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PETERSON-TYPE DIMENSION FORMULAS FOR GRADED LIE SUPERALGEBRAS

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Abstract. Let $\widehat{\Gamma}$ be a free abelian group of finite rank, let Γ be a sub-semigroup of $\widehat{\Gamma}$ satisfying certain finiteness conditions, and let $\mathfrak{L} = \bigoplus_{(\alpha,a)\in\Gamma\times\mathbb{Z}_2} \mathfrak{L}_{(\alpha,a)}$ be a $(\Gamma \times \mathbb{Z}_2)$-graded Lie superalgebra. In this paper, by applying formal differential operators and the Laplacian to the denominator identity of $\mathfrak{L}$, we derive a new recursive formula for the dimensions of homogeneous subspaces of $\mathfrak{L}$. When applied to generalized Kac-Moody superalgebras, our formula yields a generalization of Peterson's root multiplicity formula. We also obtain a Freudenthal-type weight multiplicity formula for highest weight modules over generalized Kac-Moody superalgebras.

§1. Introduction

We first recall the *binomial expansion*

$$(1 - t)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} t^k. \tag{1.1}$$

This elementary product identity can be given a Lie-theoretic interpretation as follows: Let $\mathfrak{L} = \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n$ be an n-dimensional abelian Lie algebra with a basis $x_1, \cdots, x_n$. Since $\mathfrak{L}$ is abelian, we have $H_k(\mathfrak{L}) = \Lambda^k(\mathfrak{L})$ and

$$\dim H_k(\mathfrak{L}) = \dim \Lambda^k(\mathfrak{L}) = \binom{n}{k}.$$

Hence the binomial expansion (1.1) can be interpreted as the Euler-Poincaré principle for the abelian Lie algebra $\mathfrak{L}$.

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Next, consider the *Jacobi triple product identity*

$$\prod_{n=1}^{\infty}(1-p^nq^n)(1-p^{n-1}q^n)(1-p^nq^{n-1}) = \sum_{k\in\mathbb{Z}}(-1)^k p^{\frac{k(k-1)}{2}} q^{\frac{k(k+1)}{2}}, \tag{1.2}$$

which arises from the theory of modular forms and theta functions. In [8], V. G. Kac discovered a character formula, called the *Weyl-Kac formula*, for irreducible highest weight modules over symmetrizable Kac-Moody algebras with dominant integral highest weights. When applied to the 1-dimensional trivial representation, the Weyl-Kac formula yields the *denominator identity*, and it was shown in [8] that the Macdonald identities ([19]) are equivalent to the denominator identities for affine Kac-Moody algebras. In particular, the denominator identity for the affine Kac-Moody algebra $A_1^{(1)}$ is equal to the Jacobi triple product identity.

In [2], R. E. Borcherds proved the product identity

$$p^{-1}\prod_{\substack{m>0\\ n\in\mathbb{Z}}}(1-p^mq^n)^{c(mn)} = j(p)-j(q) \tag{1.3}$$

for the elliptic modular function

$$j(q) = \sum_{n=-1}^{\infty} c(n)q^n = q^{-1} + 196884q + 21493760q^2 + \cdots,$$

and showed that (1.3) is the denominator identity for the *Monster Lie algebra*. The Monster Lie algebra is a special case of *generalized Kac-Moody algebras* and it played a crucial role in Borcherds' proof of the Moonshine Conjecture ([2], [5]).

In this paper, we consider general product identities of the form

$$\prod_{(\alpha,a)\in\Gamma\times\mathbb{Z}_2}(1-E^{(\alpha,a)})^{\nu(\alpha,a)} = 1 + \sum_{(\beta,b)\in\Gamma\times\mathbb{Z}_2}\zeta(\beta,b)E^{(\beta,b)}, \tag{1.4}$$

where $\nu(\alpha,a), \zeta(\beta,b) \in \mathbb{Z}$ and Γ is a countable (usually infinite) subsemigroup of a free abelian group $\widehat{\Gamma}$ of finite rank with a nondegenerate symmetric bilinear form. Suppose that we have a $(\Gamma\times\mathbb{Z}_2)$-graded Lie superalgebra $\mathfrak{L} = \bigoplus_{(\alpha,a)\in\Gamma\times\mathbb{Z}_2}\mathfrak{L}_{(\alpha,a)}$ such that

$$\begin{aligned}
&\operatorname{sdim}\mathfrak{L}_{(\alpha,a)} = (-1)^a\dim\mathfrak{L}_{(\alpha,a)} = \nu(\alpha,a),\\
&\operatorname{sdim}H(\mathfrak{L})_{(\beta,b)} = (-1)^b\dim H(\mathfrak{L})_{(\beta,b)} = \zeta(\beta,b)
\end{aligned}$$

for all $\alpha, \beta \in \Gamma$, $a, b \in \mathbb{Z}_2$. Then the product identity (1.4) can be interpreted as the Euler-Poincaré principle for the Lie superalgebra $\mathfrak{L}$ (see Section 2). We call (1.4) the *denominator identity* for the $(\Gamma \times \mathbb{Z}_2)$-graded Lie superalgebra $\mathfrak{L} = \bigoplus_{(\alpha,a)\in\Gamma\times\mathbb{Z}_2} \mathfrak{L}_{(\alpha,a)}$.

In Section 3, by applying formal differential operators and the Laplacian to the denominator identity of $\mathfrak{L}$, we derive a new recursive formula for the dimensions of homogeneous subspaces of $\mathfrak{L}$ (Theorem 3.3). As an immediate application, we recover some interesting recursive relations for the colored partitions $p_r(n)$ and the Ramanujan's tau-function $\tau(n)$ which can be found in [20]. We also obtain a set of recursive relations for the coefficients $c(n)$ of the elliptic modular function j which can be used to prove the fact that the coefficients of j are completely determined by the first 3 coefficients $c(1)$, $c(2)$ and $c(3)$ (cf. [17]).

For Kac-Moody algebras, it is well-known that *Peterson's root multiplicity formula* determines the root multiplicities recursively ([22]). In Section 4, we show that when applied to generalized Kac-Moody superalgebras, our recursive dimension formula yields a generalization of Peterson's root multiplicity formula (Theorem 4.3 and Proposition 4.4). For this reason, we call our formula the *Peterson-type dimension formula* for graded Lie superalgebras. Moreover, by applying the same technique that was used in deriving our Peterson-type dimension formula, we also derive a *Freudenthal-type recursive weight multiplicity formula* for highest weight modules over generalized Kac-Moody superalgebras (Theorem 4.5).

In the final section, we illustrate how to apply our Peterson-type root multiplicity formula and Freudenthal-type weight multiplicity formula with the examples of rank 2 generalized Kac-Moody superalgebras and their irreducible highest weight modules. We also discuss the application of our Peterson-type root multiplicity formula to Monstrous Lie superalgebras. At the end of Section 5, we present some tables of root and weight multiplicities for these algebras and modules.

