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이 논문을 이학박사학위논문으로 제출함

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Abstract

In this report, we present some implications of Abelian decomposition of Einstein's theory. Two fundamental solutions for A_2- and B_2-restricted gravity will be discussed. Indeed A_2-decomposition is adapted to describe the topological property of spacetime and B_2-decomposition is appropriate to handle all solutions of gravitational waves. And we will propose a new method to solve Einstein's equation in a gauge theoretical way. Using this method some well known solutions will be derived.

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Contents

Chapter 1

Introduction

In this paper, we will discuss some implications of Abelian decomposition of Einstein's theory proposed in a recent paper[1]. Since Einstein's theory of gravity can be regarded as a gauge theory of the Lorentz group, we can do Abelian decompositions of Einstein's theory as the case of quantum chromodynamics. In this formalism, the gauge potential and the field strength are identified with the spin-connection ${\Gamma_\mu}^{ab}$ and the Riemann curvature tensor ${R_{\mu\nu}}^{ab}$, respectively. And the gauge potential will be decomposed into the "Abelian(restricted) part" $\hat{\mathbf{\Gamma}}_\mu$ and the "valence part" $\mathbf{Z}_\mu$ in a gauge-independent way. This decomposition of the gauge potential enables us to decompose the Riemann tensor into the restricted field strength which consists of the restricted gauge potential, and the remaining part, valence field strength.

The first step of Abelian decomposition is to select some isometry vector fields on the tangent space of the Lie group. Here isometry vector fields should commute each other, and they will introduce the "magnetic symmetry" to the system in the sense of Y. M. Cho. Then we define the restricted potential $\hat{\mathbf{\Gamma}}_\mu$ as the gauge potential which covariantly preserves isometry vector fields.

So Abelian decomposition depends on the choice of such vector fields. In order to obtain the simplest picture, we will choose maximal ideals of the Lorentz group. Since there are two distinct maximal ideals in the Lorentz group, there exist two types of decomposition. The one is "light-like decomposition" which fits for gravitational waves, and the other is "non-light-like decomposition" which adapted to describe the topological properties of spacetime.

When we impose that all valence connection vanish, we get a simpler gauge theory of only restricted gauge potentials. We call such restricted theories of "non-light-like" and "light-like" decompositions "A_2-" and "B_2-gravity", respectively.

There are some reasons to study Abelian decomposition of Einstein's theory of gravitation. At first, in order to make a quantum gravity theory, we should understand interactions between gravitons and spinors. But the locally Mikowskian structure will be hidden behind the general covariance, and metric tensor cannot take account for gauge degrees of freedom of the Lorentz group. In brief, the entity which interacts with spinors are vierbein, not the metric, although mathematically they are equivalent to each other. Therefore it is advantageous to approach the quantum gravity theory in a gauge theoretical way with vierbein fields.

In Abelian decomposition, the main characters are vierbein fields and spin-connections. Since they have Lorentz degrees of freedom, they can have non-trivial topology of the Lorentz group. This is the most remarkable difference between the vierbein formalism and the metric formalism. The metric tensor cannot have any non-trivial topology because it is just a measure of infinitesimal length. Vierbein can have it, however, and we need a nice theory which can handle the topology of spacetime.

Indeed "non-light-like decomposition" provides such a nice theory. As we shall see later, the restricted gauge potential of A_2-gravity inherits all topological properties of spacetime. And we can refine further that restricted potential into a "magnetic potential". This magnetic potential is the desired potential which describes the topology of spacetime. The cosmic string will show us how the magnetic potential works.

Gravitational wave is another deep problem of quantum gravity. To understand gravitons, we should understand gravitational waves first. Because of complexity and non-linearity of Einstein's equation, we couldn't fully understand the whole solution space of gravitational waves. Usual prescription for this problem is to give up the exactness and to use the "linearized gravitational waves". But this is not satisfactory, because non-linear effects of gravitational waves will enormously increase in a strong gravitational field, for example, the early universe. So we cannot give up exactness of solutions for gravitational waves.

Therefore the only way to avoid this problem is to simplify Einstein's equation to be adapted to handle gravitational waves, and "B_2-gravity" does the job. If we turn off all valence connections in the "light-like decomposition", we can obtain simpler equations of motion for B_2-gravity which are equivalent to Einstein's equation and the metricity condition. And these simpler equations governs all radiative solutions of Einstein's equation.

This is the reason why we use the terminology "Abelian dominance". A_2-gravity manifestly shows us the topological structure of spacetime and B_2-gravity governs all gravitational waves with its own equations of motion. That is to say, the Abelian parts of Einstein's theory describe the most two important characters of Einstein's theory without any loss of generality. So we call it "Abelian dominance". And the main purpose of this paper is to demonstrate the Abelian dominance of Einstein's theory.

On the other hand, the non-light-like decomposition provides us gauge theoretical methods to solve Einstein's equation. As we shall see in the chapter 5, the geodesic equation in an orthonormal frame will have the same form of the Lorentz force formula, and our gauge potential will play the role of electromagnetic field which generates the Lorentz-like force in that frame. Therefore we can easily choose a nice ansatz for each situation with that Lorentz-like force in mind. Since we determine the gauge potential first, there appear only first-order-derivatives of gauge potentials and algebraic factors of vierbein in Einstein's equation. Moreover the metricity condition gives 24 equations which give the relation between the vierbein fields and the gauge potential. And there are only first-order-derivatives of vierbein for each connection coefficients. This process looks like the Hamilton's one. Since he took nice variables, that is, conjugate momenta, he could have simpler equations than in the Lagrangian formalism. Here we take spin-connections in an orthonormal frame, as such nice variables, and we will have simpler equations than usual geometrodynamics. This is the technical reason for our formalism.

In chapter 2, we will review the Abelian decomposition of $SU(2)$ gauge theory since it would help us to briefly understand what our decomposition is, and the vacuum structure of the Lorentz group is identical with the one of $SU(2)$ group. In chapter 3, we will review the Abelian decomposition of Einstein's theory in detail. In chapter 4, then, we will study two fundamental solutions of restricted theories of gravity, cosmic string and Einstein-Rosen-Bondi's wave solution. And in chapter 5, we will show gauge theoretical techniques to obtain some well known solutions.

Chapter 2

Abelian decomposition of SU(2): A review

In the early 1980's, Y. M. Cho first pointed out that we can obtain a subset of the full gauge theory which has an additional symmetry−he called it "magnetic symmetry"− which keeps the full gauge degrees of freedom[2]. In such a restricted theory(so called RCD-restricted chromodynamics) there exist two gauge potentials; the electric and magnetic potential. The former is timelike and is not restricted by the magnetic symmetry imposed. It describes the electric flux of isocharges. But the latter is spacelike and is fixed by the magnetic symmetry. And it describes the magnetic flux of topological charges. As is common in the papers on magnetic monopoles, the magnetic potential in RCD also has the string singularity. By introducing the dual potential of the magnetic potential, however, he could remove the string singularity and obtain a satisfactory field theory for non-Abelian monopoles.

We call such a decomposition of the gauge theory "Abelian decomposition" or "Cho-decomposition". In this chapter we will study the Abelian decomposition of the $SU(2)$ gauge theory. There are two reasons to study this group. First,

$SU(2)$ is the simplest non-Abelian gauge theory so that we can see the structure of the Abelian decomposition most transparently. Second, $SU(2)$ group is a subgroup of the Lorentz group $SO(1,3)$ which consists of spatial rotation generators. Therefore the RCD of the $SU(2)$ is directly related to the gravitation without any boosting. It might imply the possibility to find the solution of Einstein's equation which corresponds a classical solution of the Yang-Mills equation for $SU(2)$ and vice versa. Keep these reasons in mind, let us study the RCD of $SU(2)$.

In order to define a magnetic symmetry on the system we should select an isometry direction. Let $\hat{n} = \hat{n}(t, x, y, z)$ be such a vector field. Namely, $\hat{n}$ selects the "Abelian direction" at each spacetime point. Then we define the magnetic symmetry of the system as the symmetry which preserves $\hat{n}$ covariantly under any parallel transport determined by the gauge potential. To get such a gauge potential, we solve the following equation;

$$D_\mu \hat{n} = \partial_\mu \hat{n} + g\vec{A}_\mu \times \hat{n} = 0. \quad (\hat{n}^2 = 1) \tag{2.1}$$

Since we are dealing with the $SU(2)$ group, the cross-product is the same one in usual 3-dimensional vector calculus. The above equation has a unique solution,

$$\hat{A}_\mu = A_\mu \hat{n} - \frac{1}{g}\hat{n} \times \partial_\mu \hat{n} \ . \tag{2.2}$$

Here $A_\mu = \hat{n} \cdot \vec{A}_\mu$ is the "electric" potential which is not restricted by the magnetic symmetry. Remark that the second term on the right hand side of (2.2) is totally determined by the magnetic symmetry so we call it "magnetic potential".

We can recover the full gauge potential by adding the gauge covariant valence potential $\vec{X}_\mu$ to the restricted potential,

$$\vec{A}_\mu = A_\mu \hat{n} - \frac{1}{g}\hat{n} \times \partial_\mu \hat{n} + \vec{X}_\mu = \hat{A}_\mu + \vec{X}_\mu,$$
$$(\hat{n}^2 = 1, \quad \hat{n} \cdot \vec{X}_\mu = 0). \tag{2.3}$$

This is the Abelian decomposition which decomposes the gauge potential into the restricted potential $\hat{A}_\mu$ and the valence potential $\vec{X}_\mu$ [2, 3].

Let $\vec{\alpha}$ be an infinitesimal gauge parameter. Under the infinitesimal gauge transformation

$$\delta \hat{n} = -\vec{\alpha} \times \hat{n}, \qquad \delta \vec{A}_\mu = \frac{1}{g} D_\mu \vec{\alpha}, \tag{2.4}$$

one has

$$\delta A_\mu = \frac{1}{g}\hat{n} \cdot \partial_\mu \vec{\alpha}, \quad \delta \hat{A}_\mu = \frac{1}{g}\hat{D}_\mu \vec{\alpha},$$
$$\delta \vec{X}_\mu = -\vec{\alpha} \times \vec{X}_\mu. \tag{2.5}$$

There are two remarkable points here. The first is that $\hat{A}_\mu$ enjoys the full gauge degrees of freedom. Hence we will not lose any generality even though we do the gauge theory only with this restricted gauge potential. And the second is that $\vec{X}_\mu$ transforms covariantly under the gauge transformation. So we can interpret it as a source of the restricted gauge theory. In the $SU(3)$-QCD, $\vec{X}_\mu$ can be regarded as a valence gluon while $\hat{A}_\mu$ be regarded as a binding gluon[3]. Notably, the above equations show that the Abelian decomposition is gauge-independent.

Let us concentrate on the magnetic potential,

$$\vec{C}_\mu = -\frac{1}{g}\hat{n} \times \partial_\mu \hat{n}. \tag{2.6}$$

and let $\hat{n}_i$ $(i = 1, 2, 3)$ with $\hat{n}_3 = \hat{n}$ be a right-handed orthonormal basis of SU(2). Then the field strength corresponds to the magnetic potential becomes,

$$\begin{aligned} \vec{H}_{\mu\nu} &= \partial_\mu \vec{C}_\nu - \partial_\nu \vec{C}_\mu + g\vec{C}_\mu \times \vec{C}_\nu \\ &= -\frac{1}{g}\partial_\mu \hat{n} \times \partial_\nu \hat{n} = H_{\mu\nu}\hat{n}, \\ H_{\mu\nu} &= -\frac{1}{g}\hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = \partial_\mu \widetilde{C}_\nu - \partial_\nu \widetilde{C}_\mu, \\ \widetilde{C}_\mu &= \frac{1}{g}\vec{n}_1 \cdot \partial_\mu \vec{n}_2\ . \end{aligned} \tag{2.7}$$

From this we obtain the restricted field strength,

$$\begin{aligned} \hat{F}_{\mu\nu} &= \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + g\hat{A}_\mu \times \hat{A}_\nu \\ &= (F_{\mu\nu} + H_{\mu\nu})\hat{n}, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \\ H_{\mu\nu} &= \partial_\mu \widetilde{C}_\nu - \partial_\nu \widetilde{C}_\mu. \end{aligned} \tag{2.8}$$

Note that the restricted field strength is parallel to the isometry direction. This is the general feature of the Abelian decomposition, because of the identity

$$[D_\mu, D_\nu]\ \hat{n} = g\vec{F}_{\mu\nu} \times \hat{n}\ . \tag{2.9}$$

Significantly, all topological properties of the gauge group are contained in the restricted potential since it has the full gauge degrees of freedom. Actually by imposing $\hat{n} = \hat{r}$, we see that $\tilde{C}_\mu$ becomes the Wu-Yang magnetic monopole potential. When G. 't Hooft obtained the magnetic monopole solution in $O(3)$-gauge theory, he needed a triplet Higgs field which selects the behavior of gauge fields on the infinite boundary[4]. But in RCD we do not need such a Higgs field, since our isometry vector $\hat{n}$ does the same role of the Higgs field of 't Hooft. Indeed, $\hat{n}$ represents the monopole topology $\pi_2(S^2)$ which describes the mapping from

S^2 in 3-dimensional space R^3 to the coset space $SU(2)/U(1)$, and the vacuum topology $\pi_3(S^3) \simeq \pi_3(S^2)$ which describes the mapping from the compactified 3-dimensional space S^3 to the group space S^3.

With (2.3) we have

$$\vec{F}_{\mu\nu} = \hat{F}_{\mu\nu} + \hat{D}_\mu \vec{X}_\nu - \hat{D}_\nu \vec{X}_\mu + g\vec{X}_\mu \times \vec{X}_\nu, \tag{2.10}$$

so that the Yang-Mills Lagrangian is expressed as

$$\begin{aligned}\mathcal{L} = -\frac{1}{4}\vec{F}_{\mu\nu}^2 = -\frac{1}{4}\hat{F}_{\mu\nu}^2 - \frac{1}{4}(\hat{D}_\mu \vec{X}_\nu - \hat{D}_\nu \vec{X}_\mu)^2 \\ -\frac{g}{2}\hat{F}_{\mu\nu} \cdot (\vec{X}_\mu \times \vec{X}_\nu) - \frac{g^2}{4}(\vec{X}_\mu \times \vec{X}_\nu)^2.\end{aligned} \tag{2.11}$$

This assures again that the non-Abelian gauge theory can be regarded as the restricted gauge theory made of the restricted gauge potential with a source due to the valence potential.

Chapter 3

Abelian Decomposition of Gravitational Connection

3.1 Basic settings and notations

Now it is time to apply Abelian decomposition to the Lorentz group. From now on we use greek letters $(\mu, \nu, \ldots)$ for spacetime indices and latin letters (a,b,...) for Lorentz indices. At first, we introduce a coordinates basis $\{\partial_\mu : \mu = 0, 1, 2, 3\}$ on the tangent space of spacetime so that they commute

$$[\ \partial_\mu,\ \partial_\nu\ ] = 0\ .$$

In order to take account locally Minkowskian structure of spacetime and its Lorentz invariance, we also introduce an orthonormal frame $\{\xi_a = e^\mu{}_a\ \partial_\mu : a = 0, 1, 2, 3\}$. Here $e^\mu{}_a$ are the vierbein fields associated with the frame. Of course, orthonormal basis vectors do not commute,

$$[\xi_a,\ \xi_b] = f_{ab}{}^c \xi_c\ .$$

and its structure coefficients ${f_{ab}}^c$ are given by

$$ {f_{ab}}^c = ({e^\mu}_a \partial_\mu {e^\nu}_b - {e^\mu}_b \partial_\mu {e^\nu}_a){e_\nu}^c. \tag{3.1} $$

As is well known, the Lorentz group is a six dimensional Lie group. Hence it has 6 independent left-invariant vector fields as a basis of each tangent space. The standard choice for the basis of the Lorentz group is,

$$ \begin{aligned} L_i &= \text{rotation generator about the i-th axis} \\ K_i &= \text{boosting generator along the i-th axis} \end{aligned} \tag{3.2} $$

for $i = 1, 2, 3$, with their Lie algebra,

$$ [L_i,\ L_j] = \epsilon_{ijk} L_k, \quad [L_i,\ K_j] = \epsilon_{ijk} K_k, \quad [K_i,\ K_j] = -\epsilon_{ijk} L_k. \tag{3.3} $$

But if we introduce $J_{12,23,31} = L_{3,1,2}$ and $J_{01,02,03} = K_{1,2,3}$ then (3.3) is written in a more compact form as following;

$$ [J_{ab},\ J_{cd}] = \eta_{ac} J_{bd} - \eta_{bc} J_{ad} + \eta_{bd} J_{ac} - \eta_{ad} J_{bc} = {f_{ab,cd}}^{\ mn}\ J_{mn}, \tag{3.4} $$

with the strucure coefficients

$$ {f_{ab,cd}}^{\ mn} = \eta_{ac} {\delta_b}^{[m} {\delta_d}^{n]} - \eta_{bc} {\delta_a}^{[m} {\delta_d}^{n]} + \eta_{bd} {\delta_a}^{[m} {\delta_c}^{n]} - \eta_{ad} {\delta_b}^{[m} {\delta_c}^{n]}. \tag{3.5} $$

Here $\eta_{ab} = diag\ (-1, 1, 1, 1)$ is the Minkowski metric. Then we have an adjoint representation for the generators,

$$ {(J_{ab})_c}^d = -\eta_{ac} {\delta_b}^d + \eta_{bc} {\delta_a}^d\ . \tag{3.6} $$

With the above setting, any sextet vector $\mathbf{P}$ can be expressed as,

$$\begin{aligned}
\mathbf{P} &= m^i\ L_i + e^j\ K_j \\
&= m^1\ L_1 + m^2\ L_2 + m^3\ L_3 + e^1\ K_1 + e^2\ K_2 + e^3\ K_3 \\
&= m^1\ J_{23} + m^2\ J_{31} + m^3\ J_{12} + e^1\ J_{01} + e^2\ J_{02} + e^3\ J_{03} \\
&= \frac{1}{2}(m^i \epsilon_{0i}{}^{ab})\ J_{ab} + \frac{1}{2} e^j (\delta_0{}^a \delta_j{}^b - \delta_0{}^b \delta_j{}^a)\ J_{ab} \qquad (3.7)
\end{aligned}$$

where $i = 1, 2, 3$ and $a, b = 0, 1, 2, 3$. Here we set $\epsilon_{\hat{0}\hat{1}\hat{2}\hat{3}} = 1$. Therefore any vector $\mathbf{P}$ can be decomposed into its rotating part and boosting part. Using the following notation,

$$m^i L_i \equiv \begin{pmatrix} \vec{m} \\ 0 \end{pmatrix}, \qquad e^j K_j \equiv \begin{pmatrix} 0 \\ \vec{e} \end{pmatrix}, \qquad (3.8)$$

$\mathbf{P}$ can be rewritten as,

$$\begin{gathered}
\mathbf{p} = \frac{1}{2} p_{ab} \mathbf{I}^{ab} = \begin{pmatrix} \vec{m} \\ \vec{e} \end{pmatrix}, \quad p^{ab} = \mathbf{p} \cdot \mathbf{I}^{ab} = \frac{1}{2} p^{mn} I_{mn}{}^{ab}, \\
\mathbf{I}^{ab} = \begin{pmatrix} \hat{m}^{ab} \\ \hat{e}^{ab} \end{pmatrix}, \\
\hat{m}_i{}^{ab} = \epsilon_{0i}{}^{ab}, \quad \hat{e}_i{}^{ab} = (\delta_0{}^a \delta_i{}^b - \delta_0{}^b \delta_i{}^a), \\
I_{mn}{}^{ab} = (\delta_m^a \delta_n{}^b - \delta_m^b \delta_n{}^a) = -(J_{mn})^{ab}. \qquad (3.9)
\end{gathered}$$

where the vector components are given by

$$m_i = \frac{1}{2} \epsilon_{ijk} p^{jk}, \qquad e_i = p^{0i}, \qquad (3.10)$$

for $i, j, k = 1, 2, 3$. Here the "dot"-product will be defined in the next paragraph. For an infinitesimal transformation α^{ab},

$$\delta\ p^{cd} = -\frac{1}{2} f_{ab,mn}{}^{cd} \alpha^{ab}\ p^{mn}. \tag{3.11}$$

This equation show that the rotation part $\vec{m}$ and the boosting part $\vec{e}$ of $\mathbf{P}$ transform as the magnetic field and electric field do. This is why we denote them by 'm' and 'e' that are initials of 'magnetic' and 'electric', respectively.

On the other hand, the Lorentz group has two invariant tensor related to two Casimir invariants. To make it clear, let us use indices $A, B, \ldots = 1, 2, \ldots, 6$ which mean $J_i = L_i$ and $J_{3+i} = K_i$ for $i = 1, 2, 3$. Then there exists an invariant metric tensor δ_{AB} of Lorentz group,

$$\delta_{AB} = -\frac{1}{4} f_{AC}{}^{D} f_{BD}{}^{C} = diag\ (+1, +1, +1, -1, -1, -1)\ . \tag{3.12}$$

This metric tensor defines an inner product of Lorentz vectors as,

$$\mathbf{P} \cdot \mathbf{P}' = \begin{pmatrix} \vec{m} \\ \vec{e} \end{pmatrix} \cdot \begin{pmatrix} \vec{m}' \\ \vec{e}' \end{pmatrix} = \vec{m} \cdot \vec{m}' - \vec{e}\ .\cdot \vec{e}' \tag{3.13}$$

Note that this inner product can have a negative value, namely, it is indefinite. One might say, we can make a theory for gravitation which is identical with the Yang-Mills theory for the Lorentz group and its Lagrangian has the second-order-form of $\mathcal{L} = \frac{1}{4}(\mathbf{F}_{\mu\nu})^2$. But (3.12) immediately tells us that such a theory will have the negative energy problem. So we should give up the typical Yang-Mills theory of the Lorentz group. As we shall see later, the first-order-formalism will give us the same theory of Einstein's one.

The Lorentz group has another invariant tensor ϵ_{AB} which is defined by

$$\epsilon_{AB} = \epsilon_{ab,cd} = \epsilon_{abcd}, \tag{3.14}$$

where ϵ_{abcd} is the totally anti-symmetric invariant tensor. This tensor gives us dual transformations on the Lorentz group, and these transformations interchange the electric part and magnetic part $(\vec{m}, \vec{e}) \longrightarrow (-\vec{e}, \vec{m})$ as in electrodynamics. The generators are transformed under the dual transformation as,

$$\begin{aligned} \tilde{J}^{12} &= \frac{1}{2}\epsilon^{12ab}J_{ab} = -J^{03}, \quad \cdots , \\ \tilde{J}^{01} &= \frac{1}{2}\epsilon^{01ab}J_{ab} = J^{23}, \quad \cdots , \end{aligned} \tag{3.15}$$

so that

$$\tilde{\mathbf{p}} = \begin{pmatrix} \vec{e} \\ -\vec{m} \end{pmatrix}, \qquad \tilde{\tilde{\mathbf{p}}} = -\mathbf{p} \ . \tag{3.16}$$

And this dual transformation gives us the other Casimir invariant,

$$\mathbf{p} \cdot \tilde{\mathbf{p}} = \frac{1}{4}\epsilon_{abcd}p^{ab}p^{cd} = 2\vec{m} \cdot \vec{e} \ . \tag{3.17}$$

In addition,

$$[p, \ \tilde{p}] = 0, \qquad \mathbf{p} \times \tilde{\mathbf{p}} = 0 \ . \tag{3.18}$$

Hence any Lorentz vector and its dual vector always commute and their cross product vanishes. Here I leave useful vector identities on the Lorentz group.

$$\begin{aligned} \mathbf{p} \cdot \mathbf{p}' &= \vec{m} \cdot \vec{m}' - \vec{e} \cdot \vec{e}', \\ \mathbf{p} \cdot \tilde{\mathbf{p}}' &= \vec{m} \cdot \vec{e}' + \vec{e} \cdot \vec{m}' = \tilde{\mathbf{p}} \cdot \mathbf{p}', \end{aligned}$$

$$
\begin{aligned}
\mathbf{p} \times \mathbf{p}' &= \begin{pmatrix} \vec{m} \times \vec{m}' - \vec{e} \times \vec{e}' \\ \vec{m} \times \vec{e}' + \vec{e} \times \vec{m}' \end{pmatrix} = -\tilde{\mathbf{p}} \times \tilde{\mathbf{p}}', \\
\mathbf{p} \times \tilde{\mathbf{p}}' &= \begin{pmatrix} \vec{m} \times \vec{e}' + \vec{e} \times \vec{m}' \\ -\vec{m} \times \vec{m}' + \vec{e} \times \vec{e}' \end{pmatrix} = \tilde{\mathbf{p}} \times \mathbf{p}', \\
\widetilde{\mathbf{p} \times \mathbf{p}'} &= \mathbf{p} \times \tilde{\mathbf{p}}' = \tilde{\mathbf{p}} \times \mathbf{p}', \\
\mathbf{p}_1 \cdot (\mathbf{p}_2 \times \mathbf{p}_3) &= \mathbf{p}_2 \cdot (\mathbf{p}_3 \times \mathbf{p}_1) = \mathbf{p}_3 \cdot (\mathbf{p}_1 \times \mathbf{p}_2), \\
\mathbf{p}_1 \times (\mathbf{p}_2 \times \mathbf{p}_3) &= [\mathbf{p}_2\ (\mathbf{p}_1 \cdot \mathbf{p}_3) - \mathbf{p}_3\ (\mathbf{p}_1 \cdot \mathbf{p}_2)] \\
&\quad -[\tilde{\mathbf{p}}_2\ (\mathbf{p}_1 \cdot \tilde{\mathbf{p}}_3) - \tilde{\mathbf{p}}_3\ (\mathbf{p}_1 \cdot \tilde{\mathbf{p}}_2)],
\end{aligned} \tag{3.19}
$$

The last equation is the "generalized BAC-CAB rule". The first two terms on the right hand side are "BAC-CAB"-terms in usual vector calculus, but the last two terms additionally appear because of the dual structure of the Lorentz group. We should not say that there appear many formulae of Lorentz algebra that are analogous with electrodynamics. Rather we should say that such analogous formulae come from the Lorentz invariant structure of electrodynamics.

To put it bluntly, we can understand the Lorentz algebra with our intuition for electric and magnetic fields. It will be convenient to introduce 3-dimensional vector fields $\hat{n}_i$ to represent the basis on the Lorentz group as

$$
\mathbf{l}_i = \begin{pmatrix} \hat{n}_i \\ 0 \end{pmatrix}, \quad \mathbf{k}_i = \begin{pmatrix} 0 \\ \hat{n}_i \end{pmatrix} = -\tilde{\mathbf{l}}_i. \tag{3.20}
$$

for $i = 1, 2, 3$. From (3.12), $\hat{n}_i$ are unit vector fields

$$
\hat{n}_i \cdot \hat{n}_j = \delta_{ij}\ . \tag{3.21}
$$

And the definition for the cross product between two Lorentz vector requires

$$\hat{n}_i \times \hat{n}_j = \epsilon_{ijk}\hat{n}_k \ . \tag{3.22}$$

Therefore $\hat{n}_i$ can be understood as a 3-dimensional right-handed orthonormal basis. Furthermore the dot and cross product of two basis vectors are summarized as,

$$\begin{aligned} &\mathbf{l}_i \cdot \mathbf{l}_j = \delta_{ij}, \quad \mathbf{l}_i \cdot \mathbf{k}_j = 0, \quad \mathbf{k}_i \cdot \mathbf{k}_j = -\delta_{ij}, \\ &\mathbf{l}_i \times \mathbf{l}_j = \epsilon_{ijk}\mathbf{l}_k, \qquad \mathbf{l}_i \times \mathbf{k}_j = \epsilon_{ijk}\mathbf{k}_k, \\ &\mathbf{k}_i \times \mathbf{k}_j = -\epsilon_{ijk}\mathbf{l}_k \ . \end{aligned} \tag{3.23}$$

As we saw in the equation (3.20), we can take a basis $(\mathbf{l}_i, \tilde{\mathbf{l}}_j)$ instead of $(\mathbf{l}_i, \mathbf{k}_j)$ and we will prefer the former to the later in order to see the dual structure of the Lorentz group more transparently.

3.2 Two different Abelian Decompositions

The rank of the Lorentz group is 2 so that its maximal ideal is 2-dimensional. Therefore we have to determine two commuting vector fields to define the magnetic symmetry of the system. The standard choices for such ideals are

$$\begin{aligned} A_2\text{-type} \quad &: \quad L_3 \ \text{ and } \ K_3 \ , \\ B_2\text{-type} \quad &: \quad \frac{1}{\sqrt{2}}(L_1 + K_2) \ \text{ and } \ \frac{1}{\sqrt{2}}(L_2 - K_1) \ , \end{aligned}$$

with

$$[L_3, K_3] = 0 \qquad [\frac{1}{\sqrt{2}}(L_1 + K_2), \frac{1}{\sqrt{2}}(L_2 - K_1)] = 0 \ . \tag{3.24}$$

We will call the first isometry "A_2-type" and the second one "B_2-type". The subscript means the maximal ideals are 2-dimensional. But remark that

$$\widetilde{L_3} = K_3 \ ,$$
$$\frac{1}{\sqrt{2}}\widetilde{(L_1 + K_2)} = -\frac{1}{\sqrt{2}}(L_2 - K_1) \ .$$

This result is consistent with the equation (3.18). Actually if we choose a non-zero sextet vector $\mathbf{P}$ on the Lorentz group, then we can make a maximal ideal with $\mathbf{P}$ and $\tilde{\mathbf{P}}$, since they do commute and they are linearly independent each other. Indeed $\mathbf{P}$ is an isometry vector if and only if $\tilde{\mathbf{P}}$ is. To prove it, let $\mathbf{P}$ be our isometry vector, that is,

$$D_\mu \mathbf{p} = (\partial_\mu + \mathbf{\Gamma}_\mu \times)\ \mathbf{p} = 0 \ , \tag{3.25}$$

where the coupling constant is 1. (We can do it without any loss of generality.) And let us do the dual transformation on the both sides, then

$$0 = \widetilde{D_\mu \mathbf{P}} = \widetilde{\partial_\mu \mathbf{P}} + \widetilde{\mathbf{\Gamma}_\mu \times \mathbf{P}} = \partial_\mu \widetilde{\mathbf{P}} + \mathbf{\Gamma}_\mu \times \widetilde{\mathbf{P}} = D_\mu \widetilde{\mathbf{P}} \ , \tag{3.26}$$

where we have used identities in (3.19). Therefore $\mathbf{P}$ being an isometry vector is equivalent to $\widetilde{\mathbf{P}}$ being an isometry vector. This fact also can be understood from the invariance of the totally anti-symmetric tensor ϵ_{abcd}.

