers $a$ and $b$ by Lemma 4.1.4. First, we consider the case when $K \cong K_{16,\mathrm{I}}(a,b)$. Then we obtain the following lemma.

**Lemma 4.4.1.** *Let $L$ be a primitive ternary lattice which is not stable over $\mathbb{Z}_2$ and let $K := \lambda_{2e}(L)$. Suppose that $|O(K)| = 16$ and $K$ is of type* I. *Then*

*$L_2$ is isomorphic to one of the following forms;*

$$\mathbb{A} \perp \langle 4\epsilon\rangle, \quad \mathbb{H} \perp \langle 4\epsilon\rangle, \quad \langle \epsilon\rangle \perp 4\mathbb{A}, \quad \langle \epsilon\rangle \perp 4\mathbb{H},$$

$$\langle \epsilon_1, \epsilon_2, 2\epsilon_3\rangle \ (\epsilon_1 \not\equiv \epsilon_2 \pmod 4), \quad \langle \epsilon_1, \epsilon_2, 2^m\epsilon_3\rangle \ (m \geqslant 2,\ \epsilon_1 \equiv \epsilon_2 \pmod 4),$$

$$\langle \epsilon_1, 2\epsilon_2, 2^m\epsilon_3\rangle \ (m = 1, 2),\ \langle \epsilon_1, 4\epsilon_2, 2^m\epsilon_3\rangle \ (m \geqslant 2),\ \langle \epsilon_1, 2^m\epsilon_2, 2^m\epsilon_3\rangle \ (m \geqslant 3),$$

*where $m$ is a positive integer and $\epsilon, \epsilon_i$ $(i = 1, 2, 3)$ are units in $\mathbb{Z}_2$.*

*Proof.* Using the Watson transformations, We may easily verify that the above forms are only possible structures of $L_2$ up to isomorphism. Note that the form $\langle \epsilon_1, \epsilon_2, 2\epsilon_3\rangle$ may always be assumed that $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$. □

Let $K = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z \cong K_{16,\mathrm{I}}(a, b)$ and suppose that the leading Jordan component of $L_2$ is odd. Then by a similar argument to the above case, we may obtain the relation (4.3.1) and also obtain all candidates of the lattice $(L')^{4s}$ for any $L' \in \gamma_2^L(K)$ as (4.3.2). By simple calculations, we may verify that

$$L_{\mathrm{I}} \cong \langle 4a, 4a, b\rangle, \quad L_{\mathrm{II}} = \tau_{x-y}(L_{\mathrm{III}}) \cong \langle a, 4a, 4b\rangle, \quad L_{\mathrm{IV}} \cong \langle 2a, 2a, 4b\rangle,$$

$$L_{\mathrm{V}} = \tau_{x-y}(L_{\mathrm{VI}}) \cong \langle 4a\rangle \perp \begin{pmatrix} a+b & 2b \\ 2b & 4b \end{pmatrix}, \quad L_{\mathrm{VII}} \cong \begin{pmatrix} 2a+b & 2a & 2b \\ 2a & 4a & 0 \\ 2b & 0 & 4b \end{pmatrix}.$$

Next, suppose that the leading Jordan component of $L_2$ is even. Then we obtain the relation (4.3.3) and all candidates of a lattice $L' \in \gamma_4^L(K)$ as (4.3.4). Further we may obtain that

$$L'_{\mathrm{I}} = \tau_{x-y}(L'_{\mathrm{II}}) \cong \langle a, 4a, b\rangle, \quad L'_{\mathrm{III}} \cong \langle 2a, 2a, b\rangle, \quad L'_{\mathrm{IV}} \cong \langle a, a, 4b\rangle,$$

$$L'_{\mathrm{V}} = \tau_{x-y}(L'_{\mathrm{VI}}) \cong \langle a\rangle \perp \begin{pmatrix} a+b & 2b \\ 2b & 4b \end{pmatrix}, \quad L'_{\mathrm{VII}} \cong \begin{pmatrix} a+b & b & 2b \\ b & a+b & 2b \\ 2b & 2b & 4b \end{pmatrix}.$$

**Theorem 4.4.2.** *Let $L$ be a primitive ternary lattice which is not stable over $\mathbb{Z}_2$ and let $K := \lambda_{2e}(L)$. Suppose that $|O(K)| = 16$ and $K$ is of type* I. *Then* *the multiset* $\mathrm{label}(\gamma_{2e}^L(K))$ *is completely determined by the label of* $K$ *and the structure of* $L_2$*.*

*Proof.* Since $\mathrm{label}(K) = [\![16, \mathrm{I}; a, a, 2a, 2a, b]\!]$ for some relatively prime integers $a$ and $b$, it is possible to determine the values of $a$ and $b$ from the label of $K$ and hence it decides the class $[K]$. To prove the theorem, we consider each case for $L_2$ given in Lemma 4.4.1.

**Case(4.4.2.1)** $L_2 \cong \mathbb{A} \perp \langle 4\epsilon \rangle$. In this case, we have $|\gamma_4^L(K)| = 1$ and $K_2 = \lambda_4(L_2) \cong \mathbb{A} \perp \langle \epsilon \rangle$. Therefore the integers $a$ and $b$ are both odd and we obtain the relation (4.3.5). Comparing the local structures, we may easily show that $\gamma_4^L(K) = \{L'_{\mathrm{VII}}\}$. Since $L'_{\mathrm{VII}} \cong K_{16,\mathrm{II}}(a, a+b)$, we obtain

$$\mathrm{label}(\gamma_4^L(K)) = \left\{[\![16, \mathrm{II}; 2a, 2a, 4a, 4a, 4b]\!]\right\}$$

by Lemma 4.1.4.

**Case(4.4.2.2)** $L_2 \cong \mathbb{H} \perp \langle 4\epsilon \rangle$. This case is similar to Case(4.4.2.1) and we obtain the same result.

**Case(4.4.2.3)** $L \cong \langle \epsilon \rangle \perp 4\mathbb{A}$. In this case, we have $|\gamma_4^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon \rangle \perp \mathbb{A}$. Therefore the integers $a$ and $b$ are both odd and we obtain the relation (4.3.7). Comparing the local structures, we may easily show that $\gamma_2^L(K) = \{L_{\mathrm{VII}}\}$. Since $L'_{\mathrm{VII}} \cong K_{16,\mathrm{II}}(2a, 2a+b)$, we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \left\{[\![16, \mathrm{II}; 4a, 4a, 8a, 8a, 4b]\!]\right\}$$

by Lemma 4.1.4.

**Case(4.4.2.4)** $L \cong \langle \epsilon \rangle \perp 4\mathbb{H}$. This case is similar to Case(4.4.2.3) and we obtain the same result.

**Case(4.4.2.5)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 2\epsilon_3 \rangle$, $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$.
**Subcase(4.4.2.5-1)** $\epsilon_1\epsilon_2 \equiv 3 \pmod 8$. In this case, we have $|\gamma_4^L(K)| = 3$ and $K_2 = \lambda_2(L_2) \cong \mathbb{A} \perp \langle \epsilon_3 \rangle$. Therefore integers $a$ and $b$ are both odd and we obtain the relation (4.3.6). Since $K_2$ is anisotropic, $a \equiv b \pmod 4$. Comparing the local structures, we may easily show that $(L_{\mathrm{IV}})_2, (L_{\mathrm{V}})_2$ and $(L_{\mathrm{VI}})_2$ are isometric to $(L_2)^2$, and the other lattices are not. By observing the symmetries of $O(L_{\mathrm{IV}})$ and $O(L_{\mathrm{VI}})$, we obtain their orders and types. Then

we have

$$\operatorname{label}(\gamma_2^L(K)) = \bigl\{\llbracket 8, \mathrm{II}; \dot{2}a, 2a, 2b\rrbracket, \llbracket 16, \mathrm{I}; a, a, 2a, 2a, 2b\rrbracket\bigr\}.$$

**Subcase(4.4.2.5-2)** $\epsilon_1\epsilon_2 \equiv 7 \pmod 8$. This case is similar to the previous case but $(L_{\mathrm{IV}})_2$ and $(L_{\mathrm{V}})_2$ are not isometric to $(L_2)^2$ since $a \not\equiv b \pmod 4$. Therefore

$$\operatorname{label}(\gamma_2^L(K)) = \bigl\{\llbracket 16, \mathrm{I}; a, a, 2a, 2a, 2b\rrbracket\bigr\}.$$

**Case(4.4.2.6)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 2)$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$.
**Subcase(4.4.2.6-1)** $m = 2$. In this case, we have $|\gamma_4^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1', \epsilon_2', 2\epsilon_3\rangle$. Therefore the integer $a$ is odd and $b \equiv 2 \pmod 4$, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that $\gamma_2^L(K) = \{(L_{\mathrm{IV}})^{\frac12}\}$. Therefore

$$\operatorname{label}(\gamma_2^L(K)) = \bigl\{\llbracket 16, \mathrm{I}; a, a, 2a, 2a, 2b\rrbracket\bigr\}.$$

**Subcase(4.4.2.6-2)** $m = 3$. In this case, we have $|\gamma_4^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1', \epsilon_2', 4\epsilon_3\rangle$ for some units $\epsilon_1'$ and $\epsilon_2'$ in $\mathbb{Z}_2$. Therefore the integer $a$ is odd and $b \equiv 4 \pmod 8$, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that $\gamma_2^L(K) = \{(L_{\mathrm{IV}})^{\frac12}\}$ if $\epsilon_1 \equiv \epsilon_2 \pmod 8$, and $\gamma_2^L(K) = \{(L_{\mathrm{VII}})^{\frac12}\}$ otherwise. Therefore

$$\operatorname{label}(\gamma_2^L(K)) = \begin{cases} \bigl\{\llbracket 16, \mathrm{I}; a, a, 2a, 2a, 2b\rrbracket\bigr\} & \text{if } \epsilon_1 \equiv \epsilon_2 \pmod 8 \\ \bigl\{\llbracket 16, \mathrm{II}; 2a, 2a, 4a, 4a, 2b\rrbracket\bigr\} & \text{if } \epsilon_1 \equiv \epsilon_2 + 4 \pmod 8. \end{cases}$$

**Subcase(4.4.2.6-3)** $m \geqslant 4$. In this case, we have $|\gamma_4^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1', \epsilon_2', 2^{m-1}\epsilon_3\rangle$. Therefore the integer $a$ is odd and $b \equiv 0 \pmod 8$, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that $(L_{\mathrm{IV}})_2$ and $(L_{\mathrm{VII}})_2$ are isometric to $(L_2)^2$ if $\epsilon_1 \equiv \epsilon_2 \pmod 8$ and the others are impossible. Therefore

$$\operatorname{label}(\gamma_2^L(K)) = \bigl\{\llbracket 16, \mathrm{I}; a, a, 2a, 2a, 2b\rrbracket, \llbracket 16, \mathrm{II}; 2a, 2a, 4a, 4a, 2b\rrbracket\bigr\}.$$

**Case(4.4.2.7)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 2^m\epsilon_3\rangle$ $(m = 1,\ 2)$.
**Subcase(4.4.2.7-1)** $m = 1$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 =$

$\lambda_2(L_2) \cong \langle 2\epsilon_1, \epsilon_2, \epsilon_3\rangle$. Therefore the integer $a$ is odd and $b \equiv 2 \pmod 4$, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that $\gamma_2^L(K) = \{(L_{\mathrm{I}})^{\frac{1}{2}}\}$. Therefore

$$\mathrm{label}(\gamma_2^L(K)) = \left\{[\![16, \mathrm{I}; 2a, 2a, 4a, 4a, \frac{b}{2}]\!]\right\}.$$

**Subcase(4.4.2.7-2)** $m = 2$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong L_2 \cong \langle 2\epsilon_1, \epsilon_2, 2\epsilon_3\rangle$. Then we have $a \equiv 2 \pmod 4$ and the integer $b$ is odd, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that $\gamma_2^L(K) = \{(L_{\mathrm{II}})^{\frac{1}{2}}, (L_{\mathrm{III}})^{\frac{1}{2}}\}$. Therefore

$$\mathrm{label}(\gamma_2^L(K)) = \left\{[\![8, \mathrm{I}; \frac{a}{2}, 2a, 2b]\!]\right\}.$$

**Case(4.4.2.8)** $L_2 \cong \langle \epsilon_1, 4\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 2)$.
**Subcase(4.4.2.8-1)** $m = 2$. In this case, we have $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, \epsilon_2, \epsilon_3\rangle$. Then the integers $a$ and $b$ are odd, and we obtain the relation (4.3.7). First, we assume that $a \equiv b \pmod 4$. Then we may show that $L_2$ is anisotropic and $\epsilon_2 \equiv \epsilon_3 \pmod 4$, and hence $|\gamma_2^L(K)| = 3$. Comparing the local structures, we may easily show that $\gamma_2^L(K) = \{L_{\mathrm{I}}, L_{\mathrm{II}}, L_{\mathrm{III}}\}$. Therefore

$$\mathrm{label}(\gamma_2^L(K)) = \left\{[\![8, \mathrm{I}; a, 4a, 4b]\!], [\![16, \mathrm{I}; 4a, 4a, 8a, 8a, b]\!]\right\}.$$

Next, suppose that $a \not\equiv b \pmod 4$. Then $L_2$ is isotropic and hence $|\gamma_2^L(K)| = 1$ if $\epsilon_2 \equiv \epsilon_3 \pmod 4$, and 2 otherwise. Comparing the local structures, we may easily show that

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \left\{[\![16, \mathrm{I}; 4a, 4a, 8a, 8a, b]\!]\right\} & \text{if } \epsilon_2 \equiv \epsilon_3 \pmod 4, \\ \left\{[\![8, \mathrm{I}; a, 4a, 4b]\!]\right\} & \text{otherwise.} \end{cases}$$

**Subcase(4.4.2.8-2)** $m = 3$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, \epsilon_2, 2\epsilon_3\rangle$. Therefore the integer $a$ is odd and $b \equiv 2 \pmod 4$, and we obtain the relation (4.3.7). Comparing the local structures, we may easily show that $\gamma_2^L(K) = \{L_{\mathrm{II}}, L_{\mathrm{III}}\}$ if $a \equiv \epsilon_1 \pmod 4$, and $\gamma_2^L(K) = \{L_{\mathrm{V}}, L_{\mathrm{VI}}\}$

otherwise. Therefore

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![8,\mathrm{I}; a, 4a, 4b]\!]\} & \text{if } a \equiv \epsilon_1 \pmod 4, \\ \{[\![8,\mathrm{II}; \dot{4a}, 4a, 4b]\!]\} & \text{otherwise.} \end{cases}$$

**Subcase(4.4.2.8-3)** $m \geqslant 4$. In this case, we have $|\gamma_2^L(K)| = 2^{1+\epsilon_{12}}$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, \epsilon_2, 2^{m-2}\epsilon_3\rangle$. Then the integer $a$ is odd and $b \equiv 0 \pmod 4$, and we obtain the relation (4.3.7). We have $|\gamma_2^L(K)| = 4$ if $\epsilon_1 \equiv \epsilon_2 \pmod 4$, and 2 otherwise. Comparing the local structures, we may easily show that

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![8,\mathrm{I}; a, 4a, 4b]\!], [\![8,\mathrm{II}; \dot{4a}, 4a, 4b]\!]\} & \text{if } \epsilon_1 \equiv \epsilon_2 \pmod 4, \\ \{[\![8,\mathrm{II}; \dot{4a}, 4a, 4b]\!]\} & \text{otherwise.} \end{cases}$$

**Case(4.4.2.9)** $L_2 \cong \langle \epsilon_1, 2^m\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 3)$.
**Subcase(4.4.2.9-1)** $m = 3$. In this case, we have $|\gamma_2^L(K)| = \frac{2}{1+\epsilon_{23}}$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, 2\epsilon_2, 2\epsilon_3\rangle$. Then we have $a \equiv 2 \pmod 4$ and the integer $b$ is odd, and we obtain the relation (4.3.7). Suppose that $\epsilon_2 \equiv \epsilon_3 \pmod 4$. Then we have $|\gamma_2^L(K)| = 1$, and $\gamma_2^L(K) = \{L_{\mathrm{I}}\}$ if $\epsilon_1 \equiv b \pmod 8$, and $\gamma_2^L(K) = \{L_{\mathrm{VII}}\}$ otherwise. Next, suppose that $\epsilon_2 \not\equiv \epsilon_3 \pmod 4$. Then we have $\gamma_2^L(K) = \{L_{\mathrm{V}}, L_{\mathrm{VI}}\}$. Consequently,

$$\mathrm{label}(\gamma_2^L(K)) =$$

$$\begin{cases} \{[\![16,\mathrm{I}; 4a, 4a, 8a, 8a, b]\!]\} & \text{if } \epsilon_2 \equiv \epsilon_3 \pmod 4 \text{ and } \epsilon_1 \equiv b \pmod 8, \\ \{[\![16,\mathrm{II}; 4a, 4a, 8a, 8a, 4b]\!]\} & \text{if } \epsilon_2 \equiv \epsilon_3 \pmod 4 \text{ and } \epsilon_1 \not\equiv b \pmod 8, \\ \{[\![8,\mathrm{II}; \dot{4a}, 4a, 4b]\!]\} & \text{otherwise.} \end{cases}$$

**Subcase(4.4.2.9-2)** $m = 4$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, 4\epsilon_2, 4\epsilon_3\rangle$. Then we have $a \equiv 4 \pmod 8$ and the integer $b$ is odd, and we obtain the relation (4.3.7). We may easily show that $\gamma_2^L(K) = \{L_{\mathrm{I}}, L_{\mathrm{VII}}\}$ if $\epsilon_1 \equiv b \pmod 8$, and $\gamma_2^L(K) = \{L_{\mathrm{V}}, L_{\mathrm{VI}}\}$ otherwise. Consequently we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![16,\mathrm{I}; 4a, 4a, 8a, 8a, b]\!], [\![16,\mathrm{II}; 4a, 4a, 8a, 8a, 4b]\!]\} & \text{if } \epsilon_1 \equiv b \pmod 8, \\ \{[\![8,\mathrm{II}; \dot{4a}, 4a, 4b]\!]\} & \text{otherwise.} \end{cases}$$

**Subcase(4.4.2.9-3)** $m \geqslant 5$. This case is similar to Subcase(4.4.2.9-2) but $L_{\mathrm{I}}$, $L_{\mathrm{V}}$, $L_{\mathrm{VI}}$, $L_{\mathrm{VII}}$ are all in $\gamma_2^L(K)$. Hence

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![8, \mathrm{II}; \dot{4a}, 4a, 4b]\!], [\![16, \mathrm{I}; 4a, 4a, 8a, 8a, b]\!], [\![16, \mathrm{II}; 4a, 4a, 8a, 8a, 4b]\!]\}.$$

□

Next, we consider the case when $K \equiv K_{16,\mathrm{II}}(a, b)$. In this case, we have the following lemma.

**Lemma 4.4.3.** *Let $L$ be a primitive ternary lattice which is not stable over $\mathbb{Z}_2$ and let $K := \lambda_{2e}(L)$. Suppose that $|O(K)| = 16$ and $K$ is of type* II*. Then $L_2$ is isomorphic to one of the following forms;*

$$\mathbb{A} \perp \langle 16\epsilon \rangle, \quad \mathbb{H} \perp \langle 16\epsilon \rangle, \quad \langle \epsilon \rangle \perp 16\mathbb{A}, \quad \langle \epsilon \rangle \perp 16\mathbb{H},$$

$$\langle \epsilon_1, \epsilon_2, 8\epsilon_3 \rangle \ (\epsilon_1 \not\equiv \epsilon_2 \pmod 4), \quad \langle \epsilon_1, \epsilon_2, 2^m\epsilon_3 \rangle \ (m \geqslant 3,\ \epsilon_1 \equiv \epsilon_2 \pmod 4),$$
$$\langle \epsilon_1, 4\epsilon_2, 2^m\epsilon_3 \rangle \ (m \geqslant 5,\ \epsilon_1 \equiv \epsilon_2 \pmod 4),$$
$$\langle \epsilon_1, 2^m\epsilon_2, 2^m\epsilon_3 \rangle \ (m \geqslant 5,\ \epsilon_2 \equiv \epsilon_3 \pmod 4),$$

*where $m$ is a positive integer and $\epsilon, \epsilon_i$ $(i = 1, 2, 3)$ are units in $\mathbb{Z}_2$.*

*Proof.* Suppose that the integer $b$ is even. Then clearly the integer $a$ is odd since they are relatively prime. In this case, we obtain

$$K_2 \cong \begin{pmatrix} 2a & -a \\ -a & b \end{pmatrix} \perp \langle \epsilon \rangle$$

for some unit $\epsilon$ in $\mathbb{Z}_2$. This local structure has an even unimodular component of rank 2. Next, suppose that the integers $a$ and $b$ are both odd. Then we have

$$K_2 \cong \langle b, ab(2b - a), 2^m\epsilon \rangle$$

for some positive integer $m \geqslant 3$. Since $a(2b - a) \equiv 1 \pmod 4$, we have $b \equiv ab(2b - a) \pmod 4$. Let $a \equiv 2 \pmod 4$ and $b$ is odd at third. Then we obtain

$$K_2 \cong \langle b \rangle \perp 2a \begin{pmatrix} 2 & 1 \\ 1 & 1 - \frac{a}{2b} \end{pmatrix}.$$

This local structure has and even 4-modular component of rank 2. Finally, we assume that $\mathrm{ord}_2 a = m \geqslant 2$ and $b$ is odd. Then we have

$$K_2 \cong \left\langle b, 2a\left(1-\frac{a}{2b}\right), 2a\epsilon\right\rangle$$

for some unit $\epsilon$ in $\mathbb{Z}_2$, and we also have $(1-\frac{a}{2b})\epsilon \equiv 1 \pmod 4$. Therefore we may descrive the local structure of $K_2$ as follows;

$$T \perp \langle 4\epsilon\rangle,\ \langle\epsilon\rangle \perp 4T,\ \langle \epsilon_1, \epsilon_2, 2^m\epsilon_3\rangle\ (m \geqslant 3,\ \epsilon_1 \equiv \epsilon_2 \pmod 4),$$

$$\langle \epsilon_1, 2^m\epsilon_2, 2^m\epsilon_3\rangle\ (m \geqslant 3,\ \epsilon_2\epsilon_3 \equiv 1 \pmod 4),$$

where $T$ is an even unimodular lattice of rank 2. Now we choose the local structures for $L_2$ such that the Watson transformations of them have above four types. $\square$

Let $K = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z \cong K_{16,\mathrm{II}}(a,b)$ and suppose that the leading Jordan component of $L_2$ is odd. By scaling $\frac{1}{4s}$ to each lattice in (4.3.2), they are the only possible candidates that may be contained in $\gamma_2^L(K)$ as above cases, where $s$ is the scaling factor of the Watson transformation of $L$. Furthermore we may obtain

$$L_{\mathrm{IV}} \cong \langle 4a, 4a, 4b-4a\rangle, \quad L_{\mathrm{II}} = \tau_{x-y}(\text{Ł}_{\mathrm{III}}) \cong \langle 2a\rangle \perp \begin{pmatrix} 8a & -4a \\ -4a & 4b-2a \end{pmatrix},$$

$$L_{\mathrm{VI}} = \tau_x(L_{\mathrm{I}}) = \tau_{x-y}(L_{\mathrm{V}}) = \tau_y(L_{\mathrm{VII}}) \cong \begin{pmatrix} b & -2a & -2a+2b \\ -2a & 8a & -4a \\ -2a+2b & -4a & 4b \end{pmatrix}.$$

Next, suppose that the leading Jordan component of $L_2$ is even. Then the lattices in (4.3.4) are only possible candidates that may be contained in $\gamma_4^L(K)$. Further we may obtain that

$$L'_{\mathrm{IV}} \cong \langle 2a, 2a, 4b-4a\rangle, \quad L'_{\mathrm{III}} = \tau_{x-y}(\text{Ł}'_{\mathrm{VII}}) \cong \begin{pmatrix} 4a & 4a & -2a \\ 4a & 8a & -2a \\ -2a & -2a & b \end{pmatrix},$$

$$L'_{\mathrm{V}} = \tau_x(L'_{\mathrm{I}}) = \tau_{x+y}(L'_{\mathrm{II}}) = \tau_{x-y}(L'_{\mathrm{VI}}) \cong \begin{pmatrix} 2a & -a & -2a \\ -a & b & 2b-2a \\ -2a & 2b-2a & 4b \end{pmatrix}.$$

**Theorem 4.4.4.** *Let $L$ be a primitive ternary lattice which is not stable over $\mathbb{Z}_2$ and let $K := \lambda_{2e}(L)$. Suppose that $|O(K)| = 16$ and $K$ is of type* $\mathrm{II}$. *Then the multiset* $\mathrm{label}(\gamma_{2e}^L(K))$ *is completely determined by the label of $K$ and the structure of $L_2$.*

*Proof.* Since $\mathrm{label}(K) = [\![16, \mathrm{II}; 2a, 2a, 4a, 4a, 4(b-a)]\!]$ for some relatively prime integers $a$ and $b$, it is possible to determine the integers $a$ and $b$ from the label of $K$ and hence it decides the class $[K]$. To prove the theorem, we consider each case given in Lemma 4.4.3.

**Case(4.4.4.1)** $L_2 \cong \mathbb{A} \perp \langle 16\epsilon\rangle$. In this case, we have $|\gamma_4^L(K)| = 4$ and $K_2 = \lambda_4(L_2) \cong \mathbb{A} \perp \langle 4\epsilon\rangle$. Therefore the integer $a$ is odd and $b \equiv 0 \pmod 2$, and we obtain the relation (4.3.5). Comparing the local structures, we may easily show that $\gamma_4^L(K) = \{L'_{\mathrm{I}}, L'_{\mathrm{II}}, L'_{\mathrm{V}}, L'_{\mathrm{VI}}\}$. Since the order of $O(L'_{\mathrm{I}})$ is 4 and its symmetry is of type 2, we obtain

$$\mathrm{label}(\gamma_4^L(K)) = \left\{[\![4, \mathrm{II}; 2a]\!]\right\}.$$

**Case(4.4.4.2)** $L_2 \cong \mathbb{H} \perp \langle 16\epsilon\rangle$. This case is similar to Case(4.4.4.1) and we obtain the same result.

**Case(4.4.4.3)** $L_2 \cong \langle\epsilon\rangle \perp 16\mathbb{A}$. In this case, we have $|\gamma_2^L(K)| = 4$ and $K_2 = \lambda_2(L_2) \cong \langle\epsilon\rangle \perp 4\mathbb{A}$. Then we may easily show that $a \equiv 2 \pmod 4$ and the integer $b$ is odd, and we obtain the relation (4.3.7). Comparing the local structures, we may easily show that $\gamma_4^L(K) = \{L_{\mathrm{I}}, L_{\mathrm{V}}, L_{\mathrm{VI}}, L_{\mathrm{VII}}\}$. Since the order of $O(L_{\mathrm{I}})$ is 4 and its symmetry is of type 2, we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \left\{[\![4, \mathrm{II}; 16a]\!]\right\}.$$

**Case(4.4.4.4)** $L_2 \cong \langle\epsilon\rangle \perp 16\mathbb{H}$. This case is similar to Case(4.4.4.3) and we obtain the same result.

**Case(4.4.4.5)** $\langle\epsilon_1, \epsilon_2, 8\epsilon_3\rangle$ ($\epsilon_1 \not\equiv \epsilon_2 \pmod 4$).

**Subcase(4.4.4.5-1)** $\epsilon_1\epsilon_2 \equiv 3 \pmod 8$. In this case, we have $|\gamma_2^L(K)| = 6$ and $K_2 = \lambda_2(L_2) \cong \mathbb{A} \perp \langle 4\epsilon_3\rangle$. Therefore the integer $a$ is odd and $b \equiv 0 \pmod 2$, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that $(L_2)^2$ is isometric to all candidates except $L_{\mathrm{IV}}$ if $b \equiv 2 \pmod 4$. Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4, \mathrm{II}; 8a]\!], [\![8, \mathrm{II}; \dot{a}, 4a, 8(b-a)]\!]\big\}.$$

**Subcase(4.4.4.5-2)** $\epsilon_1\epsilon_2 \equiv 7 \pmod 8$. This case is similar to Subcase(4.4.4.5-1) but $L_{\mathrm{I}}$, $L_{\mathrm{V}}$, $L_{\mathrm{VI}}$ and $L_{\mathrm{VII}}$ are excluded. Therefore

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![8, \mathrm{II}; \dot{a}, 4a, 8(b-a)]\!]\big\}.$$

**Case(4.4.4.6)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 4)$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$. In this case, we have $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1', \epsilon_2', 2^{m-1}\epsilon_3\rangle$ and we obtain the relation (4.3.6). Since $|\gamma_2^L(K)| = 2$, the integer $a$ is odd and so is $b$. Then we have

$$\gamma_2^L(K) = \{L_{\mathrm{II}}^{\frac{1}{2}}, L_{\mathrm{III}}^{\frac{1}{2}}\}$$

and hence

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![8, \mathrm{II}; \dot{a}, 4a, 8(b-a)]\!]\big\}.$$

**Case(4.4.4.7)** $L_2 \cong \langle \epsilon_1, 4\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 5)$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$. In this case, we have $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, \epsilon_2, 2^{m-2}\epsilon_3\rangle$ and we obtain the relation (4.3.7). Since $|\gamma_2^L(K)| = 4$, the integers $a$ and $b$ are odd, and then we have

$$\gamma_2^L(K) = \{L_{\mathrm{I}}, L_{\mathrm{V}}, L_{\mathrm{VI}}, L_{\mathrm{VII}}\},$$

and hence

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4, \mathrm{II}; 16a]\!]\big\}.$$

**Case(4.4.4.8)** $L_2 \cong \langle \epsilon_1, 2^m\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 5)$, $\epsilon_2 \equiv \epsilon_3 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 4$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, 2^{m-2}\epsilon_2, 2^{m-2}\epsilon_3\rangle$. Then we have $a \equiv 0 \pmod 4$ and the integer $b$ is odd, and we obtain the relation (4.3.7). Comparing the local structures, we obtain

$$\gamma_2^L(K) = \{L_{\mathrm{I}}, L_{\mathrm{V}}, L_{\mathrm{VI}}, L_{\mathrm{VII}}\},$$

and hence

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![4, \mathrm{II}; 16a]\!]\}.$$

$\square$

## 4.5 The case when $|O(K)| = 12$

Assume that $|O(K)| = 12$. Then $K \cong K_{12}(a, b)$ for some relatively prime positive integers $a$ and $b$ by Lemma 4.1.2. If the integer $a$ is odd, then $K_2 \cong \mathbb{A} \perp \langle 3(3b - 2a) \rangle$, and if $a$ is even and $b$ is odd, then

$$K_2 \cong \langle b \rangle \perp a \begin{pmatrix} 2 & -b \\ -b & b(2b-a) \end{pmatrix} \cong \begin{cases} \langle \epsilon \rangle \perp 2\mathbb{H} & \text{if } a \equiv 2 \bmod 4, \\ \langle \epsilon' \rangle \perp 2^m\mathbb{A} & \text{if } a \equiv 0 \bmod 4, \end{cases}$$

for some units $\epsilon$, $\epsilon' \in \mathbb{Z}_2^\times$ and an integer $m = \mathrm{ord}_2(a)$. But $\langle \epsilon \rangle \perp 2\mathbb{H} \cong \langle \epsilon'' \rangle \perp 2\mathbb{A}$ for some unit $\epsilon'' \in \mathbb{Z}_2^\times$. Therefore $L_2$ is isometric to one of the forms in Lemma 4.3.1 as the case when $|O(K)| = 24$.

Suppose that the leading Jordan component of $L_2$ is odd. By scaling $\frac{1}{4s}$ to each lattice in (4.3.2), they are the only possible candidates that may be contained in $\gamma_2^L(K)$ as above cases, where $s$ is the scaling factor of the Watson transformation of $L$. Furthermore we may obtain

$$L_{\mathrm{VI}} = \tau_x(L_{\mathrm{I}}) = \tau_y(L_{\mathrm{VII}}) \cong \begin{pmatrix} 8a & -4a & -2a \\ -4a & 8a & 0 \\ -2a & 0 & b \end{pmatrix},$$

$$L_{\mathrm{III}} = \tau_{x+y}(L_{\mathrm{II}}) = \tau_y(L_{\mathrm{IV}}) \cong \begin{pmatrix} 2a & -2a & -2a \\ -2a & 8a & 0 \\ -2a & 0 & 4b \end{pmatrix},$$

$$L_{\mathrm{V}} \cong \begin{pmatrix} 8a & -4a & -4a \\ -4a & 2a+b & 2b \\ -4a & 2b & 4b \end{pmatrix}.$$

Next, suppose that the leading Jordan component of $L_2$ is even. Then we obtain the relation (4.3.3) and all candidates of a lattice $L' \in \gamma_4^L(K)$ as

(4.3.4). It is easy to show that

$$L'_{\mathrm{VI}} = \tau_y(L'_{\mathrm{I}}) = \tau_{x+y}(L'_{\mathrm{VII}}) \cong \begin{pmatrix} 2a & -2a & -2a \\ -2a & 2a+b & 2b \\ -2a & 2b & 4b \end{pmatrix},$$

$$L'_{\mathrm{V}} = \tau_{x+y}(L'_{\mathrm{II}}) = \tau_x(L'_{\mathrm{III}}) \cong \begin{pmatrix} b & -a & -2a \\ -a & 2a & 0 \\ -2a+2b & 0 & 4b \end{pmatrix},$$

$$L'_{\mathrm{IV}} \cong \begin{pmatrix} 2a & -a & -a \\ -a & 2a & 0 \\ -a & 0 & 4b-2a \end{pmatrix}.$$

**Theorem 4.5.1.** *Let $L$ be a primitive ternary lattice which is not stable over $\mathbb{Z}_2$ and let $K := \lambda_{2e}(L)$. Suppose that $|O(K)| = 12$. Then the multiset* $\mathrm{label}(\gamma_{2e}^L(K))$ *is completely determined by the label of $K$ and the structure of $L_2$.*

*Proof.* We note that $\mathrm{label}(K) = [\![12; 2a, 2a, 2a]\!]$ for some integer $a$. Hence it is possible to determine the value $a$ from the label of $K$. By the above argument, we only have to check the assertion under the assumption that $L_2$ is isometric to one of the forms in Lemma 4.3.1.

**Case(4.5.1.1)** $L_2 \cong \mathbb{A} \perp \langle 2^m\epsilon\rangle$ $(m \geqslant 2)$.
**Subcase(4.5.1.1-1)** $m = 2$. In this case, we obtain that $|\gamma_4^L(K)| = 1$ from Table 3.1 and $K_2 = \lambda_4(L_2) \cong \mathbb{A} \perp \langle\epsilon\rangle$. Since $K \cong K_{12}(a, b)$, the integers $a$ and $b$ are odd. By (4.3.3), we obtain the relation (4.3.5).

Now we have to find all elements of $\gamma_4^L(K)$ among the seven candidates $L'_{\mathrm{I}}$,..., $L'_{\mathrm{VII}}$ by considering their local structures. In this case, only possible candidate is $L'_{\mathrm{IV}}$, for $L_2$ is an even unimodular lattice. Since $|O(L'_{\mathrm{IV}})|$ divides 12, we may conclude that $|O(L'_{\mathrm{IV}})| = 12$ by the local structure of $(L'_{\mathrm{IV}})_2$. Therefore

$$\mathrm{label}(\gamma_4^L(K)) = \{[\![12; 2a, 2a, 2a]\!]\}.$$

**Subcase(4.5.1.1-2)** $m = 3$. In this case, we have $|\gamma_4^L(K)| = 1$ and $K_2 = \lambda_4(L_2) \cong \mathbb{A} \perp \langle 2\epsilon\rangle$. Since $K = K_{12}(a, b)$, the integer $a$ is odd and $b \equiv 0$

(mod 4), and we get the same relation (4.3.5) as Subcase(4.5.1.1-1). The lattice $(L'_{\mathrm{V}})_2$ is not isometric to $L_2$ since it is isometric to $\mathbb{H} \perp \langle 8\epsilon \rangle$. We may also obtain that $(L'_{\mathrm{VI}})_2 \cong 2\mathbb{H} \perp \langle 2\epsilon \rangle$ and it is not isometric to $L_2$ either. Hence we obtain the same result as Subcase(4.5.1.1-1).

**Subcase(4.5.1.1-3)** $m \geqslant 4$. In this case, we have $|\gamma_4^L(K)| = 4$ and $K_2 = \lambda_4(L_2) \cong \mathbb{A} \perp \langle 2^{m-2}\epsilon \rangle$. Then the integer $a$ is odd and $b \equiv 2 \pmod 4$, and we get the relation (4.3.5) again. Computing each local structure, $(L'_{\mathrm{I}})_2$, $(L'_{\mathrm{VI}})_2$ and $(L'_{\mathrm{VII}})_2$ are not isometric to $L_2$ and the others are all possible. Since the number of the possible cases coincides with the order of $\gamma_4^L(K)$, all elements in $\gamma_4^L(K)$ are founded. Since $\lambda_4(L'_{\mathrm{V}}) = K$, the order $|O(L'_{\mathrm{V}})|$ divides $|O(K)|$. Then we may easily verify that the isometry group of $L'_{\mathrm{V}}$ is of order 4 and the type of its lavel is $\mathrm{I\!I}$ by observing the symmetries, structure of $L'_{\mathrm{V}}$ and Lemma 4.1.2. Therefore

$$\mathrm{label}(\gamma_4^L(K)) = \{[\![4, \mathrm{I\!I}; 2a]\!], [\![12; 2a, 2a, 2a]\!]\}.$$

**Case(4.5.1.2)** $L_2 \cong \langle \epsilon \rangle \perp 2^m\mathbb{A}$ $(m \geqslant 1)$.

**Subcase(4.5.1.2-1)** $m = 1$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle 2\epsilon \rangle \perp \mathbb{A}$. Then the integer $a$ is odd and $b \equiv 0 \pmod 4$. By (4.3.1), we obtain the relation (4.3.6). We may easily show that the only possible candidate is $L_{\mathrm{V}}$ and thus, $L = (L_{\mathrm{V}})^{\frac{1}{2}}$. Therefore

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![12; 4a, 4a, 4a]\!]\}.$$

**Subcase(4.5.1.2-2)** $m = 2$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon \rangle \perp \mathbb{A}$. Then the integers $a$ and $b$ are odd and we obtain the relation (4.3.7) by (4.3.1). Since $(L_{\mathrm{I}})_2$ has an odd 4-modular component and $\mathfrak{s}((L_{\mathrm{I\!I}})_2) = 2\mathbb{Z}_2$, the lattices $L_{\mathrm{I}}$ and $L_{\mathrm{I\!I}}$ are not isometric to $L_2$ . By the Jordan decomposition of $(L_{\mathrm{V}})_2$, we may conclude that $L_{\mathrm{V}}$ is possible. Therefore

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![12; 8a, 8a, 8a]\!]\}.$$

**Subcase(4.5.1.2-3)** $m = 3$. In this case, we may obtain the same result as Subcase(4.5.1.2-2) by a similar way.

**Subcase(4.5.1.2-4)** $m \geqslant 4$. In this case, we have $|\gamma_2^L(K)| = 4$ and $K_2 =$

$\lambda_2(L_2) \cong \langle\epsilon\rangle \perp 2^{m-2}\mathbb{A}$. Then we have $\mathrm{ord}_2(a) \geqslant 2$ and the integer $b$ is odd, and we get the relation (4.3.7). It is clear that $L_{\mathrm{II}}$, $L_{\mathrm{III}}$ and $L_{\mathrm{IV}}$ are not possible and the other candidates are all possible by their local structures. Therefore

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4,\mathrm{II};8a]\!],[\![12;8a,8a,8a]\!]\big\}.$$

**Case(4.5.1.3)** $L_2 \cong \langle 1,3,2^m\epsilon\rangle$ $(m \geqslant 1)$.
**Subcase(4.5.1.3-1)** $m = 1$. In this case, we have $|\gamma_2^L(K)| = 3$ and $K_2 = \lambda_2(L_2) \cong \langle\epsilon\rangle \perp \mathbb{A}$. Then the integers $a$, $b$ are odd and we get the relation (4.3.6). Since $\mathfrak{s}((L_{\mathrm{I}})_2) = \mathfrak{s}((L_{\mathrm{V}})_2) = \mathbb{Z}_2$, the four candidates are excluded. The rest of three lattices are all possible to be isometric to $L_2$ and hence

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4,\mathrm{I};a]\!]\big\}.$$

**Subcase(4.5.1.3-2)** $m = 2$. In this case, we obtain the same result as Subcase(4.5.1.3-1) in a similar way.
**Subcase(4.5.1.3-3)** $m \geqslant 3$. In this case, we have $|\gamma_2^L(K)| = 6$ and $K_2 = \lambda_2(L_2) \cong \mathbb{A} \perp 2^{m-1}\langle\epsilon\rangle$. Then the integer $a$ is odd and $b \equiv 2 \pmod 4$, and we get the relation (4.3.6). Among the candidates, $(L_{\mathrm{V}})^{\frac{1}{2}}$ is the only impossible case since $\mathfrak{s}((L_{\mathrm{V}})_2) = 4\mathbb{Z}_2$, and hence

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4,\mathrm{I};a]\!],[\![4,\mathrm{II};4a]\!]\big\}.$$

**Case(4.5.1.4)** $L_2 \cong \langle\epsilon_1,4\epsilon_2,4\epsilon_3\rangle$. In this case, we have

$$|\gamma_2^L(K)| = 2^{1-\epsilon_{23}}\frac{3}{2+\chi(L_2)} \quad \text{and} \quad K_2 = \lambda_2(L_2) \cong \langle\epsilon_1,\epsilon_2,\epsilon_3\rangle.$$

Then the integers $a$ and $b$ are odd and we get the relation (4.3.7). Since $\lambda_2(L_2) \cong \langle 3(3b-2a)\rangle \perp \mathbb{A}$, this is anisotropic and so is $L_2$. Therefore we have $|\gamma_2^L(K)| = 3$. Then we may easily show that $\gamma_2^L(K) = \{L_{\mathrm{I}}, L_{\mathrm{VI}}, L_{\mathrm{VII}}\}$ and hence

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4,\mathrm{II};8a]\!]\big\}.$$

**Case(4.5.1.5)** $L_2 \cong \langle\epsilon\rangle \perp 8\mathbb{H}$. In this case, we have $|\gamma_2^L(K)| = 3$ and $K_2 = \lambda_2(L_2) \cong \langle\epsilon\rangle \perp 2\mathbb{H}$. Then we have $a \equiv 2 \pmod 4$ and the integer $b$ is odd, and we get the relation (4.3.7). Considering the Jordan decompositions of all candidates, we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4, \mathrm{II}; 8a]\!]\big\}.$$

**Case(4.5.1.6)** $L_2 \cong \mathbb{H} \perp \langle 8\epsilon\rangle$. In this case, we have $|\gamma_4^L(K)| = 3$ and $K_2 = \lambda_4(L_2) \cong \mathbb{H} \perp \langle 2\epsilon\rangle$. Then the integer $a$ is odd and $b \equiv 0 \pmod 4$, and we get the relation (4.3.5). By a simple calculation as above, we obtain that $\gamma_4^L(K) = \{L'_{\mathrm{II}}, L'_{\mathrm{III}}, L'_{\mathrm{V}}\}$ and hence

$$\mathrm{label}(\gamma_4^L(K)) = \big\{[\![4, \mathrm{II}; 2a]\!]\big\}.$$

□

## 4.6 The case when $|O(K)| = 8$

Assume that $|O(K)| = 8$. Then we classified such lattices into four types in Section 1 of this chapter. First, we consider the case when $K \cong K_{8,\mathrm{I}}(a,b,c)$, where $a$, $b$, $c$ are relatively prime positive integers. Let $K = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z \cong K_{8,\mathrm{I}}(a,b,c)$. Then we may obtain

$$L_{\mathrm{I}} \cong \langle 4a, 4b, c\rangle, \quad L_{\mathrm{II}} \cong \langle 4a, b, 4c\rangle, \quad L_{\mathrm{III}} \cong \langle a, 4b, 4c\rangle,$$

$$L_{\mathrm{IV}} \cong \langle 4c\rangle \perp \begin{pmatrix} a+b & 2b \\ 2b & 4b \end{pmatrix}, \quad L_{\mathrm{V}} \cong \langle 4a\rangle \perp \begin{pmatrix} b+c & 2c \\ 2c & 4c \end{pmatrix},$$

$$L_{\mathrm{VI}} \cong \langle 4b\rangle \perp \begin{pmatrix} a+c & 2c \\ 2c & 4c \end{pmatrix}, \quad L_{\mathrm{VII}} \cong \begin{pmatrix} a+b+c & 2b & 2c \\ 2b & 4b & 0 \\ 2c & 0 & 4c \end{pmatrix},$$

and we may also obtain

$$L'_{\mathrm{I}} \cong \langle 4a, b, c\rangle, \quad L'_{\mathrm{II}} \cong \langle a, 4b, c\rangle, \quad L'_{\mathrm{III}} \cong \begin{pmatrix} a+b & 2b \\ 2b & 4b \end{pmatrix} \perp \langle c\rangle,$$

$$L'_{\mathrm{IV}} \cong \langle a, b, 4c\rangle, \quad L'_{\mathrm{V}} \cong \begin{pmatrix} a+c & 2c \\ 2c & 4c \end{pmatrix} \perp \langle b\rangle,$$

$$L'_{\mathrm{VI}} \cong \langle a\rangle \perp \begin{pmatrix} b+c & 2c \\ 2c & 4c \end{pmatrix}, \quad L'_{\mathrm{VII}} \cong \begin{pmatrix} a+c & c & 2c \\ c & b+c & 2c \\ 2c & 2c & 4c \end{pmatrix}.$$

**Theorem 4.6.1.** *Let $L$ be a primitive ternary lattice which is not stable over $\mathbb{Z}_2$ and let $K := \lambda_{2e}(L)$. Suppose that $|O(K)| = 8$ and $K$ is of type* I. *Then the multiset* $\mathrm{label}(\gamma_{2e}^L(K))$ *is completely determined by the label of $K$ and the structure of $L_2$.*

*Proof.* Since $\mathrm{label}(K) = [\![8, \mathrm{I}; a, b, c]\!]$ for some relatively prime integers $a$, $b$ and $c$, it is possible to determine the set $\{a, b, c\}$ from the label of $K$ and hence it decides the class $[K]$.

**Case(4.6.1.1)** $L_2 \cong T \perp \langle 4\epsilon\rangle$. In this case, we have $|\gamma_4^L(K)| = 1$ and $K_2 = \lambda_4(L_2) \cong T \perp \langle\epsilon\rangle$. Therefore the integers $a$, $b$ and $c$ are odd and we obtain the relation (4.3.5). Comparing the local structures, we may easily show that $\gamma_4^L(K) = \{L'_{\mathrm{VII}}\}$. Since $L'_{\mathrm{VII}} \cong K_{8,\mathrm{IV}}(c, b, a+c)$, we obtain

$$\mathrm{label}(\gamma_4^L(K)) = \big\{[\![8, \mathrm{IV}; 4a, 4b, 4c]\!]\big\}$$

by Lemma 4.1.6.

**Case(4.6.1.2)** $L_2 \cong \langle\epsilon\rangle \perp 4T$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle\epsilon\rangle \perp T$. Therefore the integers $a$, $b$ and $c$ are odd and we obtain the relation (4.3.7). Comparing the local structures, we may easily show that $\gamma_4^L(K) = \{L_{\mathrm{VII}}\}$. Since $L_{\mathrm{VII}} \cong K_{8,\mathrm{III}}(2b, 2c, a+b+c)$, we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![8, \mathrm{III}; 4a, 4b, 4c]\!]\big\}$$

by Lemma 4.1.6.

**Case(4.6.1.3)** $L_2 \cong \langle\epsilon_1, \epsilon_2, 2\epsilon_3\rangle$, $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = \frac{3}{2+\tau}$ and $K_2 = \lambda_2(L_2) \cong T \perp \langle\epsilon_3\rangle$. Therefore the integers $a$, $b$ and $c$ are odd and we obtain the relation (4.3.6). If $K_2$ is anisotropic, then we have $a \equiv b \equiv c \pmod 4$ and $|\gamma_2^L(K)| = 3$. On the other hand, if $K_2$ is isotropic, then there is one integer in $a$, $b$ and $c$ which is not equivalent modulo 4 to the others, and $|\gamma_2^L(K)| = 1$. Without loss of generality, we may assume that $a \not\equiv b \equiv c \pmod 4$. Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{IV}})^{\frac{1}{2}}, (L_{\mathrm{V}})^{\frac{1}{2}}, (L_{\mathrm{VI}})^{\frac{1}{2}}\} & \text{if } K_2 \text{ is anisotropic,} \\ \{(L_{\mathrm{V}})^{\frac{1}{2}}\} & \text{if } K_2 \text{ is isotropic and } a \not\equiv b \equiv c \pmod 4. \end{cases}$$

Therefore, if $K_2$ is anisotropic, we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![8, \mathrm{II}; \dot{2}a, 2b, 2c]\!], [\![8, \mathrm{II}; \dot{2}b, 2a, 2c]\!], [\![8, \mathrm{II}; \dot{2}c, 2a, 2b]\!]\big\},$$

and if $K_2$ is isotropic and $a \not\equiv b \equiv c \pmod 4$,

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![8, \mathrm{II}; \dot{2}a, 2b, 2c]\!]\big\}.$$

**Case(4.6.1.4)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 2\epsilon_3 \rangle$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$. This case is similar to Case(4.6.1.3) and we obtain the same result.

