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Exact Results on Higher Dimensional Quantum Field Theories

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김 준 호

Abstract

Exact Results on Higher Dimensional Quantum Field Theories

Joonho Kim
Department of Physics and Astronomy
The Graduate School
Seoul National University

This thesis aims at studying the higher dimensional quantum field theories, engineered from the string theory. These theories are genuinely strongly interactive, thus being difficult to be understood within the conventional QFT framework. In particular, I focus on those 5d / 6d QFTs which can be deformed to the weakly coupled 5d Yang-Mills theories, in which the deformation is caused either by a relevant operator or by a circle compactification. Instantons are crucial for observing the physics of 5d / 6d QFTs which correspond to the UV fixed points of certain 5d SYMs. In the first half of the thesis, I obtain the general expression for the instanton partition function of 5d SYMs and apply it to study the spectrum of various UV QFTs. The second half focuses on the 6d non-critical strings, which are key objects of 6d QFTs. Two types of 6d strings, M-strings and E-strings, are considered, for which the worldsheet gauge theories are explicitly developed.

Keywords: Yang-Mills instanton, self-dual string, partition function, 5d / 6d SCFT
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Chapter 1

Introduction

Quantum field theory (QFT) has been very successful in describing the quantum phenomena in particle physics. It is the framework about the fundamental physical entity called the field, whose quantum excitations correspond to elementary particles. It faithfully reflects both fundamental principles of modern physics: special relativity and quantum mechanics. The interaction among particles is characterized as the interaction among underlying fields. Particle creation and annihilation, which are the distinctive phenomena of the relativistic quantum physics, are naturally incorporated in this framework. Successful models in describing quantum phenomena of elementary particles, such as quantum electrodynamics (QED) or the Standard Model of particle physics, are all based on the quantum field theory.

Despite of the tremendous success of the Standard Model, the current understanding on the fundamental physics is still far from satisfactory. The current formulation of quantum field theory heavily depends on the perturbative approach which does not operate without the existence of the weakly coupled Lagrangian description. This somewhat hinders the investigation of many important phenomena that arise in the strongly interacting system in which the Lagrangian description becomes no longer effective; One particular example is the confinement, which is essential to comprehend the various phases of quark-gluon matter. In this regard, understanding the strongly interacting field theory is of vital importance for quantitative studies on those phenomena which are inaccessible via the conventional procedure of quantum field theory.

The novel and powerful approach to the quantum field theory, including the strongly interacting one, comes from the string theory. String theory is the unique quantum theory which exhibits rich dynamics, including the gravitational and gauge interactions as well as extra colorful interactions that are rather unfamiliar. Thanks to its richness, most of important field theories can be properly situated in the string theory. For example, the dynamics of D-branes under the low energy limit is governed by the supersymmetric gauge theory [1]. Surprising dualities between different gauge theories are inferred from knowledge of the brane system; These dualities often furnish the way to study the strongly coupled gauge theory. Branes within or ending on branes successfully describe solitons or defects existing in the gauge theory. All in all, the stringy viewpoint can provide complementary understanding on large classes of supersymmetric gauge theories.

String theory also accommodates a great number of novel quantum field theories. They either reside on certain types of branes, e.g., NS5-branes in the type IIA string theory, or are engineered via Calabi-Yau compactification of the string theory. These new theories preserve the superconformal symmetry and are strongly interactive, displaying exotic properties which have not been observed in the ordinary gauge theory. It is therefore not surprising that most of them do not have the Lagrangian description which enables the direct perturbative investigation. Nevertheless, the string duality occasionally proposes the weakly coupled gauge theory as the effective description for those novel field theories. One common example is the six-dimensional $\mathcal{N} = (2,0)$ field theory which contains charged strings as light objects [2, 3]. Upon compactification, its effective description can be given as the five-dimensional $\mathcal{N} = 2$ gauge theory which was once known as UV-incomplete [4, 5]. It is the very stringy prediction that the non-perturbative correction of 5d gauge theory makes it consistent in the UV regime, being uplifted to the six-dimensional $\mathcal{N} = (2,0)$ field theory.

In this thesis, I shall study superconformal field theories defined in dimensions higher than four. These theories are of great significance as tractable examples which are clearly beyond the conventional QFT framework, demonstrating that original lessons on the quantum field theory can be drawn from the string theory. First of all, the very existence of higher dimensional quantum field theory is the remarkable prediction from the string theory, contrasting with the conventional knowledge that no UV-consistent field theory can exist in dimensions higher than four. Second, the argument based on the string theory often specifies certain distinct properties of the higher dimensional field theory, e.g., symmetry, field content, and relation to other theories, the last of which turns out to be very useful for deducing the weakly coupled effective description. My main focus in this thesis will be the field theoretical computation of certain observables in the higher dimensional quantum field theory, via its effective description that is again found by the string theory. Sometimes the observable of choice confirms back that the stringy predictions on the higher dimensional field theory are indeed correct, by manifesting the expected characteristics.

The first class of quantum field theories that I shall study is the five-dimensional superconformal theory that is engineered from the M-theory wrapping on the Calabi-Yau 3-fold [6, 7, 8]. This class of theories can be relevantly deformed by turning on the gauge kinetic term

$$-\frac{1}{2g_5^2}\int d^5x\,\mathrm{tr}F^{\mu\nu}F_{\mu\nu} \tag{1.1}$$

and its supersymmetric completion. Flowing down to the infrared regime, the resulting effective theory turns out to be the 5d supersymmetric gauge theory, in which non-perturbative corrections play the crucial role for the UV consistency. Among these 5d superconformal field theories which were systematically classified in [8], one particular type of theories can be also engineered from the D4-D8-O8 brane system in the type I' string theory. Furthermore, the string duality between the type I' and heterotic theories gives the detailed expectation that this type of theories actually enjoys the E_{N_f+1} global symmetry, even if the D4-D8-O8 brane system manifests the $SO(2N_f) \subset E_{N_f+1}$ symmetry only. Here N_f denotes a number of D8-branes, restricted to be ($N_f \leq 7$) [6]. I will therefore pay special attention to verify this prediction on the type of 5d superconformal theories obtained from the D4-D8-O8 brane configuration.

The second class of quantum field theories in my interest is the 6d $\mathcal{N} = (2, 0)$ superconformal field theory engineered from the type IIB string theory wrapping on the singularity of type A, D, or E [2]. In particular, the type A theories govern the low energy dynamics of M5-branes [3], thus being essential to understand the M-theory. Upon compactification, they can also engineer a large number of novel quantum field theories in four and lower dimensions, which are again strongly interacting. The 6d (2,0) theory includes, instead of the usual gauge interaction, the peculiar interaction conveyed by the 2-form antisymmetric tensor whose field strength is self-dual in the six-dimensional sense. It therefore accommodates charged strings coupled to the self-dual 2-form tensor, whose dynamics is described by the strongly interacting 2d CFT. These strings are also known as M-strings which are induced objects from membranes suspended between a pair of M5-branes [9].

In the tensor branch of the 6d $\mathcal{N} = (2, 0)$ theory where all M5-branes are arranged to be separated from one another, the M-strings are the key objects. In [9], the effective description for the M-string CFT was found as the two-dimensional $\mathcal{N} = (0, 4)$ gauge theory. Certain observable of the M-string CFT was explicitly computed via this 2d gauge theory, turning into an observable of the grounding 6d $(2, 0)$ theory in the tensor phase. Alternatively, one can also study the 6d $(2, 0)$ theory via its circle compactification. The duality between the type IIA and M theories dictates that the circle compactification of the 6d $(2, 0)$ theory results in the five-dimensional $\mathcal{N} = 1^*$ gauge theory, of which coupling constant g_5 satisfies $g_{5\text{d}}^2 \propto R$ for the circle radius R. Putting it differently, the 6d $(2, 0)$ theory emerges in the strong coupling limit of the 5d $\mathcal{N} = 1^*$ gauge theory, of which non-perturbative phenomena must be accounted to capture the correct physics. For the 6d $(2, 0)$ theory in the conformal phase, radial quantization puts the superconformal theory on $S^5 \times S^1$, so that the 5d gauge theory on S^5 provides the effective description. One famous characteristic of the 6d $(2, 0)$ superconformal theory (of the rank N) is the

N^3 scaling behavior of the Casimir energy, predicted from the dual AdS_7 gravity solution. [10, 11, 12] carefully chose an observable which captures the Casimir energy and checked that the above prediction is correct.

The third class of quantum field theories that I will consider is the six-dimensional $\mathcal{N} = (1, 0)$ superconformal field theory with the E_8 global symmetry. This is the theory describing the low energy dynamics of multiple M5-branes probing the boundary M9-plane (also known as the Horava-Witten wall) in the heterotic M-theory. Plenty of characteristics are parallel to the 6d $(2, 0)$ theory, including the interaction conveyed by the self-dual 2-form tensor, and correspondingly, the existence of charged strings. Because the boundary M9-plane carries the E_8 symmetry, the charged strings inherit the very E_8 symmetry in case they are induced from membranes suspended between the M9- and the M5-brane. These strings are named as the E-strings [13].

In the tensor branch, where all M5-branes are separated from each other as well as the boundary M9-brane, E-strings are the core objects in understanding the 6d (1,0) theory. Their dynamics is governed by the mysterious 2d CFT which is strongly interacting. Nevertheless, the string duality can specify the two-dimensional effective gauge theory which flows down to the E-string CFT. This effective theory can be exploited for computing some observables of the E-string CFT, which are by themselves organized into the quantity of the underlying 6d (1,0) superconformal field theory. Alternative, the 6d (1,0) theory also can be studied via circle compactification. However, unlike the (2,0) theory, the naive circle reduction does not induce the weakly coupled gauge theory. What is required to obtain the effective gauge theory is turning on the E_8 Wilson line along the reduced circle, which breaks the E_8 global symmetry into the $SO(16)$ subgroup. After all, one obtains the five-dimensional $\mathcal{N} = 1$ $Sp(N)$ gauge theory with 1 antisymmetric and 8 fundamental matter fields, in which N is the rank of 6d theory [14]. Here, the effective gauge coupling g_5 satisfies the relation

$$g_{\mathrm{5d}}^2 \propto R$$

for the circle radius R. In this thesis, both effective approaches will be utilized for investigation of this 6d (1,0) theory in the tensor phase and therefore the physics of E-strings.

For all cases that are studied, the five-dimensional supersymmetric gauge theory emerges as the weakly coupled description of the UV superconformal field theory. This fact is actually unpredictable from the conventional QFT viewpoint which has regarded the 5d gauge theory as an UV-incomplete theory due to its non-renormalizability. I will anyhow accept the stringy prediction that the strong coupling limit of a certain 5d gauge theory corresponds to the five or six-dimensional quantum field theory, then aiming to study this superconformal field theory using the 5d effective gauge theory. One should therefore take into account the non-perturbative phenomenon of 5d gauge theory; Five-dimensional gauge theory contains solitonic particles called instantons, whose mass

$$m_{\text{inst}} \propto 1/g_5^2$$

becomes zero in the strong coupling limit. Being the lightest objects, instantons are essential for capturing the correct physics in the strong coupling regime.

For this reason, I will choose the observable in the 5d gauge theory which incorporates non-perturbative instanton effects. Moreover, in case the chosen observable is invariant along the renormalization group flow, that can be interpreted as the observable of UV superconformal field theory even if the actual computation is done in the weakly coupled gauge theory. In this thesis, I will focus on the instanton partition function [15] satisfying the above two requirements. It is basically the Witten index capturing all instanton corrections.

There are two related observables in the UV quantum field theory, which can be obtained from the instanton partition function in the corresponding 5d gauge theory. The first one is the Coulomb (or tensor) branch index which is the supersymmetric partition function for the Coulomb (or tensor) branch of UV CFT, defined on $R^4 \times R^1$ (or $R^5 \times R^1$). The second quantity is so-called the superconformal index, which is again the Witten index for the radially quantized UV CFT defined on $S^4 \times R^1$ (or $S^5 \times R^1$). The superconformal index enumerates allowed quantum states in the conformal phase of UV CFT, or equivalently, the local operator spectrum thanks to the state-operator correspondence in conformal field theory. For the 5d superconformal field theories, the relation between the Coulomb phase index and the superconformal index has been established in [16]. I will also study the relation between the tensor phase index and the superconformal index for the 6d superconformal theories. In any case, the Coulomb (or tensor) branch index acts like a minimal building block for construction of the superconformal index.

The outline of this thesis is as follows. Chapter 2 is devoted to the brief review of higher dimensional superconformal field theories. Included topics are the discovery of 6d $(2,0)$ SCFT, the properties of 6d $(1,0)$ SCFTs from both viewpoints of the $E_8 \times E_8$ heterotic small instantons and F-theory compactification on Calabi-Yau three-folds, and the existence and classification of 5d SCFTs from the M-theory on CY3.

In Chapter 3, I discuss the computation of the instanton partition function in the 5d gauge theory. It begins with the brief review of Yang-Mills instantons and the application of instanton counting to the 4d $\mathcal{N} = 2$ gauge theories. Then I will consider the path integral representation of the supersymmetric index. Distinguishing all quantum fluctuations of fields into massive modes and massless zero-modes, one integrates out the massive modes and takes the moduli space approximation for zero-modes. This gives the one-dimensional instanton quantum mechanics, which is the non-linear sigma model with singular target space. It has a conical singularity arising at the point where the instanton size shrinks to zero. Nevertheless, one can consistently resolve the singularity by introducing a gauge field to the instanton quantum mechanics. The resulting theory is known to be the ADHM quantum mechanics which is actually exploited for computing the 5d observable. In fact, the mechanical partition functions can be combined together into the 5d instanton partition function. The partition function for the ADHM quantum mechanics was studied in [15] more than a decade ago. However, there remained one missing step that one needs to clarify; to specify the proper contour to complete the integral over the mechanical zero-modes.

Chapter 4 discuss that the mechanical partition function often involves extra bound states apart from instantonic states in 5d gauge theory. Having examined this phenomenon, I find that additional bound states can possibly join up when the instanton quantum mechanics is resolved by introducing the gauge field. For many examples of 5d gauge theory, which correspond to the effective description of UV superconformal field theory, I separately identify the extra contribution and divide it out from the mechanical partition function by hand. After all, the result is the desired instanton partition function which can be simultaneously interpreted as the index of UV quantum field theory. For the 5d superconformal field theory, particularly the one engineered from the D4-D8-O8 brane configuration, I will compute the superconformal index to investigate the local operator spectrum of the 5d UV field theory, and furthermore, to check if the stringy prediction of the global symmetry enhancement is indeed correct.

Chapter 5 studies the self-dual strings in 6d superconformal field theories. My discussion will be limited to the 6d tensor branch physics. The 6d tensor branch index corresponds to the instanton partition function of 5d gauge theory whose details are discussed in Chapter 3. As an alternative approach, I shall exploit the string duality to find the effective description for the strongly interactive self-string CFTs. I will develop the 2d gauge theory on the string worldsheets. For the case of M-strings, which are self-dual strings arising in the $(2,0)$ SCFTs, I will briefly review the construction of [9, 17]. For the case of E-strings, which are self-dual strings living on the 6d $(1,0)$ E_8 SCFT, I explain in detail the construction of the worldsheet UV gauge theory and compute the supersymmetric partition function. The string partition functions are combined into the 6d tensor branch index, exhibiting the full E_8 global symmetry. This approach provides the better observable than the 5d $Sp(N)$ instanton partition function, because the latter only exhibits $SO(16) \subset E_8$ global symmetry due to the the E_8 Wilson line required during construction of the 5d effective gauge theory. At last, I will check the results from both effective theories and check if they are consistent to each other.

Chapter 2

Higher-dimensional QFTs

According to the Nahm's classification [18], 6 is the highest possible dimension in which the superconformal field theory can be defined. However, the first example of higher-dimensional QFTs was belatedly discovered from the string theory in 1995. In this chapter, I will review the existence and properties of various higher-dimensional QFTs, including those which are thoroughly studied in later chapters.

2.1 Six-dimensional theory

2.1.1 6d $(2,0)$ theory

The maximally supersymmetric $\mathcal{N} = (2,0)$ theory was first investigated in [2] as the type IIB string theory wrapped on $\mathbb{R}^6 \times \mathrm{K3}$. Its moduli space $\mathcal{M} = SO(21,5;\mathbb{Z})\backslash SO(21,5;\mathbb{R})/(SO(2$ $SO(5))$ has orbifold singularities, at which in the type IIA case the extra massless particle emerges and the gauge symmetry enhances [19]. However, the same cannot happen for the type IIB theory because the $(2,0)$ chiral supersymmetry does not allow a gauge multiplet. The clue comes from the T-duality holding between type IIA and IIB theories on $\mathbb{R}^5 \times S^1 \times \mathrm{K3}$, where both must get extra massless particles.

Compactifying the 6d theories on the circle of radius R, the 5d and 6d coupling constants are related as $1/\lambda_6 = R/\lambda_5$. The type IIA and IIB theories on $\mathbb{R}^5 \times S^1 \times \mathrm{K3}$ can be equivalent only if both 5d couplings are the same, $R_A/\lambda_{6,A}^2 = R_B/\lambda_{6,B}^2$, and furthermore the T-duality holds when $R_A = R_B^{-1}$. These relations altogether imply that the 6d coupling constants of both type IIA and IIB theories are related as

$$\frac{1}{\lambda_{6,A}} = \frac{R_B}{\lambda_{6,B}}. \tag{2.1}$$

Notice the type IIA theory on $\mathbb{R}^6 \times \mathrm{K3}$ has W-bosons of mass $\epsilon/\lambda_{6,A}$, where ϵ denotes the moduli space distance from a singularity. Since the mass does not change upon a circle compactification, the dual IIB theory on $\mathbb{R}^5 \times S^1 \times \mathrm{K3}$ should have the particle of mass $\epsilon R_B/\lambda_{6,B}$. This can be possible if the 6d theory has a string of tension

$$T = \epsilon/\lambda_{6,B} \tag{2.2}$$

so that the 5d massive particle can be induced from the string enclosing the circle.

Recall the D3-brane has tension of order $1/\lambda_{6,B}$. The 6d string can be therefore realized as the D3-brane wrapping the 2-sphere S_ϵ of area ϵ. This string is the source of the 2-form tensor B, satisfying the self-dual relation $H = dB = *_6 H$, which is induced from the self-dual 4-form C_4 coupled to the D3-brane as follows.

$$B = \int_{S_\epsilon} C_4 \tag{2.3}$$

The 6d string becomes tensionless if $\epsilon \to 0$, being light objects that cannot couple to the gravity. As a result, the 6d $\mathcal{N} = (2,0)$ theories are non-trivial quantum field theories which describe interacting strings even in the low energy limit. For all possible ADE types of singular K3's, distinct 6d $(2,0)$ QFTs can be constructed.

When a $(2,0)$ QFT is of the A-type, this theory can be interpreted as the low-energy theory for the stack of M5-branes. In the 11d supergravity, [3] considers the configuration that an M2-brane is stretched between two M5-branes. The charge conservation requirement, associated to the 3-form tensor A_3 in the 11d SUGRA, forces that the boundary of M2-brane must carry the charge of the self-dual 2-form tensor B. In this picture, the M2-brane boundary induces the self-dual string living in the 6d $(2,0)$ theory. This string becomes lightweight as two M5-branes approach to each other, eventually being tensionless when M5-branes coincide.

2.1.2 6d $(1,0)$ theory

Turning to the less supersymmetric $\mathcal{N}=(1,0)$ theories, I will firstly review the heterotic string theory on $\mathbb{R}^6 \times \mathrm{K3}$. There are four types of $(1,0)$ SUSY multiplets: gravity multiplet, tensor multiplet, vector multiplet, and hypermultiplet. Any scalar field in a $(1,0)$ theory belongs to either a tensor multiplet or a hypermultiplet. The $(1,0)$ supersymmetry is very restrictive such that the generation of a potential involving hypermultiplets or dilatons (scalars in tensor multiplets) is not permitted. Inspection on the vacuum moduli space is thus important to understand the $(1,0)$ theory and identify the low energy QFTs.

The hypermultiplet moduli space is structured as a quaternionic manifold, on which the metric is independent of the dilatons. A singularity in this space should be interpreted as a massive hypermultiplet going to zero mass, which is decomposed into a massless vector multiplet and a massless hypermultiplet. In particular, [20] has studied the small instanton singularity of $SO(32)$ heterotic string theory on K3, concluding that the extra $SU(2)$ gauge symmetry appears in the low energy theory, together with the massless, bifundamental hypermultiplet under $SO(32) \times SU(2)$. More generally, the emergent gauge symmetry becomes $SO(32) \times Sp(k)$ when k instantons collapse at the same point. The extra massless hypermultiplet is therefore in the bifundamental representation of $SO(32) \times Sp(k)$.

The tensor branch opens up at those small instanton singularities, on which the metric is independent of hypermultiplet scalars. It has been argued that singular points in the tensor branch corresponds to the phase transition [21, 22], where the gauge coupling g becomes divergent, and accordingly, the instanton string of 6d gauge theory becomes tensionless. For example, at the specific point in the hypermultiplet moduli space of the type I string theory on K3, where all of 24 instantons coherently shrinks down, the gauge symmetry becomes $SO(32) \times Sp(24)$. While the $Sp(24)$ anomaly vanishes, the $SO(32)$ anomaly requires a tensor multiplet to cancel itself via the Green-Schwarz mechanism [23]. The gauge kinetic term can be written as [22]

$$-(2e^{\phi})\mathrm{tr}F^{\mu\nu}F_{\mu\nu}-(e^{-\phi}-2e^{\phi})\mathrm{tr}\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} \tag{2.4}$$

where $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$ respectively denote the field strength associated to the $Sp(24)$ and $SO(32)$ gauge groups. In the following, I will focus on the instanton strings associated to each gauge group. The $Sp(24)$ instantons, which are electrically charged, are realized as the D1-branes bounded to the D5-branes under the small instanton limit [24]. The $SO(32)$ instantons are dyonic strings, being geometrically realized as the D5-branes wrapping on $K3$. At $\phi=\phi_0$ such that $e^{2\phi_0}=2$, the $SO(32)$ gauge coupling diverges, implying the $SO(32)$ instanton strings become tensionless [22]. However, in any case, the $Sp(24)$ instanton strings have finite tension. Since this theory has a definite mass scale, it is not a conformal field theory but interpreted as an $\mathcal{N}=(1,0)$ little string theory, which does not have the definite energy-momentum tensor. The $Sp(24)$ instanton strings are little strings which provide the winding modes upon circle compactification required for T-duality.

To engineer the $\mathcal{N}=(1,0)$ QFTs, one can wrap the $E_8\times E_8$ heterotic string theory on $\mathbb{R}^6\times K3$. It is required to have (n_1,n_2) instantons in the two E_8 factors, satisfying $n_1+n_2=24$. Due to the fact that $1,2,3$ E_8 instantons are not allowed, the possbilities are

$$(n_1,n_2)=(12+n,12-n)\ \text{for}\ n=0,1,\cdots,8,12. \tag{2.5}$$

They are proven to be dual of the F-theory on Hirzeburch surface F_n [25, 26]. In fact, the F-theory construction is more useful to identify the properties of 6d theories at the generic point in the moduli space, where the gauge group is maximally broken due to the presence of instantons.

1. The $n=0$ and $n=2$ theories are equivalent, corresponding to the 6d $(2,0)$ QFT which is engineered as the type IIB string theory on $\mathbb{R}^6\times K3$. This theory has no gauge symmetry [25, 27].

2. The $n=1$ theory has no gauge symmetry, since each E_8 is generically broken if there are 10 or more instantons. The Higgs branch of this theory can be

obtained by blowing down the exceptional curve in F_1, yielding the F-theory on $\mathbb{P}^2$ in which no tensor multiplet is allowed. The phase transition from $\mathbb{P}^2$ to F_1 is the same as the transition in which an M5-brane, which corresponds to a small E_8 instanton [14], is emitted from the boundary E_8 domain wall in the 11d heterotic M-theory [20].

3. For $n = 3$, the 6d theory has $SU(3)$ gauge symmetry with no matter [26].

4. For $n = 4$, the 6d theory is dual to the $SO(32)$ heterotic string theory wrapping on $\mathbb{R}^6 \times K3$. Under the presence of 24 instantons, the unbroken gauge group turns out to be $SO(8)$ with no extra matter [25, 22, 20].

5. For $n = 5$, the 6d theory has F_4 gauge symmetry with no matter [28].

6. For $n = 6$, the 6d theory has E_6 gauge symmetry with no matter [26].

7. For $n = 7$, the 6d theory has E_7 gauge symmetry with additional half-hypermultiplet $\frac{1}{2}\mathbf{56}$ [26].

8. For $n = 8$, the 6d theory has E_7 gauge symmetry with no matter [26].

9. For $n = 12$, the first E_8 is completely broken while the second E_8 remains to be unbroken [22, 26].

All these theories include a single tensor multiplet that couples to a non-critical string, which becomes tensionless at a specific singular point in the tensor branch. When 6d strings are tensionless, these theories in fact have no energy scale, corresponding to the 6d $(1,0)$ superconformal QFTs. These non-critical strings are also called self-dual strings because they are sources of $(1,0)$ tensor multiplets which satisfy the self-duality condition.

One point that should be noticed is that all above theories are connected via the strong-coupling phase transition in the $E_8 \times E_8$ heterotic string theory on $\mathbb{R}^6 \times \mathrm{K3}$. As the heterotic string coupling g_s becomes strong, what emerges is the 11d heterotic M-theory on $\mathbb{R}^6 \times \mathrm{K3} \times S^1/\mathbf{Z}_2$, in which the ten-dimensional boundary plane (known as the Hořava-Witten wall) carries one of E_8 gauge symmetries [29]. The E_8 small instanton can be viewed as emitting an extra M5-brane stuck to the boundary plane [14]. The tensor branch opens up at the small instanton singularity, allowing the M5-brane to move along the finite segment $S^1/\mathbf{Z}_2$. If the M5-brane approaches to the other boundary, it can be absorbed into the Hořava-Witten wall as a finite-scaled E_8 instanton by transiting to the Higgs branch. This process connects the 6d $n = n_0$ theory to the $n = n_0 \pm 1$ theories, and eventually to any n theory upon iteration. In terms of the F-theory, the same transition is geometrically implemented by a sequence of blow-ups and blow-downs which changes F_n to $F_{n\pm1}$.

A number of $(1, 0)$ tensor multiplet needs not to be one. For the 6d QFTs obtained from the $E_8 \times E_8$ heterotic string theory on $K3$, the consistency condition is given by

$$n_1 + n_2 + m - 1 = 24 \tag{2.6}$$

where m denotes a number of tensor multiplets. One can maximally introduce 25 tensor multiplets by making all 24 instanons point-like. Many distinct 6d QFTs which have more than one $(1, 0)$ tensor multiplet can be engineered also from branes at orbifold singularities [30, 31, 32, 33]. More recently, the full classification of possibly all 6d QFTs, possessing the tensor branch, has been completed by classifying all bases of elliptic Calabi-Yau 3-folds on which the F-theory wraps. [34] proved that all bases of Calabi-Yau 3-folds are made of non-Higgsable clusters which hosts one of

$$SU(3), SO(8), F_4, E_6, E_7, E_8, G_2 \oplus SU(2), SU(2) \oplus SO(7) \oplus SU(2) \tag{2.7}$$

gauge algebras, joined by ADE configurations of -2 or -1 curves which does not involve the gauge algebra. Using this result, [35] classified the minimal 6d QFTs and suggested how to achieve non-minimal theories. Finally, [36] classified possibly all 6d superconformal field theories including non-minimal QFTs. Here I emphasize that most building blocks (2.7) of 6d QFTs, except those which involve multiple gauge groups, are actually 6d QFTs obtained from the $E_8 \times E_8$ heterotic string theory on $K3$ with distinct instanton configurations.

2.2 Five-dimensional theory

The first examples of interacting 5d QFTs were discovered in [6]. These 5d QFTs appeared as UV fixed points of the 5d super Yang-Mills theories, which are non-renormalizable. The argument which led to the discovery of 5d QFTs is based on the dualities of type I / type I' / heterotic string theories [37]. Soon after, [7, 31] performed the geometric analysis on the existence of strong-coupling fixed points in the 5d gauge theories with classical gauge groups. Here I summarize their results:

1. $Sp(N)$ theories can come with either $n_A = 0, 1$ antisymmetric hypermultiplet. When $n_A = 1$, there can be N_f fundamental hypermultiplets with $0 \leq N_f \leq 7$. When $n_A = 0$, there can be $N_f \leq 2N + 4$ fundamental hypermultiplets. Exceptionally at $Sp(1)$, the theories with $n_A = 0$ are identical to theories with $n_A = 1$, so $N_f \leq 7$ is allowed.

2. $SU(N)$ theories can come with bare Chern-Simons term at level κ. If the theory has N_f fundamental hypermultiplets, κ is integral if N_f is even, while κ is half an odd integer if N_f is odd. 5d UV fixed point exists if $N_f + 2|\kappa| \leq 2N$. When $N \leq 8$, one can have 1 antisymmetric and N_f fundamental hypermultiplets if $N_f + 2|\kappa| \leq 8 - N$. At $N = 4$, there can be 2 antisymmetric hypermultiplets with $N_f = \kappa = 0$. The case with $N = 2$ is exceptional as the $SU(2)$ Chern-Simons term is zero. This should be treated as an $Sp(1)$ theory, admitting $N_f \leq 7$ fundamental hypers.

3. $SO(N)$ theories can come with $n_V \leq N - 4$ hypermultiplet in the vector (fundamental) representation. For $N \leq 12$, there can be $n_S \leq 2^{6-N/2}$ spinor and $n_V \leq N - 4$ vector hypermultiplets at even N, and $n_S \leq 2^{5-(N-1)/2}$ and $n_V \leq N - 4$ at odd N.

This result is obtained from the arguments that follow: the gauge group G of the 5d gauge theory is broken to $U(1)^r$ for $r = \mathrm{rank}(G)$. The effective theory has the Abelian gauge field $A = \sum_{i=1}^r A^i T_i$ where T_i are Cartan generators of G. The theory is completely determined by the holomorphic function $\mathcal{F}(A^i)$ which is called the Seiberg-Witten prepotential [38]. Since the quantum correction should also respect the gauge invariance, the exact quantum prepotential must be given as a (at most) cubic polynomial in ϕ^i which denotes the scalar component of the vector multiplet associated to A^i. Note that the 5d minimal $\mathcal{N} = 1$ supersymmetry has 8 supercharges, implying the 1-loop exactness of the prepotential, and the general form of the quantum prepotential can be written as

$$\mathcal{F} = \frac{1}{2} m_0 h_{ij} \phi^i \phi^j + \frac{c}{6} d_{ijk} \phi^i \phi^j \phi^k + \frac{1}{12} \left(\sum_{\mathbf{r} \in \mathbf{R}} |\mathbf{r} \cdot \phi|^3 - \sum_f \sum_{\mathbf{w} \in \mathbf{W}_f} |\mathbf{w} \cdot \phi + m_f|^3 \right). \tag{2.8}$$

Here $h_{ij} = \mathrm{tr}(T_i T_j)$ and $d_{ijk} = \frac{1}{2} \mathrm{tr}(T_i \{T_j, T_k\})$. $\mathbf{R}$ are roots of the gauge group G, and $\mathbf{W}_f$ are weights of the representation in the gauge group, $\mathbf{r}_f$, under which a hypermultiplet with mass m_f transforms. The first two terms are the classical prepotential, with m_0 identified to $1/g_{cl}^2$, and c to the classical Chern-Simons coefficient normalized as

$$\mathcal{L} \ni \frac{c}{24\pi^2} \mathrm{tr}(A \wedge F \wedge F - \tfrac{1}{2} A \wedge A \wedge A \wedge F + \tfrac{1}{10} A \wedge A \wedge A \wedge A \wedge A). \tag{2.9}$$

The last two terms are the 1-loop corrected prepotential, contributed from the massive charged vector and hypermultipets.

The Hessian $g(\phi)_{ij} = \partial_i \partial_j \mathcal{F}$ is actually the metric on the Coulomb moduli space. The Coulomb branch is spanned by ϕ, and possibly divided into the Weyl chambers. For each $\mathbf{r} \in \mathbf{R}$, $\mathbf{r} \cdot \phi$ is either positive or negative throughout the Weyl chamber; Zeros are only allowed on the boundaries of Weyl chamber. On the other hand, for each $\mathbf{w} \in \mathbf{W}_f$, $\mathbf{w} \cdot \phi + m_f$ can always change its sign within the Weyl chamber, sectioning the Weyl chamber into the sub-wedges in which all $\mathbf{w} \cdot \phi + m_f$ factors have the definite sign. The important point is that the prepotential $\mathcal{F}$ and its Hessian $\partial_i \partial_j \mathcal{F}$ are continuous within the Weyl chamber. Moreover, any consistent quantum theory should have the non-negative metric g_{ij} throughout the Weyl chamber, because the negativeness of g_{ij} is the reflection of non-renormalizability and implies that the theory hits a Landau pole. In fact, it is a necessary condition for the existence of a UV fixed point, at $m_0 = 0$, that g_{ij} with $m_0 = 0$ must be non-negative throughout the Weyl chamber.

One can easily rule out new fixed points associated with product gauge groups $G_1 \times G_2$. For the theory to have something new, a hypermultiplet which transforms nontrivially under both G_1 and G_2 must be introduced. Suppose that it belongs to the represenation $(\mathbf{r}_1, \mathbf{r}_2)$ of $G_1 \times G_2$. At the specific locus of Coulomb branch, G_1 is completely broken while G_2 is still alive. Having the $(\mathbf{r}_1, \mathbf{r}_2)$ hypermultiplet implies that the prepotential receives an arbitrarily large amount of negative contribution, which is proportional to the G_1 Coulomb modulus. Therefore, g_{ij} cannot be nonnegative throughout the Coulomb moduli space, signaling the absence of UV fixed points. For this reason, [8] investigated only the gauge theories with a simple (classical) Lie group G, to cover out strongly-interacting UV fixed points.

Recall that the non-negativeness of g_{ij} is only a necessary condition. Although these gauge theories listed above meet the non-negativeness condition, some of them may not be completed to consistent quantum theories in the ultraviolet regime. The authors of [8] therefore individually searched out all Calabi-Yau three-folds, which realize these gauge theories as wrapping the M-theory compactified on themselves. Having all above physical arguments translated in terms of the Kahler geometry, the non-negativeness condition turned out to be sufficient for some cases (not including all gauge theories listed above) as follows:

- $Sp(N)$ theories with $n_A = 1$, $0 \leq N_f \leq 7$ hypermultiplets.
  $Sp(N)$ theories with $n_A = 0$, $0 \leq N_f \leq 2N + 4$ hypermultiplets.

- $SU(N)$ theories at the Chern-Simons coupling $c = 0$,
  with no adjoint, $n_F \leq 2N - 2$ fundamental hypermultiplets.

- $SO(2N + 1)$ theories with no adjoint, $2N - 3$ fundamental hypermultiplets.

- $SO(2N)$ theories with no adjoint, $2N-4$ fundamental hypermultiplets.

Even if not all the cases are proven, these examples are actually enough to conclude that most of the strong-coupling fixed points that are predicted based on the non-negativeness of g_{ij} do indeed arise. This work has been extended to the expectional gauge theories, which proved the existence of strongly-interacting UV fixed points using the same arguments based on the gauge theory as well as the geometry. The following 5d SYMs are completed to the consistent UV quantum theory [39]:

4. G_2 theories with $n_7 \leq 4$ hypermultiplets.

5. F_4 theories with $n_{26} \leq 3$ hypermultiplets.

6. E_8 theories with $n_{248} \leq 0$ hypermultiplets.

7. E_6 theories with $n_{27} \leq 3$ hypermultiplets.

8. E_7 theories with $n_{\frac{1}{2}56} \leq 5$ hypermultiplets.

Chapter 3

Instanton calculus in 5d gauge theory

For the purpose of studying some higher-dimensional QFTs introduced in Chapter 2, it is important to obtain the weakly-coupled description on these theories.

Taking the circle compactification of certain 6d $(2,0)$ or $(1,0)$ superconformal field theories, some 5d Yang-Mills theories are obtained at low energy limit. A necessary condition for the 5d SYM in this class is to have vanishing 1-loop correction to the metric on the Coulomb branch. The crutial step to UV-complete the non-renormalizable 5d gauge theory is incorporating all non-perturbative effects, so-called the Yang-Mills instantons. These instantons have the mass m proportional to $m \propto \frac{1}{R}$, for R denoting the radius of compactified circle, playing the role of Kaluza-Klein momenta along the compactified circle.

Some 5d SYM theories are obtained as relevant deformations of 5d superconformal field theories. A necessary condition for such 5d Yang-Mills theories is the non-negativeness of the Coulomb branch metric, as discussed in Section 2.2. Then one can take the bare coupling g_0 to infinite, which yields 5d SCFTs. Here again, the Yang-Mills instantons are essential to capture the UV physics because they have the mass $m \propto 1/g_{\text{eff}}^2$ which becomes zero at the UV fixed point ($g_{\text{eff}} \to \infty$). Being the lightest degrees of freedom, they strongly interact to one another through the Yang-Mills gauge field.

Having certain higher-dimensional QFTs as my ultimate motivation, I will discuss the five-dimensional Yang-Mills theory and its instantons in this chapter.

3.1 Yang-Mills instantons

Consider the 4d $SU(N)$ Euclidean Yang-Mills theory which has the action

$$S = -\frac{1}{2g^2}\int d^4x \ \mathrm{tr}\, F_{mn}^2 \tag{3.1}$$

where the field strength is defined as $F_{mn} = \partial_m A_n - \partial_n A_m + [A_m, A_n]$. Yang-Mills instantons are the solutions of the Yang-Mills equation of motion

$$D_m F_{mn} = \partial_m F_{mn} + [A_m, F_{mn}] \tag{3.2}$$

for which the Euclidean action keeps being finite. The finiteness of the action requires the field strength to be vanish at infinite, faster than r^{-2}, therefore the gauge field asymptotically becomes

$$A_m \longrightarrow U^{-1}\partial_m U \tag{3.3}$$

for $U \in \mathrm{SU}(N)$. Notice that the Yang-Mills action satisfy the inequality

$$\begin{aligned} S &= -\frac{1}{4g^2}\int d^4x \ \mathrm{tr}_N\,(F \mp \star F)^2 \mp \frac{1}{2g^2}\int d^4x \ \mathrm{tr}\ \tfrac{1}{2}\epsilon^{mnkl}F_{mn}F_{kl} \\ &\leq \mp\frac{1}{2g^2}\int d^4x \ \mathrm{tr}\ \tfrac{1}{2}\epsilon^{mnkl}F_{mn}F_{kl} = \pm\frac{8\pi^2}{g^2}k \end{aligned} \tag{3.4}$$

where k denotes the topological charge defined by

$$k = \frac{1}{16\pi^2}\int d^4x \ \mathrm{tr}\ \left(\tfrac{1}{2}\epsilon^{mnkl}F_{mn}F_{kl}\right) \in \mathbb{Z}. \tag{3.5}$$

If one considers the (anti-)self-dual instanton satisfying $(\star F)_{mn} \equiv \frac{1}{2}\epsilon_{mnkl}F^{kl} = \pm F_{mn}$, this automatically saturates the inequality (3.4). Moreover, the self-dual solutions have positive k while the anti-self-dual solutions have negative k.

When the gauge group G is $SU(2)$, the explicit $k = 1$ solution is given as

$$A_m^{\mathrm{BPST}} = \frac{\sigma_{mn}(x - x_0)_n}{(x - x_0)^2 + \rho^2} \tag{3.6}$$

in the regular gauge, where $\sigma_{mn} \equiv \frac{1}{4}(\sigma_m\bar{\sigma}_n - \sigma_n\bar{\sigma}_m)$ and $\sigma_{mn} = \frac{1}{2}\epsilon_{mnkl}\sigma_{kl}$ hold. This solution is so-called the BPST instanton [40]. One point that should be noticed is that it involves five parameters x_0, ρ called the collective coordinates. These parameters denote the location and the scale of the instanton. Moreover, there are three more collective coordinates which are related to the global $SU(2)$ transformation

$$A_m^{SU(2)} = U A_m^{\mathrm{BPST}} U^\dagger. \tag{3.7}$$

The $k = 1$ instanton solution for different gauge groups can be obtained via the embedding of the $SU(2)$ instanton. For example, the most general $SU(N)$ instanton is given by

$$A_m^{SU(N)} = U \begin{pmatrix} 0 & 0 \\ 0 & A_m^{\mathrm{BPST}} \end{pmatrix} U^\dagger \quad \text{for} \quad U \in \frac{SU(N)}{S[U(N-2)\times U(2)]}. \tag{3.8}$$

Moduli space As seen in the BPST solution (3.6), which gives infinite number of distinct solutions as the collective coordinates x_0 and ρ change, there exist inequivalent solutions of the Yang-Mills equation for a given instanton number k. Here the inequivalence means solutions cannot be identified in terms of the local gauge transformation. These distinct solutions form the space of solutions, which is called the instanton moduli space. In particular, the k instanton moduli space will be denoted by $\mathcal{M}_k$. Even though the instanton moduli space has singularities, it is a manifold so that one can introduce local coordinates to label its points. These are the collective coordinates which consist of, for $k = 1$, instanton position x_0, scale ρ, and global gauge transformation parameters. Since the Yang-Mills theory is invariant under the translation, one can even divide $\mathcal{M}_k$ into $\mathcal{M}_k = \mathbb{R}^4 \times \widehat{\mathcal{M}}_k$. The subspace $\widehat{\mathcal{M}}_k$ is also known as the centered moduli space.

The instanton moduli space $\mathcal{M}_k$ has useful properties. First of all, $\mathcal{M}_k$ has a natural metric defined by the inner product of zero modes

$$g_{ab}(X) = -\frac{2}{g^2}\int d^4x \ \mathrm{tr}\left[\delta_a A_m(X)\cdot \delta_b A_m(X)\right]. \tag{3.9}$$

It is often easier to determine the metric than the explicit instanton solution. The metric g_{ab} has certain isometries inherited from the gauge theory, such as $SO(4)$ spacetime rotation and G gauge rotation. Second, $\mathcal{M}_k$ is a hyper-Kahler manifold, allowing three complex structures $J^{1,2,3}$ which obey the relation

$$J^i \cdot J^j = -\delta^{ij} + \epsilon^{ijk} J^k. \tag{3.10}$$

In fact, the complex structures J^i of $\mathcal{M}_k$ are inherited from those of $\mathbb{R}^4$, which are chosen to be $J^i = \bar{\eta}^i_{mn}$. Here, 't Hooft η-symbols are defined by $\bar{\sigma}_{mn} = \frac{i}{2}\bar{\eta}^i_{mn}\tau^i$, satisfying $\bar{\eta}_{mn} = -\bar{\eta}_{nm}$. For a given zero mode $\delta_a A_m$, one can develop three new zero modes using η-symbols, then expand the new zero modes as a linear combination of the original ones. There must exist the coefficients $(J^i)^a{}_b$ such that

$$\bar{\eta}^i_{mn}\,(\delta_a A_m) = (J^i)^b{}_a\;(\delta_b A_n). \tag{3.11}$$

These triplets $(J^i)^a{}_b$ $(i = 1, 2, 3)$ satisfy the algebra (3.10). They are indeed the complex structures of $\mathcal{M}_k$ obtained by

$$(J^i)^a{}_b = -\frac{2}{g^2} g^{ac} \int d^4x\, \bar{\eta}^i_{mn}\ \mathrm{tr}\left[\delta_b A_m \cdot \delta_c A_n\right]. \tag{3.12}$$

Moreover, there exist the unique hyper-Kahler potential χ such that the moduli space metric $g_{a\bar{b}}$ can be written as $\frac{\partial^2\chi}{\partial Z^a \partial \bar{Z}^{\bar{b}}}$. $(Z^a, \bar{Z}^{\bar{a}})$ are holomorphic coordinates on $\mathcal{M}_k$, diagonalizing a complex structure on the moduli space to

$$J^i = \begin{pmatrix} i\delta^a{}_b & 0 \\ 0 & -i\delta^{\bar{a}}{}_{\bar{b}} \end{pmatrix}. \tag{3.13}$$

Finally, the instanton moduli space $\mathcal{M}_k$ has the conical singularity at which $\rho = 0$. For example, the metric on the $SU(2)$ $\mathcal{M}_1$ is given by

$$ds^2 = d\vec{x}_0{}^2 + d\rho^2 + \rho^2 ds^2(SU(2)) \tag{3.14}$$

which becomes singular as $\rho \to 0$. This is the genuine singularity which cannot be removed by a coordinate transformation. The singular point $\rho = 0$ is called the small instanton singularity.

The dimension of the k instanton moduli space can be computed by enumerating the zero modes. Suppose there is a solution A_m satisfying $F = \star F$. For the small perturbation $A_m \to A_m + \delta A_m$, the linearized equation of motion

$$D_m \delta A_n - D_n \delta A_m = \epsilon_{mnkl}(D_k \delta A_l) \tag{3.15}$$

must be satisfied. The zero mode associated to a collective coordinate X_a are defined to be

$$\delta_a A_m = \frac{\partial A_m}{\partial X_a} + D_m \Omega_a \tag{3.16}$$

where $D_m \Omega_a$ is introduced to require the orthogonality of $\delta_a A_m$ to any gauge transformation

$$\int d^4x \ \mathrm{tr}\,(\delta_a A_m \cdot D_m \eta) = 0 \quad \text{for all } \eta. \tag{3.17}$$

This implies that the zero mode should satisfy the equation

$$D_m(\delta_a A_m) = 0. \tag{3.18}$$

One can use the index theorem to count the solutions of (3.15) and (3.18), giving the dimension of the instanton moduli space $\mathcal{M}_k$. In general, k instanton solutions for $SU(N)$, $SO(N)$, $Sp(N)$ gauge groups involve $4kN$, $4k(N-2)$, $4k(N+1)$ number of collective coordinates.

ADHM construction Since the Yang-Mills equation of motion is non-linear equation, finding the solution is in general very difficult problem. However, for the case of self-dual instantons, the ADHM construction provides a powerful method to construct the k instanton solution for any classical gauge group G and any instanton number k [41]. For simplicity, $SU(N)$ gauge group will be the focus.

The rotation symmetry group of $\mathcal{R}^4$ space is $SO(4) = SU(2)_L \times SU(2)_R$, of which doublet indices are repectively denoted as α and $\dot{\alpha}$. An $SO(4)$ vector index m is interchanged with both $SU(2)_{L,R}$ doublet indices through the Clebsch-Gordan coefficient $\sigma^m_{\alpha\dot{\alpha}}$ or $\bar{\sigma}^{\dot{\alpha}\alpha}_m$. For example, the spacetime coordinates x_n can be represented as $x_{\alpha\dot{\alpha}} = \sigma^n_{\alpha\dot{\alpha}} x_n$. The basic object in the ADHM construction is a $(N+2k) \times 2k$ complex-valued matrix $\Delta_{\dot{\alpha}}$ which depends linearly on the spacetime coordinates

$$\Delta_{\dot{\alpha}} = a_{\dot{\alpha}} + b^{\alpha} x_{\alpha\dot{\alpha}}. \tag{3.19}$$

The matrix $\Delta_{\dot{\alpha}}$ is supposed to have the maximal rank $2k$. The next ingredient is the $(N+2k) \times N$ matrix v which belongs to the null-space of $(\Delta^{\dagger})^{\dot{\alpha}}$

$$\Delta^{\dagger\dot{\alpha}} v = 0, \tag{3.20}$$

where v is normalized as $v^{\dagger} v = \mathbf{1}_N$. Having all of these, the (anti-Hermitian) ADHM gauge field A_n can be constructed in terms of $v(x)$ as follows.

$$A_n = v^{\dagger}(x) \partial_n v(x). \tag{3.21}$$

For this ADHM gauge field to be a self-dual instanton solution, the factorization condition of $\Delta_{\dot{\beta}}$

$$\Delta^{\dagger\dot{\alpha}} \Delta_{\dot{\beta}} = \delta^{\dot{\alpha}}{}_{\dot{\beta}} \, f^{-1}(x) \tag{3.22}$$

is required, where $f(x)$ is an invertible $k \times k$ Hermitian matrix. Notice that the non-degeneracy of $\Delta_{\dot{\beta}}$ is essential to guarantee the existence of $f^{-1}(x)$.

To check the self-duality of the field strength associated to (3.21), observe that the projection operator $\mathcal{P} \equiv v v^{\dagger}$ can be expressed as

$$\mathcal{P} \equiv v v^{\dagger} = \mathbf{1}_{N+2k} - \Delta_{\dot{\alpha}} f \Delta^{\dagger\dot{\alpha}} \tag{3.23}$$

by means of (3.20). The field strength F_{mn} can be arranged to

$$\begin{aligned} F_{mn} &= v^{\dagger}(\partial_m v v^{\dagger})(\partial_n v) - v^{\dagger}(\partial_n v v^{\dagger})(\partial_m v) \\ &= -v^{\dagger}(\partial_m \Delta_{\dot{\alpha}}) f (\partial_n \Delta^{\dagger\dot{\alpha}}) v + v^{\dagger}(\partial_n \Delta_{\dot{\alpha}}) f (\partial_m \Delta^{\dagger\dot{\alpha}}) v = 4 v^{\dagger} f b^{\alpha} (\sigma_{nm})_{\alpha}{}^{\beta} \bar{b}_{\beta}. \end{aligned} \tag{3.24}$$

The self-duality of the field strength follows from the self-dual property of σ_{nm}. The instanton number of the ADHM solution can be computed using the Osborn identity [42],

$$\mathrm{tr}\left(\tfrac{1}{2}\epsilon_{mnkl} F^{mn} F^{kl}\right) = -(\partial^2)^2 \log\det f. \tag{3.25}$$

Since $f(x)$ asymptotically behaves $f^{-1}(x) \xrightarrow{|x|\to\infty} \frac{1}{2}x^2\, b^\alpha \bar{b}_\alpha\, \mathbf{1}_k$, it follows that

$$\mathrm{tr}\,\Big(\tfrac{1}{2}\epsilon_{mnkl}F^{mn}F^{kl}\Big) \xrightarrow{|x|\to\infty} k(\partial^2)^2\log x^2 = 4k\Big(\partial^2\frac{1}{x^2}\Big) = -16\pi^2 k\,\delta^{(4)}(\vec{x}). \quad (3.26)$$

Therefore, the topological instanton charge (3.5) becomes k which determines the matrix size of the ADHM data. In other words, the ADHM construction provides an ansatz given in terms of the matrix-valued ADHM data, solving the Yang-Mills equation of motion for a given instanton number k.

Explicit construction of the instanton solution hinges on solving the factorization condition (3.22). It does not change by the transformation

$$\Delta_{\dot\alpha} \to U\Delta_{\dot\alpha}M^{-1}, \quad v \to Uv, \quad f \to MfM^\dagger \quad (3.27)$$

where $U \in U(N+2k)$ and $M \in \mathrm{GL}(k,\mathbb{C})$. Without loss of generality, this allows one to assume b^α to take the following canonical form.

$$\mathcal{B} \equiv (b^1, b^2) = \begin{pmatrix} 0 \\ \mathbf{1}_2 \otimes \mathbf{1}_k \end{pmatrix} \quad (3.28)$$

where i,j ranges over $1,\cdots,k$. Then the remaining variables are encoded into $a_{\dot\alpha}$ and v such that

$$\mathcal{A} \equiv (a_{\dot 1}, a_{\dot 2}) = \begin{pmatrix} S_{\dot 1} & S_{\dot 2} \\ \multicolumn{2}{c}{\sigma^m \otimes X_m} \end{pmatrix}, \quad v = \begin{pmatrix} T \\ Q_\alpha \end{pmatrix}. \quad (3.29)$$

The following is the residual freedom to transform $\mathcal{A}$ and v without changing $\mathcal{B}$

$$S_{\dot\alpha} \to V_N S_{\dot\alpha} V_k^{-1}, \quad X_m \to V_k X_m V_k^{-1}, \quad T \to V_N T, \quad Q_\alpha \to V_k Q_\alpha. \quad (3.30)$$

Within the canonical form, the factorization condition (3.22) to be satisfied if

$$X_m = X_m^\dagger, \qquad \vec{\tau}^{\dot\beta}{}_{\dot\alpha}(\bar{a}^{\dot\alpha}a_{\dot\beta}) = 0. \quad (3.31)$$

In particular, the second equation is often called the ADHM constraint. Using the convention which chooses the σ-matrix to be $\sigma^m = (i\mathbf{1}_2, \vec{\tau})$ together with $\epsilon_{0123} = 1$, the ADHM constraint can be also expressed as the following matrix equations,

$$II^\dagger - J^\dagger J + [B_1, B_1^\dagger] - [B_2, B_2^\dagger] = 0, \quad IJ + [B_1, B_2] = 0 \quad (3.32)$$

for $J = S_{\dot{1}}$, $I = S_{\dot{2}}^{\dagger}$, $B_1 = X_0 - iX_3$, and $B_2 = -iX_1 + X_2$. Solutions of the above equations must be found in order to formulate an explicit k instanton solution of the Yang-Mills equation.

3.2 Instanton counting and Seiberg-Witten solution

For the four-dimensional $\mathcal{N} = 2$ $SU(2)$ gauge theory which undergoes strong quantum effects in the infrared regime, Seiberg and Witten completely determined the low energy effective action in the Coulomb phase where $SU(2)$ gauge theory is broken to the Abelian subgroup $U(1)$ [38, 43]. The low energy effective action is determined by the prepotential $\mathcal{F}$ which is the holomorphic function of vector multiplet scalars. Using the $\mathcal{N} = 1$ superfield notation,

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi}\text{Im}\left[\int d^4\theta \frac{\partial \mathcal{F}}{\partial A}\bar{A} + \frac{1}{2}\int d^2\theta \frac{\partial^2 \mathcal{F}}{\partial A^2} W_\alpha W^\alpha\right] \tag{3.33}$$

where A is the $\mathcal{N} = 1$ chiral multiplet in the $\mathcal{N} = 2$ vector multiplet whose scalar is denoted by a. If one denotes the complexified gauge coupling by $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$, the classical prepotential is

$$\mathcal{F}_{\text{cl}} = \frac{1}{2}\tau_{\text{cl}} A^2. \tag{3.34}$$

Due to the asymptotic freedom, this formula is valid for large a if g^2 is replaced by an effective coupling. More generally, taking the one-loop effect into account, the prepotential becomes

$$\mathcal{F}_{\text{pert}} = \frac{i}{2\pi} A^2 \log \frac{A^2}{\Lambda^2} \tag{3.35}$$

where Λ is the dynamically generated scale. There exist no higher-loop correction, but one should consider the non-perturbative instanton corrections to the prepotential. The general form of the prepotential is written by

$$\mathcal{F}_{\text{pert}} = \frac{i}{2\pi} A^2 \log \frac{A^2}{\Lambda^2} + \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{A}\right) A^2. \tag{3.36}$$

$\mathcal{F}_k$ has to be determined to obtain the complete low-energy effective action.