Then the number of choice for an magnetic isometry will be infinite since there are infinitely many non-zero sextet vectors $\mathbf{P}$. But we can classify them into two classes. To do that, let us recall the Casimir invariants. If we choose the pair of isometry vectors $\mathbf{P}$ and $\widetilde{\mathbf{P}}$, then their Casimir invariants are given by,

$$\begin{aligned} \alpha &= \mathbf{p} \cdot \mathbf{p} = \vec{m}^2 - \vec{e}^2, \\ \beta &= \mathbf{p} \cdot \tilde{\mathbf{p}} = 2\vec{m} \cdot \vec{e} \ . \end{aligned} \tag{3.27}$$

And consider a vector $\mathbf{p}'$ defined by an arbitrary linear combination of $\mathbf{P}$ and $\widetilde{\mathbf{P}}$,

$$\mathbf{p}' = a\mathbf{p} + b\tilde{\mathbf{p}}, \qquad \tilde{\mathbf{p}}' = a\tilde{\mathbf{p}} - b\mathbf{p}. \tag{3.28}$$

where a and b are constants. Of course they are also isometry vectors

$$D_\mu \mathbf{p}' = 0, \qquad D_\mu \tilde{\mathbf{p}}' = 0 \ . \tag{3.29}$$

But their Casimir invariants become,

$$\begin{aligned} \alpha' &= (a^2 - b^2)\alpha + 2ab\beta, \\ \beta' &= (a^2 - b^2)\beta - 2ab\alpha. \end{aligned} \tag{3.30}$$

Therefore with the choice of

$$\begin{aligned} a &= \sqrt{\frac{(\alpha^2+\beta^2)^{1/2} \pm \alpha}{2(\alpha^2+\beta^2)}} \ , \\ b &= \pm \frac{|\beta|}{\beta} \sqrt{\frac{(\alpha^2+\beta^2)^{1/2} \mp \alpha}{2(\alpha^2+\beta^2)}} \ , \end{aligned}$$

we can normalize two Casimir invariants of $\mathbf{P}$ and $\widetilde{\mathbf{P}}$,

$$\alpha' = \pm 1, \qquad \beta' = 0 \ , \tag{3.31}$$

unless $\alpha^2 + \beta^2 = 0$. Hence we can always transform the isometry vectors with their Casimir invariants to be $(\alpha, \beta) = (\pm 1, 0)$ or $(\alpha, \beta) = (0, 0)$ without any loss of generality. This means that we can classify all magnetic isometry of the Lorentz group into two classes. Moreover, A_2 belongs to the first class, and B_2 belongs to the second class. Therefore it will be sufficient to study the Abelian decompositions of A_2 and B_2 as standard ones of the two classes of magnetic isometries on the Lorentz group.

3.3 A_2 (Non Light-like) Isometry

Let us consider the A_2-isometry first. Two commuting sextet vectors for this isometry are L_3 and K_3. Of course they are dual to each other up to the negative sign as we studied in the previous section. Simply we can put

$$\mathbf{p} = f\ \mathbf{l}_3 = f \begin{pmatrix} \hat{n} \\ 0 \end{pmatrix}, \quad \tilde{\mathbf{p}} = f\ \tilde{\mathbf{l}}_3 = f \begin{pmatrix} 0 \\ -\hat{n} \end{pmatrix}, \tag{3.32}$$

where f is an arbitrary function of space-time. Here we do not use the standard basis for the Lorentz group but use an arbitrary 3-dimensional orthonormal basis $\hat{n}_i$'s. As we saw in the $SU(2)$ case, this choice of basis enables us to understand the topological structure of gauge fields transparently. Here the function f is fixed by the isometry condition, since

$$\partial_\mu f^2 = \partial_\mu \mathbf{p}^2 = D_\mu \mathbf{p}^2 = 2\mathbf{p} \cdot D_\mu \mathbf{p} = 0. \tag{3.33}$$

Hence we do not lose any generality even if we set $f = 1$. So the isometry vectors for A_2 can be expressed as

$$\begin{gathered} \mathbf{l} = \mathbf{l}_3 = \begin{pmatrix} \hat{n} \\ 0 \end{pmatrix}, \qquad \tilde{\mathbf{l}} = \tilde{\mathbf{l}}_3 = \begin{pmatrix} 0 \\ -\hat{n} \end{pmatrix}, \\ D_\mu \mathbf{l} = 0, \qquad D_\mu \tilde{\mathbf{l}} = 0, \end{gathered} \tag{3.34}$$

whose Casimir invariants is $(1, 0)$. In the B_2-case, both the Casimir invariants and the norms of isometry vectors vanish. Therefore we call the B_2-case "light-like" and A_2-case "non-light-like". The isometry conditions (or restrictions) for the gauge potential are,

$$D_\mu \mathbf{p} = (\partial_\mu + \hat{\mathbf{\Gamma}}_\mu \times)\mathbf{p} = 0\ , \qquad D_\mu \tilde{\mathbf{p}} = (\partial_\mu + \hat{\mathbf{\Gamma}}_\mu \times)\tilde{\mathbf{p}} = 0\ . \tag{3.35}$$

The solution for these equations is,

$$\hat{\mathbf{\Gamma}}_\mu = A_\mu\, \mathbf{l} - B_\mu\, \tilde{\mathbf{l}} - \mathbf{l} \times \partial_\mu \mathbf{l},$$
$$A_\mu = \mathbf{l} \cdot \mathbf{\Gamma}_\mu, \quad B_\mu = \tilde{\mathbf{l}} \cdot \mathbf{\Gamma}_\mu, \tag{3.36}$$

where A_μ and B_μ are two Abelian connections of $\mathbf{l}$ and $\tilde{\mathbf{l}}$ components which are not restricted by the isometry condition. Two isometry vectors $\mathbf{l}$ and $\tilde{\mathbf{l}}$, however, appear in a non-symmetrical way in the above expression since the topological potential $\mathbf{l} \times \partial_\mu \mathbf{l}$ include only $\mathbf{l}$. But this is not so, because

$$\mathbf{l} \times \partial_\mu \mathbf{l} = -\tilde{\mathbf{l}} \times \partial_\mu \tilde{\mathbf{l}}, \tag{3.37}$$

so that we can rewrite the restricted connection as

$$\hat{\mathbf{\Gamma}}_\mu = A_\mu\, \mathbf{l} - B_\mu\, \tilde{\mathbf{l}} - \frac{1}{2}(\mathbf{l} \times \partial_\mu \mathbf{l} - \tilde{\mathbf{l}} \times \partial_\mu \tilde{\mathbf{l}}). \tag{3.38}$$

Here we also define the other basis vectors as,

$$\mathbf{l}_i = \begin{pmatrix} \hat{n}_i \\ 0 \end{pmatrix}, \qquad \widetilde{\mathbf{l}_i} = \begin{pmatrix} 0 \\ \hat{n}_i \end{pmatrix} \tag{3.39}$$

for $i = 1, 2$. And their dot- and cross- multiplication tables are,

$\cdot$	$\mathbf{l}_1$	$\mathbf{l}_2$	$\mathbf{l}$	$\widetilde{\mathbf{l}_1}$	$\widetilde{\mathbf{l}_2}$	$\widetilde{\mathbf{l}}$
$\mathbf{l}_1$	1	0	0	0	0	0
$\mathbf{l}_2$	0	1	0	0	0	0
$\mathbf{l}$	0	0	1	0	0	0
$\widetilde{\mathbf{l}_1}$	0	0	0	-1	0	0
$\widetilde{\mathbf{l}_2}$	0	0	0	0	-1	0
$\widetilde{\mathbf{l}}$	0	0	0	0	0	-1

$\times$	$\mathbf{l}_1$	$\mathbf{l}_2$	$\mathbf{l}$	$\widetilde{\mathbf{l}_1}$	$\widetilde{\mathbf{l}_2}$	$\widetilde{\mathbf{l}}$
$\mathbf{l}_1$	0	$\mathbf{l}$	$-\mathbf{l}_2$	0	$\widetilde{\mathbf{l}}$	$-\widetilde{\mathbf{l}_2}$
$\mathbf{l}_2$	$-\mathbf{l}$	0	$\mathbf{l}_1$	$-\widetilde{\mathbf{l}}$	0	$\widetilde{\mathbf{l}_1}$
$\mathbf{l}$	$\mathbf{l}_2$	$-\mathbf{l}_1$	0	$\widetilde{\mathbf{l}_2}$	$-\widetilde{\mathbf{l}_1}$	0
$\widetilde{\mathbf{l}_1}$	0	$\widetilde{\mathbf{l}}$	$-\widetilde{\mathbf{l}_2}$	0	$-\mathbf{l}$	$\mathbf{l}_2$
$\widetilde{\mathbf{l}_2}$	$-\widetilde{\mathbf{l}}$	0	$\widetilde{\mathbf{l}_1}$	$\mathbf{l}$	0	$-\mathbf{l}_1$
$\widetilde{\mathbf{l}}$	$\widetilde{\mathbf{l}_2}$	$-\widetilde{\mathbf{l}_1}$	0	$-\mathbf{l}_2$	$\mathbf{l}_1$	0

With the above restricted gauge potential we can make the restricted field strength $\hat{\mathbf{R}}_{\mu\nu}$ is given by

$$\hat{\mathbf{R}}_{\mu\nu} = \partial_\mu \hat{\mathbf{\Gamma}}_\nu - \partial_\nu \hat{\mathbf{\Gamma}}_\mu + \hat{\mathbf{\Gamma}}_\mu \times \hat{\mathbf{\Gamma}}_\nu = (A_{\mu\nu} + H_{\mu\nu})\ \mathbf{l} - (B_{\mu\nu} + \widetilde{H}_{\mu\nu})\ \tilde{\mathbf{l}}, \tag{3.40}$$

where

$$\begin{aligned} A_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \\ H_{\mu\nu} &= -\mathbf{l} \cdot (\partial_\mu \mathbf{l} \times \partial_\nu \mathbf{l}), \\ \widetilde{H}_{\mu\nu} &= -\tilde{\mathbf{l}} \cdot (\partial_\mu \mathbf{l} \times \partial_\nu \mathbf{l}) = \tilde{\mathbf{l}} \cdot (\partial_\mu \tilde{\mathbf{l}} \times \partial_\nu \tilde{\mathbf{l}}) = 0, \end{aligned}$$

The components of the restricted field strength are

$$\hat{R}_{\mu\nu}{}^{ab} = \hat{\mathbf{R}}_{\mu\nu} \cdot \mathbf{I}^{ab} = (A_{\mu\nu} + H_{\mu\nu})\ l^{ab} - B_{\mu\nu}\ \tilde{l}^{ab}. \tag{3.41}$$

Here remark that $\widetilde{H}_{\mu\nu}$ vanishes. In order to regard the gauge potential as a sum of its electric and magnetic part, it is convenient to express the whole process in 3-dimensional notation. Then the isometry condition (3.34) is,

$$\begin{gathered} \hat{\mathbf{\Gamma}}_\mu = \begin{pmatrix} \hat{A}_\mu \\ \hat{B}_\mu \end{pmatrix}, \\ \hat{D}_\mu \hat{n} = 0, \qquad \hat{B}_\mu \times \hat{n} = 0, \\ \hat{D}_\mu = \partial_\mu + \hat{A}_\mu \times . \end{gathered} \tag{3.42}$$

From this we have

$$\begin{gathered} \hat{A}_\mu = A_\mu \hat{n} - \hat{n} \times \partial_\mu \hat{n}, \quad \hat{B}_\mu = B_\mu \hat{n}, \\ A_\mu = \hat{n} \cdot \hat{A}_\mu, \quad B_\mu = \hat{n} \cdot \hat{B}_\mu. \end{gathered} \tag{3.43}$$

Moreover, with

$$\hat{\mathbf{R}}_{\mu\nu} = \begin{pmatrix} \hat{A}_{\mu\nu} \\ \hat{B}_{\mu\nu} \end{pmatrix}, \tag{3.44}$$

we have

$$\begin{gathered}
\hat{A}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + \hat{A}_\mu \times \hat{A}_\nu \\
= (A_{\mu\nu} + H_{\mu\nu})\hat{n} = \bar{A}_{\mu\nu}\hat{n}, \\
\hat{B}_{\mu\nu} = \partial_\mu \hat{B}_\nu - \partial_\nu \hat{B}_\mu + \hat{A}_\mu \times \hat{B}_\nu - \hat{A}_\nu \times \hat{B}_\mu \\
= \hat{D}_\mu \hat{B}_\nu - \hat{D}_\nu \hat{B}_\mu = B_{\mu\nu}\ \hat{n}, \\
H_{\mu\nu} = -\hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = \partial_\mu \widetilde{C}_\nu - \partial_\nu \widetilde{C}_\mu, \\
\widetilde{C}_\mu = \hat{n}_1 \cdot \partial_\mu \hat{n}_2, \\
\bar{A}_{\mu\nu} = \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu, \qquad \bar{A}_\mu = A_\mu + \widetilde{C}_\mu.
\end{gathered} \tag{3.45}$$

Notice that $\hat{A}_\mu$ and $\hat{A}_{\mu\nu}$ are formally identical to the restricted potential and restricted field strength of $SU(2)$ gauge theory. In particular $H_{\mu\nu}$ is identical to what we have in the chapter 2. This, together with $\widetilde{H}_{\mu\nu} = 0$, tells that the topology of this isometry is identical to that of the $SU(2)$ subgroup. The above equations are more efficiently calculated if we use following identities.

$$\begin{aligned}
\hat{D}_\mu \mathbf{l}_1 &= K_\mu \mathbf{l}_2 - B_\mu \mathbf{l}_1 &&= (K_\mu \mathbf{l} - B_\mu \tilde{\mathbf{l}}) \times \mathbf{l}_1 \\
\hat{D}_\mu \tilde{\mathbf{l}}_1 &= K_\mu \tilde{\mathbf{l}}_2 - B_\mu \tilde{\mathbf{l}}_1 &&= (K_\mu \mathbf{l} - B_\mu \tilde{\mathbf{l}}) \times \tilde{\mathbf{l}}_1 \\
\hat{D}_\mu \mathbf{l}_2 &= -K_\mu \mathbf{l}_1 + B_\mu \tilde{\mathbf{l}}_1 &&= (K_\mu \mathbf{l} - B_\mu \tilde{\mathbf{l}}) \times \mathbf{l}_2 \\
\hat{D}_\mu \tilde{\mathbf{l}}_2 &= -K_\mu \tilde{\mathbf{l}}_1 - B_\mu \mathbf{l}_1 &&= (K_\mu \mathbf{l} - B_\mu \tilde{\mathbf{l}}) \times \tilde{\mathbf{l}}_2 \\
\hat{D}_\mu \mathbf{l} &= 0, \\
\hat{D}_\mu \tilde{\mathbf{l}} &= 0
\end{aligned} \tag{3.46}$$

With this the full connection of Lorentz group is given by

$$\mathbf{\Gamma}_\mu = \hat{\mathbf{\Gamma}}_\mu + \mathbf{Z}_\mu, \quad \mathbf{l} \cdot \mathbf{Z}_\mu = \tilde{\mathbf{l}} \cdot \mathbf{Z}_\mu = 0, \tag{3.47}$$

where $\mathbf{Z}_\mu$ is the valence connection. As we saw in the previous chapter, this valence connection also transforms covariantly under the Lorentz gauge transformation. Then the full field strength $\mathbf{R}_{\mu\nu}$ is written as

$$\begin{aligned} \mathbf{R}_{\mu\nu} &= \partial_\mu \mathbf{\Gamma}_\nu - \partial_\nu \mathbf{\Gamma}_\mu + \mathbf{\Gamma}_\mu \times \mathbf{\Gamma}_\nu \\ &= \hat{\mathbf{R}}_{\mu\nu} + \mathbf{Z}_{\mu\nu}, \\ \mathbf{Z}_{\mu\nu} &= \hat{D}_\mu \mathbf{Z}_\nu - \hat{D}_\nu \mathbf{Z}_\mu + \mathbf{Z}_\mu \times \mathbf{Z}_\nu, \\ \hat{D}_\mu &= \partial_\mu + \hat{\mathbf{\Gamma}}_\mu \times, \end{aligned} \tag{3.48}$$

where $\mathbf{Z}_{\mu\nu}$ is the valence part of the field strength. This is the same picture when we decompose the Riemann-Cartan curvature tensor into the purely Riemannian part and the torsional part. Here the valence field strength can further be decomposed to the kinetic part $\dot{\mathbf{Z}}_{\mu\nu}$ and the potential part $\mathbf{Z}'_{\mu\nu}$,

$$\begin{aligned} \mathbf{Z}_{\mu\nu} &= \dot{\mathbf{Z}}_{\mu\nu} + \mathbf{Z}'_{\mu\nu}, \\ \dot{\mathbf{Z}}_{\mu\nu} &= \hat{D}_\mu \mathbf{Z}_\nu - \hat{D}_\nu \mathbf{Z}_\mu, \quad \mathbf{Z}'_{\mu\nu} = \mathbf{Z}_\mu \times \mathbf{Z}_\nu. \end{aligned} \tag{3.49}$$

Now with

$$\begin{aligned} \mathbf{Z}_\mu &= Z_\mu^1 \mathbf{l}_1 - \widetilde{Z}_\mu^1 \tilde{\mathbf{l}}_1 + Z_\mu^2 \mathbf{l}_2 - \widetilde{Z}_\mu^2 \tilde{\mathbf{l}}_2, \\ Z_\mu^1 &= \mathbf{l}_1 \cdot \mathbf{Z}_\mu, \quad \widetilde{Z}_\mu^1 = \tilde{\mathbf{l}}_1 \cdot \mathbf{Z}_\mu, \\ Z_\mu^2 &= \mathbf{l}_2 \cdot \mathbf{Z}_\mu, \quad \widetilde{Z}_\mu^2 = \tilde{\mathbf{l}}_2 \cdot \mathbf{Z}_\mu, \end{aligned} \tag{3.50}$$

we have

$$\begin{aligned}
\dot{\mathbf{Z}}_{\mu\nu} = (\mathcal{D}_\mu Z^1_\nu - \mathcal{D}_\nu Z^1_\mu)\mathbf{l}_1 - (\mathcal{D}_\mu \widetilde{Z}^1_\nu - \mathcal{D}_\nu \widetilde{Z}^1_\mu)\tilde{\mathbf{l}}_1 \\
+(\mathcal{D}_\mu Z^2_\nu - \mathcal{D}_\nu Z^2_\mu)\mathbf{l}_2 - (\mathcal{D}_\mu \widetilde{Z}^2_\nu - \mathcal{D}_\nu \widetilde{Z}^2_\mu)\tilde{\mathbf{l}}_2, \\
\mathcal{D}_\mu Z^1_\nu = \partial_\mu Z^1_\nu - \bar{A}_\mu Z^2_\nu + B_\mu \widetilde{Z}^2_\nu, \\
\mathcal{D}_\mu \widetilde{Z}^1_\nu = \partial_\mu \widetilde{Z}^1_\nu - \bar{A}_\mu \widetilde{Z}^2_\nu - B_\mu Z^2_\nu, \\
\mathcal{D}_\mu Z^2_\nu = \partial_\mu Z^2_\nu + \bar{A}_\mu Z^1_\nu - B_\mu \widetilde{Z}^1_\nu, \\
\mathcal{D}_\mu \widetilde{Z}^2_\nu = \partial_\mu \widetilde{Z}^2_\nu + \bar{A}_\mu \widetilde{Z}^1_\nu + B_\mu Z^1_\nu, \\
\mathbf{l} \cdot \dot{\mathbf{Z}}_{\mu\nu} = \tilde{\mathbf{l}} \cdot \dot{\mathbf{Z}}_{\mu\nu} = 0.
\end{aligned} \tag{3.51}$$

Moreover, we have

$$\mathbf{Z}'_{\mu\nu} = W_{\mu\nu}\mathbf{l} - \tilde{W}_{\mu\nu}\tilde{\mathbf{l}} \tag{3.52}$$

where

$$\begin{aligned}
W_{\mu\nu} &= \mathbf{l} \cdot (\mathbf{Z}_\mu \times \mathbf{Z}_\nu) = Z^1_\mu Z^2_\nu - Z^1_\nu Z^2_\mu - (\widetilde{Z}^1_\mu \widetilde{Z}^2_\nu - \widetilde{Z}^1_\nu \widetilde{Z}^2_\mu), \\
\tilde{W}_{\mu\nu} &= \tilde{\mathbf{l}} \cdot (\mathbf{Z}_\mu \times \mathbf{Z}_\nu) = Z^1_\mu \widetilde{Z}^2_\nu - Z^1_\nu \widetilde{Z}^2_\mu + \widetilde{Z}^1_\mu Z^2_\nu - \widetilde{Z}^1_\nu Z^2_\mu.
\end{aligned} \tag{3.53}$$

Here remark that components of isometry directions do not come from only restricted parts. Actually valence parts can contribute. With this we have the full field strength

$$\begin{aligned}
\mathbf{R}_{\mu\nu} &= (\bar{A}_{\mu\nu} + W_{\mu\nu})\mathbf{l} - (B_{\mu\nu} + \tilde{W}_{\mu\nu})\tilde{\mathbf{l}} + \hat{D}_\mu \mathbf{Z}_\nu - \hat{D}_\nu \mathbf{Z}_\mu \\
&= (\mathcal{D}_\mu \bar{A}_\nu - \mathcal{D}_\nu \bar{A}_\mu)\mathbf{l} - (\mathcal{D}_\mu B_\nu - \mathcal{D}_\nu B_\mu)\tilde{\mathbf{l}} \\
&+ (\mathcal{D}_\mu Z^1_\nu - \mathcal{D}_\nu Z^1_\mu)\mathbf{l}_1 - (\mathcal{D}_\mu \widetilde{Z}^1_\nu - \mathcal{D}_\nu \widetilde{Z}^1_\mu)\tilde{\mathbf{l}}_1 \\
&+ (\mathcal{D}_\mu Z^2_\nu - \mathcal{D}_\nu Z^2_\mu)\mathbf{l}_2 - (\mathcal{D}_\mu \widetilde{Z}^2_\nu - \mathcal{D}_\nu \widetilde{Z}^2_\mu)\tilde{\mathbf{l}}_2 \\
&= R^1_{\mu\nu}\ \mathbf{l}_1 - \widetilde{R}^1_{\mu\nu}\ \tilde{\mathbf{l}}_1 + R^2_{\mu\nu}\ \mathbf{l}_2 - \widetilde{R}^2_{\mu\nu}\ \tilde{\mathbf{l}}_2 + R_{\mu\nu}\ \mathbf{l} - \widetilde{R}_{\mu\nu}\ \tilde{\mathbf{l}},
\end{aligned} \tag{3.54}$$

where

$$
\begin{gathered}
\mathcal{D}_\mu \bar{A}_\nu = \partial_\mu \bar{A}_\nu + Z^1_\mu Z^2_\nu - \widetilde{Z}^1_\mu \widetilde{Z}^2_\nu, \\
\mathcal{D}_\mu B_\nu = \partial_\mu B_\nu + Z^1_\mu \widetilde{Z}^2_\nu + \widetilde{Z}^1_\mu Z^2_\nu, \\
R^1_{\mu\nu} = \mathcal{D}_\mu Z^1_\nu - \mathcal{D}_\nu Z^1_\mu, \quad \widetilde{R}^1_{\mu\nu} = \mathcal{D}_\mu \widetilde{Z}^1_\nu - \mathcal{D}_\nu \widetilde{Z}^1_\mu, \\
R^2_{\mu\nu} = \mathcal{D}_\mu Z^2_\nu - \mathcal{D}_\nu Z^2_\mu, \quad \widetilde{R}^2_{\mu\nu} = \mathcal{D}_\mu \widetilde{Z}^2_\nu - \mathcal{D}_\nu \widetilde{Z}^2_\mu, \\
R_{\mu\nu} = \mathcal{D}_\mu \bar{A}_\nu - \mathcal{D}_\nu \bar{A}_\mu = A_{\mu\nu} + H_{\mu\nu} + W_{\mu\nu}, \\
\widetilde{R}_{\mu\nu} = \mathcal{D}_\mu B_\nu - \mathcal{D}_\nu B_\mu = B_{\mu\nu} + \tilde{W}_{\mu\nu},
\end{gathered} \tag{3.55}
$$

or equivalently

$$
\begin{aligned}
R_{\mu\nu}{}^{ab} &= \mathbf{R}_{\mu\nu} \cdot \mathbf{I}^{ab} \\
&= R^1_{\mu\nu}\ l^{ab}_1 - \widetilde{R}^1_{\mu\nu}\ \tilde{l}^{ab}_1 + R^2_{\mu\nu}\ l^{ab}_2 - \widetilde{R}^2_{\mu\nu}\ \tilde{l}^{ab}_2 + R_{\mu\nu}\ l^{ab} - \widetilde{R}_{\mu\nu}\ \tilde{l}^{ab}.
\end{aligned} \tag{3.56}
$$

This is the A_2 decomposition of the field strength. Now let us express the above results in 3-dimensional notation again. Then we have

$$
\begin{gathered}
\mathbf{Z}_\mu = \begin{pmatrix} \vec{X}_\mu \\ \vec{Y}_\mu \end{pmatrix}, \\
\vec{X}_\mu = Z^1_\mu\ \hat{n}_1 + Z^2_\mu\ \hat{n}_2, \quad \vec{Y}_\mu = \widetilde{Z}^1_\mu\ \hat{n}_1 + \widetilde{Z}^2_\mu\ \hat{n}_2, \\
\hat{n} \cdot \vec{X}_\mu = 0, \qquad \hat{n} \cdot \vec{Y}_\mu = 0.
\end{gathered} \tag{3.57}
$$

Moreover, with

$$
\mathbf{Z}_{\mu\nu} = \begin{pmatrix} \vec{X}_{\mu\nu} \\ \vec{Y}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \dot{\vec{X}}_{\mu\nu} + \vec{X}'_{\mu\nu} \\ \dot{\vec{Y}}_{\mu\nu} + \vec{Y}'_{\mu\nu} \end{pmatrix}, \tag{3.58}
$$

we have

$$
\dot{\vec{X}}_{\mu\nu} = \hat{D}_\mu \vec{X}_\nu - \hat{D}_\nu \vec{X}_\mu - \vec{B}_\mu \times \vec{Y}_\nu + \vec{B}_\nu \times \vec{Y}_\mu
$$

$$
\begin{aligned}
&= R^1_{\mu\nu}\ \hat{n}_1 + R^2_{\mu\nu}\ \hat{n}_2, \\
\dot{\vec{Y}}_{\mu\nu} &= \hat{D}_\mu \vec{Y}_\nu - \hat{D}_\nu \vec{Y}_\mu + \vec{B}_\mu \times \vec{X}_\nu - \vec{B}_\nu \times \vec{X}_\mu \\
&= \widetilde{R}^1_{\mu\nu}\ \hat{n}_1 + \widetilde{R}^2_{\mu\nu}\ \hat{n}_2, \\
\vec{X}'_{\mu\nu} &= \vec{X}_\mu \times \vec{X}_\nu - \vec{Y}_\mu \times \vec{Y}_\nu = W_{\mu\nu}\ \hat{n}, \\
\vec{Y}'_{\mu\nu} &= \vec{X}_\mu \times \vec{Y}_\nu + \vec{Y}_\mu \times \vec{X}_\nu = \tilde{W}_{\mu\nu}\ \hat{n}.
\end{aligned} \tag{3.59}
$$

Notice that the kinetic part and the potential part of $\mathbf{Z}_{\mu\nu}$ are orthogonal to each other. Finally, with

$$
\mathbf{R}_{\mu\nu} = \begin{pmatrix} \vec{A}_{\mu\nu} \\ \vec{B}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \hat{A}_{\mu\nu} + \vec{X}_{\mu\nu} \\ \hat{B}_{\mu\nu} + \vec{Y}_{\mu\nu} \end{pmatrix}, \tag{3.60}
$$

we have

$$
\begin{aligned}
\vec{A}_{\mu\nu} &= R_{\mu\nu}\ \hat{n} + \dot{\vec{X}}_{\mu\nu} = R^1_{\mu\nu}\ \hat{n}_1 + R^2_{\mu\nu}\ \hat{n}_2 + R_{\mu\nu}\ \hat{n}, \\
\vec{B}_{\mu\nu} &= \widetilde{R}_{\mu\nu}\ \hat{n} + \dot{\vec{Y}}_{\mu\nu} = \widetilde{R}^1_{\mu\nu}\ \hat{n}_1 + \widetilde{R}^2_{\mu\nu}\ \hat{n}_2 + \widetilde{R}_{\mu\nu}\ \hat{n}.
\end{aligned} \tag{3.61}
$$

This completes the A_2 decomposition of the gravitational connection.