**Case(4.6.1.5)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 4\epsilon_3 \rangle$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1', \epsilon_2', 2\epsilon_3 \rangle$. Then we may assume that $a \equiv 2 \pmod 4$ and the integers $b$, $c$ are odd, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{V}})^{\frac{1}{2}}\} & \text{if } a \equiv 2,\ bc \equiv 1 \pmod 4, \\ \{(L_{\mathrm{VII}})^{\frac{1}{2}}\} & \text{if } a \equiv 2,\ bc \equiv 3 \pmod 4. \end{cases}$$

Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \big\{[\![8, \mathrm{II}; \dot{2}a, 2b, 2c]\!]\big\} & \text{if } a \equiv 2,\ bc \equiv 1 \pmod 4, \\ \big\{[\![8, \mathrm{III}; 2a, 2b, 2c]\!]\big\} & \text{if } a \equiv 2,\ bc \equiv 3 \pmod 4. \end{cases}$$

**Case(4.6.1.6)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 8\epsilon_3 \rangle$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1', \epsilon_2', 4\epsilon_3 \rangle$. Then we may assume that $a \equiv 4 \pmod 8$ and the integers $b$ and $c$ are odd, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{V}})^{\frac{1}{2}}\} & \text{if } \epsilon_1 \equiv \frac{b+c}{2} \pmod 4, \\ \{(L_{\mathrm{VII}})^{\frac{1}{2}}\} & \text{otherwise.} \end{cases}$$

Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![8, \mathrm{II}; \dot{2a}, 2b, 2c]\!]\} & \text{if } \epsilon_1 \equiv \frac{b+c}{2} \pmod 4, \\ \{[\![8, \mathrm{III}; 2a, 2b, 2c]\!]\} & \text{otherwise.} \end{cases}$$

**Case(4.6.1.7)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 4)$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$. This case is similar to Case(4.6.1.6) and we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![8, \mathrm{II}; \dot{2a}, 2b, 2c]\!], [\![8, \mathrm{III}; 2a, 2b, 2c]\!]\},$$

where $a \equiv 0 \pmod 8$ and the integers $b$ and $c$ are odd.

**Case(4.6.1.8)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 2\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle 2\epsilon_1, \epsilon_2, \epsilon_3 \rangle$. Then we may assume that $a \equiv 2 \pmod 4$ and the integers $b$ and $c$ are odd, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that $\gamma_2^L(K) = \{L_{\mathrm{III}}^{\frac{1}{2}}\}$. Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![8, \mathrm{I}; \frac{a}{2}, 2b, 2c]\!]\},$$

where $a \equiv 2 \pmod 4$ and the integers $b$ and $c$ are odd.

**Case(4.6.1.9)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 4\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle 2\epsilon_1, \epsilon_2, 2\epsilon_3 \rangle$. Then we may assume that the integer $a$ is odd and $b \equiv c \equiv 2 \pmod 4$, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that $\gamma_2^L(K) = \{L_{\mathrm{I}}^{\frac{1}{2}}, L_{\mathrm{II}}^{\frac{1}{2}}\}$. Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![8, \mathrm{I}; 2a, 2b, \frac{c}{2}]\!], [\![8, \mathrm{I}; 2a, \frac{b}{2}, 2c]\!]\},$$

where the integer $a$ is odd and $b \equiv c \equiv 2 \pmod 4$.

**Case(4.6.1.10)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 8\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle 2\epsilon_1, \epsilon_2, 4\epsilon_3 \rangle$. Then we may assume that the integer $a$ is odd, $b \equiv 2 \pmod 4$ and $c \equiv 4 \pmod 8$, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{II}})^{\frac{1}{2}}\} & \text{if } \epsilon_1\epsilon_2 \equiv \frac{ab}{2} \pmod 4, \\ \{(L_{\mathrm{V}})^{\frac{1}{2}}\} & \text{otherwise.} \end{cases}$$

Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![8, \mathrm{I}; 2a, \frac{b}{2}, 2c]\!]\} & \text{if } \epsilon_1\epsilon_2 \equiv \frac{ab}{2} \pmod 4, \\ \{[\![8, \mathrm{II}; \dot{2}a, 2b, 2c]\!]\} & \text{otherwise,} \end{cases}$$

where the integer $a$ is odd, $b \equiv 2 \pmod 4$ and $c \equiv 4 \pmod 8$.

**Case(4.6.1.11)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 16\epsilon_3 \rangle$. This case is similar to Case(4.6.1.10) and we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![8, \mathrm{I}; 2a, \frac{b}{2}, 2c]\!]\} & \text{if } \epsilon_1\epsilon_2 \equiv \frac{ab}{2} \pmod 8, \\ \{[\![8, \mathrm{II}; \dot{2}a, 2b, 2c]\!]\} & \text{otherwise,} \end{cases}$$

where the integer $a$ is odd, $b \equiv 2 \pmod 4$ and $c \equiv 8 \pmod{16}$.

**Case(4.6.1.12)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 5)$. This case is also similar to Case(4.6.1.10) and we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \left\{[\![8, \mathrm{I}; 2a, \frac{b}{2}, 2c]\!], [\![8, \mathrm{II}; \dot{2}a, 2b, 2c]\!]\right\},$$

where the integer $a$ is odd, $b \equiv 2 \pmod 4$ and $c \equiv 0 \pmod{16}$.

**Case(4.6.1.13)** $L_2 \cong \langle \epsilon_1, 4\epsilon_2, 4\epsilon_3 \rangle$. In this case, we have

$$|\gamma_2^L(K)| = 2^{1-\epsilon_{23}} \frac{3}{2 + \chi(L_2)} \quad \text{and} \quad K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, \epsilon_2, \epsilon_3 \rangle.$$

Then the integers $a$, $b$ and $c$ are odd and we obtain the relation (4.3.7). We may assume that $a \not\equiv b \equiv c \pmod 4$ if $L_2$ is isotropic. Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{I}}, L_{\mathrm{II}}, L_{\mathrm{III}}\} & \text{if } \epsilon_2 \equiv \epsilon_3 \pmod 4 \text{ and } L_2 \text{ is anisotropic,} \\ \{L_{\mathrm{III}}\} & \text{if } \epsilon_2 \equiv \epsilon_3 \pmod 4 \text{ and } L_2 \text{ is isotropic,} \\ \{L_{\mathrm{I}}, L_{\mathrm{II}}\} & \text{if } \epsilon_2 \not\equiv \epsilon_3 \pmod 4 . \end{cases}$$

Therefore, if $\epsilon_2 \equiv \epsilon_3 \pmod 4$ and $L_2$ is anisotropic, we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![8,\mathrm{I};4a,4b,c]\!],[\![8,\mathrm{I};4a,b,4c]\!],[\![8,\mathrm{I};a,4b,4c]\!]\big\}.$$

We also conclude that if $\epsilon_2 \equiv \epsilon_3 \pmod 4$ and $L_2$ is isotropic, we have

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![8,\mathrm{I};a,4b,4c]\!]\big\},$$

and if $\epsilon_2 \not\equiv \epsilon_3 \pmod 4$, we have

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![8,\mathrm{I};4a,4b,c]\!],[\![8,\mathrm{I};4a,b,4c]\!]\big\},$$

where $a \not\equiv b \equiv c \pmod 4$.

**Case(4.6.1.14)** $L_2 \cong \langle \epsilon_1, 4\epsilon_2, 8\epsilon_3\rangle$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, \epsilon_2, 2\epsilon_3\rangle$. Then we may assume that $a \equiv 2 \pmod 4$ and the integers $b$ and $c$ are odd, and we obtain the relation (4.3.7). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{I}}, L_{\mathrm{II}}\} & \text{if } b \equiv c \equiv \epsilon_1 \pmod 4\ , \\ \{L_{\mathrm{IV}}, L_{\mathrm{VI}}\} & \text{if } b \equiv c \not\equiv \epsilon_1 \pmod 4\ , \\ \{L_{\mathrm{II}}, L_{\mathrm{VI}}\} & \text{if } c \not\equiv b \equiv \epsilon_1 \pmod 4\ , \\ \{L_{\mathrm{I}}, L_{\mathrm{IV}}\} & \text{if } b \not\equiv c \equiv \epsilon_1 \pmod 4\ . \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \big\{[\![8,\mathrm{I};4a,4b,c]\!],[\![8,\mathrm{I};4a,b,4c]\!]\big\} & \text{if } b \equiv c \equiv \epsilon_1 \pmod 4\ , \\ \big\{[\![8,\mathrm{II};\dot{4b},4a,4c]\!],[\![8,\mathrm{II};\dot{4c},4a,4b]\!]\big\} & \text{if } b \equiv c \not\equiv \epsilon_1 \pmod 4\ , \\ \big\{[\![8,\mathrm{I};4a,b,4c]\!],[\![8,\mathrm{II};\dot{4b},4a,4c]\!]\big\} & \text{if } c \not\equiv b \equiv \epsilon_1 \pmod 4\ , \\ \big\{[\![8,\mathrm{I};4a,4b,c]\!],[\![8,\mathrm{II};\dot{4c},4a,4b]\!]\big\} & \text{if } b \not\equiv c \equiv \epsilon_1 \pmod 4, \end{cases}$$

where $a \equiv 2 \pmod 4$ and the integers $b$ and $c$ are odd.

**Case(4.6.1.15)** $L_2 \cong \langle \epsilon_1, 4\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 4)$.
**Subcase(4.6.1.15-1)** $\epsilon_1 \equiv \epsilon_2 \pmod 4$. This case is similar to Case(4.6.1.14) and we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![8,\mathrm{I};4a,4b,c]\!],[\![8,\mathrm{I};4a,b,4c]\!],[\![8,\mathrm{II};\dot{4c},4a,4b]\!],[\![8,\mathrm{II};\dot{4b},4a,4c]\!]\big\},$$

where $a \equiv 4 \pmod 8$ and the integers $b$ and $c$ are odd.

**Subcase(4.6.1.15-2)** $L_2 \cong \langle \epsilon_1, 4\epsilon_2, 2^m\epsilon_3 \rangle$, $m \geqslant 4$, $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$. This case is also similar to Case(4.6.1.14) and we obtain

$$\text{label}(\gamma_2^L(K)) = \big\{[\![8, \mathrm{I}; 4a, b, 4c]\!], [\![8, \mathrm{II}; \dot{4}c, 4a, 4b]\!]\big\},$$

where $a \equiv 0 \pmod 4$, the integers $b$ and $c$ are odd, and $c \not\equiv b \equiv \epsilon_1 \pmod 4$.

**Case(4.6.1.16)** $L_2 \cong \langle \epsilon_1, 8\epsilon_2, 8\epsilon_3 \rangle$. In this case, we have $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, 2\epsilon_2, 2\epsilon_3 \rangle$. Then we may assume that the integer $a$ is odd and $b \equiv c \equiv 2 \pmod 4$, and we obtain the relation (4.3.7).

**Subcase(4.6.1.16-1)** $\epsilon_2 \equiv \epsilon_3 \pmod 4$. Then we have $|\gamma_2^L(K)| = 1$. Comparing the local structures, we may easily show that

$$\text{label}(\gamma_2^L(K)) = \begin{cases} \big\{[\![8, \mathrm{I}; a, 4b, 4c]\!]\big\} & \text{if } \frac{b}{2} \equiv \frac{c}{2} \pmod 4 \text{ and } a \equiv \epsilon_1 \pmod 8, \\ \big\{[\![8, \mathrm{III}; 4a, 4b, 4c]\!]\big\} & \text{if } \frac{b}{2} \equiv \frac{c}{2} \pmod 4 \text{ and } a+b+c \equiv \epsilon_1 \pmod 8, \\ \big\{[\![8, \mathrm{II}; \dot{4}c, 4b, 4a]\!]\big\} & \text{if } \frac{b}{2} \not\equiv \frac{c}{2} \pmod 4 \text{ and } a+b \equiv \epsilon_1 \pmod 8, \\ \big\{[\![8, \mathrm{II}; \dot{4}b, 4a, c]\!]\big\} & \text{if } \frac{b}{2} \not\equiv \frac{c}{2} \pmod 4 \text{ and } a+c \equiv \epsilon_1 \pmod 8, \end{cases}$$

**Subcase(4.6.1.16-2)** $\epsilon_2 \not\equiv \epsilon_3 \pmod 4$. Then $|\gamma_2^L(K)| = 2$. Comparing the local structures, we may easily show that

$$\text{label}(\gamma_2^L(K)) = \begin{cases} \big\{[\![8, \mathrm{II}; \dot{4}b, 4a, 4c]\!], [\![8, \mathrm{II}; \dot{4}c, 4a, 4b]\!]\big\} & \text{if } \frac{b}{2} \equiv \frac{c}{2} \pmod 4, \\ \big\{[\![8, \mathrm{I}; a, 4b, 4c]\!], [\![8, \mathrm{III}; 4a, 4b, 4c]\!]\big\} & \text{otherwise.} \end{cases}$$

**Case(4.6.1.17)** $L_2 \cong \langle \epsilon_1, 16\epsilon_2, 16\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, 4\epsilon_2, 4\epsilon_3 \rangle$. Then we may assume that the integer $a$ is odd and $b \equiv c \equiv 4 \pmod 8$, and we obtain the relation (4.3.7). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{III}}, L_{\mathrm{VII}}\} & \text{if } a \equiv \epsilon_1 \pmod 8, \\ \{L_{\mathrm{IV}}, L_{\mathrm{VI}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\operatorname{label}(\gamma_2^L(K)) = \begin{cases} \{[\![8, \mathrm{I}; a, 4b, 4c]\!], [\![8, \mathrm{III}; 4a, 4b, 4c]\!]\} & \text{if } a \equiv \epsilon_1 \pmod 8, \\ \{[\![8, \mathrm{II}; \dot{4b}, 4a, 4c]\!], [\![8, \mathrm{II}; \dot{4c}, 4a, 4b]\!]\} & \text{otherwise}, \end{cases}$$

where the integer $a$ is odd and $b \equiv c \equiv 4 \pmod 8$.

**Case(4.6.1.18)** $L_2 \cong \langle \epsilon_1, 2^m\epsilon_2, 2^m\epsilon_3 \rangle$ ($m \geqslant 5$). This case is similar to Case(4.6.1.17) and we have

$$\operatorname{label}(\gamma_2^L(K)) = \{[\![8, \mathrm{I}; a, 4b, 4c]\!], [\![8, \mathrm{II}; \dot{4b}, 4a, 4c]\!], [\![8, \mathrm{II}; \dot{4c}, 4a, 4b]\!], [\![8, \mathrm{III}; 4a, 4b, 4c]\!]\},$$

where the integer $a$ is odd and $b \equiv c \equiv 0 \pmod 8$.

**Case(4.6.1.19)** $L_2 \cong \langle \epsilon_1, 8\epsilon_2, 16\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, 2\epsilon_2, 4\epsilon_3 \rangle$. Then we may assume that the integer $a$ is odd, $b \equiv 2 \pmod 4$ and $c \equiv 4 \pmod 8$, and we obtain the relation (4.3.7). Comparing the local structures, we may easily show that

$$\operatorname{label}(\gamma_2^L(K)) = \begin{cases} \{[\![8, \mathrm{I}; a, 4b, 4c]\!]\} & \text{if } a \equiv \epsilon_1 \pmod 8, \\ \{[\![8, \mathrm{II}; \dot{4c}, 4a, 4b]\!]\} & \text{if } a + b \equiv \epsilon_1 \pmod 8, \\ \{[\![8, \mathrm{II}; \dot{4b}, 4a, 4c]\!]\} & \text{if } a + c \equiv \epsilon_1 \pmod 8, \\ \{[\![8, \mathrm{III}; 4a, 4b, 4c]\!]\} & \text{if } a + b + c \equiv \epsilon_1 \pmod 8. \end{cases}$$

**Case(4.6.1.20)** $L_2 \cong \langle \epsilon_1, 16\epsilon_2, 32\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, 4\epsilon_2, 8\epsilon_3 \rangle$. Then we may assume that the integer $a$ is odd, $b \equiv 4 \pmod 8$ and $c \equiv 8 \pmod{16}$. We obtain the relation (4.3.7). Comparing the local structures, we may easily show that

$$\operatorname{label}(\gamma_2^L(K)) = \begin{cases} \{[\![8, \mathrm{I}; a, 4b, 4c]\!], [\![8, \mathrm{II}; \dot{4b}, 4a, 4c]\!]\} & \text{if } a \equiv \epsilon_1 \pmod 8, \\ \{[\![8, \mathrm{II}; \dot{4c}, 4a, 4b]\!], [\![8, \mathrm{III}; 4a, 4b, 4c]\!]\} & \text{otherwise}, \end{cases}$$

where the integer $a$ is odd, $b \equiv 4 \pmod 8$ and $c \equiv 8 \pmod{16}$.

**Case(4.6.1.21)** $L_2 \cong \langle \epsilon_1, 2^m\epsilon_2, 2^{m+1}\epsilon_3 \rangle$ ($m \geqslant 5$). This case is similar to

Case(4.6.1.20) and we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![8,\mathrm{I};a,4b,4c]\!],[\![8,\mathrm{II};\dot{4}b,4a,4c]\!],[\![8,\mathrm{II};\dot{4}c,4a,4b]\!],[\![8,\mathrm{III};4a,4b,4c]\!]\},$$

where the integer $a$ is odd, $b \equiv 0 \pmod 8$ and $c \equiv 0 \pmod{16}$.

**Case(4.6.1.22)** $L_2 \cong \langle \epsilon_1, 8\epsilon_2, 32\epsilon_3\rangle$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, 2\epsilon_2, 8\epsilon_3\rangle$. Then we may assume that the integer $a$ is odd, $b \equiv 2 \pmod 4$ and $c \equiv 8 \pmod{16}$. We obtain the relation (4.3.7). Comparing the local structures, we may easily show that

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![8,\mathrm{I};a,4b,4c]\!],[\![8,\mathrm{II};\dot{4}b,4a,4c]\!]\} & \text{if } a \equiv \epsilon_1 \pmod 8, \\ \{[\![8,\mathrm{II};\dot{4}c,4a,4b]\!],[\![8,\mathrm{III};4a,4b,4c]\!]\} & \text{otherwise,} \end{cases}$$

where the integer $a$ is odd, $b \equiv 2 \pmod 4$ and $c \equiv 8 \pmod{16}$.

**Case(4.6.1.23)** $L_2 \cong \langle \epsilon_1, 16\epsilon_2, 64\epsilon_3\rangle$. This case is similar to Case(4.6.1.22) and we obtain the same result, where $a$ is odd, $b \equiv 4 \pmod 8$ and $c \equiv 16 \pmod{32}$.

**Case(4.6.1.24)** $L_2 \cong \langle \epsilon_1, 2^m\epsilon_2, 2^{m+2}\epsilon_3\rangle$ $(m \geqslant 5)$. This case is also similar to Case(4.6.1.22) and we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![8,\mathrm{I};a,4b,4c]\!],[\![8,\mathrm{II};\dot{4}b,4a,4c]\!],[\![8,\mathrm{II};\dot{4}c,4a,4b]\!],[\![8,\mathrm{III};4a,4b,4c]\!]\},$$

where $a$ is odd.

**Case(4.6.1.25)** $L_2 \cong \langle \epsilon_1, 8\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 6)$. This case is similar to Case(4.6.1.20) and we obtain the same result, where the integer $a$ is odd, $b \equiv 2 \pmod 4$ and $c \equiv 0 \pmod{16}$.

**Case(4.6.1.26)** $L_2 \cong \langle \epsilon_1, 16\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 7)$. This case is also similar to Case(4.6.1.20) and we obtain the same result, where the integer $a$ is odd, $b \equiv 4 \pmod 8$ and $c \equiv 0 \pmod{32}$.

**Case(4.6.1.27)** $L_2 \cong \langle \epsilon_1, 2^n\epsilon_2, 2^m\epsilon_3\rangle$ $(n \geqslant 5,\ m \geqslant n+3)$. This case is similar to Case(4.6.1.24) and we obtain the same result.
Note that the other cases are impossible because of their local structures. $\square$

Next, we consider the case when $K \cong K_{8,\mathrm{II}}(a,b,c)$, where integers $a$, $b$ and $c$ are relatively prime positive integers. Then we have $dK = ab(2c-b)$. Put $c' := 2c - b$ and let $K = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z \cong K_{8,\mathrm{II}}(a,b,c)$. Then we have

$$K = \mathbb{Z}x + \mathbb{Z}(y-z) + \mathbb{Z}z \cong \langle a \rangle \perp \begin{pmatrix} \frac{b+c'}{2} & \frac{b-c'}{2} \\ \frac{b-c'}{2} & \frac{b+c'}{2} \end{pmatrix}.$$

and the label of $K$ is $[\![8, \mathrm{II}; \dot{a}, 2b, 2c']\!]$. Now we may set as follows;

$$K = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z \cong \langle a \rangle \perp \begin{pmatrix} \frac{b+c}{2} & \frac{b-c}{2} \\ \frac{b-c}{2} & \frac{b+c}{2} \end{pmatrix} =: K'_{8,\mathrm{II}}(a,b,c).$$

Then we may obtain

$$L_{\mathrm{I}} = \tau_{y+z}(L_{\mathrm{II}}) \cong \langle 4a \rangle \perp \begin{pmatrix} 2(b+c) & b-c \\ b-c & \frac{b+c}{2} \end{pmatrix}, \quad L_{\mathrm{III}} \cong \langle a \rangle \perp \begin{pmatrix} 8b & 4b \\ 4b & 2(b+c) \end{pmatrix},$$

$$L_{\mathrm{IV}} = \tau_{y+z}(L_{\mathrm{VI}}) \cong \begin{pmatrix} 4a & 2a & 0 \\ 2a & a + \frac{b+c}{2} & b-c \\ 0 & b-c & 2(b+c) \end{pmatrix}, \quad L_{\mathrm{V}} \cong \langle 4a, 2b, 2c \rangle,$$

$$L_{\mathrm{VII}} \cong \begin{pmatrix} 8a & 4b & 4b \\ 4b & 2(b+c) & 2b \\ 4b & 2b & a+2b \end{pmatrix},$$

and we may also obtain

$$L'_{\mathrm{I}} \cong \langle 4a \rangle \perp \begin{pmatrix} \frac{b+c}{2} & \frac{b-c}{2} \\ \frac{b-c}{2} & \frac{b+c}{2} \end{pmatrix}, \quad L'_{\mathrm{II}} = \tau_{y+z}(L'_{\mathrm{IV}}) \cong \langle a \rangle \perp \begin{pmatrix} 2(b+c) & b-c \\ b-c & \frac{b+c}{2} \end{pmatrix},$$

$$L'_{\mathrm{VI}} \cong \langle a, 2b, 2c \rangle, \quad L'_{\mathrm{III}} = \tau_{y+z}(L'_{\mathrm{V}}) \cong \begin{pmatrix} 4a & 2a & 0 \\ 2a & a + \frac{b+c}{2} & \frac{b-c}{2} \\ 0 & \frac{b-c}{2} & \frac{b+c}{2} \end{pmatrix},$$

$$L'_{\mathrm{VII}} \cong \begin{pmatrix} 2b & 0 & b \\ 0 & 2c & c \\ b & c & a + \frac{b+c}{2} \end{pmatrix}.$$

**Theorem 4.6.2.** *Let $L$ be a primitive ternary lattice which is not stable over $\mathbb{Z}_2$ and let $K := \lambda_{2e}(L)$. Suppose that $|O(K)| = 8$ and $K$ is of type* $\mathrm{II}$ *. Then the multiset* $\mathrm{label}(\gamma_{2e}^{L}(K))$ *is completely determined by the label of $K$ and the structure of $L_2$.*

*Proof.* Since $\mathrm{label}(K) = [\![8, \mathrm{II}; \dot{a}, 2b, 2c]\!]$ for some integers $a$, $b$ and $c$, it is possible to determine the value $a$ and the set $\{b, c\}$ from the label of $K$ and hence it decides the class $[K]$.

**Case(4.6.2.1)** $L_2 \cong T \perp \langle 4\epsilon \rangle$. In this case, we have $|\gamma_4^L(K)| = 1$ and $K_2 = \lambda_4(L_2) \cong T \perp \langle \epsilon \rangle$. Therefore the integers $a$, $b$, $c$ are all odd and we obtain the relation (4.3.5). Comparing the local structures, we may easily show that

$$\gamma_4^L(K) = \begin{cases} \{L'_{\mathrm{VII}}\} & \text{if } b \equiv c \pmod 4, \\ \{L'_{\mathrm{I}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_4^L(K)) = \begin{cases} \{[\![8, \mathrm{III}; 4a, 2b, 2c]\!]\} & \text{if } b \equiv c \pmod 4, \\ \{[\![8, \mathrm{II}; \dot{4a}, 2b, 2c]\!]\} & \text{otherwise.} \end{cases}$$

**Case(4.6.2.2)** $L_2 \cong T \perp \langle 8\epsilon \rangle$. In this case, we have $K_2 = \lambda_4(L_2) \cong T \perp \langle 2\epsilon \rangle$. Then $a \equiv 2 \pmod 4$ and the integers $b$ and $c$ are odd, and we obtain the relation (4.3.5). Comparing the local structures, we may easily show that

$$\gamma_4^L(K) = \begin{cases} \{L'_{\mathrm{I}}\} & \text{if } T = \mathbb{A} \text{ and } bc \equiv 3 \pmod 8, \\ \{L'_{\mathrm{VII}}\} & \text{if } T = \mathbb{A} \text{ and } bc \equiv 7 \pmod 8, \\ \{L'_{\mathrm{I}}, L'_{\mathrm{III}}, L'_{\mathrm{V}}\} & \text{if } T = \mathbb{H} \text{ and } bc \equiv 3 \pmod 8, \\ \{L'_{\mathrm{III}}, L'_{\mathrm{V}}, L'_{\mathrm{VII}}\} & \text{if } T = \mathbb{H} \text{ and } bc \equiv 7 \pmod 8. \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_4^L(K)) = \begin{cases} \{[\![8, \mathrm{II}; \dot{4a}, 2b, 2c]\!]\} & \text{if } T = \mathbb{A} \text{ and } bc \equiv 3 \pmod 8, \\ \{[\![8, \mathrm{III}; 4a, 2b, 2c]\!]\} & \text{if } T = \mathbb{A} \text{ and } bc \equiv 7 \pmod 8, \\ \{[\![8, \mathrm{II}; \dot{4a}, 2b, 2c]\!], [\![4, \mathrm{II}; 4a]\!]\} & \text{if } T = \mathbb{H} \text{ and } bc \equiv 3 \pmod 8, \\ \{[\![8, \mathrm{III}; 4a, 2b, 2c]\!], [\![4, \mathrm{II}; 4a]\!]\} & \text{if } T = \mathbb{H} \text{ and } bc \equiv 7 \pmod 8. \end{cases}$$

**Case(4.6.2.3)** $L_2 \cong T \perp \langle 2^m\epsilon\rangle$ $(m \geqslant 4)$. This case is similar to Case(4.6.2.2) and we obtain

$$\mathrm{label}(\gamma_4^L(K)) = \big\{[\![8, \mathrm{II}; \dot{4a}, 2b, 2c]\!], [\![8, \mathrm{III}; 4a, 2b, 2c]\!], [\![4, \mathrm{II}; 4a]\!]\big\}.$$

**Case(4.6.2.4)** $L_2 \cong \langle\epsilon\rangle \perp 2T$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong T \perp \langle 2\epsilon\rangle$. Therefore $a \equiv 2 \pmod 4$ and the integers $b$ and $c$ are odd, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that $\gamma_2^L(K) = \{(L_{\mathrm{III}})^{\frac{1}{2}}\}$. Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![8, \mathrm{II}; \frac{\dot{a}}{2}, 4b, 4c]\!]\big\}.$$

**Case(4.6.2.5)** $L_2 \cong \langle\epsilon\rangle \perp 4T$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong T \perp \langle\epsilon\rangle$. Therefore the integers $a$, $b$ and $c$ are odd, and we obtain the relation (4.3.7). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{VII}}\} & \text{if } b \equiv c \pmod 4 \\ \{L_{\mathrm{III}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \big\{[\![8, \mathrm{IV}; 4a, 8b, 8c]\!]\big\} & \text{if } b \equiv c \pmod 4 \\ \big\{[\![8, \mathrm{II}; \dot{a}, 8b, 8c]\!]\big\} & \text{otherwise.} \end{cases}$$

**Case(4.6.2.6)** $L_2 \cong \langle\epsilon\rangle \perp 8T$. In this case, we have $|\gamma_2^L(K)| = 2 + \chi(T)$ and $K_2 = \lambda_2(L_2) \cong \langle\epsilon\rangle \perp 2T$. Therefore the integer $a$ is odd and $b \equiv c \equiv 2 \pmod 4$, and we obtain the relation (4.3.7). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{III}}\} & \text{if } T \cong \mathbb{A} \text{ and } b + c \equiv 8 \pmod{16}, \\ \{L_{\mathrm{VII}}\} & \text{if } T \cong \mathbb{A} \text{ and } b + c \equiv 0 \pmod{16}, \\ \{L_{\mathrm{III}}, L_{\mathrm{IV}}, L_{\mathrm{VI}}\} & \text{if } T \cong \mathbb{H} \text{ and } b + c \equiv 0 \pmod{16}, \\ \{L_{\mathrm{VII}}, L_{\mathrm{IV}}, L_{\mathrm{VI}}\} & \text{if } T \cong \mathbb{H} \text{ and } b + c \equiv 8 \pmod{16}. \end{cases}$$

Therefore we have

$$\text{label}(\gamma_2^L(K)) = \begin{cases} \{[\![8, \mathrm{II}; \dot{a}, 8b, 8c]\!]\} & \text{if } T \cong \mathbb{A} \text{ and } b+c \equiv 8 \pmod{16}, \\ \{[\![8, \mathrm{IV}; 4a, 8b, 8c]\!]\} & \text{if } T \cong \mathbb{A} \text{ and } b+c \equiv 0 \pmod{16}, \\ \{[\![8, \mathrm{II}; \dot{a}, 8b, 8c]\!], [\![4, \mathrm{II}; 4a]\!]\} & \text{if } T \cong \mathbb{H} \text{ and } b+c \equiv 0 \pmod{16}, \\ \{[\![8, \mathrm{IV}; 4a, 8b, 8c]\!], [\![4, \mathrm{II}; 4a]\!]\} & \text{if } T \cong \mathbb{H} \text{ and } b+c \equiv 8 \pmod{16}. \end{cases}$$

**Case(4.6.2.7)** $L_2 \cong \langle \epsilon \rangle \perp 2^m T$ $(m \geqslant 4)$. This case is similar to Case(4.6.2.6) and we have

$$\text{label}(\gamma_2^L(K)) = \{[\![4, \mathrm{II}; 4a]\!], [\![8, \mathrm{II}; \dot{a}, 8b, 8c]\!], [\![8, \mathrm{IV}; 4a, 8b, 8c]\!]\}.$$

**Case(4.6.2.8)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 2\epsilon_3 \rangle$. In this case, the integers $a$, $b$ and $c$ are odd and we obtain the relation (4.3.6). Then we may show that $K_2$ is isotropic if and only if $b+c \equiv 0 \pmod 8$, or $b+c \equiv 2 \pmod 4$ and $a + \frac{b+c}{2} \equiv 0 \pmod 4$. We may also show that $|\gamma_2^L(K)| = 1$ if $K_2$ is isotropic, and 3 otherwise. Comparing the local structures, we obtain

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{V}})^{\frac{1}{2}}\} & \text{if } K_2 \text{ is isotropic}, \\ \{(L_{\mathrm{I}})^{\frac{1}{2}}, (L_{\mathrm{II}})^{\frac{1}{2}}, (L_{\mathrm{V}})^{\frac{1}{2}}\} & \text{if } b+c \equiv 4 \pmod 8, \\ \{(L_{\mathrm{IV}})^{\frac{1}{2}}, (L_{\mathrm{V}})^{\frac{1}{2}}, (L_{\mathrm{VI}})^{\frac{1}{2}}\} & \text{if } b+c \equiv 2 \text{ and } a + \frac{b+c}{2} \equiv 2 \pmod 4. \end{cases}$$

Therefore we have

$$\text{label}(\gamma_2^L(K)) = \begin{cases} \{[\![8, \mathrm{I}; 2a, b, c]\!]\} & \text{if } K_2 \text{ is isotropic}, \\ \{[\![8, \mathrm{I}; 2a, b, c]\!], [\![4, \mathrm{I}; 2a]\!]\} & \text{if } b+c \equiv 4 \pmod 8, \\ \{[\![8, \mathrm{I}; 2a, b, c]\!], [\![4, \mathrm{II}; 2a]\!]\} & \text{if } b+c \equiv 2 \text{ and } a + \frac{b+c}{2} \equiv 2 \pmod 4. \end{cases}$$

**Case(4.6.2.9)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 4\epsilon_3 \rangle$, $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 3$ and $K_2 = \lambda_2(L_2) \cong T \perp \langle 2\epsilon_3 \rangle$. Therefore $a \equiv 2 \pmod 4$ and the integers $b$ and $c$ are odd, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{I}})^{\frac{1}{2}}, (L_{\mathrm{II}})^{\frac{1}{2}}, (L_{\mathrm{V}})^{\frac{1}{2}}\} & \text{if } b+c \equiv 4 \pmod 8, \\ \{(L_{\mathrm{IV}})^{\frac{1}{2}}, (L_{\mathrm{V}})^{\frac{1}{2}}, (L_{\mathrm{VI}})^{\frac{1}{2}}\} & \text{if } b+c \equiv 0 \pmod 8. \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![8,\mathrm{I};2a,b,c]\!],[\![4,\mathrm{I};2a]\!]\} & \text{if } b+c\equiv 4 \pmod 8, \\ \{[\![8,\mathrm{I};2a,b,c]\!],[\![4,\mathrm{II};2a]\!]\} & \text{if } b+c\equiv 0 \pmod 8. \end{cases}$$

**Case(4.6.2.10)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 3)$, $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 2(2-\tau)$ and $K_2 = \lambda_2(L_2) \cong T \perp \langle 2^{m-1}\epsilon_3\rangle$. Therefore $a \equiv 0 \pmod 4$ and the integers $b$ and $c$ are odd, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{V}})^{\frac12},(L_{\mathrm{VII}})^{\frac12}\} & \text{if } \epsilon_1\epsilon_2\equiv 3 \pmod 8, \\ \{(L_{\mathrm{I}})^{\frac12},(L_{\mathrm{II}})^{\frac12},(L_{\mathrm{IV}})^{\frac12},(L_{\mathrm{V}})^{\frac12},(L_{\mathrm{VI}})^{\frac12},(L_{\mathrm{VII}})^{\frac12}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![8,\mathrm{I};2a,b,c]\!],[\![8,\mathrm{IV};2a,4b,4c]\!]\}$$

if $\epsilon_1\epsilon_2 \equiv 3 \pmod 8$,

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![8,\mathrm{I};2a,b,c]\!],[\![8,\mathrm{IV};2a,4b,4c]\!],[\![4,\mathrm{I};2a]\!],[\![4,\mathrm{II};2a]\!]\}$$

otherwise.

**Case(4.6.2.11)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 4\epsilon_3\rangle$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1', \epsilon_2', 2\epsilon_3\rangle$. Therefore we have $a \equiv 2 \pmod 4$ and the integers $b$ and $c$ are odd, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that $\gamma_2^L(K) = \{(L_{\mathrm{V}})^{\frac12}\}$. Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![8,\mathrm{I};2a,b,c]\!]\}.$$

**Case(4.6.2.12)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 8\epsilon_3\rangle$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1', \epsilon_2', 4\epsilon_3\rangle$. Therefore $a \equiv 4 \pmod 8$ and the integers $b$ and $c$ are odd, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{V}})^{\frac{1}{2}}\} & \text{if } bc \equiv \epsilon_1\epsilon_2 \pmod 8, \\ \{(L_{\mathrm{VII}})^{\frac{1}{2}}\} & \text{if } bc \equiv \epsilon_1\epsilon_2 + 4 \pmod 8. \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![8, \mathrm{I}; 2a, b, c]\!]\} & \text{if } bc \equiv \epsilon_1\epsilon_2 \pmod 8, \\ \{[\![8, \mathrm{IV}; 2a, 4b, 4c]\!]\} & \text{if } bc \equiv \epsilon_1\epsilon_2 + 4 \pmod 8. \end{cases}$$

**Case(4.6.2.13)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 4)$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1', \epsilon_2', 2^{m-1}\epsilon_3 \rangle$. Hence we obtain the relation (4.3.6). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{IV}})^{\frac{1}{2}}, (L_{\mathrm{VI}})^{\frac{1}{2}}\} & \text{if } a \text{ is odd}, \\ \{(L_{\mathrm{V}})^{\frac{1}{2}}, (L_{\mathrm{VII}})^{\frac{1}{2}}\} & \text{otherwise}. \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{II}; 2a]\!]\} & \text{if } a \text{ is odd}, \\ \{[\![8, \mathrm{I}; 2a, b, c]\!], [\![8, \mathrm{IV}; 2a, 4b, 4c]\!]\} & \text{otherwise}. \end{cases}$$

**Case(4.6.2.14)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 2\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle 2\epsilon_1, \epsilon_2, \epsilon_3 \rangle$. Therefore $a \equiv 2 \pmod 4$ and the integers $b$ and $c$ are odd, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that $\gamma_2^L(K) = \{(L_{\mathrm{III}})^{\frac{1}{2}}\}$. Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![8, \mathrm{II}; \frac{\dot{a}}{2}, 4b, 4c]\!]\}.$$

**Case(4.6.2.15)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 4\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle 2\epsilon_1, \epsilon_2, 2\epsilon_3 \rangle$. Therefore $a$ is odd and $b \equiv c \equiv 2 \pmod 4$, and we obtain the relation (4.3.6). By the structure of $K$, we may also have $b + c \equiv 4 \pmod 8$. Comparing the local structures, we may easily show that

$\gamma_2^L(K) = \{(L_{\mathrm{I}})^{\frac{1}{2}}, (L_{\mathrm{II}})^{\frac{1}{2}}\}$. Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![4, \mathrm{I}; 2a]\!]\}.$$

**Case(4.6.2.16)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 8\epsilon_3\rangle$. Comparing the structures of $K$ and $\lambda_2(L_2)$, we may easily show that this case does not occur.

**Case(4.6.2.17)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 16\epsilon_3\rangle$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle 2\epsilon_1, \epsilon_2, 8\epsilon_3\rangle$. Then we may show that $a \equiv 2 \pmod 4$, $b \equiv 2 \pmod 4$ and $c \equiv 4 \pmod 8$, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{III}})^{\frac{1}{2}}\} & \text{if } \frac{a(b+c)}{2} \equiv 2\epsilon_1\epsilon_2 \pmod{16}, \\ \{(L_{\mathrm{VII}})^{\frac{1}{2}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![8, \mathrm{II}; \dot{\tfrac{a}{2}}, 4b, 4c]\!]\} & \text{if } \frac{a(b+c)}{2} \equiv 2\epsilon_1\epsilon_2 \pmod{16}, \\ \{[\![8, \mathrm{IV}; 2a, 4b, 4c]\!]\} & \text{otherwise.} \end{cases}$$

**Case(4.6.2.18)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 5)$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle 2\epsilon_1, \epsilon_2, 2^{m-1}\epsilon_3\rangle$. Clearly we obtain the relation (4.3.6). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{I}})^{\frac{1}{2}}, (L_{\mathrm{II}})^{\frac{1}{2}}\} & \text{if } a \text{ is odd,} \\ \{(L_{\mathrm{III}})^{\frac{1}{2}}, (L_{\mathrm{VII}})^{\frac{1}{2}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{I}; 2a]\!]\} & \text{if } a \text{ is odd,} \\ \{[\![8, \mathrm{II}; \dot{\tfrac{a}{2}}, 4b, 4c]\!], [\![8, \mathrm{IV}; 2a, 4b, 4c]\!]\} & \text{otherwise.} \end{cases}$$

**Case(4.6.2.19)** $L_2 \cong \langle \epsilon_1, 4\epsilon_2, 4\epsilon_3\rangle$. In this case, we have

$$|\gamma_2^L(K)| = 2^{1-\epsilon_{23}} \frac{3}{2 + \chi(L_2)} \quad \text{and} \quad K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, \epsilon_2, \epsilon_3\rangle.$$

Then the integers $a$, $b$ and $c$ are odd, and we obtain the relation (4.3.7). Comparing the local structures, we may easily show that

$$\gamma_2^L(K)=\begin{cases}\{L_{\mathrm{III}}\} & \text{if } b+c\equiv 2 \text{ and } \epsilon_1\not\equiv\epsilon_2\equiv\epsilon_3 \pmod 4,\\ \{L_{\mathrm{I}},L_{\mathrm{II}},L_{\mathrm{III}}\} & \text{if } b+c\equiv 2 \text{ and } \epsilon_1\equiv\epsilon_2\equiv\epsilon_3 \pmod 4,\\ \{L_{\mathrm{I}},L_{\mathrm{II}}\} & \text{if } b+c\equiv 2 \text{ and } \epsilon_2\not\equiv\epsilon_3 \pmod 4,\\ \{L_{\mathrm{VII}}\} & \text{if } b+c\equiv 0 \text{ and } \epsilon_1\not\equiv\epsilon_2\equiv\epsilon_3 \pmod 4,\\ \{L_{\mathrm{IV}},L_{\mathrm{VI}}L_{\mathrm{VII}}\} & \text{if } b+c\equiv 0 \text{ and } \epsilon_1\equiv\epsilon_2\equiv\epsilon_3 \pmod 4,\\ \{L_{\mathrm{IV}},L_{\mathrm{VI}}\} & \text{if } b+c\equiv 0 \text{ and } \epsilon_2\not\equiv\epsilon_3 \pmod 4.\end{cases}$$

Therefore we have

$$\operatorname{label}(\gamma_2^L(K))=\begin{cases}\{[\![8,\mathrm{II};\dot a,8b,8c]\!]\} & \text{if } b+c\equiv 2 \text{ and } \epsilon_1\not\equiv\epsilon_2\equiv\epsilon_3 \pmod 4,\\ \{[\![4,\mathrm{I};4a]\!],[\![8,\mathrm{II};\dot a,8b,8c]\!]\} & \text{if } b+c\equiv 2 \text{ and } \epsilon_1\equiv\epsilon_2\equiv\epsilon_3 \pmod 4,\\ \{[\![4,\mathrm{I};4a]\!]\} & \text{if } b+c\equiv 2 \text{ and } \epsilon_2\not\equiv\epsilon_3 \pmod 4,\\ \{[\![8,\mathrm{IV};4a,8b,8c]\!]\} & \text{if } b+c\equiv 0 \text{ and } \epsilon_1\not\equiv\epsilon_2\equiv\epsilon_3 \pmod 4,\\ \{[\![4,\mathrm{II};4a]\!],[\![8,\mathrm{IV};4a,8b,8c]\!]\} & \text{if } b+c\equiv 0 \text{ and } \epsilon_1\equiv\epsilon_2\equiv\epsilon_3 \pmod 4,\\ \{[\![4,\mathrm{II};4a]\!]\} & \text{if } b+c\equiv 0 \text{ and } \epsilon_2\not\equiv\epsilon_3 \pmod 4.\end{cases}$$

**Case(4.6.2.20)** $L_2\cong\langle\epsilon_1,4\epsilon_2,16\epsilon_3\rangle$. In this case, we have $\epsilon_1\not\equiv\epsilon_2 \pmod 4$ and hence $|\gamma_2^L(K)|=4$, and $K_2=\lambda_2(L_2)\cong\langle\epsilon_1,\epsilon_2,4\epsilon_3\rangle$. Therefore $a\equiv 4 \pmod 8$ and the integers $b$ and $c$ are odd, and we obtain the relation (4.3.7). Comparing the local structures, we may easily show that $\gamma_2^L(K)=\{L_{\mathrm{I}},L_{\mathrm{II}},L_{\mathrm{IV}},L_{\mathrm{VI}}\}$. Therefore we obtain

$$\operatorname{label}(\gamma_2^L(K))=\{[\![4,\mathrm{I};4a]\!],[\![4,\mathrm{II};4a]\!]\}.$$

**Case(4.6.2.21)** $L_2\cong\langle\epsilon_1,4\epsilon_2,2^m\epsilon_3\rangle$, $m\geqslant 5$. In this case, we have $|\gamma_2^L(K)|=4$, and $K_2=\lambda_2(L_2)\cong\langle\epsilon_1,\epsilon_2,2^{m-2}\epsilon_3\rangle$. Then we obtain the relation (4.3.7). Comparing the local structures, we may easily show that

$$\gamma_2^L(K)=\begin{cases}\{L_{\mathrm{I}},L_{\mathrm{II}},L_{\mathrm{III}},L_{\mathrm{VII}}\} & \text{if } a \text{ is odd},\\ \{L_{\mathrm{I}},L_{\mathrm{II}},L_{\mathrm{IV}},L_{\mathrm{VI}}\} & \text{otherwise}.\end{cases}$$

Therefore we obtain

$$\text{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4,\mathrm{I};4a]\!],[\![8,\mathrm{II};\dot{a},8b,8c]\!],[\![8,\mathrm{IV};4a,8b,8c]\!]\} & \text{if } a \text{ is odd,} \\ \{[\![4,\mathrm{I};4a]\!],[\![4,\mathrm{II};4a]\!]\} & \text{otherwise.} \end{cases}$$

**Case(4.6.2.22)** $L_2 \cong \langle \epsilon_1, 8\epsilon_2, 8\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = \frac{2}{1+\epsilon_{23}}$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, 2\epsilon_2, 2\epsilon_3 \rangle$. Then the integer $a$ is odd and $b \equiv c \equiv 2 \pmod 4$, and we obtain the relation (4.3.7). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{III}}\} & \text{if } \epsilon_2 \equiv \epsilon_3 \pmod 4 \text{ and } a \equiv \epsilon_1 \pmod 8 , \\ \{L_{\mathrm{VII}}\} & \text{if } \epsilon_2 \equiv \epsilon_3 \pmod 4 \text{ and } a \not\equiv \epsilon_1 \pmod 8 , \\ \{L_{\mathrm{IV}}, L_{\mathrm{VI}}\} & \text{if } \epsilon_2 \not\equiv \epsilon_3 \pmod 4. \end{cases}$$

Therefore we obtain

$$\text{label}(\gamma_2^L(K)) = \begin{cases} \{[\![8,\mathrm{II};\dot{a},8b,8c]\!]\} & \text{if } \epsilon_2 \equiv \epsilon_3 \pmod 4 \text{ and } a \equiv \epsilon_1 \pmod 8 , \\ \{[\![8,\mathrm{IV};4a,8b,8c]\!]\} & \text{if } \epsilon_2 \equiv \epsilon_3 \pmod 4 \text{ and } a \not\equiv \epsilon_1 \pmod 8 , \\ \{[\![4,\mathrm{II};4a]\!]\} & \text{if } \epsilon_2 \not\equiv \epsilon_3 \pmod 4. \end{cases}$$

**Case(4.6.2.23)** $L_2 \cong \langle \epsilon_1, 16\epsilon_2, 16\epsilon_3 \rangle$. This case is similar to Case(4.6.2.22) and we have

$$\text{label}(\gamma_2^L(K)) = \begin{cases} \{[\![8,\mathrm{II};\dot{a},8b,8c]\!],[\![8,\mathrm{IV};4a,8b,8c]\!]\} & \text{if } a \equiv \epsilon_1 \pmod 8 , \\ \{[\![4,\mathrm{II};4a]\!]\} & \text{otherwise.} \end{cases}$$

**Case(4.6.2.24)** $L_2 \cong \langle \epsilon_1, 2^m\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 5)$. In this case, we have $|\gamma_2^L(K)| = 4$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, 2^{m-2}\epsilon_2, 2^{m-2}\epsilon_3 \rangle$. Then we obtain the relation (4.3.7). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{III}}, L_{\mathrm{IV}}, L_{\mathrm{VI}}, L_{\mathrm{VII}}\} & \text{if } a \text{ is odd,} \\ \{L_{\mathrm{I}}, L_{\mathrm{II}}, L_{\mathrm{IV}} L_{\mathrm{VI}}\} & \text{otherwise.} \end{cases}$$

Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4,\mathrm{II};4a]\!],[\![8,\mathrm{II};\dot{a},8b,8c]\!],[\![8,\mathrm{IV};4a,8b,8c]\!]\} & \text{if } a \text{ is odd,} \\ \{[\![4,\mathrm{I};4a]\!],[\![4,\mathrm{II};4a]\!]\} & \text{otherwise.} \end{cases}$$

**Case(4.6.2.25)** $L_2 \cong \langle \epsilon_1, 8\epsilon_2, 16\epsilon_3 \rangle$. We may easily show that this case does not occur.

**Case(4.6.2.26)** $L_2 \cong \langle \epsilon_1, 16\epsilon_2, 32\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, 4\epsilon_2, 8\epsilon_3 \rangle$. Then we may assume that $a \equiv c \equiv 4 \pmod 8$ and $b \equiv 2 \pmod 4$, and we obtain the relation (4.3.7). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{I}}, L_{\mathrm{II}}\} & \text{if } \frac{b+c}{2} \equiv \epsilon_1 \pmod 8, \\ \{L_{\mathrm{IV}}, L_{\mathrm{VI}}\} & \text{otherwise.} \end{cases}$$

Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4,\mathrm{I};4a]\!]\} & \text{if } \frac{b+c}{2} \equiv \epsilon_1 \pmod 8, \\ \{[\![4,\mathrm{II};4a]\!]\} & \text{otherwise.} \end{cases}$$

**Case(4.6.2.27)** $L_2 \cong \langle \epsilon_1, 2^m\epsilon_2, 2^{m+1}\epsilon_3 \rangle$ $(m \geqslant 5)$. This case is similar to Case(4.6.2.26) and we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![4,\mathrm{I};4a]\!],[\![4,\mathrm{II};4a]\!]\}.$$

**Case(4.6.2.28)** $L_2 \cong \langle \epsilon_1, 8\epsilon_2, 32\epsilon_3 \rangle$. This case is similar to Case(4.6.2.26) and we obtain the same rsult.

**Case(4.6.2.29)** $L_2 \cong \langle \epsilon_1, 16\epsilon_2, 64\epsilon_3 \rangle$. This case is similar to Case(4.6.2.26) and we obtain the same rsult.

**Case(4.6.2.30)** $L_2 \cong \langle \epsilon_1, 2^m\epsilon_2, 2^{m+2}\epsilon_3 \rangle$ $(m \geqslant 5)$. This case is similar to Case(4.6.2.27) and we obtain the same result.

**Case(4.6.2.31)** $L_2 \cong \langle \epsilon_1, 8\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 6)$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, 2\epsilon_2, 2^{m-2}\epsilon_3 \rangle$. Then we obtain the relation (4.3.7).

Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{I}}, L_{\mathrm{II}}\} & \text{if } a \equiv 2 \pmod 4 \text{ and } \frac{b+c}{2} \equiv \epsilon_1 \pmod 8, \\ \{L_{\mathrm{IV}}, L_{\mathrm{VI}}\} & \text{if } a \equiv 2 \pmod 4 \text{ and } \frac{b+c}{2} \not\equiv \epsilon_1 \pmod 8, \\ \{L_{\mathrm{III}}, L_{\mathrm{VII}}\} & \text{if } a \text{ is odd and } a \equiv \epsilon_1 \pmod 8, \\ \{L_{\mathrm{IV}}, L_{\mathrm{VI}}\} & \text{if } a \text{ is odd and } a \not\equiv \epsilon_1 \pmod 8. \end{cases}$$

Therefore we obtain that

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{I}; 4a]\!]\} & \text{if } \frac{b+c}{2} \equiv \epsilon_1 \pmod 8, \\ \{[\![4, \mathrm{II}; 4a]\!]\} & \text{if } \frac{b+c}{2} \not\equiv \epsilon_1 \pmod 8, \end{cases}$$

if the integer $a$ is even, and

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![8, \mathrm{II}; \dot{a}, 8b, 8c]\!], [\![8, \mathrm{IV}; a, 8b, 8c]\!]\} & \text{if } a \equiv \epsilon_1 \pmod 8, \\ \{[\![4, \mathrm{II}; 4a]\!]\} & \text{if } a \not\equiv \epsilon_1 \pmod 8, \end{cases}$$

otherwise.

**Case(4.6.2.32)** $L_2 \cong \langle \epsilon_1, 16\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 7)$. This case is similar to Case(4.6.2.31) and we obtain the same result.

**Case(4.6.2.33)** $L_2 \cong \langle \epsilon_1, 2^n\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant n+3 \geqslant 8)$. This case is similar to the Case(4.6.2.31) and we obtain the same result. $\square$

Third, we consider the case when $K \cong K_{8,\mathrm{III}}(a,b,c)$ for some positive integers $a$, $b$, $c$. Then we have $dK = 2ab(2c-a-b)$. Put $c' := 2c-a-b$ and rewrite the local structure of $K$ as

$$K \cong \begin{pmatrix} 2a & 0 & a \\ 0 & 2b & b \\ a & b & \frac{a+b+c'}{2} \end{pmatrix} := K'_{8,\mathrm{III}}(a,b,c').$$

Then it is clear that $2abc' \equiv 0 \pmod 4$. Note that any lattice of the form $K'_{8,\mathrm{III}}(a,b,c)$ is independent of the order of integers $a$, $b$ and $c$ up to isometry.

Let $K = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z \cong K'_{8,\mathrm{III}}(a,b,c)$. Then we may obtain

$$L_{\mathrm{I}} = \tau_y(L_{\mathrm{V}}) = \tau_x(L_{\mathrm{VI}}) = \tau_{2z-x-y}(L_{\mathrm{VII}}) \cong \begin{pmatrix} 8a & 0 & 2a \\ 0 & 8b & 2b \\ 2a & 2b & \frac{a+b+c}{2} \end{pmatrix},$$

$$L_{\mathrm{II}} \cong \langle 2b \rangle \perp \begin{pmatrix} 8a & 2a \\ 2a & 2a+2c \end{pmatrix}, \quad L_{\mathrm{III}} \cong \langle 2a \rangle \perp \begin{pmatrix} 8b & 4b \\ 4b & 2b+2c \end{pmatrix},$$

$$L_{\mathrm{IV}} \cong \langle 2c \rangle \perp \begin{pmatrix} 8b & 4b \\ 4b & 2a+2b \end{pmatrix},$$

and we may also obtain

$$L'_{\mathrm{I}} = \tau_x(L'_{\mathrm{V}}) \cong \begin{pmatrix} 8a & 0 & 2a \\ 0 & 2b & b \\ 2a & b & \frac{a+b+c}{2} \end{pmatrix}, \quad L'_{\mathrm{II}} = \tau_y(L'_{\mathrm{VI}}) \cong \begin{pmatrix} 2a & 0 & a \\ 0 & 8b & 2b \\ a & 2b & \frac{a+b+c}{2} \end{pmatrix},$$

$$L'_{\mathrm{III}} = \tau_y(L'_{\mathrm{VII}}) \cong \begin{pmatrix} 2c & 0 & c \\ 0 & 8b & 2b \\ c & 2b & \frac{a+b+c}{2} \end{pmatrix}, \quad L'_{\mathrm{IV}} \cong \langle 2a, 2b, 2c \rangle.$$

**Theorem 4.6.3.** *Let $L$ be a primitive ternary lattice which is not stable over $\mathbb{Z}_2$ and let $K := \lambda_{2e}(L)$. Suppose that $|O(K)| = 8$ and $K$ is of type* III. *Then the multiset* label$(\gamma^L_{2e}(K))$ *is completely determined by the label of $K$ and the structure of $L_2$.*

*Proof.* Since label$(K) = [\![8, \mathrm{III}; 2a, 2b, 2c]\!]$ for some positive integers $a$, $b$ and $c$, it is possible to determine the set $\{a, b, c\}$ from the label of $K$ and hence it decides the class $[K]$. We only have to check the assetion for possible local structures of $L_2$.

**Case(4.6.3.1)** $L_2 \cong T \perp \langle 2^m \epsilon \rangle$ $(m \geqslant 4)$. In this case, we have $|\gamma^L_4(K)| = 4$ and $K_2 = \lambda_4(L_2) \cong T \perp \langle 2^{m-2}\epsilon \rangle$. Then exactly two of $a$, $b$, $c$ are odd and we obtain the relation (4.3.5). Comparing the local structures, we may easily show that

$$\gamma_4^L(K) = \begin{cases} \{L'_{\mathrm{I}}, L'_{\mathrm{II}}, L'_{\mathrm{V}}, L'_{\mathrm{VI}}\} & \text{if } a,\, b \text{ are odd,} \\ \{L'_{\mathrm{I}}, L'_{\mathrm{III}}, L'_{\mathrm{V}}, L'_{\mathrm{VII}}\} & \text{if } a,\, c \text{ are odd,} \\ \{L'_{\mathrm{II}}, L'_{\mathrm{III}}, L'_{\mathrm{VI}}, L'_{\mathrm{VII}}\} & \text{if } b,\, c \text{ are odd.} \end{cases}$$

Therefore we obtain

$$\mathrm{label}(\gamma_4^L(K)) = \begin{cases} \big\{[\![4, \mathrm{II}; 2a]\!], [\![4, \mathrm{II}; 2b]\!]\big\} & \text{if } a,\, b \text{ are odd,} \\ \big\{[\![4, \mathrm{II}; 2a]\!], [\![4, \mathrm{II}; 2c]\!]\big\} & \text{if } a,\, c \text{ are odd,} \\ \big\{[\![4, \mathrm{II}; 2b]\!], [\![4, \mathrm{II}; 2c]\!]\big\} & \text{if } b,\, c \text{ are odd.} \end{cases}$$

**Case(4.6.3.2)** $L_2 \cong \langle \epsilon \rangle \perp 8T$. Observing the local structure of $K_2$, we may verify that this case does not occur.

**Case(4.6.3.3)** $L_2 \cong \langle \epsilon \rangle \perp 2^m T$ $(m \geqslant 4)$. In this case, we have $|\gamma_2^L(K)| = 4$ and we may easily show that $\gamma_2^L(K) = \{L_{\mathrm{I}}, L_{\mathrm{V}}, L_{\mathrm{VI}}, L_{\mathrm{VII}}\}$. Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![2]\!]\big\}.$$

**Case(4.6.3.4)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 3)$, $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 2(2-\tau)$ and $K_2 = \lambda_2(L_2) \cong T \perp \langle 2^{m-1}\epsilon_3 \rangle$. Then we may assume that the integers $a$ and $b$ are odd and the integer $c$ is even, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{II}}, L_{\mathrm{III}}\} & \text{if } \epsilon_1\epsilon_2 \equiv 7 \pmod 8, \\ \{L_{\mathrm{I}}, L_{\mathrm{II}}, L_{\mathrm{III}}, L_{\mathrm{V}}, L_{\mathrm{VI}}, L_{\mathrm{VII}}\} & \text{if } \epsilon_1\epsilon_2 \equiv 3 \pmod 8. \end{cases}$$

Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \big\{[\![8, \mathrm{II}; \dot{b}, 4a, 4c]\!], [\![8, \mathrm{II}; \dot{a}, 4b, 4c]\!]\big\} & \text{if } \epsilon_1\epsilon_2 \equiv 7 \pmod 8, \\ \big\{[\![2]\!], [\![8, \mathrm{II}; \dot{b}, 4a, 4c]\!], [\![8, \mathrm{II}; \dot{a}, 4b, 4c]\!]\big\} & \text{if } \epsilon_1\epsilon_2 \equiv 3 \pmod 8. \end{cases}$$

**Case(4.6.3.5)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 8\epsilon_3 \rangle$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon'_1, \epsilon'_2, 4\epsilon_3 \rangle$. Then we may assume that the integers $a$ and $b$ are odd and $c \equiv 2 \pmod 4$, and we obtain the relation

(4.3.6). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{II}}\} & \text{if } b \equiv \epsilon_1 \pmod 4, \\ \{L_{\mathrm{III}}\} & \text{if } a \equiv \epsilon_1 \pmod 4. \end{cases}$$

Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \big\{[\![8, \mathrm{II}; \dot{b}, 4a, 4c]\!]\big\} & \text{if } b \equiv \epsilon_1 \pmod 4, \\ \big\{[\![8, \mathrm{II}; \dot{a}, 4b, 4c]\!]\big\} & \text{if } a \equiv \epsilon_1 \pmod 4. \end{cases}$$

**Case(4.6.3.6)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 4)$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$. This case is similar to Case(4.6.3.5) and we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![8, \mathrm{II}; \dot{b}, 4a, 4c], [\![8, \mathrm{II}; \dot{a}, 4b, 4c]\!]\big\}.$$

**Case(4.6.3.7)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 2)$. In this case, there is no lattice which is isometric locally to $(L_2)^2$ in the seven candidates, and hence this case is impossible.

**Case(4.6.3.8)** $L_2 \cong \langle \epsilon_1, 4\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 4)$. In this case, we have $|\gamma_2^L(K)| = 2^{1+\epsilon_{12}}$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, \epsilon_2, 2^{m-2}\epsilon_3 \rangle$. Then we may assume that the integers $a$ and $b$ are odd and the integer $c$ is even, and we obtain the relation (4.3.7). By the local structure of $K_2$, we may show that $\epsilon_1\epsilon_2 \equiv 1 \pmod 4$ and so $|\gamma_2^L(K)| = 4$. Comparing the local structures, we may easily show that

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![2]\!]\big\}.$$

**Case(4.6.3.9)** $L_2 \cong \langle \epsilon_1, 8\epsilon_2, 8\epsilon_3 \rangle$ or $\langle \epsilon_1, 16\epsilon_2, 16\epsilon_3 \rangle$. These lattices have an unimodular component of rank 1. Then we may show that the integers $a$, $b$ and $c$ are all even by the local structure of $K_2$. Therefore $dK \equiv 0 \pmod{16}$ and the former lattice is impossible. If $L_2 \cong \langle \epsilon_1, 16\epsilon_2, 16\epsilon_3 \rangle$, we have $\lambda_2(L_2) \cong \langle \epsilon_1, 4\epsilon_2, 4\epsilon_3 \rangle$ and hence $a \equiv b \equiv c \equiv 2 \pmod 4$. Then $K'_{8,\mathrm{III}}(a, b, c)_2$ is not isometric to $\lambda_2(L_2)$ and this is a contradiction. Hence the latter lattice is also impossible.

**Case(4.6.3.10)** $L_2 \cong \langle \epsilon_1, 8\epsilon_2, 32\epsilon_3 \rangle$, $\langle \epsilon_1, 16\epsilon_2, 32\epsilon_3 \rangle$ or $\langle \epsilon_1, 16\epsilon_2, 64\epsilon_3 \rangle$. By the similar argument as in Case(4.6.3.9), these lattices are impossible.

**Case(4.6.3.11)** $L_2 \cong \langle \epsilon_1, 8\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 6)$ or $\langle \epsilon_1, 16\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 7)$. These cases are impossible since the possible values of $|\gamma_2^L(K)|$ are not equal to the values in Table 3.1.

**Case(4.6.3.12)** $L_2 \cong \langle \epsilon_1, 2^m\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 5)$. In this case, we have $|\gamma_2^L(K)| = 4$ and we may easily show that

$$\text{label}(\gamma_2^L(K)) = \big\{[\![2]\!]\big\}.$$

**Case(4.6.3.13)** $L_2 \cong \langle \epsilon_1, 2^m\epsilon_2, 2^{m+2}\epsilon_3\rangle$ $(m \geqslant 5)$. In this case, we have $|\gamma_2^L(K)| = 4$ and we may easily show that

$$\text{label}(\gamma_2^L(K)) = \big\{[\![2]\!]\big\}.$$

**Case(4.6.3.14)** $L_2 \cong \langle \epsilon_1, 2^n\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant n+3 \geqslant 8)$. In this case, $|\gamma_2^L(K)| = 4$ and we may easily show that

$$\text{label}(\gamma_2^L(K)) = \big\{[\![2]\!]\big\}.$$

$\square$

Finally, we consider the case when $K = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z \cong K_{8,\mathrm{IV}}(a,b,c)$ for some integers $a$, $b$ and $c$. Then we have $dK = 4abc$ and $S(K) = \{\tau_x, \tau_{x-2y}, \tau_{x-2z}\}$. In this case, we note that any lattice of this form is independent of the order of integers $a$, $b$ and $c$ up to isometry. By simple calculations, we have

$$L_{\mathrm{I}} = \tau_x(L_{\mathrm{VI}}) \cong \begin{pmatrix} 16a & 8a & 4a \\ 8a & 4(a+b) & 2a \\ 4a & 2a & a+c \end{pmatrix}, \quad L_{\mathrm{II}} = \tau_x(L_{\mathrm{IV}}) \cong \begin{pmatrix} 16a & 4a & 8a \\ 4a & a+b & 2a \\ 8a & 2a & 4(a+c) \end{pmatrix},$$

$$L_{\mathrm{V}} = \tau_{x-2y}(L_{\mathrm{VII}}) \cong \begin{pmatrix} 16a & 8a & 8a \\ 8a & 4a+b+c & 4a+2c \\ 8a & 4a+2c & 4(a+c) \end{pmatrix}, \quad L_{\mathrm{III}} \cong \langle 4a, 4b, 4c\rangle,$$

and we also have

$$L'_{\mathrm{I}} = \tau_{x-2y}(L'_{\mathrm{III}}) = \tau_{x-2z}(L'_{\mathrm{V}}) = \tau_x(L'_{\mathrm{VII}}) \cong \begin{pmatrix} 16a & 4a & 4a \\ 4a & a+b & a \\ 4a & a & a+c \end{pmatrix},$$

$$L'_{\mathrm{II}} \cong \langle 4b \rangle \perp \begin{pmatrix} a+c & a-c \\ a-c & a+c \end{pmatrix}, \quad L'_{\mathrm{IV}} \cong \langle 4c \rangle \perp \begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix},$$

$$L'_{\mathrm{VI}} \cong \langle 4a \rangle \perp \begin{pmatrix} b+c & b-c \\ b-c & b+c \end{pmatrix}.$$

**Theorem 4.6.4.** *Let $L$ be a primitive ternary lattice which is not stable over $\mathbb{Z}_2$ and let $K := \lambda_{2e}(L)$. Suppose that $|O(K)| = 8$ and $K$ is of type* IV. *Then the multiset* $\mathrm{label}(\gamma_{2e}^L(K))$ *is completely determined by the label of $K$ and the structure of $L_2$.*

*Proof.* Checking the possible local structures of $K_2$, we may reduce the possible structures of $L_2$. Let $K \cong K_{8,\mathrm{IV}}(a,b,c)$ for some integers $a$, $b$ and $c$. Suppose that the integers $a$, $b$ and $c$ are all odd. Then $K_2 \cong T \perp \langle 4\epsilon \rangle$. Next, we suppose that $a$ is even and $b$, $c$ are odd. Then we have $K_2 \cong \langle \epsilon_1, \epsilon_2, 2^{\mathrm{ord}_2 a+2}\epsilon_3 \rangle$ for some $\epsilon_i \in \mathbb{Z}_2^\times$. Finally, we suppose that $a$ is odd and $b$, $c$ are even. Then we have

$$K_2 \cong \langle a+c \rangle \perp (a+c) \begin{pmatrix} 4ac & 2ac \\ 2ac & ab+ac+bc \end{pmatrix},$$

and we may assume that $\mathrm{ord}_2(b) \leqslant \mathrm{ord}_2(c)$. Now we may consider the following five cases.
(i) If $\mathrm{ord}_2(b) < \mathrm{ord}_2(c)$, then $K_2 \cong \langle a+c, 2^{\mathrm{ord}_2(b)}\epsilon_2, 2^{\mathrm{ord}_2(c)+2}\epsilon_3 \rangle$.
(ii) If $\mathrm{ord}_2(b) = \mathrm{ord}_2(c) = 1$ and $\frac{b}{2} \equiv \frac{c}{2} \pmod 4$, then $K_2 \cong \langle a+c \rangle \perp 4T$.
(iii) If $\mathrm{ord}_2(b) = \mathrm{ord}_2(c) = 1$ and $\frac{b}{2} \not\equiv \frac{c}{2} \pmod 4$, then $K_2 \cong \langle a+c, 4\epsilon_2, 4\epsilon_3 \rangle$.
(iv) If $\mathrm{ord}_2(b) = \mathrm{ord}_2(c) \geqslant 2$ and $\frac{b}{2^{\mathrm{ord}_2(b)}} \equiv \frac{c}{2^{\mathrm{ord}_2(c)}} \pmod 4$,
then $K_2 \cong \langle a+c, 2^{\mathrm{ord}_2(b)+1}\epsilon_2, 2^{\mathrm{ord}_2(b)+1}\epsilon_3 \rangle$.
(v) If $\mathrm{ord}_2(b) = \mathrm{ord}_2(c) \geqslant 2$ and $\frac{b}{2^{\mathrm{ord}_2(b)}} \not\equiv \frac{c}{2^{\mathrm{ord}_2(c)}} \pmod 4$,
then $K_2 \cong \epsilon_1 \perp 2^{\mathrm{ord}_2(b)+1}T$.
Therefore we may conclude that if $K_2 \cong \langle \epsilon_1, 2^\alpha\epsilon_2, 2^\beta\epsilon_3 \rangle$ , then $\alpha = \beta \geqslant 2$ or $\beta \geqslant \alpha + 3$, and if $K_2$ is even, then $K_2 \cong T \perp \langle 4\epsilon \rangle$. Since $\mathrm{label}(K) =$

$[\![8, \mathrm{IV}; 4a, 4b, 4c]\!]$, it is possible to determine the set $\{a, b, c\}$ from the label of $K$. Now we only have to check the assetion for possible local structures of $L_2$.

**Case(4.6.4.1)** $L_2 \cong T \perp \langle 16\epsilon \rangle$. Then we may easily show that

$$\gamma_4^L(K) = \{L'_{\mathrm{I}}, L'_{\mathrm{III}}, L'_{\mathrm{V}}, L'_{\mathrm{VII}}\}.$$

Therefore we obtain

$$\mathrm{label}(\gamma_4^L(K)) = \big\{[\![2]\!]\big\}.$$

**Case(4.6.4.2)** $L_2 \cong \langle \epsilon \rangle \perp 2^m T$ $(m \geqslant 4)$. Then we may easily show that

$$\gamma_2^L(K) = \{L_{\mathrm{I}}, L_{\mathrm{II}}, L_{\mathrm{IV}}, L_{\mathrm{VI}}\}$$

if we assume that $a$ is odd and $b$, $c$ are even. Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4, \mathrm{II}; 16b]\!], [\![4, \mathrm{II}; 16c]\!]\big\}.$$

**Case(4.6.4.3)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 8\epsilon_3 \rangle$, $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$. Then we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{I}})^{\frac{1}{2}}, (L_{\mathrm{II}})^{\frac{1}{2}}, (L_{\mathrm{IV}})^{\frac{1}{2}}, (L_{\mathrm{V}})^{\frac{1}{2}}, (L_{\mathrm{VI}})^{\frac{1}{2}}, (L_{\mathrm{VII}})^{\frac{1}{2}}\} & \text{if } \epsilon_1\epsilon_2 \equiv 3 \pmod 8, \\ \{(L_{\mathrm{V}})^{\frac{1}{2}}, (L_{\mathrm{VII}})^{\frac{1}{2}}\} & \text{if } \epsilon_1\epsilon_2 \equiv 7 \pmod 8, \end{cases}$$

where $a \not\equiv b \equiv c \pmod 4$ in the latter case. Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \big\{[\![4, \mathrm{II}; 8a]\!], [\![4, \mathrm{II}; 8b]\!], [\![4, \mathrm{II}; 8c]\!]\big\} & \text{if } \epsilon_1\epsilon_2 \equiv 3 \pmod 8, \\ \big\{[\![4, \mathrm{II}; 8a]\!]\big\} & \text{if } \epsilon_1\epsilon_2 \equiv 7 \pmod 8. \end{cases}$$

**Case(4.6.4.4)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 4)$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$. Then we may easily show that

$$\gamma_2^L(K) = \{(L_{\mathrm{V}})^{\frac{1}{2}}, (L_{\mathrm{VII}})^{\frac{1}{2}}\},$$

where $a$ is even and $b$, $c$ are odd. Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4, \mathrm{II}; 8a]\!]\big\}.$$

**Case(4.6.4.5)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 5)$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$. Then we may easily show that

$$\gamma_2^L(K) = \{(L_{\mathrm{V}})^{\frac{1}{2}}, (L_{\mathrm{VII}})^{\frac{1}{2}}\},$$

where $a$ is odd and $b$, $c$ are even. Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4, \mathrm{II}; 8a]\!]\big\}.$$

**Case(4.6.4.6)** $L_2 \cong \langle \epsilon_1, 4\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 5)$, $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$. Then we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{I}}, L_{\mathrm{II}}, L_{\mathrm{IV}}, L_{\mathrm{VI}}\} & \text{if } \epsilon_1 \equiv \epsilon_2 \pmod 4, \\ \{L_{\mathrm{I}}, L_{\mathrm{VI}}\} & \text{if } a + c \equiv \epsilon_1 \not\equiv \epsilon_2 \pmod 4, \\ \{L_{\mathrm{II}}, L_{\mathrm{IV}}\} & \text{if } a + b \equiv \epsilon_1 \not\equiv \epsilon_2 \pmod 4, \end{cases}$$

where $a$ is even and $b$, $c$ are odd. Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \big\{[\![4, \mathrm{II}; 16b]\!], [\![4, \mathrm{II}; 16c]\!]\big\} & \text{if } \epsilon_1 \equiv \epsilon_2 \pmod 4, \\ \big\{[\![4, \mathrm{II}; 16b]\!]\big\} & \text{if } a + c \equiv \epsilon_1 \not\equiv \epsilon_2 \pmod 4, \\ \big\{[\![4, \mathrm{II}; 16c]\!]\big\} & \text{if } a + b \equiv \epsilon_1 \not\equiv \epsilon_2 \pmod 4. \end{cases}$$

**Case(4.6.4.7)** $L_2 \cong \langle \epsilon_1, 16\epsilon_2, 16\epsilon_3 \rangle$. Then we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{I}}, L_{\mathrm{VI}}\} & \text{if } a + c \equiv \epsilon_1 \pmod 8, \\ \{L_{\mathrm{II}}, L_{\mathrm{IV}}\} & \text{if } a + b \equiv \epsilon_1 \pmod 8, \end{cases}$$

where $a$ is odd and $b$, $c$ are even. Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \big\{[\![4, \mathrm{II}; 16b]\!]\big\} & \text{if } a + c \equiv \epsilon_1 \pmod 8, \\ \big\{[\![4, \mathrm{II}; 16c]\!]\big\} & \text{if } a + b \equiv \epsilon_1 \pmod 8. \end{cases}$$

**Case(4.6.4.8)** $L_2 \cong \langle \epsilon_1, 2^m\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 5)$. Then we may easily show that

$$\gamma_2^L(K) = \{L_{\mathrm{I}}, L_{\mathrm{II}}, L_{\mathrm{IV}}, L_{\mathrm{VI}}\},$$

where $a$ is odd and $b$, $c$ are even. Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![4, \mathrm{II}; 16b]\!], [\![4, \mathrm{II}; 16c]\!]\}.$$

**Case(4.6.4.9)** $L_2 \cong \langle \epsilon_1, 8\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 6)$. Then we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{I}}, L_{\mathrm{VI}}\} & \text{if } a \equiv \epsilon_1 \pmod 8, \\ \{L_{\mathrm{II}}, L_{\mathrm{IV}}\} & \text{if } a \not\equiv \epsilon_1 \pmod 8, \end{cases}$$

where $a$ is odd and $b$, $c$ are even. Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{II}; 16b]\!]\} & \text{if } a \equiv \epsilon_1 \pmod 8, \\ \{[\![4, \mathrm{II}; 16c]\!]\} & \text{if } a \not\equiv \epsilon_1 \pmod 8. \end{cases}$$

**Case(4.6.4.10)** $L_2 \cong \langle \epsilon_1, 16\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 7)$. This case is similar to Case(4.6.4.9) and we have the same result.

**Case(4.6.4.11)** $L_2 \cong \langle \epsilon_1, 2^n\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant n+3 \geqslant 8)$. This case is similar to Case(4.6.4.8) and we obtain the same result. □

## 4.7 The case when $|O(K)| = 4$

Assume that $|O(K)| = 4$. Then we classified such lattices into two types in the above section. As mentioned above, there are some exceptional cases in which our claim does not hold. These cases are listed in the last theorem of this section.

First, we consider the case when $K = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z \cong K_{4,\mathrm{I}}(a, b, c, d)$, where $a$, $b$, $c$ and $d$ are relatively prime integers. Then we may obtain

$$L_{\mathrm{I}} \cong \langle 4a\rangle \perp \begin{pmatrix} 4b & 2c \\ 2c & d \end{pmatrix}, \quad L_{\mathrm{II}} \cong \langle 4a\rangle \perp \begin{pmatrix} b & 2c \\ 2c & 4d \end{pmatrix},$$

$$L_{\mathrm{III}} \cong \langle a\rangle \perp \begin{pmatrix} 4b & 4c \\ 4c & 4d \end{pmatrix}, \quad L_{\mathrm{IV}} \cong \begin{pmatrix} a+b & 2b & 2c \\ 2b & 4b & 4c \\ 2c & 4c & 4d \end{pmatrix},$$

$$L_{\mathrm{V}} \cong \langle 4a\rangle \perp \begin{pmatrix} b+2c+d & 2(c+d) \\ 2(c+d) & 4d \end{pmatrix}, \quad L_{\mathrm{VI}} \cong \begin{pmatrix} a+d & 2c & 2d \\ 2c & 4b & 4c \\ 2d & 4c & 4d \end{pmatrix},$$

$$L_{\mathrm{VII}} \cong \begin{pmatrix} a+b+2c+d & 2(b+c) & 2(c+d) \\ 2(b+c) & 4b & 4c \\ 2(c+d) & 4c & 4d \end{pmatrix},$$

and we also have

$$L'_{\mathrm{I}} \cong \langle 4a\rangle \perp \begin{pmatrix} b & c \\ c & b \end{pmatrix}, \quad L'_{\mathrm{II}} \cong \langle a\rangle \perp \begin{pmatrix} 4b & 2c \\ 2c & d \end{pmatrix},$$

$$L'_{\mathrm{III}} \cong \begin{pmatrix} a+b & 2b & c \\ 2b & 4b & 2c \\ c & 2c & d \end{pmatrix}, \quad L'_{\mathrm{IV}} \cong \langle a\rangle \perp \begin{pmatrix} b & 2c \\ 2c & 4d \end{pmatrix},$$

$$L'_{\mathrm{V}} \cong \begin{pmatrix} a+d & c & 2d \\ c & b & 2c \\ 2d & 2c & 4d \end{pmatrix}, \quad L'_{\mathrm{VI}} \cong \langle a\rangle \perp \begin{pmatrix} b+2c+d & 2(c+d) \\ 2(c+d) & 4d \end{pmatrix},$$

$$L'_{\mathrm{VII}} \cong \begin{pmatrix} a+d & c+d & 2d \\ c+d & b+2c+d & 2(c+d) \\ 2d & 2(c+d) & 4d \end{pmatrix}.$$

**Theorem 4.7.1.** *Let $L$ be a primitive ternary lattice which is not stable over $\mathbb{Z}_2$ and let $K := \lambda_{2e}(L)$. Suppose that $|O(K)| = 4$ and $K$ is of type* I. *Then the multiset* $\mathrm{label}(\gamma_{2e}^{L}(K))$ *is completely determined by $dK$,* $\mathrm{label}(K)$, $\mathrm{label}(\lambda_2(K))$ *and the structure of $L_2$ except only the case when $K_2$ is a unimodular lattice. In the exceptional case, the multiset* $\mathrm{label}(\gamma_{2e}^{L}(K))$ *depends on whether $K'$ is even or odd, where*

$$K \cong \langle a\rangle \perp K'$$

*for some positive integer $a$ and some binary lattice $K'$.*

*Proof.* Since $K \cong K_{4,\mathrm{I}}(a,b,c,d)$ for some integers $a$, $b$, $c$ and $d$, we have

$$K \cong \langle a \rangle \perp \begin{pmatrix} b & c \\ c & d \end{pmatrix},$$

and the label of $K$ is $[\![4, \mathrm{I}; a]\!]$.

**Case(4.7.1.1)** $L_2 \cong T \perp \langle 4\epsilon \rangle$. Then we may assume that $a$, $b$, $d$ are odd and $c$ is even, or $a$, $c$ are odd and $b$, $d$ are even by a suitable base change. Then we have

$$\gamma_4^L(K) = \begin{cases} \{L'_{\mathrm{VII}}\} & \text{if } b,\ d \text{ are odd,} \\ \{L'_{\mathrm{I}}\} & \text{if } b,\ d \text{ are even.} \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_4^L(K)) = \begin{cases} \{[\![4, \mathrm{II}; 4a]\!]\} & \text{if } K' \text{ is odd,} \\ \{[\![4, \mathrm{I}; 4a]\!]\} & \text{if } K' \text{ is even.} \end{cases}$$

But we can not determine that $K'$ is either even or odd by label$(K)$. This problem is dealt in Section 2 of Chapter 5.

**Case(4.7.1.2)** $L_2 \cong \mathbb{A} \perp \langle 8\epsilon \rangle$. In this case, we have $|\gamma_4^L(K)| = 1$ and $K_2 = \lambda_4(L_2) \cong \mathbb{A} \perp \langle 2\epsilon \rangle$. Then $a \equiv 2 \pmod 4$, $b$ and $d$ are even and $c$ is odd, and we obtain the relation (4.3.5). Comparing the local structures, we may easily show that

$$\gamma_4^L(K) = \begin{cases} \{L'_{\mathrm{I}}\} & \text{if } b \equiv d \equiv 2 \pmod 4, \\ \{L'_{\mathrm{III}}\} & \text{if } b \equiv 0 \pmod 4 \text{ and } d \equiv 2 \pmod 4, \\ \{L'_{\mathrm{V}}\} & \text{if } b \equiv 2 \pmod 4 \text{ and } d \equiv 0 \pmod 4, \\ \{L'_{\mathrm{VII}}\} & \text{if } b \equiv d \equiv 0 \pmod 4. \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_4^L(K)) = \begin{cases} \{[\![4, \mathrm{I}; 4a]\!]\} & \text{if } \frac{3a}{2} \equiv \frac{dK}{2} \pmod 8, \\ \{[\![4, \mathrm{II}; 4a]\!]\} & \text{if } \frac{7a}{2} \equiv \frac{dK}{2} \pmod 8. \end{cases}$$

**Case(4.7.1.3)** $L_2 \cong \mathbb{H} \perp \langle 8\epsilon \rangle$. This case is similar to Case(4.7.1.2) and we have

$$\mathrm{label}(\gamma_4^L(K)) = \begin{cases} \{[\![4,\mathrm{II};4a]\!],[\![4,\mathrm{II};4a]\!],[\![4,\mathrm{II};4a]\!]\} & \text{if } \frac{3a}{2} \equiv \frac{dK}{2} \pmod 8, \\ \{[\![4,\mathrm{I};4a]\!],[\![4,\mathrm{II};4a]\!],[\![4,\mathrm{II};4a]\!]\} & \text{if } \frac{7a}{2} \equiv \frac{dK}{2} \pmod 8. \end{cases}$$

**Case(4.7.1.4)** $L_2 \cong T \perp \langle 2^m\epsilon\rangle$ $(m \geqslant 4)$. This case is also similar to Case(4.7.1.2) and we have

$$\mathrm{label}(\gamma_4^L(K)) = \{[\![4,\mathrm{I};4a]\!],[\![4,\mathrm{II};4a]\!],[\![4,\mathrm{II};4a]\!],[\![4,\mathrm{II};4a]\!]\}.$$

**Case(4.7.1.5)** $L_2 \cong \langle\epsilon\rangle \perp\ 2T$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle 2\epsilon\rangle \perp T$. Then $a \equiv 2 \pmod 4$, $b$ and $d$ are even and $c$ is odd, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that $\gamma_2^L(K) = \{(L_{\mathrm{III}})^{\frac{1}{2}}\}$. Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![4,\mathrm{I};\frac{a}{2}]\!]\}.$$

**Case(4.7.1.6)** $L_2 \cong \langle\epsilon\rangle \perp\ 4T$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle\epsilon\rangle \perp T$. Then $a$ is odd and we may assume that $b$, $d$ are both even or both odd. we obtain the relation (4.3.7). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{VII}}\} & \text{if } b \text{ and } d \text{ are odd}, \\ \{L_{\mathrm{III}}\} & \text{if } b \text{ and } d \text{ are even}. \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4,\mathrm{II};4a]\!]\} & \text{if } K' \text{ is odd}, \\ \{[\![4,\mathrm{I};a]\!]\} & \text{if } K' \text{ is even}. \end{cases}$$

But we can not determine whether $K'$ is even or odd by label$(K)$. This problem is dealt in Section 2 of Chapter 5.

**Case(4.7.1.7)** $L_2 \cong \langle\epsilon\rangle \perp\ 8\mathbb{A}$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle\epsilon\rangle \perp 2\mathbb{A}$. Then $a$ is odd, $b \equiv d \equiv 0 \pmod 4$ and $c \equiv 2 \pmod 4$, and we obtain the relation (4.3.7). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{III}}\} & \text{if } b \equiv d \equiv 4 \pmod 8, \\ \{L_{\mathrm{IV}}\} & \text{if } b \equiv 4 \pmod 8 \text{ and } d \equiv 0 \pmod 8, \\ \{L_{\mathrm{VI}}\} & \text{if } b \equiv 0 \pmod 8 \text{ and } d \equiv 4 \pmod 8, \\ \{L_{\mathrm{VII}}\} & \text{if } b \equiv d \equiv 0 \pmod 8. \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \big\{[\![4, \mathrm{I}; a]\!]\big\} & \text{if } \frac{dK}{4} \equiv 3a \pmod 8, \\ \big\{[\![4, \mathrm{II}; 4a]\!]\big\} & \text{if } \frac{dK}{4} \equiv 7a \pmod 8. \end{cases}$$

**Case(4.7.1.8)** $L_2 \cong \langle \epsilon \rangle \perp 8\mathbb{H}$. This case is similar to Case(4.7.1.7) and we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \big\{[\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\big\} & \text{if } \frac{dK}{4} \equiv 3a \pmod 8, \\ \big\{[\![4, \mathrm{I}; a]\!], [\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\big\} & \text{if } \frac{dK}{4} \equiv 7a \pmod 8. \end{cases}$$

**Case(4.7.1.9)** $L_2 \cong \langle \epsilon \rangle \perp 2^m T$ $(m \geqslant 4)$. This case is also similar to Case(4.7.1.7) and we have

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4, \mathrm{I}; a]\!], [\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\big\}.$$

**Case(4.7.1.10)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 2\epsilon_3 \rangle$. In this case, the integers $a$ and $bd - c^2$ are odd, and we obtain the relation (4.3.7). Then we may assume that $b$ and $d$ are odd, or $b$ and $d$ are even. Suppose that $b$ and $d$ are odd. Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{IV}})^{\frac{1}{2}}, (L_{\mathrm{V}})^{\frac{1}{2}}, (L_{\mathrm{VI}})^{\frac{1}{2}}\} & \text{if } K_2 \text{ is anisotropic}, \\ \{(L_{\mathrm{V}})^{\frac{1}{2}}\} & \text{if } K_2 \text{ is isotropic and } a \equiv dK \pmod 4, \\ \{(L_{\mathrm{IV}})^{\frac{1}{2}}\} \text{ or } \{(L_{\mathrm{VI}})^{\frac{1}{2}}\} & \text{if } K_2 \text{ is isotropic and } a \not\equiv dK \pmod 4. \end{cases}$$

Therefore if $K_2$ is anisotropic, we have

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4, \mathrm{I}; 2a]\!], [\![4, \mathrm{II}; 2a]\!], [\![4, \mathrm{II}; 2a]\!]\big\},$$

and if $K_2$ is isotropic, we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{I}; 2a]\!]\} & \text{if } a \equiv dK \pmod 4, \\ \{[\![4, \mathrm{II}; 2a]\!]\} & \text{if } a \not\equiv dK \pmod 4. \end{cases}$$

Next, suppose that $b$ and $d$ are even. Then we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{II}})^{\frac{1}{2}}\} & \text{if } K_2 \text{ is isotropic,} \\ \{(L_{\mathrm{I}})^{\frac{1}{2}}, (L_{\mathrm{II}})^{\frac{1}{2}}, (L_{\mathrm{V}})^{\frac{1}{2}}\} & \text{if } K_2 \text{ is anisotropic.} \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{I}; 2a]\!]\} & \text{if } K_2 \text{ is isotropic,} \\ \{[\![4, \mathrm{I}; 2a]\!], [\![4, \mathrm{I}; 2a]\!], [\![4, \mathrm{I}; 2a]\!]\} & \text{if } K_2 \text{ is anisotropic.} \end{cases}$$

But we can not determine whether $K'$ is even or odd by label$(K)$. This problem is dealt in Section 2 of Chapter 5.

**Case(4.7.1.11)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 4\epsilon_3 \rangle$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1', \epsilon_2', 2\epsilon_3 \rangle$. Then we may assume that $a \equiv 2 \pmod 4$, $bd \equiv 1 \pmod 4$ and $c$ is even, or $a$, $b$ are odd and $c$, $d$ are even. We obtain the relation (4.3.6). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{V}})^{\frac{1}{2}}\} & \text{if } a \equiv 2 \pmod 4, \\ \{(L_{\mathrm{IV}})^{\frac{1}{2}}\} & \text{if } a \text{ is odd and } a+b \equiv 2 \pmod 4, \\ \{(L_{\mathrm{VII}})^{\frac{1}{2}}\} & \text{if } a \text{ is odd and } a+b \equiv 0 \pmod 4. \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{I}; 2a]\!]\} & \text{if } a \equiv 2 \pmod 4, \\ \{[\![4, \mathrm{II}; 2a]\!]\} & \text{if } a \text{ is odd.} \end{cases}$$

**Case(4.7.1.12)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 4\epsilon_3 \rangle$, $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 3$ and $K_2 = \lambda_2(L_2) \cong T \perp \langle \epsilon_3 \rangle$. Then $a \equiv 2 \pmod 4$, $b \equiv d \equiv 0 \pmod 2$ and $c$ is odd, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that

$$\gamma_2^L(K)=\begin{cases}\{(L_{\mathrm{IV}})^{\frac12},(L_{\mathrm{V}})^{\frac12},(L_{\mathrm{VI}})^{\frac12}\} & \text{if } b\equiv d\equiv 0 \pmod 4,\\ \{(L_{\mathrm{II}})^{\frac12},(L_{\mathrm{VI}})^{\frac12},(L_{\mathrm{VII}})^{\frac12}\} & \text{if } b\equiv 2 \text{ and } d\equiv 0 \pmod 4,\\ \{(L_{\mathrm{I}})^{\frac12},(L_{\mathrm{IV}})^{\frac12},(L_{\mathrm{VII}})^{\frac12}\} & \text{if } b\equiv 0 \text{ and } d\equiv 2 \pmod 4,\\ \{(L_{\mathrm{I}})^{\frac12},(L_{\mathrm{II}})^{\frac12},(L_{\mathrm{V}})^{\frac12}\} & \text{if } b\equiv d\equiv 2 \pmod 4.\end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K))=\begin{cases}\{[\![4,\mathrm{I};2a]\!],[\![4,\mathrm{II};2a]\!],[\![4,\mathrm{II};2a]\!]\} & \text{if } \frac{dK}{a}\equiv 7 \pmod 8,\\ \{[\![4,\mathrm{I};2a]\!],[\![4,\mathrm{I};2a]\!],[\![4,\mathrm{I};2a]\!]\} & \text{if } \frac{dK}{a}\equiv 3 \pmod 8.\end{cases}$$

**Case(4.7.1.13)** $L_2\cong\langle\epsilon_1,\epsilon_2,8\epsilon_3\rangle$, $\epsilon_1\equiv\epsilon_2 \pmod 4$. In this case, we have $|\gamma_2^L(K)|=1$ and $K_2=\lambda_2(L_2)\cong\langle\epsilon_1',\epsilon_2',4\epsilon_3\rangle$. Then we may assume that $a\equiv 4 \pmod 8$, $bd\equiv 1 \pmod 4$ and $c$ is even, or $a$, $b$ are odd, $c$ is odd and $d\equiv 0 \pmod 4$. We obtain the relation (4.3.6). Consider the former case. Then the possible candidates are $(L_{\mathrm{V}})^{\frac12}$ and$(L_{\mathrm{VII}})^{\frac12}$ and we may show that

$$\begin{aligned}\det\begin{pmatrix}b & c\\ c & d\end{pmatrix} &\equiv \frac14\det\begin{pmatrix}b+2c+d & 2(c+d)\\ 2(c+d) & 4d\end{pmatrix}\\ &\not\equiv \frac14\det\begin{pmatrix}a+b+2c+d & 2(c+d)\\ 2(c+d) & 4d\end{pmatrix} \pmod 8.\end{aligned}$$

Therefore we have

$$\gamma_2^L(K)=\begin{cases}\{(L_{\mathrm{V}})^{\frac12}\} & \text{if } \frac{dK}{a}\equiv\epsilon_1\epsilon_2 \pmod 8,\\ \{(L_{\mathrm{VII}})^{\frac12}\} & \text{otherwise.}\end{cases}$$

Hence we obtain

$$\mathrm{label}(\gamma_2^L(K))=\begin{cases}\{[\![4,\mathrm{I};2a]\!]\} & \text{if } \frac{dK}{a}\equiv\epsilon_1\epsilon_2 \pmod 8,\\ \{[\![4,\mathrm{II};2a]\!]\} & \text{otherwise.}\end{cases}$$

Next, consider the latter case. Then the possible cadidates are $(L_{\mathrm{IV}})^{\frac12}$ and

$(L_{\mathrm{VII}})^{\frac{1}{2}}$. Therefore in any case, we have

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4, \mathrm{II}; 2a]\!]\big\}.$$

**Case(4.7.1.14)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 4)$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1', \epsilon_2', 2^{m-1}\epsilon_3 \rangle$. Then we may assume that $a$, $b$ are odd and $c \equiv d \equiv 0 \pmod 8$, or $a \equiv 0 \pmod 8$, $b$ and $d$ are odd and $c$ is even. we obtain the relation (4.3.6). First, consider the former case. Then we have $a + b \equiv 2 \pmod 4$ since $\epsilon_1 \equiv \epsilon_2 \pmod 4$. Therefore we have

$$\gamma_2^L(K) = \{(L_{\mathrm{IV}})^{\frac{1}{2}}, (L_{\mathrm{VII}})^{\frac{1}{2}}\}$$

and hence we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4, \mathrm{II}; 2a]\!], [\![4, \mathrm{II}; 2a]\!]\big\}.$$

Next, we consider the latter case. Then we have $bd \equiv 1 \pmod 4$ since $\epsilon_1' \equiv \epsilon_2' \pmod 4$. Therefore we have

$$\gamma_2^L(K) = \{(L_{\mathrm{V}})^{\frac{1}{2}}, (L_{\mathrm{VII}})^{\frac{1}{2}}\}$$

and hence we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4, \mathrm{I}; 2a]\!], [\![4, \mathrm{II}; 2a]\!]\big\}.$$

**Case(4.7.1.15)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 3)$, $\epsilon_1\epsilon_2 \equiv 7 \pmod 8$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \mathbb{H} \perp \langle 2^{m-1}\epsilon_3 \rangle$. Then $a \equiv 0 \pmod 4$, $bd \equiv 0 \pmod 8$ and $c$ is odd, and we obtain the relation (4.3.6). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{II}})^{\frac{1}{2}}, (L_{\mathrm{IV}})^{\frac{1}{2}}\} & \text{if } b \equiv 2 \text{ and } d \equiv 0 \pmod 4, \\ \{(L_{\mathrm{I}})^{\frac{1}{2}}, (L_{\mathrm{VI}})^{\frac{1}{2}}\} & \text{if } b \equiv 0 \text{ and } d \equiv 2 \pmod 4, \\ \{(L_{\mathrm{V}})^{\frac{1}{2}}, (L_{\mathrm{VII}})^{\frac{1}{2}}\} & \text{if } b \equiv d \equiv 0 \pmod 4. \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4,\mathrm{I};2a]\!],[\![4,\mathrm{II};2a]\!]\big\}.$$

**Case(4.7.1.16)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 3)$, $\epsilon_1\epsilon_2 \equiv 3 \pmod 8$. This case is similar to Case(4.7.1.15) and we have

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4,\mathrm{I};2a]\!],[\![4,\mathrm{I};2a]\!],[\![4,\mathrm{I};2a]\!],[\![4,\mathrm{II};2a]\!],[\![4,\mathrm{II};2a]\!],[\![4,\mathrm{II};2a]\!]\big\}.$$

**Case(4.7.1.17)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 2\epsilon_3\rangle$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle 2\epsilon_1, \epsilon_2, \epsilon_3\rangle$. Then we may assume that $a$, $b$ are odd and $c \equiv d \equiv 0 \pmod 2$, or $a \equiv 0 \pmod 8$, $b$ and $d$ are odd and $c$ is even. We obtain the relation (4.3.6). First, consider the former case. Then two candidates $(L_{\mathrm{IV}})^{\frac{1}{2}}$ and $(L_{\mathrm{VII}})^{\frac{1}{2}}$ are impossible since

$$\begin{pmatrix} a+b & 2b \\ 2b & 4b \end{pmatrix} \text{ and } \begin{pmatrix} a+b+2c+d & 2(b+c) \\ 2(b+c) & 4b \end{pmatrix}$$

are 2-modular matrices. Therefore only $(L_{\mathrm{I}})^{\frac{1}{2}}$ is possible and we have

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4,\mathrm{I};2a]\!]\big\}.$$

Next, we consider the latter case. Then only $(L_{\mathrm{III}})^{\frac{1}{2}}$ is possible and we have

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4,\mathrm{I};\frac{a}{2}]\!]\big\}.$$

**Case(4.7.1.18)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 4\epsilon_3\rangle$. In this case, we have

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{I}})^{\frac{1}{2}}, (L_{\mathrm{II}})^{\frac{1}{2}}\} & \text{if } a \text{ is odd , } b \equiv d \equiv 2 \text{ and } c \equiv 0 \pmod 4, \\ \{(L_{\mathrm{I}})^{\frac{1}{2}}, (L_{\mathrm{III}})^{\frac{1}{2}}\} & \text{if } a \equiv d \equiv 2 \pmod 4, b \text{ is odd and } c \text{ is even.} \end{cases}$$

Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \big\{[\![4,\mathrm{I};2a]\!],[\![4,\mathrm{I};2a]\!]\big\} & \text{if } a \text{ is odd,} \\ \big\{[\![4,\mathrm{I};2a]\!],[\![4,\mathrm{I};\frac{a}{2}]\!]\big\} & \text{otherwise.} \end{cases}$$

**Case(4.7.1.19)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 8\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle 2\epsilon_1, \epsilon_2, 4\epsilon_3 \rangle$. Clearly we obtain the relation (4.3.6).

**Subcase(4.7.1.19-1)** First, suppose that the integer $a$ is odd. Then we may assume that $b \equiv 2 \pmod 4$, $c \equiv 0 \pmod 4$ and $d \equiv 4 \pmod 8$. Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{II}})^{\frac{1}{2}}\} & \text{if } \epsilon_1\epsilon_2 \equiv \frac{ab}{2} \pmod 4, \\ \{(L_{\mathrm{V}})^{\frac{1}{2}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \left\{[\![4, \mathrm{I}; 2a]\!]\right\}.$$

**Subcase(4.7.1.19-2)** Next, suppose that $a \equiv 2 \pmod 4$. Then we may assume that $b$ is odd, $c$ is even and $d \equiv 0 \pmod 4$. Furthermore we have $bd - c^2 \equiv 4 \pmod 8$. Comparing the local structures, we have

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{III}})^{\frac{1}{2}}\} & \text{if } \epsilon_1\epsilon_2 \equiv \frac{ab}{2} \pmod 4, \\ \{(L_{\mathrm{VI}})^{\frac{1}{2}}\} & \text{otherwise.} \end{cases}$$

In order to determine the integer $b$ modulo 4, we need more consideration. Suppose that $\frac{a}{2} \equiv \frac{dK}{8} \pmod 4$. Then $S_2(K) = (2, \frac{dK}{8})(\frac{a}{2}, \frac{dK}{8})(2, \frac{a}{2})(b, b)$, where $S_2(K)$ is the Hasse symbol of $K$ at 2. Since $b \equiv \tau S_2(K) \pmod 4$, where $\tau = (2, \frac{dK}{8})(\frac{a}{2}, \frac{dK}{8})(2, \frac{a}{2})$, we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{I}; \frac{a}{2}]\!]\} & \text{if } \epsilon_1\epsilon_2 \equiv \frac{a}{2} S_2(K)\tau \pmod 4, \\ \{[\![4, \mathrm{I}; 2a]\!]\} & \text{otherwise,} \end{cases}$$

where

$$\tau = \begin{cases} 1 & \text{if } \frac{a}{2} + \frac{dK}{8} \equiv 2 \pmod 8 \\ -1 & \text{otherwise.} \end{cases}$$

Next, we suppose that $\frac{a}{2} \not\equiv \frac{dK}{8} \pmod 4$. Then we have $\frac{bd-c^2}{4} \equiv 3 \pmod 4$. By the previous results, we have $\mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{I}; \frac{a}{2}]\!]$ or $[\![8, \mathrm{II}; \frac{a}{2}, *, *]\!]$.