The surprising discovery of [38, 43] begins with the observation that the metric on the moduli space $ds^2 = \mathrm{Im}(\tau(a))\, da d\bar{a}$ cannot be positive-definite over the whole moduli space. Instead, one defines $a_D = \partial\mathcal{F}/\partial a$ and introduces an arbitrary local holomorphic coordinate u so that the metric becomes

$$ds^2 = -\frac{i}{2}\left(\frac{da_D}{du}\frac{d\bar{a}}{d\bar{u}} - \frac{da}{du}\frac{d\bar{a}_D}{d\bar{u}}\right) du d\bar{u} \tag{3.37}$$

where $\tau(a) = \partial^2\mathcal{F}/\partial a^2$ and (a, a_D) are treated as functions of u. In fact, the metric (3.37) is invariant under the transformation

$$\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} \to M \begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} + C \tag{3.38}$$

for $M \in SL(2,\mathbb{R})$. The generators of $SL(2,\mathbb{R})$ are given by

$$T_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.39}$$

the former of which translates a_D to $a_D \to a_D + ba$. This is equivalent to the θ-angle shift by $2\pi b$, thus imposing $b \in \mathbb{Z}$. On the other hand, the S-transformation corresponds to the electric-magnetic duality, which maps one description of the theory to another description of the same theory. The $SL(2,\mathbb{Z})$ transformation does not change masses of BPS particles, but these BPS particles can possibly decay into different BPS particles as crossing $\mathrm{Im}(a_D/a)(u) = 0$ in the u-space [38].

The careful analysis of the moduli space shows there are three singularities in $\mathbb{C}^1 \cup \{\infty\}$, each of which involves a monodromy matrix $M \in SL(2,\mathbb{Z})$. The one singularity (say $+1$) arises when the magnetic monopole becomes massless, and the other singularity (say -1) occurs when the dyon becomes massless.

$$M_{+1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}, \quad M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}. \tag{3.40}$$

Introducing the complex curve (known as the Seiberg-Witten curve), which is naturally interpreted as branes in the M-theory [44], the exact solution $a(u)$ and $a_D(u)$

are analytically obtained, determining the prepotential $\mathcal{F}$ as well. The surprising fact is that the prepotential given via the SW-curve has included the perturbative quantum correction as well as the non-perturbative corrections at all instanton orders. Soon after, the analysis for pure $SU(2)$ gauge theory was extended for theories with diverse gauge groups and matters.

For the microscopic understanding of the Seiberg-Witten prepotential, one should be able to derive the prepotential directly from the supersymmetric path integral, which reduces down to the sum of integrals over the k instanton moduli space $\mathcal{M}_k$ in the semiclassical limit. The immediate difficulty came from the fact that an explicit solution for the ADHM constraint is very hard to obtain for $k > 2$. Nevertheless, [45, 46] derived the relation between $\mathcal{F}_{\text{inst}}$ and the coordinate u on the moduli space, which was first noted by [47]. The general expression for the k-instanton contribution $\mathcal{F}_k$ to the prepotential as an integral over the centered moduli space $\widehat{\mathcal{M}}_k$ was derived in [48]. To deal with the moduli space integral, the supersymmetric localization technique was adopted to produce $\mathcal{F}_1$ and $\mathcal{F}_2$ [49, 50]. Finally, [15] adopted the equivariant localization technique to calculate an integral over the moduli space $\mathcal{M}_k$. Thanks to the chemical potentials $\epsilon_{1,2}$ associated to the $U(1)^2 \subset SO(4)$ isometry, the flat modulus associated to the instanton location in $\mathbb{R}^4$ gets lifted. As a result, instantons tends to be point-like and localized in space. Denoting an integral over the k-instanton moduli space $\mathcal{M}_k$ by Z_k, Nekrasov's instanton partition function Z^{inst} is written as

$$Z^{\text{inst}} = 1 + \sum_{k=1}^{\infty} q^k \cdot Z_k. \tag{3.41}$$

Under the limit where $\epsilon_{1,2} \to 0$,

$$(\mathbb{R}^4 \text{ volume})^{-1} = \epsilon_1 \epsilon_2 + \mathcal{O}(\epsilon^3) \tag{3.42}$$

so that the instanton correction to the prepotential $\mathcal{F}_{\text{inst}}$ can be given by [15]

$$\mathcal{F}_{\text{inst}} = \lim_{\epsilon_{1,2} \to 0} \epsilon_1 \epsilon_2 \log Z_{\text{inst}}. \tag{3.43}$$

The above conjecture was rigorously proven in [51], by deriving the Seiberg-Witten geometry from the $\mathcal{N}=2$ gauge theory. Especially, the Seiberg-Witten solutions for $SU(N)$ gauge theories with various matters are microscopically derived in [15, 51]. See also [52] for $SO(N)$ and $Sp(N)$ gauge groups, and [53] for hypermultiplets in various representations.

3.3 ADHM quantum mechanics

From here on, the main focus will be instantons in the 5d Yang-Mills theory. Consider the 5d $\mathcal{N}=1$ gauge theory on $\mathbb{R}^4 \times \mathbb{R}^1$ with a classical gauge group G. It has the $SO(4)$ spatial rotation symmetry which is decomposed into $SU(2)_l \times SU(2)_r \subset SO(4)$. The 5d $\mathcal{N}=1$ supersymmetry has the bosonic $SU(2)_R$ R-symmetry. The eight supersymmetry generators are symplectic-Majorana spinors divided into Q^A_α and $\bar{Q}^A_{\dot\alpha}$, where α, α', and A denote the doublet indices for $SU(2)_l \times SU(2)_r \times SU(2)_R$ symmetries, respectively. They satisfy the $\mathcal{N}=1$ supersymmetry algebra

$$\{Q^A_M, Q^B_N\} = P_\mu(\Gamma^\mu C)_{MN}\epsilon^{AB} + i\frac{4\pi^2 k}{g^2_{YM}} C_{MN}\epsilon^{AB} + i\mathrm{tr}(v\Pi)C_{MN}\epsilon^{AB}. \tag{3.44}$$

Here, M, N are Dirac spinor indices; C_{MN} is the charge conjugation matrix; v_i is the vacuum expectation value for the vector multiplet scalar; Π_i is the electric charge for the Abelian subgroup of G. Of particular importance is the topological $U(1)_I$ charge

$$k = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{tr}(F \wedge F) \in \mathbb{Z} \tag{3.45}$$

which is always integer-valued. It is non-zero only for the solitonic field configuration known as the 5d self-dual instanton which satisfies $F_{mn} = \star_4 F_{mn} = \frac{1}{2}\epsilon_{mnpq}F_{pq}$ on the spatial $\mathbb{R}^4$, thus being called the instanton charge. The instanton background only preserves half of supersymmetries $\bar{Q}^A_{\dot\alpha}$. They can form the bound states with perturbative particles called the W-bosons, which are electrically charged. These bound states preserve the same four supercharges $\bar{Q}^A_{\dot\alpha}$ as well, having the mass

$$M = \frac{4\pi^2 k}{g^2_{YM}} + \mathrm{tr}(v\Pi) \tag{3.46}$$

where the sign of electric charges are chosen to satisfy $\mathrm{Tr}(v\Pi) \geq 0$.

In the thesis, my special interest is computing the 5d instanton partition function which was first considered in [15]. It is the Witten index which enumerates all bound states of instantons and W-bosons, together with extra informations on the behavior of bound states under the global symmetries, including $SU(2)_l \times SU(2)_r \times SU(2)_R$. The explicit definition is given by

$$Z(\epsilon_1, \epsilon_2, \alpha_i, z) = \mathrm{Tr}\left[(-1)^F q^k e^{-\beta\{Q,Q^\dagger\}} e^{-\epsilon_1(J_1+J_R)} e^{-\epsilon_2(J_2+J_R)} e^{-\alpha_i \Pi_i} e^{-zF}\right] \quad (3.47)$$

where $Q \equiv \bar{Q}^1_{\dot{1}} = -\bar{Q}^{1\dot{2}}$ and $Q^\dagger \equiv \bar{Q}^2_{\dot{2}} = \bar{Q}^{2\dot{1}}$; J_l, J_r, J_R are Cartan generators of $SU(2)_l \times SU(2)_r \times SU(2)_R$, respectively; F collectively denotes Cartan generators for remaining flavor symmetries. Recall that the Witten index is protected, to which only ground states can contribute, thereby being independent of β.

The instanton partition function is represented as the supersymmetric path integral. Thanks to supersymmetry, the full integral is efficiently localized as the quantum fluctuations around instantonic backgrounds, which are decomposed into massive modes and massless modes (also known as zero modes) for a given instanton number k. The zero modes of self-dual instantons can be described by the so-called ADHM data for classical gauge groups, subject to the ADHM constraint equations. In 5d SYM, the moduli space approximation [54] of these instantons is given by a supersymmetric sigma model with the target space given by the instanton moduli space. The partition function of [15] can be understood as that of this mechanical system. As explained in Section 3.1, the moduli space has conical singularities at which the instanton scale ρ shrinks to zero. Still one can resolve singular points in the instanton moduli space by introducing a gauge field to the quantum mechanics. The resulting theory is called the ADHM quantum mechanics, being the UV description of the instanton quantum mechanics.

The ADHM quantum mechanics inherits four supercharges $\bar{Q}^A_{\dot{\alpha}}$ preserved by the 5d instanton background, enjoying the $SU(2)_r \times SU(2)_R = SO(4)$ R-symmetry. This is called $\mathcal{N} = (0,4)$ supersymmetry. For the classical gauge group $G = U(N),\ Sp(N),\ SO(N)$ of the underlying 5d theory, the mechanical gauge group

$\hat{G}$ is given by $\hat{G} = U(k),\ O(k),\ Sp(k)$, for an instanton number k. The field contents of the ADHM quantum mechanics either come from the zero-modes of the 5d path integral, or become introduced in the course of resolving the singular moduli space. First of all, the 5d vector multiplet supplies the ADHM data and their superpartners

$$
\begin{aligned}
(a_{\alpha\dot\beta}, \lambda^A_\alpha) &: \text{adjoint representation in } \hat{G} = U(k) && \text{if } G = U(N) && (3.48)\\
&\quad\ \text{symmetric representation in } \hat{G} = O(k) && \text{if } G = Sp(N) && (3.49)\\
&\quad\ \text{antisymmetric representation in } \hat{G} = Sp(k) && \text{if } G = SO(N) && (3.50)\\
(q_{\dot\alpha}, \psi^A) &: \text{bifundamental representation in } G \times \hat{G} && && (3.51)
\end{aligned}
$$

coming from zero-modes in the instantonic background. They formulate so-called $\mathcal{N} = (0,4)$ hypermultiplets, scalars of which are in the $(\mathbf{2},\mathbf{1})$ representation of $SU(2)_r \times SU(2)_R$ R-symmetry. Second, one has introduced the quantum mechanical gauge multiplet

$$
(A_t, \varphi, \bar\lambda^A_{\dot\alpha}, D_{\dot\alpha\dot\beta}) : \text{adjoint representation in } \hat{G} \tag{3.52}
$$

during resolving the small instanton singularity, of which $D_{\dot\alpha\dot\beta}$ is the auxilary field. In particular, its scalar φ and fermions $\bar\lambda^A_{\dot\alpha}$ are bringing an extra degrees of freedom compared to the instanton quantum mechanics.

Interaction between these field contents is described by the QM Lagrangian

$$
\begin{aligned}
L = \frac{1}{g_1^2} \cdot \operatorname{tr}\Bigg[& \frac{1}{2}(D_t\varphi)^2 + \frac{1}{2}(D_t a_m)^2 + D_t q_{\dot\alpha} D_t \bar q^{\dot\alpha} + \frac{1}{2}[\varphi, a_m]^2 - (\varphi \bar q^{\dot\alpha} - \bar q^{\dot\alpha} v)(q_{\dot\alpha}\varphi - v q_{\dot\alpha}) \\
& + \frac{1}{2}(D^I)^2 - D^I \left((\tau^I)^{\dot\alpha}{}_{\dot\beta} \bar q^{\dot\beta} q_{\dot\alpha} + \frac{1}{2}(\tau^I)^{\dot\alpha}{}_{\dot\beta}[a^{\dot\beta\alpha}, a_{\alpha\dot\beta}] - \zeta^I \right) + \frac{i}{2}(\bar\lambda^{A\dot\alpha})^\dagger D_t \bar\lambda^{A\dot\alpha} \\
& - \frac{1}{2}(\bar\lambda^{A\dot\alpha})^\dagger [\varphi, \bar\lambda^{A\dot\alpha}] + \frac{i}{2}(\lambda^A_\alpha)^\dagger D_t \lambda^A_\alpha + \frac{1}{2}(\lambda^A_\alpha)^\dagger [\varphi, \lambda^A_\alpha] + i(\psi^A)^\dagger D_t \psi^A \\
& - i(\lambda^A_\alpha)^\dagger (\sigma^m)_{\alpha\dot\beta}[a_m, \bar\lambda^{A\dot\beta}] + \sqrt{2} i \left((\bar\lambda^{A\dot\alpha})^\dagger \bar q^{\dot\alpha} \psi^A - (\psi^A)^\dagger q_{\dot\alpha} \bar\lambda^{A\dot\alpha} \right) \Bigg]
\end{aligned} \tag{3.53}
$$

where g_1 is the mechanical gauge coupling; ζ^I denote FI parameters; τ^I are Pauli matrices. It is obtained via truncation of the action for $\mathcal{N} = (4,4)$ ADHM quantum mechanics, whose explicit expression is given, e.g., in [55]. Moreover, the mechanical Chern-Simons term

$$L_{CS} = \kappa(\varphi + A_t) \tag{3.54}$$

can be added, in case the 5d gauge theory has a non-zero Chern-Simons level κ [56, 57]. The accompanying supersymmetry transformation can be also obtained from the $\mathcal{N} = (4,4)$ ADHM quantum mechanics. In particular, the (0,4) hypermultiplet scalar transforms as

$$\bar{Q}^{A\dot{\alpha}}\Phi_{\dot{\beta}} = \sqrt{2}\delta^{\dot{\alpha}}_{\dot{\beta}}\Psi^A. \tag{3.55}$$

Here are two notable points regarding the above action (3.53). Firstly, the ADHM data are the physical zero-modes induced from the 5d gauge multiplet, only if they are subject to the constraining matrix equation known as the ADHM constraint. This constraint is realized as the supersymmetric D-term potential in the ADHM quantum mechanics. Secondly, if one takes the strong coupling limit $g_1 \to \infty$ in the mechanical Higgs branch, where $q_{\dot{\alpha}}$ and $a_{\alpha\dot{\beta}}$ acquire non-zero VEVs, the extra degrees of freedom in the ADHM quantum mechanics become infinitely massive; The mass of φ and $\bar{\lambda}^A_{\dot{\alpha}}$ becomes $g_1|q_{\dot{\alpha}}|$ in the Higgs branch, being infinite as one takes the $g_1 \to \infty$ limit. By integrating out those heavy fields, the ADHM quantum mechanics returns back to the instanton quantum mechanics.

When the 5d gauge theory has hypermultiplets, the instanton background allows more fermionic zero-modes which can be determined by the index theorem. They are realized in the ADHM quantum mechanics as the $(0,4)$ Fermi multiplets. In addition, the ADHM quantum mechanics may include extra bosonic fields associated to the 5d hypermultiplet, depending on its representation under G. These are $(0,4)$ twisted hypermultiplets $(\Phi^A, \Psi_{\dot{\alpha}})$, of which scalars take the $(\mathbf{1},\mathbf{2})$ representation of $SU(2)_r \times SU(2)_R$ R-symmetry. The supersymmetry transformation of the (0,4) twisted hypermultiplet is given by [58]

$$\bar{Q}^{A\dot{\alpha}}\Phi_B = \sqrt{2}\delta^A_B\Psi^{\dot{\alpha}} \tag{3.56}$$

In every case, the representation of 5d hypermultiplets determines the representation of associated 1d fields under $G \times \hat{G}$ in the ADHM quantum mechanics.

In Section 3.4, I shall compute the supersymmetric partition function of this ADHM quantum mechanics. Two supercharges among the $(0,4)$ supersymmetry are utilized for the localization of the path integral, so that it turns out to be useful to decompose all $(0,4)$ supermultiplets, which are introduced above, into $(0,2)$ supermultiplets. Regarding $Q \equiv \bar{Q}^1_{\dot{1}} = -\bar{Q}^{1\dot{2}}$ and $Q^\dagger \equiv \bar{Q}^2_{\dot{2}} = \bar{Q}^{2\dot{1}}$ used in (3.47) as $\mathcal{N} = (0,2)$ supercharges, the above $(0,4)$ multiplets are decomposed as

$$\text{vector } (A_t, \varphi, \bar{\lambda}^A_{\dot{\alpha}}) \to \text{vector } (A_t, \varphi, \bar{\lambda}^1_{\dot{1}}, \bar{\lambda}^2_{\dot{2}}) \ + \ \text{Fermi } (\bar{\lambda}^1_{\dot{2}}, \bar{\lambda}^2_{\dot{1}}) \tag{3.57}$$

$$\text{hyper } (\phi^{\dot{\alpha}}, \psi^A) \to \text{chiral } (\phi^{\dot{1}}, \psi^1) \ + \ \text{chiral } (\bar{\phi}_{\dot{2}} = \bar{\phi}^{\dot{1}}, \bar{\psi}_2) \tag{3.58}$$

$$\text{twisted hyper } (\phi^A, \psi_{\dot{\alpha}}) \to \text{chiral } (\phi^2, \psi^{\dot{2}}) \ + \ \text{chiral } (\bar{\phi}_1 = -\bar{\phi}^2, \bar{\psi}_{\dot{1}}) \tag{3.59}$$

The off-shell $(0,2)$ action and supersymmetric transformation rules, which are necessary for the computation of the partition function, can be obtained via the 1d reduction of appropriate two-dimensional $\mathcal{N} = (0,2)$ theory plus the 1d Chern-Simons term (3.54).

3.4 Exact computation of the 1d index

The instanton partition function of 5d gauge theory, defined in (3.47), can be factorized as $Z = Z^{\text{pert}} Z^{\text{inst}}$. The perturbative partition function Z^{pert} enumerates the bound states without any instantons, whereas Z^{inst} captures all instantonic bound states. One can further separate Z^{inst} into $Z^{\text{inst}} = 1 + \sum_{i=1}^{\infty} q^k \cdot Z_k$, where Z_k exclusively counts the bound states with k instantons. Here, the observable of my study is the Witten index

$$Z^{\text{1d}}_k(\epsilon_1, \epsilon_2, \alpha_i, z) = \text{Tr}\left[(-1)^F e^{-\beta\{Q,Q^\dagger\}} e^{-\epsilon_1(J_1+J_R)} e^{-\epsilon_2(J_2+J_R)} e^{-\alpha_i \Pi_i} e^{-z\cdot F}\right] \tag{3.60}$$

of the ADHM quantum mechanics, in which the gauge group $\hat{G}$ is either $U(k)$, $O(k)$, or $Sp(k)$ depending on what type of G is. Roughly speaking, the combination of the ADHM QM indices

$$Z^{\text{1d}} = 1 + \sum_{k=1}^{\infty} q^k \cdot Z^{\text{1d}}_k \tag{3.61}$$

corresponds to Z^{inst} up to a possible multiplicative factor Z^{extra}, which will be discussed in Chapter 4. In most cases, the Witten index is invariant under the continuous parameter deformation. This fact is utilized in order to replace the gauge coupling $\frac{1}{g_1^2}$ to $\frac{1}{e^2}$ (for the gauge kinetic term) and to $\frac{1}{g^2}$ (for the matter kinetic term), then take the limit $e, g \to 0$. It is likely that the mechanical path integral for Z_k^{1d} is reduced down to the Gaussian integral around zero modes. For this reason, I shall go through the following procedures: to identify the zero modes, to integrate over all massive modes, and to perform the zero mode integrals.

The zero modes comprises the holonomy of gauge field A_t on the temporal circle, as well as the value of scalar φ in the vector multiplet. For convenience, I shall make these zero modes dimensionless, by multiplying a suitable power of β, and rescale them so that the eigenvalue of A_t is 2π-periodic. The space of zero mode eigenvalues, $\phi^I = \varphi^I + iA_t^I$, is the product of r cylinders where $r = |\hat{G}|$. Keeping the zero modes fixed, I perform the Gaussian integral over non-zero modes. This step results in the one-loop determinant, to which each $(0,2)$ multiplet contributes a following factor

$$(0,2) \text{ vector} : Z_V = \prod_{\alpha \in \text{root}} 2\sinh\frac{\alpha(\phi)}{2} \prod_{I=1}^{r} \frac{d\phi_I}{2\pi i} \tag{3.62}$$

$$(0,2) \text{ chiral} : Z_\Phi = \prod_{\rho \in R_\Phi} \frac{1}{2\sinh\left(\frac{\rho(\phi)+J\epsilon_+ + F\cdot z}{2}\right)} \tag{3.63}$$

$$(0,2) \text{ Fermi} : Z_\Psi = \prod_{\rho \in R_\Psi} 2\sinh\left(\frac{\rho(\phi)+J\epsilon_+ + Fz}{2}\right) \tag{3.64}$$

depending on types and charges of a given multiplet. Here, R denotes the collection of weights in a given representation of $\hat{G}$; J is defined as $J = J_r + J_R$; F and z collectively denote the rest of global charges and their chemical potentials. In particular, F refers to Cartans of G, $SU(2)_l$ and extra flavor symmetries, of which the chemical potential z corresponds to α, $\epsilon_- \equiv \frac{\epsilon_1 - \epsilon_2}{2}$, and m, respectively.

The remaining step is to integrate the above one-loop determinant over the zero-mode space. This is the contour integral

$$Z^{\rm 1d}_{k=1} = \frac{1}{|W|}\oint e^{\kappa {\rm tr}(\phi)} Z_{\text{1-loop}} = \frac{1}{|W|}\oint e^{\kappa {\rm tr}(\phi)} Z_V \prod_{\Phi} Z_\Phi \prod_{\Psi} Z_\Psi \ , \tag{3.65}$$

which can be performed only if contours are specified. In case the gauge group $\hat{G}$ is disconnected, one should sum over integrals for distinct sectors on which discrete holonomies have turned. The effect of discrete holonomies will be discussed with the $\hat{G} = O(k)$ example in Section 3.5.3. My focus here is to clarify subtle issues regarding the integral (3.65), thus describing the correct integral contour.

First of all, the non-compactness of zero-mode space indicates that the quantum spectrum may possibly develop a continuum, which would break the invariance of SUSY partition function (3.60) under the continuous deformation. This happens only if the one-loop determinant does not provide quantum suppression in asymptotic regions of the zero-mode cylinders. Continuous deformation of the coupling constant $e, g \to 0$ in case modifies the partition function (3.60), due to the change in the continuum contribution. Nevertheless, it is certain that the instanton quantum mechanics, which emerges at the strong coupling regime in the Higgs branch of ADHM quantum mechanics, cannot observe the spectral continuum for any case; The vector multiplet scalar φ, which was responsible for the asymptotic region, becomes infinitely heavy, thereby no longer being a dynamical field in the instanton quantum mechanics. All possible continuum contributions are induced from the extra degrees of freedom which are included during resolution of the instanton moduli space. I shall eventually identify and decouple out this extra contribution $Z^{\rm extra}$ from $Z^{\rm 1d}$, for achieving the 5d SYM observable $Z^{\rm inst}$ correctly. This step is treated with many examples in Section 4. At this moment, I handle the asymptotic region by introducing the cutoff $\Lambda_{1,2} \gg 0$ such that zero-mode eigenvalues are in the region of $-\Lambda_1 \leq \varphi^I = {\rm Re}\ \phi^I \leq \Lambda_2$. The limit $\Lambda_{1,2} \to \infty$ should be taken after integrating over all zero-modes.

Second, the zero-mode space has several regions in which the naive analysis in the above may fail. Gaussian integration over non-zero modes with fixed ϕ is legitimate only if non-zero modes are massive. However, this assumption is broken around the singular point at which the one-loop determinant diverges. Let us denote the set of singular points by M_{sing}. These singular points are supplied by $(0,2)$ chiral multiplets, each of which contributes a factor (3.63) that becomes divergent at $\phi = \phi^*$ satisfying $\rho(\phi^*) + J\epsilon_+ + Fz = 0$. The analysis should be more cautious in neighborhoods of M_{sing}, so I roll (3.65) back to the expression prior to setting $e, g \to 0$ and integrating over D, the zero-mode of auxiliary scalar in the gauge multiplet. The 1-loop determinant factor (3.63) from an $(0,2)$ chiral multiplet converts back to

$$Z_\Phi(\phi, \epsilon_+, z, D) = \prod_{\rho \in R_\Phi} \prod_{n=-\infty}^{\infty} \frac{-2\pi i n + \rho(\bar{\phi}) + J\bar{\epsilon}_+ + F\bar{z}}{|2\pi i n + \rho(\phi) + J\epsilon_+ + Fz|^2 + i\rho(D)}. \tag{3.66}$$

Notice that the second kind of dangerous regions also appears in the computation of Witten indices in the 2d gauge theory, where [59, 60, 61] successfully resolved the subtle issues. For the computation of 1d indices, I will imitate the two-dimensional analysis in [60, 61] to handle the second type of dangerous regions.

3.4.1 Rank-1 gauge group

In this subsection, the discussion will be mainly focused on the rank-1 gauge theory. Generalization for the higher-rank gauge theories is discussed in Section 3.4.2. When e, g are small but non-zero, the path integral Z takes the form of

$$Z = \int_{\mathbb{R}} dD \int_M d^2\phi f_{e,g}(\phi, \bar{\phi}, D) \exp\left(-\frac{D^2}{2e^2} - i\zeta D\right) \tag{3.67}$$

where ζ is the Fayet-Iliopoulos parameter that can be turned on if $\hat{G}$ has an $U(1)$ subgroup; M denotes the zero-mode space; $f_{e,g}$ is obtained via the path integral except ϕ and D.

After performing the integral over D, (3.67) becomes

$$Z = \int_M d^2\phi \; F_{e,g}(\phi, \bar{\phi}). \tag{3.68}$$

As e remains to be finite, the $g \to \infty$ limit behaves regularly for any value of ϕ, because of an induced potential of the form $e^2(|\phi|^2 - \zeta)^2$ suppressing the integrand exponentially. Let us denote the ε-neighborhood of M_{sing} by Δ_ε. If one divides the ϕ integral into two regions $M\backslash\Delta_\varepsilon$ and Δ_ε,

$$Z = \left(\int_{M\backslash\Delta_\varepsilon} d^2\phi + \int_{\Delta_\varepsilon} d^2\phi \right) F_{e,0} \tag{3.69}$$

It is possible to take the $e \to 0$ limit in such a way that the integral over Δ_ε does not contribute. One should set $\epsilon \to 0$ much faster than a certain positive power of e, thus the volume factor of Δ_ε dominating over the divergence of $F_{e,0}$ as $e \to 0$. The precise limit of $e, \epsilon \to 0$ is specified in [60].

What remains after recovering D is the following integral.

$$Z = \lim_{e,\varepsilon\to 0} \int_{\mathbb{R}} dD \int_{M\backslash\Delta_\varepsilon} d^2\phi \; f_{\text{e}}(\phi, \bar{\phi}, D) \exp\left(-\frac{D^2}{2e^2} - i\zeta D \right) \tag{3.70}$$

If the imaginary part of D keeps very small, one can take the $e \to 0$ limit of $f_{e,0}$ which behaves as

$$\begin{aligned} f_{\text{e}}(\phi, \bar{\phi}, D) \quad &\overset{e\to 0}{\longrightarrow} \quad \int d\lambda_0 d\bar{\lambda}_0 \left\langle \int d^2x \lambda \sum_i Q_i \bar{\Psi}_i \phi_i \int d^2x \bar{\lambda} \sum_i Q_i \psi_i \bar{\phi}_o \right\rangle_{\text{free}} \\ &= \quad h(\phi, \bar{\phi}, D, \epsilon_+, z) g(\phi, \bar{\phi}, D, \epsilon_+, z) \end{aligned} \tag{3.71}$$

where

$$g(\phi, \bar{\phi}, D, \epsilon_+, z) = e^{\kappa \text{tr}(\phi)} Z_{\text{v}}(\phi) \prod_{\Phi} Z_\Phi(\phi, \bar{\phi}, D, \epsilon_+, z) \prod_{\Psi} Z_\Psi(\phi, \epsilon_+, z) \tag{3.72}$$

is the 1-loop determinant;

$$h(\phi, \bar{\phi}, D, \epsilon_+, z) = c \sum_{i,n} \frac{Q_i^2}{\left(|2\pi i n + Q_i\phi + J\epsilon_+ + Fz|^2 + iQ_i D \right) \left(-2\pi i n + Q_i\bar{\phi} + J\bar{\epsilon}_+ + F\bar{z} \right)} \tag{3.73}$$

comes from saturating the gaugino zero-modes, denoted by λ_0 and $\bar{\lambda}_0$. Below is the identity involving h and g which turns (3.70) into the holomorphic integral over ϕ.

$$h(\phi, \bar{\phi}, D, \epsilon_+, z) g(\phi, \bar{\phi}, D, \epsilon_+, z) = \frac{1}{\pi D} \frac{\partial}{\partial \bar{\phi}} g(\phi, \bar{\phi}, D, \epsilon_+, z) \tag{3.74}$$

To apply (3.74), one first deforms the D-contour slightly away from the real line to either Γ_+ or Γ_-, each of which corresponds to the contour along $\mathbb{R} + i\delta$ with

$0 < \pm\delta < \varepsilon^2$. Since $\partial(M\backslash\Delta_\varepsilon) = -\partial\Delta_\varepsilon - \partial M_0 - \partial M_\infty$ in which the boundaries of asymptotic regions are denoted as $\partial M_{0,\infty}$, the integral becomes

$$Z = -\lim_{e,\varepsilon\to 0}\int_{\Gamma_\pm}\frac{dD}{2\pi i D}\exp\left(-\frac{D^2}{2e^2} - i\zeta D\right)\oint_{\partial\Delta_\varepsilon+\partial M_0+\partial M_\infty}\frac{d\phi}{2\pi i}\, g(\phi, D, \epsilon_+, z). \tag{3.75}$$

This integrand has the poles of D located at $D = 0$ and, from (3.66),

$$D = \frac{i}{Q_i}\left|2\pi i n + Q_i\phi + J\epsilon_+ + Fz\right|^2. \tag{3.76}$$

Focus on the expression (3.75) over the contour $\phi \in \partial\Delta_\varepsilon$. If $\varepsilon \ll 1$, the location of D-poles is on the imaginary axis, of which distance from $D = 0$ is proportional to $(\operatorname{sgn} Q_i)\ \varepsilon^2$. One can further classify $\partial\Delta_\varepsilon$ into $\partial\Delta_\varepsilon^{(+)}$ and $\partial\Delta_\varepsilon^{(-)}$, depending on the signature of Q_i. If the integral contour is given as $D \in \Gamma_\pm$ and $\phi \in \Delta_\varepsilon^{(\mp)}$ whose signs are oppositely correlated, no singularity touches the D-contour while I relax the condition $0 < |\delta| < \varepsilon^2$. I therefore take the limit $\varepsilon \to 0$ as keeping δ fixed, resulting in

$$-\lim_{e,\varepsilon\to 0}\int_{\Gamma_\pm}\frac{dD}{2\pi i D}\exp\left(-\frac{D^2}{2e^2} - i\zeta D\right)\oint_{\partial\Delta_\varepsilon^{(\mp)}}\frac{d\phi}{2\pi i}\, g(\phi, D, \epsilon_+, z) = 0. \tag{3.77}$$

because the value of g with non-zero D is bounded while the integral region $\Delta_\varepsilon^{(\mp)}$ shrinks as $\varepsilon \to 0$. On the other hand, when the integral contour is given as $D \in \Gamma_\pm$ and $\phi \in \Delta_\varepsilon^{(\pm)}$ whose signs coincide, deform the D-contour to be $\Gamma_\pm \to \Gamma_\mp \mp C_0$. C_0 is the counterclockwise contour enclosing $D = 0$. Just as before, the integral over $D \in \Gamma_\mp$ vanishes so that

$$-\lim_{e,\varepsilon\to 0}\int_{\Gamma_\pm}\frac{dD}{2\pi i D}\exp\left(-\frac{D^2}{2e^2} - i\zeta D\right)\oint_{\partial\Delta_\varepsilon^{(\mp)}}\frac{d\phi}{2\pi i}\, g(\phi, D, \epsilon_+, z) = 0. \tag{3.78}$$

Notice that $g(D = 0)$ reduces to the holomorphic 1-loop determinant introduced in (3.62)-(3.64). As the final step, move to the integral (3.75) over the contour $\phi \in \partial\Delta_0 \cup \partial\Delta_\infty$. Along the contour, $|\mathrm{Re}\,\phi|$ becomes very large so that (3.66) can be treated as if $D = 0$. Replacing $g(D)$ with $g(D = 0)$, (3.75) becomes

$$-\lim_{e,\varepsilon\to 0} f_\pm(e\zeta)\oint_{\partial M_0+\partial M_\infty}\frac{d\phi}{2\pi i} g(\phi, \epsilon_+, z) \tag{3.79}$$

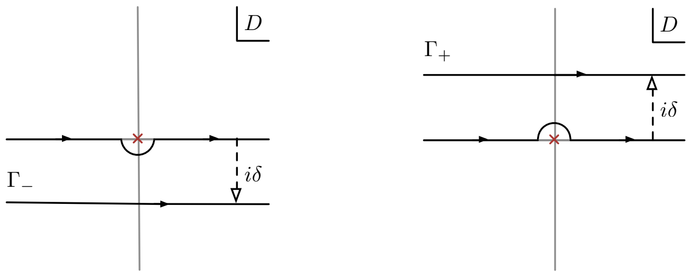

Figure 3.1: Deforming the contour $\Gamma_\pm$ for computing $f_\pm(0)$

where the function

$$f_\pm(e\zeta) = \int_{\Gamma_\pm} \frac{dD}{2\pi i D} \exp\left(-\frac{D^2}{2e^2} - i\zeta D\right) \tag{3.80}$$

is computed as follows. The derivative of $f_\pm(e\zeta)$ by ζ,

$$\frac{df_\pm}{d\zeta} = -\frac{1}{2\pi}\int_{\Gamma_\pm} dD\ \exp\left(-\frac{D^2}{2e^2} - i\zeta D\right) = -\sqrt{\frac{e^2}{2\pi}}\exp\left(-\frac{\zeta^2 e^2}{2}\right), \tag{3.81}$$

implies that the function $f_\pm(e\zeta)$ is given as

$$f_\pm(e\zeta) = f_\pm(0) - \frac{1}{2}\mathrm{erf}\left(\frac{e\zeta}{\sqrt{2}}\right) \tag{3.82}$$

in terms of the error function $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x dt\ e^{-t^2}$. Moreover, $f_\pm(0)$ can be obtained from the definition (3.80),

$$f_\pm(0) = \int_{\Gamma_\pm} \frac{dD}{2\pi i D}\ \exp\left(-\frac{D^2}{2e^2}\right) = \mp\frac{1}{2} + \int_{-\infty}^{\infty} dD\ \mathrm{Pr}\left[\frac{1}{D}e^{-\frac{D^2}{2e^2}}\right] = \mp\frac{1}{2} \tag{3.83}$$

by deforming the contour as in Fig. 3.1. The second term involving the Cauchy principal value vanishes since the integrand is an odd function of D. Therefore, one concludes that the integral (3.79) happens to be

$$Z^\pm_{\partial M_0+\partial M_\infty} = \frac{1}{2}(R_0 + R_\infty)\left[\pm 1 + \lim_{e,\varepsilon\to 0}\mathrm{erf}\left(\frac{e\zeta}{\sqrt{2}}\right)\right]. \tag{3.84}$$

Adding all the contributions $Z^\pm_{\partial\Delta_\varepsilon}$ and $Z^\pm_{\partial M_0+\partial M_\infty}$, the index Z is either of

$$Z^+ \quad = \quad +\sum_{Q_i>0} R_i + \frac{1}{2}(R_0 + R_\infty)\left[1 + \lim_{e,\varepsilon\to 0}\mathrm{erf}\left(\frac{e\zeta}{\sqrt{2}}\right)\right]$$

$$Z^- \;=\; -\sum_{Q_i<0} R_i - \frac{1}{2}(R_0+R_\infty)\left[1-\lim_{e,\varepsilon\to 0}\operatorname{erf}\left(\frac{e\zeta}{\sqrt{2}}\right)\right] \;. \tag{3.85}$$

Both expressions are actually equivalent, i.e.,

$$Z^+ - Z^- = \sum_{Q_i>0} R_i + \sum_{Q_i<0} R_i + R_0 + R_\infty = 0 \;, \tag{3.86}$$

where the last equation holds because the sum of all residues on the cylinder is zero if one includes those at infinities. So from here on $Z \equiv Z^+ = Z^-$. The FI term ζ does affect the result. For instance, at $e\zeta = -\infty$, 0 and $+\infty$, one obtains

$$Z(e\zeta=0) = \sum_{Q_i>0} R_i + \frac{1}{2}(R_0+R_\infty)\,,\, Z(e\zeta=\infty) = -\sum_{Q_i<0} R_i\,,\, Z(e\zeta=-\infty) = \sum_{Q_i>0} R_i\,. \tag{3.87}$$

Since ζ is a parameter of the theory, the index can depend on it only when there is a continuum contribution from the Coulomb branch (with nonzero φ). In particular, $Z(e\zeta)$ depends continuously on $e\zeta$, so that $Z(e\zeta)$ expanded in the fugacities cannot generally have integral coefficients. This is also expected in general, with continuum contributions. The point $\zeta = 0$ is where the Coulomb branch with nonzero ϕ meets the Higgs branch. Nonzero ζ generates a mass gap for the Coulomb branch degrees with the mass proportional to $e^2\zeta$, so that the continuum cannot affect the Witten index. Since the index is computed in the $e \to 0$ limit, the above result as a function of finite $e\zeta$ generates vanishing mass gap $e^2\zeta \to 0$. The indices with finite gaps are thus obtained in the $e\zeta \to \pm\infty$ limit. The functions $f_\pm(e\zeta)$ satisfy $f_\pm(e\zeta = \mp\infty) = 0$. When δ and ζ have opposite signs, the contribution (3.84) from the IR cutoff contours vanishes. Therefore,

$$Z(\zeta) = \operatorname{sign}(-\zeta) \sum_{\operatorname{sign}(-\zeta)Q_i>0} R_i \;, \tag{3.88}$$

which does not refer to the contributions from residues at infinities. The possible difference between the Witten indices in two limits $e\zeta = \pm\infty$,

$$Z(\zeta>0) - Z(\zeta<0) = R_0 + R_\infty \;, \tag{3.89}$$

implies a wall crossing of the index across $\zeta = 0$, at which the system of interest meets a continuum from the Coulomb branch.

All above discussions are applicable for general quantum mechanical index with $(0,2)$ SUSY. However, for the ADHM quantum mechanics, ζ-dependence of the index is unacceptable. This is because the ζ dependence incurs a strange dependence of Z on $\epsilon_+ \equiv \frac{\epsilon_1+\epsilon_2}{2}$, conjugate to the Cartan of the diagonal combination of $SU(2)_r \times SU(2)_R$. Nekrasov's partition function counts half-BPS states of the 5d $\mathcal{N} = 1$ gauge theory, preserving 4 Hermitian supercharges. Their BPS multiplets break neither $SU(2)_r$ nor $SU(2)_R$ symmetry. So although the ADHM index only refers to 2 SUSY, it should be an even function of ϵ_+ if it only captures the spectrum of these half-BPS states with a further refinement with ϵ_+. Since both ζ and ϵ_+ break $SU(2)_R$, the sign flip of ζ effectively induces the sign flip of ϵ_+ in the partition function. So if $Z(\zeta \lessgtr 0)$ are same, it is an even function of ϵ_+. However, if $R_0 + R_\infty \neq 0$, this measures the failure of $Z(\zeta > 0)$ and $Z(\zeta < 0)$ being even in ϵ_+. Thus, ζ dependence and $\epsilon_+ \to -\epsilon_+$ asymmetry is unreasonable if this index is counting half-BPS bound states of instantons and W-bosons in the 5d super-Yang-Mills theory.

To find the resolution to this puzzle, one should understand the true nature of Z. The possible wall crossing happens because the Coulomb branch continuum appears at $\zeta = 0$. From the ADHM quantum mechanics, the Coulomb branch degrees appear only as going to the UV complete gauge theory description of the instanton quantum mechanics. So there may appear contribution to the index from the extra UV degrees of the ADHM quantum mechanics which do not belong to the QFT Hilbert space. This can either be a fractional contribution from the continuum, or integral contribution coming from marginal bound states involving the extra stringy states. In any case, these have to factorize into

$$Z^{\text{1d}} = Z^{\text{inst}} \cdot Z^{\text{string}} \tag{3.90}$$

since the field theories that I shall study from string theory are obtained by taking suitable decoupling limits. One should identify Z^{string} and factor it out to study the true QFT index. Whether it is possible depends on one's knowledge on the string theory background which engineers the QFT. This point will be explained further in Section 4. Coming back to the ζ dependence and $\epsilon_+ \to -\epsilon_+$ asymmetry, the wall crossing of the index should happen only in Z^{string}, but not in Z^{inst}.

When $R_0 = R_\infty = 0$ separately, the continuum from φ is lifted by quantum effects. In this case, the index is independent of the continuous parameters of the theory, and its coefficients are integers. When $R_0 + R_\infty = 0$ but $R_0 = -R_\infty \neq 0$, there is no wall crossing but still is a continuum from φ. In this case, since a continuum is attached to the Higgs branch, the index may have fractional coefficients. However, these non-integral contributions will go to Z^{string}, and not to Z^{inst}.

3.4.2 Higher-rank gauge group

When the quantum mechanical gauge group $\hat{G}$ has rank $r > 1$, the multi-dimensional version of the integral (3.70) can be written as

$$Z = \frac{1}{|W|} \lim_{e,\varepsilon \to 0} \int_{\mathfrak{h}} d^r D \int_{M \backslash \Delta_\varepsilon} d^{2r}\phi \; f_{\text{e}}(\phi, \bar{\phi}, D) \exp\left(-\frac{D^2}{2e^2} - i\zeta(D)\right) \tag{3.91}$$

for $\mathfrak{h}$ denoting the Cartan subalgebra of $\hat{G}$. In the limit $e \to 0$, $f_{\text{e}}(\phi, \bar{\phi}, D)$ becomes

$$\begin{aligned} f_{\text{e}}(\phi, \bar{\phi}, D) &\xrightarrow{e \to 0} \int \prod_{c=1}^{r} d\lambda_{c,0} d\bar{\lambda}_{c,0} \left\langle \prod_{a,b=1}^{r} \int d^2x \lambda_a \sum_i Q_i^a \bar{\Psi}_i \phi_i \int d^2x \bar{\lambda}_b \sum_j Q_j^b \psi_j \bar{\phi}_j \right\rangle_{\text{free}} \\ &= \det\left[h^{ab}(\phi, \bar{\phi}, D, \epsilon_+, z)\right] \cdot g(\phi, \bar{\phi}, D, \epsilon_+, z) \end{aligned} \tag{3.92}$$

where h^{ab} and g are given by

$$h^{ab} = c \sum_{i,n} \frac{Q_i^a Q_i^b}{\left(|2\pi i n + Q_i(\phi) + J\epsilon_+ + Fz|^2 + iQ_i(D)\right)\left(-2\pi i n + Q_i(\bar{\phi}) + J\bar{\epsilon}_+ + F\bar{z}\right)} \tag{3.93}$$

$$g = e^{\kappa \text{tr}(\phi)} Z_{\text{v}}(\phi) \prod_{\Phi} Z_\Phi(\phi, \bar{\phi}, D, \epsilon_+, z) \prod_{\Psi} Z_\Psi(\phi, \epsilon_+, z). \tag{3.94}$$

The matrix-valued function h^{ab} satisfies the following properties

$$\frac{\partial h^{ab}}{\partial \bar{\phi}_c} = \frac{\partial h^{cb}}{\partial \bar{\phi}_a}, \qquad \frac{\partial g}{\partial \bar{\phi}_a} = \frac{i}{c} h^{ab} D_b g. \tag{3.95}$$

In terms of the $(0,1)$-forms $\nu^b \equiv d\bar{\phi}_a \, h^{ab}$ on M, the integral (3.91) is written as

$$Z = \frac{1}{|W|} \lim_{e,\varepsilon \to 0} \int_{\mathfrak{h} \times (M \backslash \Delta_\varepsilon)} \mu \tag{3.96}$$

where

$$\mu \equiv g \, \exp\left(-\frac{D^2}{2e^2} - i\zeta(D) \right) d^r\phi \wedge (\nu(dD))^{\wedge r} \, . \tag{3.97}$$

Using the same logic as the rank-1 case, the D-contour can be deformed to $\mathfrak{h} \to \Gamma = \mathfrak{h} + i\delta$ without changing Z, as long as δ is sufficiently small and satisfying $Q_i(\delta) \neq 0$ for all i's.

For a given set of charge vectors $\{Q_1, \cdots, Q_s\} \subset \mathfrak{h}^*$, the differential form $\mu_{Q_1,\cdots,Q_s}$ is defined as

$$\begin{aligned} \mu_{Q_1,\cdots,Q_s} \equiv \frac{(ic)^s}{(r-s)!} \, g \, \exp\Big[-\frac{1}{2e^2} D^2 - i\xi(D) \Big] \\ d^r u \wedge \big(\nu(dD)\big)^{\wedge(r-s)} \wedge \frac{dQ_1(D)}{Q_1(D)} \wedge \cdots \wedge \frac{dQ_s(D)}{Q_s(D)} . \end{aligned} \tag{3.98}$$

This is the $(r, r-s)$-form in ϕ-space as well as the s-form in D-space. For this not to vanish, the set $\{Q_1, \cdots, Q_s\}$ must be linearly independent. The remaining step is to apply iteratively the following identity, derived from (3.95),

$$d\mu_{Q_0,\cdots,Q_s} = \sum_{i=0}^{s} (-1)^{s-i} \mu_{Q_0,\cdots,\widehat{Q}_i,\cdots,Q_s} . \tag{3.99}$$

where "hat" means omission. One needs to construct the cell decomposition of $M \backslash \Delta_\epsilon$: Assume that singular hyperplanes considered here are non-degenerate: there is no point where distinct r hyperplanes meet. Denoting the set of indices for singular hyperplanes by $\mathfrak{H}_S$,

$$\Delta_\varepsilon = \bigcup_{i \in \mathfrak{H}_S} \Delta_\varepsilon(H_i) \tag{3.100}$$

where its boundary $\partial\Delta_\varepsilon$ is decomposed into

$$S_i = \partial\Delta_\varepsilon \cap \partial\Delta_\varepsilon(H_i). \tag{3.101}$$

Taking into account the introduction of a cutoff at infinity, the space M is replaced to $M \to M_\Lambda$ and S_∞ is the associated boundary at infinity. Define $\mathfrak{H} = \mathfrak{H}_S \cup \{\infty\}$. For $S_{i_1,\cdots,i_s} = S_{i_1} \cap \cdots \cap S_{i_s}$, the boundary operation on $S_{i_1,\cdots,i_s}$ gives

$$S_{i_1,\cdots,i_s} = -\sum_{j\in\mathfrak{H}} S_{i_1,\cdots,i_s j}. \tag{3.102}$$

The cell decomposition of $M_\Lambda \backslash \Delta_\varepsilon$ satisfies three conditions. Firstly, each cell has an interior which is in the interior of $M_\Lambda \backslash \Delta_\varepsilon$ or exactly one $S_{i_1,\cdots,i_s}$. Secondly, the cell decomposition is good in the interior of $M_\Lambda \backslash \Delta_\varepsilon$, in the sense that any codimension-k cell is lying at the intersection of $(k+1)$ codimension-$(k-1)$ cells. To describe the final condition, notice that a neighborhood in $M_\Lambda \backslash \Delta_\varepsilon$ of an interior point of $S_{i_1,\cdots,i_s}$ is in the form $(\mathbb{R}_+)^s \times \mathbb{R}^{2l-s}$. Introducing the coordinate $\{(x_{i_1}, \cdots, x_{i_s}, y_{s+1}, \cdots, y_{2l})\}$ defined by $x_{i_*} \geq 0$, the cell $C_{j_1,\cdots,j_p}$ is defined by

$$C_{j_1,\cdots,j_p} = \{0 \leq x_{j_1} = \cdots = x_{j_p} \leq \text{all other } x_i\text{'s}\} \subset (\mathbb{R}_+)^s \times \mathbb{R}^{2l-s} \tag{3.103}$$

for $\{j_1, \cdots, j_p\} \subset \{1, \cdots s\}$. The boundary of $C_{j_1,\cdots,j_p}$ is given as

$$\partial C_{j_1,\cdots,j_p} = -S_{j_1,\cdots,j_p} + \sum_i C_{j_1,\cdots,j_p i}. \tag{3.104}$$

The third condition requires that a cell touching the interior of $S_{i_1,\cdots,i_s}$ coincides with one of $C_{j_1,\cdots,j_p}$ or its subdivision in the $\mathbb{R}^{2l-s}$ direction. Using the cell decomposition, the integral over the cell in the interior region does not contribute. Therefore, the integral (3.96) becomes, for $i_* \in \mathfrak{H}$,

$$\int_{\Gamma\times(M\backslash\Delta_\varepsilon)} \mu = -\sum_{p=1}^{r} \sum_{i_1<\cdots<l_p} \int_{\Gamma\times S_{i_1\cdots i_p}} \mu_{Q_{i_1},\cdots,Q_{i_p}} \tag{3.105}$$

in which the identity (3.95) and Stokes' theorem have been repeatedly used [61].

Consider the D-integration. The D-integral contour was shifted as $\mathfrak{h} \to \mathfrak{h} + i\delta$. For each integral expression $I_{i_1\cdots i_p} = \int_{\Gamma\times S_{i_1\cdots i_p}} \mu_{Q_{i_1},\cdots,Q_{i_p}}$, the integrand is regular in the $\varepsilon \to 0$ limit if $Q_{i_a}(\delta) < 0$ for any $i_a \in \mathfrak{H}$. If $Q_{i_a}(\delta) > 0$ for all $i_a \in \mathfrak{H}$, one can deform Γ downward in each $Q_{i_a}(D)$-plane so that the contour decomposes into the infinite line and the circle around the origin. The infinite line contour does not contribute as $\varepsilon \to 0$, while the contour encircling the origin picks up the simple pole at $Q_{i_a}(D) = 0$. This can be summarized as

$$I_{i_1\cdots i_p} = \prod_{a=1}^{p} \theta(Q_{i_a}(\delta)) \int_{\Gamma_{i_1\cdots i_p}\times S_{i_1\cdots i_p}} \mu_{Q_{i_1},\cdots,Q_{i_p}} \tag{3.106}$$

where $\Gamma_{i_1\cdots i_p} = C_{i_1\cdots i_p} \times \operatorname{Ker} Q_{i_1\cdots i_p}$. Moreover, terms like

$$\int_{\Gamma_{i_1\cdots i_p}\times S_{i_1\cdots i_p}} \mu_{Q_{i_1},\cdots,Q_{i_p},Q_{j_1},\cdots,Q_{j_q}} \tag{3.107}$$

are further massaged by shifting a constant $i\delta_{i_1,\cdots,i_p} \in \operatorname{Ker} Q_{i_1\cdots i_p}$. If $Q_j(\delta_{i_1,\cdots,i_p}) < 0$ for $j \in \{j_1, \cdots, j_q\}$, the integral vanishes to zero as $\varepsilon \to 0$. One can deform the contour into the infinite line and the circle around the origin in each $Q_j(D)$ plane, just the same as before. Every term in (3.105) can be written in terms of

$$\int_{\Gamma_{i_1\cdots i_p}\times S_{i_1\cdots i_p}} \mu_{Q_{i_1},\cdots,Q_{i_p}} \quad \text{and} \quad \int_{\Gamma_{i_1\cdots i_p}\times S_{i_1\cdots i_q\infty}} \mu_{Q_{i_1},\cdots,Q_\infty} \tag{3.108}$$

for $0 \le p \le q < r$. For systematic investigation, choose a covector $\eta \in \mathfrak{h}^*$ such that all contour shifts are done in the way satisfying $\eta(\delta) > 0$ and $\eta(\delta_{i_1,\cdots,i_p}) > 0$. In particular, when the FI parameter ξ is $\mathfrak{H}_S$-generic, the choice of $\eta = -\xi$ and $Q_\infty = \xi$ is allowed. Moreover, terms related to Q_∞ just vanish in the limit of $e \to 0$ and $\xi \to \infty$ which keeps $e^2\xi$ finite. In this case, the inductive argument of [61] shows that the integral is concisely organized into the JK residue

$$\text{JK-Res}(\mathbf{Q}_*, \eta)\frac{d\phi_1 \wedge \cdots \wedge d\phi_r}{Q_{j_1}(\phi)\cdots Q_{j_r}(\phi)} = \begin{cases} |\det(Q_{j_1}, \cdots, Q_{j_r})|^{-1} & \text{if } \eta \in \text{Cone}(Q_{j_1}, \cdots, Q_{j_r}) \\ 0 & \text{otherwise} \end{cases} . \tag{3.109}$$

where ‘Cone’ denotes the cone spanned by the r independent vectors. Namely, $\eta \in \text{Cone}(Q_1, \cdots, Q_r)$ if $\eta = \sum_{i=1}^{r} a_i Q_i$ with positive coefficients a_i. Although this definition apparently looks over-determining, it is known to be consistent: see [61] and references therein. The index is given by

$$Z = \frac{1}{|W|} \sum_{\phi_*} \text{JK-Res}(\mathbf{Q}(\phi_*), \eta)\ Z_{\text{1-loop}}(\phi, \epsilon_+, z)\ . \tag{3.110}$$

Note that the result depends on the choice of η when the residue sum at the two infinities of a cylinder (spanned by any $\rho(\phi)$ appearing in $Z_{\text{1-loop}}$) does not vanish. In this case, the above expression is understood with $\eta = -\zeta$. So Z in this case is a piecewise constant function of ζ.

For non-Abelian gauge group $\hat{G}$, the covector ζ is restricted to be along the overall $U(1)$ factors only. η is chosen in [61] not to coincide with the weights Q_i associated with the poles. (More generally η is taken not to lie at the boundary of the 'chambers.' See [61] for explanations.) So one might think that it would be troublesome to impose $\eta = -\zeta$ if ζ is aligned along the forbidden direction for $-\eta$, e.g. being proportional to a weight in the problem. For Abelian theories, such as $U(1)^r$ theories, ζ can be a generic vector in h^* so that ζ on the boundary of a chamber is potentially a wall-crossing point. ζ can be displaced from such a value in the Abelian theories, and one obtains different results across the wall by setting $\eta = -\zeta$ for these displaced ζ. However, for $U(r)$, its FI term is along a fixed direction on $h^* = \mathbb{R}^r$, proportional to $(1, 1, \cdots, 1)$. This might be at the boundary of chambers. For instance, the weight $(1, 1, \cdots, 1)$ appears with the rank r totally symmetric representation of $U(r)$, or a totally antisymmetric representation of it. If ζ is at the boundary, then one can remove the ambiguity by slightly shifting η away from $-\zeta(1, 1, \cdots, 1)$. Different shifts may leave η in different chambers. However, since $-\zeta(1, \cdots, 1)$ is a Weyl invariant point of $U(r)$, these different chambers map to one another by Weyl reflection. Due to this symmetry, different shifts of η should yield the same result.