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§2. Graded Lie superalgebras

We recall some basic facts about Lie superalgebras. A $\mathbb{Z}_2$-graded vector space $\mathfrak{L} = \mathfrak{L}_{\overline{0}} \oplus \mathfrak{L}_{\overline{1}}$ is called a *Lie superalgebra* if there exists a bilinear map $[\ ,\ ] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$, called the *bracket*, such that

$$\begin{aligned} &[\mathfrak{L}_a, \mathfrak{L}_b] \subset \mathfrak{L}_{a+b}, \\ &[x, y] = -(-1)^{ab}[y, x], \\ &[x, [y, z]] = [[x, y], z] + (-1)^{ab}[y, [x, z]] \end{aligned} \tag{2.1}$$

for all $x \in \mathfrak{L}_a,\ y \in \mathfrak{L}_b,\ a, b \in \mathbb{Z}_2$. The homogeneous elements of $\mathfrak{L}_{\overline{0}}$ (resp. $\mathfrak{L}_{\overline{1}}$) are called *even* (resp. *odd*).

Let Γ be a countable (usually infinite) abelian semigroup such that every element $\alpha \in \Gamma$ can be written as a sum of elements in Γ in only finitely many ways. Consider a $(\Gamma \times \mathbb{Z}_2)$-graded Lie superalgebra $\mathfrak{L} = \bigoplus_{(\alpha,a)\in\Gamma\times\mathbb{Z}_2} \mathfrak{L}_{(\alpha,a)}$ with $\dim \mathfrak{L}_{(\alpha,a)} < \infty$ for all $(\alpha, a) \in \Gamma \times \mathbb{Z}_2$. We define the *character* of $\mathfrak{L}$ to be

$$\operatorname{ch} \mathfrak{L} = \sum_{(\alpha,a)\in\Gamma\times\mathbb{Z}_2} (\dim \mathfrak{L}_{(\alpha,a)})\, e^{(\alpha,a)},$$

where $e^{(\alpha,a)}$ are the basis elements of the semigroup algebra $\mathbb{C}[\Gamma \times \mathbb{Z}_2]$ with the multiplication given by $e^{(\alpha,a)}e^{(\beta,b)} = e^{(\alpha+\beta,a+b)}$.

On the other hand, we define the *superdimension* of the homogeneous subspace $\mathfrak{L}_{(\alpha,a)}$ by

$$\operatorname{sdim} \mathfrak{L}_{(\alpha,a)} = (-1)^a \dim \mathfrak{L}_{(\alpha,a)} \quad (\alpha \in \Gamma, a \in \mathbb{Z}_2).$$

We introduce another basis elements of $\mathbb{C}[\Gamma \times \mathbb{Z}_2]$ by setting $E^{(\alpha,a)} = (-1)^a e^{(\alpha,a)}$. Clearly, $E^{(\alpha,a)}E^{(\beta,b)} = E^{(\alpha+\beta,a+b)}$. We define the *supercharacter* of $\mathfrak{L}$ to be

$$\operatorname{sch} \mathfrak{L} = \sum_{(\alpha,a)\in\Gamma\times\mathbb{Z}_2} (\operatorname{sdim} \mathfrak{L}_{(\alpha,a)})\, E^{(\alpha,a)}.$$

Note that $\operatorname{ch} \mathfrak{L} = \operatorname{sch} \mathfrak{L}$. The only difference is that, in the supercharacter, we allow the negative coefficients. Usually, they are the elements of $\mathbb{C}[[\Gamma \times \mathbb{Z}_2]]$, the completion of $\mathbb{C}[\Gamma \times \mathbb{Z}_2]$.

The main purpose of this paper is to find an efficient recursive formula for sdim $\mathfrak{L}_{(\alpha,a)}$ ($\alpha \in \Gamma, a \in \mathbb{Z}_2$). In deriving our superdimension formula, the crucial role is played by the *denominator identity* for the Lie superalgebra $\mathfrak{L} = \bigoplus_{(\alpha,a)\in\Gamma\times\mathbb{Z}_2} \mathfrak{L}_{(\alpha,a)}$. In the following, we briefly recall how to derive the denominator identity from the *Euler-Poincaré principle* for Lie superalgebra homology (see [8] and [20] for more details).

Let $\mathbb{C}$ be the trivial 1-dimensional $\mathfrak{L}$-module. The homology modules $H_k(\mathfrak{L}) = H_k(\mathfrak{L}, \mathbb{C})$ are determined from the following *standard complex*:

$$\cdots \to C_k(\mathfrak{L}) \xrightarrow{d_k} C_{k-1}(\mathfrak{L}) \xrightarrow{d_{k-1}} \cdots \to C_1(\mathfrak{L}) \xrightarrow{d_1} C_0(\mathfrak{L}) \to 0,$$

where the $C_k(\mathfrak{L})$ are defined by

$$C_k(\mathfrak{L}) = \bigoplus_{p+q=k} \Lambda^p(\mathfrak{L}_{\bar{0}}) \otimes S^q(\mathfrak{L}_{\bar{1}})$$

and the differentials $d_k : C_k(\mathfrak{L}) \to C_{k-1}(\mathfrak{L})$ are given by

$$\begin{aligned} & d_k((x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q)) \\ & = \sum_{1\le s<t\le p} (-1)^{s+t}([x_s, x_t] \wedge x_1 \wedge \cdots \wedge \widehat{x_s} \wedge \cdots \wedge \widehat{x_t} \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q) \\ & \quad + \sum_{s=1}^{p}\sum_{t=1}^{q} (-1)^s (x_1 \wedge \cdots \wedge \widehat{x_s} \wedge \cdots \wedge x_p) \otimes ([x_s, y_t] y_1 \cdots \widehat{y_t} \cdots y_q) \\ & \quad - \sum_{1\le s<t\le q} ([y_s, y_t] \wedge x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots \widehat{y_s} \cdots \widehat{y_t} \cdots y_q) \end{aligned}$$

for $k \ge 2$, $x_i \in \mathfrak{L}_{\bar{0}}$, $y_j \in \mathfrak{L}_{\bar{1}}$ and $d_1 = 0$ (cf. [4], [6]). Since the spaces $C_k(\mathfrak{L})$ and the homology modules $H_k(\mathfrak{L})$ inherit the $(\Gamma \times \mathbb{Z}_2)$-gradation from that of $\mathfrak{L}$, the supercharacters of $C_k(\mathfrak{L})$ and $H_k(\mathfrak{L})$ are well-defined. Hence by the Euler-Poincaré principle, we obtain

$$\sum_{k=0}^{\infty} (-1)^k \operatorname{sch} C_k(\mathfrak{L}) = \sum_{k=0}^{\infty} (-1)^k \operatorname{sch} H_k(\mathfrak{L}) \quad .$$