3.4 B_2 (Light-like) Isometry

Now let us consider the B_2-isometry. The isometry vectors for this case are $(L_1 + K_2)/\sqrt{2}$ and $(L_2 - K_1)/\sqrt{2}$. Let $\mathbf{p}$ and $\tilde{\mathbf{p}}$ represent the two isometry vectors in order. As we did in the A_2-case they can be written,

$$
\begin{aligned}
\mathbf{p} &= f\Big(\frac{\mathbf{l}_1 + \mathbf{k}_2}{\sqrt{2}}\Big) = \frac{f}{\sqrt{2}} \begin{pmatrix} \hat{n}_1 \\ \hat{n}_2 \end{pmatrix}, \\
\tilde{\mathbf{p}} &= f\Big(\frac{\mathbf{l}_2 - \mathbf{k}_1}{\sqrt{2}}\Big) = \frac{f}{\sqrt{2}} \begin{pmatrix} \hat{n}_2 \\ -\hat{n}_1 \end{pmatrix}.
\end{aligned} \tag{3.62}
$$

But the norms of two isometry vectors vanish regardless of what the function f is, since the isometry condition does not restrict the function f. And the Casimir invariants (α, β) are given by $(0, 0)$ independent of f. Let us put $f = e^{\lambda}$. Then we can express the B_2 isometry by

$$
\begin{aligned}
\mathbf{j} &= \frac{e^{\lambda}}{\sqrt{2}}(\mathbf{l}_1 + \mathbf{k}_2) = \frac{e^{\lambda}}{\sqrt{2}} \begin{pmatrix} \hat{n}_1 \\ \hat{n}_2 \end{pmatrix}, \\
\tilde{\mathbf{j}} &= \frac{e^{\lambda}}{\sqrt{2}}(\mathbf{l}_2 - \mathbf{k}_1) = \frac{e^{\lambda}}{\sqrt{2}} \begin{pmatrix} \hat{n}_2 \\ -\hat{n}_1 \end{pmatrix}, \\
& D_\mu \mathbf{j} = 0, \quad D_\mu \tilde{\mathbf{j}} = 0,
\end{aligned} \tag{3.63}
$$

In order to get the restricted connection $\hat{\mathbf{\Gamma}}$ from the isometry conditions, let us introduce 4 more basis vectors on the Lorentz group.

$$
\begin{aligned}
\mathbf{k} &= \frac{e^{-\lambda}}{\sqrt{2}}(\mathbf{l}_1 - \mathbf{k}_2) = \frac{e^{-\lambda}}{\sqrt{2}} \begin{pmatrix} \hat{n}_1 \\ -\hat{n}_2 \end{pmatrix}, \quad \tilde{\mathbf{k}} = -\frac{e^{-\lambda}}{\sqrt{2}}(\mathbf{l}_2 + \mathbf{k}_1) = \frac{e^{-\lambda}}{\sqrt{2}} \begin{pmatrix} -\hat{n}_2 \\ -\hat{n}_1 \end{pmatrix}, \\
\mathbf{l} &= -\mathbf{j} \times \tilde{\mathbf{k}} = -\tilde{\mathbf{j}} \times \mathbf{k} = \begin{pmatrix} \hat{n}_3 \\ 0 \end{pmatrix}, \quad \tilde{\mathbf{l}} = \mathbf{j} \times \mathbf{k} = -\tilde{\mathbf{j}} \times \tilde{\mathbf{k}} = \begin{pmatrix} 0 \\ -\hat{n}_3 \end{pmatrix}.
\end{aligned} \tag{3.64}
$$

The dot- and cross-multiplication tables are following;

$\cdot$	$\mathbf{j}$	$\mathbf{k}$	$\mathbf{l}$	$\tilde{\mathbf{j}}$	$\tilde{\mathbf{k}}$	$\tilde{\mathbf{l}}$
$\mathbf{j}$	0	1	0	0	0	0
$\mathbf{k}$	1	0	0	0	0	0
$\mathbf{l}$	0	0	1	0	0	0
$\tilde{\mathbf{j}}$	0	0	0	0	-1	0
$\tilde{\mathbf{k}}$	0	0	0	-1	0	0
$\tilde{\mathbf{l}}$	0	0	0	0	0	-1

$\times$	$\mathbf{j}$	$\mathbf{k}$	$\mathbf{l}$	$\tilde{\mathbf{j}}$	$\tilde{\mathbf{k}}$	$\tilde{\mathbf{l}}$
$\mathbf{j}$	0	$\tilde{\mathbf{l}}$	$-\tilde{\mathbf{j}}$	0	$-\mathbf{l}$	$\mathbf{j}$
$\mathbf{k}$	$-\tilde{\mathbf{l}}$	0	$\tilde{\mathbf{k}}$	$\mathbf{l}$	0	$-\mathbf{k}$
$\mathbf{l}$	$\tilde{\mathbf{j}}$	$-\tilde{\mathbf{k}}$	0	$-\mathbf{j}$	$\mathbf{k}$	0
$\tilde{\mathbf{j}}$	0	$-\mathbf{l}$	$\mathbf{j}$	0	$-\tilde{\mathbf{l}}$	$\tilde{\mathbf{j}}$
$\tilde{\mathbf{k}}$	$\mathbf{l}$	0	$-\mathbf{k}$	$\tilde{\mathbf{l}}$	0	$-\tilde{\mathbf{k}}$
$\tilde{\mathbf{l}}$	$-\mathbf{j}$	$\mathbf{k}$	0	$-\tilde{\mathbf{j}}$	$\tilde{\mathbf{k}}$	0

Notice that four of them are null vectors, so this basis vectors fit to describe gravitational waves. From this we obtain the following restricted connection for the B_2 isometry,

$$\begin{aligned}\hat{\mathbf{\Gamma}}_\mu &= \Gamma_\mu\,\mathbf{j} - \widetilde{\Gamma}_\mu\,\tilde{\mathbf{j}} - \mathbf{k}\times\partial_\mu\mathbf{j} \\ &= \Gamma_\mu\,\mathbf{j} - \widetilde{\Gamma}_\mu\,\tilde{\mathbf{j}} - \frac{1}{2}(\mathbf{k}\times\partial_\mu\mathbf{j} - \tilde{\mathbf{k}}\times\partial_\mu\tilde{\mathbf{j}}),\end{aligned} \tag{3.65}$$

with

$$\Gamma_\mu = \mathbf{k}\cdot\mathbf{\Gamma}_\mu, \quad \widetilde{\Gamma}_\mu = \tilde{\mathbf{k}}\cdot\mathbf{\Gamma}_\mu, \quad \mathbf{k}\times\partial_\mu\mathbf{j} = -\tilde{\mathbf{k}}\times\partial_\mu\tilde{\mathbf{j}}\ . \tag{3.66}$$

Here Γ_μ and $\widetilde{\Gamma}_\mu$ are two Abelian gauge potentials of $\mathbf{j}$- and $\tilde{\mathbf{j}}$- components which are not restricted by the isometry condition. The restricted field strength $\hat{\mathbf{R}}_{\mu\nu}$ is expressed only in terms of the restricted connection,

$$\hat{\mathbf{R}}_{\mu\nu} = \partial_\mu\hat{\mathbf{\Gamma}}_\nu - \partial_\nu\hat{\mathbf{\Gamma}}_\mu + \hat{\mathbf{\Gamma}}_\mu\times\hat{\mathbf{\Gamma}}_\nu = (\Gamma_{\mu\nu} + H_{\mu\nu})\mathbf{j} - (\widetilde{\Gamma}_{\mu\nu} + \widetilde{H}_{\mu\nu})\tilde{\mathbf{j}}, \tag{3.67}$$

with following field strengths

$$\begin{aligned}\Gamma_{\mu\nu} = \partial_\mu\Gamma_\nu - \partial_\nu\Gamma_\mu, &\quad \widetilde{\Gamma}_{\mu\nu} = \partial_\mu\widetilde{\Gamma}_\nu - \partial_\nu\widetilde{\Gamma}_\mu, \\ H_{\mu\nu} &= -\mathbf{k}\cdot(\partial_\mu\mathbf{j}\times\partial_\nu\mathbf{k} - \partial_\nu\mathbf{j}\times\partial_\mu\mathbf{k}), \\ \widetilde{H}_{\mu\nu} &= -\tilde{\mathbf{k}}\cdot(\partial_\mu\mathbf{j}\times\partial_\nu\mathbf{k} - \partial_\nu\mathbf{j}\times\partial_\mu\mathbf{k}),\end{aligned} \tag{3.68}$$

so that

$$\hat{R}_{\mu\nu}^{\ \ ab} = (\Gamma_{\mu\nu} + H_{\mu\nu})j^{ab} - (\widetilde{\Gamma}_{\mu\nu} + \widetilde{H}_{\mu\nu})\tilde{j}^{ab}. \tag{3.69}$$

Notice that $\hat{\mathbf{R}}_{\mu\nu}$ is orthogonal to $\mathbf{l}$ and $\tilde{\mathbf{l}}$. This should be contrasted with the restricted field strength (3.41) of the A_2 isometry. In 3-dimensional notation the isometry condition (3.63) is written as

$$
\begin{aligned}
\hat{\mathbf{\Gamma}}_\mu &= \begin{pmatrix} \hat{A}_\mu \\ \hat{B}_\mu \end{pmatrix}, \\
\hat{D}_\mu \hat{n}_1 &= \hat{B}_\mu \times \hat{n}_2 - (\partial_\mu \lambda)\hat{n}_1, \\
\hat{D}_\mu \hat{n}_2 &= -\hat{B}_\mu \times \hat{n}_1 - (\partial_\mu \lambda)\hat{n}_2. \qquad (3.70)
\end{aligned}
$$

From this we have

$$
\begin{aligned}
\hat{A}_\mu &= A^1_\mu \hat{n}_1 + A^2_\mu \hat{n}_2 + (\hat{n}_1 \cdot \partial_\mu \hat{n}_2)\hat{n}_3 \\
&= \Big(\frac{e^\lambda}{\sqrt{2}}\Gamma_\mu + \frac{\hat{n}_2 \cdot \partial_\mu \hat{n}_3}{2}\Big)\hat{n}_1 - \Big(\frac{e^\lambda}{\sqrt{2}}\widetilde{\Gamma}_\mu - \frac{\hat{n}_3 \cdot \partial_\mu \hat{n}_1}{2}\Big)\hat{n}_2 + (\hat{n}_1 \cdot \partial_\mu \hat{n}_2)\hat{n}_3, \\
\hat{B}_\mu &= B^1_\mu \hat{n}_1 + B^2_\mu \hat{n}_2 - (\partial_\mu \lambda)\hat{n}_3 \\
&= \Big(\frac{e^\lambda}{\sqrt{2}}\widetilde{\Gamma}_\mu + \frac{\hat{n}_3 \cdot \partial_\mu \hat{n}_1}{2}\Big)\hat{n}_1 + \Big(\frac{e^\lambda}{\sqrt{2}}\Gamma_\mu - \frac{\hat{n}_2 \cdot \partial_\mu \hat{n}_3}{2}\Big)\hat{n}_2 - (\partial_\mu \lambda)\hat{n}_3, \\
A^1_\mu &= \frac{e^\lambda}{\sqrt{2}}(\Gamma_\mu - \widetilde{C}^1_\mu), \quad A^2_\mu = -\frac{e^\lambda}{\sqrt{2}}(\widetilde{\Gamma}_\mu - \widetilde{C}^2_\mu), \\
B^1_\mu &= \frac{e^\lambda}{\sqrt{2}}(\widetilde{\Gamma}_\mu + \widetilde{C}^2_\mu), \quad B^2_\mu = \frac{e^\lambda}{\sqrt{2}}(\Gamma_\mu + \widetilde{C}^1_\mu), \\
\widetilde{C}^1_\mu &= -\frac{e^{-\lambda}}{\sqrt{2}}\hat{n}_2 \cdot \partial_\mu \hat{n}_3, \quad \widetilde{C}^2_\mu = -\frac{e^{-\lambda}}{\sqrt{2}}\hat{n}_1 \cdot \partial_\mu \hat{n}_3, \qquad (3.71)
\end{aligned}
$$

so that

$$
\begin{aligned}
\hat{A}_\mu &= -\hat{n}_3 \times \hat{B}_\mu + \frac{1}{2}\epsilon_{ijk}(\hat{n}_i \cdot \partial_\mu \hat{n}_j)\hat{n}_k = B^2_\mu \hat{n}_1 - B^1_\mu \hat{n}_2 + \frac{1}{2}\epsilon_{ijk}(\hat{n}_i \cdot \partial_\mu \hat{n}_j)\hat{n}_k, \\
\hat{B}_\mu &= \hat{n}_3 \times \hat{A}_\mu - \partial_\mu \hat{n}_3 - (\partial_\mu \lambda)\hat{n}_3 = -A^2_\mu \hat{n}_1 + A^1_\mu \hat{n}_2 - \partial_\mu \hat{n}_3 - (\partial_\mu \lambda)\hat{n}_3. \quad (3.72)
\end{aligned}
$$

Notice that both $\hat{A}_\mu$ and $\hat{B}_\mu$ have non-vanishing $\hat{n}_3$ components.

With

$$
\hat{\mathbf{R}}_{\mu\nu} = \begin{pmatrix} \hat{A}_{\mu\nu} \\ \hat{B}_{\mu\nu} \end{pmatrix}
$$

we have

$$
\begin{aligned}
\hat{A}_{\mu\nu} &= \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + \hat{A}_\mu \times \hat{A}_\nu - \hat{B}_\mu \times \hat{B}_\nu \\
&= \frac{e^\lambda}{\sqrt{2}}(\Gamma_{\mu\nu} + H_{\mu\nu})\hat{n}_1 - \frac{e^\lambda}{\sqrt{2}}(\widetilde{\Gamma}_{\mu\nu} + \widetilde{H}_{\mu\nu})\hat{n}_2 \\
&= A^1_{\mu\nu}\hat{n}_1 + A^2_{\mu\nu}\hat{n}_2, \\
\hat{B}_{\mu\nu} &= \partial_\mu \hat{B}_\nu - \partial_\nu \hat{B}_\mu + \hat{A}_\mu \times \hat{B}_\nu - \hat{A}_\nu \times \hat{B}_\mu \\
&= \hat{D}_\mu \hat{B}_\nu - \hat{D}_\nu \hat{B}_\mu \\
&= \frac{e^\lambda}{\sqrt{2}}(\widetilde{\Gamma}_{\mu\nu} + \widetilde{H}_{\mu\nu})\hat{n}_1 + \frac{e^\lambda}{\sqrt{2}}(\Gamma_{\mu\nu} + H_{\mu\nu})\hat{n}_2 \\
&= B^1_{\mu\nu}\hat{n}_1 + B^2_{\mu\nu}\hat{n}_2, \\
\hat{A}_{\mu\nu} &= -\hat{n}_3 \times \hat{B}_{\mu\nu}, \quad \hat{B}_{\mu\nu} = \hat{n}_3 \times \hat{A}_{\mu\nu},
\end{aligned}
\tag{3.73}
$$

where

$$
\begin{aligned}
H_{\mu\nu} &= \partial_\mu \widetilde{C}^1_\nu - \partial_\nu \widetilde{C}^1_\mu \\
&= \frac{e^{-\lambda}}{\sqrt{2}}\Big(- \hat{n}_1 \cdot (\partial_\mu \hat{n}_1 \times \partial_\nu \hat{n}_1) + \hat{n}_2 \cdot (\partial_\mu \lambda \partial_\nu \hat{n}_3 - \partial_\nu \lambda \partial_\mu \hat{n}_3)\Big), \\
\widetilde{H}_{\mu\nu} &= \partial_\mu \widetilde{C}^2_\nu - \partial_\nu \widetilde{C}^2_\mu \\
&= \frac{e^{-\lambda}}{\sqrt{2}}\Big(\hat{n}_2 \cdot (\partial_\mu \hat{n}_2 \times \partial_\nu \hat{n}_2) + \hat{n}_1 \cdot (\partial_\mu \lambda \partial_\nu \hat{n}_3 - \partial_\nu \lambda \partial_\mu \hat{n}_3)\Big), \\
A^1_{\mu\nu} &= B^2_{\mu\nu} = \frac{e^\lambda}{\sqrt{2}}(\partial_\mu K_\nu - \partial_\nu K_\mu), \\
A^2_{\mu\nu} &= -B^1_{\mu\nu} = -\frac{e^\lambda}{\sqrt{2}}(\partial_\mu \widetilde{K}_\nu - \partial_\nu \widetilde{K}_\mu), \\
K_\mu &= \Gamma_\mu + \widetilde{C}^1_\mu, \quad \widetilde{K}_\mu = \widetilde{\Gamma}_\mu + \widetilde{C}^2_\mu.
\end{aligned}
\tag{3.74}
$$

Notice that both $\hat{A}_{\mu\nu}$ and $\hat{B}_{\mu\nu}$ are orthogonal to $\hat{n}_3$, although $\hat{A}_\mu$ and $\hat{B}_\mu$ are not. Here we leave again the useful formulae to obtain the above results;

$$
\begin{aligned}
\hat{D}_\mu \mathbf{j} &= 0 \qquad \hat{D}_\mu \tilde{\mathbf{j}} = 0 \\
\hat{D}_\mu \mathbf{k} &= \quad K_\mu \tilde{\mathbf{l}} + \widetilde{K}_\mu \mathbf{l} \quad = \quad (K_\mu \mathbf{j} - \widetilde{K}_\mu \tilde{\mathbf{j}}) \times \mathbf{k}
\end{aligned}
$$

$$\begin{aligned}
\hat{D}_\mu \tilde{\mathbf{k}} &= -K_\mu \mathbf{l} + \widetilde{K}_\mu \tilde{\mathbf{l}} &= (K_\mu \mathbf{j} - \widetilde{K}_\mu \tilde{\mathbf{j}}) \times \tilde{\mathbf{k}} \\
\hat{D}_\mu \mathbf{l} &= -K_\mu \tilde{\mathbf{j}} - \widetilde{K}_\mu \mathbf{j} &= (K_\mu \mathbf{j} - \widetilde{K}_\mu \tilde{\mathbf{j}}) \times \mathbf{l} \\
\hat{D}_\mu \tilde{\mathbf{l}} &= K_\mu \mathbf{j} - \widetilde{K}_\mu \tilde{\mathbf{j}} &= (K_\mu \mathbf{j} - \widetilde{K}_\mu \tilde{\mathbf{j}}) \times \tilde{\mathbf{l}}
\end{aligned} \tag{3.75}$$

In order to obtain the full gauge potential, it is sufficient to add the valence connection $\mathbf{Z}_\mu$ to the restricted potential as we did in the A_2-case;

$$\begin{gathered}
\mathbf{\Gamma}_\mu = \hat{\mathbf{\Gamma}}_\mu + \mathbf{Z}_\mu, \\
\mathbf{k} \cdot \mathbf{Z}_\mu = \tilde{\mathbf{k}} \cdot \mathbf{Z}_\mu = 0.
\end{gathered} \tag{3.76}$$

With

$$\begin{gathered}
\mathbf{Z}_\mu = J_\mu \mathbf{k} - \widetilde{J}_\mu \tilde{\mathbf{k}} + L_\mu \mathbf{l} - \widetilde{L}_\mu \tilde{\mathbf{l}}, \\
J_\mu = \mathbf{j} \cdot \mathbf{Z}_\mu, \quad \widetilde{J}_\mu = \tilde{\mathbf{j}} \cdot \mathbf{Z}_\mu, \quad L_\mu = \mathbf{l} \cdot \mathbf{Z}_\mu, \quad \widetilde{L}_\mu = \tilde{\mathbf{l}} \cdot \mathbf{Z}_\mu,
\end{gathered} \tag{3.77}$$

we have

$$\begin{gathered}
\dot{\mathbf{Z}}_{\mu\nu} = \hat{D}_\mu \mathbf{Z}_\nu - \hat{D}_\nu \mathbf{Z}_\mu \\
= U_{\mu\nu} \mathbf{j} - \widetilde{U}_{\mu\nu} \tilde{\mathbf{j}} + (\partial_\mu J_\nu - \partial_\nu J_\mu) \mathbf{k} - (\partial_\mu \widetilde{J}_\nu - \partial_\nu \widetilde{J}_\mu) \tilde{\mathbf{k}} \\
+ (\mathcal{D}_\mu L_\nu - \mathcal{D}_\nu L_\mu) \mathbf{l} - (\mathcal{D}_\mu \widetilde{L}_\nu - \mathcal{D}_\nu \widetilde{L}_\mu) \tilde{\mathbf{l}}, \\
U_{\mu\nu} = -K_\mu \widetilde{L}_\nu - \widetilde{K}_\mu L_\nu + (K_\nu \widetilde{L}_\mu + \widetilde{K}_\nu L_\mu), \\
\widetilde{U}_{\mu\nu} = K_\mu L_\nu - \widetilde{K}_\mu \widetilde{L}_\nu - (K_\nu L_\mu - \widetilde{K}_\nu \widetilde{L}_\mu), \\
\mathcal{D}_\mu L_\nu = \partial_\mu L_\nu + K_\mu \widetilde{J}_\nu + \widetilde{K}_\mu J_\nu, \\
\mathcal{D}_\mu \widetilde{L}_\nu = \partial_\mu \widetilde{L}_\nu - K_\mu J_\nu + \widetilde{K}_\mu \widetilde{J}_\nu, \\
\mathbf{Z}'_{\mu\nu} = \mathbf{Z}_\mu \times \mathbf{Z}_\nu = V_{\mu\nu} \mathbf{k} - \widetilde{V}_{\mu\nu} \tilde{\mathbf{k}}, \\
V_{\mu\nu} = J_\mu \widetilde{L}_\nu + \widetilde{J}_\mu L_\nu - (J_\nu \widetilde{L}_\mu + \widetilde{J}_\nu L_\mu), \\
\widetilde{V}_{\mu\nu} = \widetilde{J}_\mu \widetilde{L}_\nu - J_\mu L_\nu - (\widetilde{J}_\nu \widetilde{L}_\mu - J_\nu L_\mu),
\end{gathered} \tag{3.78}$$

so that

$$\begin{aligned}
\mathbf{Z}_{\mu\nu} = \dot{\mathbf{Z}}_{\mu\nu} + \mathbf{Z}'_{\mu\nu} = U_{\mu\nu}\mathbf{j} - \widetilde{U}_{\mu\nu}\tilde{\mathbf{j}} \\
+(\mathcal{D}_\mu J_\nu - \mathcal{D}_\nu J_\mu)\ \mathbf{k} - (\mathcal{D}_\mu \widetilde{J}_\nu - \mathcal{D}_\nu \widetilde{J}_\mu)\ \tilde{\mathbf{k}} \\
+(\mathcal{D}_\mu L_\nu - \mathcal{D}_\nu L_\mu)\ \mathbf{l} - (\mathcal{D}_\mu \widetilde{L}_\nu - \mathcal{D}_\nu \widetilde{L}_\mu)\ \tilde{\mathbf{l}}, \\
\mathcal{D}_\mu J_\nu = \partial_\mu J_\nu - \widetilde{L}_\mu J_\nu - L_\mu \widetilde{J}_\nu, \\
\mathcal{D}_\mu \widetilde{J}_\nu = \partial_\mu \widetilde{J}_\nu - \widetilde{L}_\mu \widetilde{J}_\nu + L_\mu J_\nu.
\end{aligned} \tag{3.79}$$

Notice that in this case the kinetic part $\dot{\mathbf{Z}}_{\mu\nu}$ contains all six components, but the potential part $\mathbf{Z}'_{\mu\nu}$ has only $\mathbf{k}$ and $\tilde{\mathbf{k}}$ components. With this we have the full field strength

$$\begin{aligned}
\mathbf{R}_{\mu\nu} = \hat{\mathbf{R}}_{\mu\nu} + \dot{\mathbf{Z}}_{\mu\nu} + \mathbf{Z}'_{\mu\nu} \\
= (\Gamma_{\mu\nu} + H_{\mu\nu} + U_{\mu\nu})\mathbf{j} - (\widetilde{\Gamma}_{\mu\nu} + \widetilde{H}_{\mu\nu} + \widetilde{U}_{\mu\nu})\tilde{\mathbf{j}} \\
+(\mathcal{D}_\mu J_\nu - \mathcal{D}_\nu J_\mu)\mathbf{k} - (\mathcal{D}_\mu \widetilde{J}_\nu - \mathcal{D}_\nu \widetilde{J}_\mu)\tilde{\mathbf{k}} \\
+(\mathcal{D}_\mu L_\nu - \mathcal{D}_\nu L_\mu)\mathbf{l} - (\mathcal{D}_\mu \widetilde{L}_\nu - \mathcal{D}_\nu \widetilde{L}_\mu)\tilde{\mathbf{l}} \\
= (\mathcal{D}_\mu K_\nu - \mathcal{D}_\nu K_\mu)\mathbf{j} - (\mathcal{D}_\mu \widetilde{K}_\nu - \mathcal{D}_\nu \widetilde{K}_\mu)\tilde{\mathbf{j}} \\
+(\mathcal{D}_\mu J_\nu - \mathcal{D}_\nu J_\mu)\mathbf{k} - (\mathcal{D}_\mu \widetilde{J}_\nu - \mathcal{D}_\nu \widetilde{J}_\mu)\tilde{\mathbf{k}} \\
+(\mathcal{D}_\mu L_\nu - \mathcal{D}_\nu L_\mu)\mathbf{l} - (\mathcal{D}_\mu \widetilde{L}_\nu - \mathcal{D}_\nu \widetilde{L}_\mu)\tilde{\mathbf{l}} \\
= K_{\mu\nu}\mathbf{j} - \widetilde{K}_{\mu\nu}\tilde{\mathbf{j}} + J_{\mu\nu}\mathbf{k} - \widetilde{J}_{\mu\nu}\tilde{\mathbf{k}} + L_{\mu\nu}\mathbf{l} - \widetilde{L}_{\mu\nu}\tilde{\mathbf{l}}, \\
\mathcal{D}_\mu K_\nu = \partial_\mu K_\nu + \widetilde{L}_\mu K_\nu + L_\mu \widetilde{K}_\nu, \\
\mathcal{D}_\mu \widetilde{K}_\nu = \partial_\mu \widetilde{K}_\nu + \widetilde{L}_\mu \widetilde{K}_\nu - L_\mu K_\nu, \\
K_{\mu\nu} = \Gamma_{\mu\nu} + H_{\mu\nu} + U_{\mu\nu} = \mathcal{D}_\mu K_\nu - \mathcal{D}_\nu K_\mu, \\
\widetilde{K}_{\mu\nu} = \widetilde{\Gamma}_{\mu\nu} + \widetilde{H}_{\mu\nu} + \widetilde{U}_{\mu\nu} = \mathcal{D}_\mu \widetilde{K}_\nu - \mathcal{D}_\nu \widetilde{K}_\mu, \\
J_{\mu\nu} = \mathcal{D}_\mu J_\nu - \mathcal{D}_\nu J_\mu, \quad \widetilde{J}_{\mu\nu} = \mathcal{D}_\mu \widetilde{J}_\nu - \mathcal{D}_\nu \widetilde{J}_\mu, \\
L_{\mu\nu} = \mathcal{D}_\mu L_\nu - \mathcal{D}_\nu L_\mu, \quad \widetilde{L}_{\mu\nu} = \mathcal{D}_\mu \widetilde{L}_\nu - \mathcal{D}_\nu \widetilde{L}_\mu,
\end{aligned} \tag{3.80}$$

or equivalently

$$\begin{aligned} R_{\mu\nu}{}^{ab} &= \mathbf{R}_{\mu\nu} \cdot \mathbf{I}^{ab} \\ &= K_{\mu\nu} j^{ab} - \widetilde{K}_{\mu\nu} \tilde{j}^{ab} + J_{\mu\nu} k^{ab} - \widetilde{J}_{\mu\nu} \tilde{k}^{ab} + L_{\mu\nu} l^{ab} - \widetilde{L}_{\mu\nu} \tilde{l}^{ab} \ . \end{aligned} \tag{3.81}$$

This is the B_2 decomposition of the curvature tensor.

In 3-dimensional notation, we have

$$\begin{gathered} \mathbf{Z}_\mu = \begin{pmatrix} \vec{X}_\mu \\ \vec{Y}_\mu \end{pmatrix}, \\ \vec{X}_\mu = \frac{e^{-\lambda}}{\sqrt{2}} (J_\mu \hat{n}_1 + \widetilde{J}_\mu \hat{n}_2) + L_\mu \hat{n}_3, \quad \vec{Y}_\mu = \frac{e^{-\lambda}}{\sqrt{2}} (\widetilde{J}_\mu \hat{n}_1 - J_\mu \hat{n}_2) + \widetilde{L}_\mu \hat{n}_3, \\ \hat{n}_1 \cdot \vec{X}_\mu + \hat{n}_2 \cdot \vec{Y}_\mu = 0, \quad \hat{n}_2 \cdot \vec{X}_\mu - \hat{n}_1 \cdot \vec{Y}_\mu = 0. \end{gathered} \tag{3.82}$$

Moreover, with

$$\mathbf{Z}_{\mu\nu} = \begin{pmatrix} \vec{X}_{\mu\nu} \\ \vec{Y}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \dot{\vec{X}}_{\mu\nu} + \vec{X}'_{\mu\nu} \\ \dot{\vec{Y}}_{\mu\nu} + \vec{Y}'_{\mu\nu} \end{pmatrix}, \tag{3.83}$$

we have

$$\begin{aligned} \dot{\vec{X}}_{\mu\nu} &= \Big\{ \frac{e^{\lambda}}{\sqrt{2}} U_{\mu\nu} + \frac{e^{-\lambda}}{\sqrt{2}} (\partial_\mu J_\nu - \partial_\nu J_\mu) \Big\} \hat{n}_1 - \Big\{ \frac{e^{\lambda}}{\sqrt{2}} \widetilde{U}_{\mu\nu} - \frac{e^{-\lambda}}{\sqrt{2}} (\partial_\mu \widetilde{J}_\nu - \partial_\nu \widetilde{J}_\mu) \Big\} \hat{n}_2 \\ &\qquad + L_{\mu\nu} \hat{n}_3, \\ \dot{\vec{Y}}_{\mu\nu} &= \Big\{ \frac{e^{\lambda}}{\sqrt{2}} \widetilde{U}_{\mu\nu} + \frac{e^{-\lambda}}{\sqrt{2}} (\partial_\mu \widetilde{J}_\nu - \partial_\nu \widetilde{J}_\mu) \Big\} \hat{n}_1 + \Big\{ \frac{e^{\lambda}}{\sqrt{2}} U_{\mu\nu} - \frac{e^{-\lambda}}{\sqrt{2}} (\partial_\mu J_\nu - \partial_\nu J_\mu) \Big\} \hat{n}_2 \\ &\qquad + \widetilde{L}_{\mu\nu} \hat{n}_3, \\ \vec{X}'_{\mu\nu} &= \frac{e^{-\lambda}}{\sqrt{2}} (V_{\mu\nu} \hat{n}_1 + \widetilde{V}_{\mu\nu} \hat{n}_2), \\ \vec{Y}'_{\mu\nu} &= \frac{e^{-\lambda}}{\sqrt{2}} (\widetilde{V}_{\mu\nu} \hat{n}_1 - V_{\mu\nu} \hat{n}_2), \end{aligned} \tag{3.84}$$

so that

$$
\begin{aligned}
\vec{X}_{\mu\nu} &= \Big(\frac{e^{\lambda}}{\sqrt{2}}U_{\mu\nu} + \frac{e^{-\lambda}}{\sqrt{2}}J_{\mu\nu}\Big)\hat{n}_1 - \Big(\frac{e^{\lambda}}{\sqrt{2}}\widetilde{U}_{\mu\nu} - \frac{e^{-\lambda}}{\sqrt{2}}\widetilde{J}_{\mu\nu}\Big)\hat{n}_2 + L_{\mu\nu}\hat{n}_3, \\
\vec{Y}_{\mu\nu} &= \Big(\frac{e^{\lambda}}{\sqrt{2}}\widetilde{U}_{\mu\nu} + \frac{e^{-\lambda}}{\sqrt{2}}\widetilde{J}_{\mu\nu}\Big)\hat{n}_1 + \Big(\frac{e^{\lambda}}{\sqrt{2}}U_{\mu\nu} - \frac{e^{-\lambda}}{\sqrt{2}}J_{\mu\nu}\Big)\hat{n}_2 + \widetilde{L}_{\mu\nu}\hat{n}_3. \qquad (3.85)
\end{aligned}
$$

Finally with

$$
\mathbf{R}_{\mu\nu} = \begin{pmatrix} \vec{A}_{\mu\nu} \\ \vec{B}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \hat{A}_{\mu\nu} + \vec{X}_{\mu\nu} \\ \hat{B}_{\mu\nu} + \vec{Y}_{\mu\nu} \end{pmatrix}, \qquad (3.86)
$$

we have

$$
\begin{aligned}
\vec{A}_{\mu\nu} &= \Big(\frac{e^{\lambda}}{\sqrt{2}}K_{\mu\nu} + \frac{e^{-\lambda}}{\sqrt{2}}J_{\mu\nu}\Big)\hat{n}_1 - \Big(\frac{e^{\lambda}}{\sqrt{2}}\widetilde{K}_{\mu\nu} - \frac{e^{-\lambda}}{\sqrt{2}}\widetilde{J}_{\mu\nu}\Big)\hat{n}_2 + L_{\mu\nu}\hat{n}_3, \\
\vec{B}_{\mu\nu} &= \Big(\frac{e^{\lambda}}{\sqrt{2}}\widetilde{K}_{\mu\nu} + \frac{e^{-\lambda}}{\sqrt{2}}\widetilde{J}_{\mu\nu}\Big)\hat{n}_1 + \Big(\frac{e^{\lambda}}{\sqrt{2}}K_{\mu\nu} - \frac{e^{-\lambda}}{\sqrt{2}}J_{\mu\nu}\Big)\hat{n}_2 + \widetilde{L}_{\mu\nu}\hat{n}_3. \qquad (3.87)
\end{aligned}
$$

This completes the B_2 decomposition of the gravitational connection.

3.5 Abelian Decomposition of Einstein's Theory

Now we are at the position to study the Abelian decomposition of Einstein's theory. As has been noted earlier, the typical Lagrangian of Yang-Mills theory for the Lorentz group makes a negative energy problem. Hence it is natural to consider the first order formalism. Let us take following lagrangian of the first order;

$$
\begin{aligned}
S[e^{\mu}{}_{a},\ \mathbf{\Gamma}_{\mu}] &= \frac{1}{16\pi G_N}\int e\ \Big(e^{\mu}{}_{a}\ e^{\nu}{}_{b}\ \mathbf{I}^{ab}\cdot\mathbf{R}_{\mu\nu}\Big)\ d^4x \\
&= \frac{1}{16\pi G_N}\int e\Big(\mathbf{g}_{\mu\nu}\cdot\mathbf{R}^{\mu\nu}\Big)\ d^4x, \\
e &= \mathrm{Det}\ (e_{\mu a}), \qquad \mathbf{g}_{\mu\nu} = e_{\mu}{}^{a}\ e_{\nu}{}^{b}\ \mathbf{I}_{ab}, \\
g_{\mu\nu}{}^{ab} &= (e_{\mu}{}^{a}e_{\nu}{}^{b} - e_{\nu}{}^{a}e_{\mu}{}^{b}) = g_{[\mu\nu]}{}^{[ab]}. \qquad (3.88)
\end{aligned}
$$

Here we have introduced the Lorentz covariant four index metric tensor $\mathbf{g}_{\mu\nu}$ (which should not be confused with the usual two index space-time metric $g_{\mu\nu}$). Since we do a gauge theory, not geometrodynamics, the most fundamental quantity is the gauge potential, not the metric $g_{\mu\nu}$. The metric $g_{\mu\nu}$ will be calculated from the vierbein fields if it needed.