In all examples that are studied hereafter, I found that the Jeffrey-Kirwan residue rule is equivalent to the following prescription, which is well known in the instanton calculus for some theories. One should perform the contour integral over r variables ϕ_I one by one. The contour integral takes the form of

$$\frac{1}{|W|} \oint \prod_{I=1}^{r} \frac{d\phi_I}{2\pi i}\ Z_{\text{v}}(\phi, \epsilon_+, z) \prod_{\Phi} Z_\Phi(\phi, \alpha, \epsilon_+, z) \prod_{\Psi} Z_\Psi(\phi, \alpha, \epsilon_+, z)\ . \tag{3.111}$$

The poles from a chiral multiplet factor takes the form

$$\frac{1}{2\sinh\left(\frac{Q_i(\phi)+J_i\epsilon_++F_iz}{2}\right)} \sim \frac{1}{e^{Q_i(\phi)} - e^{-J_i\epsilon_+}e^{-F_iz}}\ , \tag{3.112}$$

where Q_i is the charge vector of the chiral multiplet Φ_i. So the pole one picks up for the $z_I = e^{\phi_I}$ variables are determined by r different equations of the kind $e^{Q_i(\phi)} = (e^{-\epsilon_+})^{J_i}e^{-F_iz}$. In the instanton calculus, the value of charge J conjugate to ϵ_+ is always positive when the quantum mechanical chiral multiplet comes from the 5d SYM theory's vector multiplet (namely, the ADHM degrees). On the other hand, one always finds that $J < 0$ for the mechanical chiral multiplet coming from 5d hypermultiplets. At this point, I temporarily treat the $e^{-\epsilon_+}$ factors appearing in the Z_Φ's from the 5d vector multiplet and those from the 5d hypermultiplet independently. Namely, I substitute $e^{-\epsilon_+} \to t < 1$ for the measure coming from 5d vector multiplet, and $e^{-\epsilon_+} \to T > 1$ for the measure from 5d hypermultiplet. This makes the pole for $e^{Q_i(\phi)}$ to be all inside the unit circle in (3.112).

With these understood, the alternative residue prescription which turns out to give the same result is obtained by regarding each integral variable z_I as living on the unit circle on the complex plane. Then integrate over these variables one by one, for which one has to pick up poles inside the contour and sum over their residues (assuming $t < 1$, $T > 1$). After the residue sums of all r integrals, I set t, T back to the same value $t = T = e^{-\epsilon_+}$. This yields the same result as the index obtained by the sum of JK-Res. Of course, to see the agreement most clearly, one should choose η carefully, related to the order of integral for $\phi_1, \phi_2, \cdots, \phi_r$ in the prescription. Whenever one encounters a pole at $z_I = 0$, one does not include their residues, as part of the prescription. Also, one occasionally encounters poles which are not clearly inside or outside the unit circle with $t \ll 1$, $T \gg 1$ only. Here, one may choose other fugacities arbitrarily to shift such poles away from the unit circle. The arbitrary shift will not affect the result, as illustrated later. So far, this is the prescription which works when theindex has zero residue sums at infinities over a cylinder. For some $U(k)$ instanton calculus for which the sign of FI term matters, I chose $\eta = -\zeta$ and summed over JK-Res. In the alternative prescription, ζ

dependence appears as a failure of the index to be an even function of ϵ_+. So the two different indices are obtained by either running through the above prescription as explained above, or alternatively taking $t = e^{\epsilon_+} < 1$ and $T = e^{\epsilon_+} > 1$ temporarily and going through the unit circle contour prescription. This yields results which are related to each other by flipping $\epsilon_+ \to -\epsilon_+$, or equivalently $\zeta \to -\zeta$.

In the rank 1 case, it is immediate that the alternative prescription yields the same result as the result of Section 3.4.1. This is because (3.112) is given by

$$\frac{1}{z^{Q_i} - t^{J_i} e^{-F_i z}} \ , \quad \frac{1}{z^{Q_i} - T^{J_i} e^{-F_i z}} \tag{3.113}$$

for the chiral multiplet originating from 5d vector and hypermultiplet, respectively. The rule in Section 3.4.1 was to sum over the residues with $Q_i > 0$. The poles to be kept are

$$z = t^{J_i/Q_i} e^{-F_i/Q_i z} \ , \quad z = T^{J_i/Q_i} e^{-F_i/Q_i z} \qquad (Q_i > 0) \ . \tag{3.114}$$

These are all inside the unit circle $|z| < 1$ since $J_i \gtrless 0$ respective for chiral multiplets from 5d vector/hypermultiplet, and $t < 1, T > 1$. So this is agrees with the unit circle contour prescription. In fact the temporary relaxation $e^{-\epsilon_+} \to t < 1$ and $e^{-\epsilon_+} \to T > 1$ can be understood as pushing all poles with nonzero JK-Res inside the unit circles. So even for the higher rank case, this prescription is quite heuristic but I am not aware of a general proof that the two are equivalent. I will just provide comparisons of the two rules with higher rank examples in Section 3.5. Although the final result is the same, the latter prescription picks more poles and residues in the intermediate stage compared to the JK-Res rule: the extra residues however all cancel out in pairs, as explained in later examples. Such an alternative rule is known and widely used in the instanton calculus, starting from [15]. Comparisons of the two rules above in Section 3.5 will thus rigorously justify the existing prescriptions from the JK-Res rules. When there are subtleties due to the poles at the infinities of the cylinders, the JK residue rule also justifies various vague parts in the existing prescriptions. Note that temporarily substituting $t < 1$ and $T > 1$ for the vector/hypermultiplet poles is essentially the '$i\epsilon$' and '$-i\epsilon$' prescriptions given to the vector and hypermultiplet poles, observed in [62]. I can rephrase the alternative prescription as picking all the poles/residues inside the unit circles from the 5d vector multiplets, and picking all of them outside the unit circles from the 5d hypermultiplets, assuming $e^{-\epsilon_+} \ll 1$.

3.5 Examples

3.5.1 $\mathcal{N} = 1^*$ theories

Here I first discuss the SYM theory with one adjoint hypermultiplet, with arbitrary classical gauge group $G = U(N), Sp(N), SO(N)$. The contour integral takes the form of

$$\frac{1}{k!}\oint \left[\prod_{I=1}^{k} \frac{d\phi_I}{2\pi i}\right] Z_{\text{vec}}(\phi, \alpha, \epsilon_{1,2}) Z_{\text{adj}}(\phi, \alpha, \epsilon_{1,2}, m) \ . \tag{3.115}$$

Z_{vec} is the 1-loop determiant for the quantum mechanical modes which come from the 5d gauge theory's vector multiplet. For brevity, introduce the notation $2\sinh(a \pm b) \equiv 2\sinh(a+b) \cdot 2\sinh(a-b)$ and so on.

$$Z_{\text{vec}} = \frac{\prod_{\alpha\in\text{root}(\hat{\text{G}})} 2\sinh\left(\frac{\alpha(\phi)}{2}\right) \cdot \prod_{\alpha\in\text{adj}(\hat{\text{G}})} 2\sinh\left(\frac{\alpha(\phi)+2\epsilon_+}{2}\right)}{\prod_{\hat{\rho}\in\text{fund}(\hat{\text{G}})}\prod_{\rho\in\text{fund(G)}} 2\sinh\left(\frac{\pm(\hat{\rho}(\phi)-\rho(\alpha))+\epsilon_+}{2}\right) \prod_{\alpha\in\text{adj}(\hat{G})} 2\sinh\left(\frac{\alpha(\phi)\pm\epsilon_-+\epsilon_+}{2}\right)} \tag{3.116}$$

For $G = U(N)$ and $\hat{G} = U(k)$, Z_{vec} is given by

$$\frac{\prod_{I\neq J} 2\sinh\left(\frac{\phi_{IJ}}{2}\right) \cdot \prod_{I,J=1}^{k} 2\sinh\left(\frac{\phi_{IJ}+2\epsilon_+}{2}\right)}{\prod_{I=1}^{k}\prod_{i=1}^{N} 2\sinh\left(\frac{\phi_I-\alpha_i+\epsilon_+}{2}\right) \cdot 2\sinh\left(\frac{\alpha_i-\phi_I+\epsilon_+}{2}\right) \prod_{I,J=1}^{k} 2\sinh\left(\frac{\phi_{IJ}+\epsilon_1}{2}\right) \cdot 2\sinh\left(\frac{\phi_{IJ}+\epsilon_2}{2}\right)} \tag{3.117}$$

where $\phi_{IJ} \equiv \phi_I - \phi_J$, and 'adj' in the product means that all modes in the adjoint representation including Cartans are included. For $G = Sp(N), \hat{G} = O(k)_+$ and $G = SO(N), \hat{G} = Sp(k)$, under reality conditions on mechanical degrees, Z_{vec} is

$$\frac{\prod_{\alpha\in\text{root}(\hat{\text{G}})} 2\sinh\left(\frac{\alpha(\phi)}{2}\right) \cdot \prod_{\alpha\in\text{adj}(\hat{\text{G}})} 2\sinh\left(\frac{\alpha(\phi)+2\epsilon_+}{2}\right)}{\prod_{\hat{\rho}\in\text{fund}(\hat{\text{G}})}\prod_{\rho\in\text{fund(G)}} 2\sinh\left(\frac{\hat{\rho}(\phi)-\rho(\alpha)+\epsilon_+}{2}\right) \prod_{\rho\in\text{R}(\hat{G})} 2\sinh\left(\frac{\rho(\phi)+\epsilon_1}{2}\right) \cdot 2\sinh\left(\frac{\rho(\phi)+\epsilon_2}{2}\right)} \tag{3.118}$$

where R is symmetric/antisymmetric representation of $O(k)_+/Sp(k)$, respectively. The result for $O(k)_-$ is more complicated [63], which is reviewed in Section 3.5.3. Z_{adj} is the 1-loop determiant for the quantum mechanical modes coming from the 5d theory's adjoint hypermultiplet. For $G = U(N), \hat{G} = U(k)$, it is given by

$$Z_{\text{adj}} = \frac{\prod_{I=1}^{k}\prod_{i=1}^{N} 2\sinh\left(\frac{\phi_I-\alpha_i+m}{2}\right)\cdot 2\sinh\left(\frac{\alpha_i-\phi_I+m}{2}\right)\prod_{I,J=1}^{k} 2\sinh\left(\frac{\phi_{IJ}\pm m-\epsilon_-}{2}\right)}{\prod_{I,J=1}^{k} 2\sinh\left(\frac{\phi_{IJ}\pm m-\epsilon_+}{2}\right)}, \tag{3.119}$$

and for other groups,

$$Z_{\text{adj}} = \frac{\prod_{\hat{\rho}\in\text{fund}(\hat{\text{G}})}\prod_{\rho\in\text{fund(G)}} 2\sinh\left(\frac{\hat{\rho}(\phi)-\rho(\alpha)+m}{2}\right)\prod_{\rho\in\text{R}(\hat{G})} 2\sinh\left(\frac{\rho(\phi)\pm m-\epsilon_-}{2}\right)}{\prod_{\alpha\in\text{adj}(\hat{\text{G}})} 2\sinh\left(\frac{\alpha(\phi)\pm m-\epsilon_+}{2}\right)} \tag{3.120}$$

where R is chosen in the same way for each group as in Z_{vec}.

Turning to the contour integral, one can show that the sums of two residues at the infinities of cylinders are always zero, so η can be arbitrarily chosen without referring to ζ. Here I start from the well-known case with $G = U(N)$, $\hat{G} = U(k)$ [15]. For k instantons, the covector space h^* for charges is $\mathbb{R}^k$. Choose $\eta = (1, 1, \cdots, 1) = e_1+e_2+\cdots+e_k$. Let me first explain all possible choices of k charges $\{Q_{i_1}, Q_{i_2}, \cdots, Q_{i_k}\}$ satisfying $\eta \in \text{Cone}(Q_{i_1}, \cdots, Q_{i_k})$. They determine $\mathbf{Q}(\phi_*)$ for poles with nonzero JK-Res, both in non-degenerate cases ($n = k$) and in degenerate cases ($n > k$) where the k charges form a subset of $\mathbf{Q}(\phi_*)$.

Possible Q_i's are $\{\pm e_I\}$ from the fundamental/anti-fundamental weights, and $\{e_I - e_J\}$ from adjoint. With η having all positive components, note that $-e_I$ can never be chosen in the k charges which contain η in their cone. Using the Weyl invariance of $U(k)$ which permutes k e_I's, it suffices to show that $Q_1 = -e_1 = (-1, 0, \cdots, 0)$ cannot be chosen. Suppose that it could be. Then one must choose the remaining $k-1$ charge vectors which satisfy

$$\eta = (1, 1, \cdots, 1) = (-a_1, 0, \cdots, 0) + \sum_{I=2}^{k} a_I Q_I \tag{3.121}$$

with $a_1, a_2, \cdots, a_k > 0$. For this to be true, at least one of the $k-1$ Q_I's should have positive first component, which one takes to be Q_2. $Q_2 = e_1$ is impossible, because then Q_1, Q_2 are linearly dependent. Other choices are $Q_2 = e_1 - e_I$ for $I \neq 1$, in which $I = 2$ is taken using Weyl symmetry. Then nonzero second component of $a_1 Q_1 + a_2 Q_2 = (a_2 - a_1, -a_2)$ requires that $Q_3 = e_2 - e_3$ up to Weyl reflection, and so on. This step repeats, until one finds (up to Weyl reflections) all the k vectors given by

$$(Q_1, Q_2, \cdots, Q_k) = (-e_1, e_1 - e_2, e_2 - e_3, \cdots, e_{k-1} - e_k), \tag{3.122}$$

for the first $k-1$ components of (3.121) to be positive. Then one finds that the last component of (3.121) is $-a_k < 0$, arriving at a contradiction.

So I choose k charges among $\{e_I\}$ and $\{e_I - e_J\}$ only. Using the arguments similar to the previous paragraph based on positivity and linear independence, the allowed charges are given as follows. Firstly, there should be one or more charges chosen among $\{e_I\}$, since the latter set $\{e_I - e_J\}$ only generates $k-1$ dimensional subspace of $\mathbb{R}^k$. Let me choose $p(\leq k)$ of them, which can be taken as $e_1, e_2, \cdots, e_p$ again up to Weyl reflections. For each chosen e_I with $1 \leq I \leq p$, the other charges can be divided into p groups, each group containing exactly one e_I.

As an example, I will pick the group containing e_1 and explain its structure, as other groups will be similar. First, charges of the type $e_1 - e_{J_a}$ should not be selected. Once both e_1 and $e_1 - e_{J_a}$ are chosen, say $J_a = k$, it is required that

$$(a_1 + a_2, 0, \cdots, 0, -a_2) + \sum_{I=3}^{k} a_I Q_I = (1, 1, \cdots, 1). \tag{3.123}$$

Since the first component can always be a unity by adjusting a_1, one can simply drop it so that the problem gets reduced to picking $k-2$ additional charges in which $-e_k$ has been chosen. The previous paragraph showed that this is impossible. Second, there are charges of the form $e_{J_a} - e_1$ with $J_a \neq 2, \cdots, p$. J_a's have to be different from $2, \cdots, p$, since otherwise there will be a linearly dependent combination of charge vectors. One can say that these make a tree graph, with a branch $e_{J_a} - e_1$ attached to e_1. Then, with e_1 and (possibly more than one) $e_{J_a} - e_1$

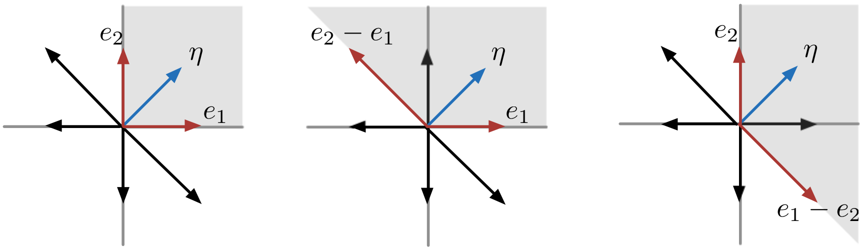

Figure 3.2: Choice of charge vectors for $U(N)$ index at $k = 2$ with $\eta = (1, 1)$

chosen, one can further find Q_i vectors which branch out from one of e_{J_a}'s, taking the form of $e_{K_b} - e_{J_a}$. K_b are again different from all the subscripts which appeared so far ($I = 1, 2, \cdots, p$, J_a's), to avoid linear relations among selected Q vectors. This procedure can be repeated, attaching adjoint charge vectors to e_{K_b}, and so on. This forms a tree graph originating from e_1. The same tree graph can be formed starting from e_2. It starts from $e_{L_c} - e_2$, with L_c being different from all indices that appeared so far. In this way, one can make p possible trees with k charges. This tree structure will be further constrained below, by considering whether there actually exist poles which refer to these charges in $\mathbf{Q}(\phi_*)$.

For instance, for $k = 2$ with $\eta = (1, 1)$, the selected charge vectors are

$$\{e_1; e_2\} \ , \quad \{e_1, e_2 - e_1\} \ , \tag{3.124}$$

and other charges obtained from above by permuting e_I's: here $\{e_2, e_1 - e_2\}$. These can also be immediately found from Fig. 3.2. For $k = 3$ with $\eta = (1, 1, 1)$, one finds

$$\{e_1; e_2; e_3\} \ , \quad \{e_1, e_2 - e_1; e_3\} \ , \quad \{e_1, e_2 - e_1, e_3 - e_1\} \ , \quad \{e_1, e_2 - e_1, e_3 - e_2\} \ , \tag{3.125}$$

and others obtained by permuting e_I's. For $k = 4$ with $\eta = (1, 1, 1, 1)$, one finds

$$\begin{aligned} &\{e_1; e_2; e_3; e_4\} \ , \ \{e_1, e_2 - e_1; e_3; e_4\} \ , \ \{e_1, e_2 - e_1, e_3 - e_1; e_4\}, \\ &\{e_1, e_2 - e_1, e_3 - e_2; e_4\} \ , \ \{e_1, e_2 - e_1; e_3, e_4 - e_3\} \ , \ \{e_1, e_2 - e_1, e_3 - e_1, e_4 - e_1\}, \\ &\{e_1, e_2 - e_1, e_3 - e_1, e_4 - e_2\} \ , \ \{e_1, e_2 - e_1, e_3 - e_2, e_4 - e_3\} \end{aligned} \tag{3.126}$$

and their Weyl reflections.

Now consider the pole ϕ_* whose $\mathbf{Q}_*$ forms a tree that was just explained (non-degenerate), or contains it (degenerate). The poles ϕ_* that actually arise from the integrand are labeled as follows, which will be proven below by induction. It is the famous colored Young diagram rule [15]. For each element Q_i in the chosen $\{Q_1, Q_2, \cdots, Q_n\}$, one assigns a hyperplane equation which constrains ϕ_*. When Q_i is one of the fundamental weights, $\{e_I\}$, one imposes an equation of the form

$$\phi_I - \alpha_i + \epsilon_+ = 0 \ , \tag{3.127}$$

with $i = 1, \cdots, N$. When Q_i belongs to the type of $e_I - e_J$, one should impose one of the following equations,

$$\phi_I - \phi_J + \epsilon_1 = 0 \ , \ \ \phi_I - \phi_J + \epsilon_2 = 0 \ , \ \ \phi_I - \phi_J - \epsilon_+ + m = 0 \ , \ \ \phi_I - \phi_J - \epsilon_+ - m = 0 \ , \tag{3.128}$$

where the first two come from Z_{vec} and the latter two come from Z_{adj}. When $n > k$, $n - k$ of them should be redundant for deciding ϕ_*. So pick k independent hyperplane equations which will be used to define ϕ_*. Since I am interested in the poles with nonzero JK-Res, there should be at least one choice $\{Q_1, Q_2, \cdots, Q_k\}$ in $\mathbf{Q}_*$ which contains η in their cone. I will work with k hyperplane equations picked in this way, whenever necessary.

The 'Young diagram rule' first states that there are no poles with nonzero JK-Res which refer to any of the hyperplane equation of the latter two types in (3.128) (containing m). Namely, [15] asserts that the poles coming from the 5d hypermultiplet measure Z_{adj} can be completely ignored when classifying relevant JK-Res. Then [15] focuses on the hyperplanes (3.127) and the first two types of hyperplanes in (3.128), all coming from Z_{vec}. The set of hyperplanes from the poles of Z_{vec} with nonzero residues are classified by the N-colored Young diagrams with k boxes. A colored Young diagram consists of N Young diagrams $Y = (Y_1, \cdots, Y_N)$ which satisfy $|Y_1| + \cdots + |Y_N| = k$, where $k_i = |Y_i|$ is the number of boxes of the Young diagram. Each box in the diagram $(Y_1, \cdots, Y_N)$ corresponds to a hyperplane among

(3.127) and the first two of (3.128). Hereafter I explain how $n_i \geq k_i$ hyperplanes are chosen for a given Young diagram Y_i. Firstly, assign to each of the k_i boxes one of the k_i variables $\phi_{I_1}, \cdots, \phi_{I_{k_i}}$. Let me say that ϕ_{I_1} maps to the box at the upper-left corner. The corresponding hyperplane is given by

$$\phi_{I_1} - \alpha_i + \epsilon_+ = 0 \ . \tag{3.129}$$

Then, consider all possible pairs of boxes one can form in Y_i, by grouping horizontally attached boxes or vertically attached boxes. For a horizontal pair, with ϕ_{I_1} and ϕ_{I_2} mapping to the left and right box respectively, I assign the hyperplane

$$\phi_{I_2} - \phi_{I_1} + \epsilon_1 = 0 \ . \tag{3.130}$$

For a vertical pair, with ϕ_{I_1} and ϕ_{I_2} mapping to the upper and lower box respectively, I assign the hyperplane

$$\phi_{I_2} - \phi_{I_1} + \epsilon_2 = 0 \ . \tag{3.131}$$

One obtains at least k_i independent hyperplanes this way. For instance, the diagram $Y_i = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ defines $k_i = 3$ hyperplanes

$$\phi_1 - \alpha_i + \epsilon_+ = 0 \ , \quad \phi_{21} + \epsilon_1 = 0 \ , \quad \phi_{31} + \epsilon_2 = 0 \ , \tag{3.132}$$

while the diagram $Y_i = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$ with $k_i = 4$ defines $n_i = 5 > k_i$ hyperplanes

$$\phi_1 - \alpha_i + \epsilon_+ = 0 \ , \ \ \phi_{21} + \epsilon_1 = 0 \ , \ \ \phi_{31} + \epsilon_2 = 0 \ , \ \ \phi_{43} + \epsilon_1 = 0 \ , \ \ \phi_{42} + \epsilon_2 = 0 \ . \tag{3.133}$$

In all hyperplane assignments, one can easily see that $n_i \geq k_i$ equations determine unique ϕ_* and never over-determine it. Repeating the process for all N Young diagrams, one picks $n = \sum_{i=1}^N n_i \geq k$ independent hyperplanes. By taking a look, one can convince oneself that the chosen $\mathbf{Q}(\phi_*)$ is always projective. For instance, the 5 charges responsible for (3.133) are $e_1, e_{21}, e_{31}, e_{43}, e_{42}$ on $\mathbb{R}^4$. They are projective, since they are contained in the half-space $x_4 + \epsilon(x_2 + x_3) + \epsilon^2 x_1 > 0$ with small enough ϵ. The mapping of ϕ_I variables to the k boxes of Y can be done in a unique way, by eating up the Weyl symmetry factor $\frac{1}{k!}$.

To derive the above 'colored Young diagram rules,' I will make the inductive argument. Firstly, it has been shown that this is true at $k = 1$. At $k = 1$, there is no pole from Z_{adj} so that ignoring all possible poles from Z_{adj} is trivially true. Then, one only has to choose the pole value of single ϕ variable. By the JK residue rule, or equivalently the rank 1 residue rule of Section 3.4.1, this is given by choosing one of the N equations $\phi_I - \alpha_i + \epsilon_+ = 0$ for the pole. These choices correspond to N different colored Young diagrams with 1 box, confirming the rule at $k = 1$.

Now assume that the 'Young diagram rule' is true at rank $k-1$. To use induction, one picks the k independent hyperplane equations with $\eta \in \text{Cone}(Q_1, \cdots, Q_k)$ in $\mathbf{Q}(\phi_*)$. Here, recall the 'tree structure' of these k charge vectors. Apart from the case with k independent trees without any branches from e_I's, corresponding to the colored Young diagram in which each Y_i contains only a single box, there are always one or more charge vectors of the form $Q = e_I - e_J$ which are at the end of a tree (not having further branches attached to them). The hyperplane equation $Q(\phi) + \cdots = 0$ with such a Q is the only one which refers to ϕ_I coordinate among the k hyperplane equations. Using the Weyl symmetry, I take ϕ_k to be such a coordinate which appears only once in the k hyperplane equations. So take the set of k hyperplanes to be

$$(k-1 \text{ hyperplanes referring to } \phi_1, \cdots, \phi_{k-1} \text{ only}) \cap H_k \ . \tag{3.134}$$

H_k is the only hyperplane whose equation contains the ϕ_k coordinate, so that the other $k-1$ hyperplanes refer to $\phi_1, \cdots, \phi_{k-1}$ only. The charges appearing in these $k-1$ hyperplanes are those for the $U(k-1)$. Also, the integrand which contains $\phi_1, \cdots, \phi_{k-1}$ but not ϕ_k is the integrand for the $k-1$ instantons in the $U(N)$ theory. Finally, if $\{Q_1, Q_2, \cdots, Q_k\}$ is the set of charges which contains k dimensional $\eta^{(k)} = (1, \cdots, 1)$ in their cone, then $\{Q_1, \cdots, Q_{k-1}\}$ contains $\eta^{k-1} = (1, \cdots, 1, 0)$ in $\mathbb{R}^{k-1} \subset \mathbb{R}^k$. This is obvious from the fact that the charge $Q = \phi_k - \phi_J$ is at the end of the tree, so dropping it yields a tree with $k-1$ charges. So the poles on $\mathbb{R}^k$ with nonzero JK-Res are obtained by first studying the poles on $\mathbb{R}^{k-1}$ for $\phi_1, \cdots, \phi_{k-1}$ with nonzero JK-Res, and then determining the values of ϕ_k by

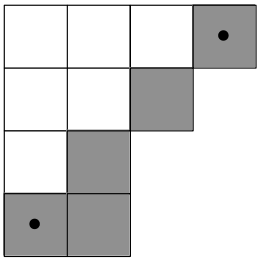

Figure 3.3: Shaded boxes form the border, dotted boxes are at the corners.

considering possible H_k's.

By the assumption of the induction, $k-1$ dimensional poles with nonzero JK-Res are classified by colored Young diagrams with $k-1$ boxes, denoted as $Y^{(k-1)}$. I will show that all possible extra hyperplane conditions H_k with nonzero k dimensional JK-Res map to the possibilities of adding one more boxes to $Y^{(k-1)}$ which makes all possible $Y^{(k)}$'s. Collect all possible hyperplane equations for H_k, which could be

$$\phi_k - \alpha_i + \epsilon_+ = 0 \tag{3.135}$$

only if the i'th Young diagram Y_i is empty in $Y^{(k-1)}$. (If Y_i is already occupied with $\phi_J - \alpha_i + \epsilon_+ = 0$, then $\sinh \frac{\phi_{IJ}}{2}$ in the numerator of (3.117) vanish.) This configuration by definition forms a colored Young diagram with k boxes, where a new nonempty diagram Y_i with one box is created. Other possible equations could be

$$\phi_k - \phi_I + \epsilon_1 = 0 \ , \quad \phi_k - \phi_I + \epsilon_2 = 0 \ , \quad \phi_k - \phi_I - \epsilon_+ \pm m = 0 \ . \tag{3.136}$$

Firstly, the hyperplanes $\phi_k - \phi_I - \epsilon_+ \pm m = 0$ yield zero residues. If ϕ_I maps to the box at the upper-left corner of a Young diagram, then $\phi_I = \alpha_j - \epsilon_+$ for some j. Then, the factor $2\sinh\left(\frac{\phi_k - \alpha_j \pm m}{2}\right)$ in the numerator of Z_{adj} vanishes so that the pole does not exist. If ϕ_I does not map to the box at the upper-left corner, then there should be a box with ϕ_J which is left-adjacent or upper-adjacent to the box with ϕ_I, namely $\phi_J = \phi_I - \epsilon_{1,2}$ for either ϵ_1 or ϵ_2. Here I note that there are factors

$$\prod_{\pm} 2\sinh\left(\frac{\phi_{kJ} \pm \epsilon_- - m}{2}\right) \cdot 2\sinh\left(\frac{\phi_{kJ} \pm \epsilon_- + m}{2}\right) \tag{3.137}$$

in the numerator of (3.119). Inserting either of $\phi_J = \phi_I - \epsilon_{1,2}$, one finds that the factor

$$\prod_{\pm} 2\sinh\left(\frac{\phi_{kI} - \epsilon_+ \pm m}{2}\right) \tag{3.138}$$

is always contained in the numerator, which vanishes due to the hyperplane condition $\phi_{kI} - \epsilon_+ \pm m = 0$ for one of the two signs. This shows that the corresponding poles do not exist. Next, I consider the first two types of hyperplanes in (3.136). The hyperplane of the first two sorts will correspond to adding a box to $Y^{(k-1)}$ when the box corresponding to ϕ_I is at the 'border' of $Y^{(k-1)}$. See Figure 3.3 for what I mean by the boxes at the border of a Young diagram.

The first two equations with I not at the border has zero residue, shown as follows. The box ϕ_I not at the border always has a right-adjacent and lower-adjacent boxes, which I call ϕ_{J_1}, ϕ_{J_2}, respectively. These variables are determined by the hyperplane equations $\phi_{J_1 I} + \epsilon_1 = 0$ and $\phi_{J_2 I} + \epsilon_2 = 0$. So if ϕ_I is not at the border of $Y^{(k-1)}$, the factor $\sinh\frac{\phi_{kJ_1}}{2}$ or $\sinh\frac{\phi_{kJ_2}}{2}$ in the numerator of (3.117) vanishes, yielding zero residue. The remaining hyperplane conditions in (3.136) that are not ruled out are $\phi_{kI} + \epsilon_{1,2} = 0$ with ϕ_I at the border. Now using the 'box' language, the box ϕ_k may either attach to two boxes ϕ_I, ϕ_J of $Y^{(k-1)}$ like $\begin{array}{|c|c|}\hline P & I \\ \hline J & k \\ \hline\end{array}$, attach to one box at the 'corners' of the Young diagram like $\begin{array}{|c|c|}\hline I & k \\ \hline\end{array}$ or $\begin{array}{|c|}\hline I \\ \hline k \\ \hline\end{array}$ (see Figure 3.3), or attach to one box ϕ_I in the middle of the border of $Y^{(k-1)}$ like $\begin{array}{|c|c}\hline P & \\ \hline I & \multicolumn{1}{c|}{k} \\ \hline\end{array}$ and $\begin{array}{c|c|}\hline \multicolumn{1}{|c|}{P} & I \\ \hline & k \\ \cline{2-2}\end{array}$. The first three are stacking the k'th box to form a colored Young diagram $Y^{(k)}$, while the last two are not. In the last two cases, the factor $2\sinh\left(\frac{\phi_{kP}+2\epsilon_+}{2}\right)$ in the numerator of Z_{vec} vanishes so that the corresponding poles do not exist. In the first case, the factor

$$\frac{\sinh\frac{\phi_{kP}+2\epsilon_+}{2}}{\sinh\frac{\phi_{kI}+\epsilon_1}{2}\sinh\frac{\phi_{kJ}+\epsilon_2}{2}} \tag{3.139}$$

partly cancels to keep a simple pole. The second and third cases also develop poles. So only the first three types of hyperplanes survive, exhausting all possible ways of putting the k'th box to $Y^{(k-1)}$ to make $Y^{(k)}$. This finishes the inductive proof of the map between poles with nonzero JK-Res and colored Young diagrams.

Having identified the poles, one can compute the JK-Res at these poles. For this, one expands the integrand in the Laurent series of $Q_i(\phi-\phi_*)$, and the computation boils down to knowing various JK-Res$(\mathbf{Q}(\phi_*),\eta)\frac{d\phi_1\wedge\cdots\wedge d\phi_k}{Q_{j_1}(\phi-\phi_*)\cdots Q_{j_k}(\phi-\phi_*)}$ given by (3.109). In particular, all JK-Res for the poles labeled by the colored Young diagrams can be regarded as iterated contour integrals. Firstly, JK-Res factorizes into N groups, each group mapping to a Young diagram Y_i. Within a given Young diagram Y_i, the iterated integral goes in the reverse order of stacking the boxes. For instance, for the Young diagram $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array}$, the integral over the relevant pole terms is

$$\begin{aligned}
&\text{JK-Res}\frac{\bigwedge_{i=1}^{6}d\phi_i}{\phi_1\phi_{21}\phi_{32}\phi_{41}}\cdot\frac{\phi_{51}}{\phi_{52}\phi_{54}}\cdot\frac{\phi_{62}}{\phi_{63}\phi_{65}}\\
&=\text{JK-Res}\frac{\bigwedge_{i=1}^{6}d\phi_i}{\phi_1\phi_{21}\phi_{32}\phi_{41}}\left(\frac{1}{2\phi_{52}}+\frac{1}{2\phi_{54}}\right)\left(\frac{1}{2\phi_{63}}+\frac{1}{2\phi_{65}}\right)\\
&=1=\frac{1}{(2\pi i)^6}\oint\frac{d\phi_1}{\phi_1}\oint\frac{d\phi_2}{\phi_{21}}\oint\frac{d\phi_3}{\phi_{32}}\oint\frac{d\phi_4}{\phi_{41}}\oint d\phi_5\frac{\phi_{51}}{\phi_{52}\phi_{54}}\oint d\phi_6\frac{\phi_{62}}{\phi_{63}\phi_{65}}\ ,
\end{aligned}\tag{3.140}$$

where $\oint$ for each ϕ_I is done around a small counterclockwise circle surrounding the pole. Such iterated integral formula holds for all poles labeled by Young diagrams. This yields the expression for the $U(N)$ instanton partition function [64, 65]:

$$Z_k=\sum_{\sum_i|Y_i|=k}\prod_{i,j=1}^{N}\prod_{s\in Y_i}\frac{\sinh\frac{E_{ij}+m-\epsilon_+}{2}\sinh\frac{E_{ij}-m-\epsilon_+}{2}}{\sinh\frac{E_{ij}}{2}\sinh\frac{E_{ij}-2\epsilon_+}{2}}\tag{3.141}$$

where

$$E_{ij}=\alpha_i-\alpha_j-\epsilon_1 h_i(s)+\epsilon_2(v_j(s)+1)\ .\tag{3.142}$$

Here, s runs over the boxes in the i'th Young diagram Y_i. $h_i(s)$ is the distance from the box s to the edge of the right side of Y_i that one reaches by moving to the right. $v_j(s)$ is the distance from s to the edge on the bottom side of Y_j that one reaches by moving down [64, 65, 55].

Now consider the alternative prescriptions for the $U(N)$ contour integral as stated at the end of Section 3.4.2. Namely, with the relaxation understood, $e^{-\epsilon_+}\to$

Integral over z_1	Integral over z_2	$(\lvert z_1\rvert, \lvert z_2\rvert)$	JK-res
$\phi_1 + \epsilon_+ - \alpha_i = 0$	$\phi_2 + \epsilon_+ - \alpha_j = 0$	(tw_i, tw_j)	Yes
	$\phi_1 - \phi_2 + \epsilon_+ \pm \epsilon_- = 0$	$(tw_i, u^{\mp 1} w_i)^a$	No
	$\phi_1 - \phi_2 - \epsilon_+ \pm m = 0$	$(tw_i, tTv^{\mp 1} w_i)^b$	No
	$\phi_2 - \phi_1 + \epsilon_+ \pm \epsilon_- = 0$	$(tw_i, t^2 u^{\pm 1} w_i)$	Yes
	$\phi_2 - \phi_1 - \epsilon_+ \pm m = 0$	$(tw_i, tT^{-1} v^{\pm 1} w_i)$	Yes
$\phi_1 - \phi_2 + \epsilon_+ \pm \epsilon_- = 0$	$\phi_1 + \epsilon_+ - \alpha_i = 0$	$(tu^{\pm 1}, u^{\mp 1} w_i)^c$	No
	$\phi_2 + \epsilon_+ - \alpha_i = 0$	$(tu^{\pm 1}, tw_i)$	Yes
$\phi_1 - \phi_2 - \epsilon_+ \pm m = 0$	$\phi_1 + \epsilon_+ - \alpha_i = 0$	$(T^{-1} v^{\pm 1}, tTv^{\mp 1} w_i)^d$	No
	$\phi_2 + \epsilon_+ - \alpha_i = 0$	$(T^{-1} v^{\pm 1}, tw_i)$	Yes

Table 3.1: Poles for $U(N)$ instantons at $k = 2$: $w_i \equiv e^{\alpha_i}$, $u \equiv e^{-\epsilon_-}$, $v \equiv e^{-m}$.

$t \ll 1$ and $e^{-\epsilon_+} \to T \gg 1$, one takes all the e^{ϕ_I} variables to live on the unit circles on the complex plane. Multiple unit circle integrals can be done in any order. In fact, this should be the original method used by [15, 65] to get the result (3.141). For the purpose of illustrating how the alternative contour prescription works, I will repeat it for $U(N)$ index at $k = 2$.

First integrate over $z_1 = e^{\phi_1}$. One sums over all residues for poles in $|z_1| < 1$ inside the unit circle, keeping z_2 fixed with $|z_2| = 1$. Then integrate over $z_2 = e^{\phi_2}$, again picking all residues for poles in $|z_2| < 1$. The rule excludes all the poles at the origin, $z_1 = 0$ or $z_2 = 0$. The possible poles in this procedure are shown on the first two columns of Table 3.1. At a given row, one first chooses an equation from the left column, which gives the poles for z_1 inside the unit contour. Then one moves on to the second column on the same row, which gives possible poles for z_2 inside its unit contour. The third column shows the values of $|z_1|, |z_2|$ which decides whether the pole is within the unit circle or not. For z_1, it does not necessarily agree with its actual value after the pole for z_2 is selected, since $|z_2| = 1$ while integrating over z_1. Table 3.1 contains only those selected by the z_1 unit contour rule. Some of them evidently stay inside the z_2 unit contour, while the four cases which are labeled by the superscripts $a, \cdots, d$ are rather ambiguous with the unit contour rule for z_2. One finds that all the poles which are unambiguously inside the unit contour $T^2 = S^1 \times S^1$ map to the poles which are chosen by the JK-Res rule. (Of course, I observed that some of the residues within this class can be zero, by using extra structures of the $U(N)$ index.)

As for the four ambiguous cases, whether they are inside or outside the unit contour for z_2 may depend on the scales of other fugacities which are unspecified. But one can notice that there always exists a pair of poles at an ambiguous location of z_2. a, c and b, d are such pairs. So the paired poles are either simultaneously inside or outside the unit contour of z_2. When they are outside the z_2 unit contour, they provide no contribution so that the result is consistent with the JK-Res rule. When they are both inside the z_2 unit contour, the two residues cancel. The sum of two paired residues is actually a result of doing the contour integral of the form:

$$\oint \frac{d\phi_2}{2\pi i} \oint \frac{d\phi_1}{2\pi i} \frac{f(\phi_1, \phi_2)}{(\phi_1 - a)(\phi_1 - \phi_2 - b)} , \tag{3.143}$$

with $f(\phi_1, \phi_2)$ being regular at $\phi_1 = a, \phi_1 - \phi_2 = b$. The integral is given by

$$\oint \frac{d\phi_2}{2\pi i} \left(\frac{f(a, \phi_2)}{a - b - \phi_2} + \frac{f(\phi_2 + b, \phi_2)}{\phi_2 + b - a} \right) = -f(a, a - b) + f(a, a - b) = 0 \ . \tag{3.144}$$

The two terms $-f(a, a-b)$ and $f(a, a-b)$ are precisely the pair of residues in all the above cases, guaranteeing cancelation when they are in the unit contour. This illustrates that the unit contour rule with $e^{-\epsilon_+} \to t \ll 1$ and $e^{-\epsilon_+} \to T \gg 1$ yields the same result as the JK-Res rule: although the unit contour rule may appear to keep more residues, after pairwise cancelations the two rules become equivalent. I confirmed that similar pairwise cancelations happen for the $Sp(N)$ indices at $k = 4$, as summarized in Section 3.5.3. In some other cases, such as the $Sp(1)$ index at $k = 5$ in Section 4.3, I just used the iterated integral rule along unit contour without checking its equivalence with the Jeffrey-Kirwan rule. Possibly, one could be able to prove the equivalence in full generality.

I emphasize here that the above type of pole classification goes through for $U(N)$ instanton partition functions with other matters. For fundamental hypermultiplets, there are no extra poles incurred by the hypermultiplets. Then the above arguments can be reused, simply ignoring all the discussions involving the hyperplanes from Z_{adj}. I also checked that the bi-fundamental hypermultiplets in the $U(n)$ quiver theories do not provide any poles with nonzero JK-Res at two instanton order, as derived in [66]. The absence of poles coming from the hypermultiplet factor Z_{adj} is an accidental property of the $U(N)$ theory. This simplification does not happened for the $\mathcal{N} = 1^*$ theory with other gauge groups. One should just use the Jeffrey-Kirwan residue rule, or alternatively use the unit contour integration rule after suitably replacing $e^{-\epsilon_+}$ by $t \ll 1$ and $T \gg 1$. Let me leave the studies on these indices to the future.

3.5.2 $U(N)$ theories with matters and Chern-Simons term

In this section, I will consider the instanton partition function of 5d $U(N)$ SYM, with N_f fundamental matters and nonzero Chern-Simons term at level κ. Consider only the theories which are related to the 5d SCFTs at the UV fixed points. The contour integral has the same structure as what I explained in the previous section, picking up poles and residues labeled by the colored Young diagrams. However, there occasionally arise subtleties in this class of theories. The residue sums at the two ends of cylinders will not be zero when $N_f + 2|\kappa| = 2N$. The nonzero residues at the infinity regions of φ_I imply a continuum in the ADHM quantum mechanics. The nonzero sum of two residues at the infinites of a cylinder implies a wall crossing as the FI parameter changes. These can be naturally understood with the string theory realizations of these 5d SYMs and the UV SCFTs. Before proceeding, let me emphasize that most of the studies in this section and Section 4.2 are already done in [67, 68, 69, 70, 71, 72]. Mostly, I will just reproduce their results, sometimes filling the missing values of N, N_f, κ not checked by them, to illustrate the (absence of unphysical) wall crossing issue.

In Figure 3.4, various (p, q) 5-brane webs are shown which engineers the $U(2)$

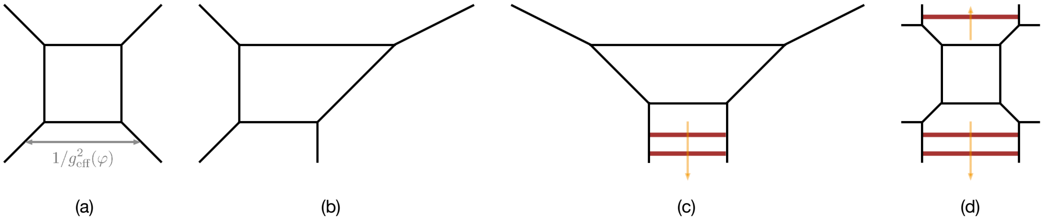

Figure 3.4: (a) 5-brane web for the pure $SU(2)$ theory; (b) $SU(2)$ at $\kappa = 1$; (c) $SU(2)$ at $\kappa = 2$; (d) $SU(2)$ with $N_f = 4$ at $\kappa = 0$. Horizontal lines are D5-braens on which 5d QFTs live. Red horizontal lines denote D1-branes which can escape to infinity by developing a continuum.

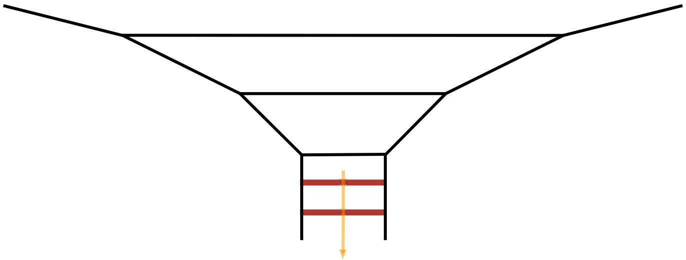

Figure 3.5: 5-brane web for the pure $SU(3)$ theory at $\kappa = 3$

gauge theory with fundamental hypermultiplets and/or bare Chern-Simons term

$$S_{\rm CS} = \frac{\kappa}{24\pi^2}\int {\rm tr}\left(A\wedge F\wedge F + \frac{i}{2}A^3\wedge F - \frac{1}{10}A^5\right)\ . \qquad (3.145)$$

κ is integral when the number N_f of fundamental hypermultiplets is even, and is half an odd integer when N_f is odd. The overall $U(1)$ of the $U(2)$ is non-dynamical in QFT. Note that there cannot be a bare Chern-Simons term for the $SU(2)$ theory. Thus the bare $U(2)$ Chern-Simons term means the mixed Chern-Simons term for the $U(1)$-$SU(2)$-$SU(2)$, inducing the background $U(1)$ electric charge to the $SU(2)$ instantons. The two horizontal lines are D5-branes on which the $U(2)$ theory lives. The overall $U(1)$ has infinite inertia, as the overall displacement of the two D5-branes induce translations of the asymptotic branes. When $N_f + 2|\kappa| = 2N$, one finds horizontal D1-branes stretched between the two parallel vertical lines (NS5-branes). These D1-branes, shown by the red lines in Fig. 3.4(c),(d), can escape up/down from the D5-branes on which the 5d QFT is defined. This implies that the ADHM quantum mechanics for the D1-D5 system (UV completing the instanton mechanics) has a continuum in the Coulomb branch. In the contour integrand for the instanton index, this continuum causes a nonzero pole at one or two ends of the cylinder. The case with $N = 3, \kappa = 3, N_f = 0$ is shown in Figure 3.5. So in these examples, the interpretation of the poles at infinities is the continuum developed by the stringy D1-brane states which appear in the ADHM mechanics, which are not in the QFT spectrum. These were studied well in [67, 68, 69, 70, 71, 72].

The k instanton partition function, which could possibly include the extra decoupled Z_{string} factor when $N_f + 2|\kappa| = 2N$, is given by the following contour integral (see, e.g. [63])

$$Z_k = \frac{(-1)^{\kappa+\frac{N_f}{2}}}{k!} \oint \left[\frac{d\phi_I}{2\pi i}\right] e^{\kappa \sum_{I=1}^{k} \phi_I} Z_{\text{vec}}(\phi, \alpha, \epsilon_{1,2}) Z_{\text{fund}}(\phi, m_a) \tag{3.146}$$

where Z_{vec} takes the same form as (3.117), and

$$Z_{\text{fund}} = \prod_{I=1}^{k} \prod_{a=1}^{N_f} 2\sinh\left(\frac{\phi_I + m_a}{2}\right) . \tag{3.147}$$

The overall sign $(-1)^{\kappa+N_f/2}$ was found in [69, 70] to be the physically sensible one, from various indirect evidences. [69, 70] conjectured that it will have to do with the effect of 5d Chern-Simons term, but its microscopic derivation seems to be unavailable yet. The pole selection derived from the JK-Res rule is exactly the same as what I derived for the $\mathcal{N} = 1^*$ theory in the previous subsection, labeled by the colored Young diagram. In the previous subsection, I chose $\eta = (1, \cdots, 1)$. Here, note in foresight that the index may depend on ζ. The choice of η in Section 3.5.1 is for $\zeta < 0$. For the theories considered here, the result is given by

$$Z_k = (-1)^{\kappa+N_f/2} \sum_{\sum_i |Y_i|=k} \prod_{i=1}^{N} \prod_{s\in Y_i} \frac{e^{\kappa\phi(s)} \prod_{l=1}^{N_f} 2\sinh\frac{\phi(s)+m_l}{2}}{\prod_{j=1}^{N} 2\sinh\frac{E_{ij}}{2} \cdot 2\sinh\frac{E_{ij}-2\epsilon_+}{2}} . \tag{3.148}$$

E_{ij} is defined by (3.142), and $\phi(s)$ is given by

$$\phi(s) = \alpha_i - \epsilon_+ - (m-1)\epsilon_1 - (n-1)\epsilon_2 \ , \tag{3.149}$$

where $s = (m,n) \in Y_i$ with m, n being the vertical and horizontal positions of the box s from the upper-left corner of Y_i [63]. When $\zeta > 0$, one would have to choose $\eta = -(1, \cdots, 1)$ and use the JK-Res rule. It is easy to get the result for $\zeta > 0$. Since $\zeta \to -\zeta$ can be undone by the $SU(2)_r$ Weyl reflection, or the Weyl reflection of the diagonal of $SU(2)_r \times SU(2)_R$, the sign flip of ζ is equivalent to that of ϵ_+. So by flipping all signs of ϵ_+ in the above result, one obtains the index for $\zeta > 0$. The two results will be the same unless $N_f + 2|\kappa| = 2N$.

At $N_f + 2|\kappa| = 2N$, Z^k_{QM} factorizes into $Z_{\text{inst}} Z_{\text{string}}$ with nontrivial Z_{string}, and furthermore Z_{string} exhibits a wall crossing as ζ flips sign. Nontrivial Z_{string} was analyzed and factored out from Z_{QM} in [67, 68, 69, 70, 71, 72]. I will explain and review these indices and their ζ dependence in Section 4.2.

3.5.3 $Sp(N)$ theories

In this subsection, I study the instanton partition function for the $Sp(N)$ gauge theories with N_f fundamental and $n_A = 0, 1$ antisymmetric hypermultiplets.

Let me first write down the contour integral expression. The integral variables are the zero modes of the ADHM quantum mechanics for the $Sp(N)$ instantons. Part of the zero modes is the holonomy of $\hat{G}$ on the temporal circle. For k instantons, they come with $\hat{G} = O(k)$ gauge group. Since $O(k)$ has two components $O(k)_+$ and $O(k)_-$, one should also turn on discrete holonomies for e^{iA_τ}. These can all be labeled by the complexified group element $U = e^{\phi} = e^{\varphi + iA_\tau}$, which can be taken as [63]

$$U_+ = e^{\phi_+} = \begin{cases} \text{diag}(e^{i\sigma_2\phi_1}, \cdots, e^{i\sigma_2\phi_n}) & \text{for even } k = 2n \\ \text{diag}(e^{i\sigma_2\phi_1}, \cdots, e^{i\sigma_2\phi_n}, 1) & \text{for odd } k = 2n+1 \end{cases} \tag{3.150}$$

for $O(k)_+$, and

$$U_- = e^{\phi_-} = \begin{cases} \text{diag}(e^{i\sigma_2\phi_1}, \cdots, e^{i\sigma_2\phi_{n-1}}, \sigma_3) & \text{for even } k = 2n \\ \text{diag}(e^{i\sigma_2\phi_1}, \cdots, e^{i\sigma_2\phi_n}, -1) & \text{for odd } k = 2n+1 \end{cases} \tag{3.151}$$

for $O(k)_-$. The above expressions with imaginary ϕ_I are the $O(k)_\pm$ group elements, while their complexifications come with $\varphi_I = \mathrm{Re}(\phi_I)$. Writing $k = 2n + \chi$, with $\chi = 0, 1$, one will get two intermediate indices $Z^k_\pm$ from the path integral. Each of them is obtained by taking the complexified holonomy in either $U_\pm$, performing Gaussian integration over non-zero modes, and then exactly summing or integrating over $U_\pm$ (with contours explained in section 2.3). The final index is given by [63]

$$Z^k = \frac{Z^k_+ + Z^k_-}{2} \ . \tag{3.152}$$

There is a variation of this result due to nontrivial $\pi_4(Sp(N)) = \mathbb{Z}_2$, which often defines new 5d SCFTs. With nontrivial $\mathbb{Z}_2$ element, one would have to take [69]

$$Z^k = (-1)^k \frac{Z^k_+ - Z^k_-}{2} \ . \tag{3.153}$$

This will be discussed more in section 3.4.4. $Z^k_\pm$ are given by

$$Z^k_\pm = \frac{1}{|W|} \oint [d\phi] Z^\pm_{\mathrm{vec}} Z^\pm_{\mathrm{fund}} Z^\pm_{\mathrm{anti}} \ . \tag{3.154}$$

The Weyl factors $|W|$ for $O(k)_\pm$ are

$$|W|^{\chi=0}_+ = \frac{1}{2^{n-1} n!}, \ |W|^{\chi=1}_+ = \frac{1}{2^n n!}, \ |W|^{\chi=0}_- = \frac{1}{2^{n-1}(n-1)!}, \ |W|^{\chi=1}_- = \frac{1}{2^n n!}. \tag{3.155}$$

The integrands are given as follows:

$$\begin{aligned} Z^+_{\mathrm{vec}} = & \left[\prod_{I<J}^n 2\sinh\tfrac{\pm\phi_I \pm \phi_J}{2} \left(\prod_I^n 2\sinh\tfrac{\pm\phi_I}{2} \right)^\chi \right] \prod_{I<J}^n \frac{2\sinh\frac{\pm\phi_I\pm\phi_J+2\epsilon_+}{2}}{2\sinh\frac{\pm\phi_I\pm\phi_J\pm\epsilon_-+\epsilon_+}{2}} \\ & \times \left(\frac{1}{2\sinh\frac{\pm\epsilon_-+\epsilon_+}{2} \prod_{i=1}^N 2\sinh\frac{\pm\alpha_i+\epsilon_+}{2}} \cdot \prod_{I=1}^n \frac{2\sinh\frac{\pm\phi_I+2\epsilon_+}{2}}{2\sinh\frac{\pm\phi_I\pm\epsilon_-+\epsilon_+}{2}} \right)^\chi \\ & \times \prod_{I=1}^n \frac{2\sinh\epsilon_+}{2\sinh\frac{\pm\epsilon_-+\epsilon_+}{2} 2\sinh\frac{\pm2\phi_I\pm\epsilon_-+\epsilon_+}{2} \prod_{i=1}^N 2\sinh\frac{\pm\phi_I\pm\alpha_i+\epsilon_+}{2}} \end{aligned} \tag{3.156}$$

from the ADHM data of 5d vector multiplet with $O(k)_+$,

$$Z^-_{\mathrm{vec}} = \left[\prod_{I<J}^n 2\sinh\tfrac{\pm\phi_I\pm\phi_J}{2} \prod_I^n 2\cosh\tfrac{\pm\phi_I}{2} \right] \prod_{I<J}^n \frac{2\sinh\frac{\pm\phi_I\pm\phi_J+2\epsilon_+}{2}}{2\sinh\frac{\pm\phi_I\pm\phi_J\pm\epsilon_-+\epsilon_+}{2}} \tag{3.157}$$

$$\times \frac{1}{2\sinh\frac{\pm\epsilon_-+\epsilon_+}{2}\prod_{i=1}^N 2\cosh\frac{\pm\alpha_i+\epsilon_+}{2}} \cdot \prod_{I=1}^n \frac{2\cosh\frac{\pm\phi_I+2\epsilon_+}{2}}{2\cosh\frac{\pm\phi_I\pm\epsilon_-+\epsilon_+}{2}}$$

$$\times \prod_{I=1}^n \frac{2\sinh\epsilon_+}{2\sinh\frac{\pm\epsilon_-+\epsilon_+}{2} 2\sinh\frac{\pm 2\phi_I\pm\epsilon_-+\epsilon_+}{2}\prod_{i=1}^N 2\sinh\frac{\pm\phi_I\pm\alpha_i+\epsilon_+}{2}}$$

with $O(k)_-$ when $k = 2n+1$;

$$Z^-_{\text{vec}} = \left[\prod_{I<J}^{n-1} 2\sinh\frac{\pm\phi_I\pm\phi_J}{2}\prod_I^{n-1} 2\sinh(\pm\phi_I)\right]\prod_{I<J}^{n-1}\frac{2\sinh\frac{\pm\phi_I\pm\phi_J+2\epsilon_+}{2}}{2\sinh\frac{\pm\phi_I\pm\phi_J\pm\epsilon_-+\epsilon_+}{2}} \tag{3.158}$$

$$\times \frac{2\cosh\epsilon_+}{2\sinh\frac{\pm\epsilon_-+\epsilon_+}{2} 2\sinh(\pm\epsilon_-+\epsilon_+)\prod_{i=1}^N 2\sinh(\pm\alpha_i+\epsilon_+)} \cdot \prod_{I=1}^{n-1}\frac{2\sinh(\pm\phi_I+2\epsilon_+)}{2\sinh(\pm\phi_I\pm\epsilon_-+\epsilon_+)}$$

$$\times \prod_{I=1}^{n-1} \frac{2\sinh\epsilon_+}{2\sinh\frac{\pm\epsilon_-+\epsilon_+}{2} 2\sinh\frac{\pm 2\phi_I\pm\epsilon_-+\epsilon_+}{2}\prod_{i=1}^N 2\sinh\frac{\pm\phi_I\pm\alpha_i+\epsilon_+}{2}}$$

with $O(k)_-$ when $k = 2n$;

$$Z^+_{\text{anti}} = \left(\frac{\prod_{i=1}^N 2\sinh\frac{m\pm\alpha_i}{2}}{2\sinh\frac{m\pm\epsilon_+}{2}}\prod_{I=1}^n\frac{2\sinh\frac{\pm\phi_I\pm m-\epsilon_-}{2}}{2\sinh\frac{\pm\phi_I\pm m-\epsilon_+}{2}}\right)^\chi \prod_{I=1}^n \frac{2\sinh\frac{\pm m-\epsilon_-}{2}\prod_{i=1}^N 2\sinh\frac{\pm\phi_I\pm\alpha_i-m}{2}}{2\sinh\frac{\pm m-\epsilon_+}{2}\sinh\frac{\pm 2\phi_I\pm m-\epsilon_+}{2}}$$

$$\times \prod_{I<J}^n \frac{2\sinh\frac{\pm\phi_I\pm\phi_J\pm m-\epsilon_-}{2}}{2\sinh\frac{\pm\phi_I\pm\phi_J\pm m-\epsilon_+}{2}} \tag{3.159}$$

from 5d antisymmetric hypermultiplet for $O(k)_+$;

$$Z^-_{\text{anti}} = \frac{\prod_{i=1}^N 2\cosh\frac{m\pm\alpha_i}{2}}{2\sinh\frac{m\pm\epsilon_+}{2}} \cdot \prod_{I=1}^n \frac{2\cosh\frac{\pm\phi_I\pm m-\epsilon_-}{2}}{2\cosh\frac{\pm\phi_I\pm m-\epsilon_+}{2}}\frac{2\sinh\frac{\pm m-\epsilon_-}{2}\prod_{i=1}^N 2\sinh\frac{\pm\phi_I\pm\alpha_i-m}{2}}{2\sinh\frac{\pm m-\epsilon_+}{2}\sinh\frac{\pm 2\phi_I\pm m-\epsilon_+}{2}}$$

$$\times \prod_{I<J}^n \frac{2\sinh\frac{\pm\phi_I\pm\phi_J\pm m-\epsilon_-}{2}}{2\sinh\frac{\pm\phi_I\pm\phi_J\pm m-\epsilon_+}{2}} \tag{3.160}$$

for $O(k)_-$ when $k = 2n+1$;

$$Z^-_{\text{anti}} = \frac{2\cosh\frac{\pm m-\epsilon_-}{2}\prod_{i=1}^N 2\sinh(m\pm\alpha_i)}{2\sinh\frac{m\pm\epsilon_+}{2} 2\sinh(m\pm\epsilon_+)}\prod_{I<J}^{n-1}\frac{2\sinh\frac{\pm\phi_I\pm\phi_J\pm m-\epsilon_-}{2}}{2\sinh\frac{\pm\phi_I\pm\phi_J\pm m-\epsilon_+}{2}} \tag{3.161}$$

$$\times \prod_{I=1}^{n-1}\frac{2\sinh(\pm\phi_I\pm m-\epsilon_-)}{2\sinh(\pm\phi_I\pm m-\epsilon_+)}\frac{2\sinh\frac{\pm m-\epsilon_-}{2}\prod_{i=1}^N 2\sinh\frac{\pm\phi_I\pm\alpha_i-m}{2}}{2\sinh\frac{\pm m-\epsilon_+}{2}\sinh\frac{\pm 2\phi_I\pm m-\epsilon_+}{2}}.$$

for $O(k)_-$ when $k = 2n$;

$$Z^+_{\text{fund}} = \prod_{l=1}^{N_f}\left(\left(2\sinh\frac{m_l}{2}\right)^\chi \prod_{I=1}^n 2\sinh\frac{\pm\phi_I+m_l}{2}\right) \tag{3.162}$$

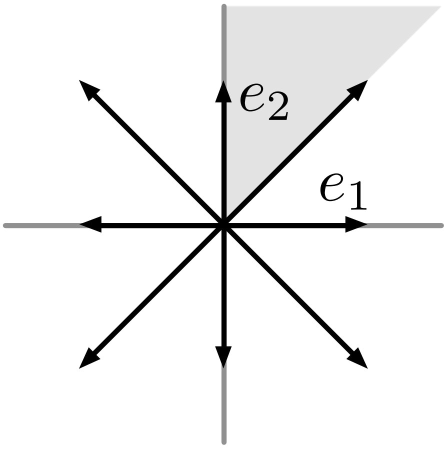

Figure 3.6: The charges for the $Sp(1)$ index at $k = 4$. The charges $\pm 2e_1$, $\pm 2e_2$ are not shown. I chose η in the shaded chamber.

from N_f fundamental hypermultiplets for $O(k)_+$;

$$Z^{-}_{\text{fund}} = \prod_{l=1}^{N_f} \left(2\cosh\frac{m_l}{2} \prod_{I=1}^{n} 2\sinh\frac{\pm\phi_I + m_l}{2} \right) \tag{3.163}$$

for $O(k)_-$ when $k = 2n+1$;

$$Z^{-}_{\text{fund}} = \prod_{l=1}^{N_f} \left(2\sinh m_l \prod_{I=1}^{n-1} 2\sinh\frac{\pm\phi_I + m_l}{2} \right) \tag{3.164}$$

for $O(k)_-$ when $k = 2n$. When one considers the index with $n_A = 0$, of course $Z^{\pm}_{\text{anti}}$ factors are dropped from the integrand.