Let

$$C(\mathfrak{L}) = \sum_{k=0}^{\infty}(-1)^k C_k(\mathfrak{L}) = \mathbb{C} \ominus \mathfrak{L} \oplus C_2(\mathfrak{L}) \ominus \cdots,$$

$$\Lambda(\mathfrak{L}_{\overline{0}}) = \sum_{k=0}^{\infty}(-1)^k \Lambda^k(\mathfrak{L}_{\overline{0}}) = \mathbb{C} \ominus \mathfrak{L}_{\overline{0}} \oplus \Lambda^2(\mathfrak{L}_{\overline{0}}) \ominus \cdots,$$

$$S(\mathfrak{L}_{\overline{1}}) = \sum_{k=0}^{\infty}(-1)^k S^k(\mathfrak{L}_{\overline{1}}) = \mathbb{C} \ominus \mathfrak{L}_{\overline{1}} \oplus S^2(\mathfrak{L}_{\overline{1}}) \ominus \cdots,$$

$$H(\mathfrak{L}) = \sum_{k=1}^{\infty}(-1)^k H_k(\mathfrak{L}) = -H_1(\mathfrak{L}) \oplus H_2(\mathfrak{L}) \ominus H_3(\mathfrak{L}) \oplus \cdots,$$

the alternating direct sum of superspaces. Then it is easy to see that

$$C(\mathfrak{L}) = \Lambda(\mathfrak{L}_{\overline{0}}) \otimes S(\mathfrak{L}_{\overline{1}})$$

and that

$$\operatorname{sch} C(\mathfrak{L}) = \prod_{(\alpha,a)\in\Gamma\times\mathbb{Z}_2} \left(1 - E^{(\alpha,a)}\right)^{\operatorname{sdim} \mathfrak{L}_{(\alpha,a)}}.$$

Therefore, we obtain the *denominator identity* for the $(\Gamma \times \mathbb{Z}_2)$- graded Lie superalgebra $\mathfrak{L} = \bigoplus_{(\alpha,a)\in\Gamma\times\mathbb{Z}_2} \mathfrak{L}_{(\alpha,a)}$:

$$\prod_{(\alpha,a)\in\Gamma\times\mathbb{Z}_2} \left(1 - E^{(\alpha,a)}\right)^{\operatorname{sdim} \mathfrak{L}_{(\alpha,a)}} = 1 + \operatorname{sch} H(\mathfrak{L}). \tag{2.2}$$

We often deal with the Γ-grading on $\mathfrak{L}$ defined by

$$\mathfrak{L}_\alpha = \mathfrak{L}_{(\alpha,\overline{0})} \oplus \mathfrak{L}_{(\alpha,\overline{1})} \quad (\alpha \in \Gamma).$$

In this case, by setting $\operatorname{sdim} \mathfrak{L}_\alpha = \dim \mathfrak{L}_{(\alpha,\overline{0})} - \dim \mathfrak{L}_{(\alpha,\overline{1})}$ and $E^{(\alpha,a)} = E^\alpha$, we have

$$\operatorname{sch} \mathfrak{L} = (\sum_{\alpha\in\Gamma} \operatorname{sdim} \mathfrak{L}_\alpha)\, E^\alpha,$$

and (2.2) yields the denominator identity for Γ-graded Lie superalgebra $\mathfrak{L} = \bigoplus_{\alpha\in\Gamma} \mathfrak{L}_\alpha$:

$$\prod_{\alpha\in\Gamma} (1 - E^\alpha)^{\operatorname{sdim} \mathfrak{L}_\alpha} = 1 + \operatorname{sch} H(\mathfrak{L}). \tag{2.3}$$

In the next section, we will derive *Peterson-type recursive formulas* for $\operatorname{sdim} \mathfrak{L}_{(\alpha,a)}$ and $\operatorname{sdim} \mathfrak{L}_\alpha$ $(\alpha \in \Gamma, a \in \mathbb{Z}_2)$. The main idea is to *differentiate* both sides of the denominator identities (2.2) and (2.3).

§3. Peterson-type formulas

Let $\widehat{\Gamma}$ be a free abelian group with finite rank and let Γ be a countable (usually infinite) semi-subgroup in $\widehat{\Gamma}$ such that every element $\alpha \in \Gamma$ can be written as a sum of elements of Γ in only finitely many ways. Let $\widehat{\Gamma}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} \widehat{\Gamma}$ be the complexification of $\widehat{\Gamma}$. Choose a nondegenerate symmetric bilinear form $(\ |\ )$ on $\widehat{\Gamma}_{\mathbb{C}}$ and fix a pair of dual bases $\{u_i\}$ and $\{u^i\}$. Define a partial ordering on Γ by $\lambda \geq \mu$ if and only if $\lambda - \mu \in \Gamma$ or $\lambda = \mu$. We will denote by $\lambda > \mu$ if $\lambda \geq \mu$ and $\lambda \neq \mu$.

Suppose that we have a product identity of the form

$$\prod_{(\alpha,a)\in\Gamma\times\mathbb{Z}_2} (1 - E^{(\alpha,a)})^{\nu(\alpha,a)} = 1 + \sum_{(\beta,b)\in\Gamma\times\mathbb{Z}_2} \zeta(\beta,b) E^{(\beta,b)}, \tag{3.1}$$

where $\nu(\alpha,a), \zeta(\beta,b) \in \mathbb{Z}$. First, we define the *partial differential operators* by

$$\partial_i(E^{(\lambda,a)}) = (\lambda|u_i)E^{(\lambda,a)}, \quad \partial^i(E^{(\lambda,a)}) = (\lambda|u^i)E^{(\lambda,a)}.$$

Fix an element $\rho \in \widehat{\Gamma}_{\mathbb{C}}$. We define the *$\rho$-directional derivative* by

$$\nabla_\rho(E^{(\lambda,a)}) = \sum (\rho|u_i)\partial^i(E^{(\lambda,a)}) = (\rho|\lambda)E^{(\lambda,a)}$$

and the *Laplacian* by

$$\Delta(E^{(\lambda,a)}) = \sum \partial^i \partial_i(E^{(\lambda,a)}) = (\lambda|\lambda)E^{(\lambda,a)}.$$