From variations on the Lagrangian in the equation (3.88) we have the following equation of motion

$$\begin{aligned} \delta e_{\mu a} \quad &; \qquad \mathbf{g}_{\mu\nu} \cdot \mathbf{R}^{\nu\rho} e_{\rho a} = 0 \\ \delta \mathbf{\Gamma}_\mu \quad &; \qquad \mathscr{D}_\mu \mathbf{g}^{\mu\nu} = (\nabla_\mu + \mathbf{\Gamma}_\mu \times) \mathbf{g}^{\mu\nu} = 0, \end{aligned} \tag{3.89}$$

where $\mathscr{D}_\mu$ is generally and gauge covariant derivative.

The second equation means that our four-index metric is invariant under the parallel transport along the ∂_μ direction defined by the gauge potential $\mathbf{\Gamma}_\mu$. Imposing the torsionless condition, we can solve this equation and the solution is,

$$\begin{aligned} \mathbf{\Gamma}_\mu \cdot \mathbf{I}^{ab} = \frac{1}{2}&(e^{\nu a}\partial_\nu e_\mu{}^b + e^{\nu b}\partial_\mu e_\nu{}^a + e^{\nu a} e^{\rho b} e_{\mu c} \partial_\nu e_\rho{}^c \\ &- e^{\nu b}\partial_\nu e_\mu{}^a - e^{\nu a}\partial_\mu e_\nu{}^b - e^{\nu b} e^{\rho a} e_{\mu c} \partial_\nu e_\rho{}^c) \\ =& \frac{1}{2} e_{\mu c}(f^{cab} - f^{cba} - f^{abc}) = \Gamma_\mu{}^{ab}. \end{aligned} \tag{3.90}$$

where $f_{ab}{}^c$ is the commutation coefficients of the orthonormal basis in (3.1). This is nothing but the expression of the Levi-Civita connection in terms of vierbein fields on the tetrad basis. Therefore our gauge potential is just a spin connection of geometrodynamics. Then the meaning of the first equation in (3.89) becomes clear. Since our gauge potential is the spin connection of the chosen tetrad basis, the left hand side of the first equation becomes Ricci tensor and the equation becomes

$$R_{\mu a} = \mathbf{g}_{\mu\nu} \cdot \mathbf{R}^{\nu\rho} e_{\rho a} = 0 \ .$$

This is the Einstein's equation for a vacuum. We can recover the full Einstein's equation if we introduce the Lagrangian of matter. Therefore the equation (3.89) confirms that our theory is really Einstein's theory and we are doing the Abelian decomposition of Einstein's theory. But we cannot always impose the torsionless condition, since the spinor field generates the torsion on spacetime. It would be interested if one generalize our theory to the torsional case.

Therefore the condition that $\mathbf{g}^{\mu\nu}$ being invariant under any parallel transformation associated with gauge potential is equivalent to the metricity condition in geometrodynamics.

$$\mathscr{D}_\mu \mathbf{g}^{\mu\nu} = 0 \Longleftrightarrow \nabla_\alpha g_{\mu\nu} = 0. \tag{3.91}$$

But this condition was imposed from the beginning of our formalism. Because we have

$$D_\mu \eta_{ab} = 0. \tag{3.92}$$

Combining the above identity with following identity

$$\mathscr{D}_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a - \Gamma_{\mu\nu}{}^\alpha e_\alpha{}^a + \Gamma_\mu{}^a{}_b e_\nu{}^b = 0, \tag{3.93}$$

then, the equation $\mathscr{D}_\mu \mathbf{g}^{\mu\nu} = 0$ is reduced to

$$D_\mu \mathbf{I}^{ab} = 0, \tag{3.94}$$

So the second equation of (3.89) can actually be viewed as an identity.

3.5.1 A_2 (Non Light-like) Decomposition

Using the Abelian decomposition of the gravitational connection discussed earlier, we can express the Einstein's equation (3.89) in an adapted form for the A_2-gravity. In order to do this, let us introduce two projection operators,

$$
\begin{aligned}
\mathbf{\Sigma}_{ab} &= l_{ab}\, \mathbf{l} - \tilde{l}_{ab}\, \tilde{\mathbf{l}}, \\
\mathbf{\Pi}_{ab} = \mathbf{I}_{ab} - \mathbf{\Sigma}_{ab} &= l^1_{ab}\, \mathbf{l}_1 - \tilde{l}^1_{ab}\, \tilde{\mathbf{l}}_1 + l^2_{ab}\, \mathbf{l}_2 - \tilde{l}^2_{ab}\, \tilde{\mathbf{l}}_2, \\
\Sigma_{ab}{}^{cd} = l_{ab} l^{cd} - \tilde{l}_{ab} \tilde{l}^{cd}, &\quad \Pi_{ab}{}^{cd} = I_{ab}{}^{cd} - \Sigma_{ab}{}^{cd}, \\
\mathbf{Z}_\mu \cdot \mathbf{\Sigma}_{ab} = 0, &\quad \mathbf{Z}_\mu \cdot \mathbf{\Pi}_{ab} = Z_\mu{}^{ab}.
\end{aligned} \tag{3.95}
$$

Clearly $\mathbf{\Sigma}_{ab}$ and $\mathbf{\Pi}_{ab}$ become projection operators because

$$
\begin{aligned}
\mathbf{\Sigma}_{ab} \cdot \mathbf{\Sigma}^{cd} &= \frac{1}{2} \Sigma_{ab}{}^{mn} \Sigma_{mn}{}^{cd} = \Sigma_{ab}{}^{cd}, \\
\mathbf{\Pi}_{ab} \cdot \mathbf{\Pi}^{cd} &= \frac{1}{2} \Pi_{ab}{}^{mn} \Pi_{mn}{}^{cd} = \Pi_{ab}{}^{cd}, \\
\mathbf{\Sigma}_{ab} \cdot \mathbf{\Pi}^{cd} &= 0\ .
\end{aligned} \tag{3.96}
$$

Now we can rewrite the Einstein-Hilbert action as

$$
\begin{aligned}
S[e^\mu_a,\ A_\mu,\ B_\mu,\ \mathbf{Z}_\mu] &= \frac{1}{16\pi G_N} \int e\, \Big\{ \mathbf{g}_{\mu\nu} \cdot \mathbf{R}^{\mu\nu} \\
&+ \lambda(\mathbf{l}^2 - 1) + \tilde{\lambda}(\mathbf{l} \cdot \tilde{\mathbf{l}}) + \lambda_\mu(\mathbf{l} \cdot \mathbf{Z}^\mu) + \tilde{\lambda}_\mu(\tilde{\mathbf{l}} \cdot \mathbf{Z}^\mu) \Big\}\, d^4x, \\
\mathbf{R}_{\mu\nu} &= \hat{\mathbf{R}}_{\mu\nu} + (\hat{D}_\mu \mathbf{Z}_\nu - \hat{D}_\nu \mathbf{Z}_\mu) + \mathbf{Z}_\mu \times \mathbf{Z}_\nu \\
&= (\mathcal{D}_\mu \bar{A}_\nu - \mathcal{D}_\nu \bar{A}_\mu)\mathbf{l} - (\mathcal{D}_\mu B_\nu - \mathcal{D}_\nu B_\mu)\tilde{\mathbf{l}} + (\hat{D}_\mu \mathbf{Z}_\nu - \hat{D}_\nu \mathbf{Z}_\mu),
\end{aligned} \tag{3.97}
$$

where $\lambda's$ are the Lagrange multipliers. From this we get the following equations of motion

$$
\begin{aligned}
\delta e_{\mu c};\quad & (e^a_\mu\ e^b_\nu)[(\mathcal{D}^\nu \bar{A}^\rho - \mathcal{D}^\rho \bar{A}^\nu)\ l_{ab} - (\mathcal{D}^\nu B^\rho - \mathcal{D}^\rho B^\nu)\tilde{l}_{ab} \\
& +(\hat{D}^\nu \mathbf{Z}^\rho - \hat{D}^\rho \mathbf{Z}^\nu)\cdot \mathbf{\Pi}_{ab}]e_{\rho c} = 0, \\
\delta A_\nu;\quad & \nabla_\mu(e^\mu_a\ e^\nu_b\ l^{ab}) + \mathbf{l}\cdot(\mathbf{Z}_\mu \times \mathbf{g}^{\mu\nu}) = 0, \\
\delta B_\nu;\quad & \nabla_\mu(e^\mu_a\ e^\nu_b\ \tilde{l}^{ab}) + \tilde{\mathbf{l}}\cdot(\mathbf{Z}_\mu \times \mathbf{g}^{\mu\nu}) = 0, \\
\delta \mathbf{Z}_\nu;\quad & \hat{\mathscr{D}}_\mu(e^\mu_a\ e^\nu_b\ \mathbf{\Pi}^{ab}) + (e^\mu_a e^\nu_b)[(\mathbf{Z}_\mu \times \mathbf{l})l^{ab} \\
& -(\mathbf{Z}_\mu \times \tilde{\mathbf{l}})\tilde{l}^{ab}] = 0. \\
& \hat{\mathscr{D}}_\mu = \nabla_\mu + \hat{\mathbf{\Gamma}}_\mu \times .
\end{aligned}
\tag{3.98}
$$

Remark here that, using the isometry (3.34), the last three equations can be combined into a single equation,

$$
\mathscr{D}_\mu \mathbf{g}^{\mu\nu} = 0. \tag{3.99}
$$

In order to make the above equation clearer, let us decompose the four-index metric into the projected parts by $\mathbf{\Sigma}_{ab}$ and $\mathbf{\Pi}_{ab}$, respectively. I will call $\hat{\mathbf{g}}_{\mu\nu}$ the restricted metric of $\mathbf{g}_{\mu\nu}$ which is the projected part of $\mathbf{g}_{\mu\nu}$ by $\mathbf{\Sigma}_{ab}$. And I let $\mathbf{G}_{\mu\nu}$ be the projected part of $\mathbf{g}_{\mu\nu}$ by $\mathbf{\Pi}_{ab}$. Then

$$
\begin{aligned}
& \mathbf{g}_{\mu\nu} = \hat{\mathbf{g}}_{\mu\nu} + \mathbf{G}_{\mu\nu}, \\
& \hat{\mathbf{g}}_{\mu\nu} = e^a_\mu\ e^b_\nu\ \mathbf{\Sigma}_{ab} = G_{\mu\nu}\ \mathbf{l} - \widetilde{G}_{\mu\nu}\tilde{\mathbf{l}}, \\
& \mathbf{G}_{\mu\nu} = e^a_\mu\ e^b_\nu\ \mathbf{\Pi}_{ab} = G^1_{\mu\nu}\mathbf{l}_1 - \widetilde{G}^1_{\mu\nu}\tilde{\mathbf{l}}_1 + G^2_{\mu\nu}\mathbf{l}_2 - \widetilde{G}^2_{\mu\nu}\tilde{\mathbf{l}}_2, \\
& G_{\mu\nu} = e^a_\mu\ e^b_\nu\ l_{ab}, \quad \widetilde{G}_{\mu\nu} = e^a_\mu\ e^b_\nu\ \tilde{l}_{ab}, \\
& G^1_{\mu\nu} = e^a_\mu\ e^b_\nu\ l^1_{ab}, \quad \widetilde{G}^1_{\mu\nu} = e^a_\mu\ e^b_\nu\ \tilde{l}^1_{ab}, \\
& G^2_{\mu\nu} = e^a_\mu\ e^b_\nu\ l^2_{ab}, \quad \widetilde{G}^2_{\mu\nu} = e^a_\mu\ e^b_\nu\ \tilde{l}^2_{ab}.
\end{aligned}
\tag{3.100}
$$

Notice that

$$\widetilde{G}_{\mu\nu} = \frac{1}{2}\epsilon_{abcd}e_\mu^{\ a}\ e_\nu^{\ b}\ l^{cd} = \frac{1}{2}\epsilon_{\mu\nu cd}l^{cd} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^{\rho\sigma} = G^d_{\mu\nu},$$
$$\widetilde{G}^1_{\mu\nu} = G^{1\ d}_{\mu\nu}, \qquad \widetilde{G}^2_{\mu\nu} = G^{2\ d}_{\mu\nu}. \tag{3.101}$$

Remark that $G^{(i)}_{\mu\nu}$ and $\widetilde{G}^{(i)}_{\mu\nu}$ are dual to each other for $i = 1, 2$, so do $G_{\mu\nu}$ and $\widetilde{G}_{\mu\nu}$. With this, (3.98) has the following compact expression

$$\begin{aligned}
G_{\mu\nu}(\mathcal{D}^\nu \bar{A}^\rho - \mathcal{D}^\rho \bar{A}^\nu) - \widetilde{G}_{\mu\nu}(\mathcal{D}^\nu B^\rho - \mathcal{D}^\rho B^\nu)& \\
+\mathbf{G}_{\mu\nu} \cdot (\hat{D}^\nu \mathbf{Z}^\rho - \hat{D}^\rho \mathbf{Z}^\nu) = 0,& \\
\nabla_\mu G^{\mu\nu} + \mathbf{l} \cdot (\mathbf{Z}_\mu \times \mathbf{G}^{\mu\nu}) = 0,& \\
\nabla_\mu \widetilde{G}^{\mu\nu} + \tilde{\mathbf{l}} \cdot (\mathbf{Z}_\mu \times \mathbf{G}^{\mu\nu}) = 0,& \\
\hat{\mathscr{D}}_\mu \mathbf{G}^{\mu\nu} + \mathbf{Z}_\mu \times \hat{\mathbf{g}}^{\mu\nu} = 0,&
\end{aligned} \tag{3.102}$$

or equivalently

$$\begin{aligned}
G_{\mu\nu}(\mathcal{D}^\nu \bar{A}^\rho - \mathcal{D}^\rho \bar{A}^\nu) - \widetilde{G}_{\mu\nu}(\mathcal{D}^\nu B^\rho - \mathcal{D}^\rho B^\nu)& \\
+G^i_{\mu\nu}(\mathcal{D}^\nu Z^\rho_i - \mathcal{D}^\rho Z^\nu_i) - \widetilde{G}^i_{\mu\nu}(\mathcal{D}^\nu \widetilde{Z}^\rho_i - \mathcal{D}^\rho \widetilde{Z}^\nu_i) = 0,& \\
\nabla_\mu G^{\mu\nu} + \epsilon_{ij}(Z^i_\mu G^{\mu\nu}_j - \widetilde{Z}^i_\mu \widetilde{G}^{\mu\nu}_j) = 0,& \\
\nabla_\mu \widetilde{G}^{\mu\nu} + \epsilon_{ij}(Z^i_\mu \widetilde{G}^{\mu\nu}_j + \widetilde{Z}^i_\mu G^{\mu\nu}_j) = 0,& \\
\nabla_\mu G^{\mu\nu}_i - \epsilon_{ij}(\bar{A}_\mu G^{\mu\nu}_j - B_\mu \widetilde{G}^{\mu\nu}_j - Z^j_\mu G^{\mu\nu} + \widetilde{Z}^j_\mu \widetilde{G}^{\mu\nu}) = 0,& \\
\nabla_\mu \widetilde{G}^{\mu\nu}_i - \epsilon_{ij}(\bar{A}_\mu \widetilde{G}^{\mu\nu}_j + B_\mu G^{\mu\nu}_j - Z^j_\mu \widetilde{G}^{\mu\nu} - \widetilde{Z}^j_\mu G^{\mu\nu}) = 0.& \\
(i, j = 1, 2, \quad \epsilon_{12} = -\epsilon_{21} = 1)&
\end{aligned} \tag{3.103}$$

This suggests that the valence connection $\mathbf{Z}_\mu$ plays the role of the gravitational source of the restricted metric.

In 3-dimensional notation we have

$$
\begin{aligned}
\hat{\mathbf{g}}_{\mu\nu} &= \begin{pmatrix} \hat{m}_{\mu\nu} \\ \hat{e}_{\mu\nu} \end{pmatrix}, \quad \mathbf{G}_{\mu\nu} = \begin{pmatrix} \vec{M}_{\mu\nu} \\ \vec{E}_{\mu\nu} \end{pmatrix}, \\
\mathbf{g}_{\mu\nu} &= \begin{pmatrix} \hat{m}_{\mu\nu} + \vec{M}_{\mu\nu} \\ \hat{e}_{\mu\nu} + \vec{E}_{\mu\nu} \end{pmatrix}, \\
\hat{m}_{\mu\nu} &= G_{\mu\nu}\hat{n}, \quad \hat{e}_{\mu\nu} = \widetilde{G}_{\mu\nu}\hat{n}, \\
\vec{M}_{\mu\nu} &= G^1_{\mu\nu}\hat{n}_1 + G^2_{\mu\nu}\hat{n}_2, \\
\vec{E}_{\mu\nu} &= \widetilde{G}^1_{\mu\nu}\hat{n}_1 + \widetilde{G}^2_{\mu\nu}\hat{n}_2,
\end{aligned} \tag{3.104}
$$

so that the Einstein-Hilbert action (3.97) acquires the following form

$$
\begin{aligned}
S[e_a^\mu,\ A_\mu,\ B_\mu,\ Z^i_\mu,\ \widetilde{Z}^i_\mu] = \frac{1}{16\pi G_N}\int e\, \Big\{ & G_{\mu\nu}(\mathcal{D}^\mu \bar{A}^\nu - \mathcal{D}^\nu \bar{A}^\mu) \\
& -\widetilde{G}_{\mu\nu}(\mathcal{D}^\mu B^\nu - \mathcal{D}^\nu B^\mu) + G^i_{\mu\nu}(\mathcal{D}^\mu Z^\nu_i - \mathcal{D}^\nu Z^\mu_i) \\
& -\widetilde{G}^i_{\mu\nu}(\mathcal{D}^\mu \widetilde{Z}^\nu_i - \mathcal{D}^\nu \widetilde{Z}^\mu_i)\Big\} d^4x.
\end{aligned} \tag{3.105}
$$

From this we can reproduce (3.103). This completes the A_2 decomposition (the space-like decomposition) of Einstein's theory.

3.5.2 B_2 (Light-like) Decomposition

We can repeat the same procedure with the B_2 isometry to obtain the desired decomposition of Einstein's equation. With the Einstein-Hilbert action

$$
\begin{aligned}
S[e_a^\mu,\ \Gamma_\mu,\ \widetilde{\Gamma}_\mu,\ \mathbf{Z}_\mu] = \frac{1}{16\pi G_N}\int e\, \Big\{ & \mathbf{g}_{\mu\nu}\cdot\ \mathbf{R}_{\mu\nu} \\
& +\lambda\ \mathbf{j}^2 + \tilde{\lambda}(\mathbf{j}\cdot\tilde{\mathbf{j}}) + \lambda_\mu(\mathbf{k}\cdot\mathbf{Z}^\mu) + \tilde{\lambda}_\mu(\tilde{\mathbf{k}}\cdot\mathbf{Z}^\mu)\Big\}\ d^4x, \\
\mathbf{R}_{\mu\nu} = \hat{\mathbf{R}}_{\mu\nu} + (\hat{D}_\mu\mathbf{Z}_\nu - \hat{D}_\nu\mathbf{Z}_\mu) + \mathbf{Z}_\mu\times\mathbf{Z}_\nu & \\
= (\mathcal{D}_\mu K_\nu - \mathcal{D}_\nu K_\mu)\ \mathbf{j} - (\mathcal{D}_\mu\widetilde{K}_\nu - \mathcal{D}_\nu\widetilde{K}_\mu)\ \tilde{\mathbf{j}} & \\
+(\mathcal{D}_\mu J_\nu - \mathcal{D}_\nu J_\mu)\mathbf{k} - (\mathcal{D}_\mu\widetilde{J}_\nu - \mathcal{D}_\nu\widetilde{J}_\mu)\tilde{\mathbf{k}} &
\end{aligned}
$$

$$+(\mathcal{D}_\mu L_\nu - \mathcal{D}_\nu L_\mu)\mathbf{l} - (\mathcal{D}_\mu \widetilde{L}_\nu - \mathcal{D}_\nu \widetilde{L}_\mu)\tilde{\mathbf{l}}, \tag{3.106}$$

we get following equations of motion

$$\begin{aligned}
&\delta e_{\mu c}\ ;\ (e^a_\mu\ e^b_\nu)\Big[(\mathcal{D}^\nu K^\rho - \mathcal{D}^\rho K^\nu)\ j_{ab} - (\mathcal{D}^\nu \widetilde{K}^\rho - \mathcal{D}^\rho \widetilde{K}^\nu)\ \tilde{j}_{ab} + \mathbf{Z}^{\nu\rho}\cdot \mathbf{\Pi}_{ab}\Big] e_{\rho c} = 0,\\
&\delta\Gamma_\nu\ ;\ \nabla_\mu(e^\mu_a\ e^\nu_b\ j^{ab}) + \mathbf{j}\cdot(\mathbf{Z}_\mu \times \mathbf{g}^{\mu\nu}) = 0,\\
&\delta\widetilde{\Gamma}_\nu\ ;\ \nabla_\mu(e^\mu_a\ e^\nu_b\ \tilde{j}^{ab}) + \tilde{\mathbf{j}}\cdot(\mathbf{Z}_\mu \times \mathbf{g}^{\mu\nu}) = 0,\\
&\delta\mathbf{Z}_\nu\ ;\ \hat{\mathscr{D}}_\mu(e^\mu_a\ e^\nu_b\ \mathbf{\Pi}^{ab}) + \mathbf{Z}_\mu \times (e^\mu_a\ e^\nu_b)(k^{ab}\mathbf{j} - \tilde{k}^{ab}\tilde{\mathbf{j}})\\
&\qquad = (e^\mu_a\ e^\nu_b)(j^{ab}\hat{D}_\mu \mathbf{k} - \tilde{j}^{ab}\hat{D}_\mu\tilde{\mathbf{k}}),
\end{aligned} \tag{3.107}$$

where now

$$\begin{aligned}
&\mathbf{\Sigma}_{ab} = j_{ab}\ \mathbf{k} - \tilde{j}_{ab}\ \tilde{\mathbf{k}},\\
&\mathbf{\Pi}_{ab} = k_{ab}\ \mathbf{j} - \tilde{k}_{ab}\ \tilde{\mathbf{j}} + l_{ab}\ \mathbf{l} - \tilde{l}_{ab}\ \tilde{\mathbf{l}} = \mathbf{I}_{ab} - \mathbf{\Sigma}_{ab},\\
&\mathbf{Z}_\mu \cdot \mathbf{\Sigma}^{ab} = 0, \quad \mathbf{Z}_\mu \cdot \mathbf{\Pi}^{ab} = Z_\mu{}^{ab}.
\end{aligned} \tag{3.108}$$

But notice here that $\mathbf{\Pi}_{ab}$ and $\mathbf{\Sigma}_{ab}$ do not make projection operators, because

$$\mathbf{\Pi}_{ab}\cdot \mathbf{\Sigma}^{cd} = k_{ab}\ j^{cd} - \tilde{k}_{ab}\ \tilde{j}^{cd} \neq 0. \tag{3.109}$$

Now, again we can combine the last three equations of (3.107) into a single equation with the isometry (3.63),

$$\mathscr{D}_\mu \mathbf{g}^{\mu\nu} = 0.$$

This confirms that (3.107) is equivalent to (3.89), which tells that (3.106) describes the Einstein's gravity.

Now, with

$$\mathbf{g}_{\mu\nu} = \hat{\mathbf{g}}_{\mu\nu} + \mathbf{G}_{\mu\nu},$$

$$
\begin{aligned}
\hat{\mathbf{g}}_{\mu\nu} &= e_\mu^a\ e_\nu^b\ \mathbf{\Sigma}^{ab} = \mathcal{J}_{\mu\nu}\ \mathbf{k} - \widetilde{\mathcal{J}}_{\mu\nu}\ \tilde{\mathbf{k}}, \\
\mathbf{G}_{\mu\nu} &= e_\mu^a\ e_\nu^b\ \mathbf{\Pi}^{ab} = \mathcal{K}_{\mu\nu}\ \mathbf{j} - \widetilde{\mathcal{K}}_{\mu\nu}\ \tilde{\mathbf{j}} + \mathcal{L}_{\mu\nu}\ \mathbf{l} - \widetilde{\mathcal{L}}_{\mu\nu}\ \tilde{\mathbf{l}}, \\
\mathcal{J}_{\mu\nu} &= e_\mu^a\ e_\nu^b\ j_{ab}, \quad \widetilde{\mathcal{J}}_{\mu\nu} = e_\mu^a\ e_\nu^b\ \tilde{j}_{ab}, \\
\mathcal{K}_{\mu\nu} &= e_\mu^a\ e_\nu^b\ k_{ab}, \quad \widetilde{\mathcal{K}}_{\mu\nu} = e_\mu^a\ e_\nu^b\ \tilde{k}_{ab}, \\
\mathcal{L}_{\mu\nu} &= e_\mu^a\ e_\nu^b\ l_{ab}, \quad \widetilde{\mathcal{L}}_{\mu\nu} = e_\mu^a\ e_\nu^b\ \tilde{l}_{ab},
\end{aligned} \tag{3.110}
$$

the equation (3.107) is written as

$$
\begin{gathered}
\mathcal{J}_{\mu\nu}(\mathcal{D}^\nu K^\rho - \mathcal{D}^\rho K^\nu) - \widetilde{\mathcal{J}}_{\mu\nu}(\mathcal{D}^\nu \widetilde{K}^\rho - \mathcal{D}^\rho \widetilde{K}^\nu) + \mathbf{G}_{\mu\nu} \cdot \mathbf{Z}^{\nu\rho} = 0, \\
\nabla_\mu \mathcal{J}^{\mu\nu} + \mathbf{j} \cdot (\mathbf{Z}_\mu \times \mathbf{g}^{\mu\nu}) = 0, \\
\nabla_\mu \widetilde{\mathcal{J}}^{\mu\nu} + \tilde{\mathbf{j}} \cdot (\mathbf{Z}_\mu \times \mathbf{g}^{\mu\nu}) = 0, \\
\hat{\mathscr{D}}_\mu \mathbf{G}^{\mu\nu} + \mathbf{Z}_\mu \times (\mathcal{K}^{\mu\nu}\ \mathbf{j} - \widetilde{\mathcal{K}}^{\mu\nu}\ \tilde{\mathbf{j}}) = -\mathcal{J}^{\mu\nu} \hat{D}_\mu \mathbf{k} + \widetilde{\mathcal{J}}^{\mu\nu} \hat{D}_\mu \tilde{\mathbf{k}},
\end{gathered} \tag{3.111}
$$

or equivalently

$$
\begin{gathered}
\mathcal{J}_{\mu\nu}(\mathcal{D}^\nu K^\rho - \mathcal{D}^\rho K^\nu) - \widetilde{\mathcal{J}}_{\mu\nu}(\mathcal{D}^\nu \widetilde{K}^\rho - \mathcal{D}^\rho \widetilde{K}^\nu) \\
+\mathcal{K}_{\mu\nu}(\mathcal{D}^\nu J^\rho - \mathcal{D}^\rho J^\nu) - \widetilde{\mathcal{K}}_{\mu\nu}(\mathcal{D}^\nu \widetilde{J}^\rho - \mathcal{D}^\rho \widetilde{J}^\nu) \\
+\mathcal{L}_{\mu\nu}(\mathcal{D}^\nu L^\rho - \mathcal{D}^\rho L^\nu) - \widetilde{\mathcal{L}}_{\mu\nu}(\mathcal{D}^\nu \widetilde{L}^\rho - \mathcal{D}^\rho \widetilde{L}^\nu) = 0, \\
\nabla_\mu \mathcal{J}^{\mu\nu} - L_\mu \widetilde{\mathcal{J}}^{\mu\nu} - \widetilde{L}_\mu \mathcal{J}^{\mu\nu} + J_\mu \widetilde{\mathcal{L}}^{\mu\nu} + \widetilde{J}_\mu \mathcal{L}^{\mu\nu} = 0, \\
\nabla_\mu \widetilde{\mathcal{J}}^{\mu\nu} + L_\mu \mathcal{J}^{\mu\nu} - \widetilde{L}_\mu \widetilde{\mathcal{J}}^{\mu\nu} - J_\mu \mathcal{L}^{\mu\nu} + \widetilde{J}_\mu \widetilde{\mathcal{L}}^{\mu\nu} = 0, \\
\nabla_\mu \mathcal{K}^{\mu\nu} + L_\mu \widetilde{\mathcal{K}}^{\mu\nu} + \widetilde{L}_\mu \mathcal{K}^{\mu\nu} = K_\mu \widetilde{\mathcal{L}}^{\mu\nu} + \widetilde{K}_\mu \mathcal{L}^{\mu\nu}, \\
\nabla_\mu \widetilde{\mathcal{K}}^{\mu\nu} - L_\mu \mathcal{K}^{\mu\nu} + \widetilde{L}_\mu \widetilde{\mathcal{K}}^{\mu\nu} = -K_\mu \mathcal{L}^{\mu\nu} + \widetilde{K}_\mu \widetilde{\mathcal{L}}^{\mu\nu}, \\
\nabla_\mu \mathcal{L}^{\mu\nu} - J_\mu \widetilde{\mathcal{K}}^{\mu\nu} - \widetilde{J}_\mu \mathcal{K}^{\mu\nu} = -K_\mu \widetilde{\mathcal{J}}^{\mu\nu} - \widetilde{K}_\mu \mathcal{J}^{\mu\nu}, \\
\nabla_\mu \widetilde{\mathcal{L}}^{\mu\nu} + J_\mu \mathcal{K}^{\mu\nu} - \widetilde{J}_\mu \widetilde{\mathcal{K}}^{\mu\nu} = K_\mu \mathcal{J}^{\mu\nu} - \widetilde{K}_\mu \widetilde{\mathcal{J}}^{\mu\nu}.
\end{gathered} \tag{3.112}
$$

Remember that $\mathcal{J}_{\mu\nu}$, $\mathcal{K}_{\mu\nu}$, $\mathcal{L}_{\mu\nu}$ and $\widetilde{\mathcal{J}}_{\mu\nu}$, $\widetilde{\mathcal{K}}_{\mu\nu}$, $\widetilde{\mathcal{L}}_{\mu\nu}$ are dual to each other. Here again the valence connection becomes the gravitational source of the restricted metric.