In all of the above integrands, the arguments are written in the form of $\sinh\left(\frac{Q(\phi)+\cdots}{2}\right)$, where Q is the weight of the chiral or Fermi multiplet responsible for this factor. The contour integral is understood as the sum of Jeffrey-Kirwan residues with a chosen η. Here, any choice of η will provide the same result. I checked the behavior of poles carefully for the $Sp(1)$ theory with one antisymmetric hypermultiplet, up to $k = 4$ instanton order. The case with $k = 1$ has no integral. The case with $k = 2$ either has rank 1 for $O(2)_+$, where the formulae of section 2.2 applies, or has no integral for $O(2)_-$. The case with $k=3$ again has at most 1 integral. The case with $k=4$ has rank 2 for $O(4)_+$. In 2 dimensional h^*, I take η in the shaded chamber in Figure 3.6. One can show that all hyperplane arrangements are projective, fulfilling the condition for the results of [61] to be applicable. To see this, it suffices to check all the degenerate hyperplane arrangements. I find that they are either

$$2\phi_1+\epsilon_+\pm\epsilon_-=0\ ,\quad 2\phi_2+\epsilon_+\pm\epsilon_-=0\ ,\quad \phi_1+\phi_2+\epsilon_+\pm\epsilon_-=0 \tag{3.165}$$

or

$$2\phi_1-\epsilon_+\pm m=0\ ,\quad 2\phi_2-\epsilon_+\pm m=0\ ,\quad \phi_1+\phi_2-\epsilon_+\pm m=0 \tag{3.166}$$

with $\pm$ signs correlated, or others obtained by making Weyl reflections on all the charges Q_i appearing in the hyperplane equations. These are obviously projective.

In the analysis of Section 4.3, I used the iterated integrals over $z_I=e^{\phi_I}$ with $e^{-\epsilon_+}\to t\ll 1, T\gg 1$ replacements. I have checked the equivalence of the two rules for $Sp(1)$ theory till $k=4$, similar to what I explained for $U(N)$ $k=2$ in Section 3.5.3. With η chosen in the shaded chamber shown in Figure 3.6, I integrated over $z_1=e^{\phi_1}$ first and then over $z_2=e^{\phi_2}$. From the integrals over unit circles, one encounters $372=292+80$ possible poles. 292 poles are unambiguously inside the unit circle, and are those kept from the Jeffrey-Kirwan rule. The 80 extra poles are ambiguous but show pairwise cancelations, as explained around (3.144), proving the equivalence. In $Z_{\text{1-loop}}$, some poles are actually absent because sinh factors in the numerators vanish at the poles. (Similar phenomena were repeatedly observed for the $U(N)$ case, while deriving the Young diagram rules.) Taking these into account, there are 324 nonzero residues from the unit circle integrations, and 260 nonzero Jeffrey-Kirwan residues: 64 extra residues from the former cancel pairwise. Finally, identifying t and T at the final stage, 188 nonzero poles remain. Similar structures are found for $Sp(N)$ at $O(4)_+$, although there are more poles.

Chapter 4

Application of instanton calculus

In this chapter, I will explain in various examples how one can factor out Z_{string} from the index of ADHM quantum mechanics, and obtain $Z_{\text{inst}} = \frac{Z_{\text{QM}}}{Z_{\text{string}}}$ of my interest. The examples that I shall mainly discuss are $U(N)$ gauge theories with N_f fundamental hypermultiplets and 5d Chern-Simons level κ satisfying $N_f + 2|\kappa| \leq 2N$, and $Sp(N)$ gauge theories with $0 \leq N_f \leq 8$ fundamental and $n_A = 1$ antisymmetric hypermultiplets.

4.1 6d (2,0) SCFT

Before considering the main examples of this chapter, let me first discuss about the 6d maximally supersymmetric $\mathcal{N} = (2,0)$ SCFTs. They are engineered from the type IIB string theory wrapping on the singularity of type A, D, or E. In particular, the type A theories govern the low energy dynamics of M5-branes, thus being essential to understand the M-theory. The 6d (2,0) theory includes, instead of the usual gauge interaction, the peculiar interaction conveyed by the 2-form antisymmetric tensor whose field strength is self-dual in the six-dimensional sense. It therefore accommodates charged strings coupled to the self-dual 2-form tensor, whose dynamics is described by the strongly interacting 2d CFT. These strings are also known as M-strings which are induced objects from membranes suspended between a pair of M5-branes. I will review these M-strings studied by [9, 17] in Section 5.1.

If one reduces the $\mathcal{N} = (2,0)$ theory on a circle, obtained is the 5d maximal super Yang-Mills theory which describes the low energy physics of D4-branes. The 5d theory is the $(2,0)$ theory without all Kaluza-Klein momentum modes. However, instanton solitons of 5d SYM turn out to carry all KK momenta along the circle [73, 74]. Even though the 5d gauge theory is non-renormalizable, it is likely to have an UV fixed point which corresponds to 6d theory. No further degrees of freedom are required, but non-perturbative instantons and monopole strings in the 5d gauge theory play an important role to reach the UV fixed point [4, 5]. In particular, since instanton solitons have the mass proportional to

$$m \propto \frac{1}{g_{5d}^2} \propto \frac{1}{R}, \tag{4.1}$$

those particles corresponds to the Kaluza-Klein momenta of the compactified 6d circle. For the 6d $(2,0)$ theory in the tensor phase, the resulting 5d Yang-Mills theory is in the Coulomb phase. In fact, [55] studied the Kaluza-Klein spectrum of the circle compactified 6d $(2,0)$ SCFT via the 5d $\mathcal{N} = 1^*$ instanton partition function, which is the Coulomb branch observable computed in Section 3.5.1.

However, the Kaluza-Klein spectrum in [55] does not exhibit the superconformal symmetry because it has been explicitly broken in the tensor phase. For understanding the 6d (2,0) theory in the conformal phase, one first needs to perform the radial quantization, which puts the superconformal theory on $S^5 \times S^1$. Upon circle compactification, the effective description becomes the 5d gauge theory on S^5. The S^5 instanton partition function, which corresponds to the superconformal index of 6d $(2,0)$ SCFT, was studied in [10, 75, 11, 12]. One famous characteristic of the 6d $(2,0)$ superconformal theory (of the rank N) is the N^3 scaling of the Casimir energy, predicted from the dual AdS$_7$ gravity solution. The S^5 partition function indeed observes that the Casimir energy is proportional to N^3 for large N.

4.2 $U(N)$ theories for 5d SCFTs

The example that will be considered here is the $U(N)$ SYM with N_f fundamental hypermultiplets and bare Chern-Simons term at level κ, satisfying $N_f + 2|\kappa| \leq 2N$. The indices for the theories saturating the inequality have Z_{string} contributions. These partition functions are studied in great detail in [67, 68, 69, 70, 71, 72].

Let me first discuss the theories with $U(2)$ gauge group, with $N_f \leq 4$ fundamental matters and CS level κ satisfying $N_f + 2|\kappa| \leq 2N$. The 5-brane webs engineering some of these theories are shown in Figure 3.4. The $SU(2)$ part of the $U(2)$ gauge group is identified with the $Sp(1)$ gauge group, while the overall $U(1)$ is non-dynamical. The information on the overall $U(1)$, especially the CS level κ, should be irrelevant for Z_{inst}, since the QFT is just the $Sp(1)$ theory coupled to N_f fundamental hypermultiplets. So one expects

$$\frac{Z_{\text{QM}}^{U(2)}(N_f, \kappa)}{Z_{\text{QFT}}^{Sp(1)}(N_f)} = Z_{\text{string}}^{U(2)}(N_f, \kappa) \tag{4.2}$$

for all κ, where $Z_{\text{QFT}}^{Sp(1)}$ is the QFT index that one obtains by dividing $Z_{\text{QM}}^{U(2)}$ by $Z_{\text{string}}^{U(2)}$. (I suppressed the $\alpha_i, \epsilon_{1,2}, m, y_i, \zeta$ dependence.) At $N_f + 2|\kappa| < 2N$, there is no continuum from the string theory which are attached to the instanton quantum mechanics, and the right hand side is 1. At $N_f + 2|\kappa| = 2N$, the right hand side is not 1 and further experiences a wall crossing as the FI parameter ζ changes.

Before explaining the results, one should realize that the 5d $Sp(N)$ theories can be classified into two [76], labeled by two discrete theta angles. Namely, there are two topologically distinct configurations due to $\pi_4(Sp(N)) = \mathbb{Z}_2$. This also descends to the two topologically distinct configurations in the $O(k)$ ADHM quantum mechanics, due to $\pi_0(O(k)) = \mathbb{Z}_2$ [69]. In both 5d/1d cases, the sector with non-trivial element of $\mathbb{Z}_2$ has a relative -1 sign in the path integral. So the instanton calculus rule $Z^k_{\theta=0} = \frac{Z^k_+ + Z^k_-}{2}$ changes to [69]

$$Z^k_{\theta=\pi} = (-1)^k \frac{Z^k_+ - Z^k_-}{2} \ . \tag{4.3}$$

The overall factor of $(-1)^k$ was argued in [69] at $k = 1, 2$ in a somewhat indirect way. At $N_f = 0$, the two cases with $\theta = 0, \pi$ were shown (based on the instanton partition function calculus) in [69] to uplift to the so-called E_1 and $\tilde{E}_1$ theories, respectively [7]. With $N_f \geq 1$, the relative minus signs from $\mathbb{Z}_2$ nontrivial sector can be canceled by flipping the sign of a mass parameter. In the following, I will stick to our previous definition of m_i parameters, which implies that one should insert the relative minus sign for the Z_-^k when explicitly writing $\theta = \pi$. But this is related to new SCFT only when $N_f = 0$. In other cases, inserting extra minus sign is simply changing the convention for m_i. [69] finds that $Z_{\rm QM}^{U(2)}(N_f, \kappa)$ is related to $Z_{\rm QFT}^{Sp(1)}(N_f, \theta = 0)$ when $N - (\kappa + \frac{N_f}{2})$ is even, while it is related to $Z_{\rm QFT}^{Sp(1)}(N_f, \theta = \pi)$ when $N - (\kappa + \frac{N_f}{2})$ is odd. To make the comparison between the $U(2)$ and $Sp(1)$ observables, I will identify $\alpha_1 + \alpha_2 = 0$ in the $U(2)$ result below.

One first finds [69, 70]

$$\frac{Z_{\rm QM}^{U(2)}(N_f, \kappa)}{Z_{\rm inst}^{Sp(1)}(N_f, e^{i\theta} = \pm 1)} = 1 \tag{4.4}$$

when $N_f + 2|\kappa| < 2N$, with $e^{i\theta} = \pm 1$ if $N - (\kappa + \frac{N_f}{2})$ is even / odd, respectively. I checked this fact for $N = 2$, and all possible N_f, κ satisfying $N_f + 2|\kappa| < 2N$ up to q^3 order. This was already analyzed in [69, 70]. In proving this, it is crucial to insert the factor $(-1)^{k(\kappa + N_f/2)}$ in the k instanton index of the $U(2)$ theory, as explained in [69, 70]. Secondly, one finds

$$\frac{Z_{\rm QM}^{U(2)}(N_f = 2N - 2|\kappa|, \kappa, \zeta)}{Z_{\rm inst}^{Sp(1)}(N_f = 2N - 2|\kappa|, e^{i\theta} = \pm 1)} = Z_{\rm string}(\zeta) \tag{4.5}$$

when the 5d SCFT bound $N_f + 2|\kappa| \leq 2N$ is saturated. The theta angle is chosen between $e^{i\theta} = \pm 1$ depending on whether $N - (\kappa + \frac{N_f}{2})$ is even or odd. Namely, when $\kappa \geq 0$ and saturates 5d SCFT bound $\kappa = N - \frac{N_f}{2}$, one takes $e^{i\theta} = +1$. On the other hand, when $\kappa < 0$, one takes $e^{i\theta} = (-1)^{N_f}$. Note here that whenever $N_f \neq 0$, the choice of θ is purely a convention on the masses m_i. At $N_f = 0$ when θ acquires meaning, I find $e^{i\theta} = 1$ for both $\kappa = \pm 2$.[1]

[1]More generally, $U(N)$ theory with $N_f = 0$ have the same value of $e^{i\theta}$ at CS levels $\pm\kappa$.

The extra Z_{string} factor is naturally expected, since there always exist D1-branes which can be separated from the QFT system in this case, as explained in Section 3.5.2. This is given by

$$Z_{\text{string}}(\zeta) = \begin{cases} \text{PE}\left[-\frac{qt}{(1-tu)(1-t/u)}(ty_1\cdots y_{N_f})\right] & \text{when } \zeta < 0 \\ \text{PE}\left[-\frac{qt}{(1-tu)(1-t/u)}(t^{-1}y_1\cdots y_{N_f})\right] & \text{when } \zeta > 0 \end{cases} \tag{4.6}$$

for $\kappa > 0$,

$$Z_{\text{string}}(\zeta) = \begin{cases} \text{PE}\left[-\frac{qt}{(1-tu)(1-t/u)}(ty_1\cdots y_{N_f})^{-1}\right] & \text{when } \zeta < 0 \\ \text{PE}\left[-\frac{qt}{(1-tu)(1-t/u)}(t^{-1}y_1\cdots y_{N_f})^{-1}\right] & \text{when } \zeta > 0 \end{cases} \tag{4.7}$$

for $\kappa < 0$, and

$$Z_{\text{string}}(\zeta) = \begin{cases} \text{PE}\left[-\frac{qt}{(1-tu)(1-t/u)}\left(ty_1\cdots y_{N_f} + \frac{1}{ty_1\cdots y_{N_f}}\right)\right] & \text{when } \zeta < 0 \\ \text{PE}\left[-\frac{qt}{(1-tu)(1-t/u)}\left(\frac{y_1\cdots y_{N_f}}{t} + \frac{t}{y_1\cdots y_{N_f}}\right)\right] & \text{when } \zeta > 0 \end{cases} \tag{4.8}$$

for $\kappa = 0$ and $N_f = 2N$. Here y_i is defined to be $y_i \equiv e^{m_i/2}$. I have checked these results for $N = 2$ and $(N_f, \kappa) = (0, \pm 2)$, $(1, \pm\frac{3}{2})$, $(2, \pm 1)$, $(3, \pm\frac{1}{2})$, $(4, 0)$ up to q^3 order. These results are known from [68, 67, 69, 70, 71, 72]. In particular, [68] explains that it is consistent with the structure of the index for M2-branes wrapping 2-cycles in CY$_3$ which can escape from the QFT. Note that $Z_{\text{QM}}^{U(N)}$ at $N_f + 2|\kappa| = 2N$ lacks the $\epsilon_+ \to -\epsilon_+$ (or $t \to t^{-1}$) invariance, which is inconsistent either as a half-BPS index of 5d SYM (as explained in Section 3.4) or the index of 5d SCFT with $SU(2)_R$ symmetry [69]. This asymmetry all goes to Z_{string}, leaving $Z_{\text{inst}}^{U(2)} = Z_{\text{inst}}^{Sp(1)}$ invariant under the sign flip of ϵ_+. The bulk contribution is not invariant under $\epsilon_+ \to -\epsilon_+$. I do not recognize the half-BPS state interpretation of this part of the index: i.e. the index is well-defined only at $\zeta \neq 0$, and the spectrum has a continuum only at $\zeta = 0$ with unbroken $SU(2)_R$ symmetry. So there appears to be no contradiction. Also, the bulk spectrum is not constrained by the 5d superconformal symmetry, so $\epsilon_+ \to -\epsilon_+$ asymmetry is fine.

Note that the ratios $\frac{Z_{\text{QM}}^{U(2)}(\zeta<0)}{Z_{\text{QM}}^{U(2)}(\zeta>0)}$ are always given by

$$\text{PE}\left[-R_0-R_\infty\right]=\begin{cases}\text{PE}\left[\frac{\text{sign}(\kappa)qt(t^{-1}-t)}{(1-tu)(1-t/u)}\left(w_1w_2y_1\cdots y_{N_f}\right)^{\text{sign}(\kappa)}\right] & \text{when } \kappa\neq 0\\ \text{PE}\left[\frac{qt(t^{-1}-t)}{(1-tu)(1-t/u)}\left(w_1w_2y_1\cdots y_{N_f}-\frac{1}{w_1w_2y_1\cdots y_{N_f}}\right)\right] & \text{when } \kappa=0\end{cases}\tag{4.9}$$

where $w_i\equiv e^{\alpha_i}$, and the results are listed without taking $w_1w_2=1$. R_0, R_∞ are the residues of the holomorphic measure for the rank 1 integrand. This is consistent with what was found for the rank 1 case in Section 3.4.1. At $w_1w_2=1$, it just reduces to the ratio of two Z_{string} factors at $\zeta\lessgtr 0$ that I found by comparing $Z_{\text{QM}}^{U(2)}$ with $Z_{\text{QFT}}^{Sp(1)}$. For $U(N)$ with $N\geq 3$, one cannot directly disentangle $Z_{\text{QM}}^{U(N)}=Z_{\text{inst}}^{SU(N)}Z_{\text{string}}$. However, from the 5-brane web diagram, one could naturally expect that the structure of D1-branes escaping from the QFT is exactly the same as those for the $U(2)$ theory. For instance, see Figure 3.5 where horizontal D1-branes can escape from the QFT by moving downwards. [70] used this strategy to extract $Z_{\text{inst}}^{SU(3)}$, by dividing out the Z_{string} that one could get from the $U(2)$ theory at $\kappa=2$. This is also consistent with the ratio of $Z_{\text{QM}}^{U(3)}$ at $\zeta<0$ and $\zeta>0$, which is

$$\frac{Z_{\text{QM}}^{U(3)}(\zeta<0)}{Z_{\text{QM}}^{U(3)}(\zeta>ß0)}=\text{PE}\left[\frac{\text{sign}(\kappa)qt}{(1-tu)(1-t/u)}(t^{-1}-t)\left(w_1w_2w_3y_1\cdots y_{N_f}\right)^{\text{sign}(\kappa)}\right]\tag{4.10}$$

for $\kappa\neq 0$, or which is the product of two expressions (4.10) for positive / negative κ if $\kappa=0$. Setting $U(1)\subset U(3)$ fugacity to $w_1w_2w_3=1$, the right hand side equals $\frac{Z_{\text{QM}}^{U(2)}(\zeta<0)}{Z_{\text{QM}}^{U(2)}(\zeta>0)}$ at $w_1w_2=1$.

4.3 5d SCFT from D4-D8-O8 configuration

Here I will discuss the $Sp(N)$ theories with $N_f\leq 7$ fundamental and 1 antisymmetric hypermultiplets. The case with $N_f=8$ fundamental hypermultiplets is discussed in the next section separately. The ADHM quantum mechanics describes the k D0-branes along 0 direction, N D4-branes along 01234 directions, N_f D8-branes and one O8-plane along $0\cdots 8$ directions. The scalars φ_I from the ADHM vector multiplet represent D0-branes' positions along the 9 direction, transverse to all D-branes. The general analysis in Section 3.4 says that there is no pole at infinities of φ_I. One can expect this, since $N_f \leq 7$ D8-branes do not completely cancel the 8-brane charge of the O8-plane, so that the dilaton runs along the 9 direction. D0-brane's mass increases linearly in φ_I, explaining the absence of the continuum for φ_I. However, there is an extra contribution Z_{string} from D0-branes which are unbound to D4-branes, but are bound to D8-O8 only. Since the motion of D0's along the worldvolume of D8-O8 is fully gapped by the chemical potentials $\epsilon_1, \epsilon_2, m$, one could compute the multi-particle index for the D0-particles in $8+1$ dimensions. These D0-D8-O8 bound states' index will never refer to the electric charge fugacities α_i on D4. So to detect the possible Z_{string} factor, it suffices to examine the expansion of Z_{QM} in the Coulomb VEV $e^{-\alpha_i}$ with $\alpha_1 > \alpha_2 > \cdots > 0$, and study the sector which carries zero electric charges. The index can be written as

$$Z_{\text{QM}}(\alpha, \epsilon_{1,2}, v, q) = Z^{(0)}(\epsilon_{1,2}, v, q) Z^{(1)}(\alpha, \epsilon_{1,2}, v, q) \ . \tag{4.11}$$

$v = e^{-m}$ is the flavor fugacity rotating the antisymmetric hypermultiplet. $Z^{(1)}$ is given by

$$Z^{(1)} = 1 + \sum_{n_i} Z_{n_i} e^{-n_i \alpha_i} \ . \tag{4.12}$$

One can write

$$Z^{(0)}_{N_f} = PE\left[f_{N_f}(t, u, v, y_i, q)\right] \equiv \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} f_{N_f}(t^n, u^n, v^n, y_i^n, q^n)\right] \ , \tag{4.13}$$

where $t = e^{-\epsilon_+}$, $u = e^{-\epsilon_-}$, $v = e^{-m}$, $y_i = e^{m_i/2}$ with $i = 1, \cdots, N_f$. f_{N_f} is the single particle index. One finds

$$\begin{aligned}
f_0 &= -\frac{t^2}{(1-tu)(1-t/u)(1-tv)(1-t/v)}\, q && \text{for } N_f = 0 \\
f_{N_f} &= -\frac{t^2}{(1-tu)(1-t/u)(1-tv)(1-t/v)}\, q\chi(y_i)^{SO(2N_f)}_{\mathbf{2^{N_f-1}}} && \text{for } 1 \leq N_f \leq 5 \\
f_6 &= -\frac{t^2}{(1-tu)(1-t/u)(1-tv)(1-t/v)} \left[q\chi(y_i)^{SO(12)}_{\mathbf{32}} + q^2\right]
\end{aligned} \tag{4.14}$$

$$f_7 \;=\; -\frac{t^2}{(1-tu)(1-t/u)(1-tv)(1-t/v)}\left[q\chi(y_i)^{SO(14)}_{\mathbf{64}}+q^2\chi(y_i)^{SO(14)}_{\mathbf{14}}\right]\ .$$

$\mathbf{2}^{N_f-1}$ is the chiral spinor representation of $SO(2N_f)$, whose highest weight state contributes $y_1y_2\cdots y_{N_f}$ to the character. I have checked these forms of f_{N_f} up to q^4 order from the $Sp(1)$ index with all fugacities kept, and the same result up to q^3 from the $Sp(2)$ index. In Subsection 4.3.1, I will derive these indices from the D0-D8-O8 system, which proves that $Z^{(0)}_{N_f}=PE[f_{N_f}]$ is indeed $Z_{\rm string}$.

In the remaining part of this section, I will show that this $Z^{(0)}$ is precisely what one expects from the type I' string theory with N_f D8-branes, which should exhibit E_{N_f+1} gauge symmetry on the 8-branes' worldvolume (from its duality to heterotic strings [6]). To see this, one has to combine $Z^{(0)}$ with the contribution to the index from perturbative type I' string theory. N_f D8-branes and an O8-plane host massless degrees given by the 9d SYM theory with $SO(2N_f)$ gauge group. Nonperturbative enhancement $SO(2N_f)\to E_{N_f+1}$ is expected from string duality, where E_{N_f+1} includes the D0-brane charge in its Cartan [6]. So the nonperturbative index of the type I' theory should be that of the 9d E_{N_f+1} SYM theory.

Let me explain the perturbative index first. The index of the 9d $SO(2N_f)$ SYM is defined referring to the same 2 supercharges that I used to define the ADHM QM index. The 16 supercharges preserved by the D8-O8 system can be decomposed according to their representations of $SO(4)\times SO(4)=SU(2)_l\times SU(2)_r\times SU(2)_R\times SU(2)_F$ symmetry. The first $SO(4)=SU(2)_l\times SU(2)_r$ is the spatial rotation on the common worldvolume of D4-D8-O8. Second $SO(4)=SU(2)_R\times SU(2)_F$ is the rotation on D8-O8 worldvolume transverse to D4. $SU(2)_R$ was the R-symmetry of 5d $\mathcal{N}=1$ theory. $SU(2)_F$ with the chemical potential m rotates the $Sp(N)$ antisymmetric hypermultiplet. Denoting by $a=1,2$ the doublet index for $SU(2)_F$, the 16 supercharges can be written as

$$Q^a_\alpha\ ,\quad Q^A_\alpha\ ,\quad \bar{Q}^a_{\dot\alpha}\ ,\quad \bar{Q}^A_{\dot\alpha}\ , \tag{4.15}$$

where $\alpha,\dot\alpha,A$ indices are for $SU(2)_l,SU(2)_r,SU(2)_R$ doublets as before, and each $SU(2)\times SU(2)$ doublet satisfies symplectic-Majorana condition. 9d SYM has half-

BPS W-bosons and their superpartners in their BPS spectrum, in the Coulomb branch where one real scalar is given nonzero VEV. The $SO(2N_f)$ electric charges have fugacities $y_i \equiv e^{m_i/2}$, which were introduced in 5d SYM as flavor fugacities. Write the 32×32 gamma matrix in 10d as $(\Gamma^0, \Gamma^9) = \mathbf{1}_8 \otimes (\sigma_2, \sigma_1)$, $\Gamma^i = \gamma^i \otimes \sigma_3$, with γ^i given by the $SO(8)$ gamma matrices ($i = 1, \cdots, 8$). The BPS condition for the half-BPS W-boson is one of $\Gamma^{09}\epsilon = \pm i\epsilon$ in the 10d chiral Majorana spinor notation, where 9 stands for the scalar direction. The SUSY parameter ϵ satisfies the 10d chirality condition $\gamma^{1\cdots 8} \otimes \sigma_3 \epsilon = \epsilon$. W-bosons' BPS condition says that ϵ is either chiral or anti-chiral $SO(8)$ spinors. In my notation, the preserved supercharges are either $Q^a_\alpha, \bar{Q}^A_{\dot\alpha}$ or $Q^A_\alpha, \bar{Q}^a_{\dot\alpha}$. Since the indices are always defined using $\bar{Q}^{A=1}_{\dot\alpha=\dot 1}$ and $\bar{Q}^{A=2}_{\dot\alpha=\dot 2}$, the sector which is captured by the index contains W-bosons preserving the former. The broken supercharges $Q^A_\alpha, \bar{Q}^a_{\dot\alpha}$ provide Goldstone fermion zero modes, which contribute to the single particle index of 9d W-bosons. The 4 pairs of fermionic oscillator from these Goldstinos provide a factor

$$2\sinh\frac{\epsilon_1}{2}\cdot 2\sinh\frac{\epsilon_2}{2}\cdot 2\sinh\frac{m+\epsilon_+}{2}\cdot 2\sinh\frac{m-\epsilon_+}{2} = \chi^{SO(8)}(\mathbf{8}_v) - \chi^{SO(8)}(\mathbf{8}_c)\ , \quad (4.16)$$

where

$$\chi^{SO(8)}(\mathbf{8}_v) = \chi^{SO(8)}(\mathbf{8}_s) \equiv (t+t^{-1})(u+u^{-1}+v+v^{-1}) \quad (4.17)$$

$$\chi^{SO(8)}(\mathbf{8}_c) \equiv t^2+2+t^{-2}+(u+u^{-1})(v+v^{-1}) \quad (4.18)$$

are the $SO(8)$ characters of the vector, spinor, conjugate spinor representations. $\mathbf{8}_v$ and $\mathbf{8}_c$ are for the W-bosons A_μ and superpartner fermions Ψ in $(8+1)$d SYM. The index also acquires contribution from 8 bosonic zero modes for the translation on $\mathbb{R}^8$. They contribute the factor

$$\frac{1}{\left(2\sinh\frac{\epsilon_1}{2}\cdot 2\sinh\frac{\epsilon_2}{2}\cdot 2\sinh\frac{m+\epsilon_+}{2}\cdot 2\sinh\frac{m-\epsilon_+}{2}\right)^2} \quad (4.19)$$

to the index. One should also consider $\chi^{SO(2N_f)}_{\mathbf{adj}}(y_i)^+$ factor for the the W-bosons, where the $+$ superscript denotes that only the positive roots contribute to this character. This is because one is counting only W-bosons and their superpartners in the Coulomb branch of the 9d theory, without anti-W-bosons or the massless Cartans. So one obtains

$$
\begin{aligned}
f_{\text{9d SYM}} &= \frac{\chi^{SO(2N_f)}_{\mathbf{adj}}(y_i)^+}{2\sinh\frac{\epsilon_1}{2}\cdot 2\sinh\frac{\epsilon_2}{2}\cdot 2\sinh\frac{m+\epsilon_+}{2}\cdot 2\sinh\frac{m-\epsilon_+}{2}} \\
&= -\frac{t^2\chi^{SO(2N_f)}_{\mathbf{adj}}(y_i)^+}{(1-tu)(1-t/u)(1-tv)(1-t/v)} .
\end{aligned}
\tag{4.20}
$$

The four factors in the denominator can be understood as the four complex zero modes on $\mathbb{C}^4 = \mathbb{R}^8$, indicating that this is coming from 8 dimensional particles.

Combining (4.20) and (4.14) together, one obtains the single particle index for the W-bosons of 9d E_{N_f+1} SYM. One first finds that at $N_f = 0$, $E_1 = SU(2)$ adjoint decomposes into 3 states which have $U(1)_I$ instanton charges $0, +1, -1$, respectively. The latter two are the non-perturbative enhanced symmetry generators. Adjoint representation of $E_2 = SU(2)\times U(1)$, which is $\mathbf{3}+\mathbf{1}$ in $SU(2)$, decomposes in $SO(2)\times U(1)_I$ to two neutral generators, and two non-perturbative generators carrying $q^{\pm 1}y_1^{\pm 1}$. $E_3 = SU(3)\times SU(2)$ contains the perturbative $SO(4)\times U(1)_I = SU(2)\times SU(2)\times U(1)_I$ in the following way. The second $SU(2)$ of $SO(4)$ is the same as the $SU(2)$ factor of E_3, while $SU(3)$ adjoint branches to the remaining $SU(2)\times U(1)_I$ irreps as

$$
\mathbf{8} \to \mathbf{1}_0 + \mathbf{3}_0 + \mathbf{2}_1 + \mathbf{2}_{-1} \ . \tag{4.21}
$$

The branching rules of the E_{N_f+1} adjoints, with $N_f \geq 4$, to $SO(2N_f)\times U(1)_I$ irreps are

$$
\begin{aligned}
E_4 = SU(5) \ &: \ \mathbf{24} \to \mathbf{1}_0 + \mathbf{15}_0 + \mathbf{4}_1 + \overline{\mathbf{4}}_{-1} \\
E_5 = SO(10) \ &: \ \mathbf{45} \to \mathbf{1}_0 + \mathbf{28}_0 + (\mathbf{8_s})_1 + (\mathbf{8_s})_{-1} \\
E_6 \ &: \ \mathbf{78} \to \mathbf{1}_0 + \mathbf{45}_0 + \mathbf{16}_1 + \overline{\mathbf{16}}_{-1} \\
E_7 \ &: \ \mathbf{133} \to \mathbf{1}_0 + \mathbf{66}_0 + \mathbf{32}_1 + \mathbf{32}_{-1} + \mathbf{1}_2 + \mathbf{1}_{-2} \\
E_8 \ &: \ \mathbf{248} \to \mathbf{1}_0 + \mathbf{91}_0 + \mathbf{64}_1 + \overline{\mathbf{64}}_{-1} + \mathbf{14}_2 + \mathbf{14}_{-2} \ .
\end{aligned}
\tag{4.22}
$$

The subscripts all denote the $U(1)_I$ instanton number. The first $\mathbf{1}_0$'s all denote the generator of the $U(1)_I$, while the next $U(1)_I$ singlets are all adjoints of $SO(2N_f)$.

As explained around (4.20), only the positive roots from the $SO(2N_f)$ adjoints contribute to the index. Among the remaining non-singlets on the right hand side, only the states which have positive $U(1)_I$ charge will contribute to the index, as the index counts instantons but not anti-instantons. The instanton contribution to the 9d E_{N_f+1} SYM index required from (4.22) and the preceding branching rules indeed appear in (4.14) for all N_f. So the addition of (4.20) and (4.14) precisely captures the contribution from the W-bosons of 9d E_{N_f+1} SYM.

So one concludes that $Z^{(0)}_{N_f} = PE[f_{N_f}]$ with f_{N_f} given by (4.14) is precisely Z_{string}, the extra string theory or UV contribution to Z_{QM}. This will be reconfirmed in Subsection 4.3.1 by a direct computation of the D0-D8-O8 index, without assuming type I'-heterotic duality. The index for the 5d SCFT is given by $Z_{\text{inst}} = \frac{Z_{\text{QM}}}{Z_{\text{string}}}$, which shall be used in Section 4.3.2.

Before closing this section, let me comment on the $Sp(N)$ partition function with N_f fundamental hypermultiplets at $n_A = 0$. This engineers another class of 5d SCFTs, which can be realized by M-theory on suitable CY_3 [8]. For $Sp(1)$, this should yield the same 5d SCFT indices as those obtained from the quantum mechanics with $n_A = 1$. The only issue is that the two descriptions may have different Z_{string} factors. At all N, including $N = 1$, the condition for the contour integrand $Z_{\text{1-loop}}$ to vanish at $|\varphi| \to \infty$ is $N_f < 2N + 4$. $Z_{\text{1-loop}}$ approaches a constant asymptotically for $N_f = 2N+4$. So one studies the $Sp(1)$ ADHM instanton calculus at $N_f \leq 6$. In the Calabi-Yau engineering, instantons are realized as M2-branes wrapping certain 2-cycles. When $Z_{\text{1-loop}}$ does not vanish at the infinity of φ, the BPS 2-cycle for these M2-brane worldvolume has a noncompact modulus so that this M2-brane can continuously move to infinity. When $N_f \leq 5$, there is no noncompact moduli so that $Z_{\text{string}} = 1$. This is supported by the analysis of [63]. When $N_f = 6$, one finds $Z_{\text{QM}} = Z_{\text{inst}} Z_{\text{string}}$, where Z_{inst} is the SCFT partition function that I derived with $n_A = 1$, and

$$Z_{\text{string}} = \text{PE}\left[-\frac{(1+t^2)q^2}{2(1-tu)(1-t/u)}\right] . \tag{4.23}$$

This fact was confirmed up to q^4 order. Since (4.23) comes with a fractional coefficient, it clearly has to do with the continuum. It may be possible to explain the t, u, q dependence by understanding the CY$_3$ geometry of [8], which I do not attempt in this thesis. In all cases with $N_f \leq 6$, I confirmed that $Z_{\rm inst}$ computed from the ADHM mechanics with $n_A = 0$ and $n_A = 1$ are the same, up to q^4 order.

4.3.1 Direct computations of the D0-D8-O8 indices

The computations reported in this short subsection supplement the discussions of Section 4.3. There I extracted out the neutral part $Z^{(0)}$ of the D0-D4-D8-O8 index and argued that this contains $Z_{\rm string}$ which is deducible from string dualities, etc. Instead, one can simply derive the $Z_{\rm string}$ factors of the previous subsections directly from the D0-D8-O8 quantum mechanics. One can start from the gauged quantum mechanics for the open strings connecting D0-D8-O8 with $O(k)$ gauge group. The field contents can be easily obtained from the previous D0-D4-D8-O8 fields by dropping all $N \times k$ bi-fundamental fields. The index is also obvious: one just uses the index in Section 3.5.3 after dropping all the determinant factors for the fields charged in $Sp(N)$. (In fact, the expansion in $e^{-\alpha_i}$ that was discussed in Section 4.3 is almost the same as doing this.) So computing these indices, in all examples up to q^4 order, I obtained

$$\begin{aligned}
Z_{N_f=0} &= \mathrm{PE}\left[-\frac{t^2 q}{(1-tu)(1-t/u)(1-tv)(1-t/v)}\right] \\
Z_{1\leq N_f\leq 5} &= \mathrm{PE}\left[-\frac{t^2}{(1-tu)(1-t/u)(1-tv)(1-t/v)} q\chi(y_i)^{SO(2N_f)}_{\mathbf{2^{N_f-1}}}\right] \\
Z_{N_f=6} &= \mathrm{PE}\left[-\frac{t^2}{(1-tu)(1-t/u)(1-tv)(1-t/v)}\left(q\chi(y_i)^{SO(12)}_{\mathbf{32}} + q^2\right)\right] \\
Z_{N_f=7} &= \mathrm{PE}\left[-\frac{t^2}{(1-tu)(1-t/u)(1-tv)(1-t/v)}\left(q\chi(y_i)^{SO(14)}_{\mathbf{64}} + q^2\chi(y_i)^{SO(14)}_{\mathbf{14}}\right)\right]
\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}
Z_{N_f=8} =\mathrm{PE}\Bigg[&-\frac{(t+t^3)(u+u^{-1}+v+v^{-1})}{2(1-tu)(1-t/u)(1-tv)(1-t/v)}\frac{q^2}{1-q^2} \\
&-\frac{t^2}{(1-tu)(1-t/u)(1-tv)(1-t/v)}\left(\chi(y_i)^{SO(16)}_{\mathbf{120}}\frac{q^2}{1-q^2} + \chi(y_i)^{SO(16)}_{\mathbf{128}}\frac{q}{1-q^2}\right)\Bigg] .
\end{aligned} \tag{4.25}$$

These all directly justify the Z_{string} factors that were argued using string dualities. In particular, (4.24) supports the non-perturbative duality between the type I' and heterotic strings by finding a spectrum which allows E_{N_f+1} enhancement. (4.25) supports that non-perturbative physics of type I' strings reconstructs the physics of M9-plane compactified on a circle.

4.3.2 Superconformal indices

Here I will use the QFT instanton partition function $Z_{\text{inst}} = \frac{Z_{\text{QM}}}{Z_{\text{string}}}$ for the $Sp(N)$ theory with 1 antisymmetric and $N_f \leq 7$ fundamental hypermultiplets to study the 5d SCFT of [6]. The relevant Z_{string} factors have been all identified so far. In particular, I would like to study the superconformal index [77, 78] for the 5d SCFTs. This index is a supersymmetric partition function on $S^4 \times S^1$. When the 5d SCFT admits a relevant deformation to a 5d SYM, [63] studied this quantity in detail. One can define it by

$$I(t, u, m_i, q) = \text{Tr}\left[(-1)^F e^{-\beta\{Q,S\}} t^{2(J_r+J_R)} u^{2J_l} e^{-F\cdot m} q^k\right] . \tag{4.26}$$

J_r, J_l are rotations of $SO(4) \subset SO(5)$ on S^4, being the Cartans of $SU(2)_r \times SU(2)_l \subset SO(4)$. The J_r, J_l have two fixed points at the north and south poles of S^4. J_R is the Cartan of the $SU(2)_R$ symmetry of the $F(4)$ superconformal symmetry. F are the global symmetries of the SCFT which are visible in the 5d SYM as Noether charges. k is the instanton number in 5d SYM. This index counts BPS local operators on $\mathbb{R}^5$, or BPS states on $S^4 \times \mathbb{R}$, which saturate the following bound

$$\{Q, S\} = E - 2J_r - 3J_R \geq 0 \tag{4.27}$$

for the scale dimension (or energy) E.

In 5d SYM, [63] showed that this index can be expressed as a unitary matrix integral with group G, the gauge group of 5d SYM. The measure of the integrand is given by a product of two instanton partition functions of the 5d gauge theory, or more abstractly the partition function of 5d SCFT on Omega-deformed $\mathbb{R}^4 \times S^1$.

Especially in the latter abstract viewpoint, one should be using Z_{inst} rather than Z_{QM}. The precise form is given by

$$I(t,u,m_i,q) = \int [da] Z_{\text{pert}}(ia,t,u,m_i) Z_{\text{inst}}(ia,t,u,m_i,q) Z_{\text{inst}}(-ia,t,u,-m_i,q^{-1}) \ . \tag{4.28}$$

$[da]$ is the integral over holonomies of G, including its Haar measure. Z_{pert} is

$$Z_{\text{pert}} = \text{PE}\left[f_{\text{vec}}(t,u,e^{ia}) + f_{\text{fund}}(t,u,e^{ia},e^{m_l}) + f_{\text{anti}}(t,u,e^{ia},e^{m})\right] \ , \tag{4.29}$$

where

$$\begin{aligned}
f_{\text{vec}} &= -\frac{t(u+u^{-1})}{(1-tu)(1-t/u)}\left[\sum_{i<j}^{N} e^{\pm ia_i \pm ia_j} + \sum_{i=1}^{N} e^{\pm 2ia_i} + N\right] \\
f_{\text{fund}} &= \frac{t}{(1-tu)(1-t/u)} \sum_{i=1}^{N}\sum_{l=1}^{N_f} e^{\pm ia_i \pm m_l} \\
f_{\text{anti}} &= \frac{t(e^{m}+e^{-m})}{(1-tu)(1-t/u)}\left[\sum_{i<j}^{N} e^{\pm ia_i \pm ia_j} + N\right] \ .
\end{aligned} \tag{4.30}$$

Here I used the notation $e^{\pm x} = e^{+x} + e^{-x}$, and so on. Of course for $Sp(1)$, one should not include f_{anti} in Z_{pert}. Each Z_{inst} is the instanton contribution, which is given by Z_{inst} given in Section 4.3.

$Sp(1)$ indices

Since it was checked that Z_{inst} from the ADHM quantum mechanics with $n_A = 1$ (our work) and with $n_A = 0$ computed in [63] are same for $N_f \leq 5$, one does not have to compute the superconformal indices again. For $N_f = 0$, one obtains

$$\begin{aligned}
I = &\ 1 + \chi_{\mathbf{3}}^{E_1} t^2 + \chi_2(u)\big[1+\chi_{\mathbf{3}}^{E_1}\big] t^3 + \Big(\chi_3(u)\big[1+\chi_{\mathbf{3}}^{E_1}\big] + 1 + \chi_{\mathbf{5}}^{E_1}\Big) t^4 \\
&+ \Big(\chi_4(u)\big[1+\chi_{\mathbf{3}}^{E_1}\big] + \chi_2(u)\big[1+\chi_{\mathbf{3}}^{E_1}+\chi_{\mathbf{5}}^{E_1}\big]\Big) t^5 \\
&+ \Big(\chi_5(u)\big[1+\chi_{\mathbf{3}}^{E_1}\big] + \chi_3(u)\big[1+\chi_{\mathbf{3}}^{E_1}+\chi_{\mathbf{5}}^{E_1}+\chi_{\mathbf{3}}^{E_1}\chi_{\mathbf{3}}^{E_1}\big] + \chi_{\mathbf{3}}^{E_1} + \chi_{\mathbf{7}}^{E_1} - 1\Big) t^6 \\
&+ \Big(\chi_6(u)\big[1+\chi_{\mathbf{3}}^{E_1}\big] + \chi_4(u)\big[2+4\chi_{\mathbf{3}}^{E_1}+2\chi_{\mathbf{5}}^{E_1}\big] + \chi_2(u)\big[1+3\chi_{\mathbf{3}}^{E_1}+2\chi_{\mathbf{5}}^{E_1}+\chi_{\mathbf{7}}^{E_1}\big]\Big) t^7
\end{aligned}$$

$$+\Big(\chi_7(u)\big[1+\chi_{\mathbf{3}}^{E_1}\big]+\chi_5(u)\big[3\chi_{\mathbf{5}}^{E_1}+5\chi_{\mathbf{3}}^{E_1}+4\big]+\chi_3(u)\big[2\chi_{\mathbf{7}}^{E_1}+3\chi_{\mathbf{5}}^{E_1}+7\chi_{\mathbf{3}}^{E_1}+2\big]$$
$$+\chi_{\mathbf{9}}^{E_1}+2\chi_{\mathbf{5}}^{E_1}+2\chi_{\mathbf{3}}^{E_1}+3\Big)t^8+\mathcal{O}(t^9),$$

where $\chi_n(u)$ is the character of n-dimensional representation of $SU(2)$. The enhanced symmetry $E_1 = SU(2)$ appears rather trivially, as the superconformal index is manifestly invariant under the $q \to q^{-1}$ Weyl symmetry. For $N_f = 1$, one obtains

$$I = 1+\chi_{\mathbf{4}}^{E_2}t^2+\chi_2(u)\big[1+\chi_{\mathbf{4}}^{E_2}\big]t^3+\Big(\chi_3(u)\big[1+\chi_{\mathbf{4}}^{E_2}\big]+1+\chi_{\mathbf{5}}^{SU(2)}-\chi_{\mathbf{4}}(f)\Big)t^4$$
$$+\Big(\chi_4(u)\big[1+\chi_{\mathbf{4}}^{E_2}\big]+\chi_2(u)\big[\chi_{\mathbf{4}}^{E_2}+\chi_{\mathbf{3}}^{SU(2)}+\chi_{\mathbf{5}}^{SU(2)}-\chi_{\mathbf{4}}(f)\big]\Big)t^5$$
$$+\Big(\chi_5(u)\big[1+\chi_{\mathbf{4}}^{E_2}\big]+\chi_3(u)\big[4\chi_{\mathbf{4}}^{E_2}+2\chi_{\mathbf{5}}^{SU(2)}-\chi_{\mathbf{4}}(f)\big]+\chi_{\mathbf{7}}^{SU(2)}+3\chi_{\mathbf{3}}^{SU(2)}+1\Big)t^6$$
$$+\Big(\chi_6(u)\big[1+\chi_{\mathbf{4}}^{E_2}\big]+\chi_4(u)\big[5\chi_{\mathbf{4}}^{E_2}+2\chi_{\mathbf{3}}^{SU(2)}+2\chi_{\mathbf{5}}^{SU(2)}-\chi_{\mathbf{4}}(f)\big]$$
$$+\chi_2(u)\big[6\chi_{\mathbf{4}}^{E_2}+2\chi_{\mathbf{5}}^{SU(2)}+\chi_{\mathbf{7}}^{SU(2)}-\chi_{\mathbf{3}}^{SU(2)}\chi_{\mathbf{4}}(f)\big]\Big)t^7$$
$$+\Big(\chi_7(u)\big[1+\chi_{\mathbf{4}}^{E_2}\big]+\chi_5(u)\big[9\chi_{\mathbf{4}}^{E_2}+3\chi_{\mathbf{5}}^{SU(2)}-\chi_{\mathbf{4}}(f)\big]+\chi_3(u)\big[9\chi_{\mathbf{4}}^{E_2}+2\chi_{\mathbf{7}}^{SU(2)}+4\chi_{\mathbf{5}}^{SU(2)}$$
$$+2\chi_{\mathbf{3}}^{SU(2)}-(\chi_{\mathbf{4}}^{E_2}+\chi_{\mathbf{3}}^{SU(2)})\chi_{\mathbf{4}}(f)\big]+3\chi_{\mathbf{4}}^{E_2}+\chi_{\mathbf{9}}^{SU(2)}+2\chi_{\mathbf{5}}^{SU(2)}+2-\chi_{\mathbf{4}}^{E_2}\chi_{\mathbf{4}}(f)\Big)t^8+\mathcal{O}(t^9),$$

with $E_2 = SU(2)\times U(1)$. $\chi_{\mathbf{4}}^{E_2}$ is the adjoint character $1+\chi_{\mathbf{3}}^{SU(2)}$ of E_2, while other $SU(2)$ characters with boldfaced subscripts are for its $SU(2)$ subgroup. $\chi_{\mathbf{4}}(f)$ is given by [63]

$$\chi_{\mathbf{4}}(f) = \left(e^{i\frac{\rho}{2}}+e^{-i\frac{\rho}{2}}\right)\chi_{\mathbf{2}}\ , \tag{4.31}$$

where $\chi_{\mathbf{2}}$ is the $SU(2)$ character and ρ is the $U(1)$ chemical potential in $E_2 = SU(2)\times U(1)$. The embedding of $SO(2)\times U(1)_I$ into E_2 is given by

$$E_2 = SU(2)_{\frac{1}{2}(m_1+w)}\times U(1)_{\frac{1}{2}(7m_1-w)} \supset SO(2)_{m_1}\times U(1)_{I_w}. \tag{4.32}$$

Therefore, $\chi_{\mathbf{2}}$ and $e^{i\frac{\rho}{2}}$ are written in terms of $SO(2)\times U(1)_I$ fugacities $y_1 = e^{m_1/2}, q = e^{w/2}$ by

$$\chi_{\mathbf{2}} = y_1^{\frac{1}{2}}q^{\frac{1}{2}}+y_1^{-\frac{1}{2}}q^{-\frac{1}{2}}\quad ,\quad e^{i\frac{\rho}{2}} = y_1^{7/2}q^{-1/2}\ . \tag{4.33}$$

For $2 \leq N_f \leq 5$, one obtains

$$I = 1 + \chi_{\mathbf{adj}}\, t^2 + \chi_2(u)\big[1 + \chi_{\mathbf{adj}}\big] t^3 + \Big(\chi_3(u)\big[1 + \chi_{\mathbf{adj}}\big] + 1 + \chi_{\mathbf{adj^2}}\Big) t^4$$
$$+ \Big(\chi_4(u)\big[1 + \chi_{\mathbf{adj}}\big] + \chi_2(u)\big[1 + \chi_{\mathbf{adj^2}} + \chi_{(\mathbf{adj}\otimes\mathbf{adj})_A}\big]\Big) t^5$$
$$+ \Big(\chi_5(u)\big[1 + \chi_{\mathbf{adj}}\big] + \chi_3(u)\big[1 + \chi_{\mathbf{adj}} + \chi_{\mathbf{adj^2}} + \chi_{\mathbf{adj}\otimes\mathbf{adj}}\big] + \chi_{\mathbf{adj}} + \chi_{\mathbf{adj^3}} + \chi_{(\mathbf{adj}\otimes\mathbf{adj})_A}\Big) t^6 + \mathcal{O}(t$$

where **adj** denotes the adjoint representation of E_{N_f+1}, and $(\mathbf{adj} \otimes \mathbf{adj})_A$ denotes antisymmetrized tensor product of two adjoint representations. A brief explanation of E_n characters is provided at Appendix A.