Let

$$C = \sum_{(\gamma,a)\in\Gamma\times\mathbb{Z}_2} C(\gamma,a)E^{(\gamma,a)} \text{ and } C^* = \sum_{(\gamma,a)\in\Gamma\times\mathbb{Z}_2} C^*(\gamma,a)E^{(\gamma,a)}$$

be the formal power series whose coefficients are given by

$$C(\gamma,\overline{0}) = \sum_{d|\gamma} \frac{1}{d}\nu\left(\frac{\gamma}{d},\overline{0}\right) + \sum_{\substack{d|\gamma \\ d:\ \text{even}}} \frac{1}{d}\nu\left(\frac{\gamma}{d},\overline{1}\right),$$

$$C(\gamma,\overline{1}) = \sum_{\substack{d|\gamma \\ d:\ \text{odd}}} \frac{1}{d}\nu\left(\frac{\gamma}{d},\overline{1}\right),$$

$$C^*(\gamma,a) = (\gamma|\gamma)C(\gamma,a) - \sum_{(\gamma,a)=(\gamma',a')+(\gamma'',a'')} (\gamma'|\gamma'')C(\gamma',a')C(\gamma'',a'').$$

PROPOSITION 3.1. *Let*

$$D = \prod_{(\alpha,a)\in\Gamma\times\mathbb{Z}_2} (1 - E^{(\alpha,a)})^{\nu(\alpha,a)} = 1 + \sum_{(\beta,b)\in\Gamma\times\mathbb{Z}_2} \zeta(\beta,b)E^{(\beta,b)}.$$

Then the following differential equations hold:

$$\nabla_\rho(D) = -\nabla_\rho(C)D, \qquad \Delta(D) = -C^* D. \tag{3.2}$$

Proof. First, note the following basic fact:

$$(\alpha|\beta) = \sum_i (\alpha|u_i)(\beta|u^i) \text{ for all } \alpha, \beta \in \widehat{\Gamma}_{\mathbb{C}}.$$

Using the formal power series $\log(1-t) = -\sum_{k=1}^{\infty} \frac{1}{k} t^k$, we obtain

$$\begin{aligned} \nabla_\rho(D) &= D\nabla_\rho(\log D) \\ &= D\nabla_\rho\Big(- \sum_{\substack{(\alpha,a)\in\Gamma\times\mathbb{Z}_2 \\ k\geq 1}} \frac{1}{k}\nu(\alpha,a)E^{k(\alpha,a)}\Big) \\ &= -D\nabla_\rho(C), \end{aligned}$$

which yields the first equation. For the second one, observe the following differential equation

$$\begin{aligned} \frac{\Delta(D)}{D} &= \sum_i \partial_i \left(\frac{\partial^i D}{D}\right) + \sum_i \left(\frac{\partial_i D}{D}\right)\left(\frac{\partial^i D}{D}\right) \\ &= \Delta(\log D) + \sum_i \partial_i(\log D)\partial^i(\log D). \end{aligned}$$

On the other hand, note that

$$\Delta(\log D) = - \sum_{\substack{(\alpha,a)\in\Gamma\times\mathbb{Z}_2 \\ k\geq 1}} (k\alpha|\alpha)\nu(\alpha,a)E^{k(\alpha,a)}$$

and that

$$\begin{aligned} &\sum_i \partial_i(\log D)\partial^i(\log D) \\ &= \sum_i \Big\{ - \sum_{\substack{(\alpha',a')\in\Gamma\times\mathbb{Z}_2 \\ p\geq 1}} (\alpha'|u_i)\nu(\alpha',a')E^{p(\alpha',a')}\Big\} \end{aligned}$$

$$\times\Big\{-\sum_{\substack{(\alpha'',a'')\in\Gamma\times\mathbb{Z}_2\\ q\geq 1}}(\alpha''|u^i)\nu(\alpha'',a'')E^{q(\alpha'',a'')}\Big\}$$

$$=\sum_{\substack{(\alpha',a'),(\alpha'',a'')\in\Gamma\times\mathbb{Z}_2\\ p,q\geq 1}}(\alpha'|\alpha'')\nu(\alpha',a')\nu(\alpha',a'')E^{p(\alpha',a')+q(\alpha'',a'')}.$$

Combining the above identities, we have

$$\begin{aligned}\frac{\Delta(D)}{D}&=-\sum_{\substack{(\alpha,a)\in\Gamma\times\mathbb{Z}_2\\ k\geq 1}}k\nu(\alpha,a)(\alpha|\alpha)E^{k(\alpha,a)}\\ &\quad+\sum_{\substack{(\alpha',a'),(\alpha'',a'')\in\Gamma\times\mathbb{Z}_2\\ p,q\geq 1}}\nu(\alpha',a')\nu(\alpha'',a'')(\alpha'|\alpha'')E^{p(\alpha',a')+q(\alpha'',a'')}\\ &=-\sum_{(\beta,b)\in\Gamma\times\mathbb{Z}_2}(\beta|\beta)C(\beta,b)E^{(\beta,b)}\\ &\quad+\sum_{(\beta',b'),(\beta'',b'')\in\Gamma\times\mathbb{Z}_2}(\beta'|\beta'')C(\beta',b')C(\beta'',b'')E^{(\beta'+\beta'',b'+b'')}.\end{aligned}$$

It follows that

$$\Delta(D)=-C^*D.$$

□

By comparing the coefficients of both sides in Proposition 3.1, we obtain the following recursive relations between $\nu(\alpha,a)$ and $\zeta(\alpha,a)$ $(\alpha\in\Gamma, a\in\mathbb{Z}_2)$.

THEOREM 3.2. *For $(\alpha,a)\in\Gamma\times\mathbb{Z}_2$, we have*

$$(\rho|\alpha)C(\alpha,a)+\sum_{\substack{\beta<\alpha\\ b\in\mathbb{Z}_2}}(\rho|\beta)C(\beta,b)\zeta(\alpha-\beta,a-b)=-(\rho|\alpha)\zeta(\alpha,a),\tag{3.3}$$

$$C^*(\alpha,a)+\sum_{\substack{\beta<\alpha\\ b\in\mathbb{Z}_2}}C^*(\beta,b)\zeta(\alpha-\beta,a-b)=-(\alpha|\alpha)\zeta(\alpha,a).\tag{3.4}$$

□

As a direct consequence of (2.2) and (3.2), we obtain the main result of this paper: the *Peterson-type recursive superdimension formulas* for graded Lie superalgebras.