Suppose that $O(\lambda_2(K))$ is of order 4. Then we may write

$$\lambda_2(K) \cong \left\langle \frac{a}{2} \right\rangle \perp \begin{pmatrix} b' & c' \\ c' & d' \end{pmatrix},$$

where $b' \equiv d' \equiv 2 \pmod 4$ and $c' \equiv 0 \pmod 4$. Then we have $\frac{b'd'-c'^2}{4} \equiv \frac{bd-c^2}{4} \equiv 3 \pmod 4$ and hence $\frac{b'}{2} \not\equiv \frac{d'}{2} \pmod 4$. By Case(4.7.1.18), we obtain

$$\gamma_2^K(\lambda_2(K))/\sim = \left\{[M], [\widetilde{M}]\right\},$$

where

$$M = \langle a \rangle \perp \begin{pmatrix} 2b' & c' \\ c' & \frac{d'}{2} \end{pmatrix}, \quad \widetilde{M} = \langle a \rangle \perp \begin{pmatrix} \frac{b'}{2} & c' \\ c' & 2d' \end{pmatrix}.$$

Therefore we have

$$\left\{\mathrm{label}(\gamma_2^L(M)), \mathrm{label}(\gamma_2^L(\widetilde{M}))\right\} = \left\{\{[\![4, \mathrm{I}; \frac{a}{2}]\!]\}, \{[\![4, \mathrm{II}; 2a]\!]\}\right\}.$$

Here, it is clear that $[K] \in \left\{[M], [\widetilde{M}]\right\}$. Actually, this result does not mean that we determined the multiset label$(\gamma_2^L(K))$. But this kind of determination enable us to excute Step (II) of our procedure successfully. Next, suppose that $|O(\lambda_2(K))| = 8$. Then we may write

$$\lambda_2(K) \cong \left\langle \frac{a}{2} \right\rangle \perp \begin{pmatrix} \frac{b'+c'}{2} & \frac{b'-c'}{2} \\ \frac{b'-c'}{2} & \frac{b'+c'}{2} \end{pmatrix},$$

where $b' \equiv c' \equiv 2 \pmod 4$ and $b' + c' \equiv 4 \pmod 8$ by Case(4.6.2.15). But since $\frac{ab'c'}{8} \equiv \frac{dK}{8} \pmod 8$, the integer $\frac{b'+c'}{4}$ is even and this is a contradiction. Therefore there is no such a case.

**Subcase(4.7.1.19-3)** Finally, suppose that $a \equiv 4 \pmod 8$. Then we may assume that $b$ is odd, $c$ is even and $d \equiv 2 \pmod 4$. We may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{I}})^{\frac{1}{2}}\} & \text{if } \frac{a}{4} \cdot \frac{dK}{8} \equiv \epsilon_1\epsilon_2 \pmod 4, \\ \{(L_{\mathrm{VI}})^{\frac{1}{2}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\operatorname{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{I}; 2a]\!]\} & \text{if } \frac{a}{4} \cdot \frac{dK}{8} \equiv \epsilon_1\epsilon_2 \pmod 4, \\ \{[\![4, \mathrm{II}; 2a]\!]\} & \text{otherwise.} \end{cases}$$

**Case(4.7.1.20)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 16\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle 2\epsilon_1, \epsilon_2, 8\epsilon_3 \rangle$. Then the integer $a$ is odd or $a \equiv 2 \pmod 4$ or $a \equiv 8 \pmod{16}$, and we obtain the relation (4.3.6). First, assume that the integer $a$ is odd. Then we may assume that $b \equiv 2 \pmod 4$, $c \equiv 8 \pmod{16}$ and $d \equiv 4 \pmod 8$. Then we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{II}})^{\frac{1}{2}}\} & \text{if } \frac{ab}{2} \equiv \epsilon_1\epsilon_2 \pmod 4, \\ \{(L_{\mathrm{V}})^{\frac{1}{2}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\operatorname{label}(\gamma_2^L(K)) = \{[\![4, \mathrm{I}; 2a]\!]\}.$$

Next, suppose that $a \equiv 2 \pmod 4$. Then we may assume that $b$ is odd, $c \equiv 4 \pmod 8$ and $d \equiv 8 \pmod{16}$. Then we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{III}})^{\frac{1}{2}}\} & \text{if } \frac{ab}{2} \equiv \epsilon_1\epsilon_2 \pmod 8, \\ \{(L_{\mathrm{VI}})^{\frac{1}{2}}\} & \text{otherwise.} \end{cases}$$

Therefore we obtain

$$\operatorname{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{I}; \frac{a}{2}]\!]\} & \text{if } \frac{ab}{2} \equiv \epsilon_1\epsilon_2 \pmod 8, \\ \{[\![4, \mathrm{II}; 2a]\!]\} & \text{otherwise.} \end{cases}$$

We may determine $b$ modulo 8 by the congruent relation $\frac{ab}{2} \equiv \epsilon_1\epsilon_2 \pmod 4$ and Table 4.1 for $S_2(K)$. Finally, suppose that $a \equiv 8 \pmod{16}$. Then we may assume that $b$ is odd, $d \equiv 2 \pmod 4$ and $c$ is even. We may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{I}})^{\frac{1}{2}}\} & \text{if } \frac{dK}{16} \cdot \frac{a}{8} \equiv \epsilon_1\epsilon_2 \pmod 8, \\ \{(L_{\mathrm{VI}})^{\frac{1}{2}}\} & \text{otherwise.} \end{cases}$$

Table 4.1: Values of $S_2(K)$

| | $\frac{a}{2} \equiv 1 \pmod 4$ | $\frac{a}{2} \equiv 3 \pmod 4$ |
|---|---|---|
| $\frac{dK}{16} \equiv 1 \pmod 8$ | $(b,-2)$ | $-(b,2)$ |
| $\frac{dK}{16} \equiv 3 \pmod 8$ | $(b,2)$ | $(b,-2)$ |
| $\frac{dK}{16} \equiv 5 \pmod 8$ | $-(b,-2)$ | $(b,2)$ |
| $\frac{dK}{16} \equiv 7 \pmod 8$ | $-(b,2)$ | $-(b,-2)$ |

Therefore we obtain

$$\operatorname{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4,\mathrm{I};2a]\!]\} & \text{if } \frac{dK}{16}\cdot\frac{a}{8} \equiv \epsilon_1\epsilon_2 \pmod 8, \\ \{[\![4,\mathrm{II};2a]\!]\} & \text{otherwise.} \end{cases}$$

**Case(4.7.1.21)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 5)$. In this case, we have $|\gamma_2^L(K)| = 2$. This case is similar to Case(4.7.1.20) and we may easily obtain

$$\operatorname{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4,\mathrm{I};2a]\!],[\![4,\mathrm{I};2a]\!]\} & \text{if } a \text{ is odd,} \\ \{[\![4,\mathrm{I};\frac{a}{2}]\!],[\![4,\mathrm{II};2a]\!]\} & \text{if } a \equiv 2 \pmod 4, \\ \{[\![4,\mathrm{I};2a]\!],[\![4,\mathrm{II};2a]\!]\} & \text{if } a \equiv 0 \pmod{16}. \end{cases}$$

**Case(4.7.1.22)** $L_2 \cong \langle \epsilon_1, 4\epsilon_2, 4\epsilon_3\rangle$. Then $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, \epsilon_2, \epsilon_3\rangle$ and we obtain the relation (4.3.7). Clearly the integer $a$ is odd, and we may assume that $b$ and $d$ are both odd or both even.

**Subcase(4.7.1.22-1)** Suppose that $K_2$ is isotropic and $\epsilon_2 \equiv \epsilon_2 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 1$. Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{III}}\} & \text{if } b,\ d \text{ are odd and } a \equiv dK \pmod 4, \\ \{L_{\mathrm{I}}\} \text{ or } \{L_{\mathrm{II}}\} & \text{if } b,\ d \text{ are odd and } a \not\equiv dK \pmod 4, \\ \{L_{\mathrm{IV}}\} \text{ or } \{L_{\mathrm{VI}}\} \text{ or } \{L_{\mathrm{VII}}\} & \text{if } b,\ d \text{ are even.} \end{cases}$$

Therefore we have

$$\text{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{I}; a]\!]\} & \text{if } K' \text{ is odd and } a \equiv dK \pmod 4, \\ \{[\![4, \mathrm{I}; 4a]\!]\} & \text{if } K' \text{ is odd and } a \not\equiv dK \pmod 4, \\ \{[\![4, \mathrm{II}; 4a]\!]\} & \text{if } K' \text{ is even,} \end{cases}$$

**Subcase(4.7.1.22-2)** Suppose that $K_2$ is isotropic and $\epsilon_2 \not\equiv \epsilon_2 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 2$ and $\epsilon_1 \not\equiv dK \pmod 4$. Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{I}}, L_{\mathrm{II}}\} & \text{if } b,\ d \text{ are odd and } a \equiv dK \pmod 4, \\ \{L_{\mathrm{I}}, L_{\mathrm{III}}\} \text{ or } \{L_{\mathrm{II}}, L_{\mathrm{III}}\} & \text{if } b,\ d \text{ are odd and } a \not\equiv dK \pmod 4, \\ \{L_{\mathrm{IV}}, L_{\mathrm{VI}}\} & \text{if } b \equiv d \equiv 0 \pmod 4, \\ \{L_{\mathrm{VI}}, L_{\mathrm{VII}}\} & \text{if } b \equiv 2 \text{ and } d \equiv 0 \pmod 4, \\ \{L_{\mathrm{IV}}, L_{\mathrm{VII}}\} & \text{if } b \equiv 0 \text{ and } d \equiv 2 \pmod 4, \end{cases}$$

Therefore we have

$$\text{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{I}; 4a]\!], [\![4, \mathrm{I}; 4a]\!]\} & \text{if } K' \text{ is odd and } a \equiv dK \pmod 4, \\ \{[\![4, \mathrm{I}; 4a]\!], [\![4, \mathrm{I}; a]\!]\} & \text{if } K' \text{ is odd and } a \not\equiv dK \pmod 4, \\ \{[\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\} & \text{if } K' \text{ is even.} \end{cases}$$

**Subcase(4.7.1.22-3)** Suppose that $K_2$ is anisotropic. In this case, we have $|\gamma_2^L(K)| = 3$. Then we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{I}}, L_{\mathrm{II}}, L_{\mathrm{III}}\} & \text{if } b,\ d \text{ are odd,} \\ \{L_{\mathrm{IV}}, L_{\mathrm{VI}}, L_{\mathrm{VII}}\} & \text{if } b,\ d \text{ are even.} \end{cases}$$

Therefore we have

$$\text{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{I}; a]\!], [\![4, \mathrm{I}; 4a]\!], [\![4, \mathrm{I}; 4a]\!]\} & \text{if } K' \text{ is odd,} \\ \{[\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\} & \text{if } K' \text{ is even.} \end{cases}$$

But we can not determine whether $K'$ is even or odd by label$(K)$. This problem is dealt in Section 2 of Chapter 5.

**Case(4.7.1.23)** $L_2 \cong \langle \epsilon_1, 4\epsilon_2, 8\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, \epsilon_2, 2\epsilon_3 \rangle$. Then the integer $a$ is odd or $a \equiv 2 \pmod 4$, and we obtain the relation (4.3.7).

**Subcase(4.7.1.23-1)** Suppose that the integer $a$ is odd. Then we may assume that $b$ is odd, $c$ is even and $d \equiv 2 \pmod 4$. Then we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{II}}, L_{\mathrm{III}}\} \text{ or } \{L_{\mathrm{III}}, L_{\mathrm{V}}\} & \text{if } a \equiv \epsilon_1 \pmod 4, \\ \{L_{\mathrm{II}}, L_{\mathrm{VI}}\} \text{ or } \{L_{\mathrm{V}}, L_{\mathrm{VI}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{I}; a]\!], [\![4, \mathrm{I}; 4a]\!]\} & \text{if } a \equiv \epsilon_1 \pmod 4, \\ \{[\![4, \mathrm{I}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\} & \text{otherwise.} \end{cases}$$

**Subcase(4.7.1.23-2)** Suppose that $a \equiv 2 \pmod 4$. Then we may assume that $b$ and $d$ are odd and $c$ is even. If $\frac{dK}{2} \not\equiv \frac{a}{2} \pmod 4$, then $b \not\equiv d \pmod 4$ and we may easily show that

$$\gamma_2^L(K) = \{L_{\mathrm{I}} \text{ or } L_{\mathrm{II}}, L_{\mathrm{IV}} \text{ or } L_{\mathrm{VI}}\}.$$

If $\frac{dK}{2} \equiv \frac{a}{2} \pmod 4$, then $b \equiv d \pmod 4$ and we have $b \equiv (-1)^\eta S_2(K) \pmod 4$, where

$$\eta = \begin{cases} 1 & \text{if } \frac{a}{2} + \frac{dK}{2} \equiv 6 \pmod 8, \\ 0 & \text{otherwise.} \end{cases}$$

Comparing the local structures, we may show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{I}}, L_{\mathrm{II}}\} & \text{if } \epsilon_1 \equiv (-1)^\eta S_2(K) \pmod 4, \\ \{L_{\mathrm{IV}}, L_{\mathrm{VI}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{I}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\} & \text{if } \frac{dK}{2} \not\equiv \frac{a}{2} \pmod 4, \\ \{[\![4, \mathrm{I}; 4a]\!], [\![4, \mathrm{I}; 4a]\!]\} & \text{if } \frac{dK}{2} \equiv \frac{a}{2} \text{ and } \epsilon_1 \equiv (-1)^\eta S_2(K) \pmod 4, \\ \{[\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\} & \text{otherwise.} \end{cases}$$

**Case(4.7.1.24)** $L_2 \cong \langle \epsilon_1, 4\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 4)$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 4$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, \epsilon_2, 2^{m-2}\epsilon_3 \rangle$. Then the integer $a$ is odd or $a \equiv 0 \pmod 4$, and we obtain the relation (4.3.7).
**Subcase(4.7.1.24-1)** Suppose that the integer $a$ is odd. We may assume that $b$ is odd, $c$ is even and $d \equiv 0 \pmod 4$. Then we may easily show that

$$\gamma_2^L(K) = \{L_{\mathrm{II}}, L_{\mathrm{III}}, L_{\mathrm{V}}, L_{\mathrm{VI}}\}.$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4, \mathrm{I}; a]\!], [\![4, \mathrm{I}; 4a]\!], [\![4, \mathrm{I}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\big\}.$$

**Subcase(4.7.1.24-2)** Suppose that $a \equiv 0 \pmod 4$. We may assume that $b$ and $d$ are odd and $c$ is even. Then we may easily show that

$$\gamma_2^L(K) = \{L_{\mathrm{I}}, L_{\mathrm{II}}, L_{\mathrm{IV}}, L_{\mathrm{VI}}\}.$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4, \mathrm{I}; 4a]\!], [\![4, \mathrm{I}; 4a]\!], [\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\big\}.$$

**Case(4.7.1.25)** $L_2 \cong \langle \epsilon_1, 4\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 4)$, $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 2$ and we obtain the relation (4.3.7).
**Subcase(4.7.1.25-1)** Suppose that the integer $a$ is odd. We may assume that $b$ is odd, $c$ is even and $d \equiv 0 \pmod 4$. Since $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$, we have $a \not\equiv b \pmod 4$. Then we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{III}}, L_{\mathrm{VI}}\} & \text{if } a \equiv \epsilon_1 \pmod 4, \\ \{L_{\mathrm{II}}, L_{\mathrm{V}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \big\{[\![4, \mathrm{I}; a]\!], [\![4, \mathrm{II}; 4a]\!]\big\} & \text{if } a \equiv \epsilon_1 \pmod 4, \\ \big\{[\![4, \mathrm{I}; 4a]\!], [\![4, \mathrm{I}; 4a]\!]\big\} & \text{otherwise.} \end{cases}$$

**Subcase(4.7.1.25-2)** Suppose that $a \equiv 0 \pmod 4$. We may assume that $b$

and $d$ are odd and $c$ is even. Since $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$, we have $b \not\equiv d \pmod 4$. Then we may easily show that

$$\gamma_2^L(K) = \{L_{\mathrm{I}} \text{ or } L_{\mathrm{II}}, L_{\mathrm{IV}} \text{ or } L_{\mathrm{VI}}\}.$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4, \mathrm{I}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\big\}.$$

**Case(4.7.1.26)** $L_2 \cong \langle \epsilon_1, 8\epsilon_2, 8\epsilon_3 \rangle$, $\epsilon_2 \equiv \epsilon_3 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 1$ and the integer $a$ is odd or $a \equiv 2 \pmod 4$. We obtain the relation (4.3.7).

**Subcase(4.7.1.26-1)** Suppose that the integer $a$ is odd. We may assume that $b \equiv d \equiv 2 \pmod 4$ and $c \equiv 0 \pmod 4$. Then we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{III}}\} & \text{if } a \equiv \epsilon_1 \pmod 8, \\ \{L_{\mathrm{VII}}\} & \text{if } a \equiv \epsilon_1 + 4 \pmod 8, \\ \{L_{\mathrm{IV}}\} \text{ or } \{L_{\mathrm{VI}}\} & \text{if } a \not\equiv \epsilon_1 \pmod 4. \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \big\{[\![4, \mathrm{I}; a]\!]\big\} & \text{if } a \equiv \epsilon_1 \pmod 8, \\ \big\{[\![4, \mathrm{II}; 4a]\!]\big\} & \text{otherwise.} \end{cases}$$

**Subcase(4.7.1.26-2)** Suppose that $a \equiv 2 \pmod 4$. We may assume that $b$ is odd, $c \equiv 0 \pmod 4$ and $d \equiv 2 \pmod 4$. Then we have $K_2 \cong \langle a, b, ab \cdot \frac{dK}{4} \rangle$, $d \equiv ab \cdot \frac{dK}{4} \pmod 8$ and $\epsilon_1 \equiv \frac{dK}{4} \pmod 4$. Consider the case that $\frac{a}{2} \equiv 1 \pmod 4$ and $\frac{dK}{4} \equiv 1 \pmod 8$. Then $a \equiv 2 \pmod 8$, $d \equiv 2b \pmod 8$ and $S_2(K) = (b, b)(b, 2)$. We may show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{II}}\} \text{ or } \{L_{\mathrm{V}}\} & \text{if } \epsilon_1 \equiv 1 \pmod 8 \text{ and } S_2(K) = 1, \\ \{L_{\mathrm{IV}}\} \text{ or } \{L_{\mathrm{VII}}\} & \text{if } \epsilon_1 \equiv 1 \pmod 8 \text{ and } S_2(K) = -1, \\ \{L_{\mathrm{IV}}\} \text{ or } \{L_{\mathrm{VII}}\} & \text{if } \epsilon_1 \not\equiv 1 \pmod 8 \text{ and } S_2(K) = 1, \\ \{L_{\mathrm{II}}\} \text{ or } \{L_{\mathrm{V}}\} & \text{if } \epsilon_1 \not\equiv 1 \pmod 8 \text{ and } S_2(K) = -1. \end{cases}$$

Ovserving the other cases, we have

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{II}}\} \text{ or } \{L_{\mathrm{V}}\} & \text{if } \epsilon_1 \equiv \frac{dK}{4} \pmod 8 \text{ and } S_2(K) \equiv \frac{a}{2} \cdot \frac{dK}{4} \pmod 4, \\ \{L_{\mathrm{IV}}\} \text{ or } \{L_{\mathrm{VII}}\} & \text{if } \epsilon_1 \equiv \frac{dK}{4} \pmod 8 \text{ and } S_2(K) \not\equiv \frac{a}{2} \cdot \frac{dK}{4} \pmod 4, \\ \{L_{\mathrm{IV}}\} \text{ or } \{L_{\mathrm{VII}}\} & \text{if } \epsilon_1 \not\equiv \frac{dK}{4} \pmod 8 \text{ and} S_2(K) \equiv \frac{a}{2} \cdot \frac{dK}{4} \pmod 4, \\ \{L_{\mathrm{II}}\} \text{ or } \{L_{\mathrm{V}}\} & \text{if } \epsilon_1 \not\equiv \frac{dK}{4} \pmod 8 \text{ and } S_2(K) \not\equiv \frac{a}{2} \cdot \frac{dK}{4} \pmod 4. \end{cases}$$

Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4,\mathrm{I};4a]\!]\} & \text{if } \epsilon_1 \equiv \frac{dK}{4} \pmod 8 \text{ and } S_2(K) \equiv \frac{a}{2} \cdot \frac{dK}{4} \pmod 4, \\ \{[\![4,\mathrm{II};4a]\!]\} & \text{if } \epsilon_1 \equiv \frac{dK}{4} \pmod 8 \text{ and } S_2(K) \not\equiv \frac{a}{2} \cdot \frac{dK}{4} \pmod 4, \\ \{[\![4,\mathrm{II};4a]\!]\} & \text{if } \epsilon_1 \not\equiv \frac{dK}{4} \pmod 8 \text{ and} S_2(K) \equiv \frac{a}{2} \cdot \frac{dK}{4} \pmod 4, \\ \{[\![4,\mathrm{I};4a]\!]\} & \text{if } \epsilon_1 \not\equiv \frac{dK}{4} \pmod 8 \text{ and } S_2(K) \not\equiv \frac{a}{2} \cdot \frac{dK}{4} \pmod 4. \end{cases}$$

**Case(4.7.1.27)** $L_2 \cong \langle \epsilon_1, 8\epsilon_2, 8\epsilon_3 \rangle$, $\epsilon_2 \not\equiv \epsilon_3 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 2$ and the integer $a$ is odd or $a \equiv 2 \pmod 4$. We obtain the relation (4.3.7).

**Subcase(4.7.1.27-1)** Suppose that the integer $a$ is odd. We may assume that $b \equiv d \equiv 2 \pmod 4$ and $c \equiv 0 \pmod 4$. Then we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{III}}, L_{\mathrm{VII}}\} & \text{if } a \equiv \epsilon_1 \pmod 4, \\ \{L_{\mathrm{IV}}, L_{\mathrm{VI}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4,\mathrm{I};a]\!], [\![4,\mathrm{II};4a]\!]\} & \text{if } a \equiv \epsilon_1 \pmod 4, \\ \{[\![4,\mathrm{II};4a]\!], [\![4,\mathrm{II};4a]\!]\} & \text{otherwise.} \end{cases}$$

**Subcase(4.7.1.27-2)** Suppose that $a \equiv 2 \pmod 4$. We may assume that $b$ is odd, $c$ is even and $d \equiv 2 \pmod 4$. Then we may easily show that

$$\gamma_2^L(K) = \{L_{\mathrm{II}} \text{ or } L_{\mathrm{V}}, L_{\mathrm{IV}} \text{ or } L_{\mathrm{VII}}\}.$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![4, \mathrm{I}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\}.$$

**Case(4.7.1.28)** $L_2 \cong \langle \epsilon_1, 16\epsilon_2, 16\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = 2$ and the integer $a$ is odd or $a \equiv 4 \pmod 8$. We obtain the relation (4.3.7).
**Subcase(4.7.1.28-1)** Suppose that the integer $a$ is odd. We may assume that $b \equiv d \equiv 4 \pmod 8$ and $c \equiv 0 \pmod 8$. Then we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{III}}, L_{\mathrm{VII}}\} & \text{if } a \equiv \epsilon_1 \pmod 8, \\ \{L_{\mathrm{IV}}, L_{\mathrm{VI}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{I}; a]\!], [\![4, \mathrm{II}; 4a]\!]\} & \text{if } a \equiv \epsilon_1 \pmod 8, \\ \{[\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\} & \text{otherwise.} \end{cases}$$

**Subcase(4.7.1.28-2)** Suppose that $a \equiv 4 \pmod 8$. We may assume that $b$ is odd, $c \equiv 0 \pmod 4$ and $d \equiv 4 \pmod 8$. Then we may easily show that

$$\gamma_2^L(K) = \{L_{\mathrm{II}}, L_{\mathrm{VII}}\} \text{ or } \{L_{\mathrm{IV}}, L_{\mathrm{V}}\}.$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![4, \mathrm{I}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\}.$$

**Case(4.7.1.29)** $L_2 \cong \langle \epsilon_1, 2^m\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 5)$. In this case, we have $|\gamma_2^L(K)| = 4$ and the integer $a$ is odd or $a \equiv 0 \pmod 8$. We obtain the relation (4.3.7).
**Subcase(4.7.1.29-1)** Suppose that the integer $a$ is odd. Then we may easily show that

$$\gamma_2^L(K) = \{L_{\mathrm{III}}, L_{\mathrm{IV}}, L_{\mathrm{VI}}, L_{\mathrm{VII}}\}.$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![4, \mathrm{I}; a]\!], [\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\}.$$

**Subcase(4.7.1.29-2)** Suppose that $a \equiv 0 \pmod 8$. We may assume that $b$ is odd, $c$ is even. Then we may easily show that

$$\gamma_2^L(K) = \{L_{\mathrm{II}}, L_{\mathrm{IV}}, L_{\mathrm{V}}, L_{\mathrm{VII}}\}.$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4, \mathrm{I}; 4a]\!], [\![4, \mathrm{I}; 4a]\!], [\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\big\}.$$

**Case(4.7.1.30)** $L_2 \cong \langle \epsilon_1, 8\epsilon_2, 16\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = 1$, and the integer $a$ is odd, $a \equiv 2 \pmod 4$ or $a \equiv 4 \pmod 8$. We obtain the relation (4.3.7).

**Subcase(4.7.1.30-1)** Suppose that the integer $a$ is odd. Then we may assume that $b \equiv 2 \pmod 4$, $c \equiv 0 \pmod 4$ and $d \equiv 4 \pmod 8$. We may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{III}}\} & \text{if } a \equiv \epsilon_1 \pmod 8, \\ \{L_{\mathrm{IV}}\} \text{ or } \{L_{\mathrm{VI}}\} \text{ or } \{L_{\mathrm{VII}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \big\{[\![4, \mathrm{I}; a]\!]\big\} & \text{if } a \equiv \epsilon_1 \pmod 8, \\ \big\{[\![4, \mathrm{II}; 4a]\!]\big\} & \text{otherwise.} \end{cases}$$

**Subcase(4.7.1.30-2)** Suppose that $a \equiv 2 \pmod 4$. We may assume that $b$ is odd, $c \equiv 0 \pmod 4$ and $d \equiv 4 \pmod 8$. Then we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{II}}\} \text{ or } \{L_{\mathrm{V}}\} & \text{if } b \equiv \epsilon_1 \pmod 4, \\ \{L_{\mathrm{IV}}\} \text{ or } \{L_{\mathrm{VII}}\} & \text{otherwise.} \end{cases}$$

Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \big\{[\![4, \mathrm{I}; 4a]\!]\big\} & \text{if } b \equiv \epsilon_1 \pmod 4, \\ \big\{[\![4, \mathrm{II}; 4a]\!]\big\} & \text{otherwise.} \end{cases}$$

In order to determine $\mathrm{label}(\gamma_2^L(K))$, we have to compute the value $b$ modulo

4. Suppose that $\frac{a}{2} \equiv \frac{dK}{8} \pmod 4$. Then $S_2(K) = (a, \frac{a}{2})(a, \frac{dK}{8})(\frac{dK}{8}, \frac{dK}{8})(b, b)$. Therefore we have

$$b \equiv \begin{cases} S_2(K) \pmod 4 & \text{if } \frac{a}{2} \equiv 1 \pmod 4 \text{ and } \frac{a}{2} \equiv \frac{dK}{8} \pmod 8, \\ -S_2(K) \pmod 4 & \text{if } \frac{a}{2} \equiv 1 \pmod 4 \text{ and } \frac{a}{2} \not\equiv \frac{dK}{8} \pmod 8, \\ S_2(K) \pmod 4 & \text{if } \frac{a}{2} \equiv 3 \pmod 4 \text{ and } \frac{a}{2} \not\equiv \frac{dK}{8} \pmod 8, \\ -S_2(K) \pmod 4 & \text{if } \frac{a}{2} \equiv 3 \pmod 4 \text{ and } \frac{a}{2} \equiv \frac{dK}{8} \pmod 8. \end{cases}$$

Next, suppose that $\frac{a}{2} \not\equiv \frac{dK}{8} \pmod 4$. Then we use a similar argument as Case(4.7.1.19-2). We may show that $\mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{I}; \frac{a}{2}]\!]$ by the above results and if we put $\gamma_2^K(\lambda_2(K))/\sim = \{[K], [\widetilde{K}]\}$, we have

$$\left\{\mathrm{label}(\gamma_2^L(K)), \mathrm{label}(\gamma_2^L(\widetilde{K}))\right\} = \left\{\{[\![4, \mathrm{I}; 4a]\!]\}, \{[\![4, \mathrm{II}; 4a]\!]\}\right\}.$$

**Subcase(4.7.1.30-3)** Suppose that $a \equiv 4 \pmod 8$. We may assume that $b$ is odd, $c \equiv 0 \pmod 8$ and $d \equiv 2 \pmod 4$. Then we have $K_2 \cong \langle a, b, b\cdot\frac{a}{2}\cdot\frac{dK}{8}\rangle$ and $d \equiv 2b\cdot\frac{a}{4}\cdot\frac{dK}{8} \pmod 8$. Consider the case that $\frac{a}{4} \equiv 1 \pmod 8$ and $\frac{dK}{8} \equiv 1 \pmod 4$. Then $a \equiv 4 \pmod 8$, $d \equiv 2b \pmod 8$ and $S_2(K) = (b, b)(b, 2)$. We may show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{II}}\} \text{ or } \{L_{\mathrm{V}}\} & \text{if } \epsilon_1 \equiv 1 \pmod 8 \text{ and } S_2(K) = 1, \\ \{L_{\mathrm{IV}}\} \text{ or } \{L_{\mathrm{VII}}\} & \text{if } \epsilon_1 \equiv 1 \pmod 8 \text{ and } S_2(K) = -1, \\ \{L_{\mathrm{II}}\} \text{ or } \{L_{\mathrm{V}}\} & \text{if } \epsilon_1 \equiv 3 \pmod 8 \text{ and } S_2(K) = 1, \\ \{L_{\mathrm{IV}}\} \text{ or } \{L_{\mathrm{VII}}\} & \text{if } \epsilon_1 \equiv 3 \pmod 8 \text{ and } S_2(K) = -1, \\ \{L_{\mathrm{IV}}\} \text{ or } \{L_{\mathrm{VII}}\} & \text{if } \epsilon_1 \equiv 5 \pmod 8 \text{ and } S_2(K) = 1, \\ \{L_{\mathrm{II}}\} \text{ or } \{L_{\mathrm{V}}\} & \text{if } \epsilon_1 \equiv 5 \pmod 8 \text{ and } S_2(K) = -1, \\ \{L_{\mathrm{II}}\} \text{ or } \{L_{\mathrm{V}}\} & \text{if } \epsilon_1 \equiv 7 \pmod 8 \text{ and } S_2(K) = 1, \\ \{L_{\mathrm{IV}}\} \text{ or } \{L_{\mathrm{VII}}\} & \text{if } \epsilon_1 \equiv 7 \pmod 8 \text{ and } S_2(K) = -1. \end{cases}$$

Observing all cases, we may obtain

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{I}; 4a]\!]\} & \text{if } (\frac{a}{4}, \frac{dK}{4})(\frac{a}{4}\cdot\frac{dK}{8}, -\epsilon_1)(\epsilon_1, -2) = S_2(K), \\ \{[\![4, \mathrm{II}; 4a]\!]\} & \text{otherwise}. \end{cases}$$

**Case(4.7.1.31)** $L_2 \cong \langle \epsilon_1, 16\epsilon_2, 32\epsilon_3\rangle$. In this case, we have $|\gamma_2^L(K)| = 2$ and the integer $a$ is odd or $a \equiv 4 \pmod 8$ or $a \equiv 8 \pmod{16}$. We obtain the relation (4.3.7).

**Subcase(4.7.1.31-1)** Suppose that the integer $a$ is odd. Then we may assume that $b \equiv 4 \pmod 8$, $c \equiv 0 \pmod 8$ and $d \equiv 8 \pmod{16}$. We may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{III}}, L_{\mathrm{VI}}\} & \text{if } a \equiv \epsilon_1 \pmod 8, \\ \{L_{\mathrm{IV}}, L_{\mathrm{VII}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\text{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{I}; a]\!], [\![4, \mathrm{II}; 4a]\!]\} & \text{if } a \equiv \epsilon_1 \pmod 8, \\ \{[\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\} & \text{otherwise.} \end{cases}$$

**Subcase(4.7.1.31-2)** Suppose that $a \equiv 4 \pmod 8$. We may assume that $b$ is odd, $c \equiv 0 \pmod 4$ and $d \equiv 8 \pmod{16}$ . Then we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{II}}, L_{\mathrm{V}}\} & \text{if } b \equiv \epsilon_1 \pmod 8, \\ \{L_{\mathrm{IV}}, L_{\mathrm{VII}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\text{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{I}; 4a]\!], [\![4, \mathrm{I}; 4a]\!]\} & \text{if } b \equiv \epsilon_1 \pmod 8, \\ \{[\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\} & \text{otherwise.} \end{cases}$$

We may determine $b$ modulo 8 by the congruent relation $b \equiv \epsilon_1 \pmod 4$ and Table 4.2 for $S_2(K)$.

Table 4.2: Values of $S_2(K)$

| | $\frac{a}{2} \equiv 1 \pmod 8$ | $\frac{a}{2} \equiv 3 \pmod 8$ | $\frac{a}{2} \equiv 5 \pmod 8$ | $\frac{a}{2} \equiv 7 \pmod 8$ |
|---|---|---|---|---|
| $\frac{dK}{32} \equiv 1 \pmod 4$ | $(b, -2)$ | $(b, 2)$ | -(b,-2) | -(b,2) |
| $\frac{dK}{32} \equiv 3 \pmod 4$ | $-(b, 2)$ | $(b, -2)$ | (b,2) | -(b,-2) |

**Subcase(4.7.1.31-3)** Suppose that $a \equiv 8 \pmod{16}$. We may assume that $b$ is odd, $c \equiv 0 \pmod 4$ and $d \equiv 4 \pmod 8$ . Then we may easily show that

$$\gamma_2^L(K) = \{L_{\mathrm{II}}, L_{\mathrm{IV}}\} \text{ or } \{L_{\mathrm{V}}, L_{\mathrm{VII}}\}.$$

Therefore we have $\mathrm{label}(\gamma_2^L(K)) = \big\{\llbracket 4, \mathrm{I}; 4a \rrbracket, \llbracket 4, \mathrm{II}; 4a \rrbracket\big\}$.

**Case(4.7.1.32)** $L_2 \cong \langle \epsilon_1, 2^m\epsilon_2, 2^{m+1}\epsilon_3 \rangle$ $(m \geqslant 5)$. In this case, we have $|\gamma_2^L(K)| = 4$ and the integer $a$ is odd or $a \equiv 0 \pmod 8$. We obtain the relation (4.3.7). Then we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{III}}, L_{\mathrm{IV}}, L_{\mathrm{VI}}, L_{\mathrm{VII}}\} & \text{if } a \text{ is odd,} \\ \{L_{\mathrm{II}}, L_{\mathrm{IV}}, L_{\mathrm{V}}, L_{\mathrm{VII}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \big\{\llbracket 4, \mathrm{I}; a \rrbracket, \llbracket 4, \mathrm{II}; 4a], \llbracket 4, \mathrm{II}; 4a], \llbracket 4, \mathrm{II}; 4a \rrbracket\big\} & \text{if } a \text{ is odd,} \\ \big\{\llbracket 4, \mathrm{I}; 4a \rrbracket, \llbracket 4, \mathrm{I}; 4a], \llbracket 4, \mathrm{II}; 4a], \llbracket 4, \mathrm{II}; 4a \rrbracket\big\} & \text{otherwise.} \end{cases}$$

**Case(4.7.1.33)** $L_2 \cong \langle \epsilon_1, 8\epsilon_2, 32\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = 2$ and the integer $a$ is odd or $a \equiv 2 \pmod 4$ or $a \equiv 8 \pmod{16}$. We obtain the relation (4.3.7).

**Subcase(4.7.1.33-1)** Suppose that the integer $a$ is odd. Then we may assume that $b \equiv 2 \pmod 4$, $c \equiv 0 \pmod 8$ and $d \equiv 8 \pmod{16}$. We may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{III}}, L_{\mathrm{VI}}\} & \text{if } a \equiv \epsilon_1 \pmod 8, \\ \{L_{\mathrm{IV}}, L_{\mathrm{VII}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \big\{\llbracket 4, \mathrm{I}; a \rrbracket, \llbracket 4, \mathrm{II}; 4a \rrbracket\big\} & \text{if } a \equiv \epsilon_1 \pmod 8, \\ \big\{\llbracket 4, \mathrm{II}; 4a \rrbracket, \llbracket 4, \mathrm{II}; 4a \rrbracket\big\} & \text{otherwise.} \end{cases}$$

**Subcase(4.7.1.33-2)** Suppose that $a \equiv 2 \pmod 4$. Then we may assume that $b$ is odd, $c \equiv 0 \pmod 4$ and $d \equiv 8 \pmod{16}$. Since $b \equiv \frac{a}{2}\epsilon_1\epsilon_2 \pmod 4$, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{II}}, L_{\mathrm{V}}\} & \text{if } \frac{a}{2} \equiv \epsilon_2 \pmod 4, \\ \{L_{\mathrm{IV}}, L_{\mathrm{VII}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4,\mathrm{I};4a]\!],[\![4,\mathrm{I};4a]\!]\} & \text{if } \frac{a}{2} \equiv \epsilon_2 \pmod 4, \\ \{[\![4,\mathrm{II};4a]\!],[\![4,\mathrm{II};4a]\!]\} & \text{otherwise.} \end{cases}$$

**Subcase(4.7.1.33-3)** Suppose that $a \equiv 8 \pmod{16}$. Then we may assume that $b$ is odd, $c \equiv 0 \pmod 2$ and $d \equiv 2 \pmod 4$. Since $b \equiv \frac{a}{2}\epsilon_1\epsilon_2 \pmod 4$, we may easily show that

$$\gamma_2^L(K) = \{L_{\mathrm{II}}, L_{\mathrm{IV}}\} \text{ or } \{L_{\mathrm{V}}, L_{\mathrm{VII}}\}.$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4,\mathrm{I};4a]\!],[\![4,\mathrm{II};4a]\!]\big\}.$$

**Case(4.7.1.34)** $L_2 \cong \langle \epsilon_1, 16\epsilon_2, 64\epsilon_3\rangle$. In this case, we have $|\gamma_2^L(K)| = 2$ and the integer $a$ is odd or $a \equiv 4 \pmod 8$ or $a \equiv 16 \pmod{32}$. We obtain the relation (4.3.7).

**Subcase(4.7.1.34-1)** Suppose that the integer $a$ is odd. This case is similar to Subcase
(4.7.1.33- 1) and we obtain the same result.

**Subcase(4.7.1.34-2)** Suppose that $a \equiv 4 \pmod 8$. This case is similar to Subcase(4.7.1.33-2) but we can not determine $b$ modulo 8. By the similar argument as Subcase(4.7.1.19-2), if we put $\gamma_2^K(\lambda_2(K))/\sim = \big\{[K],[\widetilde{K}]\big\}$, we have

$$\Big\{\mathrm{label}(\gamma_2^L(K)), \mathrm{label}(\gamma_2^L(\widetilde{K}))\Big\} = \Big\{\{[\![4,\mathrm{I};4a]\!],[\![4,\mathrm{I};4a]\!]\},\{[\![4,\mathrm{II};4a]\!],[\![4,\mathrm{II};4a]\!]\}\Big\}.$$

**Subcase(4.7.1.34-3)** Suppose that $a \equiv 16 \pmod{32}$. This case is similar to Subcase(4.7.1.33-3) and we obtain the same result.

**Case(4.7.1.35)** $L_2 \cong \langle \epsilon_1, 2^m\epsilon_2, 2^{m+2}\epsilon_3\rangle$ $(m \geqslant 5)$. In this case, we have

$|\gamma_2^L(K)| = 4$ and the integer $a$ is odd or even. We obtain the relation (4.3.7). We may easily show that

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4,\mathrm{I};a]\!],[\![4,\mathrm{II};4a]\!],[\![4,\mathrm{II};4a]\!],[\![4,\mathrm{II};4a]\!]\} & \text{if } a \text{ is odd,} \\ \{[\![4,\mathrm{I};4a]\!],[\![4,\mathrm{I};4a]\!],[\![4,\mathrm{II};4a]\!],[\![4,\mathrm{II};4a]\!]\} & \text{otherwise.} \end{cases}$$

**Case(4.7.1.36)** $L_2 \cong \langle \epsilon_1, 8\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 6)$. In this case, we have $|\gamma_2^L(K)| = 2$ and the integer $a$ is odd or $a \equiv 2 \pmod 4$ or $a \equiv 0 \pmod{16}$. We obtain the relation (4.3.7). We may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{III}}, L_{\mathrm{VI}}\} & \text{if } a \equiv \epsilon_1 \pmod 8, \\ \{L_{\mathrm{IV}}, L_{\mathrm{VII}}\} & \text{if } a \text{ is odd and } a \not\equiv \epsilon_1 \pmod 8, \\ \{L_{\mathrm{II}}, L_{\mathrm{V}}\} & \text{if } \frac{a}{2} \equiv \epsilon_2 \pmod 8, \\ \{L_{\mathrm{IV}}, L_{\mathrm{VII}}\} & \text{if } a \equiv 2 \pmod 4 \text{ and } \frac{a}{2} \not\equiv \epsilon_2 \pmod 8, \\ \{L_{\mathrm{II}}, L_{\mathrm{IV}}\} \text{ or } \{L_{\mathrm{V}}, L_{\mathrm{VII}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4,\mathrm{I};a]\!],[\![4,\mathrm{II};4a]\!]\} & \text{if } a \equiv \epsilon_1 \pmod 8, \\ \{[\![4,\mathrm{II};4a]\!],[\![4,\mathrm{II};4a]\!]\} & \text{if } a \text{ is odd and } a \not\equiv \epsilon_1 \pmod 8, \\ \{[\![4,\mathrm{I};4a]\!],[\![4,\mathrm{I};4a]\!]\} & \text{if } \frac{a}{2} \equiv \epsilon_2 \pmod 8, \\ \{[\![4,\mathrm{II};4a]\!],[\![4,\mathrm{II};4a]\!]\} & \text{if } a \equiv 2 \pmod 4 \text{ and } \frac{a}{2} \not\equiv \epsilon_2 \pmod 8, \\ \{[\![4,\mathrm{I};4a]\!],[\![4,\mathrm{II};4a]\!]\} & \text{otherwise.} \end{cases}$$

**Case(4.7.1.37)** $L_2 \cong \langle \epsilon_1, 16\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 7)$. This case is similar to Case(4.7.1.36) and we obtain the same result.

**Case(4.7.1.38)** $L_2 \cong \langle \epsilon_1, 2^n\epsilon_2, 2^m\epsilon_3\rangle$ $(8 \leqslant n+3 \leqslant m)$. This case is also similar to Case(4.7.1.36) and we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4,\mathrm{I};a]\!],[\![4,\mathrm{II};4a]\!],[\![4,\mathrm{II};4a]\!],[\![4,\mathrm{II};4a]\!]\} & \text{if } a \text{ is odd,} \\ \{[\![4,\mathrm{I};4a]\!],[\![4,\mathrm{I};4a]\!],[\![4,\mathrm{II};4a]\!],[\![4,\mathrm{II};4a]\!]\} & \text{otherwise.} \end{cases}$$

$\square$

We need to determine whether the binary lattice $K'$ is even or odd in the four cases Case(4.7.1.1), Case(4.7.1.6), Case(4.7.1.10) and Case(4.7.1.22). We are going to deal this problem in Section 2 of Chapter 5.

Next, we consider the case when $K = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z \cong K_{4,\mathrm{II}}(a,b,c,d)$. Then we may obtain

$$L_{\mathrm{I}} \cong \begin{pmatrix} 8a & 4a & 0 \\ 4a & 4b & 2c \\ 0 & 2c & d \end{pmatrix}, \quad L_{\mathrm{II}} = \tau_x(L_{\mathrm{IV}}) \cong \begin{pmatrix} 8a & 2a & 0 \\ 2a & b & 2c \\ 0 & 2c & 4d \end{pmatrix},$$

$$L_{\mathrm{III}} \cong \langle 2a \rangle \perp \begin{pmatrix} 4b-2a & -4c \\ -4c & 4d \end{pmatrix}, \quad L_{\mathrm{VI}} \cong \begin{pmatrix} 2a+d & 2(a+c) & 2d \\ 2(a+c) & 4b & 4c \\ 2d & 4c & 4d \end{pmatrix},$$

$$L_{\mathrm{V}} = \tau_x(L_{\mathrm{VII}}) \cong \begin{pmatrix} 8a & 2a & 0 \\ 2a & b+2c+d & 2(c+d) \\ 0 & 2(c+d) & 4d \end{pmatrix},$$

and

$$L'_{\mathrm{I}} = \tau_x(L'_{\mathrm{III}}) \cong \begin{pmatrix} 8a & 2a & 0 \\ 2a & b & c \\ 0 & c & d \end{pmatrix}, \quad L'_{\mathrm{II}} \cong \langle 2a \rangle \perp \begin{pmatrix} 4b-2a & 2c \\ 2c & d \end{pmatrix},$$

$$L'_{\mathrm{IV}} \cong \begin{pmatrix} 2a & a & 0 \\ a & b & 2c \\ 0 & 2c & 4d \end{pmatrix}, \quad L'_{\mathrm{V}} = \tau_x(L'_{\mathrm{VII}}) \cong \begin{pmatrix} 2a+d & a+c & 2d \\ a+c & b & 2c \\ 2d & 2c & 4d \end{pmatrix},$$

$$L'_{\mathrm{VI}} \cong \begin{pmatrix} 2a & a & 0 \\ a & b+2c+d & 2(c+d) \\ 0 & 2(c+d) & 4d \end{pmatrix}.$$

**Theorem 4.7.2.** *Let $L$ be a primitive ternary lattice which is not stable over $\mathbb{Z}_2$ and let $K := \lambda_{2e}(L)$. Suppose that $|O(K)| = 4$ and $K$ is of type $\mathrm{II}$. Then the multiset* label$(\gamma^L_{2e}(K))$ *is completely determined by $dK$,* label$(K)$, label$(\lambda_2(K))$ *and the structure of $L_2$.*

*Proof.* Since $K \cong K_{4,\mathrm{II}}(a,b,c,d)$ for some integers $a$, $b$, $c$ and $d$, we have

$$K \cong \begin{pmatrix} 2a & a & a \\ a & b & c \\ a & c & d \end{pmatrix},$$

and the label of $K$ is $[\![4,\mathrm{II};2a]\!]$.