Before proceeding, let me comment on the calculations of the superconformal index in series expansion. Unlike Nekrasov's partition function in which the instanton fugacity q is the main expansion parameter, the superconformal index is expanded in $t = e^{-\epsilon_+}$, and comes in both positive and negative powers in q. One should first fix the order t^n to which one wishes to expand I. Then one investigates the q expansion or q^{-1} expansion of the two Z_{inst}'s, and see how many instantons one has to keep.

Now move to the $Sp(1)$ theory with $N_f = 6$ matters. One obtains

$$\begin{aligned} I = 1 &+ \chi^{E_7}_{\mathbf{133}} t^2 + \chi_2(u)\left[1 + \chi^{E_7}_{\mathbf{133}}\right] t^3 + \left[1 + \chi^{E_7}_{\mathbf{7371}} + \chi_3(u)\left(1 + \chi^{E_7}_{\mathbf{133}}\right)\right] t^4 \\ &+ \left[\chi_2(u)\left(1 + \chi^{E_7}_{\mathbf{133}} + \chi^{E_7}_{\mathbf{7371}} + \chi^{E_7}_{\mathbf{8645}}\right) + \chi_4(u)\left(1 + \chi^{E_7}_{\mathbf{133}}\right)\right] t^5 \\ &+ \Big[2\chi^{E_7}_{\mathbf{133}} + \chi^{E_7}_{\mathbf{8645}} + \chi^{E_7}_{\mathbf{238602}} + \chi_3(u)\left(2 + 2\chi^{E_7}_{\mathbf{133}} + \chi^{E_7}_{\mathbf{1539}} + 2\chi^{E_7}_{\mathbf{7371}} + \chi^{E_7}_{\mathbf{8645}}\right) \\ &\qquad + \chi_5(u)\left(1 + \chi^{E_7}_{\mathbf{133}}\right)\Big] t^6 + \mathcal{O}\left(t^7\right) \,, \end{aligned} \tag{4.34}$$

showing the E_7 enhancement. The branching rules for $E_7 \to SO(12) \times U(1)$ are[2]

$$\mathbf{133} = \mathbf{1}_2 + \mathbf{1}_0 + \mathbf{1}_{-2} + \overline{\mathbf{32}}_1 + \overline{\mathbf{32}}_{-1} + \mathbf{66}_0, \tag{4.35}$$
$$\mathbf{1539} = \mathbf{1}_0 + \overline{\mathbf{32}}_1 + \overline{\mathbf{32}}_{-1} + \mathbf{66}_2 + \mathbf{66}_0 + \mathbf{66}_{-2} + \mathbf{77}_0 + \overline{\mathbf{352}}_1 + \overline{\mathbf{352}}_{-1} + \mathbf{495}_0,$$

[2]The names of representations displayed on the right hand sides, especially the barred ones, follow the chirality convention in [79]. For instance, the (unbarred) chiral spinors used in (4.14) are anti-chiral spinors for $N_f = 2, 3, 6, 7$ in [79] and (4.35), (4.37), while they are still chiral spinors for $N_f = 4, 5$ in [79].

$$\begin{aligned}\mathbf{7371} &= \mathbf{1}_4 + \mathbf{1}_2 + 2\times\mathbf{1}_0 + \mathbf{1}_{-2} + \mathbf{1}_{-4} + \overline{\mathbf{32}}_3 + 2\times\overline{\mathbf{32}}_1 + 2\times\overline{\mathbf{32}}_{-1} + \overline{\mathbf{32}}_{-3}\\
&\quad + \mathbf{66}_2 + \mathbf{66}_0 + \mathbf{66}_{-2} + \overline{\mathbf{462}}_2 + \overline{\mathbf{462}}_0 + \overline{\mathbf{462}}_{-2} + \mathbf{495}_0 + \mathbf{1638}_0 + \overline{\mathbf{1728}}_1 + \overline{\mathbf{1728}}_{-1},\\
\mathbf{8645} &= \mathbf{1}_2 + \mathbf{1}_0 + \mathbf{1}_{-2} + \overline{\mathbf{32}}_3 + 2\times\overline{\mathbf{32}}_1 + 2\times\overline{\mathbf{32}}_{-1} + \overline{\mathbf{32}}_{-3} + \mathbf{66}_2 + 2\times\mathbf{66}_0 + \mathbf{66}_{-2}\\
&\quad + \overline{\mathbf{352}}_1 + \overline{\mathbf{352}}_{-1} + \overline{\mathbf{462}}_0 + \mathbf{495}_2 + \mathbf{495}_0 + \mathbf{495}_{-2} + \overline{\mathbf{1728}}_1 + \overline{\mathbf{1728}}_{-1} + \mathbf{2079}_0,\\
\mathbf{238602} &= \mathbf{1}_6 + \mathbf{1}_4 + 2\times\mathbf{1}_2 + 2\times\mathbf{1}_0 + 2\times\mathbf{1}_{-2} + \mathbf{1}_{-4} + \mathbf{1}_{-6}\\
&\quad + \overline{\mathbf{32}}_5 + 2\times\overline{\mathbf{32}}_3 + 3\times\overline{\mathbf{32}}_1 + 3\times\overline{\mathbf{32}}_{-1} + 2\times\overline{\mathbf{32}}_{-3} + \overline{\mathbf{32}}_{-5}\\
&\quad + \mathbf{66}_4 + \mathbf{66}_2 + 2\times\mathbf{66}_0 + \mathbf{66}_{-2} + \mathbf{66}_{-4} + \overline{\mathbf{462}}_4 + 2\times\overline{\mathbf{462}}_2 + 3\times\overline{\mathbf{462}}_0\\
&\quad + 2\times\overline{\mathbf{462}}_{-2} + \overline{\mathbf{462}}_{-4} + \mathbf{495}_2 + \mathbf{495}_0 + \mathbf{495}_{-2} + \mathbf{1638}_2 + \mathbf{1638}_0 + \mathbf{1638}_{-2}\\
&\quad + \overline{\mathbf{1728}}_3 + 2\times\overline{\mathbf{1728}}_1 + 2\times\overline{\mathbf{1728}}_{-1} + \overline{\mathbf{1728}}_{-3}\\
&\quad + \overline{\mathbf{4224}}_3 + \overline{\mathbf{4224}}_1 + \overline{\mathbf{4224}}_{-1} + \overline{\mathbf{4224}}_{-3} + \overline{\mathbf{8800}}_1 + \overline{\mathbf{8800}}_{-1}\\
&\quad + \mathbf{21021}_0 + \overline{\mathbf{21450}}_2 + \overline{\mathbf{21450}}_0 + \overline{\mathbf{21450}}_{-2} + \mathbf{23100}_0 + \overline{\mathbf{36960}}_1 + \overline{\mathbf{36960}}_{-1}.\end{aligned}$$

To completely obtain all contributions up to t^6 order, one must count the orders as follows. Firstly, one can check that Z_{QM} at 4-instanton order starts from t^6, while Z_{QM} at 5-instanton order starts from t^9. So it may appear that the result up to t^6 will be consistently obtained by making a 4-instanton expansion in both Z_{inst}'s in (4.28). However, note that Z_{inst} in (4.28) should be $Z_{\text{inst}} = \frac{Z_{\text{QM}}}{Z_{\text{string}}}$, and Z_{string} obeys a different upper bound on instanton number with given order in t. Namely, in (4.14), the single particle index f_6 contains t^2q^2. So in $Z_{\text{string}} = PE[f_6]$, t^6 can come with $t^6q^6 = (t^2q^2)^3$, which contain more than 4-instanton order at t^6. Actually this is the reason why the branching rule of **238602** contains $5, 6$ instanton contributions. However, since I know Z_{string} exactly, all contributions at $k > 4$ can be easily traced. I expanded Z_{QM} that appear in (4.28) up to 4-instantons, and Z_{string} up to 6-instantons, which consistently yields all contributions till t^6 order.

At last, consider the $Sp(1)$ index at $N_f = 7$. The superconformal index is

$$\begin{aligned}I &= 1 + \chi^{E_8}_{\mathbf{248}}t^2 + \chi_2(u)\left[1 + \chi^{E_8}_{\mathbf{248}}\right]t^3 + \left[1 + \chi^{E_8}_{\mathbf{27000}} + \chi_3(u)\left(1 + \chi^{E_8}_{\mathbf{248}}\right)\right]t^4\\
&\quad + \left[\chi_2(u)\left(1 + \chi^{E_8}_{\mathbf{248}} + \chi^{E_8}_{\mathbf{27000}} + \chi^{E_8}_{\mathbf{30380}}\right) + \chi_4(u)\left(1 + \chi^{E_8}_{\mathbf{248}}\right)\right]t^5\\
&\quad + \left[2\chi^{E_8}_{\mathbf{248}} + \chi^{E_8}_{\mathbf{30380}} + \chi^{E_8}_{\mathbf{1763125}} + \chi_3(u)\left(2 + 2\chi^{E_8}_{\mathbf{133}} + \chi^{E_8}_{\mathbf{3875}} + 2\chi^{E_8}_{\mathbf{27000}} + \chi^{E_8}_{\mathbf{30380}}\right)\right.\end{aligned}$$

$$+\chi_5(u)\left(1+\chi^{E_8}_{\mathbf{248}}\right)\Big]\,t^6+\mathcal{O}\left(t^7\right)\ , \tag{4.36}$$

with E_8 enhancement. The relevant $E_8 \to SO(14)\times U(1)$ branching rules are

$$\begin{aligned}
\mathbf{248} &= \mathbf{1}_0+\mathbf{14}_2+\mathbf{14}_{-2}+\mathbf{64}_{-1}+\overline{\mathbf{64}}_1+\mathbf{91}_0,\\
\mathbf{3875} &= \mathbf{1}_4+\mathbf{1}_0+\mathbf{1}_{-4}+\mathbf{14}_2+\mathbf{14}_{-2}+\mathbf{64}_3+\mathbf{64}_{-1}+\overline{\mathbf{64}}_1+\overline{\mathbf{64}}_{-3}+\mathbf{91}_0\\
&\quad+\mathbf{104}_0+\mathbf{364}_2+\mathbf{364}_{-2}+\mathbf{832}_{-1}+\overline{\mathbf{832}}_1+\mathbf{1001}_0,\\
\mathbf{27000} &= 2\times\mathbf{1}_0+\mathbf{14}_2+\mathbf{14}_{-2}+2\times\mathbf{64}_{-1}+2\times\overline{\mathbf{64}}_1+2\times\mathbf{91}_0\\
&\quad+\mathbf{104}_4+\mathbf{104}_0+\mathbf{104}_{-4}+\mathbf{364}_2+\mathbf{364}_{-2}\\
&\quad+\mathbf{832}_3+\mathbf{832}_{-1}+\overline{\mathbf{832}}_1+\overline{\mathbf{832}}_{-3}+\mathbf{896}_2+\mathbf{896}_{-2}+\mathbf{1001}_0\\
&\quad+\mathbf{1716}_{-2}+\overline{\mathbf{1716}}_2+\mathbf{3003}_0+\mathbf{3080}_0+\mathbf{4928}_{-1}+\overline{\mathbf{4928}}_1,\\
\mathbf{30380} &= \mathbf{1}_0+2\times\mathbf{14}_2+2\times\mathbf{14}_{-2}+\mathbf{64}_3+2\times\mathbf{64}_{-1}+2\times\overline{\mathbf{64}}_1+\overline{\mathbf{64}}_{-3}\\
&\quad+\mathbf{91}_4+3\times\mathbf{91}_0+\mathbf{91}_{-4}+\mathbf{104}_0+\mathbf{364}_2+\mathbf{364}_{-2}\\
&\quad+\mathbf{832}_3+2\times\mathbf{832}_{-1}+2\times\overline{\mathbf{832}}_1+\overline{\mathbf{832}}_{-3}+\mathbf{896}_2+\mathbf{896}_{-2}\\
&\quad+\mathbf{1001}_0+\mathbf{2002}_2+\mathbf{2002}_{-2}+\mathbf{3003}_0+\mathbf{4004}_0+\mathbf{4928}_{-1}+\overline{\mathbf{4928}}_1,\\
\mathbf{1763125} &= 2\times\mathbf{1}_0+2\times\mathbf{14}_2+2\times\mathbf{14}_{-2}+3\times\mathbf{64}_{-1}+3\times\overline{\mathbf{64}}_1+3\times\mathbf{91}_0\\
&\quad+\mathbf{104}_4+\mathbf{104}_0+\mathbf{104}_{-4}+\mathbf{364}_2+\mathbf{364}_{-2}+\mathbf{546}_6+\mathbf{546}_2+\mathbf{546}_{-2}+\mathbf{546}_{-6}\\
&\quad+2\times\mathbf{832}_3+2\times\mathbf{832}_{-1}+2\times\overline{\mathbf{832}}_1+2\times\overline{\mathbf{832}}_{-3}+2\times\mathbf{896}_2+2\times\mathbf{896}_{-2}\\
&\quad+2\times\mathbf{1001}_0+2\times\mathbf{1716}_{-2}+2\times\overline{\mathbf{1716}}_2+\mathbf{2002}_2+\mathbf{2002}_{-2}\\
&\quad+3\times\mathbf{3003}_0+2\times\mathbf{3080}_0+\mathbf{4004}_4+2\times\mathbf{4004}_0+\mathbf{4004}_{-4}\\
&\quad+3\times\mathbf{4928}_{-1}+3\times\overline{\mathbf{4928}}_1+\mathbf{5625}_4+\mathbf{5625}_0+\mathbf{5625}_{-4}\\
&\quad+\mathbf{5824}_3+\mathbf{5824}_{-1}+\mathbf{5824}_{-5}+\overline{\mathbf{5824}}_5+\overline{\mathbf{5824}}_1+\overline{\mathbf{5824}}_{-3}\\
&\quad+\mathbf{11648}_2+\mathbf{11648}_{-2}+\mathbf{17472}_3+\mathbf{17472}_{-1}+\overline{\mathbf{17472}}_1+\overline{\mathbf{17472}}_{-3}\\
&\quad+\mathbf{18200}_2+\mathbf{18200}_{-2}+\mathbf{21021}_0+\mathbf{21021}_{-4}+\overline{\mathbf{21021}}_4+\overline{\mathbf{21021}}_0\\
&\quad+\mathbf{24024'}_2+\mathbf{24024'}_{-2}+\mathbf{27456}_3+\overline{\mathbf{27456}}_{-3}+\mathbf{36608}_2+\mathbf{36608}_{-2}\\
&\quad+\mathbf{40768}_{-1}+\overline{\mathbf{40768}}_1+\mathbf{45760}_3+\mathbf{45760}_{-1}+\overline{\mathbf{45760}}_1+\overline{\mathbf{45760}}_{-3}\\
&\quad+\mathbf{58344}_0+\mathbf{58968}_0+\mathbf{64064'}_{-1}+\overline{\mathbf{64064'}}_1+\mathbf{115830}_{-2}+\overline{\mathbf{115830}}_2
\end{aligned}$$

$$+\,\mathbf{146432}_{-1}+\overline{\mathbf{146432}}_{1}+\mathbf{200200}_{0}. \tag{4.37}$$

The instanton order counting for the t expansion up to t^6 goes as follows. I computed $Z_{\rm QM}$ up to 5-instantons to get these results. 5-instanton results start at t^6, so assuming that higher instantons come with higher powers in t, the above result should be reliable up to t^6 order.[3] Again $Z_{\rm string}$ up to t^6 order can come with higher instantons. Since f_7 in (4.14) comes with t^2q^2, one can maximally have q^6 from $Z_{\rm string}=PE[f_7]$ at t^6. This is the reason why one finds contribution at $k=\pm 6$ in the branching rule of **1763125**. Again, since one knows $Z_{\rm string}$ exactly, it can be expanded up to t^6 as well as $Z_{\rm QM}$ up to 5-instantons to consistently get all terms up to t^6.

This finishes my illustration that the $Sp(1)$ index at $N_f=6,7$ exhibits E_7 and E_8 enhancement, respectively, complementing the results of [63] at $N_f\leq 5$. Let me close this subsection by a few comments on related works. The first line of the index (4.34) was obtained in [68], by computing $Z_{\rm inst}$ from a suitablly Higgsed 5d T_4 theory [80]. The microscopic computation of the index (4.36) with E_8 symmetry appears to be new.

$Sp(2)$ indices

By following the same procedures, one can use $Z_{\rm inst}=Z_{\rm QM}/Z_{\rm string}$ for the $Sp(2)$ theories as $Z_{\rm inst}$ and compute the superconformal indices. For $0\leq N_f\leq 7$, I simply note that the superconformal index up to t^6 order takes the following form:

$$\begin{aligned}
I = {} & 1+\chi_2(e^m)\,t+\Big(\chi_2(u)\chi_2(e^m)+2\chi_3(e^m)+\chi_{\mathbf{adj}}\Big)t^2 \\
&+\Big(\chi_3(u)\chi_2(e^m)+\chi_2(u)\left[2\chi_3(e^m)+2+\chi_{\mathbf{adj}}\right]+2\chi_4(e^m)+\chi_2(e^m)(1+2\chi_{\mathbf{adj}})\Big)t^3 \\
&+\Big(\chi_4(u)\chi_2(e^m)+\chi_3(u)\left[3\chi_3(e^m)+2+\chi_{\mathbf{adj}}\right]+\chi_2(u)\left[3\chi_4(e^m)+\chi_2(e^m)(5+3\chi_{\mathbf{adj}})\right] \\
&\quad+3\chi_5(e^m)+\chi_3(e^m)(1+3\chi_{\mathbf{adj}})+3+\chi_{\mathbf{adj}}+\chi_{(\mathbf{adj}\otimes\mathbf{adj})_S}\Big)t^4
\end{aligned} \tag{4.38}$$

[3] Here I made a small assumption that 6 and higher instantons do not contribute till t^6 order. I could not check this due to large computational time at $k=6$. So the E_8 enhancement at t^6 found from 5 instanton calculus is justified with this assumption.

$$
\begin{aligned}
&+\Big(\chi_5(u)\chi_2(e^m)+\chi_4(u)\left[3\chi_3(e^m)+3+\chi_{\mathbf{adj}}\right]+\chi_3(u)\left[5\chi_4(e^m)+\chi_2(e^m)(8+4\chi_{\mathbf{adj}})\right]\\
&+\chi_2(u)\left[4\chi_5(e^m)+\chi_3(e^m)(9+6\chi_{\mathbf{adj}})+5+4\chi_{\mathbf{adj}}+\chi_{\mathbf{adj}\otimes\mathbf{adj}}\right]\\
&\qquad+3\chi_6(e^m)+\chi_4(e^m)(3+4\chi_{\mathbf{adj}})+\chi_2(e^m)(6+3\chi_{\mathbf{adj}}+\chi_{\mathbf{adj}^2}+\chi_{\mathbf{adj}\otimes\mathbf{adj}}-\chi_{\text{fer}}^{N_f})\Big)t^5\\
&+\Big(\chi_6(u)\chi_2(e^m)+\chi_5(u)\left[4\chi_3(e^m)+3+\chi_{\mathbf{adj}}\right]+\chi_4(u)\left[7\chi_4(e^m)+\chi_2(e^m)(11+5\chi_{\mathbf{adj}})\right]\\
&+\chi_3(u)\left[8\chi_5(e^m)+\chi_3(e^m)(16+10\chi_{\mathbf{adj}})+13+7\chi_{\mathbf{adj}}+2\chi_{(\mathbf{adj}\otimes\mathbf{adj})_S}+\chi_{(\mathbf{adj}\otimes\mathbf{adj})_A}\right]\\
&+\chi_2(u)\left[5\chi_6(e^m)+\chi_4(e^m)(14+9\chi_{\mathbf{adj}})+\chi_2(e^m)(16+12\chi_{\mathbf{adj}}+\chi_{\mathbf{adj}^2}+3\chi_{\mathbf{adj}\otimes\mathbf{adj}}-\chi_{\text{fer}}^{N_f})\right]\\
&+4\chi_7(e^m)+\chi_5(e^m)(3+5\chi_{\mathbf{adj}})+\chi_3(e^m)(14+6\chi_{\mathbf{adj}}+2\chi_{\mathbf{adj}^2}+2\chi_{(\mathbf{adj}\otimes\mathbf{adj})_S}+\chi_{(\mathbf{adj}\otimes\mathbf{adj})_A})\\
&+4+6\chi_{\mathbf{adj}}+\chi_{\mathbf{adj}^2}+2\chi_{(\mathbf{adj}\otimes\mathbf{adj})_A}+\chi_{\text{res}}^{N_f}-2\chi_{\text{fer}}^{N_f}\Big)t^6+\mathcal{O}\left(t^7\right).
\end{aligned}
$$

m is the chemical potential for $SU(2)_F$ global symmetry rotating the anti-symmetric $Sp(N)$ hypermultiplet. **adj** denotes the adjoint representation of E_{N_f+1} The terms $\chi_{\text{res}}^{N_f}$ and $-\chi_{\text{fer}}^{N_f}$ are non-universal terms which depend on N_f. $-\chi_{\text{fer}}^{N_f}$ is nonzero only for $N_f=1$, given by

$$
\chi_{\text{fer}}^{N_f=1}=1+\chi_{\mathbf{4}}(f)=1+\left(e^{i\frac{\varrho}{2}}+e^{-i\frac{\varrho}{2}}\right)\chi_{\mathbf{2}}\ . \tag{4.39}
$$

$\chi_4(f)$ and the fugacities in it are explained around (4.31). $\chi_{\text{res}}^{N_f}$ is given by

$$
\begin{aligned}
\chi_{\text{res}}^0&=\chi_{\mathbf{3}}+\chi_{\mathbf{7}}=\chi_{(\mathbf{3}\times\mathbf{3}\times\mathbf{3})_S},\\
\chi_{\text{res}}^1&=1+\chi_{\mathbf{3}}+\chi_{\mathbf{5}}+\chi_{\mathbf{7}},\\
\chi_{\text{res}}^2&=\chi_3+\chi_7+\chi_{\mathbf{8}}(1+\chi_5)+\chi_{\mathbf{10}}+\chi_{\overline{\mathbf{10}}}+\chi_{\mathbf{27}}(1+\chi_3)+\chi_{\mathbf{64}},\\
\chi_{\text{res}}^3&=\chi_{\mathbf{24}}+\chi_{\mathbf{126}}+\chi_{\overline{\mathbf{126}}}+\chi_{\mathbf{200}}+\chi_{\mathbf{1000}}+\chi_{\mathbf{1024}},\\
\chi_{\text{res}}^4&=\chi_{\mathbf{45}}+\chi_{\mathbf{945}}+\chi_{\mathbf{1386}}+\chi_{\mathbf{5940}}+\chi_{\mathbf{7644}},\\
\chi_{\text{res}}^5&=\chi_{\mathbf{78}}+\chi_{\mathbf{2925}}+\chi_{\mathbf{34749}}+\chi_{\mathbf{43758}},\\
\chi_{\text{res}}^6&=\chi_{\mathbf{133}}+\chi_{\mathbf{8645}}+\chi_{\mathbf{152152}}+\chi_{\mathbf{238602}},\\
\chi_{\text{res}}^7&=\chi_{\mathbf{248}}+\chi_{\mathbf{30380}}+\chi_{\mathbf{779247}}+\chi_{\mathbf{1763125}}=\chi_{(\mathbf{248}\otimes\mathbf{248}\otimes\mathbf{248})_S}
\end{aligned}\tag{4.40}
$$

where $\chi_{\mathbf{n}}$ is the character of the **n** dimensional irrep of E_{N_f+1} for $N_f\neq 1,2$. For $N_f=1$, $\chi_{\mathbf{n}}$ is a character of $SU(2)$ in $E_2=SU(2)\times U(1)$. For $N_f=2$, $\chi_{\mathbf{n}}$ is the character of $SU(3)$ and χ_n is the character of $SU(2)$ in $E_3 = SU(3) \times SU(2)$. The $Sp(2)$ superconformal indices all show the E_{N_f+1} symmetry enhancements to the t^6 order that I checked.

4.4 6d (1,0) SCFT with E_8 flavor symmetry

Now turn to the case with $N_f = 8$, for D0-branes probing N D4, 8 D8's and an O8. Again the t^9 direction is a half-line $\mathbb{R}^+$. The difference from the cases with $N_f \leq 7$ is that the D8-brane charges completely cancel between 8 D8's and one O8. The dilaton asymptotically becomes a constant as one moves away from the brane system along t^9. So this system uplifts to M-theory on $\mathbb{R}^{8,1} \times \mathbb{R}^+ \times S^1$ at strong coupling. The 5d SYM is thus a low energy description of circle compactified 6d $(1,0)$ theory for the M5-M9 system. In this case, there are poles at infinities of cylinders in $Z_{\text{1-loop}}$, since D0's can move away from the 8-branes with a continuum. Following the same strategy as the cases with $N_f \leq 7$, one must first extract out $Z^{(0)}$ as this should contain all possible Z_{string} factors. Again writing $Z^{(0)} = PE[f]$, f is given by

$$\begin{aligned} f \;=\; & \left[\frac{t(v+v^{-1}-u-u^{-1})}{(1-tu)(1-t/u)} - \frac{(t+t^3)(u+u^{-1}+v+v^{-1})}{2(1-tu)(1-t/u)(1-tv)(1-t/v)}\right]\frac{q^2}{1-q^2} \\ & - \frac{t^2}{(1-tu)(1-t/u)(1-tv)(1-t/v)}\left[\chi(y_i)^{SO(16)}_{\mathbf{120}}\frac{q^2}{1-q^2} + \chi(y_i)^{SO(16)}_{\mathbf{128}}\frac{q}{1-q^2}\right], \end{aligned} \tag{4.41}$$

where I checked the q dependence up to 4-instanton order from the $Sp(1)$ theory. Namely, the above expression is obtained with $\frac{q^2}{1-q^2} \to q^2 + q^4$ and $\frac{q}{1-q^2} \to q + q^3$. So all properties shown below are proven up to this order. **120** and **128** are the adjoint and chiral spinor representations of $SO(16)$. I will now explain the terms in (4.41) which should go to Z_{string}.

Consider first the second line of (4.41). This provides a single particle index for certain $8+1$ dimensional particles, thus should go to the factorized Z_{string} from bulk degrees. Let me first explain what is expected from the string dualities and heterotic M-theory. Heterotic M-theory was proposed in [81] as a strong coupling limit of $E_8 \times E_8$ heterotic string theory. It has a low energy limit described by 11d supergravity on $\mathbb{R}^{9,1} \times I$, where $I = S^1/\mathbb{Z}_2$ is an interval. There are two fixed planes of the $\mathbb{Z}_2$ action at both ends of I, which are called the M9-planes. Each M9-plane hosts E_8 gauge symmetry, having a massless sector of 10d E_8 super-Yang-Mills theory. One can compactify the heterotic M-theory on a small circle with radius R to $\mathbb{R}^{8,1} \times I$. The circle compactification can be made with nonzero $E_8 \times E_8$ Wilson lines on two 10d SYM theories on M9-planes. In particular, consider the following Wilson line

$$RA^{E_8} = (0, 0, 0, 0, 0, 0, 0, 1) \tag{4.42}$$

for each E_8 SYM. My convention is to pick 8 Cartans of $SO(16) \subset E_8$ which rotate 8 orthogonal 2-planes of $SO(16)$. The adjoint representation **248** of E_8 decomposes in $SO(16)$ to

$$\mathbf{248} \to \mathbf{120} + \mathbf{128} \ . \tag{4.43}$$

The holonomy (4.42) is such that $e^{2\pi i RA}$ leaves **120** invariant, while giving -1 sign to the spinors. So the compactification with this holonomy yields a 10d theory with $SO(16) \times SO(16)$ symmetry. This is the type I' string theory on $\mathbb{R}^{8,1} \times I$, which has two orientifold 8-planes (O8-planes) at the two ends of I. Each O8-plane has 8 D8-branes on top of it. A crucial part of this identification is that the nonperturbative D0-brane physics of type I' theory should enable us to see the 11th circle's KK modes. I will show that the second line of (4.41) achieves it.

Note that the fugacities q, y_i in (4.41), especially on the second line, probe the momentum and $SO(16)$ charges in the background of Wilson line (4.42). The charges of the type I' theory and the heterotic M-theory are related by [82]

$$k = 2P - RA^{E_8} \cdot F^{E_8} = 2P - F_8 \ , \tag{4.44}$$

where k is the type I' instanton charge, P is the circle momentum of heterotic M-theory, F^{E^8} are the E_8 charges, A^{E_8} is the holonomy (4.42). The expression in [82] has more shifts to k on the right hand side, depending on the string winding number, which is zero for all states captured by $Z^{(0)}$. The fugacities conjugate to

k, F_8 are more naturally viewed in the heterotic M-theory as

$$q^k y_8^{F_8} = q^{2P}(y_8 q^{-1})^{F_8} \ . \tag{4.45}$$

If one replaces all y_8's in $Z^{(0)}$ by $y_8 q$, this effectively turns off the background holonomy (4.42). Namely, q^2 and y_i should be regarded as the fugacities conjugate to P and E_8 charges after the replacement. Note that this replacement $y_8 \to y_8 q$ makes a rearrangement of the instanton series expansion, as the original expansion is made with $q \ll y_i, y_i^{-1}$. After the replacement, one temporarily decomposes the $SO(16)$ characters into $SO(14)$ characters. The characters appearing on the second line of (4.41) are decomposed as

$$\chi_{\mathbf{120}}^{SO(16)} \to 1 + \chi_{\mathbf{91}}^{SO(14)} + (y_8^2 q^2 + y_8^{-2} q^{-2})\chi_{\mathbf{14}}^{SO(14)} \tag{4.46}$$

$$\chi_{\mathbf{128}}^{SO(16)} \to y_8 q \chi_{\mathbf{64}}^{SO(14)} + y_8^{-1} q^{-1} \chi_{\overline{\mathbf{64}}}^{SO(14)} \tag{4.47}$$

after the replacement $y_8 \to y_8 q$. The second line of (4.41) thus rearranges as

$$-\frac{t^2}{(1-tu)(1-t/u)(1-tv)(1-t/v)} \times \left[\chi_{\mathbf{248}}^{E_8}(y_i)\frac{q^2}{1-q^2} + \chi_{\mathbf{14}}^{SO(14)} y_8^{-2} + \chi_{\mathbf{64}}^{SO(14)} y_8^{-1} - \chi_{\mathbf{14}}^{SO(14)} y_8^2 q^2\right] \tag{4.48}$$

where $\chi_{\mathbf{248}}^{E_8} = \chi_{\mathbf{120}}^{SO(16)} + \chi_{\mathbf{128}}^{SO(16)}$. The first term in the square bracket is exactly what one expects from the heterotic M-theory with nonzero momentum P, since there should be contributions from the 10d E_8 SYM with $P = 1, 2, 3, \cdots$. Note also that the prefactor is also the index for the vector multiplet in the $\mathbb{R}^8$ part, as computed around (4.20). Therefore, the replacement $y_8 \to y_8 q$ makes $SO(16) \to E_8$ enhancement visible by turning off the background Wilson line.

It only remains to understand the last three terms in the square bracket. These can be understood again by combining it with the perturbative 9d $SO(16)$ SYM of type I' theory. Depending on the Coulomb VEV conjugate to F_8, the W-boson and anti-W-boson indices would be one of

$$f_{\text{pert}}^{\pm} = -\frac{t^2}{(1-tu)(1-t/u)(1-tv)(1-t/v)} \left[\chi_{\mathbf{91}}^{\pm} + y_8^{\pm 2}\chi_{\mathbf{14}}^{SO(14)}\right] \ . \tag{4.49}$$

After the replacement $y_8 \to y_8 q$ in $f^{\pm}_{\text{pert}}$, the second term of f^{+}_{pert} becomes

$$-\frac{t^2}{(1-tu)(1-t/u)(1-tv)(1-t/v)} y_8^2 q^2 \chi^{SO(14)}_{\mathbf{14}} \ . \tag{4.50}$$

This precisely cancels with the last term of (4.48). The second and third terms of (4.48) combine with $\chi^{\pm}_{\mathbf{91}}$ in (4.49), to provide the positive/negative roots of E_8 at $\mathcal{O}(q^0)$. (Other roots at q^0 are provided by shifts from anti-instanton sector.) Thus, the KK tower of 10d E_8 SYM with $k > 0$ is completely reproduced by the second line of (4.41), with the last term of (4.48) provided by the W-bosons of 9d $SO(16)$ SYM. This states that the second line of (4.41) should go to Z_{string}.

Then in (4.41), consider the term

$$-\frac{(t+t^3)(u+u^{-1}+v+v^{-1})}{2(1-tu)(1-t/u)(1-tv)(1-t/v)} \frac{q^2}{1-q^2} \tag{4.51}$$

on the first line. The overall coefficient $\frac{1}{2}$ shows that this is clearly the continuum contribution. In fact, there is no way to turn on the FI term with $O(k)$ gauge group, so that one cannot decouple the continuum from the Witten index calculus. Although I do not have an account for the factor $\frac{1}{2}$, in a way similar to [83, 84], it can be shown that all the dependence on the fugacities is that for the continuum states in our problem.

To show this, let me investigate the 11d supergravity spectrum on $\mathbb{R}^{8,1} \times S^1 \times \mathbb{R}^+$. The continuum is formed by the states which propagate along $\mathbb{R}^+$. The $\mathbb{R}^8 \times S^1$ part of the space has a fully gapped spectrum, either by having compact space or by having nonzero chemical potentials for the rotations. So the gapped part of the spectrum can be computed by investigating the supergravity multiplet, setting aside an overall fractional coefficient which can only be determined by knowing the dynamics along $\mathbb{R}^+$ (and the deformations in the index computation). The factor $\frac{q^2}{1-q^2}$ simply shows that the KK fields of the 11d gravity on circle have all same spin contents in 10d. So the t, u, v dependence of this term can be computed from the 10d type I' supergravity. Also, since one is only paying attention to the $\mathbb{R}^8$ part of the spectrum, one can replace $\mathbb{R}^+$ by $I = S^2/\mathbb{Z}_2$ and apply T-duality along this direction, after which the well known type I supergravity spectrum will be relevant.

The type I supergravity contains a dilaton ϕ, RR 2-form C_2, graviton $g_{\mu\nu}$, dilatino λ, and the gravitino ψ_μ. All of them are in the following representation of $SO(8)$, rotating $\mathbb{R}^8$:

$$(\mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35}_v)_{\text{boson}}\oplus(\mathbf{8}_s\oplus\mathbf{56}_s)_{\text{fermion}} = (\mathbf{8}_v\otimes\mathbf{8}_v)_{\text{sym}}\oplus(\mathbf{8}_v\otimes\mathbf{8}_c)\oplus(\mathbf{8}_c\otimes\mathbf{8}_c)_{\text{anti}}\ . \quad (4.52)$$

The $SU(2)^4$ characters of $\mathbf{8}_v, \mathbf{8}_s, \mathbf{8}_c$ on the right hand sides (with $(-1)^F$ signs for $\mathbf{8}_s$, $\mathbf{8}_c$) are

$$\begin{aligned}
\chi(\mathbf{8}_v) &= (t+t^{-1})(u+u^{-1}+v+v^{-1}) \\
\chi(\mathbf{8}_c) &= -t^2-2-t^{-2}-(u+u^{-1})(v+v^{-1}) \\
\chi(\mathbf{8}_s) &= -(t+t^{-1})(u+u^{-1}+v+v^{-1})\ .
\end{aligned} \quad (4.53)$$

From this, one can compute the index for the right hand side of (4.52). Note that the symmetrized and anti-symmetrized characters are given by $\frac{f(t,u,v)^2\pm f(t^2,u^2,v^2)}{2}$, where f is the character of $\mathbf{8}_{v,c}$ appearing in the (anti)symmetrization. Multiplying this with the factor $\frac{t^4}{(1-tu)^2(1-t/u)^2(1-tv)^2(1-t/v)^2}$ which comes from the translation zero modes on $\mathbb{R}^8$, one obtains

$$-\frac{(t+t^3)(u+u^{-1}+v+v^{-1})}{(1-tu)(1-t/u)(1-tv)(1-t/v)}\ . \quad (4.54)$$

So this proves that (4.51) is the continuum contribution from 11d supergravity KK modes.

Having explained the second line of (4.41) and the t,u,v,q dependence of (4.51), there is no ambiguity in the separation of the 4d index and the higher dimensional index in $Z^{(0)}$. So $Z_{\text{string}} = PE[f_{\text{string}}]$ with

$$\begin{aligned}
f_{\text{string}} = &-\frac{(t+t^3)(u+u^{-1}+v+v^{-1})}{2(1-tu)(1-t/u)(1-tv)(1-t/v)}\frac{q^2}{1-q^2} \\
&-\frac{t^2}{(1-tu)(1-t/u)(1-tv)(1-t/v)}\left[\chi(y_i)^{SO(16)}_{\mathbf{120}}\frac{q^2}{1-q^2}+\chi(y_i)^{SO(16)}_{\mathbf{128}}\frac{q}{1-q^2}\right]
\end{aligned} \quad (4.55)$$

is the contribution from the string theory or UV sector. The first term of (4.41),

$$\frac{t(v+v^{-1}-u-u^{-1})}{(1-tu)(1-t/u)}\frac{q^2}{1-q^2} = \frac{\sinh\frac{m+\epsilon_-}{2}\sinh\frac{m-\epsilon_-}{2}}{\sinh\frac{\epsilon_1}{2}\sinh\frac{\epsilon_2}{2}}\frac{q^2}{1-q^2} \equiv I_-(\epsilon_{1,2},m)\frac{q^2}{1-q^2}\ , \quad (4.56)$$

is not included in Z_{string}. It is part of the 6d QFT spectrum.

Again, one can directly compute Z_{string} from the index of D0-D8-O8 quantum mechanics, without relying on unproved properties. See Section 4.3.1. So far I explained a clear recipe to compute the index of the circle compactified 6d $(1,0)$ SCFT on the M5-M9 system, $Z_{\text{inst}} = \frac{Z_{\text{QM}}}{Z_{\text{string}}}$, with replacement $y_8 \to y_8 q$, etc. The circle compactified 6d $(1,0)$ SCFT will be more thoroughly investigated in Section 5.2, via the self-dual strings it contains.

Chapter 5

Non-critical strings in 6d QFTs

Six dimensional superconformal theories with $(2,0)$ and $(1,0)$ supersymmetry enjoy a special status among all superconformal theories: they are at the highest possible dimension. They play a key role in various aspects of string dualities as well as in obtaining lower dimensional supersymmetric systems upon compactification. They have the interaction conveyed by the self-dual 2-form tensor which implies the existence of charged strings (often called the self-dual strings). In the tensor branch, self-dual strings are the core objects in understanding the 6d SCFT. For the 6d SCFTs without a gauge group, these strings are only possible BPS excitations. Thus studying the string spectrum often implies that one can understand the full BPS spectrum of six-dimensional QFTs. This is indeed the case for the 6d (2.0) SCFTs and $(1,0)$ SCFTs with E_8 global symmetry, which one can compare the self-dual string spectra and the instanton partition functions of 5d Yang-Mills theories obtained as the circle compactification of 6d QFTs.

The worldsheet dynamics are governed by the 2d CFT which is strongly interactive. The key to access such strongly interacting theories is to find out the weakly-coupled gauge theories which provide the UV description of 2d CFTs. The string duality was crutial to construct the gauge theory, and the recent progress was made on M-strings [9, 17], which are self-dual strings in $(2,0)$ SCFTs. In this chapter I will briefly review the recent progress made on M-strings, then construct the UV gauge theory of E-strings which dictate the physics of an M5-brane approaches the M9-brane boundary, or equivalently, the small instantons in $E_8 \times E_8$ heterotic string theory [20, 14, 22].

5.1 M-strings in 6d $(2,0)$ SCFT

Consider M5-branes and the $(2,0)$ SCFTs living on those 5-branes. It is allowed for M2-branes to end on those 5-branes, inducing the stringy objects in the six-dimensional worldvolume [3]. These strings are called M-strings, of which the worldsheet 2d SCFT preserves $\mathcal{N} = (4,4)$ supersymmetry. This SCFT is strongly interactive, which are difficult to be dealt with. Instead of dealing with $\mathcal{N} = (4,4)$ SCFT, [9, 17] successfully constructed the 2d gauge theory which preserves $\mathcal{N} = (0,4)$ chiral supersymmetry. Since the 2d Yang-Mills gauge theory is super-renormalizable, the gauge coupling $g_{2\text{d}}$ goes to infinite as one takes the infrared limit. Therefore, one expects that the $\mathcal{N} = (0,4)$ gauge theory might have the enhanced $(4,4)$ supersymmetry and describe the worldsheet physics of M-strings in the infrared limit.

Here I will briefly review the construction of the $(0,4)$ gauge theory explained in [17]. For simplicity, let me restrict the discussion to the case of two M5-branes. In the tensor branch of 6d $(2,0)$ SCFT, both M5-branes are separated from one another. M2-branes suspended between both M5-branes acquires a string tension which is proportional to the distance between two M5-branes. One then place the whole brane system at the center of the Taub-NUT geometry.

To obtain the weakly coupled system in the string theory background, the duality between type IIA string theory and M-theory is critically used. One reduces the Taub-NUT circle to be very small, for going down to the type IIA string theory. The resulting system is the type IIA brane set-up given in Figure 5.1. What lives on the D2-brane segment is the 3d Yang-Mills gauge theory, which gets reduced to the 2d theory as one takes both NS5-branes to be close to each other. The 2d gauge theory lives on the intersection of D2 and NS5-branes, which were M2 and M5-branes before reducing the M-theory circle. This theory preserves the $(0,4)$

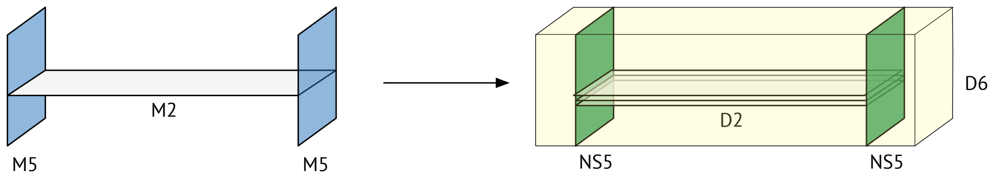

Figure 5.1: The M5-M2 system is mapped to the type IIA D2-NS5-D6 brane system.

supersymmetry, but here I will not explain why because the very same analysis will be given for E-strings in the next section.

One important property of this 2d gauge theory is that the 2d gauge coupling g_{2d} has the following property

$$g_{\text{2d}}^2 \propto g_s \propto R_M \tag{5.1}$$

where R_M denotes the reduced Taub-NUT circle. Therefore, the D2-NS5-D6 brane system goes back to the M2-M5 brane set-up probing the Taub-NUT geometry in the strong coupling regime, which corresponds to the infrared limit of the 2d gauge theory. Summing up, this 2d gauge theory is expected to be the UV gauge theory describing M-strings.

The details of this 2d gauge theory is given as follows: for k M-strings, it has the $U(k)$ gauge group. It has the $SO(4)$ global symmetry which is related to the rotation longitudinal to the NS5-brane worldvolume. Decomposing $SO(4)_1 = SU(2)_{1L} \times SU(2)_{1R}$, α and $\dot{\alpha}$ represent doublet indices for both $SU(2)$s. The $(0,4)$ supersymmetry has $SO(4)_2$ R-symmetry which is also decomposed into $SU(2)_{2L} \times SU(2)_{2R}$. Field contents are summarized in the quiver diagram (Figure 5.2) and Table 5.1, i.e., solid lines denote the $(0,4)$ hypermultiplets, while dotted lines represent the $(0,4)$ Fermi multiplets. Three $U(1)$'s are global symmetries, two of which will be eventually locked in such that $U(1)_m \subset U(1) \times U(1)$ is alive. This $U(1)_m$ is a Cartan subgroup of $SU(2)_{2L}$, whose double indices are denoted by a. For the remaining $SU(2)_{2R}$, I adopt the dotted $\dot{a}$ for denoting its double indices. The lines (2), (3), (4) connecting the global $U(1)$ and the gauge $U(k)$ nodes represent the $U(k) \times U(1)$ bifundamental fields, while the line (1) denotes the $U(k)$ adjoint fields.

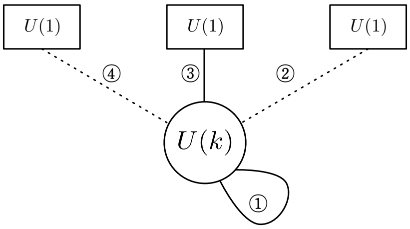

Figure 5.2: (0, 4) quiver diagram for M-string gauge theory

No.	Field	Multiplet	$U(k)$ representation
1	$(a^{\alpha}{}_{\dot\beta}, \lambda_-^{\alpha A})$	hyper.	adjoint
2 and 4	(ψ_-^a)	Fermi.	fund.
3	$(q_{\dot\alpha}, \psi_+^A)$	hyper.	fund.

Table 5.1: (0, 4) superfields in M-string gauge theory

To consider the spectrum of the M-string SCFT, one should perform the radial quantization which puts the IR SCFT on T^2. By putting the above 2d gauge theory on T^2, any infrared observable can be computed in principle if one can somehow trace all quantum corrections. Moreover, if one considers a protected SUSY observable such as the Witten index, it can be immediately interpreted as the observable of the IR SCFT. The Witten index for the 2d gauge theory on a torus is often called the elliptic genus. [9, 17] computed the refined elliptic genus of M-strings via the UV gauge theory, which can be combined to the tensor branch index of 6d (2, 0) SCFT. Their results are compatible with the index of 6d (2, 0) SCFT studied from the instanton partition function [55].

$$Z_{6\mathrm{d}} = 1 + \sum_{i=1}^{\infty} w^n Z_{\mathrm{M-str}}^{(n)}(q) = 1 + \sum_{i=1}^{\infty} q^n Z_{\mathrm{inst}}^{(n)}(w) \tag{5.2}$$

This provides the strong evidence that the above (0, 4) gauge theory indeed describes the M-string worldsheet physics. The discussion up to here will be elaborated in detail for more challenging case, the 6d (1, 0) E_8 SCFT.

5.2 E-strings in 6d $(1,0)$ E_8 SCFT

The most basic $(1,0)$ E_8 superconformal theory is known to arise in for small instantons in $E_8 \times E_8$ heterotic string theory or when an M5-brane approaches to a boundary M9-brane [20, 14, 22]. It also has an F-theory dual description given by blowing up a point on $\mathbb{C}^2$ base of F-theory [27, 25, 26]. This superconformal theory has an E_8 global symmetry. It also has a one dimensional Coulomb branch, parameterized by a real scalar in the $(1,0)$ tensor multiplet. In the M-theory setup, the scalar parameterizes the distance between M5 and M9 branes [81]. In F-theory setup, it parameterizes the size of the $\mathbb{P}^1$ obtained by blowing up a point. On the Coulomb branch this theory has light strings, known as E-strings [13]. In the M-theory setup they arise by M2 branes stretched between M5 brane and M9 brane. In F-theory setup they arise by wrapping D3 branes on the blown up $\mathbb{P}^1$. It is natural to ask whether one can find a nice description of E-strings.

If one is computing supersymmetry protected quantities, such as elliptic genus, one can change parameters to make the computation easy. In particular one can change parameters and use string dualities to find a suitable description of the resulting strings. This strategy was employed in particular for M-strings and their orbifolds [9, 17]. Two basic ways were used to compute the elliptic genus of the M-strings: one was to use string dualities to map the 2d theory to a super-Yang-Mills type gauge theory and use the technique developed recently [59, 60, 61] to compute their elliptic genera. The other way was to use the relation of the elliptic genus to BPS quantities upon circle compactification of these theories, that can in principle be computed using topological strings.

I will employ the former method and identify the gauge theory which captures their low energy physics. This is done by considering the duality of M-theory with type IIA, by introducing a circle transverse to M5 brane, leading to a system involving NS5-brane and where the M9 brane is replaced by O8 plane with 8 D8 branes on it. The M2 branes suspended between M5 and M9 branes map to D2 branes suspended between NS5-brane and O8-D8 pair. I find a simple $(0,4)$ supersymmetric quiver describing this system with $O(n)$ gauge symmetry, where n denotes the number of suspended M2 branes. I will use it to compute the elliptic genus of n E-strings by employing the techniques developed in [60, 61].

The other method of computing the elliptic genus of E-string involves the F-theory picture. Namely, one compactifies the theory on a circle leading to an M-theory description, and consider the BPS states of wrapped M2 branes, which correspond to E-strings wound around S^1 [85]. M-theory geometry involves the canonical bundle over $\frac{1}{2}K3$. As is well known, the BPS states of M2 branes wrapped on it, are captured by topological string amplitudes [86, 87]. In this context the (refined) topological string for $\frac{1}{2}K3$ has been computed to a high genus [88, 89], though an all genus answer is not available. The method adopted in this thesis will lead to a complete answer for refined topological string on $\frac{1}{2}K3$.

5.2.1 The brane setup and the 2d $(0, 4)$ gauge theories

Let me construct a brane system in the type IIA string theory, which at low energy engineers the 6d E_8 SCFT and the 2d CFT for E-strings. Take an NS5-brane to wrap the 013456 directions, located at $x^2 = L\ (> 0)$, $x^7 = x^8 = x^9 = 0$. An O8-plane and 8 D8-branes (or 16 D8-branes in the covering space of orientifold) wrap 013456789 directions, located at $x^2 = 0$. To describe E-strings, n D2-branes are stretched between the NS5 and 8-brane system ($0 < x^2 < L$), occupying 012 directions. x^1 direction is compactified to a circle. This brane system has $SO(4)_1 \times SO(3)_2 = SU(2)_L \times SU(2)_R \times SU(2)_I$ symmetry which rotates 3456 and 789 directions. Let me denote by $\alpha, \beta, \cdots = 1, 2$, $\dot{\alpha}, \dot{\beta}, \cdots = 1, 2$ and $A, B, \cdots = 1, 2$ the doublet indices of these three $SU(2)$ symmetries. See Table 5.2 and Fig. 5.3.

The M-theory uplift of this brane configuration, with extra circle direction labeled by x^{10}, is given as follows. The NS5-brane lifts to the M5-brane transverse to the x^{10} direction. The D8-O8 system uplifts to an M9-plane, or the Horava-Witten wall [81], longitudinal in x^{10} direction. In order to get a weakly-coupled type IIA string theory at low energy, one has to turn on suitable E_8 Wilson line

	0	1	2	3	4	5	6	7	8	9
NS5	•	•		•	•	•	•			
D8-O8	•	•		•	•	•	•	•	•	•
D2	•	•	•							

Table 5.2: Brane configuration for the E-strings

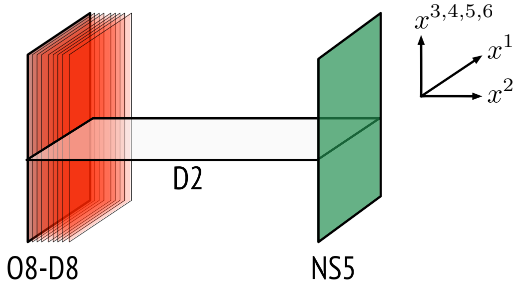

Figure 5.3: The type IIA brane configuration for the E-strings.

along x^{10} to break $E_8 \to SO(16)$ [14]. See Sections 4.4 and 5.2.3 for more details. D2-branes uplift to M2-branes transverse in x^{10}. In the strong coupling limit of the type IIA theory, the radius of the M-theory circle becomes large. The geometry $\mathbb{R}^3 \times S^1$ transverse to the 5-brane is replaced by $\mathbb{R}^4$. So the brane configuration contains the M5-M9 system, in the Coulomb branch of the 6d E_8 CFT. M2-branes suspended between them are the E-strings.

At an energy scale much lower than L^{-1}, one obtains a 2d QFT living at the intersection of these branes. At $g_{\rm YM} \ll E \ll L^{-1}$ with $g_{\rm YM}^2 \sim \frac{g_s}{L\ell_s}$, where ℓ_s, g_s are the string scale and the coupling constant, one obtains a weakly coupled 2d Yang-Mills description with coupling constant g_{YM}. (One can take g_s to be sufficiently small, and L to be sufficiently larger than ℓ_s.) When $E \ll g_{\rm YM}$, the 2d Yang-Mills theory is strongly coupled and is expected to flow to an interacting SCFT. In terms of the Planck scale $\ell_P \sim g_s^{1/3}\ell_s$ of M-theory and the radius $R \sim g_s\ell_s$ of the x^{10} circle, the strong coupling regime of the 2d Yang-Mills theory is $E \ll \frac{R}{L^{1/2}\ell_P^{3/2}}$. L

is related to the VEV v of the scalar in the 6d tensor multiplet by $L \sim v\ell_P^3$. So the low energy limit is $E \ll \frac{R}{v^{1/2}\ell_P^3}$. In the Coulomb branch with fixed v, this low energy limit of the 2d theory is obtained by taking the M-theory limit $R \to \infty$. Thus our 2d gauge theory describes E-strings at its strong coupling fixed point.

Let me comment on the enhanced IR symmetries. Consider the $SO(3) \times U(1)$ acting on $\mathbb{R}^3 \times S^1$. In the M-theory limit, this enhances to $SO(4) \sim SU(2)_l \times SU(2)_r$ of $\mathbb{R}^4$. $SO(3)$ is identified as the diagonally locked combination of $SU(2)_r$ and $SU(2)_l$. On the other hand, from the viewpoint of 6d superconformal symmetry, $SU(2)_r$ is the R-symmetry of the 6d $(1,0)$ SCFT and $SU(2)_l$ is a flavor symmetry. So it might appear that our 2d gauge theory is probing only a combination of the R-symmetry and a flavor symmetry. However, in the rank 1 system with only one M5-brane, the extra flavor $SU(2)_l$ completely decouples with the 6d CFT. For instance, these can be seen by studying the instanton partition functions of circle reduced 5d SYM (see Section 4.4), which will also be the subject of Section 5.2.3. Thus one can identify $SO(3)$ visible in the UV theory as the superconformal R-symmetry of the 6d CFT. Generalization to the higher rank CFT is given in [90].

The UV theory exhibits $SO(16)$ symmetry only. This should enhance to E_8 in the IR, which is naturally expected from the brane perspective. Namely, the type IIA brane system is obtained by compactifying M-theory brane system with an E_8 Wilson line which breaks E_8 to $SO(16)$. The IR limit on the 2d gauge theory is the strong coupling limit, which is the decompactification limit of the M-theory circle. So in this limit, the information on the Wilson line will be invisible, making us to expect an IR E_8 enhancement. In Section 5.2.2, I will compute the elliptic genera of these gauge theories at various values of n, which will be invariant under the E_8 Weyl symmetry and support the E_8 enhancement.