THEOREM 3.3. *Let* $\mathfrak{L} = \bigoplus_{(\alpha,a)\in\Gamma\times\mathbb{Z}_2} \mathfrak{L}_{(\alpha,a)}$ *be a* $(\Gamma\times\mathbb{Z}_2)$*-graded Lie superalgebra with finite dimensional homogeneous subspaces and define*

$$C(\gamma,\overline{0}) = \sum_{d|\gamma} \frac{1}{d}\operatorname{sdim}\mathfrak{L}_{(\frac{\gamma}{d},\overline{0})} + \sum_{\substack{d|\gamma \\ d:\ \text{even}}} \frac{1}{d}\operatorname{sdim}\mathfrak{L}_{(\frac{\gamma}{d},\overline{1})},$$

$$C(\gamma,\overline{1}) = \sum_{\substack{d|\gamma \\ d:\ \text{odd}}} \frac{1}{d}\operatorname{sdim}\mathfrak{L}_{(\frac{\gamma}{d},\overline{1})},$$

$$C^*(\gamma,a) = (\gamma|\gamma)C(\gamma,a) - \sum_{(\gamma,a)=(\gamma',a')+(\gamma'',a'')} (\gamma'|\gamma'')C(\gamma',a')C(\gamma'',a'').$$

Then we have

$$(3.5)\qquad (\rho|\alpha)C(\alpha,a) = -\sum_{\substack{\beta<\alpha \\ b\in\mathbb{Z}_2}} (\rho|\beta)C(\beta,b)\operatorname{sdim} H(\mathfrak{L})_{(\alpha-\beta,a-b)} - (\rho|\alpha)\operatorname{sdim} H(\mathfrak{L})_{(\alpha,a)},$$

$$(3.6)\qquad C^*(\alpha,a) = -\sum_{\substack{\beta<\alpha \\ b\in\mathbb{Z}_2}} C^*(\beta,b)\operatorname{sdim} H(\mathfrak{L})_{(\alpha-\beta,a-b)} - (\alpha|\alpha)\operatorname{sdim} H(\mathfrak{L})_{(\alpha,a)}.$$

□

We return to the product identity (3.1). By setting $\nu(\alpha) = \nu(\alpha,\overline{0}) + \nu(\alpha,\overline{1})$, $\zeta(\beta) = \zeta(\beta,\overline{0}) + \zeta(\beta,\overline{1})$, and $E^{\alpha} = E^{(\alpha,\overline{0})} = E^{(\alpha,\overline{1})}$, the product identity (3.1) gives rise to another product identity of the form

$$(3.7)\qquad \prod_{\alpha\in\Gamma}(1-E^{\alpha})^{\nu(\alpha)} = 1 + \sum_{\beta\in\Gamma}\zeta(\beta)E^{\beta}.$$

Let

$$C = \sum_{\gamma\in\Gamma} C(\gamma)E^{\gamma} \quad \text{and} \quad C^* = \sum_{\gamma\in\Gamma} C^*(\gamma)E^{\gamma}$$

be the formal power series whose coefficients are given by

$$C(\gamma) = C(\gamma,\overline{0}) + C(\gamma,\overline{1}) = \sum_{d|\gamma}\frac{1}{d}\nu\left(\frac{\gamma}{d}\right),$$

$$C^*(\gamma) = (\gamma|\gamma)C(\gamma) - \sum_{\gamma=\gamma'+\gamma''}(\gamma'|\gamma'')C(\gamma')C(\gamma''),$$

and define the *partial differential operators*, the *ρ-directional derivatives*, and the *Laplacian* by

$$\begin{aligned}\partial_i(E^\lambda) &= (\lambda|u_i)E^\lambda, \quad \partial^i(E^\lambda) = (\lambda|u^i)E^\lambda,\\ \nabla_\rho(E^\lambda) &= \sum(\rho|u_i)\partial^i(E^\lambda) = (\rho|\lambda)E^\lambda,\\ \Delta(E^\lambda) &= \sum \partial^i\partial_i(E^\lambda) = (\lambda|\lambda)E^\lambda.\end{aligned} \tag{3.8}$$

Then we can derive the Peterson-type recursive superdimension formulas for Γ-graded Lie superalgebras.

THEOREM 3.4. (a) *Let*

$$D = \prod_{\alpha\in\Gamma}(1-E^\alpha)^{\nu(\alpha)} = 1 + \sum_{\beta\in\Gamma}\zeta(\beta)E^\beta.$$

Then the following differential equations hold:

$$\nabla_\rho(D) = -\nabla_\rho(C)D, \qquad \Delta(D) = -C^*D. \tag{3.9}$$

(b) *For any $\alpha \in \Gamma$, we have*

$$(\rho|\alpha)C(\alpha) + \sum_{\beta<\alpha}(\rho|\beta)C(\beta)\zeta(\alpha-\beta) = -(\rho|\alpha)\zeta(\alpha), \tag{3.10}$$

$$C^*(\alpha) + \sum_{\beta<\alpha}C^*(\beta)\zeta(\alpha-\beta) = -(\alpha|\alpha)\zeta(\alpha). \tag{3.11}$$

□

THEOREM 3.5. *Let $\mathfrak{L} = \bigoplus_{\alpha\in\Gamma}\mathfrak{L}_\alpha$ be a Γ-graded Lie superalgebra with finite dimensional homogeneous subspaces and set*

$$C(\gamma) = \sum_{d|\gamma}\frac{1}{d}\operatorname{sdim}\mathfrak{L}_{\frac{\gamma}{d}},$$

$$C^*(\gamma) = (\gamma|\gamma)C(\gamma) - \sum_{\gamma=\gamma'+\gamma''}(\gamma'|\gamma'')C(\gamma')C(\gamma'').$$

Then we have

$$(\rho|\alpha)C(\alpha) = -\sum_{\beta<\alpha}(\rho|\beta)C(\beta)\operatorname{sdim}H(\mathfrak{L})_{\alpha-\beta} - (\rho|\alpha)\operatorname{sdim}H(\mathfrak{L})_\alpha, \tag{3.12}$$

$$C^*(\alpha) = -\sum_{\beta<\alpha} C^*(\beta)\,\mathrm{sdim}\, H(\mathfrak{L})_{\alpha-\beta} - (\alpha|\alpha)\,\mathrm{sdim}\, H(\mathfrak{L})_{\alpha}. \tag{3.13}$$

□

Remark. We can determine the exponents $\nu(\alpha, a)$ (resp. $\nu(\alpha)$) in the left-hand side of (3.1) (resp. (3.7)) recursively from the coefficients $\zeta(\beta, b)$ (resp. $\zeta(\beta)$) in the right-hand side of (3.1) (resp. (3.7)), and vice versa. In other words, we can determine the superdimensions of homogeneous subspaces of graded Lie superalgebras recursively from the superdimensions of homogeneous subspaces of its homology modules, and vice versa.

EXAMPLE 3.6. (a) Suppose that rank$\widehat{\Gamma} = 1$. Take $\widehat{\Gamma} = \mathbb{Z}$, $\Gamma = \mathbb{Z}_{>0}$, and consider the product identity of the form

$$\prod_{n=1}^{\infty}(1-q^n)^{\nu(n)} = \sum_{n=0}^{\infty}\zeta(n)q^n, \tag{3.14}$$

where $\nu(n), \zeta(n) \in \mathbb{Z}$ and $\zeta(0) = 1$. Let $V = \bigoplus_{n=1}^{\infty} V_n$ be a $\mathbb{Z}_{>0}$-graded superspace with $\mathrm{sdim}\, V_n = -\zeta(n)$ for all $n \in \mathbb{Z}_{>0}$, and let $\mathfrak{L} = \bigoplus_{n=1}^{\infty} \mathfrak{L}_n$ be the free Lie superalgebra generated by V. Then the identity (3.14) can be interpreted as the denominator identity for the free Lie superalgebra $\mathfrak{L}$ and hence we have $\mathrm{sdim}\,\mathfrak{L}_n = \nu(n)$ ($n \in \mathbb{Z}_{>0}$) (see [13]).