In 3-dimensional notation we have

$$
\begin{aligned}
\hat{\mathbf{g}}_{\mu\nu} &= \begin{pmatrix} \hat{m}_{\mu\nu} \\ \hat{e}_{\mu\nu} \end{pmatrix}, \quad \mathbf{G}_{\mu\nu} = \begin{pmatrix} \vec{M}_{\mu\nu} \\ \vec{E}_{\mu\nu} \end{pmatrix}, \\
\mathbf{g}_{\mu\nu} &= \begin{pmatrix} \hat{m}_{\mu\nu} + \vec{M}_{\mu\nu} \\ \hat{e}_{\mu\nu} + \vec{E}_{\mu\nu} \end{pmatrix}, \\
\hat{m}_{\mu\nu} &= \frac{e^{-\lambda}}{\sqrt{2}}(\mathcal{J}_{\mu\nu}\hat{n}_1 + \widetilde{\mathcal{J}}_{\mu\nu}\hat{n}_2) = \hat{n}_3 \times \hat{e}_{\mu\nu}, \\
\hat{e}_{\mu\nu} &= \frac{e^{-\lambda}}{\sqrt{2}}(\widetilde{\mathcal{J}}_{\mu\nu}\hat{n}_1 - \mathcal{J}_{\mu\nu}\hat{n}_2) = -\hat{n}_3 \times \hat{m}_{\mu\nu}, \\
\vec{M}_{\mu\nu} &= \frac{e^{\lambda}}{\sqrt{2}}(\mathcal{K}_{\mu\nu}\hat{n}_1 - \widetilde{\mathcal{K}}_{\mu\nu}\hat{n}_2) + \mathcal{L}_{\mu\nu}\hat{n}_3, \\
\vec{E}_{\mu\nu} &= \frac{e^{\lambda}}{\sqrt{2}}(\widetilde{\mathcal{K}}_{\mu\nu}\hat{n}_1 + \mathcal{K}_{\mu\nu}\hat{n}_2) + \widetilde{\mathcal{L}}_{\mu\nu}\hat{n}_3,
\end{aligned}
\tag{3.113}
$$

so that the Einstein-Hilbert action (3.106) is expressed as

$$
\begin{aligned}
S[e_a^\mu,\ K_\mu,\ \widetilde{K}_\mu,\ J_\mu,\ \widetilde{J}_\mu,\ L_\mu,\ \widetilde{L}_\mu] = \frac{1}{16\pi G_N}\int e\ \Big\{&\mathcal{J}_{\mu\nu}(\mathcal{D}^\mu K^\nu - \mathcal{D}^\nu K^\mu) \\
&-\widetilde{\mathcal{J}}_{\mu\nu}(\mathcal{D}^\mu \widetilde{K}^\nu - \mathcal{D}^\nu \widetilde{K}^\mu) + \mathcal{K}_{\mu\nu}(\mathcal{D}^\mu J^\nu - \mathcal{D}^\nu J^\mu) \\
&-\widetilde{\mathcal{K}}_{\mu\nu}(\mathcal{D}^\mu \widetilde{J}^\nu - \mathcal{D}^\nu \widetilde{J}^\mu) + \mathcal{L}_{\mu\nu}(\mathcal{D}^\mu L^\nu - \mathcal{D}^\nu L^\mu) \\
&-\widetilde{\mathcal{L}}_{\mu\nu}(\mathcal{D}^\mu \widetilde{L}^\nu - \mathcal{D}^\nu \widetilde{L}^\mu)\Big\}\ d^4x.
\end{aligned}
\tag{3.114}
$$

From this we can reproduce (3.112). This completes the B_2 decomposition (the light-like decomposition) of Einstein's theory.

3.6 Restricted Gravity

As we discussed in the introduction, the most difficulty of Einstein's theory is caused by the non-linearity of its field equation. Since the superposition principle does not hold any more, we cannot be sure that the superposition of two gravitational wave becomes wave. Indeed this is not true. Two head-on colliding gravitational waves can make the Schwarzchild solution. Unfortunately, we do not have any successful theory which enables us to understand gravitational waves on the same level of electromagnetic waves yet.

Therefore it would be helpful if we have a simpler equations of motion for gravitational waves with the full general covariance, equivalently, the full gauge degrees of freedom. The main purpose of this section is to provide such minimal equations of motion for gravitational waves. What we need to get the minimal equations is just a restriction $\mathbf{Z}_\mu = 0$ in both restricted theories A_2 and B_2. Since this restriction filters out unnecessary parts of the full equations of motion, we get the equations of motion which contain only the connections related to the gravitational waves, not the sources. Then we obtain the Maxwell-Einstein's equations for gravitational waves, and we can introduce 1-form potentials which describe gravitational waves.

An interesting point is that the same Maxwell-Einstein equations appear in the A_2-gravity as well as B_2-gravity. This looks strange because A_2 has a non-light-like isometry. We can understand this situation if we recognize that the vacuum itself also satisfies the wave equation. Therefore there is a possibility that the fundamental solution for the vacuous A_2-gravity is just a Minkowskian spacetime. B_2-gravity, however, has non-trivial solutions of gravitational waves as we shall see in the chapter 4. Furthermore all Petrov N-type solutions are contained in the B_2-gravity. Therefore we can say that the restricted equations of motion in B_2-gravity is the desired equations for gravitational waves. But as of now it is not clear whether there exists a solution for a gravitational wave which can be described by A_2-gravity.

3.6.1 A_2 Gravity

Let us consider the A_2 decomposition first. As mentioned earlier, the valence potential $\mathbf{Z}_\mu$ can be interpreted as a source for the restricted potential since it transforms covariantly under the gauge transformation, that is, the Lorentz transformation of the orthonormal frame. In order to obtain the restricted gravity for the A_2 gravity, it is sufficient to impose $\mathbf{Z}_\mu = 0$ in (3.98);

$$
\begin{aligned}
G_{\mu\nu}(\partial^\nu \bar{A}^\rho - \partial^\rho \bar{A}^\nu) - \widetilde{G}_{\mu\nu}(\partial^\nu B^\rho - \partial^\rho B^\nu) = 0, \\
\nabla_\mu G^{\mu\nu} = 0, \qquad \nabla_\mu \widetilde{G}^{\mu\nu} = 0, \\
\hat{\mathscr{D}}_\mu \mathbf{G}^{\mu\nu} = 0.
\end{aligned}
\tag{3.115}
$$

These are the equations of motion for the A_2 gravity.

Remark that the equations on the first and third lines of (3.115) are first-ordered so that they cannot describe any dynamical graviton. Hence they become the constraint equations for the system. But the equations on the second line are Einstein-Maxwell equations. They can describe the graviton in the following way. Since $G_{\mu\nu}$ and $\tilde{G}_{\mu\nu}$ are dual to each other and the 4-dimensional divergence of $\tilde{G}_{\mu\nu}$ vanishes, we can introduce a 1-form vector potential G_μ which satisfies,

$$
G_{\mu\nu} = \nabla_\mu G_\nu - \nabla_\nu G_\mu = \partial_\mu G_\nu - \partial_\nu G_\mu, \tag{3.116}
$$

In the same way, there exists a 1-form vector potential $\tilde{G}_\mu$ which satisfies,

$$
\widetilde{G}_{\mu\nu} = \nabla_\mu \tilde{G}_\nu - \nabla_\nu \tilde{G}_\mu = \partial_\mu \tilde{G}_\nu - \partial_\nu \tilde{G}_\mu, \tag{3.117}
$$

So we can express the equations of the restricted metric $G_{\mu\nu}$ and $\widetilde{G}_{\mu\nu}$ (the $\mathbf{l}$ and $\tilde{\mathbf{l}}$ components of the Lorentz covariant metric $\mathbf{g}_{\mu\nu}$) as a Maxwell-type second order differential equation in terms of the potential G_μ,

$$\nabla_\mu G^{\mu\nu} = 0, \qquad G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu. \tag{3.118}$$

This is really remarkable and surprising, because this shows that the dynamical part of A_2 gravity can be described by an Abelian gauge theory.

3.6.2 B_2 Gravity

In this section we repeat the same procedure of the previous section for the B_2-gravity. With $\mathbf{Z}_\mu = 0$, we get the restricted equations of motion for the B_2-gravity as following;

$$\begin{gathered}
\mathcal{J}_{\mu\nu}(\partial^\nu K^\rho - \partial^\rho K^\nu) - \widetilde{\mathcal{J}}_{\mu\nu}(\partial^\nu \widetilde{K}^\rho - \partial^\rho \widetilde{K}^\nu) = 0, \\
\nabla_\mu \mathcal{J}^{\mu\nu} = 0, \qquad \nabla_\mu \widetilde{\mathcal{J}}^{\mu\nu} = 0, \\
\hat{\mathscr{D}}_\mu \mathbf{G}^{\mu\nu} + \mathcal{J}^{\mu\nu} \hat{D}_\mu \mathbf{k} - \widetilde{\mathcal{J}}^{\mu\nu} \hat{D}_\mu \tilde{\mathbf{k}} = 0 \ .
\end{gathered} \tag{3.119}$$

Here again equations on the first and the third lines can be viewed as the constraint equations for the system. But the two equations for $\mathcal{J}_{\mu\nu}$ and $\widetilde{\mathcal{J}}_{\mu\nu}$ on the second line enables us to introduce a 1-form potential $\mathcal{J}_\mu$ for $\mathcal{J}_{\mu\nu}$

$$\mathcal{J}_{\mu\nu} = \partial_\mu \mathcal{J}_\nu - \partial_\nu \mathcal{J}_\mu, \tag{3.120}$$

and $\widetilde{\mathcal{J}}_\mu$ for $\widetilde{\mathcal{J}}_{\mu\nu}$

$$\widetilde{\mathcal{J}}_{\mu\nu} = \partial_\mu \widetilde{\mathcal{J}}_\nu - \partial_\nu \widetilde{\mathcal{J}}_\mu \ . \tag{3.121}$$

With this we can express the equations of the restricted metric $\mathcal{J}_{\mu\nu}$ and $\widetilde{\mathcal{J}}_{\mu\nu}$ (the $\mathbf{j}$ and $\tilde{\mathbf{j}}$ components of the Lorentz covariant metric $\mathbf{g}_{\mu\nu}$) as a Maxwell-type second order differential equation in terms of the potential $\mathcal{J}_\mu$,

$$\nabla_\mu \mathcal{J}^{\mu\nu} = 0, \qquad \mathcal{J}_{\mu\nu} = \partial_\mu \mathcal{J}_\nu - \partial_\nu \mathcal{J}_\mu. \tag{3.122}$$

This shows that the dynamical part of B_2 gravity can also be described by an Abelian gauge theory.

Chapter 4

Fundamental Solutions of Restricted Gravity

In this chapter we will investigate two fundamental solutions of restricted gravity. Supposing the simplest trial solutions for each type of restricted gravity, we will get the cosmic string solution for A_2-case and the Einstein-Rosen-Bondi wave for B_2-case. The cosmic string solution is the simplest example which shows us how the magnetic potential of a restricted connection describes topological properties of spacetime. And the Einstein-Rosen-Bondi wave gives us a new method to describe gravitational waves by a 1-form vector potential. In the last section of this chapter we will generalize the above result and prove that general pp-waves will contained in B_2-gravity. And we will see that all radiative solutions of Einstein's equation(Petrov N-type) will contained in $B-2$-gravity.

4.1 A fundamental solution for A_2-gravity : Cosmic String

Since the isometry vectors of A_2 gravity are the rotation generator about z-axis and the boosting generator along the z-axis, it is natural to take the cylindrical coordinates system (t, z, ρ, ϕ) to get a fundamental solution of A_2-gravity. And let us take following tetrads.

$$\mathbf{e}_{\hat{t}} = e^{-\tau}\partial_t, \quad \mathbf{e}_{\hat{\rho}} = e^{-R}\partial_\rho, \quad \mathbf{e}_{\hat{\phi}} = \frac{e^{-\Phi}}{\rho}\partial_\phi, \quad \mathbf{e}_{\hat{z}} = e^{-\zeta}\partial_z. \tag{4.1}$$

From the equation (3.90), spin-connection coefficients for the above tetrads are,

$$\begin{aligned}
\Gamma_\mu{}^{\hat{\rho}\hat{\phi}} &= -\Gamma_\mu{}^{\hat{\phi}\hat{\rho}} = \frac{e^{R-\Phi}\partial_\phi R}{\rho}\partial_\mu\rho - e^{\Phi-R}(1+\rho\partial_\rho\Phi)\ \partial_\mu\phi, \\
\Gamma_\mu{}^{\hat{\phi}\hat{z}} &= -\Gamma_\mu{}^{\hat{z}\hat{\phi}} = e^{\Phi-\zeta}\rho\ \partial_z\Phi\ \partial_\mu\phi - \frac{e^{\zeta-\Phi}\partial_\phi\zeta}{\rho}\ \partial_\mu z, \\
\Gamma_\mu{}^{\hat{z}\hat{\rho}} &= -\Gamma_\mu{}^{\hat{\rho}\hat{z}} = e^{\zeta-R}\ \partial_\rho\zeta\ \partial_\mu z - e^{R-\zeta}\ \partial_z R\ \partial_\mu\rho, \\
\Gamma_\mu{}^{\hat{t}\hat{\rho}} &= -\Gamma_\mu{}^{\hat{\rho}\hat{t}} = e^{\tau-R}\ \partial_\rho\tau\ \partial_\mu t - e^{R-\tau}\ \partial_t R\ \partial_\mu\rho, \\
\Gamma_\mu{}^{\hat{t}\hat{\phi}} &= -\Gamma_\mu{}^{\hat{\phi}\hat{t}} = \frac{e^{\tau-R}\ \partial_\phi\tau}{\rho}\ \partial_\mu t - e^{\Phi-\tau}\rho\ \partial_t\Phi\ \partial_\mu\phi, \\
\Gamma_\mu{}^{\hat{t}\hat{z}} &= -\Gamma_\mu{}^{\hat{z}\hat{t}} = e^{\tau-\zeta}\ \partial_z\tau\ \partial_\mu t - e^{\zeta-\tau}\ \partial_t\zeta\ \partial_\mu z.
\end{aligned} \tag{4.2}$$

Now let's concentrate on the cosmic string, a uniform mass distribution along an infinitely long and straight line, say, z-axis. For the sake of simplicity we will suppose a uniform mass density for the internal solution, so our energy-momentum tensor for the internal and external cases become;

$$T_\mu{}^\nu = \begin{cases} -\epsilon(\partial_\mu t\ \partial^\nu t + \partial_\mu z\ \partial^\nu z) & \text{if } \rho < \rho_0\ , \\ 0 & \text{if } \rho > \rho_0\ . \end{cases} \tag{4.3}$$

where ϵ is the mass density of the string.

4.1.1 Exterior solution for the cosmic string

Let us solve the exterior case first. To get the most transparent picture, let us choose following three dimensional basis for the Lorentz group,

$$\hat{\mathbf{n}}_1 = \hat{z} \ , \quad \hat{\mathbf{n}}_2 = \hat{\rho} \ , \quad \hat{\mathbf{n}} = \hat{\mathbf{n}}_3 = \hat{\phi} = \begin{pmatrix} -\sin a\phi \\ \cos a\phi \\ 0 \end{pmatrix} . \tag{4.4}$$

with a constant a. Here, remark that $\hat{\phi}$ is not parametrized by just ϕ, but $a\phi$. This constant a will be related to the mass density, that is, the tension of the string. Then our restricted gauge potential is,

$$\hat{\mathbf{\Gamma}}_\mu = \begin{pmatrix} A_\mu \hat{\phi} - \hat{\phi} \times \partial_\mu \hat{\phi} \\ B_\mu \hat{\phi} \end{pmatrix} . \tag{4.5}$$

From the equation(3.9), we can calculate spin-connections,

$$\begin{aligned} \Gamma_\mu{}^{\hat{z}\hat{\rho}} &= -\Gamma_\mu{}^{\hat{\rho}\hat{z}} = A_\mu, \\ \Gamma_\mu{}^{\hat{\rho}\hat{\phi}} &= -\Gamma_\mu{}^{\hat{\phi}\hat{\rho}} = a\ \partial_\mu \phi, \\ \Gamma_\mu{}^{\hat{t}\hat{\phi}} &= -\Gamma_\mu{}^{\hat{\phi}\hat{t}} = -B_\mu, \end{aligned} \tag{4.6}$$

and the other coefficients vanish. As the simplest ansatz, let's assume,

$$A_\mu = B_\mu = 0. \tag{4.7}$$

Using (4.7) and comparing (4.2) with (130),

$$R = R(\rho), \quad \Phi = \Phi(\rho), \quad \zeta = \zeta(z), \quad \tau = \tau(t) \ , \tag{4.8}$$

and

$$e^{\Phi-R}(1+\rho\partial_\rho\Phi)=a. \tag{4.9}$$

Because of the equations in (4.8), we can change variables to rescale line elements without loss of any generality;

$$\tau=R=\zeta=0. \tag{4.10}$$

And the equation (4.9) is easy to solve, the solution is,

$$e^\Phi=a \tag{4.11}$$

Then our line element is reduced into,

$$ds^2=-dt^2+d\rho^2+a^2\rho^2d\phi^2+dz^2. \tag{4.12}$$

Since our gauge potentials are just constant, all field strength vanish and so is the Ricci tensor. Therefore this solution automatically satisfies the Einstein's equation for the exterior case. This is the same result of Hiscock(1985). Even though the above solution describes a Minkowskian spacetime, but it has a topological singularity. Here the constant a determines the intensity of conical singularity and it will be related to the mass density of the cosmic string by junction conditions.

4.1.2 Interior solution for the cosmic string

As we studied in the previous chapter, our valence potentials play a role of gravitational source to the restricted part. So one might guess that the interior solution cannot be included in the restricted gravity. But this is not so. Because whether some part of full gauge connection is a valence connection or not depends on our choice of the isometry direction. So if there exists a choice of the isometry vector in which basis the potential under consideration has the form of restricted potential (3.38), then that potential surely can be described by a restricted theory. At the end of this section we will recognize that the interior solution of cosmic string can be described by the restricted theory. But it is instructive to introduce a valence potential for a later discussion. Of course, we will see how to choose the isometry vector to include this valence connection in its restricted part.

The choice of three dimensional basis for the Lorentz group (4.4) is same for the exterior case. But now we consider a special ansatz for the valence connection;

$$\mathbf{Z}_\mu = \begin{pmatrix} (1 - f(\rho))\ \hat{\phi} \times \partial_\mu \hat{\phi} \\ 0 \end{pmatrix} \tag{4.13}$$

Remark the rotation part of the above ansatz $(1 - f(\rho))\ \hat{\phi} \times \partial_\mu \hat{\phi}$. This kind of potential appears in many papers in gauge theories. For example, G. 't Hooft used this ansatz to obtain a magnetic monopole solution in $O(3)$-gauge theory, and Y. M. Cho and D. Maison used this ansatz to obtain a magnetic monopole solution in Weinberg-Salam model[5]. Especially this potential is very useful to analyze a topological property of the system. Later this kind of potential will be very important when we obtain the Schwarzchild solution. Our full gauge potential is,

$$\mathbf{\Gamma}_\mu = \begin{pmatrix} A_\mu \hat{\phi} + f(\rho)\ \hat{\phi} \times \partial_\mu \hat{\phi} \\ B_\mu \hat{\phi} \end{pmatrix} = \begin{pmatrix} f(\rho)\ \hat{\phi} \times \partial_\mu \hat{\phi} \\ 0 \end{pmatrix} . \tag{4.14}$$

For the sake of simplicity, let's assume that $A_\mu = B_\mu = 0$ as the exterior case. And with the basis (4.4) and the above ansatz, our connection coefficients become

$$\Gamma_\mu{}^{\hat{\rho}\hat{\phi}} = -\Gamma_\mu{}^{\hat{\phi}\hat{\rho}} = a f(\rho)\ \partial_\mu \phi \ . \tag{4.15}$$

and all other coefficients vanish. Here the constant a can be absorbed into the function $f(\rho)$. So we can set $a = 1$ simply. Comparing (4.15) with coefficients in (4.2), we get

$$R = R(\rho), \quad \Phi = \Phi(\rho), \quad \zeta = \zeta(z), \quad \tau = \tau(z) \ , \tag{4.16}$$

and

$$f(\rho) = e^{\Phi - R}(1 + \rho \frac{d}{d\rho}\Phi) \ . \tag{4.17}$$

Here we can set $\tau = R = \zeta = 0$ without loss of any generality, and our line element is reduced into,

$$ds^2 = -dt^2 + d\rho^2 + e^{2\Phi}\rho^2 d\phi^2 + dz^2, \tag{4.18}$$

with

$$f(\rho) = e^{\Phi}(1 + \rho\partial_\rho\Phi) = \frac{d}{d\rho}(\rho e^{\Phi}) \equiv \frac{d}{d\rho}g(\rho) \ . \tag{4.19}$$

where $g(\rho) = \rho e^{\Phi}$. From the equation (4.13) we have,

$$\begin{aligned} \mathbf{R}_{\mu\nu} &= \partial_\mu \mathbf{\Gamma}_\nu - \partial_\nu \mathbf{\Gamma}_\mu + \mathbf{\Gamma}_\mu \times \mathbf{\Gamma}_\nu \\ &= (\partial_\mu \rho \ \partial_\nu \phi - \partial_\mu \phi \ \partial_\nu \rho) g''(\rho) \mathbf{l}_1 \end{aligned} \tag{4.20}$$

where the prime denotes $\frac{d}{d\rho}$. Then the Ricci tensor is,

$$R_{\mu\nu} = g''(\partial_\mu \rho \ \partial_\nu \rho + \frac{g^2}{\rho} \partial_\mu \phi \ \partial_\nu \phi) \ . \tag{4.21}$$

Then the Einstein's equation with the interior energy-momentum tensor in (4.3) is reduced into a single equation as following;

$$g'' + 8\pi\epsilon g = 0 \ . \tag{4.22}$$

This is nothing but the equation of motion for a 1-dimensional simple harmonic oscillator. So,

$$g(\rho) = A \sin\left(\frac{\rho}{\rho_*}\right) + B \cos\left(\frac{\rho}{\rho_*}\right) \tag{4.23}$$

where $\rho_* = 1/\sqrt{8\pi\epsilon}$. Since the metric on the axis will be flat, that is, there is no cone singularity,

$$A = \rho_* \ , \quad B = 0 \ . \tag{4.24}$$

Finally our metric becomes,

$$ds^2 = -dt^2 + d\rho^2 + \rho_*^2 \sin^2\left(\frac{\rho}{\rho_*}\right) d\phi^2 + dz^2. \tag{4.25}$$

According to the argument of Hiscock[11], the condition to have no stress-energy on the boundary surface is,

$$a = \cos\left(\frac{\rho_0}{\rho_*}\right) \ . \tag{4.26}$$

Then the mass per unit length μ is,

$$\mu = \int_0^{\rho_0} \int_0^{2\pi} \epsilon \rho_* \sin(\rho/\rho_*) d\phi d\rho = 2\pi\epsilon\rho_*^2 (1 - \cos(\rho/\rho_*)) \ . \tag{4.27}$$

From the above equation with $\rho_* = 1/\sqrt{8\pi\epsilon}$, we get the relation between the constant a and the tension μ,

$$a = 1 - 4\mu \ . \tag{4.28}$$

so that the deficit angle is $\delta\phi = 2\pi(1-a)$.

Here one might think that our source (4.14) has a singularity on the $z-$axis. Indeed there is no singularity, however, in the valence potential. Because we know,

$$f = \cos(\frac{\rho}{\rho_*}) \ . \tag{4.29}$$

from (4.19), (4.23) and (4.24). So $f = 1$ on the z-axis, our valence potential vanishes on the z-axis, that is to say,

$$\mathbf{Z}_\mu = \begin{pmatrix} (1-f(\rho))\ \hat{\phi}\times\partial_\mu\hat{\phi} \\ 0 \end{pmatrix} = 0 \ , \qquad \text{if } \rho = 0 \ . \tag{4.30}$$

Therefore $f(\rho)$ regularizes the singularity on the symmetry axis but it transparently shows us topological property of our system. This is why so many physicists use this form of ansatz. Of course there exists a singularity in the restricted potential, $\hat{\mathbf{A}}_\mu = -\hat{\phi}\times\partial_\mu\hat{\phi}$. But this singularity is a kind of the coordinates singularity, and is not a physical one.

4.1.3 Gauge potential of the interior solution of cosmic string as a restricted potential

The following simple equation helps us to recognize that the interior solution can be described by A_2-gravity.

$$-\hat{\phi}\times\partial_\mu\hat{\phi} = \partial_\mu\phi\ \hat{\mathbf{z}} \ . \tag{4.31}$$

Now let us describe the interior solution by the restricted theory. At first, let us take the isometry direction as $\hat{\mathbf{z}}$;

$$\hat{\mathbf{n}}_1 = \hat{\rho}\ , \quad \hat{\mathbf{n}}_2 = \hat{\phi}\ , \quad \hat{\mathbf{n}} = \hat{\mathbf{n}}_3 = \hat{z}\ . \tag{4.32}$$

From now on all arguments are similar to what we did in the previous section. The restricted gauge potential is,

$$\hat{\mathbf{\Gamma}}_\mu = \begin{pmatrix} A_\mu \hat{\mathbf{z}} \\ B_\mu \hat{\mathbf{z}} \end{pmatrix}. \tag{4.33}$$

This reduces our spin-connections as,

$${\Gamma_\mu}^{\hat{\rho}\hat{\phi}} = -{\Gamma_\mu}^{\hat{\phi}\hat{\rho}} = A_\mu, \quad {\Gamma_\mu}^{\hat{t}\hat{z}} = {\Gamma_\mu}^{\hat{z}\hat{t}} = B_\mu, \tag{4.34}$$

and the other coefficients vanish. Comparing with (4.2) we have,

$$R = R(\rho, \phi), \quad \Phi = \Phi(\rho, \phi), \quad \zeta = \zeta(t, z), \quad \tau = \tau(t, z)\ . \tag{4.35}$$

The cylindrical symmetry implies ϕ-independence, so $R = R(\rho), \quad \Phi = \Phi(\rho)$. And $B_\mu = 0$ since, we do not have any boosting along the z-axis. Then we have $\zeta = \zeta(z)$ and $\tau = \tau(t)$. Therefore without loss of any generality, we put

$$\tau = R = \zeta = 0. \tag{4.36}$$

and,

$$ds^2 = -dt^2 + d\rho^2 + e^{2\Phi}\rho^2 d\phi^2 + dz^2 \tag{4.37}$$

with

$$A_\mu = {\Gamma_\mu}^{\hat{\rho}\hat{\phi}} = -e^{\Phi}(1 + \rho\ \partial_\rho \Phi)\ \partial_\mu \phi\ , \tag{4.38}$$

and the other gauge potentials (spin-connections) vanish. From the equation (3.72) with connection coefficients (4.38), we have the Abelian field strength;

$$\begin{aligned}\bar{A}_{\mu\nu} &= H_{\mu\nu} = -e^{\Phi}(\rho\Phi'' + \rho\Phi'^2 + 2\Phi')\,(\partial_\mu\rho\;\partial_\nu\phi - \partial_\mu\phi\;\partial_\nu\rho), \qquad (4.39)\\ B_{\mu\nu} &= 0\ .\end{aligned}$$

where $\Phi' = \dfrac{d\Phi}{d\rho}$. And from the equation (3.100) with tetrads for the line element (4.37) we get the Abelian component of the four-indexed metric tensor,

$$G_{\mu\nu} = \rho e^{\Phi}(\partial_\mu\rho\;\partial_\nu\phi - \partial_\mu\phi\;\partial_\nu\rho) \tag{4.40}$$

Then the Ricci tensor of our system is,

$$R_{\mu\nu} = A_{\mu\alpha}g^{\sigma\beta}G_{\beta\nu} = \rho e^{2\Phi}(\rho\Phi'' + \rho\Phi'^2 + 2\Phi')\;(\frac{e^{-2\Phi}}{\rho^2}\;\partial_\mu\rho\;\partial_\nu\rho + \partial_\mu\phi\;\partial_\nu\phi)\ . \tag{4.41}$$

Now the equations of motion $A_{\mu\alpha}g^{\sigma\beta}G_{\beta\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})$ reduces into a single equation,

$$\Phi'' + \Phi'^2 + \frac{2}{\rho}\Phi' = -8\pi\epsilon\ . \tag{4.42}$$

Using the function defined in the last section $g = \rho e^{\Phi}$, then the above equation becomes the same equation in (4.22). For the interior case, we can introduce vector potentials A_μ and G_μ which satisfy

$$A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu\ . \tag{4.43}$$

They are

$$A_\mu = -\cos(\frac{\rho}{\rho_*})\;\partial_\mu\phi\ , \quad G_\mu = -\rho_*^2\cos(\frac{\rho}{\rho_*})\;\partial_\mu\phi\ . \tag{4.44}$$

Therefore we can describe the interior solution of the cosmic string by the restricted A_2-gravity. Actually this solution was discovered by J. R. Gott III(1984) and Hiscock(1985) independently. But there are some differences between our and their derivation. First, they got the solution for cosmic string using Weyl's canonical coordinates and Levi-Civita solution. But we got the same solution without them, we just used the Abelian restriction for gauge potentials. Second, we could study gauge theoretical properties of the cosmic string and could check that some technique frequently used in gauge theories can be applied to the gravitational theory.

4.2 A fundamental solution for B_2-gravity : Einstein-Rosen-Bondi wave solution

In 1959, H. Bondi, F. A. E. Pirani and I. Robinson pointed out that the Einstein's theory allows an exact wave solution. Actually their solution was first suggested by Einstein and Rosen in 1937, but Rosen concluded that such a solution was unphysical. The reason was that this solution has a region in spacetime where the determinant of the metric vanishes. This is due to the fact that most physicists at that time including Rosen could not recognize the necessity to distinguish the physical singularity and the coordinates singularity. Therefore, after some developments of the mathematical foundation for general relativity, Bondi, Pirani and Robinson could show that the Einstein-Rosen's solution was really physical[6]. In this report, I will call this solution Einstein-Rosen-Bondi wave solution or shortly Bondi's solution.