Let me study the SUSY of this system. The D2, D8 SUSY are associated with the projectors Γ^{012}, $\Gamma^{0134567889}\Gamma^{11} \sim \Gamma^2$, while the NS5-brane projector is $\Gamma^{01}\Gamma^{3456}$. Various combinations of branes share different SUSY. I list the following projectors which should assume definite eigenvalues, for various combinations of branes:

$$\text{D2-D8-NS5} \;:\; \Gamma^{01}\;,\;\; \Gamma^{2}\;,\;\; \Gamma^{3456} \tag{5.3}$$

$$\text{D2-NS5} \;:\; \Gamma^{01}\Gamma^{2}\;,\;\; \Gamma^{01}\Gamma^{3456} \tag{5.4}$$

$$\text{D2-D8-O8} \;:\; \Gamma^{01}\;,\;\; \Gamma^{2}\;. \tag{5.5}$$

The projectors (5.3) will yield the SUSY preserved by the brane system. The SUSY given by (5.4) and (5.5) will constrain the boundary conditions of the 3d D2-brane fields at the two ends of the segment along x^2. Let me investigate them in more detail. The type IIA supercharges with 32 components can be arranged to be eigenstates of $\Gamma^{01}, \Gamma^{3456}, \Gamma^{2}$. The eigenspinors of Γ^{01} are 2d chiral spinors, while those of Γ^{3456} belong to either $(\mathbf{2},\mathbf{1})$ or $(\mathbf{1},\mathbf{2})$ representations of $SU(2)_L \times SU(2)_R$. The 32 supercharges decompose into the sum of the $(\mathbf{2},\mathbf{1},\mathbf{2})_{\pm\pm} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_{\pm\pm}$ representations of $SU(2)_L \times SU(2)_R \times SU(2)_I$ with all four possible choices of $\pm\pm$, where the first/second $\pm$ subscripts denote 2d chirality and Γ^2 eigenvalues, respectively. The SUSY preserved by various combinations of branes are given by

$$\text{D2-D8-NS5} \;:\; (\mathbf{1},\mathbf{2},\mathbf{2})_{-+} \tag{5.6}$$

$$\text{D2-NS5} \;:\; (\mathbf{2},\mathbf{1},\mathbf{2})_{+-} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_{-+} \tag{5.7}$$

$$\text{D2-D8-O8} \;:\; (\mathbf{2},\mathbf{1},\mathbf{2})_{-+} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_{-+}\;. \tag{5.8}$$

(5.6) yields the 2d $(0,4)$ SUSY, which I write as $Q_-^{\dot\alpha A}$. (5.7) yields 2d $(4,4)$ SUSY $Q_+^{\alpha A}$, $Q_-^{\dot\alpha A}$. (5.8) yields 2d $(0,8)$ SUSY $Q_-^{\alpha A}$, $Q_-^{\dot\alpha A}$. $\pm$ subscripts of Q denote 2d left/right spinors.

Now study the field contents of the 2d $\mathcal{N}=(0,4)$ gauge theory. This is obtained by starting from the 3d field theory living on D2-branes, ogether with the boundary degrees at $x^2 = 0, L$, and then taking a 2d limit when $E \ll L^{-1}$. The 3d fields living in the region $0 < x^2 < L$ are

$$\begin{aligned} \text{D2-D2} \;:\;\; & A_\mu \;\; (\mu = 0,1,2)\;;\;\; X^I \sim \varphi^{\alpha\dot\beta} \;\; (I = 3,4,5,6)\;;\;\; X^{I'} \;\; (I' = 7,8,9) \\ & \lambda \;\; (\text{16 component spinor satisfying } \; \Gamma^{012}\lambda = \lambda)\;. \end{aligned} \tag{5.9}$$

The D2-D2 fields are in adjoint representation of $U(n)$. One also finds boundary degrees at the brane intersections. At the intersection of D2-D8, open strings provide 2d $(0,8)$ Fermi multiplet fields which I write as Ψ_l ($l = 1, \cdots, 16$). They will be in the bi-fundamental representation of $O(n) \times SO(16)$ (after introducing the orientifold boundary condition on D2-D8). Ψ_l are Majorana-Weyl spinors.

Consider the boundary conditions of the 3d fields. At the two ends $x^2 = 0, L$, there are separate boundary conditions. As the goal is to obtain the 2d theory, I shall only keep the zero modes of the 3d fields along the x^2 direction. This means that I will keep the bosonic fields satisfying the Neumann boundary conditions on both ends, and the fermionic fields which survive suitable projection conditions at both ends. The SUSY conditions for D2-D2 fields at $x^2 = 0, L$ take the form of

$$(x^2 \text{ component of supercurrent}) = \text{tr}\left(\bar{\epsilon}\Gamma^{MN}F_{MN}\Gamma_2\lambda\right) = 0 \tag{5.10}$$

in the 10d notation with $M, N = 0, \cdots, 9$. ϵ is chosen to be $(4,4)$ on D2-NS5 ($x^2 = L$), and $(0,8)$ on D2-D8 ($x^2 = 0$). One can follow the strategy of [91] to obtain the SUSY boundary conditions. With given SUSY ϵ, one imposes suitable bosonic boundary condition, depending on which brane D2's are ending on. Then the condition (5.10) would determine the boundary condition for the fermions λ.

Let me study the D2-NS5 boundary condition first. Choosing the supercharge Q to be in $(\mathbf{2},\mathbf{1},\mathbf{2})_{+-} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_{-+}$, $\bar{\epsilon} = \epsilon^\dagger\Gamma^0$ should be chosen to have nonzero overlap with it. The D2-D2 fermion λ has a definite Γ^{012} eigenvalue (same as that of the supercharges), so is in

$$(\mathbf{2},\mathbf{1},\mathbf{2})_{+-} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_{+-} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_{-+} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_{-+} \ . \tag{5.11}$$

Start from the boundary conditions for the bosonic fields that I know for D2-NS5:

$$F_{\mu 2} = 0 \ , \quad D_2 X^I = 0 \ , \quad X^{I'} = 0 \tag{5.12}$$

with $\mu = 0, 1$, $I = 3, 4, 5, 6$, $I' = 7, 8, 9$. This gives the following constraints on λ:

$$0 = \bar{\epsilon}\lambda = \bar{\epsilon}\Gamma^{\mu 2I}\lambda = \bar{\epsilon}\Gamma^{IJ}\Gamma^2\lambda = \bar{\epsilon}\Gamma^{I'}\lambda \ . \tag{5.13}$$

This requires λ to be in

$$(\mathbf{2},\mathbf{1},\mathbf{2})_{-+}\oplus(\mathbf{1},\mathbf{2},\mathbf{2})_{+-}\ , \tag{5.14}$$

namely, with a right mover $\lambda_-^{\alpha A}$ and a left mover $\lambda_+^{\dot\alpha A}$. (The former will belong to a 2d $(0,4)$ hypermultiplet and the latter will belong to a 2d $(0,4)$ vector multiplet.)

Now consider the D2-D8-O8 boundary conditions. The effect of having 8 D8-branes is simply adding $(0,8)$ Fermi multiplet fields as explained above. So I will focus on the effect of the O8-plane. Following [91], consider the covering space of $x^2>0$ and consider the 3d SYM on $\mathbb{R}^{2,1}$. The reflection $x^2\to-x^2$ of space is accompanied by an outer automorphism τ acting on $G=U(n)$ gauge group. The algebra g of G decomposes into $g^{(+)}\oplus g^{(-)}$, where τ acts on $g^{(\pm)}$ as ±1. In this case, $g^{(+)}$ is the algebra of $O(n)\subset U(n)$, and $g^{(-)}$ forms a rank 2 symmetric representation of $O(n)$. So any adjoint-valued field Φ can be written as $\Phi=\Phi^{(+)}+\Phi^{(-)}$. The reflection is further accompanied by $X^I\to-X^I$ for $I=3,\cdots,9$. The fields are required to be invariant under the net reflection:

$$A_\mu(x^2)=A_\mu^\tau(-x^2),\ A_2(x^2)=-A_2^\tau(-x^2),\ X_I(x^2)=-X_I^\tau(-x^2) \tag{5.15}$$

where $\Phi^\tau=\tau\Phi\tau^{-1}$, $\mu=0,1$ and $I=3,\cdots,9$. So at the fixed plane $x^2=0$, the boundary condition is given by

$$F_{\mu2}^{(+)}=0\ ,\quad F_{\mu\nu}^{(-)}=0\ ,\quad D_2X_I^{(-)}=0\ ,\quad X_I^{(+)}=0\qquad(I=3,\cdots,9)\ . \tag{5.16}$$

One can again find the fermionic boundary conditions from (5.10). This requires

$$0=\bar\epsilon\lambda^{(+)}=\bar\epsilon\Gamma^I\lambda^{(+)}=\bar\epsilon\Gamma^{IJ9}\lambda^{(+)}\ ,\quad 0=\bar\epsilon\Gamma^\mu\lambda^{(-)}=\bar\epsilon\Gamma^{\mu I9}\lambda^{(-)} \tag{5.17}$$

with $\mu=0,1$ and $I,J=3,\cdots,9$. ϵ is chosen so that $\bar\epsilon$ has nonzero overlap with the $(0,8)$ SUSY (5.8), given by $(\mathbf{2},\mathbf{1},\mathbf{2})_{-+}\oplus(\mathbf{1},\mathbf{2},\mathbf{2})_{-+}$. Solving these constraints, the $O(n)$ adjoint fermion $\lambda^{(+)}$ and the $O(n)$ symmetric fermion $\lambda^{(-)}$ are required to be in

$$\lambda^{(+)}\ :\ (\mathbf{2},\mathbf{1},\mathbf{2})_{+-}\oplus(\mathbf{1},\mathbf{2},\mathbf{2})_{+-}$$

$$\lambda^{(-)} \quad : \quad (\mathbf{2},\mathbf{1},\mathbf{2})_{-+} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_{-+} \ . \tag{5.18}$$

Combine the D2-NS5 and D2-O8 boundary conditions to read off the 2d field contents. For bosons, requiring (5.12) and (5.16) yields the following 2d degrees:

$$A_\mu^{(+)}\ , \quad X_I^{(-)} \sim \varphi_{\alpha\dot\beta} \quad (I = 3,4,5,6)\ . \tag{5.19}$$

Although $F_{\mu 2}^{(+)}$ is not required to be zero in the above consideration, one can make an x^2-dependent gauge transformation to set $A_2^{(+)} = 0$. For fermions, requiring (5.14) and (5.18) together, one finds that $\lambda_-^{\alpha A} \sim (\mathbf{2},\mathbf{1},\mathbf{2})_{-+}$ is in the symmetric representation of $O(n)$, while $\lambda_+^{\dot\alpha A} \sim (\mathbf{1},\mathbf{2},\mathbf{2})_{+-}$ is in the adjoint (i.e. antisymmetric) representation. So from the D2-D2 modes, one obtains the $(0,4)$ vector multiplet A_μ, $\lambda_+^{\dot\alpha A}$ of $O(n)$, and also a $(0,4)$ hypermultiplet $\varphi_{\alpha\dot\beta}$, $\lambda_-^{\alpha A}$ in the symmetric representation of $O(n)$.

So to summarize, one obtains the following 2d $\mathcal{N} = (0,4)$ field contents:

$$\begin{aligned} \text{vector} &\ :\ O(n) \text{ antisymmetric} \quad (A_\mu,\ \lambda_+^{\dot\alpha A}) \\ \text{hyper} &\ :\ O(n) \text{ symmetric} \quad (\varphi_{\alpha\dot\beta},\ \lambda_-^{\alpha A}) \\ \text{Fermi} &\ :\ O(n)\times SO(16) \text{ bifundamental} \quad \Psi_l\ . \end{aligned} \tag{5.20}$$

Figure 5.4 shows the quiver diagram of this gauge theory. One can check the $SO(n)$ gauge anomaly cancellation of this chiral matter content. Note that there are no twisted hypermultiplets, whose scalars form doublets of $SU(2)_I$ and fermions form doublets of $SU(2)_R$.

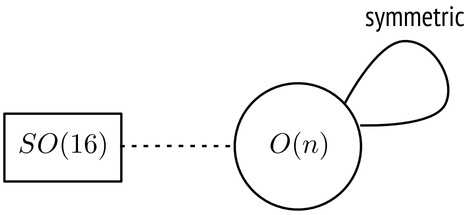

Figure 5.4: Quiver diagram of the 2d $\mathcal{N} = (0,4)$ gauge theory for E-strings.

Let me also explain how to get the full Lagrangian of this system. Viewing this as a special case of $\mathcal{N}=(0,2)$ supersymmetric system, it suffices to determine the two holomorphic functions $E_\Psi(\Phi_i)$, $J^\Psi(\Phi_i)$ for each Fermi multiplet Ψ, depending on the $(0,2)$ chiral multiplet fields Φ_i. I choose $Q \equiv Q_1^{\dot{1}}$ and $Q^\dagger$ as the $(0,2)$ subset, for further explanations. To have $(0,4)$ SUSY, the E, J functions for the $(0,2)$ Fermi multiplet $\Theta \equiv (\lambda_+^{\dot{1}2}, \lambda_+^{\dot{2}1})$ in the $(0,4)$ vector multiplet are constrained [92] as

$$J_\Theta = \varphi\tilde{\varphi} - \tilde{\varphi}\varphi \ , \quad E_\Theta = 0 \ , \tag{5.21}$$

where $\varphi \equiv \varphi_{1\dot{1}}$, $\tilde{\varphi} \equiv \varphi_{2\dot{1}}$ are $(0,2)$ chiral multiplet scalars which transform under $Q \equiv Q_1^{\dot{1}}$. Note that, if the $(0,4)$ theory has both hypermultiplets and twisted hypermultiplets, the full interaction has to be more complicated [92]. Without twisted hypermultiplets in our system, (5.21) provides the full interactions associated with Θ. This induces a bosonic potential of the form $|J_\Theta|^2$, as well as the Yukawa interaction. Extra Fermi multiplets in the $(0,2)$ viewpoint are Ψ_l from D2-D8-O8 modes, so one should also determine their E, J. E_{Ψ_l}, J^{Ψ_l} are simply zero, from $SO(16)$ symmetry. With all the E, J functions determined, the supersymmetric action can be written down if $E^a J_a = 0$, where the index a runs over all $(0,2)$ Fermi multiplets. This condition is clearly met. With these data, the full action can be written down in a standard manner: see, for instance, [93, 92]. In the current case, the bosonic potential consists of $|J_\Theta|^2$ and the usual D-term potential, making the D-term potential from the '$SU(2)_R$ triplet' of D-terms. The classical Higgs branch moduli space, given by nonzero $\varphi, \tilde{\varphi}$, is real $4n$ dimensional. Semi-classically, these are the positions of n E-strings.

One can also compute the central charges of the IR CFT from the UV gauge theory. The gauge theory in principle could flow to more than one decoupled CFTs in IR, which will be explained shortly. Once one knows the correct superconformal R-symmetry of the IR SCFT, the (right-moving) central charge of the IR CFT can be computed in UV by the anomaly of the superconformal R-symmetry. I closely follow [94, 93, 92], which use the $(0,2)$ superconformal R-symmetry to determine the central charges.

In the $(0,4)$ system, at least there will be one CFT in IR, which admits a semi-classical description when $\varphi^{\alpha\dot{\beta}}$ scalars are large. This is the CFT associated with the classical Higgs branch [95]. In this CFT, the superconformal R-symmetry can only come from $SU(2)_I$ in the UV theory. This is because the right sector contains the $O(n)$ symmetric scalar $\varphi_{\alpha\dot{\beta}}$, and the superconformal R-symmetry should not act on it [95]. Following [92], let me choose the supercharge $Q \equiv Q^{\dot{1}2}$ and use the $(0,2)$ superconformal symmetry to determine the central charge. The right-moving central charge c_R is given by

$$c_R = 3\mathrm{Tr}(\gamma^3 R^2) \ , \tag{5.22}$$

with $\gamma^3 = \pm 1$ for the right/left moving fermions, respectively, and the trace acquires an extra $\frac{1}{2}$ factor for real fermions. The $(0,2)$ R-charge R is normalized so that $R[Q] = -1$. In the Higgs branch CFT, this should be proportional to the Cartan of $SU(2)_I$, so I set $R = 2J_I$. Collecting the contribution from $O(n)$ symmetric $\lambda^{\alpha A}$ in the right sector and adjoint $\lambda^{\dot{\alpha}A}$ in the left sector, one obtains

$$c_R = 3 \times \frac{1}{2} \times \frac{n^2+n}{2} \times (4 \times 1^2) - 3 \times \frac{1}{2} \times \frac{n(n-1)}{2} \times (4 \times 1^2) = 6n \ . \tag{5.23}$$

The left moving central charge c_L is determined from c_R by the gravitational anomaly [93]:

$$c_R - c_L = \mathrm{Tr}(\gamma^3) = \frac{1}{2} \times 4\frac{n^2+n}{2} - \frac{1}{2} \times 4\frac{n^2-n}{2} - \frac{1}{2} \times 16n = -6n \ \rightarrow \quad c_L = 12n \ . \tag{5.24}$$

$c_L = 12n$ is consistent with the result obtained in [96] (where $c_L = 12n - 4$ was found after eliminating 4 from the decoupled center-of-mass degrees.) One can semiclassically understand some of these results, by studying the region with large value of the Higgs scalar $\varphi^{\alpha\dot{\beta}}$. $c_R = 6n$ comes from the n pairs of 4 scalars and 4 fermions for n E-strings. As for $c_L = 12n$, the $4n$ scalars in the left moving sector accounts for $4n$, and the $16n$ real fermions Ψ_l accounts for $8n$. For $n = 1$, I know that the last 8 is given by the $G = E_8$ current algebra at level $k = 1$ (with dual Coxeter number $c_2 = 30$) [14, 13], whose central charge is indeed $\frac{k|G|}{k+c_2} = \frac{248}{1+30} = 8$.

One may also try to explore if the UV theory could flow to more than one decoupled conformal field theories in the IR. For instance, it happens in the $\mathcal{N} = (4,4)$ SCFT with both Higgs and Coulomb branches [95]. Another type of example is a recently analyzed $(0,4)$ gauge theory for the D1-D5-D5' system [92]. This theory was proposed to have a 'localized CFT' whose ground state wavefunction is localized at the intersection of the two Higgs branches, which was suggested to be the holographic dual of type IIB strings on $AdS_3 \times S^3 \times S^3 \times S^1$. Morally, the last localized CFT should be coming from the D1-branes forming threshold bounds with D5 and D5' branes. As the current system will also exhibit threshold bounds of E-strings, it would be interesting to know if similar decoupled 'localized CFTs' exist like those of [92], other than the 'Higgs branch CFT' that I explained in the previous paragraph. If there exist localized CFTs with all E-strings fully bound, they will not have a regime which allows a semi-classical description (large $\varphi^{\alpha\dot{\beta}}$). So the argument of [95] does not apply, and both $SU(2)_R$ and $SU(2)_I$ can participate in the superconformal R-symmetry [92]. Following [92], I will first determine the correct superconformal R-symmetry in this case (if it exists), again within the context of $(0,2)$ superconformal symmetry as in the previous paragraph. Take the two Cartans J_R, J_I of $SU(2)_R$, $SU(2)_I$, and consider their linear combination $R = -aJ_R + bJ_I$ which is taken as a trial $U(1)$ R-symmetry. $R[Q] = -1$ demands $a + b = 2$. The superconformal R-symmetry should have no mixed anomaly with flavor charges, i.e. all global symmetries commuting with $Q, Q^\dagger$ [94]. In this case, one only needs to consider the mixing with $V \equiv J_R + J_I$, chosen in $SU(2)_R \times SU(2)_I$ with $V[Q] = 0$. By demanding $\mathrm{Tr}(\gamma^3 VR) = 0$, one finds $na + 2b = 0$. While computing this, one should exclude the decoupled center-of-mass modes for the n E-strings, provided by $\mathrm{tr}(\varphi_{\alpha\dot{\beta}})$. These decoupled modes always live in a 'Higgs branch' in which the R-symmetry is $SU(2)_I$. So if there is no accidental IR symmetry, the R-symmetry is given by

$$a + b = 2 \ , \quad na + 2b = 0 \ . \tag{5.25}$$

Note that these equations do not have solutions if $n = 2$, for two E-strings. This could be implying the absence of the localized CFT, decoupled to the Higgs branch CFT. In other cases, one finds $a = -\frac{4}{n-2}$, $b = \frac{2n}{n-2}$. The right central charge is again given by (5.22) with the new R, again without including the contributions from the center-of-mass modes. It is given by $c_R = \frac{6n(n-1)}{n-2}$. The left central charge is given by $c_L = c_R + 6n$. As emphasized, this result could be meaningful only at $n \neq 2$. When $n = 1$, one finds $c_R = 0$, which is consistent with the absence of the extra localized CFT for a single E-string. For $n \geq 3$, it will be interesting to know whether such CFTs actually exist (when consistent with the c-theorem).

For $n = 2$, unless there are accidental IR symmetries, this study implies that there are no more decoupled CFTs. If this is true, one should be able to understand the elliptic genus of the 2 E-strings solely from the Higgs branch CFT. In the regime with large $\varphi^{\alpha\dot{\beta}}$, one can employ a semi-classical approximation to study the Higgs branch CFT. This requires us to study a free QFT, with $4n = 8$ bosonic fields given by eigenvalues $\varphi_i^{\alpha\dot{\beta}}$ and $16n = 32$ fermions Ψ_{il} $(i = 1, 2)$. The spectrum of this QFT is subject to a gauge singlet condition for a discrete D_4 subgroup of $O(2)$ gauge symmetry, surviving in the Higgs branch. The two generators of D_4 are given by

$$\begin{aligned} x &: \quad (\varphi_1, \varphi_2) \to (\varphi_2, \varphi_1) \ , \quad (\Psi_{1l}, \Psi_{2l}) \to (-\Psi_{2l}, \Psi_{1l}) \\ y &: \quad (\varphi_1, \varphi_2) \to (\varphi_1, \varphi_2) \ , \quad (\Psi_{1l}, \Psi_{2l}) \to (\Psi_{1l}, -\Psi_{2l}) \ , \end{aligned} \tag{5.26}$$

which satisfy $x^4 = 1$, $y^2 = 1$, $yxy^{-1} = x^3$ and define D_4. One can show that the index for the gauge invariant states, after adding twisted sectors, is simply given by the Hecke transformation of the single E-string index. This does not agree with the correct two E-string index [97], which I shall compute in Section 5.2.2. This implies that the Higgs branch CFT for two E-strings should be more nontrivial than what one sees in the semi-classical regime. It will be interesting to understand this Higgs branch CFT better.

5.2.2 E-string elliptic genera from 2d gauge theories

Consider the elliptic genus of the 2d $(0,4)$ $O(n)$ gauge theory, constructed in the previous section. Pick a $(0,2)$ SUSY and define the elliptic genus as follows:

$$Z_n(q,\epsilon_{1,2},m_l) = \mathrm{Tr}_{\mathrm{RR}}\left[(-1)^F q^{H_L}\bar{q}^{H_R} e^{2\pi i\epsilon_1(J_1+J_I)} e^{2\pi i\epsilon_2(J_2+J_I)} \prod_{l=1}^{8} e^{2\pi i m_l F_l}\right] . \tag{5.27}$$

J_1, J_2 are the Cartans of $SO(4)_{3456} = SU(2)_L \times SU(2)_R$ which rotate the 34 and 56 orthogonal 2-planes, and J_I is the Cartan of $SU(2)_{789}$. F_l are the Cartans of $SO(16)$, which one expects to be the Cartans of enhanced E_8 in IR. Note that $H_R \sim \{Q, Q^\dagger\}$ with $Q = Q^{\dot{1}}_1$ and $Q^\dagger = -Q^{\dot{2}}_2$, and the remaining factors inside the trace commute with $Q, Q^\dagger$. Note also that, the 2d gauge theory itself has a noncompact Higgs branch spanned by $\varphi^{\alpha\dot{\beta}}$. They are given nonzero masses by turning on ϵ_1, ϵ_2, so that the path integral for this index does not have any noncompact zero modes. The interpretation of the zero modes from $\varphi^{\alpha\dot{\beta}}$ at $\epsilon_1, \epsilon_2 = 0$ is clearly the multi-particle positions, so by keeping nonzero $\epsilon_{1,2}$ one is computing the multi-particle index, as usual. The single particle spectrum can be extracted from the multi-particle index.

The general form of the index (5.27) for $\mathcal{N} = (0,2)$ gauge theories was studied in [60, 61], by computing the path integral of the gauge theory on T^2. There appear compact zero modes from the path integral, coming from the flat connections on T^2. [60, 61] first fix the flat connections, integrate over the nonzero modes, and then integrate over the flat connections to obtain their final expression for the index, as I have done in Section 3.4 to obtain the quantum mechanical indices.

Let me first explain the possible flat connections of our $O(n)$ gauge theories on T^2. These are given by two commuting $O(n)$ group elements U_1, U_2, the Wilson lines along the temporal and spatial circles of T^2. Note that $O(n)$ is a disconnected group so that U_1 and U_2 can each have two disconnected sectors, depending on whether their determinants are 1 or -1. The general $O(n)$ holonomies on T^2, up to conjugation, can be derived using a D-brane picture [98]. The $O(n)$ flat connections are the zero energy configurations of the n D2-branes and an O2-plane wrapping

T^2. By T-dualizing twice along the torus, one obtains n D0-branes moving along the covering space T^2 of $T^2/\mathbb{Z}_2$ orientifold. The flat connections T-dualize to the positions of D0-branes on $T^2/\mathbb{Z}_2$. There are four O0-plane fixed points on the covering space T^2. It suffices for us to classify all possible positions of D0-branes. When two D0-branes on the covering space are paired as $\mathbb{Z}_2$ images of each other, they have one complex parameter u as their position. Some D0-branes can also be stuck at the $\mathbb{Z}_2$ fixed points without a pair: they are fractional branes on $T^2/\mathbb{Z}_2$, whose positions are freezed at the fixed points. So the classification of $O(n)$ flat connections reduces to classifying the possible fractional brane configurations.

When $n = 2p$ is even, one can first have all $2p$ D0-branes to make p pairs. In this branch, one finds p complex moduli u_i $(i = 1, \cdots, p)$. Another possibility is to form $p-1$ pairs to freely move, while having 2 fractional D-branes stuck at two of the 4 fixed points. Note that the two fractional branes have to be stuck at different fixed points: otherwise they can pair and leave the fixed point, being a special case of the first branch. There are 6 ways of choosing 2 fixed points among 4, so one obtains 6 more sectors. Finally, one finds a sector in which $p-2$ pairs freely move, while 4 fractional D-branes are stuck at 4 different fixed points (when $p \geq 2$). After T-dualizing, U_1, U_2 are exponentials of the D0-brane positions. The above 8 sectors are summarized by the following pairs of Wilson lines U_1, U_2, for $O(2p)$ with $p \geq 2$:

$$
\begin{aligned}
\text{(ee)} \quad &: \quad U_1 = \text{diag}(e^{iu_{1i}\sigma_2})_p \ , \quad U_2 = \text{diag}(e^{iu_{2i}\sigma_2})_p \ ; \\
&\quad\ \ U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, 1, -1, -1, 1)_{p-2} \ , \quad U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, 1, 1, -1, -1)_{p-2}; \\
\text{(eo)} \quad &: \quad U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, 1, 1)_{p-1} \ , \quad U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, 1, -1)_{p-1} \ ; \\
&\quad\ \ U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, -1, -1)_{p-1} \ , \quad U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, 1, -1)_{p-1}; \\
\text{(oe)} \quad &: \quad U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, 1, -1)_{p-1} \ , \quad U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, 1, 1)_{p-1} \ ; \\
&\quad\ \ U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, 1, -1)_{p-1} \ , \quad U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, -1, -1)_{p-1}; \\
\text{(oo)} \quad &: \quad U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, 1, -1)_{p-1} \ , \quad U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, 1, -1)_{p-1} \ ; \\
&\quad\ \ U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, 1, -1)_{p-1} \ , \quad U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, -1, 1)_{p-1} \ .
\end{aligned}
\tag{5.28}
$$

(ee), (eo), (oe), (oo) are for U_1, U_2 in the even or odd elements of $O(n)$. The symbol 'diag' denotes a block-diagonalized matrix. The subscripts are the number of independent complex parameters. The parameters live on $u_i = u_{1i} + \tau u_{2i} \in \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, where τ is related to our fugacity q by $q = e^{2\pi i\tau}$. For odd $n = 2p+1$ with $n \geq 3$, one can make a similar analysis. There are 4 cases in which one has 1 fractional brane stuck at one of the 4 fixed points, and 4 more cases (when $p \geq 1$) in which 3 fractional branes are stuck at three of the 4 fixed points. So one obtains the following 8 sectors, for $p \geq 1$:

$$\begin{aligned}
\text{(ee)} \;\; &: \;\; U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, 1)_p \;, \;\; U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, 1)_p \;; \\
& \quad\;\; U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, -1, -1, 1)_{p-1} \;, \;\; U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, 1, -1, -1)_{p-1}; \\
\text{(eo)} \;\; &: \;\; U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, 1)_p \;, \;\; U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, -1)_p \;; \\
& \quad\;\; U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, -1, -1, 1)_{p-1} \;, \;\; U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, 1, -1, 1)_{p-1}; \\
\text{(oe)} \;\; &: \;\; U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, -1)_p \;, \;\; U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, 1)_p \;; \\
& \quad\;\; U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, 1, -1, 1)_{p-1} \;, \;\; U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, -1, -1, 1)_{p-1}; \\
\text{(oo)} \;\; &: \;\; U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, -1)_p \;, \;\; U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, -1)_p \;; \\
& \quad\;\; U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, 1, 1, -1)_{p-1} \;, \;\; U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, 1, -1, 1)_{p-1} \;.
\end{aligned} \tag{5.29}$$

There are two exceptional cases. For $O(1)$, the four sectors in (5.29) with rank $p-1$ are absent. So one only has four rank 0 sectors

$$(U_1, U_2) = (1,1),\; (1,-1),\; (-1,1),\; (-1,-1) \;. \tag{5.30}$$

For $O(2)$, the second sector in (5.28) with rank $p-2$ is absent. So there are seven sectors

$$(U_1, U_2) = (e^{iu_1\sigma_2}, e^{iu_2\sigma_2}),\; (1,\sigma_3),\; (-1,\sigma_3),\; (\sigma_3,1),\; (\sigma_3,-1),\; (\sigma_3,\sigma_3),\; (\sigma_3,-\sigma_3) \;. \tag{5.31}$$

The Wilson lines can be more conveniently labeled by their exponents, which I call $u = (u_1, \cdots, u_n)$ for $O(n)$. In the 2×2 blocks $e^{iu_{1i}\sigma_2}, e^{iu_{2i}\sigma_2}$ with continuous elements, the associated two u parameters are given by the two eigenvalues $\pm(u_{1i} + \tau u_{2i})$. In the blocks with discrete numbers, let me assign $u_i = 0$ for each eigenvalue

$(1,1)$ of the Wilson line U_1, U_2, $u_i = \frac{1}{2}$ for each eigenvalue $(-1,1)$, $u_i = \frac{\tau}{2}$ for $(1,-1)$, and $u_i = \frac{1+\tau}{2}$ for $(-1,-1)$. For the above 8 sectors, one thus obtains

$$\begin{aligned}
(\text{ee}) \quad &: \quad u = (\pm u_1, \cdots, \pm u_p) \ ; \quad u = (\pm u_1, \cdots, \pm u_{p-2}, 0, \frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}) \\
(\text{eo}) \quad &: \quad u = (\pm u_1, \cdots, \pm u_{p-1}, 0, \frac{\tau}{2}) \ ; \quad u = (\pm u_1, \cdots, \pm u_{p-1}, \frac{1}{2}, \frac{1+\tau}{2}) \\
(\text{oe}) \quad &: \quad u = (\pm u_1, \cdots, \pm u_{p-1}, 0, \frac{1}{2}) \ ; \quad u = (\pm u_1, \cdots, \pm u_{p-1}, 0, \frac{\tau}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}) \\
(\text{oo}) \quad &: \quad u = (\pm u_1, \cdots, \pm u_{p-1}, 0, \frac{1+\tau}{2}) \ ; \quad u = (\pm u_1, \cdots, \pm u_{p-1}, \frac{\tau}{2}, \frac{1}{2})
\end{aligned} \tag{5.32}$$

for $O(2p)$, and

$$\begin{aligned}
(\text{ee}) \quad &: \quad u = (\pm u_1, \cdots, \pm u_p, 0) \ ; \quad u = (\pm u_1, \cdots, \pm u_{p-1}, \frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}) \\
(\text{eo}) \quad &: \quad u = (\pm u_1, \cdots, \pm u_p, \frac{\tau}{2}) \ ; \quad u = (\pm u_1, \cdots, \pm u_{p-1}, \frac{1}{2}, \frac{1+\tau}{2}, 0) \\
(\text{oe}) \quad &: \quad u = (\pm u_1, \cdots, \pm u_p, \frac{1}{2}) \ ; \quad u = (\pm u_1, \cdots, \pm u_{p-1}, \frac{\tau}{2}, \frac{1+\tau}{2}, 0) \\
(\text{oo}) \quad &: \quad u = (\pm u_1, \cdots, \pm u_p, \frac{1+\tau}{2}) \ ; \quad u = (\pm u_1, \cdots, \pm u_{p-1}, 0, \frac{\tau}{2}, \frac{1}{2})
\end{aligned} \tag{5.33}$$

for $O(2p+1)$. These u couple minimally to the matters in the fundamental representation. The parameters coupling to fields in a different representation of $SO(n)$ are given by $\rho(u)$, where ρ runs over the weights of the representation.

With the Wilson line backgrounds identified, let me study the subgroup of $O(n)$ gauge symmetry which acts within the U_1, U_2 specified above. This is the 'Weyl group,' defined in each disconnected sector of (U_1, U_2). When U_1, U_2 are given by r 2×2 blocks and an $s \times s$ diagonal matrix with ± 1 eigenvalues (with $2r + s = n$ and $s \leq 4$), the Weyl group is given by

$$[\text{Weyl group of } O(2r)] \times [O(s) \text{ elements commuting with the } s \times s \text{ block}] \ . \tag{5.34}$$

The former part has order $2^r r!$, and the latter has order 2^s coming from $\text{diag}_s(\pm 1, \cdots, \pm 1)$. So the order of the Weyl group $W(O(n))_s$, with given U_1, U_2, is given by

$$\begin{aligned}
&|W(O(2p))_0| = 2^p p! \ , \quad |W(O(2p))_2| = 2^{p+1}(p-1)! \ , \quad |W(O(2p))_4| = 2^{p+2}(p-2)! \\
&|W(O(2p+1))_1| = 2^{p+1} p! \ , \quad |W(O(2p+1)_3)| = 2^{p+2}(p-1)! \ ,
\end{aligned} \tag{5.35}$$

where the subscript denotes the value of s for U_1, U_2.

In the above background, the Gaussian path integral of non-zero modes yields $Z_{\text{1-loop}}$, which is the product of the following 1-loop determinants for various supermultiplets [61]:[1]

$$
\begin{aligned}
Z_{\text{sym. hyper}} &= \prod_{\rho\in\text{sym}} \frac{i\eta(q)}{\theta_1(q,\epsilon_1+\rho(u))}\cdot\frac{i\eta(q)}{\theta_1(q,\epsilon_2+\rho(u))} \\
Z_{SO(16)\text{ Fermi}} &= \prod_{\rho\in\text{fund}}\prod_{l=1}^{8}\frac{\theta_1(q,m_l+\rho(u))}{i\eta(q)} \\
Z_{\text{vector}} &= \prod_{i=1}^{r}\left(\frac{2\pi\eta^2 du_i}{i}\cdot\frac{\theta_1(\epsilon_1+\epsilon_2)}{i\eta}\right)\cdot\prod_{\alpha\in\text{root}}\frac{\theta_1(\alpha(u))\theta_1(\epsilon_1+\epsilon_2+\alpha(u))}{i^2\eta^2}\ .
\end{aligned}
\tag{5.36}
$$

Whenever I omit the modular parameters of the theta functions, it is understood as τ. The 'rank' r is the number of continuous complex parameters in U_1, U_2. α runs over the roots of $SO(n)$. Multiplying all these factors, one finally has to integrate over the continuous parameters in u and then sum over distinct sectors of flat connections. The result is

$$
\sum_a \frac{1}{|W_a|}\cdot\frac{1}{(2\pi i)^r}\oint Z^{(a)}_{\text{1-loop}}\ ,\quad Z^{(a)}_{\text{1-loop}}\equiv Z^{(a)}_{\text{vector}}Z^{(a)}_{\text{sym. hyper}}Z^{(a)}_{SO(16)\text{ Fermi}}\ , \tag{5.37}
$$

a labels the disconnected sectors of the flat connection U_1, U_2. The integral is a suitable 'contour integral' over the continuous parameters u, to be explained shortly. W_a is the Weyl group with given U_1, U_2 explained above.

Before proceeding, let me comment on the periodicity of (5.36) in u. Each u_i (for $i = 1, \cdots, p$) lives on $T^2/\mathbb{Z}_2$, due to large gauge transformations on T^2, so is a periodic variable $u_i \sim u_i+1 \sim u_i+\tau$. However, since $\theta_1(u,\tau)$ is only a quasi-periodic

[1] One difference from [61] is that there is a factor i in the denominator of the contribution $\frac{\theta_1(q,z)}{i\eta(q)}$ from each Fermi multiplet. Of course this only affects the overall sign of the index, which is ambiguous in 2d without knowing the spin-statistics relation inherited from higher dimensional physics. This choice is compatible with the physics of circle compactified 6d CFT, by comparing with some known results. Collecting all the factors of i in $Z_{\text{1-loop}}$, one obtains $(-1)^n$.

function,

$$\theta_1(z+1) = -\theta_1(z),\ \theta_1(z+\tau) = -q^{-1/2}y^{-1}\theta_1(z),\ \theta_1(z+1+\tau) = q^{-1/2}y^{-1}\theta_1(z)\ , \tag{5.38}$$

each measure in (5.36) is not invariant under these shifts. The failure of periodicity is related to the gauge anomaly of the chiral theory. The extra factors spoiling the periodicity cancel in the combination (5.37), due to the anomaly cancelation of our gauge theory.

Another subtlety is the determinant of the real scalars and Majorana fermions. Each real scalar or fermion contributes to a square-root of θ_1 factor. Equivalently, each charge conjugate pair of fermion modes contributes a factor of $\frac{\theta_1(z)}{i\eta}$, while such a pair of bosons contributes $\frac{i\eta}{\theta_1(z)}$ in (5.36). In particular, on these modes, the discrete shifts on the holonomy (5.32), (5.33) given by $u_i = \frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}$ has to be understood with some care. When such a shift is made in the argument of θ_1 coming from a pair of real fields, one should understand it as "$\theta_1(z+u_i)$" $\sim \sqrt{\theta_1(z+u_i)\theta_1(z-u_i)}$. Having this in mind, and applying

$$\theta_1(z+\tfrac{1}{2}) = \theta_2(z)\ ,\ \ \theta_1(z+\tfrac{\tau}{2}) = iq^{-1/8}y^{-1/2}\theta_4(z)\ ,\ \ \theta_1(z+\tfrac{1+\tau}{2}) = q^{-1/8}y^{-1/2}\theta_3(z)\ , \tag{5.39}$$

one can replace $\theta_1(z+\frac{1}{2})$, $\theta_1(z+\frac{\tau+1}{2})$, $\theta_1(z+\frac{\tau}{2})$ by $\theta_2(z)$, $\theta_3(z)$, $\theta_4(z)$, respectively, apart from the extra factors appearing in (5.39). These extra factors in (5.37) again cancel to 1. So θ_1 with half-period shifts can be replaced by $\theta_2, \theta_3, \theta_4$.

Finally let me explain the meaning of the 'contour integral' in (5.37), following [60, 61]. The 'contour integral' is defined by providing a prescription for the residue sum which replaces the integral, whenever one encounters a pole on the parameter space of (U_1, U_2). The prescription is derived in [61], using the so-called Jeffrey-Kirwan residues. At each pole $u = u_*$ on the r complex dimensional u space, there are r or more hyperplanes of the form $\rho_i(u) + z_i = 0$ (mod $\mathbb{Z} + \tau\mathbb{Z}$) which passes through it, where $i = 1, \cdots, d$ ($\geq r$). z_i are linear combinations of the chemical potentials so that $\theta_1(\rho_i(u)+z_i)$ appear in the denominator of $Z_{\text{1-loop}}$. In the current problem, z_i are either ϵ_1 or ϵ_2. When exactly r hyperplanes intersect at a point

$u = u_*$ (mod $\mathbb{Z} + \tau\mathbb{Z}$), this pole is called non-degenerate. When $d > r$, the pole is called degenerate.

Before explaining the Jeffrey-Kirwan residues (or JK-Res) of the integrand at $u = u_*$, note that the results of [61] apply when the pole at u_* is 'projective.' The pole is called projective when all the weight vectors ρ_i associated with the hyperplanes meeting at $u = u_*$ are contained in a half space. Namely, the projective condition requires that there is a vector v in the Cartan $\mathfrak{h}$ so that $\rho_i(v) > 0$. Note that all non-degenerate poles are projective. In this problem, even for degenerate poles, one can generally show that all poles should be projective, thus allowing one to use the results of [61]. To see this, first note that

$$\rho_i(u_*) = -z_i + m_i + n_i\tau \ , \tag{5.40}$$

for suitable integers m_i, n_i. Since ρ_i is chosen among the weight system of the $O(n)$ symmetric representation, it is either $\pm 2e_I$ or $\pm e_I \pm e_J$ with $I, J = 1, \cdots, \left[\frac{n}{2}\right]$. Thus, one can take all m_i, n_i to be either 0 or 1 to find all possible solutions for u_*, mod $\mathbb{Z} + \tau\mathbb{Z}$. Also, z_i is either ϵ_1 or ϵ_2 for all i's. Then, taking a solution $u_*(\epsilon_1, \epsilon_2)$ which depends on $\epsilon_{1,2}$, one deforms the solution to the regime in which ϵ_1, ϵ_2 are real and negative, taken to be $-\epsilon_{1,2} \gg 1$ and $-\epsilon_{1,2} \gg \mathrm{Re}(\tau)$. Then one finds that $\rho_i \cdot \mathrm{Re}(u_*) > 0$, fulfilling the projective condition. In fact, one can always provide this kind of argument on the projective nature of poles when the system has independent flavor symmetry for each matter supermultiplet. The $\mathcal{N} = (2, 2)$ or $(0, 2)$ models may exhibit non-projective poles if there are nonzero superpotentials so that flavor symmetries are restricted. In $\mathcal{N} = (0, 4)$ models, independent flavor symmetry can be found for each hypermultiplet. This is why it is easier to apply the results of [61] to $(0, 4)$ theories.

[61] finds that the integral in (5.37) is given by

$$\frac{1}{(2\pi i)^r} \oint Z^{(a)}_{\text{1-loop}} = \sum_{u_*} \text{JK-Res}_{u_*}(\mathbf{Q}_*, \eta) Z^{(a)}_{\text{1-loop}} \ , \tag{5.41}$$

where u_* runs over all the poles in the integrand. The JK-Res appearing in this expression is defined as follows. JK-Res is a linear functional which refers to an auxiliary vector η in the charge space, and also to the set of charge vectors $\mathbf{Q}_* = (Q_1, \cdots, Q_d)$ for the hyperplanes crossing u_*. The defining property of JK-$\text{Res}_{u_*}(\mathbf{Q}_*, \eta)$ is

$$\text{JK-Res}_{u_*} \frac{dQ_{j_1}(u) \wedge \cdots \wedge dQ_{j_r}(u)}{Q_{j_1}(u-u_*) \cdots Q_{j_r}(u-u_*)} = \begin{cases} \text{sign } \det(Q_{j_1}, \cdots, Q_{j_r}) & \text{if } \eta \in \text{Cone}(\mathbf{Q}_*) \\ 0 & \text{otherwise} \end{cases}, \tag{5.42}$$

or equivalently

$$\text{JK-Res}_{u_*} \frac{du_1 \wedge \cdots \wedge du_r}{Q_{j_1}(u-u_*) \cdots Q_{j_r}(u-u_*)} = \begin{cases} |\det(Q_{j_1}, \cdots, Q_{j_r})|^{-1} & \text{if } \eta \in \text{Cone}(\mathbf{Q}_*) \\ 0 & \text{otherwise} \end{cases}. \tag{5.43}$$

To make the condition $\eta \in \text{Cone}(Q_{j_1}, \cdots, Q_{j_r})$ unambiguous, one has to put η at a sufficiently generic point, as explained in [61]. These rules are giving a definite residue when the integrand takes the form of a 'simple pole.' Although this definition apparently overdetermines JK-Res due to many relations among the forms $\bigwedge_{i=1}^r \frac{dQ_{j_i}(u)}{Q_{j_i}(u)}$, it turns out to be consistent (see [61] and references therein). As one expands the integrand $Z_{\text{1-loop}}^{(a)}$ around $u = u_*$, one will encounter not just simple poles, but also multiple poles and less singular homogeneous expressions in $u - u_*$, multiplied by $du_1 \wedge \cdots \wedge du_r$. The JK-Res of the last two classes of monomials are all (naturally) zero: this is also consistent with the alternative 'constructive definition,' which expresses JK-Res as an iterated integral over a cycle. Using this definition to compute the integral is especially simple for non-degenerate poles, in which case one can directly read off a unique integral of the form (5.43) at a given $u = u_*$. The case with degenerate poles require some more work, but of course coming with a clear rule. The final result (5.41) is independent of η [61].

In the remaining part of this section, I will analyze the elliptic genera for $n = 1, 2, 3, 4$ E-strings in great detail. In Section 5.2.2, I will then illustrate the structure of the higher E-string indices. In particular, degenerate poles start to appear from $n \geq 6$. The residue evaluations are almost as simple as the non-degenerate poles for $n = 6, 7$, all coming from simple poles. Their residues are simply given by combinations of theta functions. For $n \geq 8$, there start to appear degenerate poles which are also multiple poles. Their residues are given by theta functions and their derivatives in the elliptic parameters.

One E-string

Consider the elliptic genus for the $O(1)$ theory. Since $O(1) = \mathbb{Z}_2$, there are four different flat connections $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$. The indices in the four sectors are given by

$$Z_{1(i)} = -\,[1]_{\text{vec}} \cdot \left[\frac{\eta^2}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)}\right]_{\text{sym hyper}} \cdot \left[\prod_{l=1}^{8} \frac{\theta_i(m_l)}{\eta}\right]_{\text{Fermi}} , \tag{5.44}$$

where $i = 1,2,3,4$ for the Wilson line $(1,1),(1,-1),(-1,-1),(-1,1)$, respectively. Combining all four contributions, and dividing by the Weyl group order $|W| = 2$ in each sector, the full index is given by

$$Z_1 = \sum_{i=1}^{4} \frac{Z_{1(i)}}{2} = -\frac{\Theta(q,m_l)}{\eta^6\theta_1(\epsilon_1)\theta_1(\epsilon_2)} , \tag{5.45}$$

where the E_8 theta function Θ is given by

$$\Theta(\tau,m_l) = \frac{1}{2}\sum_{n=1}^{4}\prod_{l=1}^{8}\theta_n(\tau,m_l) . \tag{5.46}$$

Physically, $\frac{Z_{1(1)}+Z_{1(2)}}{2}$ simply imposes the $O(1) = \mathbb{Z}_2$ singlet condition, while the remainder $\frac{Z_{1(3)}+Z_{1(4)}}{2}$ is the contribution from the twisted sector.

In [13], the above result was derived using topological strings and was explained using an effective free string theory calculus, in which the left moving sector consists of the E_8 current algebra at level 1 and the right moving sector consists of a $(0,4)$ supersymmetric string with target space $\mathbb{R}^4$. The four terms of $\Theta(\tau,m_i)$ can be understood as coming from the Ramond and Neveu-Schwarz sectors of the left-moving fermions, and then truncating the Hilbert space by a GSO projection. In the UV gauge theory calculus, the twisting and GSO projection is a consequence of the $O(1)$ gauge symmetry.

Since $\Theta(q, m_l)$ is given by the summation over the E_8 root lattice, Z_1 has a manifest E_8 symmetry, and is expanded as the sum of E_8 characters. This supports the IR enhancement $SO(16) \to E_8$ of global symmetry in the $(0,4)$ gauge theory.

Two E-strings

Consider the $O(2)$ theory. There are 7 sectors of $O(2)$ Wilson lines given by (5.31). One in the (ee) sector has a complex modulus, while the other six are all discrete. I named the sectors as follows, where (a_+, a_-) are the two eigenvalues of u in the discrete sectors which act on the fundamental representation [60]:

$$\begin{aligned}
(0) \equiv (\mathrm{ee}) \quad &: \quad (U_1, U_2) = (e^{iu_1\sigma_2}, e^{iu_2\sigma_2}) \\
(1),(2) \equiv (\mathrm{oe})_\pm \quad &: \quad (\sigma_3, \pm 1) \ \to \ (a_v, a_+, a_-) = (\tfrac{1}{2}, 0, \tfrac{1}{2}) \ , \quad (\tfrac{1}{2}, \tfrac{\tau}{2}, \tfrac{1+\tau}{2}) \\
(3),(4) \equiv (\mathrm{eo})_\pm \quad &: \quad (\pm 1, \sigma_3) \ \to \ (a_v, a_+, a_-) = (\tfrac{\tau}{2}, 0, \tfrac{\tau}{2}) \ , \quad (\tfrac{\tau}{2}, \tfrac{1}{2}, \tfrac{1+\tau}{2}) \\
(5),(6) \equiv (\mathrm{oo})_\pm \quad &: \quad (\pm\sigma_3, \sigma_3) \ \to \ (a_v, a_+, a_-) = (\tfrac{1+\tau}{2}, 0, \tfrac{1+\tau}{2}) \ , \ (\tfrac{1+\tau}{2}, \tfrac{1}{2}, \tfrac{\tau}{2}) \ .
\end{aligned}$$

All eigenvalues a_+, a_- are defined mod $\mathbb{Z} + \tau\mathbb{Z}$. $a_v = a_+ + a_-$ is the eigenvalue acting on the $O(2)$ adjoint (antisymmetric) representation. The discrete holonomy eigenvalues acting on the $O(2)$ symmetric representation are $a_v = a_+ + a_-$, $2a_+$, $2a_-$. The contributions $Z_{2(a)}$ (with $a = 0, \cdots, 6$) are given by

$$\begin{aligned}
Z_{2(0)} &= \oint \eta^2 du \cdot \frac{\theta_1(2\epsilon_+)}{i\eta} \cdot \frac{\eta^6}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(\epsilon_1 \pm 2u)\theta_1(\epsilon_2 \pm 2u)} \cdot \prod_{l=1}^{8} \frac{\theta_1(m_l \pm u)}{\eta^2} \\
Z_{2(a)} &= \frac{\theta_1(a_v)\theta_1(2\epsilon_+ + a_v)}{\eta^2} \cdot \frac{\eta^6}{\theta_1(\epsilon_1 + a_v)\theta_1(\epsilon_2 + a_v)\theta_1(\epsilon_1 + 2a_\pm)\theta_1(\epsilon_2 + 2a_\pm)} \\
&\quad \times \prod_{l=1}^{8} \frac{\theta_1(m_l + a_+)\theta_1(m_l + a_-)}{\eta^2} \qquad (a = 1, \cdots, 6) \ ,
\end{aligned} \tag{5.47}$$

where $\epsilon_+ = \frac{\epsilon_1+\epsilon_2}{2}$. As explained after (5.37), $\theta_1(z + a_v)$ factors should be understood as θ_i, with $i = 1, 2, 3, 4$ for $a_v = 0, \frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}$, respectively.

The contour integral in $Z_{2(0)}$ can be done by taking residues from poles with positive $SO(2)$ electric charge only: this is the simple rule for the rank 1 theory obtained by taking $\eta = 1$ [60]. The relevant poles are at $\theta_1(\epsilon_1 + 2u) = 0$ and $\theta_1(\epsilon_2 + 2u) = 0$. Using

$$\frac{1}{2\pi i}\oint_{u=a+b\tau}\frac{du}{\theta_1(\tau|u)} = \frac{(-1)^{a+b}e^{i\pi b^2\tau}}{\theta_1'(\tau|0)} = \frac{(-1)^{a+b}e^{i\pi b^2\tau}}{2\pi\eta^3}\ , \tag{5.48}$$

one should pick the residues at $u = -\frac{\epsilon_{1,2}}{2}, -\frac{\epsilon_{1,2}}{2}+\frac{1}{2}, -\frac{\epsilon_{1,2}}{2}+\frac{\tau}{2}, -\frac{\epsilon_{1,2}}{2}+\frac{1+\tau}{2}$. The residue sum is

$$Z_{2(0)} = \frac{1}{2\eta^{12}\theta_1(\epsilon_1)\theta_1(\epsilon_2)}\sum_{i=1}^{4}\left[\frac{\prod_{l=1}^{8}\theta_i(m_l\pm\frac{\epsilon_1}{2})}{\theta_1(2\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)} + \frac{\prod_{l=1}^{8}\theta_i(m_l\pm\frac{\epsilon_2}{2})}{\theta_1(2\epsilon_2)\theta_1(\epsilon_1-\epsilon_2)}\right]. \tag{5.49}$$

Expressions with $\pm$ signs mean $\theta_i(x\pm y) \equiv \theta_i(x+y)\theta_i(x-y)$. The contributions from the other six sectors are

$$\begin{aligned}
Z_{2(1)} &= \frac{\theta_2(0)\theta_2(2\epsilon_+)\prod_{l=1}^{8}\theta_1(m_l)\theta_2(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_2(\epsilon_1)\theta_2(\epsilon_2)}, \quad Z_{2(2)} = \frac{\theta_2(0)\theta_2(2\epsilon_+)\prod_{l=1}^{8}\theta_3(m_l)\theta_4(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_2(\epsilon_1)\theta_2(\epsilon_2)}, \\
Z_{2(3)} &= \frac{\theta_4(0)\theta_4(2\epsilon_+)\prod_{l=1}^{8}\theta_1(m_l)\theta_4(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_4(\epsilon_1)\theta_4(\epsilon_2)}, \quad Z_{2(4)} = \frac{\theta_4(0)\theta_4(2\epsilon_+)\prod_{l=1}^{8}\theta_2(m_l)\theta_3(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_4(\epsilon_1)\theta_4(\epsilon_2)}, \\
Z_{2(5)} &= \frac{\theta_3(0)\theta_3(2\epsilon_+)\prod_{l=1}^{8}\theta_1(m_l)\theta_3(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_3(\epsilon_1)\theta_3(\epsilon_2)}, \quad Z_{2(6)} = \frac{\theta_3(0)\theta_3(2\epsilon_+)\prod_{l=1}^{8}\theta_2(m_l)\theta_4(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_3(\epsilon_1)\theta_3(\epsilon_2)}.
\end{aligned} \tag{5.50}$$

The two E-string index is given by

$$Z_2(\tau,\epsilon_{1,2},m_l) = \frac{1}{2}Z_{2(0)} + \frac{1}{4}\sum_{a=1}^{6}Z_{2(a)}\ , \tag{5.51}$$

by dividing the order of the 'Weyl group,' as defined around (5.35).