Take $\rho = 1$ and let $(\ |\ )$ be the multiplication in $\mathbb{C}$. Then by Theorem 3.4(b), we obtain the following recursive relations between $\nu(n)$ and $\zeta(n)$:

$$\sum_{k=1}^{n}\Big(\sum_{d|k} d\nu(d)\Big)\zeta(n-k) = -n\zeta(n). \tag{3.15}$$

In particular, we obtain

$$\mathrm{sdim}\,\mathfrak{L}_n = -\zeta(n) - \frac{1}{n}\sum_{k=1}^{n-1}\Big(\sum_{d|k} d\ \mathrm{sdim}\,\mathfrak{L}_d\Big)\zeta(n-k) - \sum_{\substack{d|n \\ d>1}} \frac{d}{n}\,\mathrm{sdim}\,\mathfrak{L}_d.$$

(b) Let r be a positive integer and set $\nu(n) = -r$ for all $n \in \mathbb{Z}_{>0}$. Then the product identity (3.14) yields the generating function for the r-*colored partitions* of positive integers:

$$\prod_{n=1}^{\infty}(1-q^n)^{-r} = \sum_{n=0}^{\infty} p_r(n)q^n.$$

Hence the relation (3.15) gives the following recursive formula for r-colored partitions:

$$p_r(n) = \frac{r}{n}\sum_{k=1}^{n}\sigma_1(k)p_r(n-k), \tag{3.16}$$

where $\sigma_1(k)$ denotes the sum of all divisors of k.

(c) If we take $\nu(n) = 24$ for all $n \in \mathbb{Z}_{>0}$, the product identity (3.14) is the definition of *Ramanujan's tau-function*:

$$\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=0}^{\infty}\tau(n+1)q^n.$$

Then the relation (3.15) yields the following recursive formula for the values of $\tau(n)$:

$$\tau(n+1) = -\frac{24}{n}\sum_{k=1}^{n}\sigma_1(k)\tau(n+1-k). \tag{3.17}$$

Remark. The recursive formulas (3.16) and (3.17) can be derived from the equation (2.10) in [20]. Our method can be regarded as a generalization of Macdonald's method to several variables.

EXAMPLE 3.7. (a) Suppose that rank$\widehat{\Gamma} = 2$. Take $\widehat{\Gamma} = \mathbb{Z}\times\mathbb{Z}$, $\Gamma = \mathbb{Z}_{>0}\times\mathbb{Z}_{>0}$ and consider the product identity of the form

$$\prod_{m,n=1}^{\infty}(1-p^mq^n)^{\nu(m,n)} = 1 + \sum_{m,n=1}^{\infty}\zeta(m,n)p^mq^n, \tag{3.18}$$

where $\nu(m,n), \zeta(m,n) \in \mathbb{Z}$. Let $V = \bigoplus_{m,n=1}^{\infty} V_{(m,n)}$ be a $(\mathbb{Z}_{>0}\times\mathbb{Z}_{>0})$-graded superspace with $\operatorname{sdim} V_{(m,n)} = -\zeta(m,n)$ for all $m,n \in \mathbb{Z}_{>0}$, and let $\mathfrak{L} = \bigoplus_{m,n=1}^{\infty}\mathfrak{L}_{(m,n)}$ be the free Lie superalgebra generated by V. Then the identity (3.18) can be interpreted as the denominator identity for the free Lie superalgebra $\mathfrak{L}$ and hence we have $\operatorname{sdim}\mathfrak{L}_{(m,n)} = \nu(m,n)$ for all $m,n \in \mathbb{Z}_{>0}$ (see [13]).

Let $\rho = (1,0)$ and let $(\ |\ )$ denote the standard inner product on $\mathbb{C}^2$. Then by Theorem 3.4(b), we obtain the following recursive relations between $\nu(m,n)$ and $\zeta(m,n)$:

$$m\sum_{d|(m,n)}\frac{1}{d}\nu\left(\frac{m}{d},\frac{n}{d}\right) \tag{3.19}$$

$$= -m\zeta(m,n) - \sum_{(m,n)=(k,l)+(s,t)} k\Big(\sum_{d|(k,l)} \frac{1}{d}\nu\Big(\frac{k}{d},\frac{l}{d}\Big)\Big)\zeta(s,t).$$

Equivalently, we have

$$\nu(m,n) = -\zeta(m,n) - \sum_{\substack{d>1 \\ d|(m,n)}} \frac{1}{d}\nu\left(\frac{m}{d},\frac{n}{d}\right) - \frac{1}{m}\sum_{(m,n)=(k,l)+(s,t)} k\left(\sum_{d|(k,l)} \frac{1}{d}\nu\left(\frac{k}{d},\frac{l}{d}\right)\right)\zeta(s,t). \tag{3.20}$$

It is worthwhile to note that the various choices of ρ and bilinear forms give many different relations. For example, if we take $\rho = (0,1)$, then we get

$$n\sum_{d|(m,n)} \frac{1}{d}\nu\left(\frac{m}{d},\frac{n}{d}\right) = -n\zeta(m,n) - \sum_{(m,n)=(k,l)+(s,t)} l\left(\sum_{d|(k,l)} \frac{1}{d}\nu\left(\frac{k}{d},\frac{l}{d}\right)\right)\zeta(s,t). \tag{3.21}$$

(b) Recall the *elliptic modular function* j defined by

$$j(q) - 744 = q^{-1} + 196884q + 21473760q^2 + \cdots = \sum_{n=-1}^{\infty} c(n)q^n. \tag{3.22}$$

In [2], Borcherds proved the product identity

$$p^{-1}\prod_{\substack{m>0 \\ n\in\mathbb{Z}}} (1-p^m q^n)^{c(mn)} = j(p) - j(q),$$

which is equivalent to

$$\prod_{m,n=1}^{\infty} (1-p^m q^n)^{c(mn)} = 1 - \sum_{m,n=1}^{\infty} c(m+n-1)p^m q^n.$$

This is the denominator identity for the (negative part of) *Monster Lie algebra* which played a crucial role in Borcherds' proof of the Moonshine Conjecture ([2], [5]).