4.2.1 Plane wave solution of Bondi's spacetime

In order to obtain the simplest solution of the B_2 gravity, we first choose the following B_2 isometry

$$\mathbf{j} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{n}_1 \\ \hat{n}_2 \end{pmatrix}, \quad \tilde{\mathbf{j}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{n}_2 \\ -\hat{n}_1 \end{pmatrix},$$
$$\hat{n}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{n}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{4.45}$$

With this, the restricted connection should have the following form, [1]

$$\hat{\mathbf{\Gamma}}_\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \Gamma_\mu \hat{n}_1 - \widetilde{\Gamma}_\mu \hat{n}_2 \\ \widetilde{\Gamma}_\mu \hat{n}_1 + \Gamma_\mu \hat{n}_2 \end{pmatrix},$$
$$\Gamma_\mu{}^{12} = \Gamma_\mu{}^{03} = 0, \quad \Gamma_\mu{}^{23} = -\Gamma_\mu{}^{02} = \frac{\Gamma_\mu}{\sqrt{2}},$$
$$\Gamma_\mu{}^{31} = \Gamma_\mu{}^{01} = -\frac{\widetilde{\Gamma}_\mu}{\sqrt{2}}. \tag{4.46}$$

The task now is to find a solution of Einstein's theory which has such connection. To do that we introduce the following tetrad basis,

$$e_0 = e^{-\tau}\partial_t, \; e_1 = e^{-\alpha}\partial_x, \; e_2 = e^{-\beta}\partial_y, \; e_3 = e^{-\gamma}\partial_z,$$
$$ds^2 = -e^{-2\tau}dt^2 + e^{-2\alpha}dx^2 + e^{-2\beta}dy^2 + e^{-2\gamma}dz^2, \tag{4.47}$$

where τ, α, β, and γ are functions of space-time coordinates (t, x, y, z). With this the torsion-free spin connections can be calculated from (3.90). We have

$$\Gamma_\mu{}^{12} = e^{\alpha-\beta}(\partial_y \alpha)\, \partial_\mu x - e^{\beta-\alpha}(\partial_x \beta)\, \partial_\mu y,$$
$$\Gamma_\mu{}^{23} = e^{\beta-\gamma}(\partial_z \beta)\, \partial_\mu y - e^{\gamma-\beta}(\partial_y \gamma)\, \partial_\mu z,$$

[1] In this section all numbers appear as indices represent the Lorentz indices. For example, $\Gamma_\mu{}^{12} = \Gamma_\mu{}^{\hat{x}\hat{y}}$, $e_\mu{}^3 = e_\mu{}^{\hat{z}}$, and so on.

$$
\begin{aligned}
\Gamma_\mu{}^{31} &= e^{\gamma-\alpha}(\partial_x\gamma)\ \partial_\mu z - e^{\alpha-\gamma}(\partial_z\alpha)\ \partial_\mu x,\\
\Gamma_\mu{}^{01} &= e^{\tau-\alpha}(\partial_x\tau)\ \partial_\mu t + e^{\alpha-\tau}(\partial_t\alpha)\ \partial_\mu x,\\
\Gamma_\mu{}^{02} &= e^{\tau-\beta}(\partial_y\tau)\ \partial_\mu t + e^{\beta-\tau}(\partial_t\beta)\ \partial_\mu y,\\
\Gamma_\mu{}^{03} &= e^{\tau-\gamma}(\partial_z\tau)\ \partial_\mu t + e^{\gamma-\tau}(\partial_t\gamma)\ \partial_\mu z.
\end{aligned} \tag{4.48}
$$

Now, comparing these with (4.46) we find that the condition (4.48) to have the B_2 isometry is given by

$$
\begin{gathered}
\alpha=\alpha(x,z,t), \qquad \beta=\beta(y,z,t), \qquad \gamma=\gamma(z), \qquad \tau=\tau(t),\\
e^{-\gamma}\partial_z\alpha + e^{-\tau}\partial_t\alpha = 0,\\
e^{-\gamma}\partial_z\beta + e^{-\tau}\partial_t\beta = 0.
\end{gathered} \tag{4.49}
$$

But since γ and τ are functions of z and t respectively, we can rescale the coordinates and let $\gamma = \tau = 0$ without loss of generality. In this case the above equations reduce into

$$
(\partial_t+\partial_z)\alpha = 0, \qquad (\partial_t+\partial_z)\beta = 0, \tag{4.50}
$$

so that we have $\alpha = \alpha(x, t-z)$ and $\beta = \beta(y, t-z)$. Next, we have to solve the B_2 gravity equation (3.119). From (3.69) and (4.45) we have $K_\mu = \Gamma_\mu$ and $\tilde{K}_\mu = \widetilde{\Gamma}_\mu$, and the first equation of (3.119) reduces to

$$
\Gamma_{\mu\nu}\mathcal{J}^{\nu\rho} = \widetilde{\Gamma}_{\mu\nu}\widetilde{\mathcal{J}}^{\nu\rho} \tag{4.51}
$$

Now, from (4.49) we have

$$
\Gamma_\mu = \sqrt{2}\ e^{\beta}(\partial_t\beta)\ \partial_\mu y, \quad \widetilde{\Gamma}_\mu = \sqrt{2}\ e^{\alpha}(\partial_t\alpha)\ \partial_\mu x,
$$

so that

$$\Gamma_{\mu\nu} = 2\sqrt{2}\ e^{\beta}[\partial_t^2\beta + (\partial_t\beta)^2] \times (\partial_{[\mu}t\ \partial_{\nu]}y + \partial_{[\mu}y\ \partial_{\nu]}z),$$
$$\widetilde{\Gamma}_{\mu\nu} = 2\sqrt{2}\ e^{\alpha}[\partial_t^2\alpha + (\partial_t\alpha)^2] \times (\partial_{[\mu}t\ \partial_{\nu]}x - \partial_{[\mu}z\ \partial_{\nu]}x). \tag{4.52}$$

Moreover, from (4.45) and (4.47) we have

$$\mathcal{J}_{\mu\nu} = e_\mu^a e_\nu^b j_{ab} = \sqrt{2}e^{-\beta}(\partial_{[\mu}y\ \partial_{\nu]}z - \partial_{[\mu}t\ \partial_{\nu]}y),$$
$$\widetilde{\mathcal{J}}_{\mu\nu} = e_\mu^a e_\nu^b \tilde{j}_{ab} = \sqrt{2}e^{-\alpha}(\partial_{[\mu}t\ \partial_{\nu]}x + \partial_{[\mu}z\ \partial_{\nu]}x). \tag{4.53}$$

For the sake of simplicity, let's assume that α and β don't depend on x nor y. Introducing a new variable $u = t - z$, we can write $\alpha = \alpha(u)$ and $\beta = \beta(u)$. With this (4.51) reduces to the single equation

$$\alpha'' + \beta'' + (\alpha')^2 + (\beta')^2 = 0 \tag{4.54}$$

where the prime denotes d/du. Let's introduce here new functions $L = L(u)$ and $\theta = \theta(u)$ with

$$e^{\alpha} = Le^{\theta}, \quad e^{\beta} = Le^{-\theta}. \tag{4.55}$$

In terms of the new functions the line element of (4.47) is expressed by

$$ds^2 = -dt^2 + L^2(e^{2\theta}dx^2 + e^{-2\theta}dy^2) + dz^2, \tag{4.56}$$

and (4.54) is transformed to

$$L'' + L\theta'^2 = 0. \tag{4.57}$$

This is exactly the Einstein-Rosen-Bondi gravitational wave equation. Moreover there exist 1-form vector potentials $\mathcal{J}_\mu$ and $\widetilde{\mathcal{J}}_\mu$ which satisfy

$$\mathcal{J}_{\mu\nu} = \partial_\mu \mathcal{J}_\nu - \partial_\nu \mathcal{J}_\mu, \qquad \widetilde{\mathcal{J}}_{\mu\nu} = \partial_\mu \widetilde{\mathcal{J}}_\nu - \partial_\nu \widetilde{\mathcal{J}}_\mu \tag{4.58}$$

since four-dimensional divergences of $\mathcal{J}_{\mu\nu}$ and $\widetilde{\mathcal{J}}_{\mu\nu}$ vanish, that is, $\nabla_\mu \mathcal{J}^{\mu\nu} = 0$ and $\nabla_\mu \widetilde{\mathcal{J}}^{\mu\nu} = 0$. In this case our 1-form vector potentials are

$$\begin{aligned}
\mathcal{J}_\mu &= \frac{1}{\sqrt{2}} \int L e^{-\beta} du \; \partial_\mu y \; , \\
\widetilde{\mathcal{J}}_\mu &= \frac{1}{\sqrt{2}} \int L e^{\beta} du \; \partial_\mu x \; .
\end{aligned} \tag{4.59}$$

In this point, one would say that there are two vector potentials to describe a gravitational plane wave solution. But this is wrong. Since they are dual to each other, they have only one physical degree of freedom. For a given function $\theta(u)$, $L(u)$ will be determined and vice versa from the equation (4.57).

4.2.2 Polarized wave solution of Bondi's spacetime

Since whole procedure to obtain the polarized Bondi's wave is very similar to the plane type, I will briefly leave just results in this section. Since we want to obtain a polarized wave from the plane one, it is natural to consider simple vierbein as following;

$$\begin{aligned}
e_\mu{}^{\hat{0}} &= \partial_\mu t \; , \\
e_\mu{}^{\hat{1}} &= L(e^{\beta} \cos\theta \; \partial_\mu x - e^{\beta} \sin\theta \; \partial_\mu y) \; , \\
e_\mu{}^{\hat{2}} &= L(e^{-\beta} \sin\theta \; \partial_\mu x + e^{-\beta} \cos\theta \; \partial_\mu y), \\
e_\mu{}^{\hat{3}} &= \partial_\mu z \; .
\end{aligned} \tag{4.60}$$

Note here again that I omit the hat sign on numbers 0, 1, 2, and 3 which indicate indices for group space as long as there is no confusion. And for spacetime indices I will use t, x, y, and z. By the metricity condition, we can calculate all spin-connection coefficients of the above vierbein,

$$
\begin{aligned}
\Gamma_\mu{}^{01} &= \Gamma_\mu{}^{31} = -\Gamma_\mu{}^{10} = -\Gamma_\mu{}^{13} \\
&= (L' + L\beta')e^{\beta}(\cos\theta\partial_\mu x + \sin\theta\partial_\mu y) - L\theta' \sinh 2\beta e^{-\beta}(\sin\theta\partial_\mu x - \cos\theta\partial_\mu y) \\
\Gamma_\mu{}^{02} &= -\Gamma_\mu{}^{23} = -\Gamma_\mu{}^{20} = \Gamma_\mu{}^{32} \\
&= (L' - L\beta')e^{-\beta}(\sin\theta\partial_\mu x + \cos\theta\partial_\mu y) - L\theta' \sinh 2\beta e^{\beta}(\cos\theta\partial_\mu x - \sin\theta\partial_\mu y) \\
\Gamma_\mu{}^{12} &= -\Gamma_\mu{}^{21} = -\theta' \cosh 2\beta \ \partial_\mu u, \\
\Gamma_\mu{}^{03} &= \Gamma_\mu{}^{30} = 0.
\end{aligned} \tag{4.61}
$$

where $u = t - z$.

Now we turn to the gauge space. At first I introduce a numerically fixed standard basis for the Lorentz space,

$$
\hat{\mathbf{e}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \hat{\mathbf{e}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \hat{\mathbf{e}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{4.62}
$$

Then we can write the identity on the Lorentz space as,

$$
\begin{aligned}
\mathbf{I}^{12} &= \begin{pmatrix} \hat{\mathbf{e}}_3 \\ \mathbf{0} \end{pmatrix}, \qquad \mathbf{I}^{23} = \begin{pmatrix} \hat{\mathbf{e}}_1 \\ \mathbf{0} \end{pmatrix}, \qquad \mathbf{I}^{31} = \begin{pmatrix} \hat{\mathbf{e}}_2 \\ \mathbf{0} \end{pmatrix}, \\
\mathbf{I}^{01} &= \begin{pmatrix} \mathbf{0} \\ \hat{\mathbf{e}}_1 \end{pmatrix}, \qquad \mathbf{I}^{02} = \begin{pmatrix} \mathbf{0} \\ \hat{\mathbf{e}}_2 \end{pmatrix}, \qquad \mathbf{I}^{03} = \begin{pmatrix} \mathbf{0} \\ \hat{\mathbf{e}}_3 \end{pmatrix} .
\end{aligned} \tag{4.63}
$$

Since our gravitational wave is polarized and it propagates along the z-axis, we can simply choose following gauge.

$$
\begin{aligned}
\hat{\mathbf{n}}_1 &= \cos\phi \ \hat{\mathbf{e}}_1 + \sin\phi \ \hat{\mathbf{e}}_2 \ , \\
\hat{\mathbf{n}}_2 &= -\sin\phi \ \hat{\mathbf{e}}_1 + \cos\phi \ \hat{\mathbf{e}}_2 \ ,
\end{aligned}
$$

$$\hat{\mathbf{n}}_3 = \hat{\mathbf{e}}_3 . \tag{4.64}$$

Now we have two rotation angle ϕ and θ, the former describes a rotation in the Lorentz space, and the latter represents a rotation in the spacetime. Actually we will obtain a non-trivial relation between them later. This gauge determines our basis for Lorentz space of B_2-type,

$$\begin{aligned}
\mathbf{j} &= \frac{e^{\lambda}}{\sqrt{2}}\begin{pmatrix} \cos\phi\ \hat{\mathbf{e}}_1 + \sin\phi\ \hat{\mathbf{e}}_2 \\ -\sin\phi\ \hat{\mathbf{e}}_1 + \cos\phi\ \hat{\mathbf{e}}_2 \end{pmatrix}, \qquad \tilde{\mathbf{j}} = \frac{e^{\lambda}}{\sqrt{2}}\begin{pmatrix} -\sin\phi\ \hat{\mathbf{e}}_1 + \cos\phi\ \hat{\mathbf{e}}_2 \\ -\cos\phi\ \hat{\mathbf{e}}_1 - \sin\phi\ \hat{\mathbf{e}}_2 \end{pmatrix}, \\
\mathbf{k} &= \frac{e^{-\lambda}}{\sqrt{2}}\begin{pmatrix} \cos\phi\ \hat{\mathbf{e}}_1 + \sin\phi\ \hat{\mathbf{e}}_2 \\ \sin\phi\ \hat{\mathbf{e}}_1 - \cos\phi\ \hat{\mathbf{e}}_2 \end{pmatrix}, \qquad \tilde{\mathbf{k}} = \frac{e^{-\lambda}}{\sqrt{2}}\begin{pmatrix} \sin\phi\ \hat{\mathbf{e}}_1 - \cos\phi\ \hat{\mathbf{e}}_2 \\ -\cos\phi\ \hat{\mathbf{e}}_1 - \sin\phi\ \hat{\mathbf{e}}_2 \end{pmatrix} \\
\mathbf{l} &= \begin{pmatrix} \hat{\mathbf{n}} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{e}}_3 \\ \mathbf{0} \end{pmatrix}, \qquad \tilde{\mathbf{l}} = \begin{pmatrix} \mathbf{0} \\ -\hat{\mathbf{n}} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ -\hat{\mathbf{e}}_3 \end{pmatrix} .
\end{aligned} \tag{4.65}$$

And these give following coefficients,

$$\begin{aligned}
j_{01} &= -j_{31} = -j_{10} = j_{13} = -\frac{e^{\lambda}}{\sqrt{2}}\sin\phi\ , \quad j_{02} = j_{23} = -j_{20} = -j_{32} = \frac{e^{\lambda}}{\sqrt{2}}\cos\phi\ , \\
\tilde{j}_{01} &= -\tilde{j}_{31} = -\tilde{j}_{10} = \tilde{j}_{13} = -\frac{e^{\lambda}}{\sqrt{2}}\cos\phi\ , \quad \tilde{j}_{02} = \tilde{j}_{23} = -\tilde{j}_{20} = -\tilde{j}_{32} = -\frac{e^{\lambda}}{\sqrt{2}}\sin\phi\ , \\
k_{01} &= k_{31} = -k_{10} = -k_{13} = \frac{e^{-\lambda}}{\sqrt{2}}\sin\phi\ , \quad k_{02} = -k_{23} = -k_{20} = k_{32} = -\frac{e^{-\lambda}}{\sqrt{2}}\cos\phi\ , \\
\tilde{k}_{01} &= \tilde{k}_{31} = -\tilde{k}_{10} = -\tilde{k}_{13} = \frac{e^{-\lambda}}{\sqrt{2}}\cos\phi\ , \quad \tilde{k}_{02} = -\tilde{k}_{23} = -\tilde{k}_{20} = \tilde{k}_{32} = -\frac{e^{-\lambda}}{\sqrt{2}}\sin\phi\ , \\
l_{12} &= -l_{21} = 1 = -\tilde{l}_{01} = \tilde{l}_{10}
\end{aligned} \tag{4.66}$$

where the other coefficients all vanish. On the other hand, our restricted gauge potential is,

$$\hat{\mathbf{\Gamma}}_\mu = \Gamma_\mu \mathbf{j} - \widetilde{\Gamma}_\mu \tilde{\mathbf{j}} - \mathbf{k} \times \partial_\mu \mathbf{j}. \tag{4.67}$$

Now I assume that $\phi = \phi(u)$ and $\lambda = \lambda(u)$. Then $\mathbf{j}$ does not depend on coordinates x and y and, it clearly satisfies,

$$\partial_t \, \mathbf{j} = (\partial_t \lambda) \, \mathbf{j} + (\partial_t \phi) \, \tilde{\mathbf{j}} = \lambda' \, \mathbf{j} + \phi' \, \tilde{\mathbf{j}} = -\partial_z \, \mathbf{j} \ . \tag{4.68}$$

Therefore the restricted gauge potential becomes,

$$\hat{\mathbf{\Gamma}}_\mu = \Gamma_\mu \mathbf{j} - \widetilde{\Gamma}_\mu \tilde{\mathbf{j}} - \phi'(\partial_\mu u)\mathbf{l} + \lambda'(\partial_\mu u)\tilde{\mathbf{l}} \ . \tag{4.69}$$

And we obtain the full gauge potential by adding the valence part to the restricted part, that is,

$$\begin{aligned} \mathbf{\Gamma}_\mu &= \hat{\mathbf{\Gamma}}_\mu + \mathbf{Z}_\mu \\ &= [\ \Gamma_\mu \mathbf{j} - \widetilde{\Gamma}_\mu \tilde{\mathbf{j}} - \phi'(\partial_\mu u)\mathbf{l} + \lambda'(\partial_\mu u)\tilde{\mathbf{l}}\ ] + [J_\mu \mathbf{k} - \widetilde{J}_\mu \tilde{\mathbf{k}} + L_\mu \mathbf{l} - \widetilde{L}_\mu \tilde{\mathbf{l}}] \\ &= \Gamma_\mu \mathbf{j} - \widetilde{\Gamma}_\mu \tilde{\mathbf{j}} + J_\mu \mathbf{k} - \widetilde{J}_\mu \tilde{\mathbf{k}} + [L_\mu - \phi'(\partial_\mu u)]\mathbf{l} - [\widetilde{L}_\mu - \lambda'(\partial_\mu u)]\tilde{\mathbf{l}}. \end{aligned} \tag{4.70}$$

Now it is time to prove that our ansatz (4.60) is included in the B_2-restricted part if it satisfies a special condition. In order to do that, let's calculate the valence connection using the equations in (4.85) and (4.66).

$$\begin{aligned} J_\mu &= \mathbf{j} \cdot \mathbf{\Gamma}_\mu = \mathbf{j} \cdot (\frac{1}{2}{\Gamma_\mu}^{ab}\mathbf{I}_{ab}) = \frac{1}{2}{\Gamma_\mu}^{ab}(\mathbf{j} \cdot \mathbf{I}_{ab}) = \frac{1}{2}{\Gamma_\mu}^{ab} j_{ab} \\ &= {\Gamma_\mu}^{01} j_{01} + {\Gamma_\mu}^{02} j_{02} + {\Gamma_\mu}^{23} j_{23} + {\Gamma_\mu}^{31} j_{31} \\ &= {\Gamma_\mu}^{01}(j_{01} + j_{31}) + {\Gamma_\mu}^{02}(j_{02} - j_{23}) = 0 \ , \end{aligned} \tag{4.71}$$

$$\begin{aligned} \widetilde{J}_\mu &= \tilde{\mathbf{j}} \cdot \mathbf{\Gamma}_\mu = \tilde{\mathbf{j}} \cdot (\frac{1}{2}{\Gamma_\mu}^{ab}\mathbf{I}_{ab}) = \frac{1}{2}{\Gamma_\mu}^{ab}(\tilde{\mathbf{j}} \cdot \mathbf{I}_{ab}) = \frac{1}{2}{\Gamma_\mu}^{ab} \tilde{j}_{ab} \\ &= {\Gamma_\mu}^{01} \tilde{j}_{01} + {\Gamma_\mu}^{02} \tilde{j}_{02} + {\Gamma_\mu}^{23} \tilde{j}_{23} + {\Gamma_\mu}^{31} \tilde{j}_{31} \\ &= {\Gamma_\mu}^{01}(\tilde{j}_{01} + \tilde{j}_{31}) + {\Gamma_\mu}^{02}(\tilde{j}_{02} - \tilde{j}_{23}) = 0 \ , \end{aligned} \tag{4.72}$$

$$L_\mu - \phi'(\partial_\mu u) = \mathbf{l} \cdot \mathbf{\Gamma}_\mu = \mathbf{l} \cdot (\frac{1}{2}{\Gamma_\mu}^{ab}\mathbf{I}_{ab}) = \frac{1}{2}{\Gamma_\mu}^{ab}(\mathbf{l} \cdot \mathbf{I}_{ab})$$

$$= \frac{1}{2}\Gamma_\mu{}^{ab} l_{ab} = \Gamma_\mu{}^{12} l_{12} = \theta' \cosh 2\beta (\partial_\mu u), \tag{4.73}$$

$$\begin{aligned} \widetilde{L}_\mu - \lambda'(\partial_\mu u) &= \tilde{\mathbf{l}} \cdot \mathbf{\Gamma}_\mu = \tilde{\mathbf{l}} \cdot (\frac{1}{2}\Gamma_\mu{}^{ab} \mathbf{I}_{ab}) = \frac{1}{2}\Gamma_\mu{}^{ab} (\tilde{\mathbf{l}} \cdot \mathbf{I}_{ab}) \\ &= \frac{1}{2}\Gamma_\mu{}^{ab} \tilde{l}_{ab} = \Gamma_\mu{}^{03} j_{03} = 0 \ . \end{aligned} \tag{4.74}$$

Remark the last two equations. If we choose the following gauge conditions,

$$\phi' = -\theta' \cosh 2\beta \ , \qquad \lambda' = 0, \tag{4.75}$$

then, we get the following equations,

$$J_\mu = \widetilde{J}_\mu = L_\mu = \widetilde{L}_\mu = 0. \tag{4.76}$$

Therefore our ansatz(4.60) is described by the restricted B_2-gravity if the equation (4.75) holds. From now on I will fix the value of λ by $\lambda = 0$, since its derivative vanishes and our basis for the Lorentz group can be always normalized. And we should note the first equation in (4.75). That is the formula which connects two rotation angles ϕ and θ. Here the condition for vanishing Ricci tensor is,

$$L'' + (\theta'^2 \sinh^2 2\beta + \beta'^2) L = 0, \tag{4.77}$$

where the prime denotes a derivative on $u = t - z$. In order to see this solution is really Bondi's one, let's change the coordinates by $dt = e^\phi d\tau$, $dz = e^\phi d\xi$, $d\eta = dx$ and $d\zeta = dy$. Defining a new function $L(u) = \int e^{-\phi} du$ for $u = t - z$, then we obtain

$$\begin{aligned} ds^2 &= e^{2\phi}(-d\tau^2 + d\xi^2) + \chi^2[\cosh 2\beta (d\eta^2 + d\zeta^2) \\ &+ \sinh 2\beta \cos 2\theta (d\eta^2 - d\zeta^2) - 2 \sinh 2\beta \sin 2\theta \ d\eta d\zeta] \end{aligned} \tag{4.78}$$

with the equation of motion changed into,

$$2\phi' = \chi(\beta'^2 + \theta'^2 \sinh^2 2\beta). \tag{4.79}$$

This confirms that we re-derived the Bondi's polarized wave solution. As we did in the previous section, we can introduce a vector potential $\mathcal{J}_\mu$ which satisfies $\mathcal{J}_{\mu\nu} = \nabla_\mu \mathcal{J}_\nu - \nabla_\nu \mathcal{J}_\mu = \partial_\mu \mathcal{J}_\nu - \partial_\nu \mathcal{J}_\mu$, since the four dimensional divergence of its dual part vanishes in the third equation in (30). Clearly we can also introduce $\tilde{\mathcal{J}}_\mu$ which satisfies, $\tilde{\mathcal{J}}_{\mu\nu} = \nabla_\mu \tilde{\mathcal{J}}_\nu - \nabla_\nu \tilde{\mathcal{J}}_\mu = \partial_\mu \tilde{\mathcal{J}}_\nu - \partial_\nu \tilde{\mathcal{J}}_\mu$. Therefore these $\mathcal{J}_\mu$ and $\tilde{\mathcal{J}}_\mu$ are good candidates which can be quantum fields of polarized Bondi's wave. To obtain them, let's calculate $\mathcal{J}_{\mu\nu}$ first.

$$\begin{aligned}
\mathcal{J}_{\mu\nu} &= e_\mu{}^a\ e_\nu{}^b j_{ab} = e_\mu{}^0\ e_\nu{}^1 j_{01} + e_\mu{}^0\ e_\nu{}^2 j_{02} + e_\mu{}^2\ e_\nu{}^3 j_{23} + e_\mu{}^3\ e_\nu{}^1 j_{31} - (\mu \leftrightarrow \nu) \\
&= -\frac{\sin\phi}{\sqrt{2}}(e_\mu{}^0\ e_\nu{}^1 - e_\mu{}^3\ e_\nu{}^1) + \frac{\cos\phi}{\sqrt{2}}(e_\mu{}^0\ e_\nu{}^2 + e_\mu{}^2\ e_\nu{}^3) - (\mu \leftrightarrow \nu) \\
&= -\frac{\sin\phi}{\sqrt{2}}(\partial_\mu t\ e_\nu{}^1 - \partial_\mu z\ e_\nu{}^1) + \frac{\cos\phi}{\sqrt{2}}(\partial_\mu t\ e_\nu{}^2 + e_\mu{}^2\ \partial_\mu z) - (\mu \leftrightarrow \nu)
\end{aligned} \tag{4.80}$$

From the equations (4.60) and (4.80), we get the following results.

$$\begin{aligned}
\mathcal{J}_{tx} &= -\mathcal{J}_{zx} = -\mathcal{J}_{xt} = \mathcal{J}_{xz} = -\frac{1}{\sqrt{2}} L(e^\beta \sin\phi\cos\theta - e^{-\beta}\cos\phi\sin\theta) \\
\mathcal{J}_{ty} &= \mathcal{J}_{yz} = -\mathcal{J}_{yt} = -\mathcal{J}_{zy} = \frac{1}{\sqrt{2}} L(e^\beta \sin\phi\sin\theta - e^{-\beta}\cos\phi\cos\theta)
\end{aligned} \tag{4.81}$$

where all the other components vanish. Here we can take a one-form vector potential $\mathcal{J}_\mu$ for $\mathcal{J}_{\mu\nu}$ as following;

$$\begin{aligned}
\mathcal{J}_\mu = &- \frac{1}{\sqrt{2}} \int L(e^\beta \sin\phi\cos\theta - e^{-\beta}\cos\phi\sin\theta) du\ \partial_\mu x \\
&+ \frac{1}{\sqrt{2}} \int L(e^\beta \sin\phi\sin\theta - e^{-\beta}\cos\phi\cos\theta) du\ \partial_\mu y
\end{aligned} \tag{4.82}$$

Of course, we can also calculate $\widetilde{\mathcal{J}}_{\mu\nu}$. But it has the same information of $\mathcal{J}_{\mu\nu}$, since it is just a dual part of $\mathcal{J}_{\mu\nu}$. So I will skip to calculate it here.

4.3 The pp-wave spacetime

The pp-wave spacetime is a family of exact solutions of Einstein's equation. This family represents radiations which propagate along one direction with the speed of light. "pp-wave" is the abbreviation for "plane-fronted waves with parallel propagation"[8]. And its standard line element is following;

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + 2f(x, y, u)du^2 \ , \tag{4.83}$$

where $u = t - z$. In this section, I will follow the tetrad system which was introduced by Peres[9],

$$\begin{aligned} \mathbf{W}^{\hat{t}} &= (1-f)dt - f dz \ , \quad \mathbf{W}^{\hat{z}} = f dt + (1+f)dz \ , \\ \mathbf{W}^{\hat{x}} &= \cos a \ dx + \sin a \ dy \ , \quad \mathbf{W}^{\hat{y}} = -\sin a \ dx + \cos a \ dy \ , \end{aligned} \tag{4.84}$$

where $\tan 2a = f_{xy}/f_{xx}$. The spin-connection coefficients of the above tetrads are,

$$\begin{aligned} \Gamma_\mu{}^{01} &= \Gamma_\mu{}^{31} = -\Gamma_\mu{}^{10} = -\Gamma_\mu{}^{13} = -\partial_\mu u \ (\cos a \ \partial_x f + \sin a \ \partial_y f) \ , \\ \Gamma_\mu{}^{02} &= -\Gamma_\mu{}^{23} = -\Gamma_\mu{}^{20} = \Gamma_\mu{}^{32} = \partial_\mu u \ (-\sin a \ \partial_x f + \cos a \ \partial_y f) \ , \\ \Gamma_\mu{}^{12} &= -(\partial_x a) \ \partial_\mu x - (\partial_y a) \ \partial_\mu y - a' \ \partial_\mu u \ , \\ \Gamma_\mu{}^{03} &= \Gamma_\mu{}^{30} = 0. \end{aligned} \tag{4.85}$$

Basic settings for the gauge space are the same for the case of the polarized Bondi's wave. Recall the equations (4.62) to (4.66). But now we allow that ϕ

and λ can depend on (x, y, u). Then (4.69) becomes,

$$\hat{\mathbf{\Gamma}}_\mu = \Gamma_\mu \mathbf{j} - \widetilde{\Gamma}_\mu \tilde{\mathbf{j}} - (\partial_\mu \phi)\ \mathbf{l} + (\partial_\mu \lambda)\ \tilde{\mathbf{l}}\ . \tag{4.86}$$

Then the full gauge potentials are,

$$\begin{aligned}
\mathbf{\Gamma}_\mu &= \hat{\mathbf{\Gamma}}_\mu + \mathbf{Z}_\mu \\
&= \Gamma_\mu \mathbf{j} - \widetilde{\Gamma}_\mu \tilde{\mathbf{j}} + J_\mu \mathbf{k} - \widetilde{J}_\mu \tilde{\mathbf{k}} + (L_\mu - \partial_\mu \phi)\mathbf{l} - (\widetilde{L}_\mu - \partial_\mu \lambda)\tilde{\mathbf{l}}.
\end{aligned} \tag{4.87}$$

Here J_μ and $\widetilde{J}_\mu$ automatically vanish because of (4.85). L_μ and $\widetilde{L}_\mu$ vanish if following equations hold;

$$\partial_\mu(\phi + a) = 0\ , \qquad \partial_\mu \lambda = 0\ . \tag{4.88}$$

Note that we always can choose the angle ϕ and the scale factor λ such that

$$\phi = -a\ , \quad \lambda = 0\ . \tag{4.89}$$

Therefore the pp-wave solutions can be described by the B_2-gravity. This shows us again that B_2-gravity contains the most important gravitational wave solutions. Indeed whole solutions of the Petrov's N-type can be contained in B_2-gravity, and it will be proved in the subsequent paper.

And the condition for vanishing Ricci tensor is,

$$(\partial_x^2 + \partial_y^2) f = 0\ . \tag{4.90}$$

Thus we know that f will be determined by the above Laplace equation and its boundary condition. And the function f has the only physical degree of the pp-wave spacetime.

Chapter 5

Gauge Theoretical Methods to Solve Einstein's Equation

From the time of Einstein, all methods to solve Einstein's equation have been dependent to Killing symmetries on spacetime since these symmetries reduce dependencies of coordinates and make problems simpler. This typical method teaches us how to deal with a gravitational system in a geometrical way.

But it is well known that Einstein's theory of gravitation can be regarded as a gauge theory of the Lorentz group and we have followed such a viewpoint in this paper. Therefore it is natural to try to solve Einstein's equation in a gauge theoretical way, not in a geometrical way. And this way will show us some common structures of Yang-Mills theory and Einstein's theory.

We need to study such common structures between them. Many quantum-theoretical developments have been achieved in Yang-Mills theory, but a few classical solutions were discovered, for example, monopoles and instantons. In contrast so many classical solutions of Einstein's theory were discovered, but there is no satisfactory theory for quantum gravity. Therefore we can more deeply understand both theories based on such common structures of them.

The purpose of this chapter is to explain such gauge theoretical techniques. In the first section we will search the most general vacuum potential in gravity for later purposes. From the second section we will re-derive well known solutions of Einstein's equation in a gauge theoretical way.

5.1 Most general vacuum potential of Einstein's spacetime

It was pointed out by many authors that gauge theory allows non-trivial vacua since its equations of motion are nonlinear. The gauge theory for the Lorentz group also have such multi-vacua and we should find the vacuum potential for each vacuum. With a proper choice of vacuum potential which adapted to a system under consideration, we can choose the simplest ansatz as the cases of 't Hooft Cho-Maison and cosmic strings. In addition it would make our equations of motion simpler since we can separate the vacuum part from the full connection and we can concentrate on the source.

As we shall see later, the Lorentz group includes the $SU(2)$ as a subgroup. This subgroup consists of spatial rotation generators and inherits the vacuum structure of the Lorentz group. So let us consider the $SU(2)$ gauge theory first. From here we just follow a recent paper of Y. M. Cho[10].