Recently, [97] obtained the 2 E-string elliptic genus. This was done by constraining its form with its modularity, the 'domain wall' ansatz of [9], and a few low orders in the genus expansion known from the topological string calculus. The result of [97] is given by

$$\begin{aligned}
Z_2 = &\frac{1}{576\eta^{12}\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(\epsilon_2-\epsilon_1)\theta_1(2\epsilon_1)}\Big[4A_1^2(\phi_{0,1}(\epsilon_1)^2 - E_4\theta_{-2,1}(\epsilon_1)^2) \\
&+ 3A_2(E_4^2\phi_{-2,1}(\epsilon_1)^2 - E_6\phi_{-2,1}(\epsilon_1)\phi_{0,1}(\epsilon_1)) + 5B_2E_6\phi_{-2,1}(\epsilon_1)^2 \\
&- 5B_2E_4\phi_{-2,1}(\epsilon_1)\phi_{0,1}(\epsilon_1)\Big] + (\epsilon_1 \leftrightarrow \epsilon_2)
\end{aligned} \tag{5.52}$$

where $E_4(\tau)$, $E_6(\tau)$ are the Eisenstein series, summarized in Appendix B,

$$\phi_{-2,1}(\epsilon,\tau) = -\frac{\theta_1(\epsilon,\tau)^2}{\eta(\tau)^6}\ ,\quad \phi_{0,1}(\epsilon,\tau) = 4\left[\frac{\theta_2(\epsilon,\tau)^2}{\theta_2(0,\tau)^2}+\frac{\theta_3(\epsilon,\tau)^2}{\theta_3(0,\tau)^2}+\frac{\theta_4(\epsilon,\tau)^2}{\theta_4(0,\tau)^2}\right]\ , \tag{5.53}$$

and $A_1(m_l)$, $A_2(m_l)$, $B_2(m_l)$ are three of the nine Jacobi forms which are invariant under the Weyl group of E_8. See, for instance, the appendix of [89] for the full list. A_1 is simply the E_8 theta function $A_1 = \Theta(m_l,\tau)$, and

$$\begin{aligned} A_2 &= \frac{8}{9}\left[\Theta(2m_l,2\tau)+\frac{\Theta(m_l,\frac{\tau}{2})+\Theta(m_l,\frac{\tau+1}{2})}{16}\right] \qquad (5.54)\\ B_2 &= \frac{8}{15}\left[(\theta_3^4+\theta_4^4)\Theta(2m_l,2\tau)-\frac{1}{16}(\theta_2^4+\theta_3^4)\Theta(m_l,\tfrac{\tau}{2})+\frac{1}{16}(\theta_2^4-\theta_4^4)\Theta(m_l,\tfrac{\tau+1}{2})\right]\ . \end{aligned}$$

I made a full analytic proof, at $\epsilon_1=-\epsilon_2$ for simplicity (but keeping all E_8 masses and $\epsilon_- = \frac{\epsilon_1-\epsilon_2}{2}$), that (5.51) and (5.52) agree with each other. See Appendix C.2 for the proof. On one side, this agreement shows that the 'domain wall ansatz' of [97] is at work. On the other hand, it also shows that the gauge theory index exhibits the Weyl symmetry of E_8, which is manifest in (5.52). So this supports the IR E_8 symmetry enhancement of the $(0,4)$ gauge theory.

Three E-strings

There are eight sectors of $O(3)$ holonomies on T^2, labeled as follows:

$$\begin{array}{ll} (\text{ee}) : \text{diag}(e^{iu_1\sigma_2},1),\ \text{diag}(e^{iu_2\sigma_2},1)\to(1) & \text{diag}(-1,-1,1),\ \text{diag}(1,-1,-1)\to(1)' \\ (\text{eo}) : \text{diag}(e^{iu_1\sigma_2},1),\ \text{diag}(e^{iu_2\sigma_2},-1)\to(4) & \text{diag}(-1,-1,1),\ \text{diag}(1,-1,1)\to(4)' \\ (\text{oe}) : \text{diag}(e^{iu_1\sigma_2},-1),\ \text{diag}(e^{iu_2\sigma_2},1)\to(2) & \text{diag}(1,-1,1),\ \text{diag}(-1,-1,1)\to(2)' \\ (\text{oo}) : \text{diag}(e^{iu_1\sigma_2},-1),\ \text{diag}(e^{iu_2\sigma_2},-1)\to(3) & \text{diag}(1,1,-1),\ \text{diag}(1,-1,1)\to(3)'. \end{array}$$

The indices in various sectors are given as follows. Firstly,

$$\begin{aligned} Z_{3(1)} &= -\oint \eta^2 du\cdot\frac{\theta_1(2\epsilon_+)\theta_1(2\epsilon_+\pm u)\theta_1(\pm u)}{i\eta^5}\cdot\frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^2\theta_1(\epsilon_{1,2}\pm u)\theta_1(\epsilon_{1,2}\pm 2u)} \\ &\quad \cdot\prod_{l=1}^{8}\frac{\theta_1(m_l)\theta_1(m_l+u)\theta_1(m_l-u)}{\eta^3} \end{aligned} \tag{5.55}$$

$$
\begin{aligned}
Z_{3(1)'} &= -\frac{\theta_2(0)\theta_3(0)\theta_4(0)\theta_2(2\epsilon_+)\theta_3(2\epsilon_+)\theta_4(2\epsilon_+)}{\eta^6} \cdot \frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^3\theta_2(\epsilon_{1,2})\theta_3(\epsilon_{1,2})\theta_4(\epsilon_{1,2})} \\
&\quad \cdot \prod_{l=1}^{8} \frac{\theta_2(m_l)\theta_3(m_l)\theta_4(m_l)}{\eta^3} \ . \qquad (5.56)
\end{aligned}
$$

$Z_{3(1)'}$ is obtained with discrete holonomy $(a_1, a_2, a_3) = (\frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2})$ acting on the fundamental, $(a_1+a_2, a_2+a_3, a_3+a_1) = (\frac{\tau}{2}, \frac{1}{2}, \frac{1+\tau}{2})$ on adjoint, and $(2a_1, 2a_2, 2a_3, a_1+a_2, a_2+a_3, a_3+a_1)$ on symmetric representations. Similarly, one obtains

$$
\begin{aligned}
Z_{3(4)} &= -\oint \eta^2 du \cdot \frac{\theta_1(2\epsilon_+)\theta_4(2\epsilon_+ \pm u)\theta_4(\pm u)}{i\eta^5} \cdot \frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^2\theta_4(\epsilon_{1,2} \pm u)\theta_1(\epsilon_{1,2} \pm 2u)} \\
&\quad \cdot \prod_{l=1}^{8} \frac{\theta_4(m_l)\theta_1(m_l+u)\theta_1(m_l-u)}{\eta^3} \qquad (5.57) \\
Z_{3(4)'} &= -\frac{\theta_2(0)\theta_3(0)\theta_4(0)\theta_2(2\epsilon_+)\theta_3(2\epsilon_+)\theta_4(2\epsilon_+)}{\eta^6} \cdot \frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^3\theta_2(\epsilon_{1,2})\theta_3(\epsilon_{1,2})\theta_4(\epsilon_{1,2})} \\
&\quad \cdot \prod_{l=1}^{8} \frac{\theta_1(m_l)\theta_2(m_l)\theta_3(m_l)}{\eta^3} \qquad (5.58)
\end{aligned}
$$

from the (eo) sectors with $(a_1, a_2, a_3) = (\frac{1}{2}, \frac{1+\tau}{2}, 0)$,

$$
\begin{aligned}
Z_{3(2)} &= -\oint \eta^2 du \cdot \frac{\theta_1(2\epsilon_+)\theta_2(2\epsilon_+ \pm u)\theta_2(\pm u)}{i\eta^5} \cdot \frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^2\theta_2(\epsilon_{1,2} \pm u)\theta_1(\epsilon_{1,2} \pm 2u)} \\
&\quad \cdot \prod_{l=1}^{8} \frac{\theta_2(m_l)\theta_1(m_l+u)\theta_1(m_l-u)}{\eta^3} \qquad (5.59) \\
Z_{3(2)'} &= -\frac{\theta_2(0)\theta_3(0)\theta_4(0)\theta_2(2\epsilon_+)\theta_3(2\epsilon_+)\theta_4(2\epsilon_+)}{\eta^6} \cdot \frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^3\theta_2(\epsilon_{1,2})\theta_3(\epsilon_{1,2})\theta_4(\epsilon_{1,2})} \\
&\quad \cdot \prod_{l=1}^{8} \frac{\theta_1(m_l)\theta_3(m_l)\theta_4(m_l)}{\eta^3} \qquad (5.60)
\end{aligned}
$$

from the (oe) sectors with $(a_1, a_2, a_3) = (\frac{\tau}{2}, \frac{1+\tau}{2}, 0)$, and

$$
\begin{aligned}
Z_{3(3)} &= -\oint \eta^2 du \cdot \frac{\theta_1(2\epsilon_+)\theta_3(2\epsilon_+ \pm u)\theta_3(\pm u)}{i\eta^5} \cdot \frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^2\theta_3(\epsilon_{1,2} \pm u)\theta_1(\epsilon_{1,2} \pm 2u)} \\
&\quad \cdot \prod_{l=1}^{8} \frac{\theta_3(m_l)\theta_1(m_l+u)\theta_1(m_l-u)}{\eta^3} \qquad (5.61) \\
Z_{3(3)'} &= -\frac{\theta_2(0)\theta_3(0)\theta_4(0)\theta_2(2\epsilon_+)\theta_3(2\epsilon_+)\theta_4(2\epsilon_+)}{\eta^6} \cdot \frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^3\theta_2(\epsilon_{1,2})\theta_3(\epsilon_{1,2})\theta_4(\epsilon_{1,2})}
\end{aligned}
$$

$$\cdot \prod_{l=1}^{8} \frac{\theta_1(m_l)\theta_2(m_l)\theta_4(m_l)}{\eta^3} \tag{5.62}$$

from the (oo) sectors with $(a_1, a_2, a_3) = (0, \frac{\tau}{2}, \frac{1}{2})$. The contour integrals in $Z_{3(i)}$ acquire residue contributions from poles $u_* = -\frac{\epsilon_{1,2}}{2}, -\frac{\epsilon_{1,2}}{2} + \frac{1}{2}, -\frac{\epsilon_{1,2}}{2} + \frac{\tau}{2}, -\frac{\epsilon_{1,2}}{2} + \frac{1+\tau}{2}$ and $u_* = -\epsilon_{1,2} + \cdots$, where $\cdots$ part is decided by $\theta_i(u + \epsilon_{1,2}) = 0$. The residue sums are given by

$$Z_{3(i)} = \frac{-\eta^4}{\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2} \left[\frac{\eta^2\theta_1(\epsilon_1)\theta_1(\epsilon_2)}{\theta_1(2\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)\theta_1(3\epsilon_1)\theta_1(\epsilon_2-2\epsilon_1)} \prod_{l=1}^{8} \frac{\theta_i(m_l)\theta_i(m_l \pm \epsilon_1)}{\eta^3} \right.$$
$$\left. + \frac{1}{2}\sum_{a=1}^{4} \frac{\eta^2\theta_{\sigma_i(a)}(\frac{3\epsilon_1}{2}+\epsilon_2)\theta_{\sigma_i(a)}(-\frac{\epsilon_1}{2})}{\theta_1(2\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)\theta_{\sigma_i(a)}(\frac{3\epsilon_1}{2})\theta_{\sigma_i(a)}(\epsilon_2-\frac{\epsilon_1}{2})} \prod_{l=1}^{8} \frac{\theta_i(m_l)\theta_a(m_l \pm \frac{\epsilon_1}{2})}{\eta^3} + (\epsilon_1 \leftrightarrow \epsilon_2) \right] \tag{5.63}$$

where the permutations are defined by

$$\begin{aligned} &\sigma_1(1,2,3,4) = (1,2,3,4)\ , \quad \sigma_2(1,2,3,4) = (2,1,4,3), \\ &\sigma_3(1,2,3,4) = (3,4,1,2)\ , \quad \sigma_4(1,2,3,4) = (4,3,2,1)\ . \end{aligned} \tag{5.64}$$

The full index is given by

$$Z_3 = \sum_{i=1}^{4} \left(\frac{1}{4} Z_{3(i)} + \frac{1}{8} Z_{3(i)'} \right) \ , \tag{5.65}$$

after dividing by the Weyl factors (5.35).

For simplicity, study the indices at $m_l = 0$, $\epsilon_1 = -\epsilon_2 \equiv \epsilon$ in detail, which are

$$Z_{3(i)} = \frac{\eta^4}{\theta_1(\epsilon)^4} \left[\frac{2\theta_1(\epsilon)^2\theta_i(0)^8\theta_i(\epsilon)^{16}}{\eta^{22}\theta_1(2\epsilon)^2\theta_1(3\epsilon)^2} + \sum_{a=1}^{4} \frac{\theta_{\sigma_i(a)}(\frac{\epsilon}{2})^2\theta_i(0)^8\theta_a(\frac{\epsilon}{2})^{16}}{\eta^{22}\theta_1(2\epsilon)^2\theta_{\sigma_i(a)}(\frac{3\epsilon}{2})^2} \right] \tag{5.66}$$

$$Z_{3(1)'} = \frac{\theta_2(0)^{10}\theta_3(0)^{10}\theta_4(0)^{10}}{\eta^{18}\theta_1(\epsilon)^6\theta_2(\epsilon)^2\theta_3(\epsilon)^2\theta_4(\epsilon)^2} = \frac{4\theta_2(0)^8\theta_3(0)^8\theta_4(0)^8}{\eta^{18}\theta_1(\epsilon)^4\theta_1(2\epsilon)^2} \ , \tag{5.67}$$

with $Z_{3(2)'} = Z_{3(3)'} = Z_{3(4)'} = 0$. Consider the genus expansion of Z_3, where genus is defined for the topological string amplitudes on the CY_3 which engineers our 6d CFT in the F-theory context. Namely, expand

$$F_3 \equiv Z_3 - Z_1 Z_2 + \frac{1}{3}{Z_1}^3 = \sum_{n\geq 0, g\geq 0} (\epsilon_1+\epsilon_2)^n (\epsilon_1\epsilon_2)^{g-1} F^{(n,g,3)}(\tau) \ . \tag{5.68}$$

Taking $\epsilon_+ = 0$, some known results on $F^{(0,g,3)}$ are summarized in (C.1), which were computed in [99] up to genus 5. This can be compared with $F^{(0,g,3)}$ obtained from the gauge theory index. Numerically, I checked the agreements for $g \leq 5$ up to first 10 terms in the q expansions, starting at $q^{-3/2}$, with the last term that was checked at $q^{15/2}$. (The two terms at $q^{-1/2}$ and $q^{1/2}$ are all zero due to a vanishing theorem.) I also analytically checked the agreements for $F^{(0,0,3)}$, $F^{(0,1,3)}$, and a refined amplitude $F^{(1,0,3)}$, against the results known from the topological string calculus. See Appendix C.1 for the details.

Four E-strings

The indices from the two sectors in the (ee) part of $O(4)$ holonomy are

$$\begin{aligned} Z_{4(1)} =& -\oint \eta^4 du_1 du_2 \cdot \frac{\theta_1(2\epsilon_+)^2\theta_1(2\epsilon_+ \pm u_1 \pm u_2)\theta_1(\pm u_1 \pm u_2)}{\eta^{10}} \\ &\cdot \frac{\eta^{20}}{\theta_1(\epsilon_{1,2})^2\theta_1(\epsilon_{1,2} \pm u_1 \pm u_2)\theta_1(\epsilon_{1,2} \pm 2u_1)\theta_1(\epsilon_{1,2} \pm 2u_2)} \cdot \prod_{l=1}^{8} \frac{\theta_1(m_l \pm u_1)\theta_1(m_l \pm u_2)}{\eta^4} \end{aligned} \tag{5.69}$$

$$\begin{aligned} Z_{4(1)'} =& \frac{\theta_2(0)^2\theta_3(0)^2\theta_4(0)^2\theta_2(2\epsilon_+)^2\theta_3(2\epsilon_+)^2\theta_4(2\epsilon_+)^2}{\eta^{12}} \\ &\cdot \frac{\eta^{20}}{\theta_1(\epsilon_{1,2})^4\theta_2(\epsilon_{1,2})^2\theta_3(\epsilon_{1,2})^2\theta_4(\epsilon_{1,2})^2} \cdot \prod_{l=1}^{8} \frac{\theta_1(m_l)\theta_2(m_l)\theta_3(m_l)\theta_4(m_l)}{\eta^4} \end{aligned} \tag{5.70}$$

where $Z_{4(1)'}$ is obtained with discrete holonomy $(a_1, a_2, a_3, a_4) = (0, \frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2})$ for the fundamental representation. The shorthand notation $\theta_i(\epsilon_{1,2}) \equiv \theta_i(\epsilon_1)\theta_i(\epsilon_2)$ is used. The indices from the two sectors in the (oe) part are

$$\begin{aligned} Z_{4(2)} =& \oint \eta^2 du \cdot \frac{\theta_1(2\epsilon_+)\theta_2(2\epsilon_+)\theta_1(2\epsilon_+ \pm u)\theta_2(2\epsilon_+ \pm u)\theta_2(0)\theta_1(\pm u)\theta_2(\pm u)}{i\eta^{11}} \\ &\cdot \frac{\eta^{20}}{\theta_1(\epsilon_{1,2} \pm 2u)\theta_1(\epsilon_{1,2})^3\theta_2(\epsilon_{1,2})\theta_1(\epsilon_{1,2} \pm u)\theta_2(\epsilon_{1,2} \pm u)} \cdot \prod_{l=1}^{8} \frac{\theta_1(m_l \pm u)\theta_1(m_l)\theta_2(m_l)}{\eta^4} \end{aligned} \tag{5.71}$$

$$Z_{4(2)'} = \oint \eta^2 du \cdot \frac{\theta_1(2\epsilon_+)\theta_2(2\epsilon_+)\theta_3(2\epsilon_+ \pm u)\theta_4(2\epsilon_+ \pm u)\theta_2(0)\theta_3(\pm u)\theta_4(\pm u)}{i\eta^{11}}$$

$$\cdot \frac{\eta^{20}}{\theta_1(\epsilon_{1,2}\pm 2u)\theta_1(\epsilon_{1,2})^3\theta_2(\epsilon_{1,2})\theta_3(\epsilon_{1,2}\pm u)\theta_4(\epsilon_{1,2}\pm u)}\cdot\prod_{l=1}^{8}\frac{\theta_1(m_l\pm u)\theta_3(m_l)\theta_4(m_l)}{\eta^4} \tag{5.72}$$

where the holonomy $(a_1,a_2,a_3,a_4)=(u,-u,0,\frac{1}{2})$ and $(u,-u,\frac{\tau}{2},\frac{1+\tau}{2})$ are used for $Z_{4(2)}$ and $Z_{4(2)'}$, respectively. The indices from the two sectors in the (oo) part are

$$Z_{4(3)}=\oint \eta^2 du\cdot\frac{\theta_1(2\epsilon_+)\theta_3(2\epsilon_+)\theta_1(2\epsilon_+\pm u)\theta_3(2\epsilon_+\pm u)\theta_3(0)\theta_1(\pm u)\theta_3(\pm u)}{i\eta^{11}}$$

$$\cdot \frac{\eta^{20}}{\theta_1(\epsilon_{1,2}\pm 2u)\theta_1(\epsilon_{1,2})^3\theta_3(\epsilon_{1,2})\theta_1(\epsilon_{1,2}\pm u)\theta_3(\epsilon_{1,2}\pm u)}\cdot\prod_{l=1}^{8}\frac{\theta_1(m_l\pm u)\theta_1(m_l)\theta_3(m_l)}{\eta^4} \tag{5.73}$$

$$Z_{4(3)'}=\oint \eta^2 du\cdot\frac{\theta_1(2\epsilon_+)\theta_3(2\epsilon_+)\theta_2(2\epsilon_+\pm u)\theta_4(2\epsilon_+\pm u)\theta_3(0)\theta_2(\pm u)\theta_4(\pm u)}{i\eta^{11}}$$

$$\cdot \frac{\eta^{20}}{\theta_1(\epsilon_{1,2}\pm 2u)\theta_1(\epsilon_{1,2})^3\theta_3(\epsilon_{1,2})\theta_2(\epsilon_{1,2}\pm u)\theta_4(\epsilon_{1,2}\pm u)}\cdot\prod_{l=1}^{8}\frac{\theta_1(m_l\pm u)\theta_2(m_l)\theta_4(m_l)}{\eta^4} \tag{5.74}$$

where the holonomy $(a_1,a_2,a_3,a_4)=(u,-u,0,\frac{1+\tau}{2})$ and $(u,-u,\frac{\tau}{2},\frac{1}{2})$ are used for $Z_{4(3)}$ and $Z_{4(3)'}$, respectively. Finally, the indices from the two (eo) sectors are

$$Z_{4(4)}=\oint \eta^2 du\cdot\frac{\theta_1(2\epsilon_+)\theta_4(2\epsilon_+)\theta_1(2\epsilon_+\pm u)\theta_4(2\epsilon_+\pm u)\theta_4(0)\theta_1(\pm u)\theta_4(\pm u)}{i\eta^{11}}$$

$$\cdot \frac{\eta^{20}}{\theta_1(\epsilon_{1,2}\pm 2u)\theta_1(\epsilon_{1,2})^3\theta_4(\epsilon_{1,2})\theta_1(\epsilon_{1,2}\pm u)\theta_4(\epsilon_{1,2}\pm u)}\cdot\prod_{l=1}^{8}\frac{\theta_1(m_l\pm u)\theta_1(m_l)\theta_4(m_l)}{\eta^4} \tag{5.75}$$

$$Z_{4(4)'}\oint \eta^2 du\cdot\frac{\theta_1(2\epsilon_+)\theta_4(2\epsilon_+)\theta_2(2\epsilon_+\pm u)\theta_3(2\epsilon_+\pm u)\theta_4(0)\theta_2(\pm u)\theta_3(\pm u)}{i\eta^{11}}$$

$$\cdot \frac{\eta^{20}}{\theta_1(\epsilon_{1,2}\pm 2u)\theta_1(\epsilon_{1,2})^3\theta_4(\epsilon_{1,2})\theta_2(\epsilon_{1,2}\pm u)\theta_3(\epsilon_{1,2}\pm u)}\cdot\prod_{l=1}^{8}\frac{\theta_1(m_l\pm u)\theta_2(m_l)\theta_3(m_l)}{\eta^4} \tag{5.76}$$

where the holonomy $(a_1,a_2,a_3,a_4)=(u,-u,0,\frac{\tau}{2})$ and $(u,-u,\frac{1}{2},\frac{1+\tau}{2})$ are used for $Z_{4(4)}$ and $Z_{4(4)'}$, respectively.

One also needs to specify the residues which contribute to the above contour integrals. For the rank 1 cases, one just keeps all poles and residues associated with positively charged chiral fields. So for $Z_{4(i)}$ with $i=2,3,4$, the relevant poles are at $u_*=-\frac{\epsilon_{1,2}}{2}+\frac{p}{2}$, where p runs over $(p_1,p_2,p_3,p_4)=(0,1,1+\tau,\tau)$, and $u_*=-\epsilon_{1,2}$, $-\epsilon_{1,2}+\frac{p_i}{2}$. For $Z_{4(i)'}$ with $i=2,3,4$, the poles are at $u_*=-\frac{\epsilon_{1,2}}{2}+\frac{p}{2}$, again with p running over $(p_1,p_2,p_3,p_4)=(0,1,1+\tau,\tau)$, and at $u_*=-\epsilon_{1,2}+p_j$ with two possible values of $j\neq 1,i$. The resulting residue sums are given by

$$\begin{aligned} Z_{4(2)} =&\sum_{i=1}^{4}\frac{\theta_2(\epsilon_1+\epsilon_2)\theta_i(\frac{3\epsilon_1}{2}+\epsilon_2)\theta_{\sigma_2(i)}(\frac{3\epsilon_1}{2}+\epsilon_2)\theta_2(0)\theta_i(-\frac{\epsilon_1}{2})\theta_{\sigma_2(i)}(-\frac{\epsilon_1}{2})\prod_l\theta_1(m_l)\theta_2(m_l)\theta_i(m_l\pm\frac{\epsilon_1}{2})}{2\eta^{24}\theta_1(2\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)\theta_1(\epsilon_{1,2})^3\theta_2(\epsilon_{1,2})\theta_i(\frac{3\epsilon_1}{2})\theta_i(\epsilon_2-\frac{\epsilon_1}{2})\theta_{\sigma_2(i)}(\frac{3\epsilon_1}{2})\theta_{\sigma_2(i)}(\epsilon_2-\frac{\epsilon_1}{2})} \\ &+\frac{\theta_2(2\epsilon_1+\epsilon_2)\theta_2(\epsilon_1)\left(\prod_l\theta_1(m_l\pm\epsilon_1)+\prod_l\theta_2(m_l\pm\epsilon_1)\right)\prod_l\theta_1(m_l)\theta_2(m_l)}{\eta^{24}\theta_1(3\epsilon_1)\theta_1(\epsilon_2-2\epsilon_1)\theta_1(\epsilon_{1,2})^2\theta_1(2\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)\theta_2(2\epsilon_1)\theta_2(\epsilon_2-\epsilon_1)}+(\epsilon_1\leftrightarrow\epsilon_2) \end{aligned} \tag{5.77}$$

$$\begin{aligned} Z_{4(2)'} =&\sum_{i=1}^{4}\frac{\theta_2(\epsilon_1+\epsilon_2)\theta_{\sigma_3(i)}(\frac{3\epsilon_1}{2}+\epsilon_2)\theta_{\sigma_4(i)}(\frac{3\epsilon_1}{2}+\epsilon_2)\theta_2(0)\theta_{\sigma_3(i)}(-\frac{\epsilon_1}{2})\theta_{\sigma_4(i)}(-\frac{\epsilon_1}{2})\prod_l\theta_3(m_l)\theta_4(m_l)\theta_i(m_l\pm\frac{\epsilon_1}{2})}{2\eta^{24}\theta_1(2\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)\theta_1(\epsilon_{1,2})^3\theta_2(\epsilon_{1,2})\theta_{\sigma_3(i)}(\frac{3\epsilon_1}{2})\theta_{\sigma_3(i)}(\epsilon_2-\frac{\epsilon_1}{2})\theta_{\sigma_4(i)}(\frac{3\epsilon_1}{2})\theta_{\sigma_4(i)}(\epsilon_2-\frac{\epsilon_1}{2})} \\ &+\frac{\theta_2(2\epsilon_1+\epsilon_2)\theta_2(\epsilon_1)\left(\prod_l\theta_3(m_l\pm\epsilon_1)+\prod_l\theta_4(m_l\pm\epsilon_1)\right)\prod_l\theta_3(m_l)\theta_4(m_l)}{\eta^{24}\theta_1(3\epsilon_1)\theta_1(\epsilon_2-2\epsilon_1)\theta_1(\epsilon_{1,2})^2\theta_1(2\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)\theta_2(2\epsilon_1)\theta_2(\epsilon_2-\epsilon_1)}+(\epsilon_1\leftrightarrow\epsilon_2) \end{aligned} \tag{5.78}$$

where σ_i are defined as (5.64). The expressions for $Z_{4(i)}$ and $Z_{4(i)'}$ with $i=3,4$ are obtained by permuting the roles of the subscripts $2,3,4$ of the θ-functions and σ_i.

The rank 2 contour integral in $Z_{4(1)}$ can be done as follows. The charges of the $(0,2)$ chiral multiplets, responsible for the poles in the integrand, are $\pm 2e_I$, $\pm e_I\pm e_J$ $(I\neq J)$ with $I,J=1,2$. I chose the vector η to be in the cone between e_1+e_2 and $2e_2$. Then, the poles with nonzero Jeffrey-Kirwan residues (after eliminating the fake poles due to vanishing numerators from Fermi multiplets) are at the following 104 positions:

$$(1)\ :\ \ 2u_2+\epsilon=0,\ u_1+u_2+\epsilon'=0\ \rightarrow\ u_2=-\frac{\epsilon}{2}+\frac{p_i}{2},\ u_1=-\epsilon'+\frac{\epsilon}{2}+\frac{p_i}{2} \tag{5.79}$$

$$(2)\ :\ \ 2u_2+\epsilon=0,\ 2u_1+\epsilon=0\ \rightarrow\ u_2=-\frac{\epsilon}{2}+\frac{p_i}{2},\ u_1=-\frac{\epsilon}{2}+\frac{p_j}{2}\quad (p_i\neq p_j)$$

$$(3)\ :\ \ 2u_2+\epsilon=0,\ 2u_1+\epsilon'=0\ \rightarrow\ u_2=-\frac{\epsilon}{2}+\frac{p_i}{2},\ u_1=-\frac{\epsilon'}{2}+\frac{p_j}{2}$$

$$(4)\ :\ \ 2u_2+\epsilon=0,\ u_1-u_2+\epsilon=0\ \rightarrow\ u_2=-\frac{\epsilon}{2}+\frac{p_i}{2},\ u_1=-\frac{3\epsilon}{2}+\frac{p_i}{2}$$

$$
\begin{aligned}
(5)\ &:\ u_2-u_1+\epsilon=0,\ u_1+u_2+\epsilon=0\ \rightarrow\ u_2=-\epsilon+\frac{p_i}{2},\ u_1=0+\frac{p_i}{2}\\
(6)\ &:\ u_2-u_1+\epsilon=0,\ u_1+u_2+\epsilon'=0\ \rightarrow\ u_2=-\frac{\epsilon+\epsilon'}{2}+\frac{p_i}{2},\ u_1=-\frac{\epsilon'-\epsilon}{2}+\frac{p_i}{2}\\
(7)\ &:\ u_2-u_1+\epsilon=0,\ 2u_1+\epsilon=0\ \rightarrow\ u_2=-\frac{3\epsilon}{2}+\frac{p_i}{2},\ u_1=-\frac{\epsilon}{2}+\frac{p_i}{2}\\
(8)\ &:\ -2u_1+\epsilon=0, u_1+u_2+\epsilon=0\ \rightarrow\ u_1=+\frac{\epsilon}{2}+\frac{p_i}{2},\ u_2=-\frac{3\epsilon}{2}+\frac{p_i}{2}\ .
\end{aligned}
$$

where $(p_1,p_2,p_3,p_4)=(0,1,1+\tau,\tau)$. ϵ can be either ϵ_1 or ϵ_2, and $\epsilon'\neq\epsilon$ is the remaining parameter. In the second case, the four cases with $p_i=p_j$ do not provide poles since there are vanishing factors in the numerator. One can check that these poles are all non-degenerate.

The residue sums from these 8 cases are given by (the sectors labeled by (4), (7), (8) yield same result, shown on the second line)

$$
\begin{aligned}
&(1):\sum_{i=1}^{4}\frac{\theta_1(2\epsilon_1+\epsilon_2)\theta_1(-\epsilon_1)\prod_l\theta_i(m_l\pm(\epsilon_1-\frac{\epsilon_2}{2}))\theta_i(m_l\pm\frac{\epsilon_2}{2})}{2\eta^{24}\theta_1(\epsilon_{1,2})^2\theta_1(2\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)\theta_1(2\epsilon_1-\epsilon_2)\theta_1(2\epsilon_2-\epsilon_1)\theta_1(3\epsilon_1-\epsilon_2)\theta_1(2\epsilon_2-2\epsilon_1)}+(\epsilon_1\leftrightarrow\epsilon_2)\\
&(4):\sum_{i=1}^{4}\frac{\prod_l\theta_i(m_l\pm\frac{\epsilon_1}{2})\theta_i(m_l\pm\frac{3\epsilon_1}{2})}{2\eta^{24}\theta_1(\epsilon_{1,2})\theta_1(2\epsilon_1)\theta_1(3\epsilon_1)\theta_1(4\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)\theta_1(\epsilon_2-2\epsilon_1)\theta_1(\epsilon_2-3\epsilon_1)}+(\epsilon_1\leftrightarrow\epsilon_2)=(7)=(8)\\
&(5):\sum_{i=1}^{4}\frac{\theta_1(2\epsilon_1+\epsilon_2)\theta_1(-\epsilon_1)\prod_l\theta_i(m_l)^2\theta_i(m_l\pm\epsilon_1)}{2\eta^{24}\theta_1(\epsilon_{1,2})^2\theta_1(2\epsilon_1)^2\theta_1(\epsilon_2-\epsilon_1)^2\theta_1(3\epsilon_1)\theta_1(\epsilon_2-2\epsilon_1)}+(\epsilon_1\leftrightarrow\epsilon_2)\\
&(6):\sum_{i=1}^{4}\frac{\prod_l\theta_i(m_l\pm\frac{\epsilon_1+\epsilon_2}{2})\theta_i(m_l\pm\frac{\epsilon_1-\epsilon_2}{2})}{\eta^{24}\theta_1(\epsilon_{1,2})\theta_1(2\epsilon_1)\theta_1(\epsilon_1-\epsilon_2)\theta_1(2\epsilon_2)\theta_1(\epsilon_2-\epsilon_1)\theta_1(2\epsilon_1-\epsilon_2)\theta_1(2\epsilon_2-\epsilon_1)}\\
&(2):\Bigg[\frac{\theta_2(0)\theta_2(-\epsilon_1)\theta_2(\epsilon_1+\epsilon_2)\theta_2(2\epsilon_1+\epsilon_2)\left(\prod_l\theta_1(m_l\pm\frac{\epsilon_1}{2})\theta_2(m_l\pm\frac{\epsilon_1}{2})+\prod_l\theta_3(m_l\pm\frac{\epsilon_1}{2})\theta_4(m_l\pm\frac{\epsilon_1}{2})\right)}{2\eta^{24}\theta_1(\epsilon_{1,2})^2\theta_1(2\epsilon_1)^2\theta_1(\epsilon_2-\epsilon_1)^2\theta_2(\epsilon_{1,2})\theta_2(2\epsilon_1)\theta_2(\epsilon_2-\epsilon_1)}\\
&\qquad+(2,3,4\rightarrow3,4,2)+(2,3,4\rightarrow4,2,3)\Bigg]+(\epsilon_1\leftrightarrow\epsilon_2)\\
&(3):\sum_{i,j=1}^{4}\frac{\prod_l\theta_j(m_l\pm\frac{\epsilon_1}{2})\theta_i(m_l\pm\frac{\epsilon_2}{2})}{2\eta^{24}\theta_1(\epsilon_{1,2})^2\theta_1(2\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)\theta_1(2\epsilon_2)\theta_1(\epsilon_1-\epsilon_2)}\ \frac{\theta_{\sigma_j(i)}(-\frac{\epsilon_1+\epsilon_2}{2})\theta_{\sigma_j(i)}(\frac{3(\epsilon_1+\epsilon_2)}{2})}{\theta_{\sigma_j(i)}(\frac{3\epsilon_1-\epsilon_2}{2})\theta_{\sigma_j(i)}(\frac{3\epsilon_2-\epsilon_1}{2})}\ .
\end{aligned}
$$

$Z_{4(1)}$ is given by the sum of eight contributions $(1),\cdots,(8)$. The full index is

$$
Z_4=\frac{1}{8}\sum_{i=1}^{4}Z_{4(i)}+\frac{1}{8}\sum_{i=2}^{4}Z_{4(i)'}+\frac{1}{16}Z_{4(1)'}\ , \tag{5.80}
$$

with the Weyl factors given by (5.35).

I tested the above results against various known ones. First consider the case in which one sets

$$\epsilon_1 = -\epsilon_2 \equiv \epsilon,\ m_1 = m_2 = 0, m_3 = m_4 = \frac{1}{2},\ m_5 = m_6 = -\frac{1+\tau}{2},\ m_7 = m_8 = \frac{\tau}{2}\ . \tag{5.81}$$

This case was considered recently in [100]. In particular, [100] wrote down the concrete forms of the elliptic genera in this limit for 2 and 4 E-strings. The case with 2 E-strings is a special case of [97], so also agrees with our results. The index of [100] at (5.81) is always zero for odd number of E-strings. By plugging in (5.81) to the 3 E-string index in the previous subsection, all $Z_{3(i)}, Z_{3(i)'}$ are identically zero, agreeing with the results of [100]. Now let me study the 4 E-string index. Plugging in (5.81), one finds that the contributions from the seven sectors are zero, and the only nonzero contribution is $Z_{4(1)}$. The surviving contributions are

$$\begin{aligned}(1) = (4) = (7) = (8) =& \frac{4\prod_{i=1}^{4}\theta_i(3\epsilon/2)^4\theta_i(\epsilon/2)^4}{\eta^{24}\theta_1(\epsilon)^2\theta_1(2\epsilon)^2\theta_1(3\epsilon)^2\theta_1(4\epsilon)^2}\\ (2) = (3) =& \frac{2\prod_i\theta_i(\epsilon/2)^8}{\eta^{24}\theta_1(\epsilon)^4\theta_1(2\epsilon)^4}\left[\frac{\theta_2(0)^2}{\theta_2(2\epsilon)^2}+\frac{\theta_3(0)^2}{\theta_3(2\epsilon)^2}+\frac{\theta_4(0)^2}{\theta_4(2\epsilon)^2}\right]\end{aligned} \tag{5.82}$$

while $(5), (6)$ become zero. So one obtains

$$\begin{aligned}Z_{4(1)} =& \frac{16\prod_{i=1}^{4}\theta_i(\frac{3\epsilon}{2})^4\theta_i(\frac{\epsilon}{2})^4}{\eta^{24}\theta_1(\epsilon)^2\theta_1(2\epsilon)^2\theta_1(3\epsilon)^2\theta_1(4\epsilon)^2}+\frac{4\prod_i\theta_i(\frac{\epsilon}{2})^8}{\eta^{24}\theta_1(\epsilon)^4\theta_1(2\epsilon)^4}\left[\frac{\theta_2(0)^2}{\theta_2(2\epsilon)^2}+\frac{\theta_3(0)^2}{\theta_3(2\epsilon)^2}+\frac{\theta_4(0)^2}{\theta_4(2\epsilon)^2}\right]\\ =& \frac{16\theta_1(\epsilon)^2\theta_1(3\epsilon)^2}{\theta_1(2\epsilon)^2\theta_1(4\epsilon)^2}+\frac{4\theta_1(\epsilon)^4}{\theta_1(2\epsilon)^4}\left[\frac{\theta_2(0)^2}{\theta_2(2\epsilon)^2}+\frac{\theta_3(0)^2}{\theta_3(2\epsilon)^2}+\frac{\theta_4(0)^2}{\theta_4(2\epsilon)^2}\right]\ .\end{aligned} \tag{5.83}$$

The four E-string index at (5.81) is given in [100] by

$$\frac{\theta_1(\epsilon)^{20}}{2\eta^{48}\theta_1(2\epsilon)^2\theta_1(4\epsilon)^2}\left[72(\wp')^4\wp^2-18(\wp'')^2(\wp')^2\wp+2\wp''(\wp')^4+(\wp'')^4\right]\ , \tag{5.84}$$

where $\wp(\tau,\epsilon)$ is the Weierstrass's elliptic function. I checked that this agrees with the index $\frac{1}{8}Z_{4(1)}$ in a serious expansion in q for the first 11 terms, up to and including $\mathcal{O}(q^{10})$.

Now compare this result with the genus expansion, at $m_l = 0$ and $\epsilon_1 = -\epsilon_2 = \epsilon$. The above indices become

$$Z_{4(1)} = \sum_{i=1}^{4}\left[\frac{4\theta_i(\frac{3\epsilon}{2})^{16}\theta_i(\frac{\epsilon}{2})^{16}}{\eta^{24}\theta_1(\epsilon)^2\theta_1(2\epsilon)^2\theta_1(3\epsilon)^2\theta_1(4\epsilon)^2}+\frac{2\theta_i(0)^{16}\theta_i(\epsilon)^{16}}{\eta^{24}\theta_1(\epsilon)^2\theta_1(2\epsilon)^4\theta_1(3\epsilon)^2}\right] \tag{5.85}$$

$$+\frac{2}{\eta^{24}\theta_1(\epsilon)^4\theta_1(2\epsilon)^4}\left[\frac{\theta_2(0)^2(\theta_1(\frac{\epsilon}{2})^{16}\theta_2(\frac{\epsilon}{2})^{16}+\theta_3(\frac{\epsilon}{2})^{16}\theta_4(\frac{\epsilon}{2})^{16})}{\theta_2(2\epsilon)^2}+(3,4,2)+(4,2,3)\right]$$

$$Z_{4(2)'}=\sum_{i=1}^{4}\frac{\theta_2(0)^2\theta_{\sigma_3(i)}(\frac{\epsilon}{2})^2\theta_{\sigma_4(i)}(\frac{\epsilon}{2})^2\theta_3(0)^8\theta_4(0)^8\theta_i(\frac{\epsilon}{2})^{16}}{\eta^{24}\theta_1(2\epsilon)^2\theta_1(\epsilon)^6\theta_2(\epsilon)^2\theta_{\sigma_3(i)}(\frac{3\epsilon}{2})^2\theta_{\sigma_4(i)}(\frac{3\epsilon}{2})^2}+\frac{2\theta_2(\epsilon)^2\theta_3(0)^8\theta_4(0)^8(\theta_3(\epsilon)^{16}+\theta_4(\epsilon)^{16})}{\eta^{24}\theta_1(3\epsilon)^2\theta_1(2\epsilon)^2\theta_1(\epsilon)^4\theta_2(2\epsilon)^2}\ ,$$

with $Z_{4(1)'}=0$, $Z_{4(2)}=Z_{4(3)}=Z_{4(4)}=0$, and $Z_{4(3)'}$, $Z_{4(4)'}$ are obtained from $Z_{4(2)'}$ by changing the roles of $2,3,4$ appearing in the subscripts of the theta functions and $\sigma_2(i),\sigma_3(i),\sigma_4(i)$. I first confirmed numerically the agreement with $F^{(0,g,4)}$ computed from topological strings for $g\leq 5$ till q^5, by checking the first 10 terms in the serious expansion in q. I also exactly checked the agreements of $F^{(0,0,4)}$, $F^{(0,1,4)}$, $F^{(0,2,4)}$. See Appendix C.1 for the details.

Higher E-strings

The computation of the elliptic genus using the methods of [61] quickly becomes complicated for higher rank gauge groups. In general, there could be a fundamental complication due to some poles failing to be projective. But I showed at the beginning of this subsection that this does not happen in our cases. With higher rank, the computational problem is that there is a large number of poles and residues to be considered. For $U(n)$ indices, the possible poles are often completely classified by the so-called 'colored Young diagrams,' with a $U(n)$ adjoint and several fundamental $(0,4)$ hypermultiplets. This classification first appeared in the context of instanton counting [15, 51], which was reproduced in Chapter 3. The resulting residues are often nicely arranged into a reasonably compact form [64, 65]. However, for other gauge groups, I am not aware of systematic classifications of poles.[2] In this subsection, I will illustrate the pole structures for some higher E-strings, with $O(5)$, $O(6)$, $O(7)$, $O(8)$ gauge groups, and also make some qualitative classifi-

[2]The pole structure of our $O(n)$ index is similar to that of the $Sp(N)$ instanton partition function, whose ADHM quantum mechanics comes with $O(n)$ group for n instantons. The poles in the E-string index could be slightly simpler, because there are only $O(n)$ symmetric hypermultiplets while the ADHM mechanics also has extra N fundamental hypermultiplets. In either case, I do not know the pole classification, apart from the basic rule given by the Jeffrey-Kirwan residues.

cations of these poles. Since the purpose is to illustrate the computations for higher ranks, I will only consider the branch of $O(n)$ holonomy with maximal number of continuous parameters, in the (ee) sector.

Let me start by studying the $O(5)$ index, for five E-strings. Taking $\eta = e_1 + \varepsilon e_2$ with $0 < \varepsilon \ll 1$, the following pair of weights $\{\rho_1, \rho_2\}$ can potentially give nonzero JK-Res:

$$\begin{aligned}
&\{2e_1, 2e_2\},\ \{2e_1, e_2\},\ \{2e_1, e_2 \pm e_1\},\ \{e_1, 2e_2\},\ \{e_1, e_2\},\ \{e_1, e_2 \pm e_1\} \\
&\{e_1 - e_2, 2e_2\},\ \{e_1 - e_2, e_1 + e_2\},\ \{e_1 - e_2, e_2\},\ \{e_1 + e_2, -2e_2\},\ \{e_1 + e_2, -e_2\}\ .
\end{aligned} \tag{5.86}$$

These poles define the pole u_* by hyperplanes $\rho_i(u_*) + z_i = 0$ for suitable z_i, chosen between ϵ_1, ϵ_2. Considering all possible values of u_*, there are 142 poles, which are all non-degenerate. The evaluation of residue sum should be marginally more laborious than the $O(4)$ case.

Next, consider the $O(6)$ contour integral. The poles come from the scalar fields with charges $\pm 2e_I$, $\pm e_I \pm e_J$. Choose η to be $\eta = e_1 + \varepsilon e_2 + \varepsilon^2 e_3$ with $0 < \varepsilon \ll 1$. The groups of 3 vectors which contain η in their cones are

$$\begin{aligned}
&\{2e_1, 2e_2, 2e_3\},\ \{2e_1, 2e_2, e_3 \pm e_{1,2}\},\ \{2e_1, 2e_3, e_2 \pm e_1\},\ \{2e_1, 2e_3, e_2 - e_3\},\ \{2e_1, -2e_3, e_2 + e_3\}, \\
&\{2e_1, e_2 \pm e_1, e_3 \pm e_1\},\ \{2e_1, e_2 \pm e_1, e_3 \pm e_2\},\ \{2e_1, e_3 \pm e_1, e_2 - e_3\},\ \{2e_1, -e_3 \pm e_1, e_2 + e_3\} \\
&\{2e_1, e_2 + e_3, e_2 - e_3\},\ \{2e_2, 2e_3, e_1 - e_{2,3}\},\ \{2e_2, -2e_3, e_1 + e_3\},\ \{2e_2, e_1 - e_2, e_3 \pm e_{1,2}\} \\
&\{2e_2, e_1 + e_3, e_1 - e_3\},\ \{2e_2, e_1 + e_3, -e_2 - e_3\},\ \{2e_2, e_1 - e_3, -e_2 + e_3\},\ \{2e_3, -2e_2, e_1 + e_2\}, \\
&\{2e_3, e_1 + e_2, e_1 - e_{2,3}\},\ \{2e_3, e_1 + e_2, -e_2 - e_3\},\ \{2e_3, e_1 - e_2, e_2 - e_3\},\ \{2e_3, e_2 - e_1, e_1 - e_3\}, \\
&\{2e_3, e_1 - e_3, e_2 \pm e_3\},\ \{-2e_2, e_1 + e_2, e_3 \pm e_{1,2}\},\ \{-2e_2, e_1 + e_3, e_2 - e_3\},\ \{-2e_2, e_1 - e_3, e_2 + e_3\}, \\
&\{-2e_3, e_1 + e_2, e_1 + e_3\},\ \{-2e_3, e_1 + e_2, -e_2 + e_3\},\ \{-2e_3, e_1 - e_2, e_2 + e_3\},\ \{-2e_3, e_2 - e_1, e_1 + e_3\}, \\
&\{-2e_3, e_1 + e_3, e_2 \pm e_3\},\ \{e_1 + e_2, e_1 - e_2, e_3 \pm e_{1,2}\},\ \{e_1 + e_2, e_1 + e_3, e_1 - e_3\}, \\
&\{e_1 + e_2, e_1 + e_3, -e_2 - e_3\},\ \{e_1 + e_2, e_1 - e_3, -e_2 + e_3\},\ \{e_1 + e_2, e_3 - e_2, -e_2 - e_3\}, \\
&\{e_1 - e_2, e_1 + e_3, e_2 - e_3\},\ \{e_1 - e_2, e_1 - e_3, e_2 + e_3\}, \{e_1 - e_2, e_2 + e_3, e_2 - e_3\}, \\
&\{e_2 - e_1, e_1 + e_3, e_1 - e_3\},\ \{e_1 + e_3, e_1 - e_3, e_2 \pm e_3\},\ \{e_1 + e_3, e_2 - e_3, -e_2 - e_3\}, \\
&\{e_1 - e_3, e_2 + e_3, e_3 - e_2\}\ .
\end{aligned} \tag{5.87}$$

With these chosen $\{\rho_1, \rho_2, \rho_3\}$, the hyperplanes $\rho_i(u_*) + z_i = 0$ with $i = 1, 2, 3$ meet at a point u_* with suitable choices of z_i, which are either ϵ_1 or ϵ_2. There may exist more than the chosen three hyperplanes which meet at the same point u_*, in which case one has degenerate poles. Also, at some u_* there could be some vanishing theta functions in the numerator. Let me call the number of vanishing theta functions from the numerator and denominator as $N_n(u_*)$ and $N_d(u_*)$, respectively. When $N_d - N_u < r = 3$, then the corresponding u_* is not a pole due to too many vanishing terms in the numerator. The list below covers all the poles which have nonzero JK-Res, also provided with some illustrations on how to evaluate the residues:

1. When $N_d = 3$, $N_n = 0$, this is a non-degenerate and simple pole. I found 1680 poles in this class. Near $u = u_*$, the integrand relevant for evaluating the residue approximately takes the form of
$$\frac{1}{\prod_{i=1}^{r}(\rho_i(u) - \rho(u_*))} \cdot F(u_*) \ , \tag{5.88}$$
where $F(u)$ denotes the rest of the integrand, with $F(u_*) \neq 0$. The integral of the first factor of (5.88) can be immediately obtained from the basic definition (3.109).

2. There could be degenerate poles with $N_d = N_n + r$, $N_n \neq 0$. The leading divergences of the integrands are simple poles in this case, since $N_d - N_n = r$. Near the pole, the integrand relevant for computing the residue approximately takes the form of
$$\frac{\prod_{i=1}^{N_n}(\rho_i(u) - \rho_i(u_*))}{\prod_{i=N_n+1}^{r+2N_n}(\rho_i(u) - \rho_i(u_*))} \cdot F(u_*) \ , \tag{5.89}$$
where $F(u)$ is the rest of the integrand. The basic rule (5.43) has to be applied to the first factor of (5.89) after decomposing it into a linear combination of the expressions appearing in (5.43). In the $O(6)$ case with $r = 3$, there are two subclasses. Firstly, I found 104 poles with $N_d = 4$, $N_n = 1$. For all the poles in this class, what happens is
$$\text{JK-Res}\frac{\rho_1(u) - \rho_1(u_*)}{\prod_{i=2}^{5}(\rho_i(u) - \rho_i(u_*))} = \frac{1}{2} \ , \tag{5.90}$$

thus all with nonzero residues. Let me illustrate how this is evaluated with an example among the 104 poles, defined with $\{\rho_1,\rho_2,\rho_3,\rho_4\} = \{e_1-e_2, e_1+e_2, e_1+e_3, -e_2-e_3, -2e_2\}$:

$$\begin{aligned}
&\text{JK-Res}\frac{\bigwedge_{a=1}^{3} du_a \cdot (\epsilon_1+\epsilon_2+u_1-u_2)}{(\epsilon_1-2u_2)(\epsilon_2+u_1+u_2)(\epsilon_2-u_2-u_3)(\epsilon_1+u_2+u_3)} \\
&= \text{JK-Res}\frac{\bigwedge_{a=1}^{3} d\tilde{u}_a}{(\tilde{u}_1+\tilde{u}_3)(-\tilde{u}_2-\tilde{u}_3)}\left(\frac{1}{\tilde{u}_1+\tilde{u}_2}+\frac{1}{-2\tilde{u}_2}\right) = \frac{1}{2}+0 = \frac{1}{2}\ , \qquad (5.91)
\end{aligned}$$

where $\tilde{u} = u - u_*$. Moreover, I found 72 poles with $N_d = 5$, $N_n = 2$, in which case one finds either

$$\begin{aligned}
&\text{JK-Res}\frac{(\rho_1(u)-\rho_1(u_*))(\rho_2(u)-\rho_2(u_*))}{\prod_{i=3}^{7}(\rho_i(u)-\rho_i(u_*))} = \qquad (5.92) \\
&\qquad 0\ (32\text{ cases}),\ -\frac{1}{4}\ (16\text{ cases}),\ \frac{1}{4},\ (16\text{ cases})\ \frac{1}{2}\ (8\text{ cases})\ .
\end{aligned}$$

Thus I found 40 more poles. There are no more poles with larger N_d, N_n.

3. In general, there could be degenerate poles with $N_d > N_n + r$. The integrand contains 'multiple poles' in this case. The integrand takes the form of

$$\frac{\prod_{i=1}^{N_n}\theta_1(\rho_i(u)-\rho_i(u_*))}{\prod_{i=N_n+1}^{N_d+N_n}\theta_1(\rho_i(u)-\rho_i(u_*))}\cdot F(u)\ , \qquad (5.93)$$

where $F(u)$ is a combination of θ_1 functions which are nonzero at u_*. Since the first factor contains multiple poles, one would have to expand both first and second factors to certain orders near $u = u_*$, until one obtains a linear combination of the functions appearing in (3.109). The residue will thus be expressed by θ_1 functions and their suitable derivatives at u_*. This class of poles do not show up in the $O(6)$ case. (They will first appear in the $O(8)$ index, explained below.)

With the above $1680+104+40 = 1824$ poles and the computational rules stated in the list, clearly the $O(6)$ elliptic genus can be computed straightforwardly, although the resulting expression will be very long.

Let me explain the pole/residue structures of $O(7)$ index, with rank $r = 3$. The poles are again classified into the above three classes. To be definite, choose $\eta = e_1 + \varepsilon e_2 + \varepsilon^2 e_3$. I will simply list the number of poles in each class.

1. non-degenerate poles ($N_d = 3$, $N_n = 0$): 2468 cases

2. degenerate (but simple) poles: With $N_d = 4$, $N_n = 1$, there are 106 degenerate and simple poles. The relevant integrals of the form of (5.90) are either $\frac{1}{2}$ or 1, depending on u_*. With $N_d = 5$, $N_n = 2$, there are 72 cases. The integral analogous to (5.92) are either $0, -\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$. There are 32 cases with zero residues. So one finds 40 poles in this class. Finally, there are 4 cases with $N_d = 6$, $N_n = 3$, and the JK-Res of the rational functions are either

$$\text{JK-Res}\frac{\bigwedge_{a=1}^{r} d\tilde{u}_a \cdot \prod_{i=1}^{3} \rho_i(\tilde{u})}{\prod_{i=4}^{r+6} \rho_i(\tilde{u})} = \frac{1}{2} \ \ (2 \text{ cases}), \ \text{ or } \ 0 \ \ (2 \text{ cases}) \ . \tag{5.94}$$

So there are 2 poles in the last class. I have not found further degenerate simple poles with larger N_n.

3. degenerate multiple poles ($N_d > N_n + 3$): I have not found any poles in this case.

So I found $2468 + 106 + 40 + 2 = 2616$ poles with nonzero JK-Res.

As a final illustration, let me consider the $O(8)$ contour integral with rank $r = 4$. The number of poles quickly increases, as follows:

1. non-degenerate poles ($N_d = 4$, $N_n = 0$): 32304 poles

2. degenerate (but simple) poles: With $N_d = 5$, $N_n = 1$, there are 4424 poles. With $N_d = 6$, $N_n = 2$, there are 1696 poles. With $N_d = 7$, $N_n = 3$, there are 88 poles. Finally, with $N_d = 8$, $N_n = 4$, there are 200 poles.

3. degenerate multiple poles ($N_d > N_n + 3$): there are 72 such poles.

So I found $32304 + 4424 + 1696 + 88 + 200 + 72 = 38784$ poles for the $O(8)$ contour integral.

5.2.3 Comparison with the instanton partition function

In this section, I will explain how the E-string elliptic genus is compatible with the instanton partition function of a 5 dimensional super-Yang-Mills theory with $Sp(1)$ gauge group considered in Section 4.4. The basic idea is that suitable circle reductions of 6d SCFTs sometimes admit 5d SYM descriptions at low energy. The latter SYM, despite being non-renormalizable, remembers the 6d KK degrees in its solitonic sector as the instanton solitons [4, 5]. The self-dual strings wrapping the circle become the W-bosons, quarks or their superpartner particles in 5d. So the Witten index for the threshold bounds of these particles with instantons in the Coulomb branch [15, 51] will carry information on the elliptic genera of wrapped self-dual strings.