By the relation (3.20), we obtain a recursive relation for the coefficients $c(n)$ of the elliptic modular function j:

$$c(mn) = c(m+n-1) - \sum_{\substack{1<d \\ d|(m,n)}} \frac{1}{d} c\left(\frac{mn}{d^2}\right) + \frac{1}{m} \sum_{(k,l)+(s,t)=(m,n)} k \left(\sum_{d|(k,l)} \frac{1}{d} c\left(\frac{kl}{d^2}\right) \right) c(s+t-1). \tag{3.23}$$

Applying the recursive relation (3.23) to the pairs $(2,n)$ and $(3,n)$, we have

$$c(2n) = c(n+1) - \frac{1}{2} c\left(\frac{n}{2}\right) + \frac{1}{2} \sum_{j=1}^{n-1} c(j)c(n-j), \tag{3.24}$$

$$c(3n) = c(n+2) - \frac{1}{3} c(\frac{n}{3}) + \frac{1}{3} \sum_{j=1}^{n-1} c(j)c(n-j+1) + \frac{2}{3} \sum_{j=1}^{n-1} \left(c(2j) + \frac{1}{2} c\left(\frac{j}{2}\right) \right) c(n-j), \tag{3.25}$$

where $c(n) = 0$ for non-integral values of n. On the other hand, by taking the pairs $(2, 2n)$ and $(4, n)$, we get

$$\begin{aligned} c(4n) &= c(2n+1) + \frac{1}{2} \left(c(n)^2 - c(n) \right) + \sum_{j=1}^{n-1} c(j)c(2n-j) \\ &= c(n+3) - \frac{\xi(n)}{2} c(n) - \frac{1}{4} c\left(\frac{n}{4}\right) \\ &\quad + \frac{1}{4} \sum_{j=1}^{n-1} c(j)c(n-j+2) + \frac{1}{2} \sum_{j=1}^{n-1} \left(c(2j) + \frac{1}{2} c\left(\frac{j}{2}\right) \right) c(n-j+1) \\ &\quad + \frac{3}{4} \sum_{j=1}^{n-1} \left(c(3j) + \frac{1}{3} c\left(\frac{j}{3}\right) \right) c(n-j), \end{aligned}$$

where $\xi(n) = 1$ if n is even and $\xi(n) = 0$ if n is odd.

Combining these relations, we obtain

$$
\begin{aligned}
c(2n+1) &= -\frac{1}{2}\left(c(n)^2 - c(n)\right) - \sum_{j=1}^{n-1} c(j)c(2n-j) + c(n+3) \\
&\quad -\frac{\xi(n)}{2}c(n) - \frac{1}{4}c\left(\frac{n}{4}\right) + \frac{1}{4}\sum_{j=1}^{n-1} c(j)c(n-j+2) \\
&\quad +\frac{1}{2}\sum_{j=1}^{n-1}\left(c(2j) + \frac{1}{2}c\left(\frac{j}{2}\right)\right)c(n-j+1) \\
&\quad +\frac{3}{4}\sum_{j=1}^{n-1}\left(c(3j) + \frac{1}{3}c\left(\frac{j}{3}\right)\right)c(n-j).
\end{aligned}
\tag{3.26}
$$

By applying (3.24) and (3.25) repeatedly, we can reduce (3.26) to a recursive formula for $c(2n+1)$. □

Remark. It is well-known that the coefficients $c(n)$ are determined by the first 4 coefficients: $c(1)$, $c(2)$, $c(3)$, and $c(5)$ (see, for example, [2]). In [17], J.-K. Koo and Y.-T. Oh showed that $c(5)$ can be expressed in terms of $c(1)$, $c(2)$, and $c(3)$. Hence the first 3 coefficients completely determine the elliptic modular function j. We can derive the same result using the Peterson-type recursive formula. Actually our method can be generalized to show that the coefficients of certain class of Thompson series are completely determined by the first three coefficients (cf. [16]).

§4. Generalized Kac-Moody superalgebras

In this section, we apply our superdimension formulas to generalized Kac-Moody superalgebras to derive *Peterson-type root multiplicity formula* and *Freudenthal-type weight multiplicity formula* for generalized Kac-Moody superalgebras and their highest weight modules. We first recall some of the basic structure theory and representation theory of generalized Kac-Moody superalgebras.

Let I be a countable (possibly infinite) index set. A real square matrix $A = (a_{ij})_{i,j\in I}$ is called a *Borcherds-Cartan matrix* if it satisfies: (i) $a_{ii} = 2$ or $a_{ii} \le 0$ for all $i \in I$, (ii) $a_{ij} \le 0$ if $i \ne j$, and $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$, (iii) $a_{ij} = 0$ implies $a_{ji} = 0$. We say that an index i is *real* if $a_{ii} = 2$ and *imaginary* if $a_{ii} \le 0$ and denote by $I^{\rm re} = \{i \in I |\ a_{ii} = 2\}$, $I^{\rm im} = \{i \in I |\ a_{ii} \le 0\}$. Let $\underline{m} = (m_i \in \mathbb{N} |\ i \in I)$ be a sequence of positive integers such that $m_i = 1$

for all $i \in I^{\text{re}}$. We call $\underline{m}$ the *charge* of A. In this paper, we assume that the Borcherds-Cartan matrix A is *symmetrizable*, i.e., there is a diagonal matrix $D = \operatorname{diag}(s_i|\ i \in I)$ with $s_i > 0$ $(i \in I)$ such that DA is symmetric.

Let I^{odd} be a subset of I and set $I^{\text{even}} = I \setminus I^{\text{odd}}$. We call $i \in I$ an *even index* (resp. *odd index*) if $i \in I^{\text{even}}$ (resp. $i \in I^{\text{odd}}$). The Borcherds-Cartan matrix A is said to be *colored by* I^{odd} if $a_{ij} \in 2\mathbb{Z}_{\leq 0}$ for all $j \in I$ whenever $i \in I^{\text{re}} \cap I^{\text{odd}}$.

Let $\mathfrak{h} = (\bigoplus_{i \in I} \mathbb{C}h_i) \oplus (\bigoplus_{i \in I} \mathbb{C}d_i)$ be a complex vector space with a basis $\{h_i, d_i|\ i \in I\}$, and for each $i \in I$ define a linear functional $\alpha_i \in \mathfrak{h}^*$ by

$$\alpha_i(h_j) = a_{ji}, \quad \alpha_i(d_j) = \delta_{ij} \ \text{ for all } \ j \in I. \tag{4.1}$$

The free abelian group $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ generated by α_i's $(i \in I)$ is called the *root lattice* associated with A. Let $\Pi = \{\alpha_i|\ i \in I\}$ and B be a basis of $\mathfrak{h}^*$ extending Π. Set $B' = B \setminus \Pi$. Since A is assumed to be symmetrizable, there is a symmetric bilinear form $(\ |\ )$ on $\mathfrak{h}^*$ defined by

$$(\alpha_i|\alpha_j) = s_i a_{ij} \ \text{ for } i, j \in I,$$

$$(\lambda|\alpha_i) = \lambda(s_i h_i) \ \text{ for } \lambda \in B',$$

$$(\lambda|\mu) = 0 \ \text{ for } \lambda, \mu \in B'.$$

We can also define a nondegenerate symmetric bilinear form on $\mathfrak{h}$ by

$$(h_i|h) = \frac{1}{s_i}\alpha_i(h) \ \text{ and } \ (d_i|d_j) = 0$$

for all $h \in \mathfrak{h}$, $i, j \in I$ (cf. [7],[11]).