Let $\hat{n}_i$ $(i = 1, 2, 3)$ be an arbitrary right-handed orthonormal basis $(\hat{n}_1 \times \hat{n}_2 = \hat{n}_3 = \hat{n})$ on the tangent space of the Lorentz group. Since any vacuum obviously preserves all vectors $\hat{n}_i$ under parallel transport defined by its vacuum potential,

$$\forall_i \quad D_\mu \hat{n}_i = 0 \ . \tag{5.1}$$

This condition is so strong that the vacuum potential can be expressed only in terms of $\hat{n}_i$'s. The equation (5.1) make the field strength vanish;

$$\forall_i \quad [D_\mu,\ D_\nu]\ \hat{n}_i = g\vec{F}_{\mu\nu} \times \hat{n}_i = 0 \to F_{\mu\nu} = 0. \tag{5.2}$$

Solving (5.1) we obtain a most general $SU(2)$ vacuum potential which satisfies the vacuum isometry

$$\begin{aligned} \vec{A}_\mu \to \hat{\Omega}_\mu &= -C_\mu^k\ \hat{n}_k = -C_\mu^3 \hat{n} - \frac{1}{g}\hat{n} \times \partial_\mu \hat{n}, \\ \frac{1}{g}\hat{n} \times \partial_\mu \hat{n} &= C_\mu^1\ \hat{n}_1 + C_\mu^2\ \hat{n}_2, \\ C_\mu^k &= -\frac{1}{2g}\epsilon_{ij}{}^k(\hat{n}_i \cdot \partial_\mu \hat{n}_j), \end{aligned} \tag{5.3}$$

One can easily check that $\hat{\Omega}_\mu$ describes a vacuum since its field strength vanishes;

$$\begin{aligned} \hat{\Omega}_{\mu\nu} &= \partial_\mu \hat{\Omega}_\nu - \partial_\nu \hat{\Omega}_\mu + g\hat{\Omega}_\mu \times \hat{\Omega}_\nu \\ &= -(\partial_\mu C_\nu^k - \partial_\nu C_\mu^k + g\epsilon_{ij}{}^k C_\mu^i C_\nu^j)\ \hat{n}_k = 0. \end{aligned} \tag{5.4}$$

This tells that $\hat{\Omega}_\mu$ or equivalently $(C_\mu^1, C_\mu^2, C_\mu^3)$ describe a classical $SU(2)$ vacuum.

Now let us go back to the Lorentz group. Clearly the vacuum condition for the Lorentz group becomes,

$$\forall_i \quad D_\mu \mathbf{l}_i = 0, \qquad D_\mu \mathbf{k}_i = 0. \tag{5.5}$$

But as we studied in the chapter 3, it is enough to impose either

$$\forall_i \quad D_\mu \mathbf{l}_i = 0, \tag{5.6}$$

or equivalently

$$\forall_i \quad D_\mu \mathbf{k}_i = -D_\mu \tilde{\mathbf{l}}_i = 0, \tag{5.7}$$

because $\mathbf{l}_i$ and $\mathbf{k}_i$ are dual to each other up to the negative sign, $\mathbf{k}_i = -\tilde{\mathbf{l}}_i$.

It is easy to solve (5.5). In 3-dimensional notation (5.6) or (5.7) is written as

$$\forall_i \quad D_\mu \hat{n}_i = \vec{B}_\mu \times \hat{n}_i, \quad D_\mu \hat{n}_i = -\vec{B}_\mu \times \hat{n}_i. \tag{5.8}$$

The solution of the above equation is,

$$\vec{A}_\mu = \hat{\Omega}_\mu, \quad \vec{B}_\mu = 0. \tag{5.9}$$

where $\hat{\Omega}_\mu$ (with $g = 1$) is identical to the vacuum potential (5.3) of the $SU(2)$ subgroup. So the most general vacuum connection $\mathbf{\Omega}_\mu$ which makes curvature tensor vanish can be expressed as

$$\mathbf{\Gamma}_\mu = \mathbf{\Omega}_\mu = \begin{pmatrix} \hat{\Omega}_\mu \\ 0 \end{pmatrix}. \tag{5.10}$$

Consequently, we would know that the vacuum structure of the Lorentz group is the same as that of the $SU(2)$ group. And the vacuum structure of the spacetime depends only on the rotational part of the gauge potential[10]. This can be understood by the equivalence principle. If there exists a non-trivial boosting part in the vacuum gauge potential, then it would make an timelike acceleration for the observer and he or she might recognize the existence of matter.

5.2 Vacuum decomposition

Now we are at the position to discuss the vacuum decomposition. The most general vacuum potential is clearly contained in a restricted part since it satisfies the equation (5.1). So our Abelian connection can be more decomposed into the following form;

$$\hat{\mathbf{\Gamma}}_\mu = \mathbf{\Omega}_\mu + \hat{\mathbf{\Gamma}}^*_\mu = \begin{pmatrix} \hat{\Omega}_\mu \\ 0 \end{pmatrix} + \begin{pmatrix} \hat{\mathbf{A}}^* \\ \hat{\mathbf{B}}^* \end{pmatrix} \tag{5.11}$$

Here the connection $\mathbf{\Gamma}^*_\mu$ represents the Abelian part of full connection separated from the vacuum potential. So we have the full connection,

$$\mathbf{\Gamma}_\mu = \hat{\mathbf{\Gamma}}_\mu + \mathbf{Z}_\mu = \mathbf{\Omega}_\mu + \hat{\mathbf{\Gamma}}^*_\mu + \mathbf{Z}_\mu \tag{5.12}$$

Recall the simple harmonic oscillator with an external driven force in classical mechanics. The general solution of this system is the sum of a homogeneous solution and a particular solution. Homogeneous solutions fully describe the kernel of the differential equation, that is, the spring system itself, and particular solutions fit the external driven force. Roughly speaking, $\mathbf{\Gamma}^*_\mu$ takes into account for fields like the spring system and $\mathbf{Z}_\mu$ represents fields like the external driven force.

In $SU(3)$-QCD, $\mathbf{\Gamma}^*_\mu$ and $\mathbf{Z}_\mu$ describe the binding gluons and the valence gluons, respectively. This is the origin of the name "valence connection". In contrast with simple harmonic oscillators, our equations of motion allow non-trivial vacua, so we need $\mathbf{\Omega}_\mu$ in the last side of the above equation. Moreover, recently it was pointed out that we can separate the contribution of gluons for the full angular momentum of proton from the contribution of quarks using the vacuum potential[7].

5.3 Geodesic equation in an orthonormal frame

From the next section I will solve the Einstein's equation in a gauge-theoretical way. Now I should clarify what did I mean by"gauge-theoretical way". The standard method to solve the Einstein's equation is based on the metric. Even though starting points of each researcher are different, the final goal of them is same, to obtain the metric. They regards the Einstein's equation as a system of second order partial differential equations of metric. Once it is solved, they would calculate all gravitational effects from the metric. So the only interest of most relativists is how to reduce the number of unknown functions in the metric. For them, connection coefficients are just tools to calculate the geodesic equation and the Riemann tensor. This kind of thinking is summarized in a famous doctrine of Misner, Thorne and Wheeler; "Metric as a foundation of all".

But in the view point of the gauge theory, a "foundation of all' is not the metric but the gauge potential, that is to say, the spin connection. Therefore in this chapter, I will determine the gauge potential of the system taking account for the symmetry of the system. Then the Einstein's equation becomes the first order equation for the gauge potential. Of course equations of motion contain vierbein fields as algebraic factors.

Then how can we obtain a nice ansatz for spin connections? In order to do so, we should understand what are the physical meaning of our gauge potentials. Indeed our three-dimensional gauge potentials $\mathbf{A}_\mu$ and $\mathbf{B}_\mu$ are directly related to the geodesic equation in an orthonormal frame. The symmetry of the system comes in this relation, so we can determine the form of the gauge potential.

Let's introduce a spacetime coordinates (t, x^1, x^2, x^3) and take a tetrad system $\hat{\mathbf{e}}_a = e^\mu{}_a \partial_\mu$. Then the spatial tetrad basis $\{\hat{\mathbf{e}}_i : i = 1, 2, 3\}$ forms a right-handed orthonormal frame. Here the inner and outer product of spatial basis are determined by δ_{ij} and $\epsilon_{ijk} = \epsilon_{\hat{0}ijk}$.[1] At the same time, $\{\hat{\mathbf{n}}_i : i = 1, 2, 3\}$ forms the same right-handed orthonormal frame on the $SU(2)$ space. They have the totally same structure. So it is natural to identify them even though they live in two different

[1]In this chapter i, j, k are Lorentz indices.

tangent spaces. With this interpretation, we can rewrite the geodesic equation as,

$$\begin{aligned} a^{\hat{0}} &= -\mathbf{B}\cdot\mathbf{v} \ , \\ \mathbf{a} &= \frac{dv^i}{d\tau}\mathbf{e}_i = -v^{\hat{0}}\mathbf{B} + \mathbf{v}\times\mathbf{A} \end{aligned} \tag{5.13}$$

where $\mathbf{B} = u^\mu \mathbf{B}_\mu$, and $\mathbf{A} = u^\mu \mathbf{A}_\mu$ for $u^\mu = \dfrac{dx^\mu}{d\tau}$ and $v^{\hat{i}} = e_\mu{}^i \dfrac{dx^\mu}{d\tau}$. This is the same form of the Lorentz force. And we can understand the direct meaning of our gauge potentials $\mathbf{A}_\mu$ and $\mathbf{B}_\mu$; they play the roles of electric and magnetic fields in an orthonormal frame, respectively. With this physical sense, we can determine what kind of forms of the gauge potential can be possible in a given system.

5.4 Potential of gauge potential

In Maxwell's theory electric and magnetic fields are determined by the Maxwell equations and trajectories of test particles are determined by the Lorentz force formula. In gravitation, the trajectories of test particles are determined by the geodesic equation and we saw that the geodesic equation has the same form of the Lorentz force in an orthonormal frame. Here one would ask what are the equations of gravitation which correspond to the Maxwell equations in electromagnetism and whether we can find the four-vector potential of $\mathbf{A}_\mu$ and $\mathbf{B}_\mu$ as we did in Maxwell's theory. The answer to the former question is very easy to find; the Einstein's equation. But it is difficult to answer to the later one.

Before we answer that question, we should clear up the ambiguity in the usage of the word "potential". As we saw in the previous section, $\mathbf{A}_\mu$ and $\mathbf{B}_\mu$ do the same role of the electromagnetic fields, not of the electromagnetic potentials. But this is natural, because $\mathbf{A}_\mu$ and $\mathbf{B}_\mu$ are dual to each other and they transform in the same way of electric and magnetic field under Lorentz transformation. So when we say that they seem to be electromagnetic field, we mean their Lorentzian structure. But when we say that they form a gauge potential, we take account for their general covariance as well as their Lorentzian structure. So the second question is rewritten in a clearer form; how we obtain the potential of $\mathbf{A}_\mu$ and $\mathbf{B}_\mu$ in their Lorentzian structure.

It is so difficult to answer this question, but in a special case we can answer it. To do so let's define a terminology.

Definition 1 *Vierbein fields are called diagonal if it can be written as*

$$\mathbf{e}_0 = \partial_{\hat{0}} \ , \quad \mathbf{e}_1 = \partial_{\hat{1}} \ , \quad \mathbf{e}_2 = \partial_{\hat{2}} \ , \quad \mathbf{e}_3 = \partial_{\hat{3}} \ ,$$

In a diagonal case we introduce following one function and three vector fields,

$$n_\mu{}^{\hat{0}} = e_\mu{}^{\hat{0}} \ , \qquad \mathbf{n}_\mu = e_\mu{}^i \mathbf{n}_i \tag{5.14}$$

for $i = 1, 2$, and 3. Then $\mathbf{A}_\mu$ and $\mathbf{B}_\mu$ are given by

$$\begin{aligned} \mathbf{A}_\mu &= \nabla \times \mathbf{n}_\mu \\ \mathbf{B}_\mu &= \partial_{\hat{t}} \ \mathbf{n}_\mu + \nabla n_\mu{}^{\hat{0}} \end{aligned} \tag{5.15}$$

where $\nabla \times \mathbf{n}_\mu \equiv \frac{1}{2}\epsilon^{\hat{0}bcd} \ (\partial_b e_{\mu c} - \partial_c e_{\mu b})\mathbf{n}_d$ and $\nabla n_\mu{}^{\hat{0}} \equiv \mathbf{n}_i \ \partial_i \ n_\mu{}^{\hat{0}}$. These differential operations are Lorentzian curl and gradient with respect to the orthonormal frame under consideration. Now for a fixed μ, $\mathbf{A}_\mu$ and $\mathbf{B}_\mu$ are given by $(n_\mu{}^{\hat{0}}, \mathbf{n}_\mu)$ in the same way of the Maxwell's theory. Therefore we can regard $(n_\mu{}^{\hat{0}}, \mathbf{n}_\mu)$ as a four vector potential of $\mathbf{A}_\mu$ and $\mathbf{B}_\mu$ for each μ.

Here we should remark one point. So far the equation (3.90) has been regarded just as a formula to calculate spin connections from given vierbein. At the same time, however, it can be also regarded as a formula to obtain vierbein from given spin connections. And this process is one important axis of our formalism. Therefore it would be great if we have such a general process to obtain vierbein fields from the given spin-connection coefficients. Unfortunately nobody couldn't do that.

5.5 Purely electric potential

The analogy between gravitation and electromagnetism discussed in the previous section leads us to many simple solutions of Einstein's equation corresponding to the solutions in electrodynamics. As a simple example, one would consider a purely electric solution and a purely magnetic solution. Here 'a purely electric potential' means a potential with $\mathbf{A}_\mu = 0$, and 'a purely magnetic one' means a potential with $\mathbf{B}_\mu = 0$. Indeed we already saw a purely magnetic potential in the chapter 4, and that potential was the cosmic string. The potential of the cosmic string was,

$$\mathbf{\Gamma}_\mu = \begin{pmatrix} \mathbf{A}_\mu \\ \mathbf{B}_\mu \end{pmatrix} = \begin{pmatrix} \vec{\phi} \times \partial_\mu \vec{\phi} \\ 0 \end{pmatrix} . \tag{5.16}$$

As in the cases of magnetic monopoles in field theories, this potential showed us a topological character of the cosmic string. Therefore it is natural to ask what solution will have a purely electric potential.

In order to obtain a solution which has a purely electric potential, let us introduce a coordinates system (t, x, y, z) and a diagonal vierbein fields as following;

$$\mathbf{W}^{\hat{t}} = \tau \; dt \; , \quad \mathbf{W}^{\hat{x}} = \alpha \; dx \; , \quad \mathbf{W}^{\hat{y}} = \beta \; dy \; , \quad \mathbf{W}^{\hat{z}} = \zeta \; dz \; , \tag{5.17}$$

where all functions depend on spacetime coordinates. And let us choose a standard basis for the Lorentz group,

$$\hat{n}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \qquad \hat{n}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \qquad \hat{n}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{5.18}$$

Since we now consider a purely electric potential, we should suppose $\mathbf{A}_\mu = 0$. This is equivalent to

$$\nabla \times \mathbf{n}_\mu = 0 \ . \tag{5.19}$$

This leads us to

$$\alpha = \alpha(t, x) \ , \qquad \beta = \beta(t, y) \ , \qquad \zeta = \zeta(t, z) \ . \tag{5.20}$$

For the simplicity, let us assume that $\nabla n_\mu{}^{\hat{0}} = \nabla \tau = 0$. Then from (5.15),

$$\mathbf{B}_\mu \ = \ \partial_{\hat{t}} \, \mathbf{n}_\mu \ , \qquad \tau = \tau(t) \ . \tag{5.21}$$

Hence we can rescale $\tau = 0$ without any loss of generality. And our final assumption is that our gauge potentials do not depend on spatial coordinates (x, y, z). Then our gauge potential becomes

$$\mathbf{\Gamma}_\mu = \begin{pmatrix} \mathbf{A}_\mu \\ \mathbf{B}_\mu \end{pmatrix} \tag{5.22}$$

with

$$\mathbf{A}_\mu = 0 \ , \qquad \mathbf{B}_\mu = \begin{pmatrix} \dot{\alpha} \ \partial_\mu x \\ \dot{\beta} \ \partial_\mu y \\ \dot{\zeta} \ \partial_\mu z \end{pmatrix} \ , \tag{5.23}$$

where the dot denotes the differentiation on t. Then we straightforwardly obtain the field strength,

$$\begin{aligned}
\mathbf{R}_{\mu\nu} &= \begin{pmatrix} -\mathbf{B}_\mu \times \mathbf{B}_\nu \\ \partial_\mu \mathbf{B}_\nu - \partial_\nu \mathbf{B}_\mu \end{pmatrix} \\
&= -\dot{\beta}\dot{\zeta}(\partial_\mu y\ \partial_\nu z - \partial_\nu y\ \partial_\mu z)\mathbf{l}_1 - \ddot{\alpha}(\partial_\mu t\ \partial_\nu x - \partial_\nu t\ \partial_\mu x)\tilde{\mathbf{l}}_1 \\
&- \dot{\zeta}\dot{\alpha}(\partial_\mu z\ \partial_\nu x - \partial_\nu x\ \partial_\mu z)\mathbf{l}_2 - \ddot{\beta}(\partial_\mu t\ \partial_\nu y - \partial_\nu t\ \partial_\mu y)\tilde{\mathbf{l}}_2 \\
&- \dot{\alpha}\dot{\beta}(\partial_\mu x\ \partial_\nu y - \partial_\nu y\ \partial_\mu x)\mathbf{l}_3 - \ddot{\zeta}(\partial_\mu t\ \partial_\nu z - \partial_\nu t\ \partial_\mu z)\tilde{\mathbf{l}}_3 \ , \qquad (5.24)
\end{aligned}$$

and finally field equations;

$$\begin{aligned}
\frac{\dot{\alpha}\dot{\beta}}{\beta} + \frac{\dot{\alpha}\dot{\zeta}}{\zeta} + \ddot{\alpha} &= 0 \ , \\
\frac{\dot{\alpha}\dot{\beta}}{\alpha} + \frac{\dot{\beta}\dot{\zeta}}{\zeta} + \ddot{\beta} &= 0 \ , \\
\frac{\dot{\alpha}\dot{\zeta}}{\alpha} + \frac{\dot{\beta}\dot{\zeta}}{\beta} + \ddot{\zeta} &= 0 \ , \\
\frac{\ddot{\alpha}}{\alpha} + \frac{\ddot{\beta}}{\beta} + \frac{\ddot{\zeta}}{\zeta} &= 0 \ . \qquad (5.25)
\end{aligned}$$

The first three equations are equivalent to

$$\frac{d}{dt}\ln(\dot{\alpha}\beta\zeta) = 0 \ , \qquad \frac{d}{dt}\ln(\alpha\dot{\beta}\zeta) = 0 \ , \qquad \frac{d}{dt}\ln(\alpha\beta\dot{\zeta}) = 0 \ . \qquad (5.26)$$

Therefore we can set

$$\dot{\alpha}\beta\zeta = C_1 \ , \qquad \alpha\dot{\beta}\zeta = C_2 \ , \qquad \alpha\beta\dot{\zeta} = C_3 \ . \qquad (5.27)$$

This gives us

$$\frac{\ddot{\alpha}}{\alpha} = -\frac{C_1}{\alpha\beta\zeta}\Big(\frac{\dot{\beta}}{\beta} + \frac{\dot{\zeta}}{\zeta}\Big) \ , \quad \frac{\ddot{\beta}}{\beta} = -\frac{C_2}{\alpha\beta\zeta}\Big(\frac{\dot{\zeta}}{\zeta} + \frac{\dot{\alpha}}{\alpha}\Big) \ , \quad \frac{\ddot{\zeta}}{\zeta} = -\frac{C_3}{\alpha\beta\zeta}\Big(\frac{\dot{\alpha}}{\alpha} + \frac{\dot{\beta}}{\beta}\Big) \ . \ (5.28)$$

Inserting (5.28) into the final equation of (5.25), we get

$$(C_2 + C_3)\frac{\dot{\alpha}}{\alpha} + (C_3 + C_1)\frac{\dot{\beta}}{\beta} + (C_1 + C_2)\frac{\dot{\zeta}}{\zeta} = 0 \ . \tag{5.29}$$

Multiplying both sides of the above equation by $\alpha\beta\zeta$, and using (5.27), the above equation reduces into

$$C_1 C_2 + C_2 C_3 + C_3 C_1 = 0 \ . \tag{5.30}$$

On the other hand, dividing the first equation by the second equation of (5.27),

$$\frac{\dot{\alpha}}{\alpha} = \frac{C_1}{C_2}\frac{\dot{\beta}}{\beta} \quad \Longrightarrow \quad \alpha = a \ \beta^{\frac{C_1}{C_2}} \ , \tag{5.31}$$

for some constant a. In the same way we get,

$$\zeta = b \ \beta^{\frac{C_3}{C_2}} \ . \tag{5.32}$$

for some constant b. Inserting (5.31) and (5.32) into the second equation of (5.27),

$$\dot{\beta} = \frac{C_2}{ab} \ \beta^{-\frac{C_1+C_3}{C_2}} \ . \tag{5.33}$$

This is a first-ordered ordinary differential equation, so its solution is,

$$\beta = \left(\frac{C_1 + C_2 + C_3}{ab}\right)^{\frac{C_1+C_2+C_3}{C_2}} \ t^{\frac{C_2}{C_1+C_2+C_3}} \ , \tag{5.34}$$

where we used an initial condition that $\beta(0) = 0$. We can obtain α and ζ in the same way. Choosing the constants a and b appropriately, we have

$$\alpha = t^{\frac{C_1}{C_1+C_2+C_3}} \ , \qquad \beta = t^{\frac{C_2}{C_1+C_2+C_3}} \ , \qquad \zeta = t^{\frac{C_3}{C_1+C_2+C_3}} \ . \tag{5.35}$$

Here the condition for vanishing Ricci tensor is the equation (5.30). Introducing new parameters $\frac{C_1}{C_1+C_2+C_3} = p_1$, $\frac{C_2}{C_1+C_2+C_3} = p_2$, and $\frac{C_3}{C_1+C_2+C_3} = p_3$, finally we get

$$ds^2 = -dt^2 + t^{2p_1}\ dx^2 + t^{2p_2}\ dy^2 + t^{2p_3}\ dz^2\ , \tag{5.36}$$

where $p_1 + p_2 + p_3 = 1$. And (5.30) becomes

$$p_1p_2 + p_2p_3 + p_3p_1 = 0 \qquad \Longleftrightarrow \qquad (p_1 + p_2 + p_3)^2 = p_1^2 + p_2^2 + p_3^2\ . \tag{5.37}$$

This is the Kasner solution which describes a spacetime which is spatially homogeneous. We should remark here that this solution has a purely electric potential which is a diametrically opposite example of the cosmic string. The gauge potentials of this solution is,

$$\mathbf{A}_\mu = 0\ , \qquad \mathbf{B}_\mu = \begin{pmatrix} p_1 t^{p_1-1}\ \partial_\mu x \\ p_2 t^{p_2-1}\ \partial_\mu y \\ p_3 t^{p_3-1}\ \partial_\mu z \end{pmatrix}\ . \tag{5.38}$$

Also this solution would be a simple example of a solution with the planar symmetry, since the metric is invariant under any spatial inversions. And note that this solution has no non-trivial topology because its magnetic potential vanishes.

5.6 Static and spherically symmetric vacuum solutions

In this section I will consider a static and spherically symmetric solution of the Einstein's equation using the A_2-decomposition picture. To do so, we have to know the vacuum potential originated from the Mikowskian background discussed in the section (5.1). As we usually did in the chapter 4, I will identify coset variables with 3-dimensional spatial coordinates and parallelize bases for two spaces. That is to say, I choose

$$\hat{n}_1 = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \qquad \hat{n}_2 = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}, \qquad \hat{n}_3 = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}. \tag{5.39}$$

Then put (5.39) into the equation (5.4), we get the vacuum potential fits our spherical system;

$$\mathbf{\Omega}_\mu = \begin{pmatrix} \hat{\Omega}_\mu \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos\theta\ \hat{r} - \hat{r} \times \partial_\mu \hat{r} \\ 0 \end{pmatrix}. \tag{5.40}$$

As we studied in the section (5.2), the vacuum structure of the Lorentz group is reduced into that of the $SU(2)$ group. Since the three-dimensional potential $\mathbf{A}_\mu$ belongs to this rotational part of the gauge connection, it is natural to consider following ansatz as 't Hooft, Cho and Maison did.

$$\mathbf{A}_\mu = \hat{\Omega}_\mu + (1 - f(r))\ \hat{r} \times \partial_\mu \hat{r}\ . \tag{5.41}$$

Since $\mathbf{B}_\mu$ plays the role of electric fields in an orthonormal frame, we take a Coulomb-type ansatz.

$$\mathbf{B}_\mu = \hat{\mathbf{B}}_\mu = \hat{\mathbf{B}}^*_\mu = g(r)\ \partial_\mu t\ \hat{r}. \tag{5.42}$$

where $g(r)$ is a function of r. Actually we decomposed the connection into,

$$\mathbf{\Gamma}_\mu = \hat{\mathbf{\Gamma}}_\mu + \mathbf{Z}_\mu = \mathbf{\Omega}_\mu + \hat{\mathbf{\Gamma}}^*_\mu + \mathbf{Z}_\mu \tag{5.43}$$

where

$$\hat{\mathbf{\Gamma}}^*_\mu = \begin{pmatrix} 0 \\ g(r)\ \partial_\mu t\ \hat{r} \end{pmatrix}, \qquad \mathbf{Z}_\mu = (1 - f(r)) \begin{pmatrix} \hat{r} \times \partial_\mu \hat{r} \\ 0 \end{pmatrix}. \tag{5.44}$$

Now we get spin connections with respect to (5.39),

$$\begin{aligned}
\Gamma_\mu{}^{\hat\theta\hat\phi} &= -\Gamma_\mu{}^{\hat\phi\hat\theta} = -\cos\theta\ \partial_\mu\phi\ , \qquad \Gamma_\mu{}^{\hat r\hat\theta} = -\Gamma_\mu{}^{\hat\theta\hat r} = -f(r)\ \partial_\mu\theta\ ,\\
\Gamma_\mu{}^{\hat\phi\hat r} &= -\Gamma_\mu{}^{\hat r\hat\phi} = f(r)\ \sin\theta\ \partial_\mu\phi\ , \quad \Gamma_\mu{}^{\hat t\hat r} = -\Gamma_\mu{}^{\hat r\hat t} = -g(r)\ \partial_\mu t\ , \qquad (5.45)
\end{aligned}$$

and all the other coefficients vanish.

Now we need another assumption for our tetrads. I assume that our tetrads is diagonal;

$$e^{\hat t} = e^\tau dt, \quad e^{\hat\theta} = e^\Theta r\ d\theta, \quad e^{\hat\phi} = e^\Phi r\sin\theta\ d\phi, \quad e^{\hat r} = e^R dr. \qquad (5.46)$$

Here τ, Θ, Φ, and R are functions of t, θ, ϕ and r. And "hat" denotes the indices of the orthonormal frame, that is, $e^{\hat a} = e_\mu{}^{\hat a} dx^\mu$. Then we can calculate spin-connections from the above tetrads,

$$\begin{aligned}
\Gamma_\mu{}^{\hat\theta\hat\phi} &= -\Gamma_\mu{}^{\hat\phi\hat\theta} = e^{\Theta-\Phi}\csc\theta(\partial_\phi\Theta)(\partial_\mu\theta) - e^{\Phi-\Theta}(\cos\theta + \sin\theta\ \partial_\theta\Phi)(\partial_\mu\phi)\\
\Gamma_\mu{}^{\hat\phi\hat r} &= -\Gamma_\mu{}^{\hat r\hat\phi} = e^{\Phi-R}\sin\theta(1 + r\partial_r\Phi)(\partial_\mu\phi) - e^{R-\Phi}\frac{1}{r}\csc\theta(\partial_\phi R)(\partial_\mu r)\\
\Gamma_\mu{}^{\hat r\hat\theta} &= -\Gamma_\mu{}^{\hat\theta\hat r} = e^{R-\Theta}\frac{1}{r}(\partial_\theta R)(\partial_\mu r) - e^{\Theta-R}(1 + r\partial_r\Theta)(\partial_\mu\theta)\\
\Gamma_\mu{}^{\hat t\hat\theta} &= -\Gamma_\mu{}^{\hat\theta\hat t} = e^{\tau-\Theta}\frac{1}{r}(\partial_\theta\tau)(\partial_\mu t) + e^{\Theta-\tau} r(\partial_t\Theta)(\partial_\mu\theta)\\
\Gamma_\mu{}^{\hat t\hat\phi} &= -\Gamma_\mu{}^{\hat\phi\hat t} = e^{\tau-\Phi}\frac{1}{r}\csc\theta(\partial_\phi\tau)(\partial_\mu t) + e^{\Phi-\tau} r\sin\theta(\partial_t\Phi)(\partial_\mu\phi)\\
\Gamma_\mu{}^{\hat t\hat r} &= -\Gamma_\mu{}^{\hat r\hat t} = e^{\tau-R}(\partial_r\tau)(\partial_\mu t) + e^{R-\tau}(\partial_t R)(\partial_\mu r)\ , \qquad (5.47)
\end{aligned}$$

and all other coefficients vanish. Comparing (5.45) and (5.47) we can easily check that τ, Θ, Φ, and R depend only on r, and

$$\Phi = \Theta, \qquad f(r)\ e^R = \partial_r(re^\Theta), \qquad g(r)\ e^R = -\partial_r e^\tau. \qquad (5.48)$$

Therefore we can normalize $\Theta = \Phi = 0$ without loss of generality. Then (12) becomes,

$$f(r) = e^{-R}, \qquad g(r) = -e^{-R}\partial_r e^{\tau}. \tag{5.49}$$

Using (5.45), (5.49), we can follow the process of A_2-type Abelian decomposition, we readily arrive at the equations $R_{\mu a} = 0$;

$$\begin{aligned} 2rff' + f^2 - 1 = 0. \\ rg' + 2g = 0, \\ f' + ge^{-\tau} = 0, \end{aligned} \tag{5.50}$$

where the prime denotes $\frac{d}{dr}$. Here we should note two points. First, there are only first order derivatives of unknown functions, and second, there is no derivative term for vierbein, as we desired. These are very remarkable, since there would exist second derivatives of metric in equations of geometrodynamics. This differences arise between Hamilton's formalism and Lagrange's formalism. Hamilton could obtain only first oder equations of motion because he took a nice variable, that is, conjugate momenta. In our formalism, we could get only first order equation since we took good variables, gauge potentials.

The first and second equation are solved directly,

$$f = \sqrt{1 - \frac{C_1}{r}}\ , \qquad g = -\frac{C_2}{r}\ . \tag{5.51}$$

where C_1 and C_2 are integration constants. Let us choose these constants as $C_1 = 2M$ and $C_2 = M$. With these we solve the third equation,

$$e^{\tau} = \sqrt{1 - \frac{2M}{r}}\ . \tag{5.52}$$

and this gives the Schwarzchild metric,

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{1}{1 - 2M/r}\, dr^2 + r^2 d\Omega^2 \ . \tag{5.53}$$

Furthermore we can obtain the Reissner-Nordstrom and Cho-Freund solution with the same ansatz. In that cases, the only change is the energy-momentum tensor $T_{\mu\nu}$. Here remark again that it is much easier to solve the problem than usual manner of geometrodynamics, since we only have the first order equations of motion.

5.7 Stationary Weyl Class

Up to now we have looked at planar or spherical solutions of Einstein's equation. So it is good time to study axisymmetric solutions. Of course the cosmic string had an axisymmetry, but it was invariant under inversions for any xy-planes, that is, it also had a planar symmetry. Furthermore it was too simple. So we need to generalize that result into stationary and axisymmetric solutions and obtain an appropriate vierbein to handle that symmetry.

Actually most objects in the universe are rotating and their gravitational effects will be understood in the frame of stationary and axisymmetric solutions. But stationary cases are much more difficult to handle than static cases, so it is worth getting a nice tetrad system which fits stationary and axisymmetric solutions. The purpose of this section is to obtain such a tetrad system of the Weyl class.

At first we should define our symmetry. A spacetime is called stationary if it admits a timelike Killing vector ξ^μ. And a spacetime is said to be axisymmetric if it has a spacelike Killing vector ψ^μ whose integral curve is closed. If a spacetime satisfies the above two conditions and two Killing vectors commutes, namely,

$$[\ \xi^\mu,\ \psi^\nu\ ] = 0\ , \tag{5.54}$$

then that spacetime is called stationary and axisymmetric.