Let me expand the discussion in Section 4.4, which considered the circle reduction of 6d $(1,0)$ SCFT with the Wilson line RA that breaks $E_8 \to SO(16)$.[3]

$$RA = (0,0,0,0,0,0,0,1)\ . \tag{5.95}$$

This is in the convention that one picks the Cartans of $SO(16)$ as rotations on the 8 orthogonal 2-planes. The circle holonomy generated by this Wilson line is $\exp(2\pi i RA\cdot F)$, with $F = (F_1, F_2, \cdots, F_8)$ being the Cartans of $SO(16) \subset E_8$ in the same basis. The normalization is $F_l = \pm\frac{1}{2}$ for $SO(16)$ spinors. The holonomy with (5.95) acts on **128** as -1, and on **120** as $+1$. So E_8 symmetry breaks down to $SO(16)$. This is the background which admits the type I' theory description for small R. In the type I' brane system, one obtains the 5d $Sp(N)$ gauge theory with 1 antisymmetric and 8 fundamental hypermultiplets which lives on the worldvolume of D4-branes. This 5d gauge theory is a low-energy description of the 6d $(1,0)$ superconformal field theory compactified on a circle with E_8 Wilson line. Note that, from the worldvolume theory on D4 or M5-branes, $SO(16)$ or E_8 act as global symmetries. So from the 5d/6d field theories, the Wilson line explained above are nondynamical background fields.

[3]Had one been reducing the M5-M9 system with zero Wilson line, one would have obtained the strongly interacting 5d SCFT with E_8 symmetry [13, 101], discovered in [6].

Consider the system consisting of single M5-brane and an M9-plane, compactified on a circle with the above Wilson line. One obtains an $Sp(1)$ gauge theory description in 5d. Taking into account the effect of the background Wilson line (5.95), one can identify various charges of the 5d SYM theory and the 6d $(1,0)$ theory on circle as follows:

$$k = 2P + n(RA \cdot RA) - 2\left(RA \cdot \tilde{F}\right) = 2P + n - 2\tilde{F}_8 \tag{5.96}$$

$$F_l = \tilde{F}_l - n(RA_l) \quad \rightarrow \quad F_8 = \tilde{F}_8 - n \ . \tag{5.97}$$

Here, k is the Yang-Mills instanton charge on D4's (i.e. D0-brane number in the type I' theory), P is the momentum on E-strings along the circle, $\tilde{F}$ is the E_8 Cartan charge in the 6d theory, and F is the $SO(16)$ Cartan charges in the 5d SYM. n is the $U(1) \subset Sp(1)$ electric charge in the Coulomb phase, which is identified as the winding number of the E-strings. This formula can be naturally inferred by starting from the charge relations of the fundamental type I' stings on $\mathbb{R}^{8+1} \times I$ and the heterotic strings on $\mathbb{R}^{8+1} \times S^1$ [102, 82], where I is a segment, and then putting an M5-brane on I to decompose a heterotic string into two E-strings [97].

Later in this section, I will consider an index for the E-strings, with the weight given by

$$q^k e^{2\pi i m_8 F_8} w^n \prod_{l=1}^{7} e^{2\pi i m_l F_l} = q^{2P} (y_8')^{\tilde{F}_8} (w')^n \prod_{l=1}^{7} e^{2\pi i m_l \tilde{F}_l} \tag{5.98}$$

with $y_i \equiv e^{2\pi i m_i}$, where

$$y_8' = y_8 q^{-2} \quad , \quad w' = w q y_8^{-1} \ . \tag{5.99}$$

The right hand side is the natural expression for the E-strings, while the instanton calculus will naturally use the expression on the left hand side. After doing the instanton calculus with the above weight, I will use the fugacities y_8', w' given by (5.99). This redefinition of fugacities plays the role of canceling the background E_8 Wilson line (5.95), which obscures the E_8 symmetry in the type I' instanton calculus.[4]

Let me consider Z_{inst} of the 5d $Sp(1)$ gauge theory, i.e., the rank 1 6d $(1,0)$ SCFT compactified on circle with E_8 Wilson line. To see the E-string physics, for instance the E_8 symmetry, one should make a replacement (5.99) in the instanton partition function. The instanton partition function takes the form of series expansion in q, $Z(q,w,y) = \sum_{k=0}^{\infty} Z_k(w,y)q^k$. So at a given order in the modular parameter q, one captures the spectrum of arbitrary number of E-strings by computing $Z_k(w,y)$ exactly in w. This is in contrast to the previous study of the E-string elliptic genus, keeping definite order $Z_n(q,y)$ in w which is exact in q. So to confirm that the two approaches yield the same result, one should make a double expansion of $Z(q,w,y)$ in q, w and compare, taking into account the shifts (5.99). One first takes the E-string indices $Z_n(q,y_8')$ and defines $\tilde{Z}_n(q,y_8') \sim Z_n(q,y_8q^{-2})$ using (5.99). While making the study of instanton partition function of the $Sp(1)$ gauge theory in Section 4.4, $Z_k(w,y)$ was computed up to $k=5$. So expanding $\tilde{Z}_n$ up to $\mathcal{O}(q^5)$, and expanding Z_{inst} computed from the instanton side to $\mathcal{O}(w^n)$ for some low n, I shall find perfect agreement of the two results.

Instanton partition function

To take into account the effect of the Wilson line which breaks E_8 down to $SO(16)$, one has to make a shift of the fugacities by (5.99). After inserting $y_8' = y_8q^{-2}$ (or $e^{2\pi i m_8} \to e^{2\pi i m_8 - 2\pi i \tau}$) to the E-string indices of Section 5.2.2, various E-string indices can be written as

$$Z_1 = \left(\frac{y_8}{q}\right)\tilde{Z}_1,\ Z_2 = \left(\frac{y_8}{q}\right)^2\tilde{Z}_2,\ Z_3 = \left(\frac{y_8}{q}\right)^3\tilde{Z}_3,\ Z_4 = \left(\frac{y_8}{q}\right)^4\tilde{Z}_4 \qquad (5.100)$$

with

$$\tilde{Z}_1 = \frac{1}{2}\left(-Z_{1(1)} + Z_{1(2)} + Z_{1(3)} - Z_{1(4)}\right) \qquad (5.101)$$

[4] In this subsection, the definition of q is given by $q = e^{\pi i \tau}$, instead of $q = e^{2\pi i \tau}$ used in all other sections of Chapter 5. This is because the single instanton carries $q^{\frac{1}{2}}$ factor in the other convention, due to the fractional Wilson line, which I want to change to q^1. This is the reason for the factor q^{2P} in (5.98).

$$\tilde{Z}_2 = \frac{1}{2}Z_{2(0)} + \frac{1}{4}\left(-Z_{2(1)} - Z_{2(2)} + Z_{2(3)} + Z_{2(4)} - Z_{2(5)} - Z_{2(6)}\right)$$
$$\tilde{Z}_3 = \frac{1}{4}\left(-Z_{3(1)} - Z_{3(2)} + Z_{3(3)} + Z_{3(4)}\right) + \frac{1}{8}\left(-Z_{3(1)'} - Z_{3(2)'} + Z_{3(3)'} + Z_{3(4)'}\right)$$
$$\tilde{Z}_4 = \frac{1}{8}\left(Z_{4(1)} - Z_{4(2)} - Z_{4(2)'} - Z_{4(3)} - Z_{4(3)'} + Z_{4(4)} + Z_{4(4)'}\right) + \frac{1}{16}Z_{4(1)'} \ ,$$

and so on, where $Z_{n(i)}$'s are all defined and computed in Section 5.2.2. In all $Z_{n(i)}$ on the right hand side, the arguments are y_8, not y_8'. The overall factors $\left(y_8 q^{-1}\right)^n$ in (5.100) cancels with the shift $w' = wqy_8^{-1}$ in $Z = \sum_{n=0}^{\infty}(w')^n Z_n$. Namely, the E_8 mass shift is inducing a different value of 2d theta angle, by changing various signs in (5.101). I computed $\tilde{f}(w, q, \epsilon_{1,2}, m_i)$ defined by

$$\tilde{Z} \equiv \sum_{n=0}^{\infty} w^n \tilde{Z}_n(q, \epsilon_{1,2}, m_i) = PE\left[\tilde{f}\right] \equiv \exp\left[\sum_{n=1}^{n} \frac{1}{n}\tilde{f}(w^n, q^n, n\epsilon_1, n\epsilon_2, nm_l)\right] \ , \tag{5.102}$$

and expand $\tilde{f} = \sum_{n=1}^{\infty} w^n \tilde{f}_n(q, \epsilon_{1,2}, m_i)$. The results up to $\mathcal{O}(q^5)$ are as follows.

Defining $t \equiv e^{i\pi\epsilon_1 + i\pi\epsilon_2}$, $u \equiv e^{i\pi\epsilon_1 - i\pi\epsilon_2}$, $\tilde{f}_1$ is given by $\frac{t}{(1-tu)(1-t/u)}$ times

$$\begin{aligned}
&+q^0 \cdot \chi^{\mathrm{SO}(16)}_{\mathbf{16}} + q^1 \cdot \chi^{\mathrm{SO}(16)}_{\mathbf{128}} \\
&+q^2\Big[(t+t^{-1})(u+u^{-1})\chi^{\mathrm{SO}(16)}_{\mathbf{16}} + \chi^{\mathrm{SO}(16)}_{\mathbf{560}} + \chi^{\mathrm{SO}(16)}_{\mathbf{16}}\Big] \\
&+q^3\Big[(t+t^{-1})(u+u^{-1})\chi^{\mathrm{SO}(16)}_{\mathbf{128}} + \chi^{\mathrm{SO}(16)}_{\mathbf{1920}} + \chi^{\mathrm{SO}(16)}_{\mathbf{128}}\Big] \\
&+q^4\Big[(t+t^{-1})(u+u^{-1})(\chi^{\mathrm{SO}(16)}_{\mathbf{560}} + 2\chi^{\mathrm{SO}(16)}_{\mathbf{16}}) + \left((t^2+1+t^{-2})(u^2+1+u^{-2})-1\right)\chi^{\mathrm{SO}(16)}_{\mathbf{16}} \\
&\quad + \chi^{\mathrm{SO}(16)}_{\mathbf{4368}} + \chi^{\mathrm{SO}(16)}_{\mathbf{1344}} + \chi^{\mathrm{SO}(16)}_{\mathbf{560}} + 4\chi^{\mathrm{SO}(16)}_{\mathbf{16}}\Big] \\
&+q^5\Big[(t+t^{-1})(u+u^{-1})(\chi^{\mathrm{SO}(16)}_{\mathbf{1920}} + 2\chi^{\mathrm{SO}(16)}_{\mathbf{128}}) + \left((t^2+1+t^{-2})(u^2+1+u^{-2})-1\right)\chi^{\mathrm{SO}(16)}_{\mathbf{128}} \\
&\quad + \chi^{\mathrm{SO}(16)}_{\mathbf{13312}} + 2\chi^{\mathrm{SO}(16)}_{\mathbf{1920}} + 4\chi^{\mathrm{SO}(16)}_{\mathbf{128}}\Big] + \mathcal{O}(q^6)
\end{aligned} \tag{5.103}$$

The boldfaced subscripts are the irreps of $SO(16) \subset E_8$ visible by the 5d $Sp(1)$ gauge theory with 8 fundamental flavors. $\chi^{\mathrm{SO}(16)}_{\mathbf{R}}$ is the $SO(16)$ character of the representation $\mathbf{R}$. I computed the Z_{inst} of the 5d SYM, following the procedures outlined above (explained in Section 4.4), up to five instantons. I further expanded it in the Coulomb VEV parameter to extract the $\mathcal{O}(w^1)$ order. This completely agrees with (5.103).

$\tilde{f}_2$ is given by $\frac{t}{(1-tu)(1-t/u)}$ times

$$
\begin{aligned}
&-q^0\cdot(t+t^{-1})-q^1\Big[(t+t^{-1})\chi^{\mathrm{SO}(16)}_{\mathbf{128}}\Big] \qquad (5.104)\\
&-q^2\Big[(t^3+t+t^{-1}+t^{-3})(u^2+1+u^{-2})+(u+u^{-1})+(t^2+1+t^{-2})(u+u^{-1})(\chi^{\mathrm{SO}(16)}_{\mathbf{120}}+1)\\
&\quad+(t+t^{-1})(\chi^{\mathrm{SO}(16)}_{\mathbf{1820}}+\chi^{\mathrm{SO}(16)}_{\mathbf{120}}+2)\Big]\\
&-q^3\Big[(t+t^{-1})((t^2+t^{-2})(u^2+1+u^{-2})-1)\chi^{\mathrm{SO}(16)}_{\mathbf{128}}+(u+u^{-1})\chi^{\mathrm{SO}(16)}_{\mathbf{128}}\\
&\quad+(t^2+1+t^{-2})(u+u^{-1})(\chi^{\mathrm{SO}(16)}_{\mathbf{1920}}+2\chi^{\mathrm{SO}(16)}_{\mathbf{128}})+(t+t^{-1})(\chi^{\mathrm{SO}(16)}_{\mathbf{13312}}+\chi^{\mathrm{SO}(16)}_{\mathbf{1920}}+4\chi^{\mathrm{SO}(16)}_{\mathbf{128}})\Big]\\
&-q^4\Big[(t^4+t^{-4})(u+u^{-1})+(t^3+t+t^{-1}+t^{-3})(u^4+u^{-4})\\
&\quad+(t^2+1+t^{-2})(u^3+u^{-3})+(t+t^{-1})(u^2+u^{-2})+(t^5+t^{-5})(u^4+u^2+1+u^{-2}+u^{-4})\\
&\quad+(u+u^{-1})(\chi^{\mathrm{SO}(16)}_{\mathbf{1820}}+2\chi^{\mathrm{SO}(16)}_{\mathbf{120}}+3)+\big((t^4+t^2+1+t^{-2}+t^{-4})(u^3+u^{-3})\\
&\quad+(t^4+t^{-4})(u+u^{-1})+(t^3+t^{-3})+(t+t^{-1})(u^2+u^{-2})\big)\,(\chi^{\mathrm{SO}(16)}_{\mathbf{120}}+1)\\
&\quad+\big((t^3+t^{-3})(u^2+1+u^{-2})+(t+t^{-1})(u^2+u^{-2})\big)\,(\chi^{\mathrm{SO}(16)}_{\mathbf{1820}}+\chi^{\mathrm{SO}(16)}_{\mathbf{135}}+2\chi^{\mathrm{SO}(16)}_{\mathbf{120}}+5)\\
&\quad+(t^2+1+t^{-2})(u+u^{-1})(\chi^{\mathrm{SO}(16)}_{\mathbf{8008}}+\chi^{\mathrm{SO}(16)}_{\mathbf{70.5}}+2\chi^{\mathrm{SO}(16)}_{\mathbf{1820}}+\chi^{\mathrm{SO}(16)}_{\mathbf{135}}+6\chi^{\mathrm{SO}(16)}_{\mathbf{120}}+8)\\
&\quad+(t+t^{-1})(\chi^{\mathrm{SO}(16)}_{\mathbf{600.5}}+\chi^{\mathrm{SO}(16)}_{\mathbf{8008}}+\chi^{\mathrm{SO}(16)}_{\mathbf{70.5}}+\chi^{\mathrm{SO}(16)}_{\overline{\mathbf{6435}}}+\chi^{\mathrm{SO}(16)}_{\mathbf{5304}})\\
&\quad+(t+t^{-1})(+4\chi^{\mathrm{SO}(16)}_{\mathbf{1820}}+3\chi^{\mathrm{SO}(16)}_{\mathbf{135}}+9\chi^{\mathrm{SO}(16)}_{\mathbf{120}}+14)\Big]\\
&-q^5\Big[\,\big((t^5+t^{-5})(u^4+u^2+1+u^{-2}+u^{-4})+(t^3+t+t^{-1}+t^{-3})(u^4+u^{-4})\\
&\quad+(t^2+1+t^{-2})(u^3+u^{-3})+(t^4+t^{-4})(u+u^{-1})+(t+t^{-1})(u^2+u^{-2})\big)\,\chi^{\mathrm{SO}(16)}_{\mathbf{128}}\\
&\quad+\big((t^3+t^{-3})(u^2+1+u^{-2})+(t+t^{-1})(u^2+u^{-2})\big)\,(\chi^{\mathrm{SO}(16)}_{\mathbf{13321}}+3\chi^{\mathrm{SO}(16)}_{\mathbf{1920}}+7\chi^{\mathrm{SO}(16)}_{\mathbf{128}})\\
&\quad+\big((t^2+t^{-2})(u+u^{-1})+(t+t^{-1})+(u+u^{-1})\big)\,(\chi^{\mathrm{SO}(16)}_{\mathbf{56320}}+\chi^{\mathrm{SO}(16)}_{\mathbf{15360}}+3\chi^{\mathrm{SO}(16)}_{\mathbf{13312}})\\
&\quad+\big((t^2+t^{-2})(u+u^{-1})+(t+t^{-1})+(u+u^{-1})\big)\,(7\chi^{\mathrm{SO}(16)}_{\mathbf{1920}}+14\chi^{\mathrm{SO}(16)}_{\mathbf{128}})\\
&\quad+(u+u^{-1})(\chi^{\mathrm{SO}(16)}_{\mathbf{13312}}+2\chi^{\mathrm{SO}(16)}_{\mathbf{1920}}+5\chi^{\mathrm{SO}(16)}_{\mathbf{128}})+\big((t^2+1+t^{-2})(u^3+u^{-3})\\
&\quad+(t^4+t^{-4})(u^3+u+u^{-1}+u^{-3})+(t+t^{-1})(u^2+u^{-2})+(t^3+t^{-3})\big)\,(\chi^{\mathrm{SO}(16)}_{\mathbf{1920}}+2\chi^{\mathrm{SO}(16)}_{\mathbf{128}})\\
&\quad+(t+t^{-1})(\chi^{\mathrm{SO}(16)}_{\mathbf{161280}}+\chi^{\mathrm{SO}(16)}_{\mathbf{141440}}+3\chi^{\mathrm{SO}(16)}_{\mathbf{13312}}+5\chi^{\mathrm{SO}(16)}_{\mathbf{1920}}+9\chi^{\mathrm{SO}(16)}_{\mathbf{128}})\Big]+\mathcal{O}(q^6)
\end{aligned}
$$

This again agrees with the result obtained in Section 4.4.

I also computed $\tilde{f}_3$ with all $SO(16)\subset E_8$ masses turned off. It again completely agrees with $\tilde{f}_3$ computed from 5d instanton calculus, up to q^5 order that I checked.

Also, for 3 and 4 E-strings, I have kept all E_8 masses and compared the 2d elliptic genus with the instanton partition function up to 1 instanton order, which all show agreements.

So I saw that the instanton calculus provides the correct index for the E_8 6d SCFT. One virtue of this approach would be that, at a given order in q, the index is computed exactly in w. In particular, the chemical potential for the E-string number (the Coulomb VEV of 5d SYM) is an integration variable in the curved space partition functions, which can be used to study the conformal field theory physics. So knowing the exact form of the partition function in w will be desirable to understand the curved space partition functions.

Appendix A

Characters of $SO(2N_f)$

$SO(2N_f)$ characters can be obtained by the Weyl character formula [103]:

$$\chi(h,m) = \frac{\det[\sinh(m_i(h_j + N_f - j))] + \det[\cosh(m_i(h_j + N_f - j))]}{\det[\cosh(m_i(N_f - j))]} \tag{A.1}$$

where h denotes the highest weight of the representation with $h_1 \geq h_2 \geq \cdots \geq h_{N_f-1} \geq |h_{N_f}|$ and m denotes chemical potential. For example, two spinor representations of the highest weights $(\frac{1}{2}, \cdots, \pm\frac{1}{2})$ have the following characters:

$$\chi_{\pm}^{N_f} = \frac{1}{2}\prod_{i=1}^{N_f}\left(y_i + y_i^{-1}\right) \pm \frac{1}{2}\prod_{i=1}^{N_f}\left(y_i - y_i^{-1}\right) \tag{A.2}$$

where $y_i = e^{m_i/2}$. In the thesis two chirality conventions are used for such spinor representations. Throughout Section 4.3 I call $(\frac{1}{2}, \cdots, \frac{1}{2})$ the chiral spinor and call $(\frac{1}{2}, \cdots, -\frac{1}{2})$ the anti-chiral spinor, which has a bar on its name. On the other hand, in Section 4.3.2 I follow the convention of [79] for computational convenience.

All of the E_n characters can be read off from the branching rules of E_n into its subgroup specified in the main text. For example, $E_5 = S0(10)$ adjoint has the following decomposition under $SO(8) \times U(1)_I$:

$$\mathbf{45} \to \mathbf{1}_0 + \mathbf{28}_0 + (\mathbf{8}_s)_1 + (\mathbf{8}_s)_{-1}. \tag{A.3}$$

The corresponding character is given by

$$\chi_{\mathbf{45}}^{E_5} = \chi_{\mathbf{1}}^{SO(8)} + \chi_{\mathbf{28}}^{SO(8)} + q\chi_{\mathbf{8}_s}^{SO(8)} + q^{-1}\chi_{\mathbf{8}_s}^{SO(8)}. \tag{A.4}$$

Appendix B

Modular forms and Jacobi forms

A modular form $f_n(\tau)$ of weight n transforms under $SL(2,\mathbb{Z})$ as

$$f_n\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^n f_n(\tau) \quad , \qquad ad-bc=1 \ . \tag{B.1}$$

An important class of modular forms is given by the Eisenstein series,

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \tag{B.2}$$

where $q = e^{2\pi i \tau}$. The Bernoulli numbers B_{2k} and the divisor functions $\sigma_k(n)$ are defined by

$$\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x - 1} \ , \quad \sigma_k(n) = \sum_{d|n} d^k. \tag{B.3}$$

$E_{2k}(\tau)$ are modular forms of weight $2k$, expect for $E_2(\tau)$ which involves an anomalous term,

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) + \frac{6}{i\pi} c(c\tau+d). \tag{B.4}$$

Another example of modular form is the Dedekind eta function $\eta(\tau)$, defined by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \ . \tag{B.5}$$

Under the modular transformation, $\eta(\tau)$ behaves as a weight $\frac{1}{2}$ form up to a phase $\epsilon(a,b,c,d)$,

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \epsilon(a,b,c,d)\cdot(c\tau+d)^{1/2}\eta(\tau). \tag{B.6}$$

Jacobi forms have a modular parameter τ and an elliptic parameter z. Modular transformation for Jacobi forms $\phi_{k,m}(\tau,z)$ of weight k and index m is given by

$$\phi_{k,m}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi imcz^2}{c\tau+d}}\phi_{k,m}(\tau,z), \tag{B.7}$$

Under the translation of the elliptic parameter z, they behave as

$$\phi_{k,m}(\tau,z+a\tau+b) = e^{-2\pi im(a^2\tau+2az)}\phi_{k,m}(\tau,z). \tag{B.8}$$

where a,b are integers.

The Jacobi theta function $\vartheta(\tau,z)$ is a Jacobi form of weight $\frac{1}{2}$ and index $\frac{1}{2}$, defined as

$$\vartheta(\tau,z) = \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}}y)(1+q^{n-\frac{1}{2}}y^{-1}) = \sum_{n\in\mathbb{Z}} q^{n^2/2}y^n \tag{B.9}$$

where $q\equiv e^{2\pi i\tau}$ and $y\equiv e^{2\pi iz}$. Let me define three other functions which are closely related to the Jacobi theta function, and define

$$\begin{aligned}
\theta_1(\tau,z) &= -iq^{1/8}y^{1/2}\vartheta(\tau,z+\tfrac{1+\tau}{2}) &&= -iq^{1/8}y^{1/2}\prod_{n=1}^{\infty}(1-q^n)(1-q^ny)(1-q^{n-1}y^{-1})\\
\theta_2(\tau,z) &= q^{1/8}y^{1/2}\vartheta(\tau,z+\tfrac{\tau}{2}) &&= q^{1/8}y^{1/2}\prod_{n=1}^{\infty}(1-q^n)(1+q^ny)(1+q^{n-1}y^{-1})\\
\theta_3(\tau,z) &= \vartheta(\tau,z) &&= \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}}y)(1+q^{n-\frac{1}{2}}y^{-1})\\
\theta_4(\tau,z) &= \vartheta(\tau,z+\tfrac{1}{2}) &&= \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-\frac{1}{2}}y)(1-q^{n-\frac{1}{2}}y^{-1}).
\end{aligned} \tag{B.10}$$

From here, when I omit the parameter in various functions, it should be understood as τ. $\theta_n(z)$'s are related to others by the half-period shifts:

$$\theta_1(z+\tfrac{1}{2}) = \theta_2(z) \qquad \theta_1(z+\tfrac{1+\tau}{2}) = q^{-1/8}y^{-1/2}\theta_3(z) \qquad \theta_1(z+\tfrac{\tau}{2}) = iq^{-1/8}y^{-1/2}\theta_4(z)$$

$$\theta_2(z+\tfrac{1}{2}) = -\theta_1(z) \quad \theta_2(z+\tfrac{1+\tau}{2}) = -iq^{-1/8}y^{-1/2}\theta_4(z) \quad \theta_2(z+\tfrac{\tau}{2}) = q^{-1/8}y^{-1/2}\theta_3(z)$$
$$\theta_3(z+\tfrac{1}{2}) = \theta_4(z) \quad \theta_3(z+\tfrac{1+\tau}{2}) = iq^{-1/8}y^{-1/2}\theta_1(z) \quad \theta_3(z+\tfrac{\tau}{2}) = q^{-1/8}y^{-1/2}\theta_2(z)$$
$$\theta_4(z+\tfrac{1}{2}) = \theta_3(z) \quad \theta_4(z+\tfrac{1+\tau}{2}) = q^{-1/8}y^{-1/2}\theta_2(z) \quad \theta_4(z+\tfrac{\tau}{2}) = iq^{-1/8}y^{-1/2}\theta_1(z) \tag{B.11}$$

Various identities: The modular forms E_4, E_6, and η can be expressed in terms of Jacobi theta functions with their elliptic parameters z set to zero:

$$\begin{aligned} E_4 &= \tfrac{1}{2}(\theta_2(0)^8+\theta_3(0)^8+\theta_4(0)^8) \\ E_6 &= \tfrac{1}{2}(\theta_2(0)^4+\theta_3(0)^4)(\theta_3(0)^4+\theta_4(0)^4)(\theta_4(0)^4-\theta_2(0)^4) \\ \eta^3 &= \theta_2(0)\theta_3(0)\theta_4(0). \end{aligned} \tag{B.12}$$

$\theta_n(z)$'s also satisfy

$$\theta_2(z)^4-\theta_1(z)^4=\theta_3(z)^4-\theta_4(z)^4\ , \quad \theta_2(0)^4=\theta_3(0)^4-\theta_4(0)^4\ . \tag{B.13}$$

Further identities of $\theta_n(z)$'s with different elliptic parameters are

$$\begin{aligned}
\theta_1(a+b)\theta_1(a-b)\theta_4(0)^2 &= \theta_3(a)^2\theta_2(b)^2-\theta_2(a)^2\theta_3(b)^2=\theta_1(a)^2\theta_4(b)^2-\theta_4(a)^2\theta_1(b)^2\\
\theta_2(a+b)\theta_2(a-b)\theta_4(0)^2 &= \theta_4(a)^2\theta_2(b)^2-\theta_1(a)^2\theta_3(b)^2=\theta_2(a)^2\theta_4(b)^2-\theta_3(a)^2\theta_1(b)^2\\
\theta_3(a+b)\theta_3(a-b)\theta_2(0)^2 &= \theta_3(a)^2\theta_2(b)^2+\theta_4(a)^2\theta_1(b)^2=\theta_2(a)^2\theta_3(b)^2+\theta_1(a)^2\theta_4(b)^2\\
\theta_3(a+b)\theta_3(a-b)\theta_3(0)^2 &= \theta_1(a)^2\theta_1(b)^2+\theta_3(a)^2\theta_3(b)^2=\theta_2(a)^2\theta_2(b)^2+\theta_4(a)^2\theta_4(b)^2\\
\theta_3(a+b)\theta_3(a-b)\theta_4(0)^2 &= \theta_4(a)^2\theta_3(b)^2-\theta_1(a)^2\theta_2(b)^2=\theta_3(a)^2\theta_4(b)^2-\theta_2(a)^2\theta_1(b)^2\\
\theta_4(a+b)\theta_4(a-b)\theta_2(0)^2 &= \theta_4(a)^2\theta_2(b)^2+\theta_3(a)^2\theta_1(b)^2=\theta_2(a)^2\theta_4(b)^2+\theta_1(a)^2\theta_3(b)^2\\
\theta_4(a+b)\theta_4(a-b)\theta_3(0)^2 &= \theta_4(a)^2\theta_3(b)^2+\theta_2(a)^2\theta_1(b)^2=\theta_3(a)^2\theta_4(b)^2+\theta_1(a)^2\theta_2(b)^2\\
\theta_4(a+b)\theta_4(a-b)\theta_4(0)^2 &= \theta_3(a)^2\theta_3(b)^2-\theta_2(a)^2\theta_2(b)^2=\theta_4(a)^2\theta_4(b)^2-\theta_1(a)^2\theta_1(b)^2
\end{aligned} \tag{B.14}$$

$$\begin{aligned}
\theta_1(a\pm b)\theta_2(a\mp b)\theta_3(0)\theta_4(0) &= \theta_1(a)\theta_2(a)\theta_3(b)\theta_4(b)\pm\theta_3(a)\theta_4(a)\theta_1(b)\theta_2(b)\\
\theta_1(a\pm b)\theta_3(a\mp b)\theta_2(0)\theta_4(0) &= \theta_1(a)\theta_3(a)\theta_2(b)\theta_4(b)\pm\theta_2(a)\theta_4(a)\theta_1(b)\theta_3(b)\\
\theta_1(a\pm b)\theta_4(a\mp b)\theta_2(0)\theta_3(0) &= \theta_1(a)\theta_4(a)\theta_2(b)\theta_3(b)\pm\theta_2(a)\theta_3(a)\theta_1(b)\theta_4(b)
\end{aligned}$$

$$
\begin{aligned}
\theta_2(a \pm b)\theta_3(a \mp b)\theta_2(0)\theta_3(0) &= \theta_2(a)\theta_3(a)\theta_2(b)\theta_3(b) \mp \theta_1(a)\theta_4(a)\theta_1(b)\theta_4(b) \\
\theta_2(a \pm b)\theta_4(a \mp b)\theta_2(0)\theta_4(0) &= \theta_2(a)\theta_4(a)\theta_2(b)\theta_4(b) \mp \theta_1(a)\theta_3(a)\theta_1(b)\theta_3(b) \\
\theta_3(a \pm b)\theta_4(a \mp b)\theta_3(0)\theta_4(0) &= \theta_3(a)\theta_4(a)\theta_3(b)\theta_4(b) \mp \theta_1(a)\theta_2(a)\theta_1(b)\theta_2(b).
\end{aligned} \tag{B.15}
$$

Under the shift of modular parameter $\tau \to \tau' = \tau + 1$, the changes are

$$
\begin{aligned}
\theta_1(\tau+1, z) &= e^{i\pi/4}\theta_1(\tau, z) \qquad & \theta_2(\tau+1, z) &= e^{i\pi/4}\theta_2(\tau, z) \\
\theta_3(\tau+1, z) &= \theta_4(\tau, z) \qquad & \theta_4(\tau+1, z) &= \theta_3(\tau, z).
\end{aligned} \tag{B.16}
$$

Watson's identities and Landen's formulas involve doubling of τ,

$$
\theta_1(\tau, z)\theta_1(\tau, w) = \theta_3(2\tau, z+w)\theta_2(2\tau, z-w) - \theta_2(2\tau, z+w)\theta_3(2\tau, z-w) \tag{B.17}
$$

$$
\theta_3(\tau, z)\theta_3(\tau, w) = \theta_3(2\tau, z+w)\theta_3(2\tau, z-w) + \theta_2(2\tau, z+w)\theta_2(2\tau, z-w)
$$

$$
\begin{aligned}
\theta_1(2\tau, 2z) &= \theta_1(\tau, z)\theta_2(\tau, z)/\theta_4(2\tau, 0) \\
\theta_4(2\tau, 2z) &= \theta_3(\tau, z)\theta_4(\tau, z)/\theta_4(2\tau, 0)
\end{aligned} \tag{B.18}
$$

Considering the case with $z = 0$, one obtains

$$
\theta_2(2\tau, 0) = \sqrt{\tfrac{\theta_3(\tau,0)^2 - \theta_4(\tau,0)^2}{2}},\ \theta_3(2\tau, 0) = \sqrt{\tfrac{\theta_3(\tau,0)^2 + \theta_4(\tau,0)^2}{2}},\ \theta_4(2\tau, 0) = \sqrt{\theta_3(\tau, 0)\theta_4(\tau, 0)}. \tag{B.19}
$$

Differentiations by τ, z: The τ derivatives of E_2, E_4, E_6 can be obtained from the Ramanujan identities

$$
q\frac{d}{dq}E_2 = \frac{1}{12}(E_2^2 - E_4),\ q\frac{d}{dq}E_4 = \frac{1}{3}(E_2E_4 - E_6),\ q\frac{d}{dq}E_6 = \frac{1}{2}(E_2E_6 - E_4^2). \tag{B.20}
$$

The τ derivative of the eta function is given by

$$
q\frac{d}{dq}\eta^3 = \frac{\eta^3}{8}E_2. \tag{B.21}
$$

As for the theta functions, first note that $\theta_n(z)$'s are solutions of

$$
\left[\frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial z^2} - \frac{1}{i\pi}\frac{\partial}{\partial \tau}\right]\theta_n(\tau, z) = \left[\frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial z^2} - 2q\frac{\partial}{\partial q}\right]\theta_n(\tau, z) = 0. \tag{B.22}
$$

θ_1 is an odd function of z, while $\theta_2, \theta_3, \theta_4$ are even functions of z. The lowest non-vanishing derivatives of θ_n's at $z = 0$ are given by

$$\begin{aligned}
&\theta_1^{(1)}(0) = 2\pi\eta^3 && \theta_2^{(2)}(0) = -\tfrac{\pi^2}{3}\theta_2(0)\left(E_2 + \theta_3(0)^4 + \theta_4(0)^4\right) \\
&\theta_3^{(2)}(0) = -\tfrac{\pi^2}{3}\theta_3(0)\left(E_2 + \theta_2(0)^4 - \theta_4(0)^4\right) && \theta_4^{(2)}(0) = -\tfrac{\pi^2}{3}\theta_4(0)\left(E_2 - \theta_2(0)^4 - \theta_3(0)^4\right) ,
\end{aligned} \tag{B.23}$$

where (n) denotes n'th derivative with respect to the elliptic parameter. Using (B.22), (B.23), (B.20) and (B.21), one can also express the higher z derivatives $\theta_1^{(2n+1)}(0)$, $\theta_2^{(2n)}(0)$, $\theta_3^{(2n)}(0)$, $\theta_4^{(2n)}(0)$ at $z = 0$ in terms of $\theta_2(0)$, $\theta_3(0)$, $\theta_4(0)$, E_2. See Appendix C.2 for more details, where this procedure will be illustrated and used to prove exact properties of the E-string indices.

Appendix C

Details of computation

C.1 Genus expansions of topological string amplitudes

In this appendix, I will summarize some low genus results that were used in Section 5.2.2. The low genus amplitudes have been studied in [13, 104, 96, 99, 89]. I list the unrefined results till $g \leq 5$ (as written in [99]), and some refined results that are used to compare with my results.

For three E-strings, the unrefined genus expansion coefficients $F^{(0,g,3)}$ are

$$\begin{aligned}
F^{(0,0,3)} =& \frac{54E_2^2E_4^3 + 216E_2E_4^2E_6 + 109E_4^4 + 197E_4E_6^2}{15552\eta^{36}} \\
F^{(0,1,3)} =& \frac{78E_2^3E_4^3 + 299E_2E_4^4 + 360E_2^2E_4^2E_6 + 472E_4^3E_6 + 439E_2E_4E_6^2 + 80E_6^3}{62208\eta^{36}} \\
F^{(0,2,3)} =& \frac{1}{2488320\eta^{36}}\left(575E_2^4E_4^3 + 3040E_2^3E_4^2E_6 + 4690E_2^2E_4E_6^2 + 3548E_2^2E_4^4\right. \\
& \left. + 1600E_6^3E_2 + 10176E_6E_4^3E_2 + 2231E_4^5 + 5244E_4^2E_6^2\right) \\
F^{(0,3,3)} =& \frac{1}{209018880\eta^{36}}\left(138104E_4^4E_6 + 224024E_6E_4^3E_2^2 + 36400E_2^4E_4^2E_6 + 224456E_4^2E_6^2E_2\right. \\
& \left. + 49584E_4E_6^3 + 68460E_2^3E_4E_6^2 + 55006E_2^3E_4^4 + 6055E_2^5E_4^3 + 97431E_4^5E_2 + 33600E_6^3E_2^2\right) \\
F^{(0,4,3)} =& \frac{1}{75246796800\eta^{36}}\left(3164700E_2^4E_4E_6^2 + 8993259E_4^5E_2^2 + 14111840E_6^2E_4^3 + 806400E_6^4\right. \\
& + 25171632E_2E_6E_4^4 + 13855280E_2^3E_6E_4^3 + 8963520E_2E_6^3E_4 + 20453520E_2^2E_6^2E_4^2 \\
& \left. + 4014627E_4^6 + 208985E_2^6E_4^3 + 2016000E_6^3E_2^3 + 1417920E_2^5E_4^2E_6 + 2638125E_2^4E_4^4\right)
\end{aligned} \tag{C.1}$$

$$
\begin{aligned}
F^{(0,5,3)} =& \frac{1}{9932577177600\eta^{36}} \big(935093824E_6^2E_4^3E_2 + 233170300E_2^4E_6E_4^3 + 296640960E_2^2E_6^3E_4 \\
&+837550728E_2^2E_6E_4^4 + 453680480E_2^3E_6^2E_4^2 + 16385600E_2^6E_4^2E_6 + 42513240E_2^5E_4E_6^2 \\
&+201151929E_4^5E_2^3 + 36275085E_2^5E_4^4 + 53222400E_6^4E_2 + 266767491E_4^6E_2 \\
&+405268284E_4^5E_6 + 268326944E_4^2E_6^3 + 33264000E_6^3E_2^4 + 2155615E_2^7E_4^3\big) \ .
\end{aligned}
$$

A refined coefficient $F^{(1,0,3)}$ that was studied in Section 5.2.2 is given by

$$
F^{(1,0,3)} = -\frac{54E_2^3E_4^3 + 235E_2E_4^4 + 216E_2^2E_4^2E_6 + 776E_4^3E_6 + 287E_2E_4E_6^2 + 160E_6^3}{124416\eta^{36}} \ . \tag{C.2}
$$

For the four E-strings, $F^{(0,g,4)}$ are given as follows [99]:

$$
\begin{aligned}
F^{(0,0,4)} =& \frac{1}{62208\eta^{48}} E_4 \big(272E_4^3E_6 + 154E_6^3 + 109E_2E_4^4 + 269E_2E_4E_6^2 + 144E_2^2E_4^2E_6 + 24E_2^3E_4^3\big) \\
F^{(0,1,4)} =& \frac{1}{11943936\eta^{48}} \big(37448E_2^2E_4^2E_6^2 + 68768E_2E_4^4E_6 + 29920E_2E_4E_6^3 + 13809E_4^6 \\
&+57750E_4^3E_6^2 + 17416E_2^2E_4^5 + 4545E_6^4 + 16704E_2^3E_4^3E_6 + 2472E_2^4E_4^4\big) \\
F^{(0,2,4)} =& \frac{1}{179159040\eta^{48}} \big(77280E_2^4E_6E_4^3 + 209200E_2^2E_6^3E_4 + 547760E_2^2E_6E_4^4 + 214811E_4^6E_2 \\
&+203900E_2^3E_6^2E_4^2 + 103252E_4^5E_2^3 + 827230E_6^2E_4^3E_2 + 10200E_2^5E_4^4 + 57375E_6^4E_2 \\
&+420616E_4^5E_6 + 314360E_4^2E_6^3\big) \\
F^{(0,3,4)} =& \frac{1}{90296156160\eta^{48}} \big(28134630E_4^7 + 151049093E_4^4E_6^2 + 25488295E_4E_6^4 + 966630E_2^6E_4^4 \\
&+189296376E_6^2E_4^3E_2^2 + 8172360E_2^5E_6E_4^3 + 31388000E_2^3E_6^3E_4 + 88718416E_2^3E_6E_4^4 \\
&+24977155E_2^4E_6^2E_4^2 + 13366787E_4^5E_2^4 + 12119625E_6^4E_2^2 + 137926976E_4^2E_6^3E_2 \\
&+51557313E_4^6E_2^2 + 192353224E_4^5E_6E_2\big) \\
F^{(0,4,4)} =& \frac{1}{5417769369600\eta^{48}} \big(3336940980E_2^3E_4^3E_6^2 + 7817234620E_2E_6^2E_4^4 + 3248768730E_6^3E_4^3 \\
&+5085796952E_2^2E_4^5E_6 + 101280375E_6^5 + 3550525000E_2^2E_4^2E_6^3 + 1290318725E_2E_4E_6^4 \\
&+936363912E_4^6E_2^3 + 1481276055E_4^7E_2 + 2912603799E_4^6E_6 + 1216807640E_2^4E_4^4E_6 \\
&+152620090E_2^5E_4^5 + 78676080E_2^6E_6E_4^3 + 410158000E_2^4E_6^3E_4 + 274844990E_2^5E_6^2E_4^2 \\
&+8381520E_2^7E_4^4 + 202702500E_6^4E_2^3\big)
\end{aligned}
$$

$$
\begin{aligned}
F^{(0,5,4)} =&\frac{1}{2860582227148800\eta^{48}}\big(12207942670E_2^6E_4^5+523849095E_2^8E_4^4+156150752805E_4^8\\
&+113811930320E_2^5E_4^4E_6+1311485716360E_4^6E_6E_2+1760563778482E_2^2E_6^2E_4^4\\
&+286289201000E_2^2E_4E_6^4+381058740370E_2^4E_4^3E_6^2+1449394307792E_6^3E_4^3E_2\\
&+1106487740990E_6^2E_4^5+44575839000E_6^5E_2+109025587484E_4^6E_2^4\\
&+774483173328E_2^3E_4^5E_6+531170439360E_2^3E_4^2E_6^3+5431290480E_2^7E_6E_4^3\\
&+37160939200E_2^5E_6^3E_4+337421738130E_4^7E_2^2+21439577390E_2^6E_6^2E_4^2\\
&+22344052500E_6^4E_2^4+344998537324E_6^4E_4^2\big)\ .
\end{aligned} \tag{C.3}
$$

C.2 Exact properties of the E-string elliptic genus

Let me explain the details on how I checked various exact properties of the E-string elliptic genera, using various identities of Appendix B. I made lots of symbolic computations using computer. Below, I will explain how one can simplify various expressions which can be put on a computer for further simplifications.

2 E-strings Compare the two expressions for the elliptic genus of 2 E-strings, (5.51) and (5.52). Denote them by Z_2 and $Z_2^{\rm E8}$ respectively, in the sense that the latter expression shows manifest E_8 symmetry. After setting $\epsilon_1=-\epsilon_2=\epsilon$ for simplicity, Z_2 is given by

$$
\begin{aligned}
Z_2 =& \sum_{n=1}^{4}\frac{\prod_{l=1}^{8}\theta_n(m_l\pm\frac{\epsilon}{2})}{2\eta^{12}\theta_1(\epsilon)^2\theta_1(2\epsilon)^2}+\frac{1}{4\eta^{12}\theta_1(\epsilon)^4}\left[\frac{\theta_2(0)^2}{\theta_2(\epsilon)^2}\Big(\prod_{l=1}^{8}\theta_1(m_l)\theta_2(m_l)+\prod_{l=1}^{8}\theta_3(m_l)\theta_4(m_l)\Big)\right.\\
&\left.+\frac{\theta_4(0)^2}{\theta_4(\epsilon)^2}\Big(\prod_{l=1}^{8}\theta_1(m_l)\theta_4(m_l)+\prod_{l=1}^{8}\theta_2(m_l)\theta_3(m_l)\Big)+\frac{\theta_3(0)^2}{\theta_3(\epsilon)^2}\Big(\prod_{l=1}^{8}\theta_1(m_l)\theta_3(m_l)+\prod_{l=1}^{8}\theta_2(m_l)\theta_4(m_l)\Big)\right].
\end{aligned} \tag{C.4}
$$

Using the identity (B.15) with $a=b$, one can write $Z_2=\frac{N(\tau,z,m_l)}{\eta^{12}\theta_1(\epsilon)^2\theta_1(2\epsilon)^2}$ with

$$
N=\sum_{n=1}^{4}\frac{1}{2}\prod_{l=1}^{8}\theta_n(m_l\pm\frac{\epsilon}{2})+\frac{\theta_3(\epsilon)^2\theta_4(\epsilon)^2}{\theta_3(0)^2\theta_4(0)^2}\Big(\prod_{l=1}^{8}\theta_1(m_l)\theta_2(m_l)+\prod_{l=1}^{8}\theta_3(m_l)\theta_4(m_l)\Big)
$$

$$+\frac{\theta_2(\epsilon)^2\theta_3(\epsilon)^2}{\theta_2(0)^2\theta_3(0)^2}\Big(\prod_{l=1}^{8}\theta_1(m_l)\theta_4(m_l)+\prod_{l=1}^{8}\theta_2(m_l)\theta_3(m_l)\Big)$$

$$+\frac{\theta_2(\epsilon)^2\theta_4(\epsilon)^2}{\theta_2(0)^2\theta_4(0)^2}\Big(\prod_{l=1}^{8}\theta_1(m_l)\theta_3(m_l)+\prod_{l=1}^{8}\theta_2(m_l)\theta_4(m_l)\Big). \tag{C.5}$$

Apply (B.14) to the first term of N, where $a = m_l$ and $b = \epsilon/2$. N can be expressed as a polynomial of $\theta_n(m_l)$, $\theta_n(\epsilon)$ and $\theta_n(\epsilon/2)$, with coefficients given by $\theta_n(0)$.

On the other side, expressing (5.52) as $Z_2^{\mathrm{E8}} = N^{\mathrm{E8}}/(\eta^{12}\theta_1(\epsilon)^2\theta_1(2\epsilon)^2)$, consider

$$\begin{aligned}N^{\mathrm{E8}} &= \frac{1}{72}A_1^2(\phi_{0,1}(\epsilon)^2 - E_4\phi_{-2,1}(\epsilon)^2) + \frac{1}{96}A_2(E_4^2\phi_{-2,1}(\epsilon)^2 - E_6\phi_{-2,1}(\epsilon)\phi_{0,1}(\epsilon))\\ &+ \frac{5}{288}B_2(E_6\phi_{-2,1}(\epsilon)^2 - E_4\phi_{-2,1}(\epsilon)\phi_{0,1}(\epsilon)).\end{aligned} \tag{C.6}$$

Let me first insert (B.12) to replace E_4, E_6, η by expressions containing $\theta_2(0)$, $\theta_3(0)$, $\theta_4(0)$ only. Looking at the definition of A_2 and B_2 in (5.54), there appear $\theta_n(\frac{\tau}{2}, m_l)$ and $\theta_n(\frac{\tau+1}{2}, m_l)$. To simplify them, consider the identities

$$\begin{aligned}\theta_1(\tfrac{\tau}{2}, m_1)\theta_1(\tfrac{\tau}{2}, m_2) &= \theta_3(\tau, m_1+m_2)\theta_2(\tau, m_1-m_2) - \theta_2(\tau, m_1+m_2)\theta_3(\tau, m_1-m_2)\\ \theta_1(\tfrac{\tau+1}{2}, m_1)\theta_1(\tfrac{\tau+1}{2}, m_2) &= e^{i\pi/4}\theta_4(\tau, m_1+m_2)\theta_2(\tau, m_1-m_2) - e^{i\pi/4}\theta_2(\tau, m_1+m_2)\theta_4(\tau, m_1-m_2)\end{aligned} \tag{C.7}$$

The first identity can be obtained by replacing τ, z, w in (B.17) by $\frac{\tau}{2}, m_1, m_2$, respectively, and the second one is obtained from the first identity by using (B.16). One can also obtain three more copies of similar identities, replacing θ_1 on the left hand sides by $\theta_2, \theta_3, \theta_4$, by using (B.11). The expressions appearing on the right hand sides of (C.7) can be written as polynomials of $\theta_n(\tau, m_l)$ by using (B.15). I apply these identities, and also those with (m_1, m_2) replaced by (m_3, m_4), (m_5, m_6), (m_7, m_8), to (C.6). Then one can express all theta functions with modular parameters $\frac{\tau}{2}$ or $\frac{\tau+1}{2}$ in terms of $\theta_n(\tau, m_l)$. Other terms including $\theta_n(2\tau, 2m_l)$ can be reorganized using (B.18) and (B.19), in terms of $\theta_n(\tau, m_l)$ and $\theta_n(\tau, 0)$. So finally, N^{E_8} is written as a polynomial of $\theta_n(\tau, m_l)$, $\theta_n(\tau, \epsilon)$, with coefficients given by $\theta_n(\tau, 0)$.

Finally, to straightforwardly compare N and N^{E_8}, I want to express $\theta_n(\epsilon)$'s in terms of $\theta_n(\epsilon/2)$'s. Plugging $b = \frac{\epsilon}{2}$ and $a = \frac{\epsilon}{2} + \frac{p}{2}$ (with $p = 0, 1, \tau, \tau+1$) into (B.14) and (B.15), one obtains the desired formulae. Then inserting them into N, N^{E_8}, one can obtain polynomials of $\theta_n(\tau, m_l)$, $\theta_n(\tau, \frac{\epsilon}{2})$ with coefficients given by $\theta_n(\tau, 0)$. Now one can evaluate $N^{E_8} - N$ on computer, by eliminating $\theta_1(m_l)$, $\theta_1(\epsilon/2)$, $\theta_2(0)$ by using (B.13). This yields zero, proving the equivalence of (5.51) and (5.52).

3 and 4 E-strings Let me compare the elliptic genera (5.65) and (5.80) against the known results summarized in Appendix C.1. The free energy is expanded as

$$F = \log Z = \sum_{n_b=1}^{\infty} w^{n_b} F_{n_b} = \sum_{n,g,n_b} (\epsilon_1 + \epsilon_2)^{2n} (\epsilon_1 \epsilon_2)^{g-1} w^{n_b} F^{(n,g,n_b)} \ , \tag{C.8}$$

where $F_1 = Z_1$, $F_2 = Z_2 - \frac{1}{2} Z_1^2$, $F_3 = Z_3 - Z_1 Z_2 + \frac{1}{3} Z_1^3$ and $F_4 = Z_4 - Z_1 Z_3 - \frac{1}{2} Z_2^2 + Z_1^2 Z_2 - \frac{1}{4} Z_1^4$. The coefficients $F^{(n,g,n_b)}$ computed from topological strings, summarized in Appendix C.1, depend on η, E_2, E_4, E_6. Using (B.12), these can be arranged into expressions involving E_2 and $\theta_n(0)$ only.

On the other hand, if one sets $m_l = 0$ and computes $F^{(n,g,n_b)}$ from the gauge theory indices, they will be rational functions of $\theta_n(0)$, η, $\theta_n^{(k)}(0)$. The derivatives $\theta_n^{(k)}(0)$ appear because one is expanding the index with ϵ_1, ϵ_2. I want to express the gauge theory expressions for $F^{(n,g,n_b)}$ in terms of $\theta_n(0)$'s and E_2 only, to compare with the results summarized in Appendix C.1. Firstly, (B.12) can be used to eliminate η. The remaining task is to write $\theta_{1,2,3,4}^{(k)}(0)$ in terms of $\theta_n(0)$'s and E_2, which can be done in the following way.

Starting from the lowest non-vanishing derivatives (B.23) at $z = 0$, one can iteratively obtain $\theta_n^{(k)}(0)$ for higher k's. For example,

$$\begin{aligned} (\partial_z)^3 \theta_1(\tau, z)|_{z=0} &= -8\pi^2 (\partial_z)(q\partial_q)\theta_1(\tau, z)|_{z=0} = -8\pi^2 (q\partial_q)(\partial_z \theta_1(\tau, z))|_{z=0} \\ &= -16\pi^3 (q\partial_q)\eta^3 = -2\pi^3 \eta^3 E_2 \end{aligned} \tag{C.9}$$

where (B.22) and (B.21) are applied in the last step. Looking at another example,

$$\begin{aligned} (\partial_z)^4 \theta_2(\tau, z)|_{z=0} &= -8\pi^2 (\partial_z)^2 (q\partial_q)\theta_2(\tau, z)|_{z=0} = -8\pi^2 (q\partial_q)(\partial_z^2 \theta_2(\tau, z))|_{z=0} \\ &= \tfrac{8}{3}\pi^4 \, q\partial_q [\theta_2(0) \cdot (E_2 + \theta_3(0)^4 + \theta_4(0)^4)] \\ &= \tfrac{1}{9}\pi^4 \theta_2(0)[\alpha_2^2 + 4\theta_3(0)^4 \alpha_3 + 4\theta_4(0)^4 \alpha_4 + \tfrac{1}{12}(E_2^2 - E_4)]. \end{aligned} \tag{C.10}$$

for $\alpha_2 \equiv E_2+\theta_3(0)^4+\theta_4(0)^4$, $\alpha_3 \equiv E_2+\theta_2(0)^4-\theta_4(0)^4$, and $\alpha_4 \equiv E_2-\theta_2(0)^4-\theta_3(0)^4$. In the last step, I applied (B.22) and (B.20). Going for higher derivatives involves no more difficulty, and this way one can always express $F^{(n,g,n_b)}$ in terms of $\theta_n(0)$'s and E_2 only.

So I found two expressions for $F^{(n,g,n_b)}$, depending on $\theta_n(0)$'s and E_2 only, one from the topological string calculus and another from the gauge theories. In particular, I focus on the 3 and 4 E-strings, obtained by expanding (5.65), (5.80). I computed the differences of the two expressions for $F^{(0,0,3)}$, $F^{(0,1,3)}$, $F^{(1,0,3)}$, $F^{(0,0,4)}$, $F^{(0,1,4)}$, $F^{(0,2,4)}$ on computer, substituting $\theta_2(0)^4 = \theta_3(0)^4 - \theta_4(0)^4$, and found zero in all cases. Of course, further analytic tests can also be easily done on computer for higher genus results.

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국문 초록

고차원에서 정의된 양자장론에 관한 정량적 이해

김 준 호
서울대학교 물리천문학부

이 논문에서는 초끈 이론을 통해 발견된 다양한 고차원 양자장론들을 정량적으로 연구한다. 고차원에서 정의된 양자장론들은 본질적으로 매우 강하게 상호작용하는 이론으로써, 기존에 통용되던 양자장론의 체계 안에서는 난해한 이론들로 취급되어졌다. 이 논문에서는 관련 연산자나 원 축소화를 통해 5차원 양-밀스 게이지 이론으로 변형될 수 있는 각종 5, 6차원 양자장론들에 초점을 맞추었다. 5차원 양-밀스 이론의 UV 고정점에서 등장하는 5, 6차원 양자장론들의 물리를 올바로 관찰하기 위해서는 5차원 양-밀스 이론의 순간자들을 반드시 고려해야 한다. 따라서 논문의 전반부에서는 5차원 게이지 이론의 순간자 분배함수에 관한 일반적인 표현식을 유도하고, 이를 통해 다양한 UV 양자장론들의 스펙트럼을 연구한다. 후반부에서는 6차원 양자장론에 존재하는 핵심적 물체인 비임계적 끈의 물리를 다룬다. 특히 두가지 종류의 끈, 즉 M-끈과 E-끈을 기술하는 세계막 게이지 이론을 명시적으로 전개하였다.

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