**Case(4.7.2.1)** $L_2 \cong T \perp \langle 4\epsilon \rangle$. In this case, we have $|\gamma_4^L(K)| = 1$ and $K_2 = \lambda_4(L_2) \cong T \perp \langle \epsilon \rangle$. Then we may assume that $a$, $b$ and $d$ are odd by a suitable base change since $dK = 2a(bd - c^2) - ad^2$. Then we have

$$\gamma_4^L(K) = \{L'_{\mathrm{IV}}\} \text{ or } \{L'_{\mathrm{VI}}\}.$$

Therefore we have

$$\mathrm{label}(\gamma_4^L(K)) = \big\{[\![4,\mathrm{II};2a]\!]\big\}.$$

**Case(4.7.2.2)** $L_2 \cong T \perp \langle 8\epsilon \rangle$. In this case, we have $|\gamma_4^L(K)| = 2 + \chi(T)$ and $K_2 = \lambda_4(L_2) \cong T \perp \langle 2\epsilon \rangle$. Then we may assume that the integer $a$ is odd, $b \equiv 0 \pmod 4$ and $d$ is even by a suitable base change. Then we have

$$\gamma_4^L(K) = \begin{cases} \{L'_{\mathrm{IV}}, L'_{\mathrm{V}}, L'_{\mathrm{VII}}\} & \text{if } d \equiv 2 \pmod 4 \text{ and } T \cong \mathbb{H}, \\ \{L'_{\mathrm{I}}, L'_{\mathrm{III}}, L'_{\mathrm{IV}}\} & \text{if } d \equiv 0 \pmod 4 \text{ and } T \cong \mathbb{H}, \\ \{L'_{\mathrm{VI}}\} & \text{if } T \cong \mathbb{A}. \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_4^L(K)) = \begin{cases} \big\{[\![2]\!], [\![4,\mathrm{II};2a]\!]\big\} & \text{if } T \cong \mathbb{H}, \\ \big\{[\![4,\mathrm{II};2a]\!]\big\} & \text{if } T \cong \mathbb{A}. \end{cases}$$

**Case(4.7.2.3)** $L_2 \cong T \perp \langle 2^m\epsilon \rangle$ $(m \geqslant 4)$. In this case, we have $|\gamma_4^L(K)| = 4$ and $K_2 = \lambda_4(L_2) \cong T \perp \langle 2^{m-2}\epsilon \rangle$. Then we may easily show that

$$\mathrm{label}(\gamma_4^L(K)) = \begin{cases} \big\{[\![2]\!], [\![4,\mathrm{II};2a]\!], [\![4,\mathrm{II};2a]\!]\big\} & \text{if } a \text{ is odd}, \\ \big\{[\![2]\!], [\![2]\!]\big\} & \text{otherwise}. \end{cases}$$

**Case(4.7.2.4)** $L_2 \cong \langle\epsilon\rangle \perp 2T$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle 2\epsilon\rangle \perp T$. Then we may assume that the integer $a$ is odd, $b \equiv 0 \pmod 4$ and $d$ is even by a suitable base change. Then we have

$$\gamma_4^L(K) = \begin{cases} \{(L_{\mathrm{I}})^{\frac{1}{2}}\} & \text{if } d \equiv 2 \pmod 4, \\ \{(L_{\mathrm{VI}})^{\frac{1}{2}}\} & \text{if } d \equiv 0 \pmod 4. \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![4, \mathrm{II}; 4a]\!]\}.$$

**Case(4.7.2.5)** $L_2 \cong \langle\epsilon\rangle \perp 4T$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle\epsilon\rangle \perp T$. Then the integers $a$ and $d$ are odd, and we may assume that $b$ and $c$ are even. Then we have

$$\gamma_4^L(K) = \{L_{\mathrm{I}}\}.$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![4, \mathrm{II}; 8a]\!]\}.$$

**Case(4.7.2.6)** $L_2 \cong \langle\epsilon\rangle \perp 8T$. In this case, we have $|\gamma_2^L(K)| = 2 + \chi(T)$ and $K_2 = \lambda_2(L_2) \cong \langle\epsilon\rangle \perp 2T$. Then we may show that $a \equiv 2 \pmod 4$ and may assume that $b \equiv 0 \pmod 4$, $c \equiv 0 \pmod{16}$ and $d$ is odd by a suitable base change using the fact that $bd - c^2 \equiv 0 \pmod 4$. Then we may easily show that

$$\gamma_4^L(K) = \begin{cases} \{L_{\mathrm{I}}, L_{\mathrm{V}}, L_{\mathrm{VII}}\} & \text{if } T \cong \mathbb{H} \text{ and } b \equiv 0 \pmod 8, \\ \{L_{\mathrm{V}}, L_{\mathrm{VI}}, L_{\mathrm{VII}}\} & \text{if } T \cong \mathbb{H} \text{ and } b \equiv 4 \pmod 8, \\ \{L_{\mathrm{VI}}\} & \text{if } T \cong \mathbb{A} \text{ and } b \equiv 0 \pmod 8, \\ \{L_{\mathrm{I}}\} & \text{if } T \cong \mathbb{A} \text{ and } b \equiv 4 \pmod 8. \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![2]\!], [\![4, \mathrm{II}; 8a]\!]\} & \text{if } T \cong \mathbb{H}, \\ \{[\![4, \mathrm{II}; 8a]\!]\} & \text{if } T \cong \mathbb{A}. \end{cases}$$

**Case(4.7.2.7)** $L_2 \cong \langle\epsilon\rangle \perp 2^m T$. In this case, we have $|\gamma_2^L(K)| = 4$ and $K_2 = \lambda_2(L_2) \cong \langle\epsilon\rangle \perp 2^{m-2}T$. Then we may show that the integer $a$ is even and may assume that $b$ is odd. By the structure of $K_2$, we have $\mathrm{ord}_2(2ab-a^2) \geqslant m-1$ and $\mathrm{ord}_2(abc) = m-2$. We may verify that if $a \equiv 2 \pmod 4$, then $c$ and $d$ are both even, and if $a \equiv 0 \pmod 4$, then $\mathrm{ord}_2(a) = m-2$ and $c$ and $d$ are both odd. In the latter case, we may assume that $b$ is even and $d$ is odd by a suitable base change. Then we may easily show that

$$\gamma_4^L(K) = \begin{cases} \{L_{\mathrm{II}}, L_{\mathrm{IV}}, L_{\mathrm{V}}, L_{\mathrm{VII}}\} & \text{if } a \equiv 2 \pmod 4, \\ \{L_{\mathrm{I}}, L_{\mathrm{V}}, L_{\mathrm{VI}}, L_{\mathrm{VII}}\} & \text{if } a \equiv 0 \pmod 4. \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![2]\!], [\![2]\!]\} & \text{if } a \equiv 2 \pmod 4, \\ \{[\![2]\!], [\![4, \mathrm{II}; 8a]\!]\} & \text{if } a \equiv 0 \pmod 4. \end{cases}$$

**Case(4.7.2.8)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 2\epsilon_3 \rangle$. In this case, the lattice $K_2$ is odd unimodular and we obtain the relation (4.3.6). Then we may show that the integers $a$ and $d$ are odd and may assume that $b$ is even by the suitable base change. Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{II}})^{\frac{1}{2}}, (L_{\mathrm{III}})^{\frac{1}{2}}, (L_{\mathrm{IV}})^{\frac{1}{2}}\} & \text{if } K_2 \text{ is anisotropic}, \\ \{(L_{\mathrm{III}})^{\frac{1}{2}}\} & \text{if } K_2 \text{ is isotropic}. \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![2]\!], [\![4, \mathrm{I}; a]\!]\} & \text{if } K_2 \text{ is anisotropic}, \\ \{[\![4, \mathrm{I}; a]\!]\} & \text{if } K_2 \text{ is isotropic}. \end{cases}$$

**Case(4.7.2.9)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 4\epsilon_3 \rangle$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1', \epsilon_2', 2\epsilon_3 \rangle$. Then the integers $a$ and $d$ are odd and $d$ is even. We obtain the relation (4.3.6). Comparing the local structures, we may easily show that $\gamma_2^L(K) = \{(L_{\mathrm{III}})^{\frac{1}{2}}\}$ and hence we have

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![4, \mathrm{I}; a]\!]\}.$$

**Case(4.7.2.10)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 8\epsilon_3 \rangle$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$. In this case, we have

$|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1', \epsilon_2', 4\epsilon_3\rangle$. Then we may assume that the integer $b$ is odd, and we obtain the relation (4.3.6).

**Subcase(4.7.2.10-1)** Suppose that the integer $a$ is odd and $d \equiv 2 \pmod 4$. Then clearly $c$ is odd since $bd - c^2$ is odd. Comparing the local structures, we may show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{III}})^{\frac{1}{2}}\} & \text{if } a \equiv 1 \pmod 4 \text{ and } S_2(\langle \epsilon_1, \epsilon_2\rangle) = 1, \text{ or} \\ & \text{if } a \equiv 3 \pmod 4 \text{ and } S_2(\langle \epsilon_1, \epsilon_2\rangle) = -1, \\ \{(L_{\mathrm{I}})^{\frac{1}{2}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{I}; a]\!]\} & \text{if } a \equiv 1 \pmod 4 \text{ and } S_2(\langle \epsilon_1, \epsilon_2\rangle) = 1, \text{ or} \\ & \text{if } a \equiv 3 \pmod 4 \text{ and } S_2(\langle \epsilon_1, \epsilon_2\rangle) = -1, \\ \{[\![4, \mathrm{II}; 4a]\!]\} & \text{otherwise.} \end{cases}$$

**Subcase(4.7.2.10-2)** Suppose that the integer $a$ is odd and $d \equiv 0 \pmod 4$. Then $c$ is even and $d \equiv 4 \pmod 8$ since $dK \equiv 4 \pmod 8$. This case is similar to the previous case and we have the same result.

**Subcase(4.7.2.10-3)** Suppose that $a \equiv 2 \pmod 4$. Then $c$ is odd and we may easily show that $\gamma_2^L(K) = \{(L_{\mathrm{I}})^{\frac{1}{2}}\}$ or $\{(L_{\mathrm{VI}})^{\frac{1}{2}}\}$. Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![4, \mathrm{II}; 4a]\!]\}.$$

**Case(4.7.2.11)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 4)$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 2$ and we may assume that $b$ is odd. Clearly we obtain the relation (4.3.6).

**Subcase(4.7.2.11-1)** Suppose that the integer $a$ is odd. Then we may easily show that

$$\gamma_2^L(K) = \{(L_{\mathrm{I}})^{\frac{1}{2}}, (L_{\mathrm{III}})^{\frac{1}{2}}\} \text{ or } \{(L_{\mathrm{III}})^{\frac{1}{2}}, (L_{\mathrm{VI}})^{\frac{1}{2}}\}.$$

Hence we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![4, \mathrm{I}; a]\!], [\![4, \mathrm{II}; 4a]\!]\}.$$

**Subcase(4.7.2.11-2)** Suppose that $a \equiv 2 \pmod 4$. Then $c$ is even and $bd \equiv 1 \pmod 4$. Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \{(L_{\mathrm{V}})^{\frac{1}{2}}, (L_{\mathrm{VII}})^{\frac{1}{2}}\}.$$

Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![2]\!]\big\}.$$

**Subcase(4.7.2.11-3)** Suppose that $a \equiv 0 \pmod 4$. Then $c$ or $d$ is odd since the unimodular component of $K_2$ has rank 2. Therefore we have $bd - c^2 \equiv 1 \pmod 4$. If $c$ is odd and $d$ is even, $\gamma_2^L(K) = \{(L_{\mathrm{I}})^{\frac{1}{2}}, (L_{\mathrm{VI}})^{\frac{1}{2}}\}$ and then $\mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{I}; a]\!]$ by the previous results. If $c$ is even and $d$ is odd, we may also show that $\gamma_2^L(K) = \{(L_{\mathrm{V}})^{\frac{1}{2}}, (L_{\mathrm{VII}})^{\frac{1}{2}}\}$ and then $\mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{II}; a]\!]$. Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \big\{[\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\big\} & \text{if } \mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{I}; a]\!], \\ \big\{[\![2]\!]\big\} & \text{if } \mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{II}; a]\!]. \end{cases}$$

**Case(4.7.2.12)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 4\epsilon_3 \rangle$, $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$. In this case, we have $|\gamma_2^L(K)| = 3$ and $K_2 = \lambda_2(L_2) \cong T \perp \langle 2\epsilon_3 \rangle$. Then $a$ is odd and $b$, $d$ are even. we obtain the relation (4.3.6). Comparing the local structures, we may easily show that

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![2]\!], [\![4, \mathrm{I}; a]\!]\big\}.$$

**Case(4.7.2.13)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 3)$, $\epsilon_1\epsilon_2 \equiv 7 \pmod 8$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \mathbb{H} \perp \langle 2^{m-1}\epsilon_3 \rangle$. We obtain the relation (4.3.6).

**Subcase(4.7.2.13-1)** Suppose that the integer $a$ is odd. Then we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{I}})^{\frac{1}{2}}, (L_{\mathrm{III}})^{\frac{1}{2}}\} & \text{if } c \text{ is odd}, \\ \{(L_{\mathrm{III}})^{\frac{1}{2}}, (L_{\mathrm{VI}})^{\frac{1}{2}}\} & \text{otherwise}. \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![4, \mathrm{I}; a]\!], [\![4, \mathrm{II}; 4a]\!]\big\}.$$

**Subcase(4.7.2.13-2)** Suppose that the integer $a$ is even. Then $c$ is odd and

$bd \equiv 0 \pmod 8$. We may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{II}})^{\frac{1}{2}}, (L_{\mathrm{IV}})^{\frac{1}{2}}\} & \text{if } b \equiv 2 \pmod 4 \text{ and } d \equiv 0 \pmod 4, \\ \{(L_{\mathrm{I}})^{\frac{1}{2}}, (L_{\mathrm{VI}})^{\frac{1}{2}}\} & \text{if } b \equiv 0 \pmod 4 \text{ and } d \equiv 2 \pmod 4, \\ \{(L_{\mathrm{V}})^{\frac{1}{2}}, (L_{\mathrm{VII}})^{\frac{1}{2}}\} & \text{if } b \equiv d \equiv 0 \pmod 4. \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![2]\!]\} & \text{if } d \equiv 0 \pmod 4, \\ \{[\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\} & \text{if } d \equiv 2 \pmod 4. \end{cases}$$

Now we have to determine the value $d$ modulo 4. Suppose that $a \equiv 2 \pmod 4$. Then we obtain $d \equiv -\frac{a}{2} - \frac{dK}{4} \pmod 8$ since $dK \equiv 4 \pmod 8$. Next, suppose that $a \equiv 4 \pmod 8$. Then we obtain $d \equiv \frac{1}{2}(-\frac{a}{4} - \frac{dK}{8}) \pmod 4$ since $dK \equiv 8 \pmod{16}$. Finally we suppose that $a \equiv 0 \pmod 8$. Then we have

$$\lambda_4(K) = \begin{pmatrix} 2a & 2a & 0 \\ 2a & 4b & 4c \\ 0 & 4c & 4d \end{pmatrix}^{\frac{1}{4}} \cong \left\langle \frac{a}{2} \right\rangle \perp \begin{pmatrix} b - \frac{a}{2} & c \\ c & d \end{pmatrix}$$

and we may show that $\mathrm{label}(\lambda_4(K)) = [\![4, \mathrm{I}; \frac{a}{2}]\!]$ or $[\![8, \mathrm{II}; \frac{a}{2}, *, *]\!]$ by the previous results. Suppose that $O(\lambda_4(K))$ is of order 4. Then we have

$$\mathrm{label}(\gamma_4^K(\lambda_4(K))) = \{[\![4, \mathrm{I}; 2a]\!], [\![4, \mathrm{II}; 2a]\!], [\![4, \mathrm{II}; 2a]\!], [\![4, \mathrm{II}; 2a]\!]\}.$$

Observing each entry corresponding to $d$ in the three Gram matrices of type II, we may verify that two of them are divisible by 4 and the rest one is not. Put $\gamma_4^K(\lambda_4(K))/\sim = \{[K], [K_1], [K_2], [K_3]\}$, where $K$, $K_1$ and $K_2$ are of type II. Then we may conclude that

$$\{\mathrm{label}(\gamma_2^L(K)), \mathrm{label}(\gamma_2^L(K_1)), \mathrm{label}(\gamma_2^L(K_2))\} = \{\{[\![2]\!]\}, \{[\![2]\!]\}, \{[\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\}\}.$$

Next, suppose that $O(\lambda_4(K))$ is of order 8. Then we have

$$\gamma_4^K(\lambda_4(K)) = \{[\![4, \mathrm{II}; 2a]\!]\},$$

and the entry corresponding to $d$ in the Gram matrix is divisible by 4. Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![2]\!]\}.$$

**Case(4.7.2.14)** $L_2 \cong \langle \epsilon_1, \epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 3)$, $\epsilon_1\epsilon_2 \equiv 3 \pmod 8$. In this case, we have $|\gamma_2^L(K)| = 6$ and $K_2 = \lambda_2(L_2) \cong \mathbb{A} \perp \langle 2^{m-1}\epsilon_3 \rangle$. Clearly we obtain the relation (4.3.6). Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{I}})^{\frac{1}{2}}, (L_{\mathrm{II}})^{\frac{1}{2}}, (L_{\mathrm{III}})^{\frac{1}{2}}, (L_{\mathrm{IV}})^{\frac{1}{2}}, (L_{\mathrm{V}})^{\frac{1}{2}}, (L_{\mathrm{VII}})^{\frac{1}{2}}\} & \text{if } a,\, c \text{ are odd,} \\ \{(L_{\mathrm{I}})^{\frac{1}{2}}, (L_{\mathrm{II}})^{\frac{1}{2}}, (L_{\mathrm{IV}})^{\frac{1}{2}}, (L_{\mathrm{V}})^{\frac{1}{2}}, (L_{\mathrm{VI}})^{\frac{1}{2}}, (L_{\mathrm{VII}})^{\frac{1}{2}}\} & \text{if } a \text{ is even,} \\ \{(L_{\mathrm{II}})^{\frac{1}{2}}, (L_{\mathrm{III}})^{\frac{1}{2}}, (L_{\mathrm{IV}})^{\frac{1}{2}}, (L_{\mathrm{V}})^{\frac{1}{2}}, (L_{\mathrm{VI}})^{\frac{1}{2}}, (L_{\mathrm{VII}})^{\frac{1}{2}}\} & \text{otherwise.} \end{cases}$$

Hence we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![2]\!], [\![2]\!], [\![4, \mathrm{I}; a]\!], [\![4, \mathrm{II}; 4a]\!]\} & \text{if } a \text{ is odd,} \\ \{[\![2]\!], [\![2]\!], [\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\} & \text{if } a \text{ is even.} \end{cases}$$

**Case(4.7.2.15)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 2\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle 2\epsilon_1, \epsilon_2, \epsilon_3 \rangle$. Clearly we obtain the relation (4.3.6). We obtain that $a$ is odd and $d$ is even. Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \{(L_{\mathrm{I}})^{\frac{1}{2}}\} \text{ or } \{(L_{\mathrm{VI}})^{\frac{1}{2}}\}.$$

Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![4, \mathrm{II}; 4a]\!]\}.$$

**Case(4.7.2.16)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 4\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle 2\epsilon_1, \epsilon_2, 2\epsilon_3 \rangle$. Clearly we obtain the relation (4.3.6). We obtain that $a \equiv 2 \pmod 4$, $b \equiv c \pmod 2$ and $d$ is odd. Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \{(L_{\mathrm{II}})^{\frac{1}{2}}, (L_{\mathrm{IV}})^{\frac{1}{2}}\} \text{ or } \{(L_{\mathrm{V}})^{\frac{1}{2}}, (L_{\mathrm{VII}})^{\frac{1}{2}}\}.$$

Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![2]\!]\}.$$

**Case(4.7.2.17)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 8\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle 2\epsilon_1, \epsilon_2, 4\epsilon_3 \rangle$. We may easily show that $a \equiv 2 \pmod 4$ since $K_2$ has no unimodular component of rank 2 and $dK \equiv 8 \pmod{16}$. Further, we may show that the integers $d$ and $c$ are even and $b$ is odd. Then we have $dK \equiv 0 \pmod{16}$ and this is a contradiction. Therefore this case does not occur.

**Case(4.7.2.18)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 16\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle 2\epsilon_1, \epsilon_2, 8\epsilon_3 \rangle$. Clearly we obtain the relation (4.3.6) and the integer $a$ is even. Comparing the local structures, we may show that the possible lattices are $(L_{\mathrm{I}})^{\frac{1}{2}}$ and $(L_{\mathrm{VI}})^{\frac{1}{2}}$. We may verify that $(L_{\mathrm{I}})_2 \not\cong (L_{\mathrm{VI}})_2$ and hence

$$\gamma_2^L(K) = \{(L_{\mathrm{I}})^{\frac{1}{2}}\} \text{ or } \{(L_{\mathrm{VI}})^{\frac{1}{2}}\}.$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \left\{[\![4, \mathrm{II}; 4a]\!]\right\}.$$

Note that $bd - c^2 \equiv 2 \pmod 4$.

**Case(4.7.2.19)** $L_2 \cong \langle \epsilon_1, 2\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 5)$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle 2\epsilon_1, \epsilon_2, 2^{m-1}\epsilon_3 \rangle$. Clearly we obtain the relation (4.3.6) and the integer $a$ is even. Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{(L_{\mathrm{I}})^{\frac{1}{2}}, (L_{\mathrm{VI}})^{\frac{1}{2}}\} & \text{if } a \equiv 2 \pmod 4, \\ \{(L_{\mathrm{II}})^{\frac{1}{2}}, (L_{\mathrm{IV}})^{\frac{1}{2}}\} \text{ or } \{(L_{\mathrm{V}})^{\frac{1}{2}}, (L_{\mathrm{VII}})^{\frac{1}{2}}\} & \text{if } a \equiv 4 \pmod 8, \\ \{(L_{\mathrm{II}})^{\frac{1}{2}}, (L_{\mathrm{IV}})^{\frac{1}{2}}\} \text{ or } \{(L_{\mathrm{V}})^{\frac{1}{2}}, (L_{\mathrm{VII}})^{\frac{1}{2}}\} & \text{if } a \equiv 0 \pmod 8 \text{ and } d \text{ is odd}, \\ \{(L_{\mathrm{I}})^{\frac{1}{2}}, (L_{\mathrm{VI}})^{\frac{1}{2}}\} & \text{if } a \equiv 0 \pmod 8 \text{ and } d \text{ is even}. \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \left\{[\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\right\} & \text{if } a \equiv 2 \pmod 4, \\ \left\{[\![2]\!]\right\} & \text{if } a \equiv 4 \pmod 8, \\ \left\{[\![2]\!]\right\} & \text{if } a \equiv 0 \pmod 8 \text{ and } d \text{ is odd}, \\ \left\{[\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\right\} & \text{if } a \equiv 0 \pmod 8 \text{ and } d \text{ is even}. \end{cases}$$

In the last two cases, we have to determine the parity of the integer $d$. Put

$K = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z \cong K_{4,\mathrm{II}}(a,b,c,d)$. If $d$ is even, we have

$$\lambda_2(K) = (\mathbb{Z}x + \mathbb{Z}(2y) + \mathbb{Z}z)^{\frac{1}{2}} \cong \begin{pmatrix} 2a & 2a & 0 \\ 2a & 4b & 2c \\ 0 & 2c & d \end{pmatrix}^{\frac{1}{2}} \cong \langle a \rangle \perp \begin{pmatrix} 2b-a & c \\ c & \frac{d}{2} \end{pmatrix}$$

and we may show that $\mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{I}; a]\!]$ by the previous results. If the integer $d$ is odd, we may assume that $b$ is odd, and we have

$$\lambda_2(K) = (\mathbb{Z}x + \mathbb{Z}2y + \mathbb{Z}(y+z))^{\frac{1}{2}} \cong \begin{pmatrix} a & a & \frac{a}{2} \\ a & 2b & b+c \\ \frac{a}{2} & b+c & \frac{b+d}{2}+c \end{pmatrix}.$$

Then we may show that $\mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{II}; a]\!]$ or $[\![8, \mathrm{IV}; a, *, *]\!]$. Therefore we conclude that

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![2]\!]\} & \text{if } a \equiv 0 \pmod 8 \text{ and} \\ & \mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{II}; a]\!] \text{ or } [\![8, \mathrm{IV}; a, *, *]\!], \\ \{[\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\} & \text{if } a \equiv 0 \pmod 8 \text{ and} \\ & \mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{I}; a]\!], \end{cases}$$

in the above two cases.

**Case(4.7.2.20)** $L_2 \cong \langle \epsilon_1, 4\epsilon_2, 4\epsilon_3 \rangle$. In this case, we have $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, \epsilon_2, \epsilon_3 \rangle$ and we obtain the relation (4.3.7). It is clear that the integers $a$ and $d$ are both even and we may assume that $b$ is odd. Since $2ab - a^2 \equiv 1 \pmod 4$, we may represent that

$$K_2 \cong \langle b, \epsilon b, \epsilon dK \rangle,$$

where $\epsilon \equiv 1 \pmod 4$. Therefore $K_2$ is isotropic if and only if $b \not\equiv dK \pmod 4$.

**Subcase(4.7.2.20-1)** Suppose that $L_2$ is isotropic and $\epsilon_2 \equiv \epsilon_3 \pmod 4$. Then we have $|\gamma_2^L(K)| = 1$ and $b \not\equiv dK \pmod 4$. We may show that $\gamma_2^L(K) =$

$\{L_{\mathrm{I}}\}$ or $\{L_{\mathrm{VI}}\}$ and we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![4, \mathrm{II}; 8a]\!]\}.$$

**Subcase(4.7.2.20-2)** Suppose that $L_2$ is anisotropic and $\epsilon_2 \equiv \epsilon_3 \pmod 4$. Then we have $|\gamma_2^L(K)| = 3$ and $b \equiv dK \pmod 4$. We may show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{I}}, L_{\mathrm{II}}, L_{\mathrm{IV}}\} & \text{if } bd - c^2 \text{ is odd,} \\ \{L_{\mathrm{II}}, L_{\mathrm{IV}}, L_{\mathrm{VI}}\} & \text{if } bd - c^2 \text{ is even.} \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![2]\!], [\![4, \mathrm{II}; 8a]\!]\}.$$

**Subcase(4.7.2.20-3)** Suppose that $\epsilon_2 \not\equiv \epsilon_3$. Then we have $|\gamma_2^L(K)| = 2$ and $b \not\equiv dK \pmod 4$. We may show that

$$\gamma_2^L(K) = \{L_{\mathrm{II}}, L_{\mathrm{IV}}\}.$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![2]\!]\}.$$

**Case(4.7.2.21)** $L_2 \cong \langle \epsilon_1, 4\epsilon_2, 8\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, \epsilon_2, 2\epsilon_3 \rangle$, and we obtain the relation (4.3.7). It is clear that $a$, $b$ are odd and $d$ is even. We may easily show that

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![2]\!]\}.$$

**Case(4.7.2.22)** $L_2 \cong \langle \epsilon_1, 4\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 4)$. In this case, we have $|\gamma_2^L(K)| = 2^{1+\epsilon_{12}}$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, \epsilon_2, 2^{m-2}\epsilon_3 \rangle$, and we obtain the relation (4.3.7).
**Subcase(4.7.2.22-1)** Suppose that the integer $a$ is odd. Then we may verify that $d$ is even and $b$ is odd. Since $2ab - a^2 \equiv 1 \pmod 4$, we have $\epsilon_1 \equiv \epsilon_2 \pmod 4$ and hence $|\gamma_2^L(K)| = 4$. Consequently, we may easily show that

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![2]\!], [\![2]\!]\}.$$

**Subcase(4.7.2.22-2)** Suppose that the integer $a$ is even. Then we may assume that $b$ is odd and then $bd-c^2$ is odd. Comparing the local structures, we may show that

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![2]\!],[\![2]\!]\} & \text{if } d \text{ is even and } \epsilon_1 \equiv \epsilon_2 \pmod 4, \\ \{[\![2]\!]\} & \text{if } d \text{ is even and } \epsilon_1 \not\equiv \epsilon_2 \pmod 4, \\ \{[\![2]\!],[\![4,\mathrm{II};8a]\!],[\![4,\mathrm{II};8a]\!]\} & \text{if } d \text{ is odd and } \epsilon_1 \equiv \epsilon_2 \pmod 4, \\ \{[\![2]\!]\} & \text{if } d \text{ is odd and } \epsilon_1 \not\equiv \epsilon_2 \equiv d \pmod 4, \\ \{[\![4,\mathrm{II};8a]\!],[\![4,\mathrm{II};8a]\!]\} & \text{if } d \text{ is odd and } d \equiv \epsilon_1 \not\equiv \epsilon_2 \pmod 4. \end{cases}$$

Now, we have to determine the parity of $d$. Observing the previous results, we may verify that

$$\mathrm{label}(\lambda_2(K)) = \begin{cases} [\![4,\mathrm{I};*]\!] \text{ or } [\![8,\mathrm{II};*]\!] & \text{if } d \text{ is even}, \\ [\![4,\mathrm{II};*]\!],[\![8,\mathrm{IV};*]\!],[\![12;*]\!] \text{ or } [\![16,\mathrm{II};*]\!] & \text{if } d \text{ is odd}. \end{cases}$$

Finally, we have to determine the value of $d$ modulo 4 when $d$ is odd and $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$. If $m = 4$, we may easily show that $d$ is even. Hence we assume that $m \geqslant 5$. If $a \equiv 2 \pmod 4$ and $m = 5$, then we have

$$bd - c^2 - \frac{ad}{2} = \frac{dK}{2a} \equiv 2 \pmod 4.$$

Therefore we obtain

$$d \equiv \frac{a}{2} \pmod 4.$$

If $a \equiv 2 \pmod 4$ and $m \geqslant 6$, we obtain $d \not\equiv \frac{a}{2} \pmod 4$ by a similar calculation as above. If $a \equiv 4 \pmod 8$, then we obtain $m = 5$ and

$$d \equiv \frac{2a(\epsilon_1\epsilon_2) - dK}{16} \pmod 4.$$

Finally, suppose that $a \equiv 0 \pmod 8$. Then we may easily show that

$$\mathrm{label}(\lambda_2(K)) = [\![4,\mathrm{II};a]\!].$$

Put $\gamma_2^K(\lambda_2(K))/\sim = \{[K],[\widetilde{K}]\}$. Then by a similar argument in Subcase

(4.7.2.13-2), we obtain

$$\Big\{\mathrm{label}(\gamma_2^L(K)), \mathrm{label}(\gamma_2^L(\widetilde{K}))\Big\} = \Big\{\{[\![2]\!]\}, \{[\![4, \mathrm{II}; 8a]\!], [\![4, \mathrm{II}; 8a]\!]\}\Big\}.$$

**Case(4.7.2.23)** $L_2 \cong \langle \epsilon_1, 8\epsilon_2, 8\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = \frac{2}{1+\epsilon_{23}}$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, 2\epsilon_2, 2\epsilon_3 \rangle$. Clearly we obtain the relation (4.3.7). We may easily show that $a \equiv 2 \pmod 4$ and may assume that $b$, $c$ and $d$ are odd. By a simple calculation, we obtain $\frac{dK}{4} \equiv d \pmod 4$ and hence we have

$$d \equiv \begin{cases} \epsilon_1 \pmod 4 & \text{if } \epsilon_2 \equiv \epsilon_3 \pmod 4, \\ \epsilon_1 + 2 \pmod 4 & \text{otherwise.} \end{cases}$$

Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{I}}\} \text{ or } \{L_{\mathrm{VI}}\} & \text{if } \epsilon_2 \equiv \epsilon_3 \pmod 4, \\ \{L_{\mathrm{II}}, L_{\mathrm{IV}}\} & \text{otherwise.} \end{cases}$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{II}; 8a]\!]\} & \text{if } \epsilon_2 \equiv \epsilon_3 \pmod 4, \\ \{[\![2]\!]\} & \text{otherwise.} \end{cases}$$

**Case(4.7.2.24)** $L_2 \cong \langle \epsilon_1, 16\epsilon_2, 16\epsilon_3 \rangle$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, 4\epsilon_2, 4\epsilon_3 \rangle$. Clearly we obtain the relation (4.3.7) and the integer $a$ is even. We may assume that $b$ is odd.

**Subcase(4.7.2.24-1)** Suppose that $a \equiv 2 \pmod 4$. Then we obtain that $bd - c^2 \equiv 4 \pmod 8$ and hence $c$ is even and $d \equiv 0 \pmod 4$. Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \{L_{\mathrm{II}}, L_{\mathrm{IV}}\} \text{ or } \{L_{\mathrm{V}}, L_{\mathrm{VII}}\}.$$

Therefore we have

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![2]\!]\}.$$

**Subcase(4.7.2.24-2)** Suppose that $a \equiv 4 \pmod 8$. Then we may show that the integers $c$ and $d$ are both odd and we have

$$K_2 \cong \langle b, b(bd-c^2), *\rangle \cong \langle \epsilon_1, 4\epsilon_2, 4\epsilon_3\rangle.$$

Since $bd - c^2 \equiv 4 \pmod 8$, we have

$$\frac{dK}{16} = \frac{a}{4}\cdot\frac{bd-c^2}{2} - \left(\frac{a}{4}\right)^2 \cdot d \equiv d \equiv \epsilon_1 \pmod 4$$

and hence $\epsilon_2 \equiv \epsilon_3 \pmod 4$. Consequently, $\frac{bd-c^2}{4}$ modulo 4 is determined by the structure of $\langle \epsilon_2, \epsilon_3\rangle$. Therefore we obtain

$$d \equiv \frac{a}{4}\cdot \epsilon_1\epsilon_2 - \frac{dK}{16} \pmod 8.$$

Comparing the local structures, we may easily show that

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{I}}, L_{\mathrm{VI}}\} & \text{if } \epsilon_1 \equiv d \pmod 8, \\ \{L_{\mathrm{II}}, L_{\mathrm{IV}}\} & \text{otherwise.} \end{cases}$$

Therefore we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4,\mathrm{II};8a]\!], [\![4,\mathrm{II};8a]\!]\} & \text{if } \epsilon_1 \equiv d \pmod 8, \\ \{[\![2]\!]\} & \text{otherwise.} \end{cases}$$

**Case(4.7.2.25)** $L_2 \cong \langle \epsilon_1, 2^m\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 5)$. In this case, we have $|\gamma_2^L(K)| = 4$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, 2^{m-2}\epsilon_2, 2^{m-2}\epsilon_3\rangle$. Clearly we obtain the relation (4.3.7) and the integer $a$ is even. Then we may assume that the integer $b$ is odd. By a simple calculation, we have

$$\gamma_2^L(K) = \begin{cases} \{L_{\mathrm{II}}, L_{\mathrm{IV}}, L_{\mathrm{V}}, L_{\mathrm{VII}}\} & \text{if } a \equiv 2 \pmod 4 \text{ or } a \equiv 4 \pmod 8, \\ \{L_{\mathrm{I}}, L_{\mathrm{V}}, L_{\mathrm{VI}}, L_{\mathrm{VII}}\} & \text{if } a \equiv 0 \pmod 8 \text{ and} \\ & \mathrm{label}(\lambda_2(K)) = [\![4,\mathrm{II};*]\!], [\![8,\mathrm{IV};*]\!] \text{ or } [\![16,\mathrm{II};*]\!], \\ \{L_{\mathrm{II}}, L_{\mathrm{IV}}, L_{\mathrm{V}}, L_{\mathrm{VII}}\} & \text{if } a \equiv 0 \pmod 8 \text{ and} \\ & \mathrm{label}(\lambda_2(K)) = [\![4,\mathrm{I};*]\!] \text{ or } [\![8,\mathrm{II};*]\!]. \end{cases}$$

The last two cases are similar to Subcase(4.7.2.22-2). Therefore we obtain

$$\text{label}(\gamma_2^L(K)) = \begin{cases} \{[\![2]\!],[\![2]\!]\} & \text{if } a \equiv 2 \pmod 4 \text{ or } a \equiv 4 \pmod 8, \\ \{[\![2]\!],[\![4,\mathrm{II};8a]\!],[\![4,\mathrm{II};8a]\!]\} & \text{if } a \equiv 0 \pmod 8 \text{ and} \\ & \text{label}(\lambda_2(K)) = [\![4,\mathrm{II};*]\!],[\![8,\mathrm{IV};*]\!] \text{ or} \\ & \qquad [\![16,\mathrm{II};*]\!], \\ \{[\![2]\!],[\![2]\!]\} & \text{if } a \equiv 0 \pmod 8 \text{ and} \\ & \text{label}(\lambda_2(K)) = [\![4,\mathrm{I};*]\!] \text{ or } [\![8,\mathrm{II};*]\!]. \end{cases}$$

**Case(4.7.2.26)** $L_2 \cong \langle \epsilon_1, 8\epsilon_2, 16\epsilon_3\rangle$. In this case, we have $|\gamma_2^L(K)| = 1$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, 2\epsilon_2, 4\epsilon_3\rangle$. Clearly we obtain the relation (4.3.7) and the integer $a$ is even. We may assume that $b$ is odd. Since $dK \equiv 8 \pmod{16}$, we may show that $a \equiv 2 \pmod 4$ and $d$ is even. Then the possible candidates are $L_{\mathrm{II}}, L_{\mathrm{IV}}, L_{\mathrm{V}}, L_{\mathrm{VII}}$. But this contradicts to the fact that $|\gamma_2^L(K)| = 1$. Therefore this case does not occur.

**Case(4.7.2.27)** $L_2 \cong \langle \epsilon_1, 16\epsilon_2, 32\epsilon_3\rangle$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, 4\epsilon_2, 8\epsilon_3\rangle$. Clearly we obtain the relation (4.3.7) and the integer $a$ is even. We may assume that $b$ is odd. Since $dK \equiv 32 \pmod{64}$, we may show that $a \equiv 2 \pmod 4$ or $a \equiv 4 \pmod 8$. Comparing the local structures, we may easily show that

$$\text{label}(\gamma_2^L(K)) = \{[\![2]\!]\}.$$

**Case(4.7.2.28)** $L_2 \cong \langle \epsilon_1, 2^m\epsilon_2, 2^{m+1}\epsilon_3\rangle$ $(m \geqslant 5)$. In this case, we have $|\gamma_2^L(K)| = 4$. We may show that

$$\text{label}(\gamma_2^L(K)) = \begin{cases} \{[\![2]\!],[\![2]\!]\} & \text{if } \mathrm{ord}_2(a) \leqslant 3, \\ \{[\![2]\!],[\![4,\mathrm{II};8a]\!],[\![4,\mathrm{II};8a]\!]\} & \text{if } \mathrm{ord}_2(a) \geqslant 4 \text{ and} \\ & \quad \text{label}(\lambda_2(K)) = [\![4,\mathrm{II};*]\!], \\ \{[\![2]\!],[\![2]\!]\} & \text{if } \mathrm{ord}_2(a) \geqslant 4 \text{ and} \\ & \quad \text{label}(\lambda_2(K)) = [\![4,\mathrm{I};*]\!] \text{ or } [\![8,\mathrm{II};*]\!]. \end{cases}$$

**Case(4.7.2.29)** $L_2 \cong \langle \epsilon_1, 8\epsilon_2, 32\epsilon_3\rangle$. In this case, we have $|\gamma_2^L(K)| = 2$. We

may show that
$$\mathrm{label}(\gamma_2^L(K)) = \{[\![2]\!]\}.$$

**Case(4.7.2.30)** $L_2 \cong \langle \epsilon_1, 16\epsilon_2, 64\epsilon_3\rangle$. In this case, we have $|\gamma_2^L(K)| = 2$. We may show that
$$\mathrm{label}(\gamma_2^L(K)) = \{[\![2]\!]\}.$$

**Case(4.7.2.31)** $L_2 \cong \langle \epsilon_1, 2^m\epsilon_2, 2^{m+2}\epsilon_3\rangle$ $(m \geqslant 5)$. In this case, we have $|\gamma_2^L(K)| = 4$. We may show that

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![2]\!], [\![2]\!]\} & \text{if } \mathrm{ord}_2(a) \leqslant 3, \\ \{[\![2]\!], [\![4, \mathrm{II}; 8a]\!], [\![4, \mathrm{II}; 8a]\!]\} & \text{if } \mathrm{ord}_2(a) \geqslant 4 \text{ and } \mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{II}; *]\!], \\ \{[\![2]\!], [\![2]\!]\} & \text{if } \mathrm{ord}_2(a) \geqslant 4 \text{ and} \\ & \quad \mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{I}; *]\!] \text{ or } [\![8, \mathrm{II}; *]\!]. \end{cases}$$

**Case(4.7.2.32)** $L_2 \cong \langle \epsilon_1, 8\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 6)$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, 2\epsilon_2, 2^{m-2}\epsilon_3\rangle$. Clearly we obtain the relation (4.3.7) and the intege $a$ is even. We may assume that $b$ is odd.

**Subcase(4.7.2.32-1)** Suppose that $a \equiv 2 \pmod 4$. Since $dK \equiv 0 \pmod{32}$, we may show that $d$ is even and we may easily show that

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![2]\!]\}.$$

**Subcase(4.7.2.32-2)** Suppose that $a \equiv 4 \pmod 8$. Then we may show that the integers $c$ and $d$ are both odd and $b \not\equiv d \pmod 4$. We may easily show that

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{II}; 8a]\!], [\![4, \mathrm{II}; 8a]\!]\} & \text{if } \epsilon_1 \equiv d \pmod 4, \\ \{[\![2]\!]\} & \text{otherwise,} \end{cases}$$

where
$$d \equiv \begin{cases} \frac{a}{4} \cdot \epsilon_1\epsilon_2 - 2 \pmod 4 & \text{if } m = 6, \\ \frac{a}{4} \cdot \epsilon_1\epsilon_2 \pmod 4 & \text{if } m \geqslant 7. \end{cases}$$

**Subcase(4.7.2.32-3)** Suppose that $a \equiv 0 \pmod 8$. Then we may show that $bd - c^2 \equiv 2 \pmod 4$ and hence $\mathrm{ord}_2(2a(bd - c^2)) = 2 + \mathrm{ord}_2(a)$. Since $a \equiv 0$

(mod 8) and $\mathrm{ord}_2(a^2d) \geqslant 2\mathrm{ord}_2(a)$, we have

$$2 + \mathrm{ord}_2(a) < 2\mathrm{ord}_2(a) \leqslant \mathrm{ord}_2(a^2d).$$

Therefore $\mathrm{ord}_2(a) = m - 3$. If $m = 6$, we may show that

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![2]\!]\} & \text{if } \mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{I}; *]\!], \\ \{[\![4, \mathrm{II}; 8a]\!], [\![4, \mathrm{II}; 8a]\!]\} & \text{if } \mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{II}; *]\!] \text{ and } \epsilon_1 \equiv d (\mathrm{mod}\ 4), \\ \{[\![2]\!]\} & \text{if } \mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{II}; *]\!] \text{ and } \epsilon_1 \not\equiv d (\mathrm{mod}\ 4), \end{cases}$$

where

$$d \equiv \frac{a\epsilon_1\epsilon_2}{16} - \frac{dK}{64} \pmod 4.$$

If $m \geqslant 7$, we may use a similar argument as in Subcase(4.7.2.13-2). If we put $\gamma_2^K(\lambda_2(K))/\sim = \{[K], [\widetilde{K}]\}$, the isometry group of $\widetilde{K}$ is of order 4 and its label is of type II. Using the fact that $\mathrm{ord}_2(a) = m - 3$, we may verify that

$$\Big\{\mathrm{label}(\gamma_2^L(K)), \mathrm{label}(\gamma_2^L(\widetilde{K}))\Big\} = \Big\{\{[\![2]\!]\}, \{[\![4, \mathrm{II}; 8a]\!], [\![4, \mathrm{II}; 8a]\!]\}\Big\}.$$

**Case(4.7.2.33)** $L_2 \cong \langle \epsilon_1, 16\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 7)$. In this case, we have $|\gamma_2^L(K)| = 2$ and $K_2 = \lambda_2(L_2) \cong \langle \epsilon_1, 4\epsilon_2, 2^{m-2}\epsilon_3 \rangle$. Clearly we obtain the relation (4.3.7) and the integer $a$ is even. We may assume that $b$ is odd.

**Subcase(4.7.2.33-1)** Suppose that $a \equiv 2 \pmod 4$ or $a \equiv 4 \pmod 8$. Then $d$ is even and we may show that

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![2]\!]\}.$$

**Subcase(4.7.2.33-2)** Suppose that $a \equiv 8 \pmod{16}$. Then $bd - c^2 \equiv 4 \pmod 8$ and $bd \equiv 5 \pmod 8$. Then we may show that

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{II}; 8a]\!], [\![4, \mathrm{II}; 8a]\!]\} & \text{if } \epsilon_1 \equiv d \pmod 8, \\ \{[\![2]\!]\} & \text{otherwise,} \end{cases}$$

where

$$d \equiv \frac{a}{8}\epsilon_1\epsilon_2 - \frac{dK}{2^6} \pmod 8.$$

**Subcase(4.7.2.33-3)** Suppose that $a \equiv 16 \pmod{32}$. Then we may easily show that $\mathrm{ord}_2(dK) = 7$ and hence

$$2d \equiv \frac{a}{16}\epsilon_1\epsilon_2 - \frac{dK}{2^7} \pmod 8.$$

If $\frac{a}{16}\epsilon_1\epsilon_2 \equiv \frac{dK}{2^7} \pmod 4$, then $d$ is even and we have

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![2]\!]\big\}.$$

Suppose that $\frac{a}{16}\epsilon_1\epsilon_2 \not\equiv \frac{dK}{2^7} \pmod 4$. If we put $\gamma_2^K(\lambda_2(K))/\sim = \big\{[K],[\widetilde{K}]\big\}$, the isometry group of $\widetilde{K}$ is of order 4 and its label is of type II. Since $d \not\equiv 4b - 2a - 4c + d \pmod 8$, we may verify that

$$\Big\{\mathrm{label}(\gamma_2^L(K)), \mathrm{label}(\gamma_2^L(\widetilde{K}))\Big\} = \Big\{\big\{[\![2]\!]\big\}, \big\{[\![4, \mathrm{II}; 8a]\!], [\![4, \mathrm{II}; 8a]\!]\big\}\Big\}.$$

**Subcase(4.7.2.33-4)** Suppose that $a \equiv 32 \pmod{64}$. This case is similar to the previous case. If $\frac{a}{32}\epsilon_1\epsilon_2 \equiv \frac{dK}{2^8} \pmod 8$, then $d$ is even and we have

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![2]\!]\big\}.$$

If $\frac{a}{32}\epsilon_1\epsilon_2 \not\equiv \frac{dK}{2^8} \pmod 8$, then we have

$$\Big\{\mathrm{label}(\gamma_2^L(K)), \mathrm{label}(\gamma_2^L(\widetilde{K}))\Big\} = \Big\{\big\{[\![2]\!]\big\}, \big\{[\![4, \mathrm{II}; 8a]\!], [\![4, \mathrm{II}; 8a]\!]\big\}\Big\},$$

where $\gamma_2^K(\lambda_2(K))/\sim = \big\{[K],[\widetilde{K}]\big\}$.

**Subcase(4.7.2.33-5)** Suppose that $a \equiv 0 \pmod{64}$. This case is similar to the previous cases. If $\mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{I}; *]\!]$ or $[\![8, \mathrm{II}; *]\!]$, then $d$ is even and we have

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![2]\!]\big\}$$

and we also have

$$\Big\{\mathrm{label}(\gamma_2^L(K)), \mathrm{label}(\gamma_2^L(\widetilde{K}))\Big\} = \Big\{\big\{[\![2]\!]\big\}, \big\{[\![4, \mathrm{II}; 8a]\!], [\![4, \mathrm{II}; 8a]\!]\big\}\Big\}.$$

where $\gamma_2^K(\lambda_2(K))/\sim = \{[K], [\widetilde{K}]\}$,

**Case(4.7.2.34)** $L_2 \cong \langle \epsilon_1, 2^n\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant n+3 \geqslant 8)$. In this case, we have $|\gamma_2^L(K)| = 4$. Clearly we obtain the relation (4.3.7). This case is similar to Case(4.7.2.28) and we obtain

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![2]\!], [\![2]\!]\} & \text{if } \mathrm{ord}_2(a) \leqslant 3, \\ \{[\![2]\!], [\![4, \mathrm{II}; 8a]\!], [\![4, \mathrm{II}; 8a]\!]\} & \text{if } \mathrm{ord}_2(a) \geqslant 4 \text{ and} \\ & \quad \mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{II}; *]\!] \text{ or } [\![8, \mathrm{IV}; *]\!], \\ \{[\![2]\!], [\![2]\!]\} & \text{if } \mathrm{ord}_2(a) \geqslant 4 \text{ and} \\ & \quad \mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{I}; *]\!] \text{ or } [\![8, \mathrm{II}; *]\!]. \end{cases}$$

□

Finally, we state a summary of all results in this chapter.