Let us take coordinates t and ϕ which are parameters of integral curves of ξ^μ and ψ^μ, respectively. And let us introduce a coordinates system (t, x^1, x^2, ϕ). The general form of tetrads is,

$$\begin{aligned}
\mathbf{W}^{\hat{t}} &= T_0\ dt + T_1\ dx^1 + T_2\ dx^2 + T_3\ d\phi\ , \\
\mathbf{W}^{\hat{1}} &= X_0\ dt + X_1\ dx^1 + X_2\ dx^2 + X_3\ d\phi\ , \\
\mathbf{W}^{\hat{2}} &= Y_0\ dt + Y_1\ dx^1 + Y_2\ dx^2 + Y_3\ d\phi\ , \\
\mathbf{W}^{\hat{\phi}} &= \Phi_0\ dt + \Phi_1\ dx^1 + \Phi_2\ dx^2 + \Phi_3\ d\phi\ ,
\end{aligned} \tag{5.55}$$

where all functions depend only on x^1 and x^2 since ξ^μ and ψ^μ are Killing vectors. Since we want to deal with rotating bodies we naturally assume that we cannot recognize any change even though both directions of time and rotating are inversed simultaneously. Therefore,

$$T_1 = T_2 = X_0 = X_3 = Y_0 = Y_3 = \Phi_1 = \Phi_2 = 0\ , \tag{5.56}$$

so that our tetrads become,

$$\begin{aligned}
\mathbf{W}^{\hat{t}} &= T_0\ dt + T_3\ d\phi\ , \\
\mathbf{W}^{\hat{1}} &= X_1\ dx^1 + X_2\ dx^2\ , \\
\mathbf{W}^{\hat{2}} &= Y_1\ dx^1 + Y_2\ dx^2\ , \\
\mathbf{W}^{\hat{\phi}} &= \Phi_0\ dt + \Phi_3\ d\phi\ .
\end{aligned} \tag{5.57}$$

Here we can diagonalize $\mathbf{W}^{\hat{1}}$ and $\mathbf{W}^{\hat{2}}$. In order to prove this fact, many people use the metric formalism. But I will prove it using the tetrad and its Lorentz invariance. Let us consider following two tetrads;

$$\begin{aligned}
\mathbf{e}_{\hat{1}} &= A\,\partial_x + B\,\partial_y\ , \\
\mathbf{e}_{\hat{2}} &= C\,\partial_x + D\,\partial_y\ ,
\end{aligned} \tag{5.58}$$

where all functions depend only on x and y. Our goal is to prove that there exists a two-dimensional coordinates transformation, or equivalently a Lorentz transformation which transforms the above tetrads into

$$\mathbf{e}_{\hat{1}'} = \alpha\,\partial_{x'}\ , \qquad \mathbf{e}_{\hat{2}'} = \beta\,\partial_{y'}\ , \tag{5.59}$$

where α and β are functions of $x' = x'(x,y)$ and $y' = y'(x,y)$. Since we are interested only in two-dimensional Lorentz transformation, let it represent a two-dimensional rotation

$$L_i^{\ j'} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{5.60}$$

with θ is a gauge parameter and $i, j = 1, 2$. Under this transformation,

$$\begin{aligned}
\mathbf{e}_{\hat{1}} &= L_1^{\ j'}\,\mathbf{e}_{\hat{j}'} = L_1^{\ 1'}\,\mathbf{e}_{\hat{1}'} + L_1^{\ 2'}\,\mathbf{e}_{\hat{2}'} \\
&= = \cos\theta\ \alpha\ \partial_{x'} + \sin\theta\ \beta\ \partial_{y'}\ , \\
\mathbf{e}_{\hat{2}} &= L_2^{\ j'}\,\mathbf{e}_{\hat{j}'} = L_2^{\ 1'}\,\mathbf{e}_{\hat{1}'} + L_2^{\ 2'}\,\mathbf{e}_{\hat{2}'} \\
&= = -\sin\theta\ \alpha\ \partial_{x'} + \cos\theta\ \beta\ \partial_{y'}\ .
\end{aligned} \tag{5.61}$$

On the other hand (5.58) can be written as

$$\mathbf{e}_{\hat{1}} = A\,(\frac{\partial x'}{\partial x}\partial_{x'} + \frac{\partial y'}{\partial x}\partial_{y'}) + B\,(\frac{\partial x'}{\partial y}\partial_{x'} + \frac{\partial y'}{\partial y}\partial_{y'})$$

$$
\begin{aligned}
&= (A\frac{\partial x'}{\partial x}+B\frac{\partial x'}{\partial y})\ \partial_{x'}+(A\frac{\partial y'}{\partial x}+B\frac{\partial y'}{\partial y})\ \partial_{y'}\ , \\
\mathbf{e}_{\hat{2}} &= C\ (\frac{\partial x'}{\partial x}\partial_{x'}+\frac{\partial y'}{\partial x}\partial_{y'})+D\ (\frac{\partial x'}{\partial y}\partial_{x'}+\frac{\partial y'}{\partial y}\partial_{y'}) \\
&= (C\frac{\partial x'}{\partial x}+D\frac{\partial x'}{\partial y})\ \partial_{x'}+(C\frac{\partial y'}{\partial x}+D\frac{\partial y'}{\partial y})\ \partial_{y'}\ . \qquad (5.62)
\end{aligned}
$$

Equating (5.61) and (5.62),

$$
\begin{aligned}
\alpha\ \cos\theta &= A\ \frac{\partial x'}{\partial x}+B\ \frac{\partial x'}{\partial y}\ , \\
-\alpha\ \sin\theta &= A\ \frac{\partial y'}{\partial x}+B\ \frac{\partial y'}{\partial y}\ , \\
\beta\ \sin\theta &= C\ \frac{\partial x'}{\partial x}+D\ \frac{\partial x'}{\partial y}\ , \\
\beta\ \cos\theta &= C\ \frac{\partial y'}{\partial x}+D\ \frac{\partial y'}{\partial y}\ . \qquad (5.63)
\end{aligned}
$$

We can easily eliminate θ from the first and second equations in (5.63),

$$
\begin{aligned}
\alpha^2 &= (A^2+C^2)\left(\frac{\partial x'}{\partial x}\right)^2+2(AB+CD)\frac{\partial x'}{\partial x}\frac{\partial x'}{\partial y}+\left(C^2+D^2\right)\left(\frac{\partial x'}{\partial y}\right)^2 \\
&= g^{11}\left(\frac{\partial x'}{\partial x}\right)^2+(g^{12}+g^{21})\frac{\partial x'}{\partial x}\frac{\partial x'}{\partial y}+g^{22}\left(\frac{\partial x'}{\partial y}\right)^2 \\
&= g^{ij}\ \partial_i\partial_j\ x'(x,y)\ . \qquad (5.64)
\end{aligned}
$$

Similarily we would get

$$
\begin{aligned}
\beta^2 &= (A^2+C^2)\left(\frac{\partial y'}{\partial x}\right)^2+2(AB+CD)\frac{\partial y'}{\partial x}\frac{\partial y'}{\partial y}+(C^2+D^2)\left(\frac{\partial y'}{\partial y}\right)^2 \\
&= g^{11}\left(\frac{\partial y'}{\partial x}\right)^2+(g^{12}+g^{21})\frac{\partial y'}{\partial x}\frac{\partial y'}{\partial y}+g^{22}\left(\frac{\partial y'}{\partial y}\right)^2 \\
&= g^{ij}\ \partial_i\partial_j\ y'(x,y)\ . \qquad (5.65)
\end{aligned}
$$

It is a natural consequence that α and β are determined once x' and y' are determined. In order to get equations determining x' and y', let us eliminate α

and β in (5.63). Then,

$$\begin{aligned}(A\sin\theta + C\cos\theta)\,\frac{\partial x'}{\partial x} + (B\sin\theta + D\cos\theta)\,\frac{\partial x'}{\partial y} &= 0\ ,\\ (A\cos\theta - C\sin\theta)\,\frac{\partial y'}{\partial x} + (B\cos\theta - D\sin\theta)\,\frac{\partial y'}{\partial y} &= 0\ .\end{aligned} \tag{5.66}$$

The first equation in (5.66) is a homogeneous and linear first-ordered partial differential equation for just one function $x' = x'(x, y)$. So there must exist a solution. By the same token, there exists a solution $y' = y'(x, y)$ for the second equation in (5.66). Therefore we can always diagonalize $\mathbf{W}^{\hat{1}}$ and $\mathbf{W}^{\hat{2}}$.

Then can we do the same procedure for the other two tetrads? The answer is no since the components of these tetrads depend on x^1 and x^2, not on t and ϕ. But it is obvious that we can set either $T_3 = 0$ or $\Phi_0 = 0$ without any loss of generality. Renaming all non-vanishing functions and spatial coordinates, our first fundamental form becomes,

$$ds^2 = -e^{2\tau}dt^2 + e^{2\Phi}(d\phi + W dt)^2 + e^{2R}d\rho^2 + e^{2\zeta}dz^2\ , \tag{5.67}$$

where all functions depend on ρ and z.

This line element is called the stationary Weyl's class or canonical Weyl's class. This is not a solution for a concrete physical situation but it provides an appropriate coordinates ansatz for a stationary and axisymmetric spacetime without loss of any generality. This is why we call it "class" rather than "a solution". Of course we can further simplify the above expression considering the field equation if we deal with vacuous spacetimes.

Now let us turn back to gauge potentials. At first, we introduce a coordinates system (t, z, ρ, ϕ) and we take following basis for the Lorentz group;

$$\hat{n}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \hat{n}_2 = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}, \quad \hat{n}_3 = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}. \tag{5.68}$$

where we identified a spacetime coordinate ϕ with an azimuthal angle of Lorentz group. From the line element (5.67), we naturally choose tetrads as,

$$\begin{aligned}
\mathbf{W}^{\hat{t}} &= e^{\tau}\, dt\ , \\
\mathbf{W}^{\hat{z}} &= e^{\zeta} dz\ , \\
\mathbf{W}^{\hat{\rho}} &= e^{R} d\rho\ , \\
\mathbf{W}^{\hat{\phi}} &= e^{\Phi}(dt + W d\phi)\ .
\end{aligned} \tag{5.69}$$

Non-vanishing spin-connections corresponding to these tetrads are,

$$\begin{aligned}
\Gamma_{\mu}{}^{\hat{z}\hat{\rho}} &= (\partial_\mu z) e^{\zeta - R} \partial_\rho \zeta - (\partial_\mu \rho) e^{R-\zeta} \partial_z R\ , \\
\Gamma_{\mu}{}^{\hat{\rho}\hat{\phi}} &= -(\partial_\mu \phi) e^{\Phi - R} \partial_\rho \Phi - \frac{1}{2}(\partial_\mu t) e^{\Phi - R}(2W\partial_\rho \Phi + \partial_\rho W)\ , \\
\Gamma_{\mu}{}^{\hat{\phi}\hat{z}} &= -(\partial_\mu \phi) e^{\Phi - \zeta} \partial_z \Phi + \frac{1}{2}(\partial_\mu t) e^{\Phi - \zeta}(2W\partial_z \Phi + \partial_z W)\ , \\
\Gamma_{\mu}{}^{\hat{t}\hat{z}} &= -\frac{1}{2}(\partial_\mu \phi) e^{2\Phi - \zeta - \tau} \partial_z W + \frac{1}{2}(\partial_\mu t) e^{-\zeta - \tau}(2e^{2\tau}\partial_z \tau - e^{2\Phi} W \partial_z W)\ , \\
\Gamma_{\mu}{}^{\hat{t}\hat{\rho}} &= -\frac{1}{2}(\partial_\mu \phi) e^{2\Phi - R - \tau} \partial_\rho W + \frac{1}{2}(\partial_\mu t) e^{-R - \tau}(2e^{2\tau}\partial_\rho \tau - e^{2\Phi} W \partial_\rho W)\ , \\
\Gamma_{\mu}{}^{\hat{t}\hat{\phi}} &= -\frac{1}{2}(\partial_\mu z) e^{2\Phi - \tau} \partial_z W - \frac{1}{2}(\partial_\mu \rho) e^{2\Phi - \tau} \partial_\rho W\ .
\end{aligned} \tag{5.70}$$

and the coefficients which can be obtained by interchanging the last two indices of the above connection. Let us introduce an orthonormal gradient in the group space,

$$\hat{\nabla} \equiv \hat{\rho}\ \partial_{\hat{\rho}} + \hat{z}\ \partial_{\hat{z}}\ , \tag{5.71}$$

where $\hat{z} = \hat{n}_1$ and $\hat{\rho} = \hat{n}_2$. Then our gauge potential becomes,

$$\begin{aligned}
\mathbf{A}_\mu &= (\partial_\mu \phi + W \partial_\mu t)(\hat{\phi} \times \hat{\nabla} e^{\Phi}) + \frac{1}{2} e^{\Phi} (\partial_\mu t)(\hat{\phi} \times \hat{\nabla} W) \\
&+ (\partial_\mu z\ \partial_{\hat{\rho}} e^{\zeta} - \partial_\mu \rho\ \partial_{\hat{z}} e^{R})\ \hat{\phi}\ , \\
\mathbf{B}_\mu &= -\frac{1}{2} e^{\Phi-\tau} \partial_\mu W\ \hat{\phi} - \frac{1}{2} (\partial_\mu \phi + W \partial_\mu t) e^{2\Phi-\tau} \hat{\nabla} W + (\partial_\mu t) \hat{\nabla} e^{\tau}\ .
\end{aligned} \tag{5.72}$$

The meaning of W can be understood if we consider (5.69). It describes the rotation of $\mathbf{e}_{\hat{\phi}}$. In addition we can calculate the inertial acceleration of a test particle from the geodesic equation (6.1) in the orthonormal frame. When a particle is momentarily at rest with respect to the orthonormal frame then its velocity is $v = \partial_{\hat{t}}$. Then the particle will feel electric and magnetic fields as,

$$\begin{aligned}
\mathbf{A} &= \mathbf{A}_\mu v^\mu = e^{-\tau}\ \hat{\phi} \times (W \hat{\nabla} e^{\Phi} + \frac{1}{2} e^{\Phi} \hat{\nabla} W)\ , \\
\mathbf{B} &= \mathbf{B}_\mu v^\mu = -\frac{1}{4} e^{2(\Phi-\tau)} \hat{\nabla} W^2 + \hat{\nabla} \tau\ .
\end{aligned} \tag{5.73}$$

Note here that the test particle will feel the magnetic field orthogonal to the ϕ-direction. And it is interesting that W can contribute to the electric filed.

5.8 Static Weyl Class

A stationary space time is a spacetime which has a timelike Killing vector ξ^μ. And a stationary spacetime is called static if its timelike Killing vector is hypersurface-orthogonal, equivalently, if has a timelike Killing vector ξ^μ which satisfies following;

$$\xi_{[\mu}\nabla_{\nu}\xi_{\sigma]} = 0 \ . \tag{5.74}$$

If we let a coordinate t be a parameter of the integral curve of ξ^{μ}, then the above condition makes $W = 0$ in (5.67) so that we have

$$ds^2 = -e^{2\tau}dt^2 + e^{2\Phi}\rho^2(d\phi)^2 + e^{2R}d\rho^2 + e^{2\zeta}dz^2 \ . \tag{5.75}$$

Here we separated ρ^2 from e^{Φ} in order to simplify equations of motion.

Let us introduce the same basis (5.68) for the Lorentz group, and the same tetrads (5.69) for our spacetime with $W = o$. Then, non-vanishing spin-connections for these tetrads are,

$$\begin{aligned}
\Gamma_{\mu}{}^{\hat{z}\hat{\rho}} &= (\partial_\mu z)e^{\zeta-R}\partial_\rho\zeta - (\partial_\mu\rho)e^{R-\zeta}\partial_z R \ , \\
\Gamma_{\mu}{}^{\hat{\rho}\hat{\phi}} &= -(\partial_\mu\phi)\ e^{\Phi-R}(1+\rho\partial_\rho\Phi) \ , \\
\Gamma_{\mu}{}^{\hat{\phi}\hat{z}} &= (\partial_\mu\phi)\ e^{\Phi-\zeta}\rho\partial_z\Phi \ , \\
\Gamma_{\mu}{}^{\hat{t}\hat{z}} &= (\partial_\mu t)\ e^{\tau-\zeta}\ \partial_z\tau \ , \\
\Gamma_{\mu}{}^{\hat{t}\hat{\rho}} &= (\partial_\mu t)\ e^{\tau-R}\ \partial_\rho\tau \ , \\
\Gamma_{\mu}{}^{\hat{t}\hat{\phi}} &= 0 \ ,
\end{aligned} \tag{5.76}$$

and the coefficients which can be obtained by interchanging the last two indices of the above connection. Then our gauge potential becomes

$$\begin{aligned}
\mathbf{A}_\mu &= (\partial_\mu\phi)\ \hat{\phi}\times\hat{\nabla}(\rho e^{\Phi}) + (\partial_\mu z\ \partial_{\hat{\rho}}e^{\zeta} - \partial_\mu\rho\ \partial_{\hat{z}}e^{R})\ \hat{\phi} \ , \\
\mathbf{B}_\mu &= (\partial_\mu t)\ \hat{\nabla}e^{\tau} \ ,
\end{aligned} \tag{5.77}$$

where $\hat{\nabla}$ was defined in (5.71). Now let us consider again a test particle which is momentarily at rest with respect to the orthonormal frame. Then it will feel the following electric and magnetic fields;

$$\mathbf{A} = \mathbf{A}_\mu v^\mu = 0 \ , \qquad \mathbf{B} = \mathbf{B}_\mu v^\mu = \hat{\nabla}\tau \ . \tag{5.78}$$

Therefore this particle cannot feel any magnetic field. And the electric field which the test particle feels has the same form of Coulomb's potential. It is natural if we remind that the systems of diagonal cases are similar to the systems of electrodynamics.

5.9 Levi-Civita Solution

In the section 5.3 we have shown that the geodesic equation in an orthonormal frame has the same form of the Lorentz force. This provides us a method to take an appropriate ansatz for the gauge potential, that is to say, by analyzing the geodesic equation we can guess a sketchy form of the gauge potential under consideration.

Now let us consider a infinitely long cylinder along the z-axis which is filled with mass. And let us assume that this mass distribution is uniform along the z-axis. Our purpose of this section is to obtain an exterior solution for that situation. To do so we introduce a coordinates system (t, z, ρ, ϕ) and study the geodesic equation of a test particle.

At first let us take tetrads for our system as,

$$\mathbf{W}^{\hat{t}} = e^{\tau}\ dt \ , \quad \mathbf{W}^{\hat{z}} = e^{\zeta} dz \ , \quad \mathbf{W}^{\hat{\rho}} = e^{R} d\rho \ , \quad \mathbf{W}^{\hat{\phi}} = \rho e^{\Phi} d\phi \ . \tag{5.79}$$

Then our spin-connections are,

$$\begin{aligned}
\Gamma_\mu{}^{\hat\rho\hat\phi} &= -\Gamma_\mu{}^{\hat\phi\hat\rho} = \frac{e^{R-\Phi}\partial_\phi R}{\rho}\partial_\mu\rho - e^{\Phi-R}(1+\rho\partial_\rho\Phi)\ \partial_\mu\phi, \\
\Gamma_\mu{}^{\hat\phi\hat z} &= -\Gamma_\mu{}^{\hat z\hat\phi} = e^{\Phi-\zeta}\rho\ \partial_z\Phi\ \partial_\mu\phi - \frac{e^{\zeta-\Phi}\partial_\phi\zeta}{\rho}\ \partial_\mu z, \\
\Gamma_\mu{}^{\hat z\hat\rho} &= -\Gamma_\mu{}^{\hat\rho\hat z} = e^{\zeta-R}\ \partial_\rho\zeta\ \partial_\mu z - e^{R-\zeta}\ \partial_z R\ \partial_\mu\rho, \\
\Gamma_\mu{}^{\hat t\hat\rho} &= -\Gamma_\mu{}^{\hat\rho\hat t} = e^{\tau-R}\ \partial_\rho\tau\ \partial_\mu t - e^{R-\tau}\ \partial_t R\ \partial_\mu\rho, \\
\Gamma_\mu{}^{\hat t\hat\phi} &= -\Gamma_\mu{}^{\hat\phi\hat t} = \frac{e^{\tau-R}\ \partial_\phi\tau}{\rho}\ \partial_\mu t - e^{\Phi-\tau}\rho\ \partial_t\Phi\ \partial_\mu\phi, \\
\Gamma_\mu{}^{\hat t\hat z} &= -\Gamma_\mu{}^{\hat z\hat t} = e^{\tau-\zeta}\ \partial_z\tau\ \partial_\mu t - e^{\zeta-\tau}\ \partial_t\zeta\ \partial_\mu z\ ,
\end{aligned} \tag{5.80}$$

where all functions depend on (t, z, ρ, ϕ). Of course we can use the static Weyl coordinates, and it will efficiently simplify all dependencies of functions. But we will show in this section that study on the geodesic equation will give us the same result without the static Weyl's coordinates.

Now observe that there exists no acceleration of the test particle except the ρ-direction. And since we are dealing with a static case, it is natural to guess

$$\mathbf{B}_\mu = h(\rho)\ (\partial_\mu t)\ \hat\rho\ . \tag{5.81}$$

We should be more careful to guess the form of $\mathbf{A}_\mu$. Let us consider a test particle which is moving along the z-axis. Then it will freely fall into the z-axis so that its rotation will be generated about $\hat\phi$-direction. Next, observe a particle which is moving along the ϕ-direction. It also will freely fall into the z-axis. So there must be a gauge potential for the rotation about z-direction. But a particle which is moving along the ρ-direction will not feel any rotation.

In summary, we can guess

$$\mathbf{A}_\mu = f(\rho)\ (\partial_\mu \phi)\ \hat{z} + g(\rho)\ (\partial_\mu z)\ \hat{\phi}\ . \tag{5.82}$$

From (5.81) and (5.82), we get gauge potentials,

$$\begin{aligned}
\Gamma_\mu{}^{\hat{\rho}\hat{\phi}} &= -\Gamma_\mu{}^{\hat{\phi}\hat{\rho}} = f(\rho)\ (\partial_\mu \phi)\ , \quad \Gamma_\mu{}^{\hat{\phi}\hat{z}} = -\Gamma_\mu{}^{\hat{z}\hat{\phi}} = 0\ , \\
\Gamma_\mu{}^{\hat{z}\hat{\rho}} &= -\Gamma_\mu{}^{\hat{\rho}\hat{z}} = g(\rho)\ (\partial_\mu z)\ , \quad \Gamma_\mu{}^{\hat{t}\hat{\rho}} = -\Gamma_\mu{}^{\hat{\rho}\hat{t}} = h(\rho)\ (\partial_\mu t)\ , \\
\Gamma_\mu{}^{\hat{t}\hat{\phi}} &= -\Gamma_\mu{}^{\hat{\phi}\hat{t}} = 0\ , \quad \Gamma_\mu{}^{\hat{t}\hat{z}} = -\Gamma_\mu{}^{\hat{z}\hat{t}} = 0\ .
\end{aligned} \tag{5.83}$$

Comparing (5.80) with (5.83), we know dependencies of functions

$$\tau = \tau(\rho)\ , \quad \zeta = \zeta(\rho)\ , \quad R = R(\rho)\ , \quad \Phi = \Phi(\rho)\ , \tag{5.84}$$

and we obtain the relations between the above functions and vierbein,

$$h(\rho) = e^{\tau - R}\ \tau'\ , \quad g(\rho) = e^{\zeta - R}\ \zeta'\ , \quad f(\rho) = -e^{\Phi - R}(1 + \rho\Phi')\ , \tag{5.85}$$

where the prime denotes $d/d\rho$. Since $R = R(\rho)$, we can rescale $R = \tau \equiv \lambda - \zeta$ without loss of any generality. And let us change the variables $\rho e^{\Phi} = \alpha e^{-\zeta}$. Later we can set the radial variable as $\rho' = \alpha$. Then non-vanishing spin-connections are written as

$$\begin{aligned}
\Gamma_\mu{}^{\hat{z}\hat{\rho}} &= e^{2\zeta - \lambda}\zeta'\ \partial_\mu z\ , \quad \Gamma_\mu{}^{\hat{\rho}\hat{\phi}} = -e^{-\lambda}(-\alpha' + \alpha\zeta')\ \partial_\mu \phi\ , \\
\Gamma_\mu{}^{\hat{t}\hat{\rho}} &= (\lambda - \zeta)'\ \partial_\mu t\ ,
\end{aligned} \tag{5.86}$$

and the coefficients which can be obtained by interchanging the last two indices of the above connection. With this our field equations reduce into,

$$\alpha'' = 0\ , \qquad (\alpha\zeta')' = 0\ , \qquad \alpha'\lambda' = \alpha\ \zeta'^2\ . \tag{5.87}$$

These equations are easy to solve, so we can obtain the Levi-Civita solution;

$$ds^2 = \rho^{2m} dt^2 + \rho^{m^2-m}(dz^2 + d\rho^2) + a^2 \rho^{2-m} d\phi^2 \ , \tag{5.88}$$

where we did some coordinates transformation. Here m and a are constants.

Chapter 6

Discussion

This paper has attempted to establish gauge theoretic characters of Einstein's theory. Our starting point was the fact that Einstein's theory has the Lorentz degrees of freedom. But it is well known that we cannot construct any satisfactory theory for gravitation in a Yang-Mills type since the metric on the Lorentz group is not positive-definite. An usual way to avoid this problem is to construct a first-ordered formulation for the field strength so that the action is identified with the Einstein-Hibert action. And we followed that trace.

The first half of this paper dealt with Abelian decomposition of Einstein's theory. Whole processes are summarized as following; Introducing an additional magnetic symmetry to the system and decompose the gauge connection into the restricted part which preserves the isometry covariantly and valence part which transforms covariantly under gauge transformation. A gauge theory is called a restricted gauge theory if it has no valence part. It is very important that the restricted part has the full gauge degrees of freedom.

Even though we can apply Abelian decomposition with any Abelian subalgebra of the Lie group under consideration, we decomposed the Lorentz group with maximal Abelian subgroup of it in order to obtain the simplest picture. But the Lorentz group was rank-two, so we could obtain two different types of Abelian decomposition. They are called A_2- and B_2- types, respectively.

The maximal Abelian subgroup of A_2-type is generated by L_3 and K_3. Since norms of isometry vectors do not vanish, this type of decomposition is called "non-light-like". We should emphasize again that the restricted connection of this isometry inherits all topological characters of spacetime by the topological potential $\vec{C}_\mu$. In chapter 4, we could see how this potential represented a topological invariant for the cosmic string.

And it is obvious that the restricted potential of the A_2-type contains all vacuum potentials, and we reviewed the most general vacuum potential for the Lorentz group in chapter 5. And the vacuum structure of the Lorentz group was same for the $SU(2)$ gauge theory. It is natural because the Lorentz group has $SU(2)$ as its subgroup and subgroups consist of boosting generators has no relevance with vacua. Therefore A_2-type's restricted theory for gravity is adapted to describe topological character of spacetime.

On the other hand, the isometry vectors of the B_2-type are $L_1 + K_2$ and $L_2 - K_1$. Since norms of isometry vectors vanish, we call it "light-like". As we discussed in chapter 4, all solutions of Einstein's equation for the radiative type, namely, Petrov N-type could be described by the B_2-theory. And we should emphasize again that the field equation for this type is much simpler than the full Einstein's equation. Indeed we found the simplest equation which governs all radiative solution. It would be helpful to study nonlinear superposition rule for gravitational waves with this equation. And it will be discussed in a consequent paper.

In chapter 4, I showed two fundamental solution for the both types of restricted theories. The cosmic string was a fundamental solution of the A_2-gravity. Its exterior solution was so trivial except the deficit angle of spacetime, and this angle was determined by the tension of the string. As we discussed, the cosmic string can be also described by non-A_2-gravity. In that case we could solve the Einstein's equation with a usual ansatz in field theories. That was the first example which showed us that some techniques in field theories can be applied to the Einstein's theory.

I showed the Einstein-Rosen-Bondi's wave was a fundamental solution for B_2-gravity. And I also showed the general pp(parallel and plane wave-fronted)-waves can be described by the B_2-gravity. As has been noticed earlier, all Petrov N-type's solution can be contained in the B_2-gravity. But some superposition of two gravitational waves are not gravitational wave again. Sometimes two gravitational waves make a Schwarzchild solution. This kind of problem frequently appears in the studies on the nonlinear superposition rule of gravitational waves. It will be greatly interesting if we can understand the change of isometry in a nonlinear superpositions of gravitational waves.

And another important point is that we obtained two Einstein-Maxwell equation in the B_2-gravity so that we could introduced the 1-form gauge potentials for gravitational waves. So far it is ambiguous that the quantum field of gravitation is the metric or not. As Robert Geroch pointed out, the first problem of quantum gravity is not how to quantize, but what we should quantize. Such a 1-form potential for gravitational waves was first introduced by Y. M. Cho. And in this paper, I gave the first concrete example of the 1-form gauge potentials for gravitational waves.

In chapter 5, we discussed some gauge theoretical methods to obtain classical solutions of Einstein's equation. The first step was to find an appropriate vacuum potential for the system. And the second step was to take a nice ansatz for the spin-connections and the vierbein fields of the system. This was the most important step. In order to choose such an ansatz, we should invest the geodesic equation in an orthonormal frame. In the orthonormal frame our gauge potentials appear as electric and magnetic fields in the Lorentz force,

$$\begin{aligned} a^{\hat{0}} &= -\mathbf{B} \cdot \mathbf{v} \ , \\ \mathbf{a} &= \frac{dv^i}{d\tau} \mathbf{e}_i = -v^{\hat{0}} \mathbf{B} + \mathbf{v} \times \mathbf{A} \end{aligned} \tag{6.1}$$

where $\mathbf{B} = u^\mu \mathbf{B}_\mu$, and $\mathbf{A} = u^\mu \mathbf{A}_\mu$ for $u^\mu = \dfrac{dx^\mu}{d\tau}$ and $v^{\hat{i}} = e_\mu{}^i \dfrac{dx^\mu}{d\tau}$. This equation teach us which forms of gauge potentials are possible. And it would enables us to find gauge potentials directly by observing the motion of the test particle in any astrophysical experiment.

As we discussed in the cases of the cosmic string and the Schwarzchild solution, some techniques in field theories can be applied to Einstein's theory. Therefore it is natural to guess whether a reverse process is possible. Indeed the number of discovered solutions of Einstein's equations are overwhelmingly larger than that of Yang-Mills equation. Therefore if we develop gauge theoretical technique for Einstein's theory, then we can apply that methods to Yang-Mills equation, reversely. It would be great if one can find a new classical solution of Yang-Mills equation in such a way.

In summary, I found some fundamental solutions of two types of restricted gravity, and suggested a topological potential for the cosmic string and a 1-from potential for the Einstein-Rosen-Bondi's wave. And I obtained some well known solutions of Einstein's equation in a new method under a gauge theoretical consideration. And in the section 5.3 and 5.4, I showed some hidden Lorentz structure of the gauge potentials of Einstein's theory that are analogous with electrodynamics. This was so, because electromagnetic fields transform covariantly under Lorentz transformations.

But several questions merit discussion. Quantization for spin-1 graviton, nonlinear superposition rules of gravitational waves, obtaining a new topological solution for gravitation and so on. It is to be hoped that this paper will serve as a platform from which such studies may be undertaken.

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국문초록

이 논문에서 우리는 아인슈타인 이론에 대한 가환분해로부터 도출되는 몇 가지 결과를 다루려 한다. 먼저 A_2와 B_2 각각의 유형에 해당하는 제한된 중력 이론에 대한 기본적인 두 개의 해를 보일 것이다. 실제로 A_2-분해는 시공간의 위상적 성질을, B_2-분해는 중력파를 다루는데 용이함을 보일 것이다. 그리고 아인슈타인 방정식을 게이지 이론적인 방법으로 푸는 새로운 방법을 제안할 것이다. 이 방법을 이용하여 몇 개의 잘 알려진 해들을 유도할 것이다.