**Theorem 4.7.3.** *Let $L$ be a primitive ternary lattice which is not stable over $\mathbb{Z}_2$, and let $K := \lambda_{2e}(L)$.*
*(1) Suppose that $|O(K)| \neq 4$. Then the multiset* $\mathrm{label}(\gamma_{2e}^L(K))$ *is completely determined by the label of $K$ and the structure of $L_2$.*
*(2) Suppose that* $\mathrm{label}(K) = [\![4, \mathrm{I}; a]\!]$ *for some positive integer $a$, and let $K \cong \langle a\rangle \perp K'$. Then the multiset* $\mathrm{label}(\gamma_{2e}^L(K))$ *is completely determined by the label of $K$, the discriminant of $K$ and the structure of $L_2$ except the following exceptional cases. ( Exceptional case 1 $\sim$ Exceptional case 7)*
*(3) Suppose that* $\mathrm{label}(K) = [\![4, \mathrm{II}; 2a]\!]$ *for some positive integer $a$. Then the multiset* $\mathrm{label}(\gamma_{2e}^L(K))$ *is completely determined by the label of $K$, the discriminant of $K$ and the structure of $L_2$ except the following exceptional cases. ( Exceptional case 8 $\sim$ Exceptional case 15)*

**Exceptional case 1.** *$L_2 = T \perp \langle 4\epsilon\rangle$. In this case, we have*

$$\mathrm{label}(\gamma_4^L(K)) = \begin{cases} \{[\![4, \mathrm{I}; 4a]\!]\} & \text{if } K' \text{ is even,} \\ \{[\![4, \mathrm{II}; 4a]\!]\} & \text{if } K' \text{ is odd.} \end{cases}$$

**Exceptional case 2.** $L_2 = \langle \epsilon \rangle \perp 4T$. *In this case, we have*

$$\operatorname{label}(\gamma_2^L(K)) = \begin{cases} \big\{\llbracket 4, \mathrm{I}; a \rrbracket\big\} & \text{if } K' \text{ is even,} \\ \big\{\llbracket 4, \mathrm{II}; 4a \rrbracket\big\} & \text{if } K' \text{ is odd.} \end{cases}$$

**Exceptional case 3.** $L_2 = \langle \epsilon_1, \epsilon_2, 2\epsilon_3 \rangle$. *If $K_2$ is isotropic, then we have*

$$\operatorname{label}(\gamma_2^L(K)) = \begin{cases} \{[4, \mathrm{I}; 2a \rrbracket\} & \text{if } K' \text{ is even or } a \equiv dK \pmod 4, \\ \big\{\llbracket 4, \mathrm{II}; 2a \rrbracket\big\} & \text{if } K' \text{ is odd and } a \not\equiv dK \pmod 4. \end{cases}$$

*If $K_2$ is anisotropic, then we have*

$$\operatorname{label}(\gamma_2^L(K)) = \begin{cases} \big\{\llbracket 4, \mathrm{I}; 2a \rrbracket, \llbracket 4, \mathrm{I}; 2a \rrbracket, \llbracket 4, \mathrm{I}; 2a \rrbracket\big\} & \text{if } K' \text{ is even,} \\ \big\{\llbracket 4, \mathrm{I}; 2a \rrbracket, \llbracket 4, \mathrm{II}; 2a \rrbracket, \llbracket 4, \mathrm{II}; 2a \rrbracket\big\} & \text{if } K' \text{ is odd.} \end{cases}$$

**Exceptional case 4.** $L_2 = \langle \epsilon_1, 4\epsilon_2, 4\epsilon_3 \rangle$. *If $K_2$ is isotropic and $\epsilon_2 \equiv \epsilon_3 \pmod 4$, then we have*

$$\operatorname{label}(\gamma_2^L(K)) = \begin{cases} \big\{\llbracket 4, \mathrm{II}; 4a \rrbracket\big\} & \text{if } K' \text{ is even,} \\ \big\{\llbracket 4, \mathrm{I}; a \rrbracket\big\} & \text{if } K' \text{ is odd and } a \equiv dK \pmod 4, \\ \big\{\llbracket 4, \mathrm{I}; 4a \rrbracket\big\} & \text{if } K' \text{ is odd and } a \not\equiv dK \pmod 4. \end{cases}$$

*If $K_2$ is isotropic and $\epsilon_2 \not\equiv \epsilon_3 \pmod 4$, then we have*

$$\operatorname{label}(\gamma_2^L(K)) = \begin{cases} \big\{\llbracket 4, \mathrm{II}; 4a \rrbracket, \llbracket 4, \mathrm{II}; 4a \rrbracket\big\} & \text{if } K' \text{ is even,} \\ \big\{\llbracket 4, \mathrm{I}; 4a \rrbracket, \llbracket 4, \mathrm{I}; 4a \rrbracket\big\} & \text{if } K' \text{ is odd and } a \equiv dK \pmod 4, \\ \big\{\llbracket 4, \mathrm{I}; a \rrbracket, \llbracket 4, \mathrm{I}; 4a \rrbracket\big\} & \text{if } K' \text{ is odd and } a \not\equiv dK \pmod 4. \end{cases}$$

*If $K_2$ is anisotropic, then we have*

$$\operatorname{label}(\gamma_2^L(K)) = \begin{cases} \big\{\llbracket 4, \mathrm{II}; 4a \rrbracket, \llbracket 4, \mathrm{II}; 4a \rrbracket, \llbracket 4, \mathrm{II}; 4a \rrbracket\big\} & \text{if } K' \text{ is even,} \\ \big\{\llbracket 4, \mathrm{I}; a \rrbracket, \llbracket 4, \mathrm{I}; 4a \rrbracket, \llbracket 4, \mathrm{I}; 4a \rrbracket\big\} & \text{if } K' \text{ is odd.} \end{cases}$$

**Exceptional case 5.** $L_2 = \langle \epsilon_1, 2\epsilon_2, 8\epsilon_3 \rangle$, $a \equiv 2 \pmod 4$, $\frac{a}{2} \not\equiv \frac{dK}{8} \pmod 4$. *In this case, we have* $\gamma_2^K(\lambda_2(K))/\sim = \big\{[K], [\widetilde{K}]\big\}$ *with* $\operatorname{label}(K) = \operatorname{label}(\widetilde{K}) =$

$[\![4, \mathrm{I}; a]\!]$, *and then*

$$\left\{\mathrm{label}(\gamma_2^L(K)), \mathrm{label}(\gamma_2^L(\widetilde{K}))\right\} = \left\{\{[\![4, \mathrm{I}; \frac{a}{2}]\!]\}, \{[\![4, \mathrm{II}; 2a]\!]\}\right\}.$$

**Exceptional case 6.** $L_2 = \langle \epsilon_1, 8\epsilon_2, 16\epsilon_3 \rangle$, $a \equiv 2 \pmod 4$, $\frac{a}{2} \not\equiv \frac{dK}{8} \pmod 4$. *In this case, we have* $\gamma_2^K(\lambda_2(K))/\sim = \{[K], [\widetilde{K}]\}$ *with* $\mathrm{label}(K) = \mathrm{label}(\widetilde{K}) = [\![4, \mathrm{I}; a]\!]$, *and then*

$$\left\{\mathrm{label}(\gamma_2^L(K)), \mathrm{label}(\gamma_2^L(\widetilde{K}))\right\} = \left\{\{[\![4, \mathrm{I}; 4a]\!]\}, \{[\![4, \mathrm{II}; 4a]\!]\}\right\}.$$

**Exceptional case 7.** $L_2 = \langle \epsilon_1, 16\epsilon_2, 64\epsilon_3 \rangle$, $a \equiv 4 \pmod 8$. *In this case, we have* $\gamma_2^K(\lambda_2(K))/\sim = \{[K], [\widetilde{K}]\}$ *with* $\mathrm{label}(K) = \mathrm{label}(\widetilde{K}) = [\![4, \mathrm{I}; a]\!]$, *and then*

$$\left\{\mathrm{label}(\gamma_2^L(K)), \mathrm{label}(\gamma_2^L(\widetilde{K}))\right\} = \left\{\{[\![4, \mathrm{I}; 4a]\!], [\![4, \mathrm{I}; 4a]\!]\}, \{[\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\}\right\}.$$

**Exceptional case 8.** $L_2 = \langle \epsilon_1, \epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 4)$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$, $a \equiv 0 \pmod 4$. *In this case, we have*

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\} & \textit{if } \mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{I}; a]\!], \\ \{[\![2]\!]\} & \textit{if } \mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{II}; a]\!]. \end{cases}$$

**Exceptional case 9.** $L_2 = \langle \epsilon_1, \epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 3)$, $\epsilon_1\epsilon_2 \equiv 7 \pmod 8$, $a \equiv 0 \pmod 8$. *In this case, we have* $\gamma_4^K(\lambda_4(K))/\sim = \{[K], [K_1], [K_2], [K_3]\}$ *with* $\mathrm{label}(K_1) = \mathrm{label}(K_2) = [\![4, \mathrm{II}, 2a]\!]$ *and* $\mathrm{label}(K_3) = [\![4, \mathrm{I}; 2a]\!]$. *Then we obtain*

$$\left\{\mathrm{label}(\gamma_2^L(K)), \mathrm{label}(\gamma_2^L(K_1)), \mathrm{label}(\gamma_2^L(K_2))\right\} = \left\{\{[\![4, \mathrm{I}; 4a]\!], [\![4, \mathrm{I}; 4a]\!]\}, \{[\![2]\!]\}, \{[\![2]\!]\}\right\}.$$

**Exceptional case 10.** $L_2 = \langle \epsilon_1, 2\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 5)$, $a \equiv 0 \pmod 8$. *In this case, we have*

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4, \mathrm{II}; 4a]\!], [\![4, \mathrm{II}; 4a]\!]\} & \textit{if } \mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{I}; a]\!], \\ \{[\![2]\!]\} & \textit{if } \mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{II}; a]\!] \textit{ or } [\![8, \mathrm{IV}; *]\!]. \end{cases}$$

**Exceptional case 11.** $L_2 = \langle \epsilon_1, 4\epsilon_2, 2^m\epsilon_3 \rangle$ *($m \geqslant 4$), $a \equiv 0 \pmod 2$. If* $\mathrm{label}(\lambda_2(K))$ *is of type* $[\![4,\mathrm{I};*]\!]$ *or of type* $[\![8,\mathrm{II};*]\!]$*, we have*

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![2]\!],[\![2]\!]\} & \text{if } \epsilon_1 \equiv \epsilon_2 \pmod 4, \\ \{[\![2]\!]\} & \text{if } \epsilon_1 \not\equiv \epsilon_2 \pmod 4. \end{cases}$$

*If* $\mathrm{label}(\lambda_2(K))$ *is neither of type* $[\![4,\mathrm{I};*]\!]$ *nor of type* $[\![8,\mathrm{II};*]\!]$*, we have*

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4,\mathrm{II};8a]\!],[\![4,\mathrm{II};8a]\!],[\![2]\!]\} & \text{if } \epsilon_1 \equiv \epsilon_2 \pmod 4, \\ \{[\![2]\!]\} & \text{if } \epsilon_1 \not\equiv \epsilon_2 \equiv d \pmod 4 \text{ and } a \not\equiv 0 \pmod 8, \\ \{[\![4,\mathrm{II};8a]\!],[\![4,\mathrm{II};8a]\!]\} & \text{if } d \equiv \epsilon_1 \not\equiv \epsilon_2 \pmod 4 \text{ and } a \not\equiv 0 \pmod 8, \end{cases}$$

$$\text{where} \qquad d \equiv \begin{cases} \frac{a}{2} \pmod 4 & \text{if } a \equiv 2 \pmod 4 \text{ and } m = 5, \\ \frac{a}{2}+2 \pmod 4 & \text{if } a \equiv 2 \pmod 4 \text{ and } m \geqslant 6, \\ \frac{2a\epsilon_1\epsilon_2 - dK}{16} \pmod 4 & \text{if } a \equiv 4 \pmod 8. \end{cases}$$

*Finally, we suppose that* $\mathrm{label}(\lambda_2(K))$ *is neither of type* $[\![4,\mathrm{I};*]\!]$ *nor of type* $[\![8,\mathrm{II};*]\!]$*,* $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$ *and* $a \equiv 0 \pmod 8$*. In this case, we obtain that* $\gamma_2^L(\lambda_2(K))/\sim = \{[K],[\widetilde{K}]\}$ *with* $\mathrm{label}(\widetilde{K}) = [\![4,\mathrm{II};2a]\!]$*, and then*

$$\Big\{\mathrm{label}(\gamma_2^L(K)), \mathrm{label}(\gamma_2^L(\widetilde{K}))\Big\} = \Big\{\{[\![4,\mathrm{II};8a]\!],[\![4,\mathrm{II};8a]\!]\},\{[\![2]\!]\}\Big\}.$$

**Exceptional case 12.** $L_2 = \langle \epsilon_1, 8\epsilon_2, 2^m\epsilon_3 \rangle$ *($m \geqslant 6$), $a \equiv 0 \pmod 8$. If* $\mathrm{label}(\lambda_2(K)) = [\![4,\mathrm{I};a]\!]$*, we have*

$$\mathrm{label}(\gamma_2^L(K)) = \{[\![2]\!]\}.$$

*If $m = 6$ and* $\mathrm{label}(\lambda_2(K)) = [\![4,\mathrm{II};a]\!]$*, we have*

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \{[\![4,\mathrm{II};8a]\!],[\![4,\mathrm{II};8a]\!]\} & \text{if } \epsilon_1 \equiv \frac{4a\epsilon_1\epsilon_2 - dK}{64} \pmod 4, \\ \{[\![2]\!]\} & \text{if } \epsilon_1 \not\equiv \frac{4a\epsilon_1\epsilon_2 - dK}{64} \pmod 4. \end{cases}$$

*If $m \geqslant 7$ and* $\mathrm{label}(\lambda_2(K)) = [\![4,\mathrm{II};a]\!]$*, we have* $\gamma_2^L(\lambda_2(K))/\sim = \{[K],[\widetilde{K}]\}$

*with* $\mathrm{label}(\widetilde{K}) = [\![4, \mathrm{II}; 2a]\!]$, *and then*

$$\Big\{\mathrm{label}(\gamma_2^L(K)), \mathrm{label}(\gamma_2^L(\widetilde{K}))\Big\} = \Big\{\big\{[\![4, \mathrm{II}; 8a]\!], [\![4, \mathrm{II}; 8a]\!]\big\}, \big\{[\![2]\!]\big\}\Big\}.$$

**Exceptional case 13.** $L_2 = \langle \epsilon_1, 16\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 7)$. *If* $a \equiv 16 \pmod{32}$ *and* $\frac{a\epsilon_1\epsilon_2}{16} \not\equiv \frac{dK}{128} \pmod 4$, *we have* $\gamma_2^L(\lambda_2(K))/\sim = \big\{[K], [\widetilde{K}]\big\}$ *with* $\mathrm{label}(\widetilde{K}) = [\![4, \mathrm{II}; 2a]\!]$. *Then we obtain*

$$\Big\{\mathrm{label}(\gamma_2^L(K)), \mathrm{label}(\gamma_2^L(\widetilde{K}))\Big\} = \Big\{\big\{[\![4, \mathrm{II}; 8a]\!], [\![4, \mathrm{II}; 8a]\!]\big\}, \big\{[\![2]\!]\big\}\Big\}.$$

*If* $a \equiv 32 \pmod{64}$ *and* $\frac{a\epsilon_1\epsilon_2}{32} \not\equiv \frac{dK}{256} \pmod 8$, *we have the same result as the above case. If* $a \equiv 0 \pmod{64}$ *and* $\mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{II}; \frac{a}{2}]\!]$, *we also have the same result as the above two cases. But if* $\mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{I}; \frac{a}{2}]\!]$, *we have*

$$\mathrm{label}(\gamma_2^L(K)) = \big\{[\![2]\!]\big\}.$$

**Exceptional case 14.** $L_2 = \langle \epsilon_1, 2^m\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 5)$, $a \equiv 0 \pmod 8$. *In this case, we have*

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \big\{[\![2]\!], [\![2]\!]\big\} & \textit{if } \mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{I}; *]\!] \textit{ or } [\![8, \mathrm{II}; *]\!], \\ \big\{[\![4, \mathrm{II}; 8a]\!], [\![4, \mathrm{II}; 8a]\!], [\![2]\!]\big\} & \textit{if } \mathrm{label}(\lambda_2(K)) \neq [\![4, \mathrm{I}; *]\!] \textit{ or } [\![8, \mathrm{II}; *]\!]. \end{cases}$$

**Exceptional case 15.** $L_2 = \langle \epsilon_1, 2^n\epsilon_2, 2^m\epsilon_3\rangle$ $(m > n \geqslant 5)$, $a \equiv 0 \pmod{16}$. *In this case, we have*

$$\mathrm{label}(\gamma_2^L(K)) = \begin{cases} \big\{[\![2]\!], [\![2]\!]\big\} & \textit{if } \mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{I}; *]\!] \textit{ or } [\![8, \mathrm{II}; *]\!], \\ \big\{[\![4, \mathrm{II}; 8a]\!], [\![4, \mathrm{II}; 8a]\!], [\![2]\!]\big\} & \textit{if } \mathrm{label}(\lambda_2(K)) = [\![4, \mathrm{II}; *]\!]. \end{cases}$$

# Chapter 5

# Stable lattices

## 5.1 Labels of stable lattices

Let $K$ be a stable ternary lattice defined in Chapter 3. The method of determining the labels of all classes in $\mathrm{gen}(K)$ is almost provided in [1]. First we are going to restate these results. In order to complete the determination of labels, we have to compute the number of labels corresponding to isometry classes whose isometry groups are of order 4, by type. A method of computing these numbers is provided in Theorem 5.1.12 and Theorem 5.1.13.

Henceforth, $K$ is always a stable ternary lattice. For any integer $a$, let $\nu(a)$ be the number of distinct prime divisors of $a$. If $q$ is a prime, we define

$$e_q(a) := \begin{cases} 1 & \text{if } q \text{ divides } a, \\ 0 & \text{otherwise.} \end{cases}$$

For any positive integer $t$, let $b_t(K)$ be the number of classes in $\mathrm{gen}(K)$ whose isometry groups are of order $t$. Let $\mathfrak{P}$ be the product of odd prime divisors $q$ of $dK$ such that $K_q$ is anisotropic, and $\mathfrak{Q}$ be the product of odd prime divisors $q$ of $dK$ such that $K_q$ is isotropic. For positive integers $\alpha$, $\beta$, $\gamma$, we define

$$\Phi_K(\alpha) = \prod_{q \mid \mathfrak{P}} \left(1 - \left(\frac{-\alpha}{q}\right)\right) \prod_{q \mid \mathfrak{Q}} \left(1 + \left(\frac{-\alpha}{q}\right)\right),$$

and

$$\Phi_K(\alpha,\beta,\gamma) = \prod_{q|\mathfrak{P}} \left(1+\left(\frac{-\beta\gamma}{q}\right)\right)\left(1+\left(\frac{-\gamma\alpha}{q}\right)\right)\left(1+\left(\frac{-\alpha\beta}{q}\right)\right)$$
$$\times \prod_{q|\mathfrak{Q}} \left(1-\left(\frac{-\beta\gamma}{q}\right)\right)\left(1-\left(\frac{-\gamma\alpha}{q}\right)\right)\left(1-\left(\frac{-\alpha\beta}{q}\right)\right).$$

**Lemma 5.1.1.** *Up to isometry, there is at most one lattice in* $\operatorname{gen}(K)$ *whose isometry group is of order* $24$*, and its label is* $[\![24; 2,2,2,6,6,6,\frac{dK}{3}]\!]$*. Furthermore,*

$$b_{24}(K) = \begin{cases} 0 & \text{if } K \text{ is odd and } S_2(K) = -1, \\ \frac{e_3(dK)}{2^{\nu(\mathfrak{PQ})-1}}\Phi_K(3) & \text{otherwise.} \end{cases}$$

*Proof.* See [1].

**Lemma 5.1.2.** *Up to isometry, there is at most one lattice in* $\operatorname{gen}(K)$ *whose isometry group is of order* $12$ *and its label is* $[\![12; 2,2,2]\!]$*. Furthermore,*

$$b_{12}(K) = \begin{cases} 0 & \text{if } K \text{ is odd and } S_2(K) = -1, \\ \frac{1-e_3(dK)}{2^{\nu(\mathfrak{PQ})}}\Phi_K(3) & \text{otherwise.} \end{cases}$$

*Proof.* See [1].

**Lemma 5.1.3.** *Up to isometry, there is at most one lattice in* $\operatorname{gen}(K)$ *whose isometry group is of order* $16$*, and its label is* $[\![16,\mathrm{I}; 1,1,2,2,dK]\!]$*. Furthermore,*

$$b_{16}(K) = \frac{1-e_2(dK)}{2^{\nu(\mathfrak{PQ})}}\Phi_K(1).$$

*Proof.* See [1].

For the convenience, let $\mathfrak{T}$ be the set of triples $(a,b,c)$ of positive integers such that $abc = dK$, and we define

$\mathfrak{R}_1 = \{(a,b,c) \in \mathfrak{T}; a > b > c\}$,
$\mathfrak{R}_2 = \{(a,b,c) \in \mathfrak{T}; b > c, (b,c) \neq (3,1),\ a \equiv 2 \pmod 4 \text{ and } bc \equiv 3 \pmod 4\}$,
$\mathfrak{R}_3 = \{(a,b,c) \in \mathfrak{T}; b > c, \text{ and } (b,c) \neq (3,1)\}$.

**Lemma 5.1.4.** *If $K$ is even, then*

$$b_8(K) = \sum_{(a,b,c)\in\mathfrak{R}_2} \frac{1}{2^{\nu(\mathfrak{P}\mathfrak{Q})}}\Phi_K(a,2b,2c),$$

*and any label of a lattice in* $\mathrm{gen}(K)$ *whose isometry group is of order* 8 *is* $[\![8,\mathrm{II};a,2b,2c]\!]$. *If $K$ is odd, then*

$$b_8(K) = \sum_{(a,b,c)\in\mathfrak{R}_1} \frac{1}{2^{\nu(\mathfrak{P}\mathfrak{Q})}}\Phi_K(a,b,c) + \sum_{(a,b,c)\in\mathfrak{R}_3} \frac{1}{2^{\nu(\mathfrak{P}\mathfrak{Q})}}\Phi_K(a,2b,2c).$$

*Here, the first term denotes the number of labels of the form* $[\![8,\mathrm{I};a,b,c]\!]$ *and the second term denotes the number of labels of the form* $[\![8,\mathrm{II};\dot{a},2b,2c]\!]$.

*Proof.* See [1].

**Remark 5.1.5.** *The label of any* $K \in \mathrm{gen}(K)$ *with* $|O(M)| = 8$ *can be determined. For, if* $K \cong K_{8,\mathrm{I}}(a,b,c)$ *with* $(a,b,c) \in \mathfrak{R}_1$*, then the label of $K$ is* $[\![8,\mathrm{I};a,b,c]\!]$. *On the other hand,* $K \cong K'_{8,\mathrm{II}}(a,b,c)$ *with* $(a,b,c) \in \mathfrak{R}_2$ *or* $\mathfrak{R}_3$*, then the label of $K$ is* $[\![8,\mathrm{II};\dot{a},2b,2c]\!]$.

From now on, $\delta$ is either 1 or 2. For any integer $t$ and lattice $L$, $r(t,L)$ denotes the number of representations of $t$ by $L$. Let $m$ be a positive odd square free integer dividing $dK$. We define

$$b_{k,\delta m} = \sum_{\substack{[\widetilde{K}]\in\mathrm{gen}(K)\\ |O(\widetilde{K})|=k}} |\{\tau_x \in O(\widetilde{K}); Q(x) = \delta m\}|.$$

**Lemma 5.1.6.** *For any positive odd square free integer $m$ which divides $dK$, we have*

$$\frac{2}{4}b_{4,\delta m} + \frac{2}{8}b_{8,\delta m} + \frac{2}{12}b_{12,\delta m} + \frac{2}{16}b_{16,\delta m} + \frac{2}{24}b_{24,\delta m} = t_{m,\delta}\cdot 2^{\nu(m)-\nu(\mathfrak{P}\mathfrak{Q})}\cdot\frac{h_E}{\mu_E},$$

*where $h_E$ is the class number of the quadratic field* $E = \mathbb{Q}(\sqrt{-\delta d(\lambda_m(K))})$*, and $\mu_E$ be the number of roots of unity in $E$. The values $t_{m,\delta}$ are displayed in Table 5.1.*

Table 5.1: Values of $t_{m,\delta}$

| $\lambda_m(K)_2$ | $\delta$ | $t_{m,\delta}$ | $\lambda_m(K)_2$ | $\delta$ | $t_{m,\delta}$ |
|---|---|---|---|---|---|
| $\langle 1,1,3\rangle$ | 1 | 3 | $\langle 3,3,3\rangle$ | 1 | 1 |
| $\langle 1,1,7\rangle$ | 1 | 2 | $dK \equiv 1 \pmod 4$ | 1 | $\frac{1}{2}$ |
| $\mathbb{A} \perp \langle 2\rangle$ | 2 | 4 | $\mathbb{A} \perp \langle 14\rangle$ | 2 | 1 |
| $\mathbb{A} \perp \langle 6\rangle$ | 2 | 1 | $\mathbb{A} \perp \langle 10\rangle$ | 2 | 2 |
| odd | 2 | $\frac{1}{2}$ | | | |

*Proof.* See [1].

By Lemma 5.1.6, we may effectively compute the number of classes of lattices in $\mathrm{gen}(K)$ with label $[\![4,\mathrm{I};\delta m]\!]$ or $[\![4,\mathrm{II};\delta m]\!]$ once we know the labels of all the classes of lattices whose isometry groups are of order greater than 4.

In order to determine the number of classes in $\mathrm{gen}(K)$ with label $[\![2]\!]$, all we need is the class number of $K$ which is given by the following lemma.

**Lemma 5.1.7.** *The class number $h(K)$ of $K$ is equal to*

$$2\mathfrak{w}(K) + \sum_{\substack{m|\mathfrak{P}\mathfrak{Q},\ \delta\in\{1,2\} \\ \delta\to\mathrm{gen}(\lambda_m(K))}} t_{m,\delta}\cdot 2^{\nu(m)-\nu(\mathfrak{P}\mathfrak{Q})}\cdot\frac{h_E}{\mu_E} + \frac{1}{3}(b_{12}(K)+b_{24}(K)) + \frac{1}{4}b_{16}(K).$$

*Proof.* See [1].

Now we have only to determine the number of labels $[\![4,\mathrm{I};\delta m]\!]$ and $[\![4,\mathrm{II};\delta m]\!]$ , respectively, for any odd integer $m$ dividing $\mathfrak{P}\mathfrak{Q}$. We will define several transformations on $\mathrm{gen}(K)$ to do this work.

Let $K$ be a ternary stable lattice. Then $dK$ is odd or $K$ is even with $dK \equiv 2 \pmod 4$. Suppose that $K$ is even, and write $dK = 2\delta$. If there exists a lattice $K' \in \mathrm{gen}(K)$ such that $\mathrm{label}(K') = [\![4.\mathrm{I};2a]\!]$ for some integer $a$, then we may write $K' = \langle 2a\rangle \perp M$ for some binary lattice $M$ such that $M_2$ is a binary even unimodular lattice. This implies that if $\frac{\delta}{a} \equiv 1 \pmod 4$, then all lattices in $\mathrm{gen}(K)$ of order 4 are of type II. Therefore we may assume that $\frac{\delta}{a} \equiv 3 \pmod 4$.

**Lemma 5.1.8.** *Let $K$ be an even stable lattice with $dK = 2\delta$ for some odd integer $\delta$ and let*

$$K = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z \cong K_{4,\mathrm{II}}(a, b, c, d)$$

*for some integers $a$, $b$, $c$, $d$. If $\frac{\delta}{a} \equiv 3 \pmod 4$, we have*

$$K \cong K_{4,\mathrm{II}}(a, b, c', d')$$

*with $c' \equiv 1 \pmod 2$ and $d' \equiv 0 \pmod 8$ by a suitable base change.*

*Proof.* It is clear that the integer $a$ is odd and $2\delta = 2a(bd - c^2 - ad')$ where $d = 2d'$. Hence we have $bd - c^2 - ad' \equiv 3 \pmod 4$. If $d'$ is even, then $d \equiv 0 \pmod 8$ and $c$ is odd. Therefore we may assume that $d'$ is odd. Then we have $ad' \equiv 1 \pmod 4$ since $b$ is even, and we obtain

$$K = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}(x - 2y + z) \cong \begin{pmatrix} 2a & a & 0 \\ a & b & a - 2b + c \\ 0 & a - 2b + c & -2a + 4b - 4c + d \end{pmatrix}.$$

This is a desired form for our assertion. $\square$

Let $K = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z$. We define a transformation $\wedge$ by

$$\widehat{K} = \mathbb{Z}x + \mathbb{Z}(x - 2y) + \mathbb{Z}\left(\frac{1}{2}z\right).$$

Then we have the following lamma.

**Lemma 5.1.9.** *Let $K$ and $K'$ be even stable lattices such that*

$$K = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z \cong K_{4,\mathrm{II}}(a, b, c, d)$$

*and*

$$K' = \mathbb{Z}x' + \mathbb{Z}y' + \mathbb{Z}z' \cong K_{4,\mathrm{II}}(a, b', c', d')$$

*with $c \equiv c' \equiv 1 \pmod 2$ and $d \equiv d' \equiv 0 \pmod 8$. If there is an isometry $\sigma$ of $K'$ onto $K$ such that $\sigma(x') = x$, then $\widehat{K} = \sigma(\widehat{K'})$.*

*Proof.* Let $K = \sigma K'$ and $\sigma(x') = x$ for some isometry $\sigma$. We put $\sigma(y') = \alpha x + \beta y + \gamma z$ and $\sigma(z') = \alpha' x + \beta' y + \gamma' z$. Then we obtain $a = B(x', y') = 2a\alpha + a\beta$ and hence $2\alpha + \beta = 1$. We also have $0 = B(x', z') = 2a\alpha' + a\beta'$ and hence $\beta' = -2\alpha'$. Since $B(y', z') = c' \equiv 1 \pmod 2$, we may easily show that $\alpha' \not\equiv \gamma' \pmod 2$. Since $Q(z') = d' \equiv 0 \pmod 8$, we obtain that $\alpha'$ is even and $\gamma'$ is odd. Therefore we have

$$\sigma(\widehat{K'}) = \mathbb{Z}x + \mathbb{Z}(x - 2(\alpha x + (1-2\alpha)y + \gamma z)) + \mathbb{Z}(\frac{1}{2}(\alpha' x - 2\alpha' y + \gamma' z)) \subseteq \widehat{K}.$$

and hence $\sigma\widehat{K'} = \widehat{K}$. $\square$

Let $K$ be an even stable lattice with $dK = 2\delta$ for some odd integer $\delta$. We assume that

$$K = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z \cong K_{4,\mathrm{I}}(2a, b, c, d)$$

for some odd integer $a$ and some integers $b$, $c$ and $d$. If $\frac{\delta}{a} \equiv 7 \pmod 8$, we may choose a basis so that

$$K = \mathbb{Z}x + \mathbb{Z}y' + \mathbb{Z}z' \cong K_{4,\mathrm{I}}(2a, b', c', d')$$

with $2a + b' \equiv 0 \pmod 8$ and $d' \equiv 0 \pmod 4$. For a lattice $K = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z$, we define a transformation $\sim$ by

$$\widetilde{K} = \mathbb{Z}x + \mathbb{Z}\frac{1}{2}(x + y) + \mathbb{Z}(2z).$$

Then we have the following lemma.

**Lemma 5.1.10.** *Let $K$ and $K'$ be even stable lattices such that*

$$K = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z \cong K_{4,\mathrm{I}}(2a, b, c, d)$$

*and*

$$K' = \mathbb{Z}x' + \mathbb{Z}y' + \mathbb{Z}z' \cong K_{4,\mathrm{I}}(2a, b', c', d')$$

*with $b \equiv b' \equiv -2a \pmod 8$ and $d \equiv d' \equiv 0 \pmod 4$. If there exists an isometry $\sigma$ of $K'$ onto $K$ such that $\sigma(x') = x$, then $\widetilde{K} = \sigma(\widetilde{K'})$.*

*Proof.* Let $K = \sigma K'$ and $\sigma(x') = x$ for some isometry $\sigma$. We may put $\sigma(y') = \alpha y + \beta z$ and $\sigma(z') = \gamma y + \delta z$. Then we obtain

$$b \equiv b' \equiv Q(y') = Q(\alpha y + \beta z) = b\alpha^2 + d\beta^2 + 2\alpha\beta c \pmod 8.$$

Therefore $\alpha$ is odd and $\beta \equiv 0 \pmod 4$ since $b \equiv 2 \pmod 4$, $d \equiv 0 \pmod 4$ and $c$ is odd. Therefore

$$\sigma(\widetilde{K'}) = \mathbb{Z}x + \mathbb{Z}\frac{1}{2}(x + \alpha y + \beta z) + \mathbb{Z}2(\gamma y + \delta z) \subseteq \mathbb{Z}x + \mathbb{Z}\frac{1}{2}(x + y) + \mathbb{Z}(2z) = \widetilde{K},$$

and hence $\widetilde{K} = \sigma(\widetilde{K'})$. $\square$

**Lemma 5.1.11.** *Let $K$ and $K'$ be even stable lattices such that*

$$K = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z \cong K_{4,\mathrm{I}}(2a, b, c, d)$$

*and*

$$K' = \mathbb{Z}x' + \mathbb{Z}y' + \mathbb{Z}z' \cong K_{4,\mathrm{II}}(a', b', c', d')$$

*with $2a + b \equiv 0 \pmod 8$, $d \equiv 0 \pmod 4$, $c'$ is odd and $d' \equiv 0 \pmod{16}$. Then we have*

$$\widehat{\widetilde{K}} = K \quad \textit{and} \quad \widetilde{\widehat{K'}} = K'.$$

*Furthermore, we have $[\widehat{K}] \in gen(K)$ and $[\widetilde{K'}] \in gen(K')$.*

*Proof.* This proof is straightforward.

Note that we may choose a basis of $K'$ such that the assumption for entries in the above lemma is satisfied, if $\frac{\delta}{a} \equiv 7 \pmod 8$.

We define notations

$$\begin{aligned}
B_{8,m}(1) &= \{[K'] \in \mathrm{gen}(K) \mid \mathrm{label}(K') = [\![8, \mathrm{II}; \dot{m}, *]\!]\},\\
B_{8,m}(2) &= \{[K'] \in \mathrm{gen}(K) \mid \mathrm{label}(K') = [\![8, \mathrm{II}; \dot{*}, m, *]\!]\},\\
B_{4,m}(1) &= \{[K'] \in \mathrm{gen}(K) \mid \mathrm{label}(K') = [\![4, \mathrm{I}; m]\!]\},\\
B_{4,m}(2) &= \{[K'] \in \mathrm{gen}(K) \mid \mathrm{label}(K') = [\![4, \mathrm{II}; m]\!]\}
\end{aligned}$$

and

$$b_{8,m}(i) = |B_{8,m}(i)|, \quad b_{4,m}(i) = |B_{4,m}(i)| \ (i = 1, 2),$$

$$B_t(K) = \{[K'] \in \mathrm{gen}(K) \mid |O(K')| = t\}.$$

**Theorem 5.1.12.** *Let $K$ be an even stable lattice, and let $a$ be any positive odd integer dividing $dK$ with $\frac{\delta}{a} \equiv 7 \pmod 8$. Then*

$$\begin{cases} b_{8,2a}(1) + b_{4,2a}(1) = b_{8,2a}(2) + b_{4,2a}(2) & \text{if } a \neq 1,\ 3, \\ b_{8,2}(1) + b_{4,2}(1) = b_{8,2}(2) + b_{4,2}(2) + b_{12}(K) + b_{24}(K) & \text{if } a = 1, \\ b_{8,6}(1) + b_{4,6}(1) = b_{8,6}(2) + b_{4,6}(2) + b_{24}(K) & \text{if } a = 3. \end{cases}$$

*Proof.* We may assume that $|O(K)| \neq 48$ and it is clear that $b_{16}(K) = 0$ by Lemma 5.1.3. Define

$$B_1 := B_{8,2a}(1) \cup B_{4,2a}(1), \quad B_2 := B_{8,2a}(2) \cup B_{4,2a}(2) \cup B_{12}(K) \cup B_{24}(K).$$

By Lemma 5.1.8, Lemma 5.1.9, Lemma 5.1.10 and Lemma 5.1.11, the transformation $\sim$ induces a bijection from $B_1$ onto $B_2$ and we have an equation

$$b_{8,2a}(1) + b_{4,2a}(1) = b_{8,2a}(2) + b_{4,2a}(2) + b_{12}(K) + b_{24}(K).$$

If $a \neq 1,\ 3$, then $b_{12}(K) = b_{24}(K) = 0$ and if $a = 3$, then $b_{12}(K) = 0$ by Lemma 5.1.1 and Lemma 5.1.2. Note that $a \neq \frac{dK}{6}$ since $\frac{\delta}{a} \equiv 7 \pmod 8$. $\square$

Since we may calculate all terms in the above equations except $b_{4,2a}(1)$ and $b_{4,2a}(2)$, and we also know the value $b_{4,2a}(1) + b_{4,2a}(2)$, we may determine the values $b_{4,2a}(1)$ and $b_{4,2a}(2)$.

Next, we suppose that $K$ is an even stable lattice and $dK = 2\delta$ with $\frac{\delta}{a} \equiv 3 \pmod 8$. If $K = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z \cong K_{4,\mathrm{I}}(2a, b, c, d)$, then the integers $a$ and $c$ are odd and $b \equiv d \equiv 2 \pmod 4$. Furthermore, we may assume that

$$b \equiv d \pmod 8, \quad 2a + b \equiv 0 \pmod 8, \quad b + 2c \equiv 0 \pmod 8 \qquad (5.1.1)$$

by a suitable base change. We define transformations

$$\begin{aligned} \widetilde{K_1} &= \mathbb{Z}x + \mathbb{Z}(\tfrac{x+y}{2}) + \mathbb{Z}(2z), \\ \widetilde{K_2} &= \mathbb{Z}x + \mathbb{Z}(\tfrac{x+z}{2}) + \mathbb{Z}(2y), \\ \widetilde{K_3} &= \mathbb{Z}x + \mathbb{Z}(\tfrac{x+y+z}{2}) + \mathbb{Z}(2z). \end{aligned}$$

Then we may easily show that $\widetilde{K_i} \in \mathrm{gen}(K)$ $(i = 1, 2, 3)$. Take any primitive vector $w = \alpha y + \beta z \in \mathbb{Z}y + \mathbb{Z}z$ such that $2a + Q(w) \equiv 0 \pmod 8$ and write $K = \mathbb{Z}x + \mathbb{Z}w + \mathbb{Z}z'$ for some vector $z' \in K$. Then we have

$$\mathbb{Z}x + \mathbb{Z}\left(\frac{x+w}{2}\right) + \mathbb{Z}(2z') = \begin{cases} \widetilde{K_1} & \text{if } \alpha \text{ is odd and } \beta \equiv 0 \pmod 4, \\ \widetilde{K_2} & \text{if } \beta \text{ is odd and } \alpha \equiv 0 \pmod 4, \\ \widetilde{K_3} & \text{if } \alpha,\ \beta \text{ are odd and } \alpha \equiv \beta \pmod 4, \end{cases}$$

and they are all cases.

**Theorem 5.1.13.** *Let $K$ be an even stable lattice, and let $a$ be any positive odd integer dividing $dK$ with $\frac{\delta}{a} \equiv 3 \pmod 8$. Then*

$$\begin{cases} 2b_{8,2a}(1) + 3b_{4,2a}(1) = b_{8,2a}(2) + b_{4,2a}(2) & \textit{if } a \neq 1,\ 3,\ \frac{dK}{6}, \\ 2b_{8,2}(1) + 3b_{4,2}(1) = b_{8,2}(2) + b_{4,2}(2) + b_{12}(K) + b_{24}(K) & \textit{if } a = 1, \\ 2b_{8,6}(1) + 3b_{4,6}(1) = b_{8,6}(2) + b_{4,6}(2) + b_{24}(K) & \textit{if } a = 3, \\ 2b_{8,2a}(1) + 3b_{4,2a}(1) + b_{24}(K) = b_{8,2a}(2) + b_{4,2a}(2) & \textit{if } a = \frac{dK}{6}. \end{cases}$$

*Proof.* If $a \neq \frac{dK}{6}$, we define a correspondence $\Phi$ from $B_1$ into $B_2$ by

$$\Omega([K']) = \{[\widetilde{K'_1}], [\widetilde{K'_2}], [\widetilde{K'_3}]\},$$

where $B_1$, $B_2$ are defined in the proof of Theorem 5.1.12. Then we may show that this correspondence is well defined. If $a = \frac{dK}{6}$, then $\Omega$ induces a well defined correspondence from $B'_1$ into $B'_2$, where $B'_1 = B_1 \cup B_{24}(K)$ and $B'_2 = B_1 \backslash B_{24}(K)$. Furthermore, we may verify that

$$\bigsqcup_{[K'] \in B_1 (or B'_1)} \Omega([K']) = B_2 \text{ ( or } B'_2),$$

by considering the map $\wedge$. Let $[K']$ be an element of $B_1$ or $B'_1$.

If $K' = K_{24}(1, 2b)$, we may verify that

$$\widetilde{K'_1} \cong \widetilde{K'_2} \cong \widetilde{K'_3} \cong \begin{cases} \langle 6 \rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & \frac{b+1}{2} \end{pmatrix} & \text{if } b \equiv 3 \pmod 4, \\ \langle 2 \rangle \perp \begin{pmatrix} 6 & 3 \\ 3 & \frac{b+3}{2} \end{pmatrix} & \text{if } b \equiv 1 \pmod 4. \end{cases}$$

Next, suppose that $|O(K')| = 8$. Then the Gram matrix of $K'$ is of the form

$$K' = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z \cong \langle 2a \rangle \perp \begin{pmatrix} \frac{b+c}{2} & \frac{b-c}{2} \\ \frac{b-c}{2} & \frac{b+c}{2} \end{pmatrix}.$$

Since $dK = 2abc$, we have $bc \equiv 3 \pmod 4$.
i) Suppose that $2a + \frac{b+c}{2} \equiv 0 \pmod 8$. Then we may assume that $\frac{b+c}{2} + b - c \equiv 0 \pmod 8$ and then this form satisfies the condition (5.1.1). We may show that $\widetilde{K'_1} \cong \widetilde{K'_2}$ and claim that $\widetilde{K'_1} \not\cong \widetilde{K'_3}$. Since

$$\widetilde{K'_3} = \mathbb{Z}x' + \mathbb{Z}y' + \mathbb{Z}z' \cong \langle 2c \rangle \perp \begin{pmatrix} 2a & a \\ a & \frac{a+b}{2} \end{pmatrix},$$

we may verify that $|O(\widetilde{K'_3})| = 8$. Hence, if $\sigma(\widetilde{K'_1}) = \widetilde{K'_3}$ for some isometry $\sigma$, then $\sigma(x) = \pm y'$, and this implies that $x^\perp$ in $\widetilde{K'_1}$ is isometric to $(y')^\perp$ in $\widetilde{K'_3}$. But we may show that two lattices are not isometric and this is a contradiction.
ii) Suppose that $2a + \frac{b+c}{2} \not\equiv 0 \pmod 8$. Then we may assume that $2a + 2b \equiv 0 \pmod 8$ and obtain

$$K' = \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}(y+z) \cong \langle 2a \rangle \perp \begin{pmatrix} 2b & 3b \\ 3b & \frac{9b+c}{2} \end{pmatrix}.$$

This form satisfies the condition (5.1.1) and we obtain $\widetilde{K'_1} \not\cong \widetilde{K'_2} \cong \widetilde{K'_3}$ by a similar argument as above.

Finally, assume that $|O(K')| = 4$ and $K' \cong K_{4,\mathrm{I}}(2a, b, c, d)$. By the above discussion, we may assume that this form satisfies the condition (5.1.1) and we obtain

$$\widetilde{K_1'} \cong \begin{pmatrix} 2a & a & 0 \\ a & \frac{2a+b}{4} & c \\ 0 & c & 4d \end{pmatrix}, \quad \widetilde{K_2'} \cong \begin{pmatrix} 2a & a & 0 \\ a & \frac{2a+d}{4} & c \\ 0 & c & 4b \end{pmatrix},$$

$$\widetilde{K_3'} \cong \begin{pmatrix} 2a & a & 0 \\ a & \frac{2a+b+2c+d}{4} & c+d \\ 0 & c+d & 4d \end{pmatrix}.$$

Since the binary lattices $x^\perp$ in $\widetilde{K_i'}$ $(i = 1, 2, 3)$ are not isometric to each other, $\widetilde{K_1'}$, $\widetilde{K_2'}$ and $\widetilde{K_3'}$ are also not isometric to each other.

By a similar argument as in Theorem 5.1.12, we may obtain the required equations. □

Finally, if $dK$ is odd, we may easily verify that

$$b_{4,a}(1) + b_{4,a}(2) = \begin{cases} b_{4,a}(1) & \text{if } a \text{ is odd,} \\ b_{4,a}(2) & \text{if } a \text{ is even} \end{cases}$$

for any positive integer $a$.

## 5.2 Information for exceptional cases

Let $K$ be an odd stable lattice and $a$ be a positive integer dividing $dL$. Then we define notations

$$\begin{aligned} B_{8,a}(3) &= \{[K'] \in \text{gen}(K) \mid \text{label}(K') = [\![8, \mathrm{I}; a, *]\!]\}, \\ B_{4,a}(3) &= \{[K'] \in \text{gen}(K) \mid \text{label}(K') = [\![4, \mathrm{I}; a]\!] \text{ and } \ell \text{ is odd}\}, \\ B_{4,a}(4) &= \{[K'] \in \text{gen}(K) \mid \text{label}(K') = [\![4, \mathrm{I}; a]\!] \text{ and } \ell \text{ is even}\}, \end{aligned}$$

and

$$b_{8,a}(3) = |B_{8,a}(3)|, \quad b_{4,a}(i) = |B_{4,a}(i)| \ (i = 3, 4),$$

where $K' = \langle a \rangle \perp \ell$ for some binary lattice $\ell$ when $\text{label}(K') = [\![4, \mathrm{I}; a]\!]$. Now we are going to calculate the values $b_{4,a}(3)$ and $b_{4,a}(4)$ for complete determination of $\text{label}(\gamma_{2e}^L(K))$ $(e = 1, 2)$ when $\text{label}(K) = [\![4, \mathrm{I}; a]\!]$. There are four cases, Case(4.7.1.1), Case(4.7.1.6), Case(4.7.1.10) and Case(4.7.1.22), which need these values.

Let $L$ be a binary lattice such that $L_2 \cong \mathbb{H}$ and let $M$ be a binary lattice such that $M_2 \cong \langle 1, 7\rangle$. Assume that $L_p \cong M_p^2$ for any odd prime $p$. Then we define a map $\phi$ from $\mathrm{gen}(M)/\sim$ into $\mathrm{gen}(L)/\sim$ as follows; for any class $[M'] \in \mathrm{gen}(M)$, $\phi([M']) = [\Lambda_2(M')^{\frac{1}{2}}]$. Then this map is clearly well defined. Conversely, for any lattice $L' \in \mathrm{gen}(L)$, choose a primitive vector $x \in L'$ such that $Q(x) \equiv 2 \pmod 4$. Let $L' = \mathbb{Z}x + \mathbb{Z}y$ and define a map $\psi$ by $\psi(L') = (\mathbb{Z}(\frac{1}{2}x) + \mathbb{Z}y)^2$. Then this map is also a well defined map from $\mathrm{gen}(L)/\sim$ into $\mathrm{gen}(M)/\sim$ and furthermore, this is a bijection with the inverse map $\phi$.

For an odd stable lattice $K$ and an odd integer $a$ dividing $dK$ with $\frac{dK}{a} \equiv 7 \pmod 8$, define

$$\begin{aligned} C_1 &= \left\{[K'] \in \mathrm{gen}(K) \mid K' \cong \langle a\rangle \perp \ell\right\}, \\ C_2 &= \left\{[K'] \in \mathrm{gen}(K) \mid K' \cong \langle a\rangle \perp \ell\right\}, \end{aligned}$$

and also define two maps $\Phi$ and $\Psi$ by

$$\Phi(K') = \langle a\rangle \perp \phi(\ell) \text{ for any } K' \in C_1$$

and

$$\Psi(K') = \langle a\rangle \perp \psi(\ell) \text{ for any } K' \in C_2,$$

respectively. Then it is clear that $\Phi : C_1 \to C_2$ is a bijection with the inverse map $\Psi$. By using these maps, we may obtain the following theorem.

**Theorem 5.2.1.** *Let $K$ be an odd stable lattice and let $a$ be any positive odd integer dividing $dK$. Then*

$$\begin{cases} b_{4,a}(3) = b_{4,a}(4) & \text{if } \frac{dL}{a} \equiv 7 \pmod 8, \\ b_{4,a}(3) = 0 & \text{if } \frac{dL}{a} \equiv 3 \pmod 8, \\ b_{4,a}(4) = 0 & \text{otherwise.} \end{cases}$$

*Proof.* If $\frac{dK}{a} \equiv 1$ or $5 \pmod 8$, then it is clear that $b_{4,a}(4) = 0$. Suppose that $\frac{dL}{a} \equiv 7 \pmod 8$ and $a \neq 1$. Then we have $C_1 = B_{4,a}(3) \cup B_{8,a}(3)$ and $C_2 =$

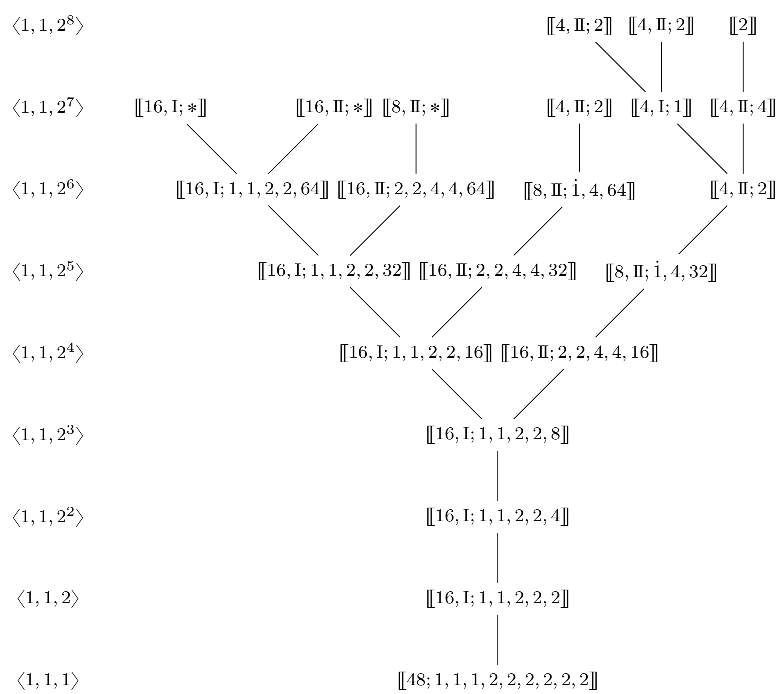
$B_{4,a}(4) \cup B_{8,a}(1)$. We may also show that $\Psi(B_{8,a}(1)) = B_{8,a}(3)$ and hence we have $\Psi(B_{4,a}(4)) = B_{4,a}(3)$. Suppose that $\frac{dL}{a} \equiv 7 \pmod 8$ and $a = 1$. Then $C_1 = B_{4,1}(3) \cup B_{8,1}(3) \cup B_{16}$ and $C_2 = B_{4,1}(4) \cup B_{8,1}(1)$, and $\Psi(B_{8,1}(1)) = B_{8,1}(3) \cup B_{16}$. Therefore we get the same result as above. Finally, we suppose that $\frac{dK}{a} \equiv 3 \pmod 8$. Then it is clear that $K_2$ is anisotropic and we have $b_{4,a}(3) = 0$. $\square$

# Chapter 6

# Applications

Finally, we provide an application of the above results. An integral lattice of the form $\langle 1, 2^m, 2^n \rangle$ $(n \geqslant m \geqslant 0)$ is called a *Bell ternary form* and we are going to give a precise formula for class number of this lattice.

First, we consider the case when $m = 0$. Then we may obtain the following tree of labels.



Observing the above tree, we may find the following rules of ramification when $n \geqslant 3$.

$$\begin{tikzcd}[column sep=small]
\llbracket 16, \mathrm{I} \rrbracket \arrow[dr, dash] & & \llbracket 16, \mathrm{I\!I} \rrbracket \arrow[dl, dash] \\
& \llbracket 16, \mathrm{I} \rrbracket &
\end{tikzcd}
\quad
\begin{tikzcd}
\llbracket 8, \mathrm{I\!I} \rrbracket \arrow[d, dash] \\
\llbracket 16, \mathrm{I\!I} \rrbracket
\end{tikzcd}
\quad
\begin{tikzcd}
\llbracket 4, \mathrm{I\!I}; 2 \rrbracket \arrow[d, dash] \\
\llbracket 8, \mathrm{I\!I} \rrbracket
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
\llbracket 4, \mathrm{I\!I}; 2 \rrbracket \arrow[dr, dash] & & \llbracket 4, \mathrm{I\!I}; 2 \rrbracket \arrow[dl, dash] \\
& \llbracket 4, \mathrm{I}; 1 \rrbracket &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
\llbracket 4, \mathrm{I}; 1 \rrbracket \arrow[dr, dash] & & \llbracket 4, \mathrm{I\!I}; 4 \rrbracket \arrow[dl, dash] \\
& \llbracket 4, \mathrm{I\!I}; 2 \rrbracket &
\end{tikzcd}$$

$$\begin{tikzcd}
\llbracket 2 \rrbracket \arrow[d, dash] \\
\llbracket 4, \mathrm{I\!I}; 4 \rrbracket
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
\llbracket 2 \rrbracket \arrow[dr, dash] & & \llbracket 2 \rrbracket \arrow[dl, dash] \\
& \llbracket 2 \rrbracket &
\end{tikzcd}$$

For a lattice $L$, we define label(gen($L$)) to be the multiset of labels which are correspond to all isometry classes in gen($L$). Then it is clear that there is only one label of type $\llbracket 16, \mathrm{I}; * \rrbracket$ in label(gen($\langle 1, 1, 2^n \rangle$)) for $n \geqslant 5$, and so are labels of types $\llbracket 16, \mathrm{I\!I}; * \rrbracket$ and $\llbracket 8, \mathrm{I\!I}; * \rrbracket$, respectively. Furthermore, observing the above tree, we may obtain

$$\sharp\llbracket 4, \mathrm{I}; 1 \rrbracket = \sharp\llbracket 4, \mathrm{I\!I}; 4 \rrbracket = \begin{cases} 2^{\frac{n-5}{2}} - 1 & \text{if } n (\geqslant 5) \text{ is odd,} \\ 2^{\frac{n-6}{2}} - 1 & \text{if } n (\geqslant 6) \text{ is even,} \end{cases}$$

$$\sharp\llbracket 4, \mathrm{I\!I}; 2 \rrbracket = \begin{cases} 2^{\frac{n-5}{2}} - 1 & \text{if } n (\geqslant 5) \text{ is odd,} \\ 2^{\frac{n-4}{2}} - 1 & \text{if } n (\geqslant 4) \text{ is even,} \end{cases}$$

$$\sharp\llbracket 2 \rrbracket = \begin{cases} 2^{n-6} - 3 \cdot 2^{\frac{n-7}{2}} + 1 & \text{if } n (\geqslant 9) \text{ is odd,} \\ 2^{n-6} - 2^{\frac{n-4}{2}} + 1 & \text{if } n (\geqslant 8) \text{ is even,} \end{cases}$$

where $\sharp\llbracket * \rrbracket$ denotes the multiplicity of the label $\llbracket * \rrbracket$ in label(gen($\langle 1, 1, 2^n \rangle$)).

Therefore for any $n \geqslant 8$, we have

$$h(\langle 1,1,2^n\rangle) = \begin{cases} 2^{n-6} + 3\cdot 2^{\frac{n-7}{2}} + 1 & \text{if } n \text{ is odd,}\\ 2^{n-6} + 2^{\frac{n-4}{2}} + 1 & \text{otherwise.}\end{cases}$$

Actually, this formula holds for any integer $n \geqslant 5$. By using the same method, we may obtain the formulas for another cases as follows;

$$h(\langle 1,1,2^n\rangle) = \begin{cases} 2^{n-6} + 3\cdot 2^{\frac{n-7}{2}} + 1 & \text{if } n(\geqslant 5) \text{ is odd,}\\ 2^{n-6} + 2^{\frac{n-4}{2}} + 1 & \text{if } n(\geqslant 6) \text{ is even,}\end{cases}$$

$$h(\langle 1,2,2^n\rangle) = \begin{cases} 2^{n-6} + 3\cdot 2^{\frac{n-7}{2}} + 1 & \text{if } n(\geqslant 7) \text{ is odd,}\\ 2^{n-6} + 2^{\frac{n-4}{2}} + 1 & \text{if } n(\geqslant 6) \text{ is even,}\end{cases}$$

$$h(\langle 1,2^2,2^n\rangle) = \begin{cases} 2^{n-6} + 3\cdot 2^{\frac{n-7}{2}} + 1 & \text{if } n(\geqslant 7) \text{ is odd,}\\ 2^{n-6} + 3\cdot 2^{\frac{n-8}{2}} + 1 & \text{if } n(\geqslant 8) \text{ is even,}\end{cases}$$

$$h(\langle 1,2^3,2^n\rangle) = \begin{cases} 2^{n-7} + 5\cdot 2^{\frac{n-9}{2}} + 1 & \text{if } n(\geqslant 9) \text{ is odd,}\\ 2^{n-7} + 3\cdot 2^{\frac{n-8}{2}} + 1 & \text{if } n(\geqslant 8) \text{ is even,}\end{cases}$$

$$h(\langle 1,2^4,2^n\rangle) = \begin{cases} 2^{n-7} + 5\cdot 2^{\frac{n-9}{2}} + 1 & \text{if } n(\geqslant 9) \text{ is odd,}\\ 2^{n-7} + 5\cdot 2^{\frac{n-10}{2}} + 1 & \text{if } n(\geqslant 10) \text{ is even,}\end{cases}$$

$$h(\langle 1,2^n,2^n\rangle) = \begin{cases} 2^{n-6} + 3\cdot 2^{\frac{n-7}{2}} + 1 & \text{if } n(\geqslant 7) \text{ is odd,}\\ 2^{n-6} + 2^{\frac{n-4}{2}} + 1 & \text{if } n(\geqslant 6) \text{ is even,}\end{cases}$$

$$h(\langle 1,2^n,2^{n+1}\rangle) = \begin{cases} 2^{n-5} + 2^{\frac{n-3}{2}} + 1 & \text{if } n(\geqslant 5) \text{ is odd,}\\ 2^{n-5} + 3\cdot 2^{\frac{n-6}{2}} + 1 & \text{if } n(\geqslant 6) \text{ is even,}\end{cases}$$

$$h(\langle 1,2^n,2^{n+2}\rangle) = \begin{cases} 2^{n-4} + 3\cdot 2^{\frac{n-5}{2}} + 1 & \text{if } n(\geqslant 5) \text{ is odd,}\\ 2^{n-4} + 3\cdot 2^{\frac{n-6}{2}} + 1 & \text{if } n(\geqslant 6) \text{ is even,}\end{cases}$$

$$h(\langle 1, 2^n, 2^{n+3}\rangle) = \begin{cases} 2^{n-4} + 3 \cdot 2^{\frac{n-5}{2}} + 1 & \text{if } n(\geqslant 5) \text{ is odd,} \\ 2^{n-4} + 5 \cdot 2^{\frac{n-6}{2}} + 1 & \text{if } n(\geqslant 6) \text{ is even,} \end{cases}$$

$$h(\langle 1, 2^n, 2^{n+4}\rangle) = \begin{cases} 2^{n-3} + 5 \cdot 2^{\frac{n-5}{2}} + 1 & \text{if } n(\geqslant 5) \text{ is odd,} \\ 2^{n-3} + 5 \cdot 2^{\frac{n-6}{2}} + 1 & \text{if } n(\geqslant 6) \text{ is even,} \end{cases}$$

$$h(\langle 1, 2^m, 2^n\rangle) = \begin{cases} 2^{n-7} + 3 \cdot 2^{\frac{n-7}{2}} + 2^{\frac{m-3}{2}} + 2^{\frac{n-m-4}{2}} & \text{if } n(\geqslant 5) \text{ is odd,} \\ 2^{n-7} + 3 \cdot 2^{\frac{n-8}{2}} + 2^{\frac{m-3}{2}} + 2^{\frac{n-m-3}{2}} & \text{if } n(\geqslant 6) \text{ is even,} \end{cases}$$

$(m(\geqslant 5)$ is odd, $n - m \geqslant 5)$

$$h(\langle 1, 2^m, 2^n\rangle) = \begin{cases} 2^{n-7} + 3 \cdot 2^{\frac{n-7}{2}} + 2^{\frac{m-4}{2}} + 2^{\frac{n-m-3}{2}} & \text{if } n(\geqslant 5) \text{ is odd,} \\ 2^{n-7} + 3 \cdot 2^{\frac{n-8}{2}} + 2^{\frac{m-4}{2}} + 2^{\frac{n-m-4}{2}} & \text{if } n(\geqslant 6) \text{ is even.} \end{cases}$$

$(m(\geqslant 5)$ is even, $n - m \geqslant 5)$

The following table represents the class numbers of Bell ternary forms in the cases which are not contained in the above closed formulas.

Table 6.1: Values of $h(\langle 1, 2^m, 2^n\rangle)$

| $m\backslash n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 2 | | | | |
| 1 | 1 | 1 | 1 | 1 | 1 | | | |
| 2 | | 1 | 1 | 2 | 2 | 3 | | |
| 3 | | | 1 | 1 | 2 | 2 | 3 | |
| 4 | | | | 2 | 2 | 3 | 3 | 4 |
| 5 | | | | | 2 | | | |

# Chapter 7

# Appendix

Table 7.1: $\operatorname{label}(K) = [\![24; 2a, 2a, 2a, 6a, 6a, 6a, b]\!]$

| $L_2$ | $\operatorname{label}(\gamma_{2e}^{L}(K))$ $(e = 1 \text{ or } 2)$ |
|---|---|
| $\mathbb{A} \perp \langle 2^m\epsilon\rangle$ $(m = 2, 3)$ | $[\![24; 2a, 2a, 2a, 6a, 6a, 6a, \dot{4b}]\!]$ |
| $\mathbb{A} \perp \langle 2^m\epsilon\rangle$ $(m \geqslant 4)$ | $[\![24; 2a, 2a, 2a, 6a, 6a, 6a, \dot{4b}]\!]$, $[\![8, \mathrm{III}; 2a, 6a, 4b]\!]$ |
| $\langle\epsilon\rangle \perp 2\mathbb{A}$ | $[\![24; 4a, 4a, 4a, 12a, 12a, 12a, \dot{\frac{b}{2}}]\!]$ |
| $\langle\epsilon\rangle \perp 2^m\mathbb{A}$ $(m = 2, 3)$ | $[\![24; 8a, 8a, 8a, 24a, 24a, 24a, \dot{b}]\!]$ |
| $\langle\epsilon\rangle \perp 2^m\mathbb{A}$ $(m \geqslant 4)$ | $[\![24; 8a, 8a, 8a, 24a, 24a, 24a, \dot{b}]\!]$, $[\![8, \mathrm{IV}; 8a, 24a, 4b]\!]$ |
| $\langle\epsilon\rangle \perp 8\mathbb{H}$ | $[\![8, \mathrm{IV}; 8a, 24a, 4b]\!]$ |
| $\mathbb{H} \perp \langle 8\epsilon\rangle$ | $[\![8, \mathrm{III}; 2a, 6a, 4b]\!]$ |
| $\langle 1, 3, 2^m\epsilon\rangle$ $(m = 1, 2)$ | $[\![8, \mathrm{I}; a, 3a, 2b]\!]$ |
| $\langle 1, 3, 2^m\epsilon\rangle$ $(m \geqslant 3)$ | $[\![8, \mathrm{I}; a, 3a, 2b]\!]$, $[\![8, \mathrm{IV}; 4a, 12a, 2b]\!]$ |
| $\langle\epsilon_1, 4\epsilon_2, 4\epsilon_3\rangle$ $(m \geqslant 3)$ | $[\![8, \mathrm{IV}; 8a, 24a, 4b]\!]$ |

Table 7.2: label$(K)$ = $[\![16, \mathrm{I}; \mathrm{a}, \mathrm{a}, 2\mathrm{a}, 2\mathrm{a}, \mathrm{b}]\!]$

| $L_2$ | | label$(\gamma_{2e}^{L}(K))$ ($e = 1$ or $2$) |
|---|---|---|
| $T \perp \langle 4\epsilon \rangle$ | | $[\![16, \mathrm{II}; 2a, 2a, 4a, 4a, 4b]\!]$ |
| $\langle \epsilon \rangle \perp 4T$ | | $[\![16, \mathrm{II}; 4a, 4a, 8a, 8a, 4b]\!]$ |
| $\langle \epsilon_1, \epsilon_2, 2\epsilon_3 \rangle$ | $\epsilon_1\epsilon_2 \equiv 3 \pmod 8$ | $[\![16, \mathrm{I}; a, a, 2a, 2a, 2b]\!]$, $[\![8, \mathrm{II}; 2a, 2a, 2b]\!]$ |
| | $\epsilon_1\epsilon_2 \equiv 7 \pmod 8$ | $[\![16, \mathrm{I}; a, a, 2a, 2a, 2b]\!] \times 2$ |
| $\langle \epsilon_1, \epsilon_2, 4\epsilon_3 \rangle$ | | $[\![16, \mathrm{I}; a, a, 2a, 2a, 2b]\!]$ |
| $\langle \epsilon_1, \epsilon_2, 8\epsilon_3 \rangle$ | $\epsilon_1 \equiv \epsilon_2 \pmod 8$ | $[\![16, \mathrm{I}; a, a, 2a, 2a, 2b]\!]$ |
| | $\epsilon_1 \not\equiv \epsilon_2 \pmod 8$ | $[\![16, \mathrm{II}; 2a, 2a, 4a, 4a, 2b]\!]$ |
| $\langle \epsilon_1, \epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 4)$ | | $[\![16, \mathrm{I}; a, a, 2a, 2a, 2b]\!]$, $[\![16, \mathrm{II}; 2a, 2a, 4a, 4a, 2b]\!]$ |
| $\langle \epsilon_1, 2\epsilon_2, 2\epsilon_3 \rangle$ | | $[\![16, \mathrm{I}; 2a, 2a, 4a, 4a, \frac{b}{2}]\!]$ |
| $\langle \epsilon_1, 2\epsilon_2, 4\epsilon_3 \rangle$ | | $[\![8, \mathrm{I}; \frac{a}{2}, 2a, 2b]\!]$ |
| $\langle \epsilon_1, 4\epsilon_2, 4\epsilon_3 \rangle$ | | |
| $a \equiv b \pmod 4$ | | $[\![8, \mathrm{I}; a, 4a, 4b]\!]$, $[\![16, \mathrm{I}; 4a, 4a, 8a, 8a, b]\!]$ |
| $a \not\equiv b \pmod 4$ | $\epsilon_2 \equiv \epsilon_3 \pmod 4$ | $[\![16, \mathrm{I}; 4a, 4a, 8a, 8a, b]\!]$ |
| | $\epsilon_2 \not\equiv \epsilon_3 \pmod 4$ | $[\![8, \mathrm{I}; a, 4a, 4b]\!]$ |
| $\langle \epsilon_1, 4\epsilon_2, 8\epsilon_3 \rangle$ | $a \equiv \epsilon_1 \pmod 4$ | $[\![8, \mathrm{I}; a, 4a, 4b]\!]$ |
| | $a \not\equiv \epsilon_1 \pmod 4$ | $[\![8, \mathrm{II}; 4a, 4a, 4b]\!]$ |
| $\langle \epsilon_1, 4\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 4)$ | $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | $[\![8, \mathrm{I}; a, 4a, 4b]\!]$, $[\![8, \mathrm{II}; \dot{4}a, 4a, 4b]\!]$ |
| | $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$ | $[\![8, \mathrm{II}; \dot{4}a, 4a, 4b]\!]$ |
| $\langle \epsilon_1, 8\epsilon_2, 8\epsilon_3 \rangle$ | | |
| $\epsilon_2 \equiv \epsilon_3 \pmod 4$ | $\epsilon_1 \equiv b \pmod 8$ | $[\![16, \mathrm{I}; 4a, 4a, 8a, 8a, b]\!]$ |
| | $\epsilon_1 \not\equiv b \pmod 8$ | $[\![8, \mathrm{II}; \dot{4}a, 4a, 4b]\!]$ |
| $\epsilon_2 \not\equiv \epsilon_3 \pmod 4$ | | $[\![8, \mathrm{II}; 4a, 4a, 4b]\!]$ |
| $\langle \epsilon_1, 16\epsilon_2, 16\epsilon_3 \rangle$ | $\epsilon_1 \equiv b \pmod 8$ | $[\![16, \mathrm{I}; 4a, 4a, 8a, 8a, b]\!]$, $[\![16, \mathrm{II}; 4a, 4a, 8a, 8a, 4b]\!]$ |
| | $\epsilon_1 \not\equiv b \pmod 8$ | $[\![8, \mathrm{II}; \dot{4}a, 4a, 4b]\!]$ |
| $\langle \epsilon_1, 2^m\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 5)$ | | $[\![8, \mathrm{II}; 4a, 4a, 4b]\!]$, $[\![16, \mathrm{I}; 4a, 4a, 8a, 8a, b]\!]$, $[\![16, \mathrm{II}; 4a, 4a, 8a, 8a, 4b]\!]$ |

Table 7.3: label$(K) = [\![16, \mathrm{II}; 2a, 2a, 4a, 4a, 4(b-a)]\!]$

| $L_2$ | | label$(\gamma_{2e}^L(K))$ ($e = 1$ or 2) |
|---|---|---|
| $T \perp \langle 16\epsilon\rangle$ | | $[\![4, \mathrm{II}; 2a]\!]$ |
| $\langle\epsilon\rangle \perp 16T$ | | $[\![4, \mathrm{II}; 16a]\!]$ |
| $\langle\epsilon_1, \epsilon_2, 8\epsilon_3\rangle$ | $\epsilon_1\epsilon_2 \equiv 3 \pmod 8$ | $[\![8, \mathrm{II}; \dot{a}, 4a, 8(b-a)]\!]$, $[\![4, \mathrm{II}; 8a]\!]$ |
| | $\epsilon_1\epsilon_2 \equiv 7 \pmod 8$ | $[\![8, \mathrm{II}; \dot{a}, 4a, 8(b-a)]\!]$ |
| $\langle\epsilon_1, \epsilon_2, 2^m\epsilon_3\rangle$ ($m \geqslant 4$) | $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | $[\![8, \mathrm{II}; \dot{a}, 4a, 8(b-a)]\!]$ |
| $\langle\epsilon_1, 4\epsilon_2, 2^m\epsilon_3\rangle$ ($m \geqslant 5$) | $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | $[\![4, \mathrm{II}; 16a]\!]$ |
| $\langle\epsilon_1, 2^m\epsilon_2, 2^m\epsilon_3\rangle$ ($m \geqslant 5$) | $\epsilon_2 \equiv \epsilon_3 \pmod 4$ | $[\![4, \mathrm{II}; 16a]\!]$ |

Table 7.4: label$(K) = [\![12; 2a, 2a, 2a]\!]$

| $L_2$ | label$(\gamma_{2e}^L(K))$ ($e = 1$ or 2) |
|---|---|
| $\mathbb{A} \perp \langle 2^m\epsilon\rangle$ ($m = 2, 3$) | $[\![12; 2a, 2a, 2a]\!]$ |
| $\mathbb{A} \perp \langle 2^m\epsilon\rangle$ ($m \geqslant 4$) | $[\![12; 2a, 2a, 2a]\!]$, $[\![4, \mathrm{II}; 2a]\!]$ |
| $\langle\epsilon\rangle \perp 2\mathbb{A}$ | $[\![12; 4a, 4a, 4a]\!]$ |
| $\langle\epsilon\rangle \perp 2^m\mathbb{A}$ ($m = 2, 3$) | $[\![12; 8a, 8a, 8a]\!]$ |
| $\langle\epsilon\rangle \perp 2^m\mathbb{A}$ ($m \geqslant 4$) | $[\![12; 8a, 8a, 8a]\!]$, $[\![4, \mathrm{II}; 8a]\!]$ |
| $\langle\epsilon\rangle \perp 8\mathbb{H}$ | $[\![4, \mathrm{II}; 8a]\!]$ |
| $\mathbb{H} \perp \langle 8\epsilon\rangle$ | $[\![4, \mathrm{II}; 2a]\!]$ |
| $\langle 1, 3, 2^m\epsilon\rangle$ ($m = 1, 2$) | $[\![4, \mathrm{I}; a]\!]$ |
| $\langle 1, 3, 2^m\epsilon\rangle$ ($m \geqslant 3$) | $[\![4, \mathrm{I}; a]\!]$, $[\![4, \mathrm{II}; 4a]\!]$ |
| $\langle\epsilon_1, 4\epsilon_2, 4\epsilon_3\rangle$ ($m \geqslant 3$) | $[\![4, \mathrm{II}; 8a]\!]$ |

Table 7.5: label($K$) = $\llbracket 8, \mathrm{I}; \mathrm{a}, \mathrm{b}, \mathrm{c} \rrbracket$

| $L_2$ | | label($\gamma^L_{2e}(K)$) ($e = 1$ or $2$) |
|---|---|---|
| $T \perp \langle 4\epsilon \rangle$ | | $\llbracket 8, \mathrm{IV}; 4a, 4b, 4c \rrbracket$ |
| $\langle \epsilon \rangle \perp 4T$ | | $\llbracket 8, \mathrm{III}; 4a, 4b, 4c \rrbracket$ |
| $\langle \epsilon_1, \epsilon_2, 4\epsilon_3 \rangle$ | $a \equiv 2 \pmod 4$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | |
| | $b + c \equiv 2 \pmod 4$ | $\llbracket 8, \mathrm{II}; \dot{2}a, 2b, 2c \rrbracket$ |
| | $b + c \equiv 0 \pmod 4$ | $\llbracket 8, \mathrm{III}; 2a, 2b, 2c \rrbracket$ |
| $\langle \epsilon_1, \epsilon_2, 8\epsilon_3 \rangle$ | $a \equiv 4 \pmod 8$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | |
| | $\epsilon_1 \equiv \frac{b+c}{2} \pmod 4$ | $\llbracket 8, \mathrm{II}; \dot{2}a, 2b, 2c \rrbracket$ |
| | $\epsilon_1 \not\equiv \frac{b+c}{2} \pmod 4$ | $\llbracket 8, \mathrm{III}; 2a, 2b, 2c \rrbracket$ |
| $\langle \epsilon_1, \epsilon_2, 2^m\epsilon_3 \rangle$ ($m \geqslant 4$) | $a \equiv 0 \pmod 8$, $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | $\llbracket 8, \mathrm{II}; \dot{2}a, 2b, 2c \rrbracket$, $\llbracket 8, \mathrm{III}; 2a, 2b, 2c \rrbracket$ |
| $\langle \epsilon_1, 2\epsilon_2, 2\epsilon_3 \rangle$ | $a \equiv 2 \pmod 4$ | $\llbracket 8, \mathrm{I}; \frac{a}{2}, 2b, 2c \rrbracket$ |
| $\langle \epsilon_1, 2\epsilon_2, 4\epsilon_3 \rangle$ | $a \equiv 1 \pmod 2$ | $\llbracket 8, \mathrm{I}; 2a, 2b, \frac{c}{2} \rrbracket$, $\llbracket 8, \mathrm{I}; 2a, 2c, \frac{b}{2} \rrbracket$ |
| $\langle \epsilon_1, 2\epsilon_2, 8\epsilon_3 \rangle$ | $a \equiv 1 \pmod 2$, $b \equiv 2 \pmod 4$ | |
| | $\epsilon_1\epsilon_2 \equiv \frac{ab}{2} \pmod 4$ | $\llbracket 8, \mathrm{I}; 2a, 2c, \frac{b}{2} \rrbracket$ |
| | $\epsilon_1\epsilon_2 \not\equiv \frac{ab}{2} \pmod 4$ | $\llbracket 8, \mathrm{II}; \dot{2}a, 2b, 2c \rrbracket$ |
| $\langle \epsilon_1, 2\epsilon_2, 16\epsilon_3 \rangle$ | $a \equiv 1 \pmod 2$, $b \equiv 2 \pmod 4$ | |
| | $\epsilon_1\epsilon_2 \equiv \frac{ab}{2} \pmod 8$ | $\llbracket 8, \mathrm{I}; 2a, 2c, \frac{b}{2} \rrbracket$ |
| | $\epsilon_1\epsilon_2 \not\equiv \frac{ab}{2} \pmod 8$ | $\llbracket 8, \mathrm{II}; \dot{2}a, 2b, 2c \rrbracket$ |
| $\langle \epsilon_1, 2\epsilon_2, 2^m\epsilon_3 \rangle$ ($m \geqslant 5$) | $a \equiv 1 \pmod 2$, $b \equiv 2 \pmod 4$ | $\llbracket 8, \mathrm{I}; 2a, 2c, \frac{b}{2} \rrbracket$, $\llbracket 8, \mathrm{II}; \dot{2}a, 2b, 2c \rrbracket$ |
| $\langle \epsilon_1, 4\epsilon_2, 4\epsilon_3 \rangle$ | $b \equiv c \pmod 4$ | |
| $\epsilon_2 \equiv \epsilon_3 \pmod 4$ | $L_2$ : isotropic | $\llbracket 8, \mathrm{I}; a, 4b, 4c \rrbracket$ |
| | $L_2$ : anisotropic | $\llbracket 8, \mathrm{I}; 4a, 4b, c \rrbracket$, $\llbracket 8, \mathrm{I}; 4a, b, 4c \rrbracket$, $\llbracket 8, \mathrm{I}; a, 4b, 4c \rrbracket$ |
| $\epsilon_2 \not\equiv \epsilon_3 \pmod 4$ | | $\llbracket 8, \mathrm{I}; 4a, 4b, c \rrbracket$, $\llbracket 8, \mathrm{I}; 4a, b, 4c \rrbracket$ |
| $\langle \epsilon_1, 4\epsilon_2, 8\epsilon_3 \rangle$ | $a \equiv 2 \pmod 4$ | |
| | $\epsilon_1 \equiv b \equiv c \pmod 4$ | $\llbracket 8, \mathrm{I}; 4a, 4b, c \rrbracket$, $\llbracket 8, \mathrm{I}; 4a, b, 4c \rrbracket$ |
| | $\epsilon_1 \not\equiv b \equiv c \pmod 4$ | $\llbracket 8, \mathrm{II}; 4a, \dot{4}b, 4c \rrbracket$, $\llbracket 8, \mathrm{II}; 4a, 4b, \dot{4}c \rrbracket$ |
| | $\epsilon_1 \equiv b \not\equiv c \pmod 4$ | $\llbracket 8, \mathrm{I}; 4a, b, 4c \rrbracket$, $\llbracket 8, \mathrm{II}; 4a, \dot{4}b, 4c \rrbracket$ |
| | $\epsilon_1 \equiv c \not\equiv b \pmod 4$ | $\llbracket 8, \mathrm{I}; 4a, 4b, c \rrbracket$, $\llbracket 8, \mathrm{II}; 4a, 4b, \dot{4}c \rrbracket$ |
| $\langle \epsilon_1, 4\epsilon_2, 2^m\epsilon_3 \rangle$ ($m \geqslant 4$) | $a \equiv 0 \pmod 4$, $b \equiv \epsilon_1 \pmod 4$ | |
| | $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | $\llbracket 8, \mathrm{I}; 4a, 4b, c \rrbracket$, $\llbracket 8, \mathrm{I}; 4a, b, 4c \rrbracket$, $\llbracket 8, \mathrm{II}; 4a, \dot{4}b, 4c \rrbracket$, $\llbracket 8, \mathrm{II}; 4a, 4b, \dot{4}c \rrbracket$ |
| | $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$ | $\llbracket 8, \mathrm{I}; 4a, b, 4c \rrbracket$, $\llbracket 8, \mathrm{II}; 4a, 4b, \dot{4}c \rrbracket$ |
| $\langle \epsilon_1, 8\epsilon_2, 8\epsilon_3 \rangle$ | $a \equiv 1 \pmod 2$, $\epsilon_2 \equiv \epsilon_3 \pmod 4$ | |
| $\frac{b}{2} \equiv \frac{c}{2} \pmod 4$ | $\epsilon_1 \equiv a \pmod 8$ | $\llbracket 8, \mathrm{I}; a, 4b, 4c \rrbracket$ |
| | $\epsilon_1 \equiv a + b + c \pmod 8$ | $\llbracket 8, \mathrm{III}; 4a, 4b, 4c \rrbracket$ |

| $L_2$ | | label$(\gamma_{2e}^{L}(K))$ $(e = 1$ or $2)$ |
|---|---|---|
| $\langle \epsilon_1, 8\epsilon_2, 8\epsilon_3 \rangle$ | $a \equiv 1 \pmod 2$, $\epsilon_2 \equiv \epsilon_3 \pmod 4$ | |
| $\frac{b}{2} \not\equiv \frac{c}{2} \pmod 4$ | $\epsilon_1 \equiv a + b \pmod 8$ | $\llbracket 8, \mathrm{II}; 4a, 4b, \dot{4c} \rrbracket$ |
| | $\epsilon_1 \equiv a + c \pmod 8$ | $\llbracket 8, \mathrm{II}; 4a, \dot{4b}, 4c \rrbracket$ |
| $\langle \epsilon_1, 8\epsilon_2, 8\epsilon_3 \rangle$ | $a \equiv 1 \pmod 2$, $\epsilon_2 \not\equiv \epsilon_3 \pmod 4$ | |
| | $\frac{b}{2} \equiv \frac{c}{2} \pmod 4$ | $\llbracket 8, \mathrm{II}; 4a, \dot{4b}, 4c \rrbracket$, $\llbracket 8, \mathrm{II}; 4a, 4b, \dot{4c} \rrbracket$ |
| | $\frac{b}{2} \not\equiv \frac{c}{2} \pmod 4$ | $\llbracket 8, \mathrm{I}; a, 4b, 4c \rrbracket$, $\llbracket 8, \mathrm{III}; 4a, 4b, 4c \rrbracket$ |
| $\langle \epsilon_1, 8\epsilon_2, 16\epsilon_3 \rangle$ | $a \equiv 1 \pmod 2$, $b \equiv 2 \pmod 4$ | |
| | $\epsilon_1 \equiv a \pmod 8$ | $\llbracket 8, \mathrm{I}; a, 4b, 4c \rrbracket$ |
| | $\epsilon_1 \equiv a + b \pmod 8$ | $\llbracket 8, \mathrm{II}; a, 4b, \dot{4c} \rrbracket$ |
| | $\epsilon_1 \equiv a + c \pmod 8$ | $\llbracket 8, \mathrm{II}; a, \dot{4b}, 4c \rrbracket$ |
| | $\epsilon_1 \equiv a + b + c \pmod 8$ | $\llbracket 8, \mathrm{III}; 4a, 4b, 4c \rrbracket$ |
| $\langle \epsilon_1, 8\epsilon_2, 32\epsilon_3 \rangle$ | $a \equiv 1 \pmod 2$, $b \equiv 2 \pmod 4$ | |
| | $\epsilon_1 \equiv a \pmod 8$ | $\llbracket 8, \mathrm{I}; a, 4b, 4c \rrbracket$, $\llbracket 8, \mathrm{II}; 4a, \dot{4b}, 4c \rrbracket$ |
| | $\epsilon_1 \not\equiv a \pmod 8$ | $\llbracket 8, \mathrm{II}; 4a, 4b, \dot{4c} \rrbracket$, $\llbracket 8, \mathrm{III}; 4a, 4b, 4c \rrbracket$ |
| $\langle \epsilon_1, 8\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 6)$ | $a \equiv 1 \pmod 2$, $b \equiv 2 \pmod 4$ | |
| | $\epsilon_1 \equiv a \pmod 8$ | $\llbracket 8, \mathrm{I}; a, 4b, 4c \rrbracket$, $\llbracket 8, \mathrm{II}; a, \dot{4b}, 4c \rrbracket$ |
| | $\epsilon_1 \not\equiv a \pmod 8$ | $\llbracket 8, \mathrm{II}; 4a, 4b, \dot{4c} \rrbracket$, $\llbracket 8, \mathrm{III}; 4a, 4b, 4c \rrbracket$ |
| $\langle \epsilon_1, 16\epsilon_2, 16\epsilon_3 \rangle$ | $a \equiv 1 \pmod 2$ | |
| | $\epsilon_1 \equiv a \pmod 8$ | $\llbracket 8, \mathrm{I}; a, 4b, 4c \rrbracket$, $\llbracket 8, \mathrm{III}; 4a, 4b, 4c \rrbracket$ |
| | $\epsilon_1 \not\equiv a \pmod 8$ | $\llbracket 8, \mathrm{II}; a, \dot{4b}, 4c \rrbracket$, $\llbracket 8, \mathrm{II}; 4a, 4b, \dot{4c} \rrbracket$ |
| $\langle \epsilon_1, 16\epsilon_2, 32\epsilon_3 \rangle$ | $a \equiv 1 \pmod 2$, $b \equiv 4 \pmod 8$ | |
| | $\epsilon_1 \equiv a \pmod 8$ | $\llbracket 8, \mathrm{I}; a, 4b, 4c \rrbracket$, $\llbracket 8, \mathrm{II}; a, \dot{4b}, 4c \rrbracket$ |
| | $\epsilon_1 \not\equiv a \pmod 8$ | $\llbracket 8, \mathrm{II}; 4a, 4b, \dot{4c} \rrbracket$, $\llbracket 8, \mathrm{III}; 4a, 4b, 4c \rrbracket$ |
| $\langle \epsilon_1, 16\epsilon_2, 64\epsilon_3 \rangle$ | $a \equiv 1 \pmod 2$, $b \equiv 4 \pmod 8$ | |
| | $\epsilon_1 \equiv a \pmod 8$ | $\llbracket 8, \mathrm{I}; a, 4b, 4c \rrbracket$, $\llbracket 8, \mathrm{II}; 4a, \dot{4b}, 4c \rrbracket$ |
| | $\epsilon_1 \not\equiv a \pmod 8$ | $\llbracket 8, \mathrm{II}; 4a, 4b, \dot{4c} \rrbracket$, $\llbracket 8, \mathrm{III}; 4a, 4b, 4c \rrbracket$ |
| $\langle \epsilon_1, 16\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 7)$ | $a \equiv 1 \pmod 2$, $b \equiv 4 \pmod 8$ | |
| | $\epsilon_1 \equiv a \pmod 8$ | $\llbracket 8, \mathrm{I}; a, 4b, 4c \rrbracket$, $\llbracket 8, \mathrm{II}; a, \dot{4b}, 4c \rrbracket$ |
| | $\epsilon_1 \not\equiv a \pmod 8$ | $\llbracket 8, \mathrm{II}; 4a, 4b, \dot{4c} \rrbracket$, $\llbracket 8, \mathrm{III}; 4a, 4b, 4c \rrbracket$ |
| $\langle \epsilon_1, 2^m\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 5)$ | $a \equiv 1 \pmod 2$ | $\llbracket 8, \mathrm{I}; a, 4b, 4c \rrbracket$, $\llbracket 8, \mathrm{III}; 4a, 4b, 4c \rrbracket$, |
| | | $\llbracket 8, \mathrm{II}; a, \dot{4b}, 4c \rrbracket$, $\llbracket 8, \mathrm{II}; 4a, 4b, \dot{4c} \rrbracket$ |
| $\langle \epsilon_1, 2^m\epsilon_2, 2^{m+1}\epsilon_3 \rangle$ $(m \geqslant 5)$ | $a \equiv 1 \pmod 2$ | $\llbracket 8, \mathrm{I}; a, 4b, 4c \rrbracket$, $\llbracket 8, \mathrm{III}; 4a, 4b, 4c \rrbracket$, |
| | | $\llbracket 8, \mathrm{II}; a, \dot{4b}, 4c \rrbracket$, $\llbracket 8, \mathrm{II}; 4a, 4b, \dot{4c} \rrbracket$ |
| $\langle \epsilon_1, 2^m\epsilon_2, 2^{m+2}\epsilon_3 \rangle$ $(m \geqslant 5)$ | $a \equiv 1 \pmod 2$ | $\llbracket 8, \mathrm{I}; a, 4b, 4c \rrbracket$, $\llbracket 8, \mathrm{III}; 4a, 4b, 4c \rrbracket$, |
| | | $\llbracket 8, \mathrm{II}; a, \dot{4b}, 4c \rrbracket$, $\llbracket 8, \mathrm{II}; 4a, 4b, \dot{4c} \rrbracket$ |
| $\langle \epsilon_1, 2^n\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant n + 3 \geqslant 8)$ | $a \equiv 1 \pmod 2$ | $\llbracket 8, \mathrm{I}; a, 4b, 4c \rrbracket$, $\llbracket 8, \mathrm{III}; 4a, 4b, 4c \rrbracket$, |
| | | $\llbracket 8, \mathrm{II}; a, \dot{4b}, 4c \rrbracket$, $\llbracket 8, \mathrm{II}; 4a, 4b, \dot{4c} \rrbracket$ |

Table 7.6: $\mathrm{label}(K) = [\![8, \mathrm{II}; \dot{a}, b, c]\!]$

| $L_2$ | | $\mathrm{label}(\gamma^L_{2e}(K))$ ($e = 1$ or $2$) |
|---|---|---|
| $T \perp \langle 4\epsilon\rangle$ | $b \equiv c \pmod 4$ | $[\![8, \mathrm{III}; 4a, 2b, 2c]\!]$ |
| | $b \not\equiv c \pmod 4$ | $[\![8, \mathrm{II}; \dot{4a}, 2b, 2c]\!]$ |
| $\mathbb{A} \perp \langle 8\epsilon\rangle$ | | $[\![8, \mathrm{II}; \dot{4a}, 2b, 2c]\!]$ |
| $\mathbb{H} \perp \langle 8\epsilon\rangle$ | | $[\![8, \mathrm{III}; 4a, 2b, 2c]\!]$, $[\![4, \mathrm{II}; 4a]\!]$ |
| $T \perp \langle 2^m\epsilon\rangle$ ($m \geqslant 4$) | | $[\![8, \mathrm{II}; \dot{4a}, 2b, 2c]\!]$, $[\![8, \mathrm{III}; 4a, 2b, 2c]\!]$, $[\![4, \mathrm{II}; 4a]\!]$ |
| $\langle\epsilon\rangle \perp 2T$ | | $[\![8, \mathrm{II}; \frac{\dot{a}}{2}, 4b, 4c]\!]$ |
| $\langle\epsilon\rangle \perp 4T$ | $b \equiv c \pmod 4$ | $[\![8, \mathrm{IV}; 4a, 8b, 8c]\!]$ |
| | $b \not\equiv c \pmod 4$ | $[\![8, \mathrm{II}; \dot{a}, 8b, 8c]\!]$ |
| $\langle\epsilon\rangle \perp 8\mathbb{A}$ | $b + c = 8 \pmod{16}$ | $[\![8, \mathrm{II}; \dot{a}, 8b, 8c]\!]$ |
| | $b + c = 0 \pmod{16}$ | $[\![8, \mathrm{IV}; 4a, 8b, 8c]\!]$ |
| $\langle\epsilon\rangle \perp 8\mathbb{H}$ | $b + c = 8 \pmod{16}$ | $[\![8, \mathrm{IV}; 4a, 8b, 8c]\!]$, $[\![4, \mathrm{II}; 4a]\!]$ |
| | $b + c = 0 \pmod{16}$ | $[\![8, \mathrm{II}; \dot{a}, 8b, 8c]\!]$, $[\![4, \mathrm{II}; 4a]\!]$ |
| $\langle\epsilon\rangle \perp 2^mT$ ($m \geqslant 4$) | | $[\![8, \mathrm{II}; \dot{a}, 8b, 8c]\!]$, $[\![8, \mathrm{IV}; 4a, 8b, 8c]\!]$, $[\![4, \mathrm{II}; 4a]\!]$ |
| $\langle\epsilon_1, \epsilon_2, 2\epsilon_3\rangle$ | | |
| $L_2$ : isotropic | | $[\![8, \mathrm{I}; 2a, b, c]\!]$ |
| $L_2$ : anisotropic | $b + c \equiv 2 \pmod 4$ | $[\![8, \mathrm{I}; 2a, b, c]\!]$, $[\![4, \mathrm{II}; 2a]\!]$ |
| | $b + c \equiv 4 \pmod 8$ | $[\![8, \mathrm{I}; 2a, b, c]\!]$, $[\![4, \mathrm{I}; 2a]\!]$ |
| $\langle\epsilon_1, \epsilon_2, 4\epsilon_3\rangle$ | | |
| $\epsilon_1 \equiv \epsilon_2$ | | $[\![8, \mathrm{I}; 2a, b, c]\!]$ |
| $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$ | $b + c = 4 \pmod 8$ | $[\![8, \mathrm{I}; 2a, b, c]\!]$, $[\![4, \mathrm{I}; 2a]\!]$ |
| | $b + c = 0 \pmod 8$ | $[\![8, \mathrm{I}; 2a, b, c]\!]$, $[\![4, \mathrm{II}; 2a]\!]$ |
| $\langle\epsilon_1, \epsilon_2, 8\epsilon_3\rangle$ | $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | |
| | $\epsilon_1\epsilon_2 \equiv bc \pmod 8$ | $[\![8, \mathrm{I}; 2a, b, c]\!]$ |
| | $\epsilon_1\epsilon_2 \not\equiv bc \pmod 8$ | $[\![8, \mathrm{IV}; 2a, 4b, 4c]\!]$ |
| $\langle\epsilon_1, \epsilon_2, 2^m\epsilon_3\rangle$ ($m \geqslant 4$) | $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | |
| | $a \equiv 1 \pmod 2$ | $[\![4, \mathrm{II}; 2a]\!]$ |
| | $a \equiv 0 \pmod 2$ | $[\![8, \mathrm{I}; 2a, b, c]\!]$,$[\![8, \mathrm{IV}; 2a, 4b, 4c]\!]$ |
| $\langle\epsilon_1, \epsilon_2, 2^m\epsilon_3\rangle$ ($m \geqslant 3$) | $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$ | |
| | $\epsilon_1\epsilon_2 \equiv 7 \pmod 8$ | $[\![8, \mathrm{I}; 2a, b, c]\!]$, $[\![4, \mathrm{II}; 2a]\!]$ |
| | $\epsilon_1\epsilon_2 \equiv 3 \pmod 8$ | $[\![8, \mathrm{I}; 2a, b, c]\!]$, $[\![8, \mathrm{IV}; 2a, 4b, 4c]\!]$, $[\![4, \mathrm{I}; 2a]\!]$, $[\![4, \mathrm{II}; 2a]\!]$ |
| $\langle\epsilon_1, 2\epsilon_2, 2\epsilon_3\rangle$ | | $[\![8, \mathrm{II}; \frac{\dot{a}}{2}, 4b, 4c]\!]$ |
| $\langle\epsilon_1, 2\epsilon_2, 4\epsilon_3\rangle$ | | $[\![4, \mathrm{I}; 2a]\!]$ |
| $\langle\epsilon_1, 2\epsilon_2, 16\epsilon_3\rangle$ | $2\epsilon_1\epsilon_2 \equiv a \cdot \frac{b+c}{2} \pmod{16}$ | $[\![8, \mathrm{II}; \frac{\dot{a}}{2}, 4b, 4c]\!]$ |
| | $2\epsilon_1\epsilon_2 \not\equiv a \cdot \frac{b+c}{2} \pmod{16}$ | $[\![8, \mathrm{IV}; 2a, 4b, 4c]\!]$ |

| $L_2$ | | label($\gamma^L_{2e}(K)$) ($e = 1$ or $2$) |
|---|---|---|
| $\langle \epsilon_1, 2\epsilon_2, 2^m\epsilon_3 \rangle$ ($m \geqslant 5$) | $a \equiv 1 \pmod 2$ | $[\![4, \mathrm{I}; 2a]\!]$ |
| | $a \equiv 2 \pmod 4$ | $[\![8, \mathrm{II}; \frac{\dot{a}}{2}, 4b, 4c]\!]$, $[\![8, \mathrm{IV}; 2a, 4b, 4c]\!]$ |
| $\langle \epsilon_1, 4\epsilon_2, 4\epsilon_3 \rangle$ | | |
| $b + c \equiv 2 \pmod 4$ | $\epsilon_1 \equiv \epsilon_2 \equiv \epsilon_3 \pmod 4$ | $[\![8, \mathrm{II}; \dot{a}, 8b, 8c]\!]$, $[\![4, \mathrm{I}; 4a]\!]$ |
| | $\epsilon_1 \not\equiv \epsilon_2 \equiv \epsilon_3 \pmod 4$ | $[\![8, \mathrm{II}; \dot{a}, 8b, 8c]\!]$ |
| | $\epsilon_2 \not\equiv \epsilon_3 \pmod 4$ | $[\![4, \mathrm{I}; 4a]\!]$ |
| $b + c \equiv 0 \pmod 4$ | $\epsilon_1 \equiv \epsilon_2 \equiv \epsilon_3 \pmod 4$ | $[\![8, \mathrm{IV}; 4a, 8b, 8c]\!]$, $[\![4, \mathrm{II}; 4a]\!]$ |
| | $\epsilon_1 \not\equiv \epsilon_2 \equiv \epsilon_3 \pmod 4$ | $[\![8, \mathrm{IV}; 4a, 8b, 8c]\!]$ |
| | $\epsilon_2 \not\equiv \epsilon_3 \pmod 4$ | $[\![4, \mathrm{II}; 4a]\!]$ |
| $\langle \epsilon_1, 4\epsilon_2, 8\epsilon_3 \rangle$ | $\epsilon_1 \equiv \frac{b+c}{2} \pmod 4$ | $[\![4, \mathrm{I}; 4a]\!]$ |
| | $\epsilon_1 \not\equiv \frac{b+c}{2} \pmod 4$ | $[\![4, \mathrm{II}; 4a]\!]$ |
| $\langle \epsilon_1, 4\epsilon_2, 16\epsilon_3 \rangle$ | $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | $[\![4, \mathrm{I}; 4a]\!]$, $[\![4, \mathrm{II}; 4a]\!]$ |
| $\langle \epsilon_1, 4\epsilon_2, 2^m\epsilon_3 \rangle$ ($m \geqslant 5$) | $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | |
| | $a \equiv 1 \pmod 2$ | $[\![8, \mathrm{II}; \dot{a}, 8b, 8c]\!]$, $[\![8, \mathrm{IV}; 4a, 8b, 8c]\!]$, $[\![4, \mathrm{I}; 4a]\!]$ |
| | $a \equiv 0 \pmod 4$ | $[\![4, \mathrm{I}; 4a]\!]$, $[\![4, \mathrm{II}; 4a]\!]$ |
| $\langle \epsilon_1, 4\epsilon_2, 2^m\epsilon_3 \rangle$ ($m \geqslant 4$) | $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$ | |
| | $\epsilon_1 \equiv a \pmod 4$ | $[\![8, \mathrm{II}; \dot{a}, 8b, 8c]\!]$, $[\![8, \mathrm{IV}; 4a, 8b, 8c]\!]$ |
| | $\epsilon_1 \not\equiv a \pmod 4$ | $[\![4, \mathrm{I}; 4a]\!]$ |
| $\langle \epsilon_1, 8\epsilon_2, 8\epsilon_3 \rangle$ | | |
| $\epsilon_2 \equiv \epsilon_3 \pmod 4$ | $\epsilon_1 \equiv a \pmod 8$ | $[\![8, \mathrm{II}; \dot{a}, 8b, 8c]\!]$ |
| | $\epsilon_1 \not\equiv a \pmod 8$ | $[\![8, \mathrm{IV}; 4a, 8b, 8c]\!]$ |
| $\epsilon_2 \not\equiv \epsilon_3 \pmod 4$ | | $[\![4, \mathrm{II}; 4a]\!]$ |
| $\langle \epsilon_1, 8\epsilon_2, 32\epsilon_3 \rangle$ | $\epsilon_1 \equiv \frac{b+c}{2} \pmod 8$ | $[\![4, \mathrm{I}; 4a]\!]$ |
| | $\epsilon_1 \not\equiv \frac{b+c}{2} \pmod 8$ | $[\![4, \mathrm{II}; 4a]\!]$ |
| $\langle \epsilon_1, 8\epsilon_2, 2^m\epsilon_3 \rangle$ ($m \geqslant 6$) | | |
| $a \equiv 1 \pmod 2$ | $\epsilon_1 \equiv a \pmod 8$ | $[\![8, \mathrm{II}; \dot{a}, 8b, 8c]\!]$, $[\![8, \mathrm{IV}; 4a, 8b, 8c]\!]$ |
| | $\epsilon_1 \not\equiv a \pmod 8$ | $[\![4, \mathrm{II}; 4a]\!]$ |
| $a \equiv 2 \pmod 4$ | $\epsilon_1 \equiv \frac{b+c}{2} \pmod 8$ | $[\![4, \mathrm{I}; 4a]\!]$ |
| | $\epsilon_1 \not\equiv \frac{b+c}{2} \pmod 8$ | $[\![4, \mathrm{II}; 4a]\!]$ |
| $\langle \epsilon_1, 16\epsilon_2, 16\epsilon_3 \rangle$ | $\epsilon_1 \equiv a \pmod 8$ | $[\![8, \mathrm{II}; \dot{a}, 8b, 8c]\!]$, $[\![8, \mathrm{IV}; 4a, 8b, 8c]\!]$ |
| | $\epsilon_1 \not\equiv a \pmod 8$ | $[\![4, \mathrm{II}; 4a]\!]$ |
| $\langle \epsilon_1, 16\epsilon_2, 2^m\epsilon_3 \rangle$ ($m = 5$, $6$) | $\epsilon_1 \equiv \frac{b+c}{2} \pmod 8$ | $[\![4, \mathrm{I}; 4a]\!]$ |
| | $\epsilon_1 \not\equiv \frac{b+c}{2} \pmod 8$ | $[\![4, \mathrm{II}; 4a]\!]$ |
| $\langle \epsilon_1, 16\epsilon_2, 2^m\epsilon_3 \rangle$ ($m \geqslant 7$) | $a \equiv 1 \pmod 2$ | |
| | $\epsilon_1 \equiv a \pmod 8$ | $[\![8, \mathrm{II}; \dot{a}, 8b, 8c]\!]$, $[\![8, \mathrm{IV}; 4a, 8b, 8c]\!]$ |
| | $\epsilon_1 \not\equiv a \pmod 8$ | $[\![4, \mathrm{II}; 4a]\!]$ |

| $L_2$ | | $\mathrm{label}(\gamma_{2e}^L(K))$ ($e = 1$ or $2$) |
|---|---|---|
| $\langle \epsilon_1, 16\epsilon_2, 2^m\epsilon_3 \rangle$ ($m \geqslant 7$) | $a \equiv 4 \pmod 8$ | |
| | $\epsilon_1 \equiv \frac{b+c}{2} \pmod 8$ | $\llbracket 4, \mathrm{I}; 4a \rrbracket$ |
| | $\epsilon_1 \not\equiv \frac{b+c}{2} \pmod 8$ | $\llbracket 4, \mathrm{II}; 4a \rrbracket$ |
| $\langle \epsilon_1, 2^m\epsilon_2, 2^m\epsilon_3 \rangle$ ($m \geqslant 5$) | $a \equiv 1 \pmod 2$ | $\llbracket 8, \mathrm{II}; \dot{a}, 8b, 8c \rrbracket$, $\llbracket 8, \mathrm{IV}; 4a, 8b, 8c \rrbracket$, $\llbracket 4, \mathrm{II}; 4a \rrbracket$ |
| | $a \equiv 0 \pmod 8$ | $\llbracket 4, \mathrm{I}; 4a \rrbracket$, $\llbracket 4, \mathrm{II}; 4a \rrbracket$ |
| $\langle \epsilon_1, 2^m\epsilon_2, 2^{m+1}\epsilon_3 \rangle$ ($m \geqslant 5$) | | $\llbracket 4, \mathrm{I}; 4a \rrbracket$, $\llbracket 4, \mathrm{II}; 4a \rrbracket$ |
| $\langle \epsilon_1, 2^m\epsilon_2, 2^{m+2}\epsilon_3 \rangle$ ($m \geqslant 5$) | | $\llbracket 4, \mathrm{I}; 4a \rrbracket$, $\llbracket 4, \mathrm{II}; 4a \rrbracket$ |
| $\langle \epsilon_1, 2^n\epsilon_2, 2^m\epsilon_3 \rangle$ ($m \geqslant n + 3 \geqslant 8$) | $a \equiv 1 \pmod 2$ | $\llbracket 8, \mathrm{II}; \dot{a}, 8b, 8c \rrbracket$, $\llbracket 8, \mathrm{IV}; 4a, 8b, 8c \rrbracket$, $\llbracket 4, \mathrm{II}; 4a \rrbracket$ |
| | $a \equiv 0 \pmod 8$ | $\llbracket 4, \mathrm{I}; 4a \rrbracket$, $\llbracket 4, \mathrm{II}; 4a \rrbracket$ |

Table 7.7: $\mathrm{label}(K) = \llbracket 8, \mathrm{III}; 2a, 2b, 2c \rrbracket$

| $L_2$ | | $\mathrm{label}(\gamma_{2e}^L(K))$ ($e = 1$ or $2$) |
|---|---|---|
| $T \perp \langle 2^m\epsilon \rangle$ ($m \geqslant 4$) | $a \equiv b \equiv 1 \pmod 2$ | $\llbracket 4, \mathrm{II}; 2a \rrbracket$, $\llbracket 4, \mathrm{II}; 2b \rrbracket$ |
| $\langle \epsilon \rangle \perp 2^m T$ ($m \geqslant 4$) | | $\llbracket 2 \rrbracket$ |
| $\langle \epsilon_1, \epsilon_2, 8\epsilon_3 \rangle$ | $\epsilon_1 \equiv \epsilon_2 \equiv a \pmod 4$ | $\llbracket 8, \mathrm{II}; \dot{a}, 4b, 4c \rrbracket$ |
| $\langle \epsilon_1, \epsilon_2, 2^m\epsilon_3 \rangle$ ($m \geqslant 4$) | $a \equiv b \equiv 1 \pmod 2$ | |
| $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | | $\llbracket 8, \mathrm{II}; \dot{a}, 4b, 4c \rrbracket$, $\llbracket 8, \mathrm{II}; 4a, \dot{b}, 4c \rrbracket$ |
| $\langle \epsilon_1, \epsilon_2, 2^m\epsilon_3 \rangle$ ($m \geqslant 3$) | $a \equiv b \equiv 1 \pmod 2$ | |
| $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$ | $\epsilon_1\epsilon_2 \equiv 7 \pmod 8$ | $\llbracket 8, \mathrm{II}; \dot{a}, 4b, 4c \rrbracket$, $\llbracket 8, \mathrm{II}; 4a, \dot{b}, 4c \rrbracket$ |
| | $\epsilon_1\epsilon_2 \equiv 3 \pmod 8$ | $\llbracket 8, \mathrm{II}; \dot{a}, 4b, 4c \rrbracket$, $\llbracket 8, \mathrm{II}; 4a, \dot{b}, 4c \rrbracket$, $\llbracket 2 \rrbracket$ |
| $\langle \epsilon_1, 4\epsilon_2, 2^m\epsilon_3 \rangle$ ($m \geqslant 4$) | | $\llbracket 2 \rrbracket$ |
| $\langle \epsilon_1, 2^m\epsilon_2, 2^m\epsilon_3 \rangle$ ($m \geqslant 5$) | | $\llbracket 2 \rrbracket$ |
| $\langle \epsilon_1, 2^m\epsilon_2, 2^{m+1}\epsilon_3 \rangle$ ($m \geqslant 5$) | | $\llbracket 2 \rrbracket$ |
| $\langle \epsilon_1, 2^n\epsilon_2, 2^m\epsilon_3 \rangle$ ($m \geqslant n + 3 \geqslant 8$) | | $\llbracket 2 \rrbracket$ |

Table 7.8: label$(K) = [\![8, \mathrm{IV}; 4a, 4b, 4c]\!]$

| $L_2$ | | label$(\gamma^L_{2e}(K))$ ($e = 1$ or $2$) |
|---|---|---|
| $T \perp \langle 16\epsilon\rangle$ | | $[\![2]\!]$ |
| $\langle\epsilon\rangle \perp 2^m T$ ($m \geqslant 4$) | $a \equiv 1 \pmod 2$ | $[\![4, \mathrm{II}; 16b]\!]$, $[\![4, \mathrm{II}; 16c]\!]$ |
| $\langle\epsilon_1, \epsilon_2, 8\epsilon_3\rangle$ | $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$ | |
| $\epsilon_1\epsilon_2 \equiv 7 \pmod 8$ | $a \not\equiv b \equiv c \pmod 4$ | $[\![4, \mathrm{II}; 8a]\!]$ |
| $\epsilon_1\epsilon_2 \equiv 3 \pmod 8$ | | $[\![4, \mathrm{II}; 8a]\!]$, $[\![4, \mathrm{II}; 8b]\!]$, $[\![4, \mathrm{II}; 8c]\!]$ |
| $\langle\epsilon_1, \epsilon_2, 2^m\epsilon_3\rangle$ ($m \geqslant 4$) | $a \equiv 0 \pmod 2$ | $[\![4, \mathrm{II}; 8a]\!]$ |
| $\langle\epsilon_1, 2\epsilon_2, 2^m\epsilon_3\rangle$ ($m \geqslant 5$) | $a \equiv 1 \pmod 2$ | $[\![4, \mathrm{II}; 8a]\!]$ |
| $\langle\epsilon_1, 4\epsilon_2, 2^m\epsilon_3\rangle$ ($m \geqslant 5$) | $b \equiv c \equiv 1 \pmod 2$ | |
| $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | | $[\![4, \mathrm{II}; 16b]\!]$, $[\![4, \mathrm{II}; 16c]\!]$ |
| $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$ | $a + c \equiv \epsilon_1 \pmod 4$ | $[\![4, \mathrm{II}; 16b]\!]$ |
| $\langle\epsilon_1, 8\epsilon_2, 2^m\epsilon_3\rangle$ ($m \geqslant 6$) | $a \equiv 1 \pmod 2$, $b \equiv 2 \pmod 4$ | |
| | $\epsilon_1 \equiv a \pmod 4$ | $[\![4, \mathrm{II}; 16b]\!]$ |
| | $\epsilon_1 \not\equiv a \pmod 4$ | $[\![4, \mathrm{II}; 16c]\!]$ |
| $\langle\epsilon_1, 16\epsilon_2, 16\epsilon_3\rangle$ | | |
| $a \equiv 1 \pmod 2$ | $\epsilon_1 \equiv a + c \pmod 8$ | $[\![4, \mathrm{II}; 16b]\!]$ |
| $\langle\epsilon_1, 16\epsilon_2, 2^m\epsilon_3\rangle$ ($m \geqslant 7$) | $a \equiv 1 \pmod 2$, $b \equiv 4 \pmod 8$ | |
| | $\epsilon_1 \equiv a \pmod 8$ | $[\![4, \mathrm{II}; 16b]\!]$ |
| | $\epsilon_1 \not\equiv a \pmod 8$ | $[\![4, \mathrm{II}; 16c]\!]$ |
| $\langle\epsilon_1, 2^m\epsilon_2, 2^m\epsilon_3\rangle$ ($m \geqslant 5$) | $a \equiv 1 \pmod 2$ | $[\![4, \mathrm{II}; 16b]\!]$, $[\![4, \mathrm{II}; 16c]\!]$ |
| $\langle\epsilon_1, 2^n\epsilon_2, 2^m\epsilon_3\rangle$ ($m \geqslant n + 3 \geqslant 8$) | $a \equiv 1 \pmod 2$ | $[\![4, \mathrm{II}; 16b]\!]$, $[\![4, \mathrm{II}; 16c]\!]$ |

Table 7.9: label$(K) = [\![4, \mathrm{I}; a]\!]$, $K \cong \langle a\rangle \perp K'$

| $L_2$ | | label$(\gamma^L_{2e}(K))$ ($e = 1$ or $2$) |
|---|---|---|
| $T \perp \langle 4\epsilon\rangle$ | $K'$ : odd | $[\![4, \mathrm{II}; 4a]\!]$ |
| | $K'$ : even | $[\![4, \mathrm{I}; 4a]\!]$ |
| $\mathbb{A} \perp \langle 8\epsilon\rangle$ | $\frac{a}{2} \cdot 3 \equiv \frac{dK}{2} \pmod 8$ | $[\![4, \mathrm{I}; 4a]\!]$ |
| | $\frac{a}{2} \cdot 7 \equiv \frac{dK}{2} \pmod 8$ | $[\![4, \mathrm{II}; 4a]\!]$ |
| $\mathbb{H} \perp \langle 8\epsilon\rangle$ | $\frac{a}{2} \cdot 3 \equiv \frac{dK}{2} \pmod 8$ | $[\![4, \mathrm{II}; 4a]\!] \times 3$ |
| | $\frac{a}{2} \cdot 7 \equiv \frac{dK}{2} \pmod 8$ | $[\![4, \mathrm{I}; 4a]\!]$, $[\![4, \mathrm{II}; 4a]\!] \times 2$ |
| $T \perp \langle 2^m\epsilon\rangle$ ($m \geqslant 4$) | | $[\![4, \mathrm{I}; 4a]\!]$, $[\![4, \mathrm{II}; 4a]\!] \times 3$ |
| $\langle\epsilon\rangle \perp 2T$ | | $[\![4, \mathrm{I}; \frac{a}{2}]\!]$ |

| $L_2$ | | label$(\gamma_{2e}^{L}(K))$ $(e = 1$ or $2)$ |
|---|---|---|
| $\langle \epsilon \rangle \perp 4T$ | $K'$ : odd | $\llbracket 4, \mathrm{II}; 4a \rrbracket$ |
| | $K'$ : even | $\llbracket 4, \mathrm{I}; a \rrbracket$ |
| $\langle \epsilon \rangle \perp 8\mathbb{A}$ | $3a \equiv \frac{dK}{4} \pmod 8$ | $\llbracket 4, \mathrm{I}; a \rrbracket$ |
| | $7a \equiv \frac{dK}{4} \pmod 8$ | $\llbracket 4, \mathrm{II}; 4a \rrbracket$ |
| $\langle \epsilon \rangle \perp 8\mathbb{H}$ | $3a \equiv \frac{dK}{4} \pmod 8$ | $\llbracket 4, \mathrm{II}; 4a \rrbracket \times 3$ |
| | $7a \equiv \frac{dK}{4} \pmod 8$ | $\llbracket 4, \mathrm{I}; a \rrbracket$, $\llbracket 4, \mathrm{II}; 4a \rrbracket \times 2$ |
| $\langle \epsilon \rangle \perp 2^m T$ $(m \geqslant 4)$ | | $\llbracket 4, \mathrm{I}; a \rrbracket$, $\llbracket 4, \mathrm{II}; 4a \rrbracket \times 3$ |
| $\langle \epsilon_1, \epsilon_2, 2\epsilon_3 \rangle$ | | |
| $K_2$ : anisotropic | $K'$ : even | $\llbracket 4, \mathrm{I}; 2a \rrbracket \times 3$ |
| | $K'$ : odd | $\llbracket 4, \mathrm{I}; 2a \rrbracket$, $\llbracket 4, \mathrm{II}; 2a \rrbracket \times 2$ |
| $K_2$ : isotropic | $K'$ : even | $\llbracket 4, \mathrm{I}; 2a \rrbracket$ |
| | $K'$ : odd, $a \equiv dK \pmod 4$ | $\llbracket 4, \mathrm{I}; 2a \rrbracket$ |
| | $K'$ : odd, $a \not\equiv dK \pmod 4$ | $\llbracket 4, \mathrm{II}; 2a \rrbracket$ |
| $\langle \epsilon_1, \epsilon_2, 4\epsilon_3 \rangle$ | | |
| $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | $a \equiv 1 \pmod 2$ | $\llbracket 4, \mathrm{II}; 2a \rrbracket$ |
| | $a \equiv 2 \pmod 4$ | $\llbracket 4, \mathrm{I}; 2a \rrbracket$ |
| $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$ | $\frac{dK}{a} \equiv 3 \pmod 8$ | $\llbracket 4, \mathrm{I}; 2a \rrbracket \times 3$ |
| | $\frac{dK}{a} \equiv 7 \pmod 8$ | $\llbracket 4, \mathrm{I}; 2a \rrbracket$, $\llbracket 4, \mathrm{II}; 2a \rrbracket \times 2$ |
| $\langle \epsilon_1, \epsilon_2, 8\epsilon_3 \rangle$ | $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | |
| $a \equiv 1 \pmod 2$ | | $\llbracket 4, \mathrm{II}; 2a \rrbracket$ |
| $a \equiv 4 \pmod 8$ | $\epsilon_1\epsilon_2 \equiv \frac{dK}{a} \pmod 8$ | $\llbracket 4, \mathrm{I}; 2a \rrbracket$ |
| | $\epsilon_1\epsilon_2 \not\equiv \frac{dK}{a} \pmod 8$ | $\llbracket 4, \mathrm{II}; 2a \rrbracket$ |
| $\langle \epsilon_1, \epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 4)$ | $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | |
| | $a \equiv 1 \pmod 2$ | $\llbracket 4, \mathrm{II}; 2a \rrbracket \times 2$ |
| | $a \equiv 0 \pmod 8$ | $\llbracket 4, \mathrm{I}; 2a \rrbracket$, $\llbracket 4, \mathrm{II}; 2a \rrbracket$ |
| $\langle \epsilon_1, \epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 3)$ | $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$ | |
| | $\epsilon_1\epsilon_2 \equiv 3 \pmod 8$ | $\llbracket 4, \mathrm{I}; 2a \rrbracket \times 3$, $\llbracket 4, \mathrm{II}; 2a \rrbracket \times 3$ |
| | $\epsilon_1\epsilon_2 \equiv 7 \pmod 8$ | $\llbracket 4, \mathrm{I}; 2a \rrbracket$, $\llbracket 4, \mathrm{II}; 2a \rrbracket$ |
| $\langle \epsilon_1, 2\epsilon_2, 2\epsilon_3 \rangle$ | $a \equiv 1 \pmod 2$ | $\llbracket 4, \mathrm{I}; 2a \rrbracket$ |
| | $a \equiv 2 \pmod 4$ | $\llbracket 4, \mathrm{I}; \frac{a}{2} \rrbracket$ |
| $\langle \epsilon_1, 2\epsilon_2, 4\epsilon_3 \rangle$ | $a \equiv 1 \pmod 2$ | $\llbracket 4, \mathrm{I}; 2a \rrbracket \times 2$ |
| | $a \equiv 2 \pmod 4$ | $\llbracket 4, \mathrm{I}; 2a \rrbracket$, $\llbracket 4, \mathrm{I}; \frac{a}{2} \rrbracket$ |
| $\langle \epsilon_1, 2\epsilon_2, 8\epsilon_3 \rangle$ | | |
| $a \equiv 1 \pmod 2$ | | $\llbracket 4, \mathrm{I}; 2a \rrbracket$ |
| $a \equiv 2 \pmod 4$ | $\begin{cases} \frac{a}{2} \equiv \frac{dK}{8} \pmod 4 \\ \epsilon_1\epsilon_2 \equiv \frac{a}{2} S_2(K)\tau \pmod 4 \end{cases}$ | $\llbracket 4, \mathrm{I}; \frac{a}{2} \rrbracket$ |

| $L_2$ | | label($\gamma^L_{2e}(K)$) ($e = 1$ or $2$) |
|---|---|---|
| $\langle \epsilon_1, 2\epsilon_2, 8\epsilon_3 \rangle$ | | |
| $a \equiv 2 \pmod 4$ | $\begin{cases} \frac{a}{2} \equiv \frac{dK}{8} \pmod 4 \\ \epsilon_1\epsilon_2 \not\equiv \frac{a}{2} S_2(K)\tau \pmod 4 \end{cases}$ | $\llbracket 4, \mathrm{II}; 2a \rrbracket$ |
| | $\tau = \begin{cases} 1 & \text{if } \frac{a}{2} + \frac{dK}{8} \equiv 2 \pmod 8, \\ -1 & \text{if } \frac{a}{2} + \frac{dK}{8} \equiv 6 \pmod 8 \end{cases}$ | |
| | $\frac{a}{2} \not\equiv \frac{dK}{8} \pmod 4$ | See Subcase(4.7.1.19-2). |
| $a \equiv 4 \pmod 8$ | $\epsilon_1\epsilon_2 \equiv \frac{a}{4} \cdot \frac{dK}{8} \pmod 4$ | $\llbracket 4, \mathrm{I}; 2a \rrbracket$ |
| | $\epsilon_1\epsilon_2 \not\equiv \frac{a}{4} \cdot \frac{dK}{8} \pmod 4$ | $\llbracket 4, \mathrm{II}; 2a \rrbracket$ |
| $\langle \epsilon_1, 2\epsilon_2, 16\epsilon_3 \rangle$ | | |
| $a \equiv 1 \pmod 2$ | | $\llbracket 4, \mathrm{I}; 2a \rrbracket$ |
| $a \equiv 2 \pmod 4$ | $\epsilon_1\epsilon_2 \equiv \frac{ab}{2} \pmod 8$ | $\llbracket 4, \mathrm{I}; \frac{a}{2} \rrbracket$ |
| | $\epsilon_1\epsilon_2 \not\equiv \frac{ab}{2} \pmod 8$ | $\llbracket 4, \mathrm{II}; 2a \rrbracket$ |
| | ($b :$ See Subcase(4.7.1.20).) | |
| $a \equiv 8 \pmod{16}$ | $\epsilon_1\epsilon_2 \equiv \frac{dK}{16} \cdot \frac{a}{8} \pmod 8$ | $\llbracket 4, \mathrm{I}; 2a \rrbracket$ |
| | $\epsilon_1\epsilon_2 \not\equiv \frac{dK}{16} \cdot \frac{a}{8} \pmod 8$ | $\llbracket 4, \mathrm{II}; 2a \rrbracket$ |
| $\langle \epsilon_1, 2\epsilon_2, 2^m\epsilon_3 \rangle$ ($m \geqslant 5$) | $a \equiv 1 \pmod 2$ | $\llbracket 4, \mathrm{I}; 2a \rrbracket \times 2$ |
| | $a \equiv 2 \pmod 4$ | $\llbracket 4, \mathrm{I}; \frac{a}{2} \rrbracket$, $\llbracket 4, \mathrm{II}; 2a \rrbracket$ |
| | $a \equiv 0 \pmod{16}$ | $\llbracket 4, \mathrm{I}; 2a \rrbracket$, $\llbracket 4, \mathrm{II}; 2a \rrbracket$ |
| $\langle \epsilon_1, 4\epsilon_2, 4\epsilon_3 \rangle$ | $K_2 :$ isotropic | |
| $\epsilon_2 \equiv \epsilon_2 \pmod 4$ | $K' :$ odd, $a \equiv dK \pmod 4$ | $\llbracket 4, \mathrm{I}; a \rrbracket$ |
| | $K' :$ odd, $a \not\equiv dK \pmod 4$ | $\llbracket 4, \mathrm{I}; 4a \rrbracket$ |
| | $K' :$ even | $\llbracket 4, \mathrm{II}; 4a \rrbracket$ |
| $\epsilon_2 \not\equiv \epsilon_2 \pmod 4$ | $K' :$ odd, $a \equiv dK \pmod 4$ | $\llbracket 4, \mathrm{I}; 4a \rrbracket \times 2$ |
| | $K' :$ odd, $a \not\equiv dK \pmod 4$ | $\llbracket 4, \mathrm{I}; a \rrbracket$, $\llbracket 4, \mathrm{I}; 4a \rrbracket$ |
| | $K' :$ even | $\llbracket 4, \mathrm{II}; 4a \rrbracket \times 2$ |
| $\langle \epsilon_1, 4\epsilon_2, 4\epsilon_3 \rangle$ | $K_2 :$ anisotropic | |
| | $K' :$ odd | $\llbracket 4, \mathrm{I}; a \rrbracket$, $\llbracket 4, \mathrm{I}; 4a \rrbracket \times 2$ |
| | $K' :$ even | $\llbracket 4, \mathrm{II}; 4a \rrbracket \times 3$ |
| $\langle \epsilon_1, 4\epsilon_2, 8\epsilon_3 \rangle$ | | |
| $a \equiv 1 \pmod 2$ | $\epsilon_1 \equiv a \pmod 4$ | $\llbracket 4, \mathrm{I}; a \rrbracket$, $\llbracket 4, \mathrm{I}; 4a \rrbracket$ |
| | $\epsilon_1 \not\equiv a \pmod 4$ | $\llbracket 4, \mathrm{I}; 4a \rrbracket$, $\llbracket 4, \mathrm{II}; 4a \rrbracket$ |
| $a \equiv 2 \pmod 4$ | $\begin{cases} \frac{dK}{2} \equiv \frac{a}{2} \pmod 4 \\ \epsilon_1 \equiv \eta S_2(K) \pmod 4 \end{cases}$ | $\llbracket 4, \mathrm{I}; 4a \rrbracket \times 2$ |

| $L_2$ | | label$(\gamma_{2e}^{L}(K))$ $(e=1$ or $2)$ |
|---|---|---|
| $\langle \epsilon_1, 4\epsilon_2, 8\epsilon_3\rangle$ | | |
| $a \equiv 2 \pmod 4$ | $\begin{cases} \frac{dK}{2} \equiv \frac{a}{2} \pmod 4 \\ \epsilon_1 \not\equiv \eta S_2(K) \pmod 4 \end{cases}$ | $[\![4, \mathrm{II}; 4a]\!] \times 2$ |
| | $\eta = \begin{cases} 1 & \text{if } \frac{a}{2} + \frac{dK}{2} \equiv 2 \pmod 8 \\ -1 & \text{if } \frac{a}{2} + \frac{dK}{2} \equiv 6 \pmod 8 \end{cases}$ | |
| | $\frac{dK}{2} \not\equiv \frac{a}{2} \pmod 4$ | $[\![4, \mathrm{I}; 4a]\!]$, $[\![4, \mathrm{II}; 4a]\!]$ |
| $\langle \epsilon_1, 4\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 4)$ | | |
| $a \equiv 1 \pmod 2$ | $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | $[\![4, \mathrm{I}; a]\!]$, $[\![4, \mathrm{I}; 4a]\!] \times 2$, $[\![4, \mathrm{II}; 4a]\!]$ |
| | $a \equiv \epsilon_1 \not\equiv \epsilon_2 \pmod 4$ | $[\![4, \mathrm{I}; a]\!]$, $[\![4, \mathrm{II}; 4a]\!]$ |
| | $\epsilon_1 \not\equiv \epsilon_2 \equiv a \pmod 4$ | $[\![4, \mathrm{I}; 4a]\!] \times 2$ |
| $a \equiv 0 \pmod 4$ | $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | $[\![4, \mathrm{I}; 4a]\!] \times 2$, $[\![4, \mathrm{II}; 4a]\!] \times 2$ |
| | $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$ | $[\![4, \mathrm{I}; 4a]\!]$, $[\![4, \mathrm{II}; 4a]\!]$ |
| $\langle \epsilon_1, 8\epsilon_2, 8\epsilon_3\rangle$ | $\epsilon_2 \equiv \epsilon_3 \pmod 4$ | |
| $a \equiv 1 \pmod 2$ | $\epsilon_1 \equiv a \pmod 8$ | $[\![4, \mathrm{I}; a]\!]$ |
| | $\epsilon_1 \not\equiv a \pmod 8$ | $[\![4, \mathrm{II}; 4a]\!]$ |
| $a \equiv 2 \pmod 4$ | $\epsilon_1 \equiv \frac{dK}{4} \pmod 8$, $S_2(K) \equiv \frac{a \cdot dK}{8} \pmod 4$ | $[\![4, \mathrm{I}; 4a]\!]$ |
| | $\epsilon_1 \equiv \frac{dK}{4} \pmod 8$, $S_2(K) \not\equiv \frac{a \cdot dK}{8} \pmod 4$ | $[\![4, \mathrm{II}; 4a]\!]$ |
| | $\epsilon_1 \not\equiv \frac{dK}{4} \pmod 8$, $S_2(K) \equiv \frac{a \cdot dK}{8} \pmod 4$ | $[\![4, \mathrm{II}; 4a]\!]$ |
| | $\epsilon_1 \not\equiv \frac{dK}{4} \pmod 8$, $S_2(K) \not\equiv \frac{a \cdot dK}{8} \pmod 4$ | $[\![4, \mathrm{I}; 4a]\!]$ |
| $\langle \epsilon_1, 8\epsilon_2, 8\epsilon_3\rangle$ | $\epsilon_2 \not\equiv \epsilon_3 \pmod 4$ | |
| $a \equiv 1 \pmod 2$ | $\epsilon_1 \equiv a \pmod 4$ | $[\![4, \mathrm{I}; a]\!]$, $[\![4, \mathrm{II}; 4a]\!]$ |
| | $\epsilon_1 \not\equiv a \pmod 4$ | $[\![4, \mathrm{II}; 4a]\!] \times 2$ |
| $a \equiv 2 \pmod 4$ | | $[\![4, \mathrm{I}; 4a]\!]$, $[\![4, \mathrm{II}; 4a]\!]$ |
| $\langle \epsilon_1, 8\epsilon_2, 16\epsilon_3\rangle$ | | |
| $a \equiv 1 \pmod 2$ | $\epsilon_1 \equiv a \pmod 8$ | $[\![4, \mathrm{I}; a]\!]$ |
| | $\epsilon_1 \not\equiv a \pmod 8$ | $[\![4, \mathrm{II}; 4a]\!]$ |
| $a \equiv 2 \pmod 4$ | $\frac{a}{2} \equiv \frac{dK}{8} \pmod 4$, $\epsilon_1 \equiv S_2(K) \pmod 4$ | $[\![4, \mathrm{I}; 4a]\!]$ |
| | $\frac{a}{2} \equiv \frac{dK}{8} \pmod 4$, $\epsilon_1 \not\equiv S_2(K) \pmod 4$ | $[\![4, \mathrm{II}; 4a]\!]$ |
| | $\frac{a}{2} \not\equiv \frac{dK}{8} \pmod 4$ | See Subcase(4.7.1.30-2). |
| $a \equiv 4 \pmod 8$ | $S_2(K) \equiv (\frac{a}{4}, \frac{dK}{4})(\frac{a \cdot dK}{32}, -\epsilon_1)(\epsilon_1, -2)$ | $[\![4, \mathrm{I}; 4a]\!]$ |
| | $S_2(K) \not\equiv (\frac{a}{4}, \frac{dK}{4})(\frac{a \cdot dK}{32}, -\epsilon_1)(\epsilon_1, -2)$ | $[\![4, \mathrm{II}; 4a]\!]$ |
| $\langle \epsilon_1, 8\epsilon_2, 32\epsilon_3\rangle$ | | |
| $a \equiv 1 \pmod 2$ | $\epsilon_1 \equiv a \pmod 8$ | $[\![4, \mathrm{I}; a]\!]$, $[\![4, \mathrm{II}; 4a]\!]$ |
| | $\epsilon_1 \not\equiv a \pmod 8$ | $[\![4, \mathrm{II}; 4a]\!] \times 2$ |
| $a \equiv 2 \pmod 4$ | $\epsilon_2 \equiv \frac{a}{2} \pmod 4$ | $[\![4, \mathrm{I}; 4a]\!] \times 2$ |
| | $\epsilon_2 \not\equiv \frac{a}{2} \pmod 4$ | $[\![4, \mathrm{II}; 4a]\!] \times 2$ |
| $a \equiv 8 \pmod{16}$ | | $[\![4, \mathrm{I}; 4a]\!]$, $[\![4, \mathrm{II}; 4a]\!]$ |

| $L_2$ | | | $\mathrm{label}(\gamma^L_{2e}(K))$ $(e = 1$ or $2)$ |
|---|---|---|---|
| $\langle \epsilon_1, 8\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 6)$ | | | |
| | $a \equiv 1 \pmod 2$ | $\epsilon_1 \equiv a \pmod 8$ | $\llbracket 4, \mathrm{I}; a \rrbracket$, $\llbracket 4, \mathrm{II}; 4a \rrbracket$ |
| | | $\epsilon_1 \not\equiv a \pmod 8$ | $\llbracket 4, \mathrm{II}; 4a \rrbracket \times 2$ |
| | $a \equiv 2 \pmod 4$ | $\epsilon_2 \equiv \frac{a}{2} \pmod 4$ | $\llbracket 4, \mathrm{I}; 4a \rrbracket \times 2$ |
| | | $\epsilon_2 \not\equiv \frac{a}{2} \pmod 4$ | $\llbracket 4, \mathrm{II}; 4a \rrbracket \times 2$ |
| | $a \equiv 0 \pmod{16}$ | | $\llbracket 4, \mathrm{I}; 4a \rrbracket$, $\llbracket 4, \mathrm{II}; 4a \rrbracket$ |
| $\langle \epsilon_1, 16\epsilon_2, 16\epsilon_3 \rangle$ | | $a \equiv 1 \pmod 2$ | $\llbracket 4, \mathrm{II}; 4a \rrbracket \times 2$ |
| | | $a \equiv 4 \pmod 8$ | $\llbracket 4, \mathrm{I}; 4a \rrbracket$, $\llbracket 4, \mathrm{II}; 4a \rrbracket$ |
| $\langle \epsilon_1, 16\epsilon_2, 32\epsilon_3 \rangle$ | | | |
| | $a \equiv 1 \pmod 2$ | $\epsilon_1 \equiv a \pmod 8$ | $\llbracket 4, \mathrm{I}; a \rrbracket$, $\llbracket 4, \mathrm{II}; 4a \rrbracket$ |
| | | $\epsilon_1 \not\equiv a \pmod 8$ | $\llbracket 4, \mathrm{II}; 4a \rrbracket \times 2$ |
| | $a \equiv 4 \pmod 8$ | $\epsilon_1 \equiv b \pmod 8$ | $\llbracket 4, \mathrm{I}; 4a \rrbracket \times 2$ |
| | | $\epsilon_1 \not\equiv b \pmod 8$ | $\llbracket 4, \mathrm{II}; 4a \rrbracket \times 2$ |
| | | ($b$ : See Subcase(4.7.1.31-2).) | |
| | $a \equiv 8 \pmod{16}$ | | $\llbracket 4, \mathrm{I}; 4a \rrbracket$, $\llbracket 4, \mathrm{II}; 4a \rrbracket$ |
| $\langle \epsilon_1, 16\epsilon_2, 64\epsilon_3 \rangle$ | | | |
| | $a \equiv 1 \pmod 2$ | $\epsilon_1 \equiv a \pmod 8$ | $\llbracket 4, \mathrm{I}; a \rrbracket$, $\llbracket 4, \mathrm{II}; 4a \rrbracket$ |
| | | $\epsilon_1 \not\equiv a \pmod 8$ | $\llbracket 4, \mathrm{II}; 4a \rrbracket \times 2$ |
| | $a \equiv 4 \pmod 8$ | | See Subcase(4.7.1.34-2). |
| | $a \equiv 16 \pmod{32}$ | | $\llbracket 4, \mathrm{I}; 4a \rrbracket$, $\llbracket 4, \mathrm{II}; 4a \rrbracket$ |
| $\langle \epsilon_1, 16\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant 7)$ | | | |
| | $a \equiv 1 \pmod 2$ | $\epsilon_1 \equiv a \pmod 8$ | $\llbracket 4, \mathrm{I}; a \rrbracket$, $\llbracket 4, \mathrm{II}; 4a \rrbracket$ |
| | | $\epsilon_1 \not\equiv a \pmod 8$ | $\llbracket 4, \mathrm{II}; 4a \rrbracket \times 2$ |
| | $a \equiv 4 \pmod 8$ | $\epsilon_2 \equiv \frac{a}{4} \pmod 8$ | $\llbracket 4, \mathrm{I}; 4a \rrbracket \times 2$ |
| | | $\epsilon_2 \not\equiv \frac{a}{4} \pmod 8$ | $\llbracket 4, \mathrm{II}; 4a \rrbracket \times 2$ |
| | $a \equiv 0 \pmod{32}$ | | $\llbracket 4, \mathrm{I}; 4a \rrbracket$, $\llbracket 4, \mathrm{II}; 4a \rrbracket$ |
| $\langle \epsilon_1, 2^n\epsilon_2, 2^m\epsilon_3 \rangle$ $(m \geqslant n \geqslant 5)$ | | $a \equiv 1 \pmod 2$ | $\llbracket 4, \mathrm{I}; a \rrbracket$, $\llbracket 4, \mathrm{II}; 4a \rrbracket \times 3$ |
| | | $a \equiv 0 \pmod 8$ | $\llbracket 4, \mathrm{I}; 4a \rrbracket \times 2$, $\llbracket 4, \mathrm{II}; 4a \rrbracket \times 2$ |

Table 7.10: $\mathrm{label}(K) = \llbracket 4, \mathrm{II}; 2a \rrbracket$

| $L_2$ | $\mathrm{label}(\gamma^L_{2e}(K))$ $(e = 1$ or $2)$ |
|---|---|
| $T \perp \langle 4\epsilon \rangle$ | $\llbracket 4, \mathrm{II}; 2a \rrbracket$ |
| $\mathbb{H} \perp \langle 8\epsilon \rangle$ | $\llbracket 4, \mathrm{II}; 2a \rrbracket$, $\llbracket 2 \rrbracket$ |
| $\mathbb{A} \perp \langle 8\epsilon \rangle$ | $\llbracket 4, \mathrm{II}; 2a \rrbracket$ |

| $L_2$ | | label$(\gamma^L_{2e}(K))$ $(e = 1$ or $2)$ |
|---|---|---|
| $T \perp \langle 2^m\epsilon\rangle$ $(m \geqslant 4)$ | $a \equiv 1 \pmod 2$ | $[\![4, \mathrm{II}; 2a]\!]$, $[\![2]\!]$ |
| | $a \equiv 0 \pmod 2$ | $[\![2]\!] \times 2$ |
| $\langle\epsilon\rangle \perp 2T$ | | $[\![4, \mathrm{II}; 4a]\!]$ |
| $\langle\epsilon\rangle \perp 4T$ | | $[\![4, \mathrm{II}; 8a]\!]$ |
| $\langle\epsilon\rangle \perp 8\mathbb{H}$ | | $[\![4, \mathrm{II}; 8a]\!]$, $[\![2]\!]$ |
| $\langle\epsilon\rangle \perp 8\mathbb{A}$ | | $[\![4, \mathrm{II}; 8a]\!]$ |
| $\langle\epsilon\rangle \perp 2^mT$ $(m \geqslant 4)$ | $a \equiv 2 \pmod 4$ | $[\![2]\!] \times 2$ |
| | $a \equiv 0 \pmod 4$ | $[\![4, \mathrm{II}; 8a]\!] \times 2$, $[\![2]\!]$ |
| $\langle\epsilon_1, \epsilon_2, 2\epsilon_3\rangle$ | $K_2$ : isotropic | $[\![4, \mathrm{I}; a]\!]$ |
| | $K_2$ : anisotropic | $[\![4, \mathrm{I}; a]\!]$, $[\![2]\!]$ |
| $\langle\epsilon_1, \epsilon_2, 4\epsilon_3\rangle$ | $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | $[\![4, \mathrm{I}; a]\!]$ |
| | $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$ | $[\![4, \mathrm{I}; a]\!]$, $[\![2]\!]$ |
| $\langle\epsilon_1, \epsilon_2, 8\epsilon_3\rangle$ | $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | |
| $a \equiv 1 \pmod 4$ | $S_2(\langle\epsilon_1, \epsilon_2\rangle) = 1$ | $[\![4, \mathrm{I}; a]\!]$ |
| | $S_2(\langle\epsilon_1, \epsilon_2\rangle) = -1$ | $[\![4, \mathrm{II}; 4a]\!]$ |
| $a \equiv 3 \pmod 4$ | $S_2(\langle\epsilon_1, \epsilon_2\rangle) = 1$ | $[\![4, \mathrm{II}; 4a]\!]$ |
| | $S_2(\langle\epsilon_1, \epsilon_2\rangle) = -1$ | $[\![4, \mathrm{I}; a]\!]$ |
| $a \equiv 2 \pmod 4$ | | $[\![4, \mathrm{II}; 4a]\!]$ |
| $\langle\epsilon_1, \epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 4)$ | $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | |
| $a \equiv 1 \pmod 2$ | | $[\![4, \mathrm{I}; a]\!]$, $[\![4, \mathrm{II}; 4a]\!]$ |
| $a \equiv 2 \pmod 4$ | | $[\![2]\!]$ |
| $a \equiv 0 \pmod 4$ | label$(\lambda_2(K)) = [\![4, \mathrm{I}; a]\!]$ | $[\![4, \mathrm{II}; 4a]\!] \times 2$ |
| | label$(\lambda_2(K)) = [\![4, \mathrm{II}; a]\!]$ | $[\![2]\!]$ |
| $\langle\epsilon_1, \epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 3)$ | $\epsilon_1\epsilon_2 \equiv 3 \pmod 8$ | |
| | $a \equiv 1 \pmod 2$ | $[\![4, \mathrm{I}; a]\!]$, $[\![4, \mathrm{II}; 4a]\!]$, $[\![2]\!] \times 2$ |
| | $a \equiv 0 \pmod 2$ | $[\![4, \mathrm{II}; 4a]\!] \times 2$, $[\![2]\!] \times 2$ |
| $\langle\epsilon_1, \epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 3)$ | $\epsilon_1\epsilon_2 \equiv 7 \pmod 8$ | |
| $a \equiv 1 \pmod 2$ | | $[\![4, \mathrm{I}; a]\!]$, $[\![4, \mathrm{II}; 4a]\!]$ |
| $a \equiv 2 \pmod 4$ | $-\frac{a}{2} - \frac{dK}{4} \equiv 2 \pmod 4$ | $[\![4, \mathrm{II}; 4a]\!] \times 2$ |
| | $-\frac{a}{2} - \frac{dK}{4} \equiv 0 \pmod 4$ | $[\![2]\!] \times 2$ |
| $a \equiv 4 \pmod 8$ | $-\frac{a}{8} - \frac{dK}{16} \equiv 2 \pmod 4$ | $[\![4, \mathrm{II}; 4a]\!] \times 2$ |
| | $-\frac{a}{8} - \frac{dK}{16} \equiv 0 \pmod 4$ | $[\![2]\!] \times 2$ |
| $a \equiv 0 \pmod 8$ | | See Subcase(4.7.2.13-2). |
| $\langle\epsilon_1, 2\epsilon_2, 2\epsilon_3\rangle$ | | $[\![4, \mathrm{II}; 4a]\!]$ |
| $\langle\epsilon_1, 2\epsilon_2, 4\epsilon_3\rangle$ | | $[\![2]\!]$ |
| $\langle\epsilon_1, 2\epsilon_2, 16\epsilon_3\rangle$ | | $[\![4, \mathrm{II}; 4a]\!]$ |

| $L_2$ | | label$(\gamma^L_{2e}(K))$ $(e=1$ or $2)$ |
|---|---|---|
| $\langle \epsilon_1, 2\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 5)$ | | |
| $a \equiv 2 \pmod 4$ | | $[\![4, \mathrm{II}; 4a]\!] \times 2$ |
| $a \equiv 4 \pmod 8$ | | $[\![2]\!]$ |
| $a \equiv 0 \pmod 8$ | label$(\lambda_2(K)) = [\![4, \mathrm{II}; *]\!]$ or $[\![8, \mathrm{IV}; *]\!]$ | $[\![2]\!]$ |
| | label$(\lambda_2(K)) = [\![4, \mathrm{I}; *]\!]$ | $[\![4, \mathrm{II}; 4a]\!] \times 2$ |
| $\langle \epsilon_1, 4\epsilon_2, 4\epsilon_3\rangle$ | | |
| $\epsilon_2 \equiv \epsilon_3 \pmod 4$ | $L_2$ : isotropic | $[\![4, \mathrm{II}; 8a]\!]$ |
| | $L_2$ : anisotropic | $[\![4, \mathrm{II}; 8a]\!]$, $[\![2]\!]$ |
| $\epsilon_2 \not\equiv \epsilon_3 \pmod 4$ | | $[\![2]\!]$ |
| $\langle \epsilon_1, 4\epsilon_2, 8\epsilon_3\rangle$ | | $[\![2]\!]$ |
| $\langle \epsilon_1, 4\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 4)$ | $a \equiv 1 \pmod 2$ | $[\![2]\!] \times 2$ |
| $\langle \epsilon_1, 4\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 4)$ | $a \equiv 0 \pmod 2$ | |
| label$(\lambda_2(K)) = [\![4, \mathrm{I}]\!], [\![8, \mathrm{II}]\!]$ | $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | $[\![2]\!] \times 2$ |
| | $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$ | $[\![2]\!]$ |
| label$(\lambda_2(K)) \neq [\![4, \mathrm{I}]\!], [\![8, \mathrm{II}]\!]$ | $\epsilon_1 \equiv \epsilon_2 \pmod 4$ | $[\![4, \mathrm{II}; 8a]\!] \times 2$, $[\![2]\!]$ |
| | $\epsilon_1 \not\equiv \epsilon_2 \equiv d \pmod 4$, $a \not\equiv 0 \pmod 8$ | $[\![2]\!]$ |
| | $d \equiv \epsilon_1 \not\equiv \epsilon_2 \pmod 4$, $a \not\equiv 0 \pmod 8$ | $[\![4, \mathrm{II}; 8a]\!] \times 2$ |
| | $d$ : See Subcase(4.7.2.22-2). | |
| | $\epsilon_1 \not\equiv \epsilon_2 \pmod 4$, $a \equiv 0 \pmod 8$ | See Subcase(4.7.2.22-2). |
| $\langle \epsilon_1, 8\epsilon_2, 8\epsilon_3\rangle$ | $\epsilon_2 \equiv \epsilon_3 \pmod 4$ | $[\![4, \mathrm{II}; 8a]\!]$ |
| | $\epsilon_2 \not\equiv \epsilon_3 \pmod 4$ | $[\![2]\!]$ |
| $\langle \epsilon_1, 8\epsilon_2, 32\epsilon_3\rangle$ | | $[\![2]\!]$ |
| $\langle \epsilon_1, 8\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 6)$ | $a \equiv 2 \pmod 4$ | $[\![2]\!]$ |
| $\langle \epsilon_1, 8\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 6)$ | $a \equiv 4 \pmod 8$ | |
| $m = 6$ | $\frac{a}{4} \equiv \epsilon_2 \pmod 4$ | $[\![2]\!]$ |
| | $\frac{a}{4} \not\equiv \epsilon_2 \pmod 4$ | $[\![4, \mathrm{II}; 8a]\!] \times 2$ |
| $m \geqslant 7$ | $\frac{a}{4} \equiv \epsilon_2 \pmod 4$ | $[\![4, \mathrm{II}; 8a]\!] \times 2$ |
| | $\frac{a}{4} \not\equiv \epsilon_2 \pmod 4$ | $[\![2]\!]$ |
| $\langle \epsilon_1, 8\epsilon_2, 2^m\epsilon_3\rangle$ $(m \geqslant 6)$ | $a \equiv 0 \pmod 8$ | |
| label$(\lambda_2(K)) = [\![4, \mathrm{I}]\!]$ | | $[\![2]\!]$ |
| label$(\lambda_2(K)) = [\![4, \mathrm{II}]\!]$ | $m = 6$, $\epsilon_1 \equiv \frac{4a\epsilon_1\epsilon_2 - dK}{64} \pmod 4$ | $[\![4, \mathrm{II}; 8a]\!] \times 2$ |
| | $m = 6$, $\epsilon_1 \not\equiv \frac{4a\epsilon_1\epsilon_2 - dK}{64} \pmod 4$ | $[\![2]\!]$ |
| | $m \geqslant 7$ | See Subcase(4.7.2.32-3). |
| $\langle \epsilon_1, 16\epsilon_2, 16\epsilon_3\rangle$ | | |
| $a \equiv 2 \pmod 4$ | | $[\![2]\!]$ |
| $a \equiv 4 \pmod 8$ | $\epsilon_1 \equiv \frac{a}{2}\epsilon_1\epsilon_2 - \frac{dK}{16} \pmod 8$ | $[\![4, \mathrm{II}; 8a]\!] \times 2$ |
| | $\epsilon_1 \not\equiv \frac{a}{2}\epsilon_1\epsilon_2 - \frac{dK}{16} \pmod 8$ | $[\![2]\!]$ |

| $L_2$ | | label($\gamma_{2e}^L(K)$) ($e = 1$ or 2) |
|---|---|---|
| $\langle \epsilon_1, 16\epsilon_2, 32\epsilon_3 \rangle$ | | $[\![2]\!]$ |
| $\langle \epsilon_1, 16\epsilon_2, 64\epsilon_3 \rangle$ | | $[\![2]\!]$ |
| $\langle \epsilon_1, 16\epsilon_2, 2^m\epsilon_3 \rangle$ ($m \geqslant 7$) | | |
| $a \equiv 2 \pmod 4$ | | $[\![2]\!]$ |
| $a \equiv 4 \pmod 8$ | | $[\![2]\!]$ |
| $a \equiv 8 \pmod{16}$ | $\epsilon_1 \equiv \frac{a}{8}\epsilon_1\epsilon_2 - \frac{dK}{64} \pmod 8$ | $[\![4, \mathrm{II}; 8a]\!] \times 2$ |
| | $\epsilon_1 \not\equiv \frac{a}{8}\epsilon_1\epsilon_2 - \frac{dK}{64} \pmod 8$ | $[\![2]\!]$ |
| $a \equiv 16 \pmod{32}$ | $\frac{a}{16}\epsilon_1\epsilon_2 \equiv \frac{dK}{128} \pmod 4$ | $[\![2]\!]$ |
| | $\frac{a}{16}\epsilon_1\epsilon_2 \not\equiv \frac{dK}{128} \pmod 4$ | See Subcase(4.7.2.33-3) |
| $a \equiv 32 \pmod{64}$ | $\frac{a}{32}\epsilon_1\epsilon_2 \equiv \frac{dK}{256} \pmod 8$ | $[\![2]\!]$ |
| | $\frac{a}{32}\epsilon_1\epsilon_2 \not\equiv \frac{dK}{256} \pmod 8$ | See Subcase(4.7.2.33-4) |
| $a \equiv 0 \pmod{64}$ | label($\lambda_2(K)$) $= [\![4, \mathrm{I}; *]\!]$ | $[\![2]\!]$ |
| | label($\lambda_2(K)$) $= [\![4, \mathrm{II}; *]\!]$ | See Subcase(4.7.2.33-5) |
| $\langle \epsilon_1, 2^m\epsilon_2, 2^m\epsilon_3 \rangle$ ($m \geqslant 5$) | | |
| $a \equiv 2 \pmod 4$ | | $[\![2]\!] \times 2$ |
| $a \equiv 4 \pmod 8$ | | $[\![2]\!] \times 2$ |
| $a \equiv 0 \pmod 8$ | label($\lambda_2(K)$) $= [\![4, \mathrm{I}]\!], [\![8, \mathrm{II}]\!]$ | $[\![2]\!] \times 2$ |
| | label($\lambda_2(K)$) $\neq [\![4, \mathrm{I}]\!], [\![8, \mathrm{II}]\!]$ | $[\![4, \mathrm{II}; 8a]\!] \times 2$, $[\![2]\!]$ |
| $\langle \epsilon_1, 2^n\epsilon_2, 2^m\epsilon_3 \rangle$ ($m > n \geqslant 5$) | | |
| $a \equiv 2 \pmod 4$ | | $[\![2]\!] \times 2$ |
| $a \equiv 4 \pmod 8$ | | $[\![2]\!] \times 2$ |
| $a \equiv 8 \pmod{16}$ | | $[\![2]\!] \times 2$ |
| $a \equiv 0 \pmod{16}$ | label($\lambda_2(K)$) $= [\![4, \mathrm{I}]\!], [\![8, \mathrm{II}]\!]$ | $[\![2]\!] \times 2$ |
| | label($\lambda_2(K)$) $= [\![4, \mathrm{II}]\!]$ | $[\![4, \mathrm{II}; 8a]\!] \times 2$, $[\![2]\!]$ |

# Bibliography


[1] W. K. Chan and B.-K. Oh, *Class numbers of ternary quadratic forms*, J. Number Theory **135**(2014), 221-261.

[2] W. K. Chan and B.-K. Oh, *Finiteness theorems for positive definite $n$-regular quadratic forms*, Trans. Amer. Math. Soc. **355**(2003), 2385-2396.

[3] W. K. Chan and A. G. Earnest, *Discriminant bounds for spinor regular ternary quadratic lattices*, J. London Math. Soc. (2) **69**(2004), 545-561.

[4] C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Interscience, New York, 1962.

[5] H. Hijikata, *Explicit formula of the traces of Hecke operators for $\Gamma_0(N)$*, J. Math. Soc. Japan **26**(1974), 56-82.

[6] Y. Kitaoka, *Arithmetic of quadratic forms*, Cambridge Univ. Press, 1993.

[7] O. Körner, *Traces of Eichler-Brandt matrices and type numbers of quaternion orders*, Proc. Indian Acad. Sci. **97**(1987), 189-199.

[8] O. T. O'Meara, *Introduction to quadratic forms*, Springer-Verlag, New York, 1963.

[9] H. Minkowski, *Über positive quadratische Formen*, Collected Works, vol. I, B. G. Teubner, 1911.

[10] M. Peters, *Ternäre und quaternäre quadratische Formen und Quaternionenalgebren*, Acta Arith. **15**(1969), 329-365.

[11] A. Pizer, *Type numbers of Eichler orders*, J. Reine Angew. Math. **264**(1973), 76-102.

[12] A. Pizer, *On the arithmetic of quaternion algebras*, Acta Arith. **31**(1976), 61-89.

[13] K. Tahara, *On the finite subgroups of* $\mathrm{GL}(3, \mathbb{Z})$, Nagoya Math. J. **41**(1971), 169-209.

[14] G. L . Watson, *Some problems in the theory of numbers*, Ph. D. thesis, University of London, 1953.

[15] G. L. Watson, *Transformations of a quadratic form which do not increase the class number*, Proc. London Math. Soc. **12**(1962), 577-587.

[16] T. Yang, *An explicit formula for local densities of quadratic forms*, J. Number Theory **72**(1998), 309-356.

# 국문초록

정수계수 이차형식의 류수는 그것의 종수에 포함된 동치류들의 개수로 정의된다. 최근에 Wai Kiu Chan 박사와 오병권 박사는 삼변수 정수계수 이차형식(양의 정부호)의 류수와 그것의 홀수인 소수에 관한 왓슨변환에 대한 류수 사이의 명확한 관계를 제시하였다. 이 논문에서는 2 또는 4에 관한 왓슨변환이 임의의 삼변수 이차형식에 적용된 경우에 대해 두 류수 사이의 관계를 명확하게 제시하고, 최종적으로 삼변수 이차형식의 류수를 계산하는 효과적인 귀납적 방법을 제시한다. 또한 그것의 한 예로서 임의의 벨 삼변수 이차형식의 류수를 계산하는 닫힌 공식을 제공한다.

**주요어휘:** 삼변수 이차형식, 류수, 왓슨 변환
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