Let $Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ and $Q^- = -Q^+$. We define a group homomorphism $\deg :\ Q \to \mathbb{Z}_2$ by

$$\deg \alpha_i = \begin{cases} \bar{0} & \text{if } \ i \in I^{\text{even}}, \\ \bar{1} & \text{if } \ i \in I^{\text{odd}}. \end{cases}$$

An element $\alpha \in Q$ is said to be *even* (resp. *odd*) if $\deg \alpha = \bar{0}$ (resp. $\deg \alpha = \bar{1}$).

The *generalized Kac-Moody superalgebra* $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, I^{\text{odd}})$ associated with a symmetrizable Borcherds-Cartan matrix $A = (a_{ij})_{i,j \in I}$ of charge $\underline{m} = (m_i|\ i \in I)$ colored by I^{odd} is the Lie superalgebra generated by the elements h_i, d_i $(i \in I)$, e_{ik}, f_{ik} $(i \in I,\ k = 1, 2, \cdots, m_i)$ with the following defining relations:

$$\begin{aligned}
&[h_i, h_j] = [h_i, d_j] = [d_i, d_j] = 0,\\
&[h_i, e_{jl}] = a_{ij} e_{jl}, \quad [h_i, f_{jl}] = -a_{ij} f_{jl},\\
&[d_i, e_{jl}] = \delta_{ij} e_{jl}, \quad [d_i, f_{jl}] = -\delta_{ij} f_{jl},\\
&[e_{ik}, f_{jl}] = \delta_{ij}\delta_{kl} h_i,\\
&(ade_{ik})^{1-a_{ij}}(e_{jl}) = (adf_{ik})^{1-a_{ij}}(f_{jl}) = 0 \ \text{ if } a_{ii} = 2 \text{ and } i \neq j,\\
&[e_{ik}, e_{jl}] = [f_{ik}, f_{jl}] = 0 \ \text{ if } a_{ij} = 0,\\
&\deg h_i = \deg d_i = \bar{0},\\
&\deg e_i = \deg f_i = \bar{0} \qquad \text{if } i \in I^{\text{even}},\\
&\deg e_i = \deg f_i = \bar{1} \qquad \text{if } i \in I^{\text{odd}},
\end{aligned}$$

for $i, j \in I,\ k = 1, \cdots, m_i,\ l = 1, \cdots, m_j$.

The abelian subalgebra $\mathfrak{h} = \left(\bigoplus_{i \in I} \mathbb{C} h_i\right) \bigoplus \left(\bigoplus_{i \in I} \mathbb{C} d_i\right)$ is called the *Cartan subalgebra* of $\mathfrak{g}$, and the linear functionals $\alpha_i \in \mathfrak{h}^*$ $(i \in I)$ defined by (4.1) are called the *simple roots* of $\mathfrak{g}$. For each $i \in I^{\text{re}}$, let $r_i \in GL(\mathfrak{h}^*)$ be the *simple reflection* on $\mathfrak{h}^*$ defined by

$$r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i \quad \text{for} \quad \lambda \in \mathfrak{h}^*.$$

The subgroup W of $GL(\mathfrak{h}^*)$ generated by the r_i's $(i \in I^{\text{re}})$ is called the *Weyl group* of $\mathfrak{g}$.

The generalized Kac-Moody superalgebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, I^{\text{odd}})$ has the *root space decomposition* $\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$, where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} |\ [h, x] = \alpha(h)x \ \text{ for all } h \in \mathfrak{h}\}.$$

If $\mathfrak{g}_\alpha \neq 0$, then α is called a *root* of $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ is called the *root space* of $\mathfrak{g}$ attached to α. We say that a root α is *real* if $(\alpha|\alpha) > 0$ and *imaginary* if $(\alpha|\alpha) \leq 0$. A root $\alpha > 0$ (resp. $\alpha < 0$) is called *positive* (resp. *negative*). One can show that all the roots are either positive or negative. We denote by Φ, Φ^+ and Φ^- the set of all roots, positive roots and negative roots, respectively. We also denote by $\Phi_{\bar{0}}$ (resp. $\Phi_{\bar{1}}$) the set of all even (resp.

odd) roots of $\mathfrak{g}$. Define the subalgebras $\mathfrak{g}^{\pm} = \bigoplus_{\alpha\in\Phi^{\pm}} \mathfrak{g}_\alpha$. Then we have the *triangular decomposition*:

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+.$$

For each $\alpha \in Q$, we define the *superdimension* of $\mathfrak{g}_\alpha$ to be

$$\operatorname{sdim} \mathfrak{g}_\alpha = (-1)^{\deg \alpha} \dim \mathfrak{g}_\alpha.$$

A $\mathfrak{g}$-module V is called $\mathfrak{h}$-*diagonalizable* if it admits a *weight space decomposition* $V = \bigoplus_{\mu\in\mathfrak{h}^*} V_\mu$, where

$$V_\mu = \{v \in V |\ h \cdot v = \mu(h)v \ \text{ for all } h \in \mathfrak{h}\}.$$

If $V_\mu \neq 0$, then μ is called a *weight* of V and V_μ is called the μ-*weight space.* We denote by $P(V)$ the set of all weights of V.

For $\lambda, \nu \in \mathfrak{h}^*$, we define $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$. We denote by $\mathcal{O}$ the category of $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-modules with finite dimensional weight spaces such that there exist a finite number of linear functionals $\lambda_1, \cdots, \lambda_s$ satisfying $P(V) \subset \cup_{i=1}^{s} D(\lambda_i)$, where $D(\lambda) = \{\mu \in \mathfrak{h}^* |\ \mu \leq \lambda\}$. The morphisms in $\mathcal{O}$ are $\mathfrak{g}$-module homomorphisms.

An $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-module V is called a *highest weight module* with highest weight $\lambda \in \mathfrak{h}^*$ if there is a nonzero vector $v_\lambda \in V$ such that (i) $e_{ik} \cdot v_\lambda = 0$ for all $i \in I$, $k = 1, \cdots, m_i$, (ii) $h \cdot v_\lambda = \lambda(h)v_\lambda$ for all $h \in \mathfrak{h}$, (iii) $V = U(\mathfrak{g}) \cdot v_\lambda$, where $U(\mathfrak{g})$ is the universal enveloping superalgebra of $\mathfrak{g}$. The vector v_λ is called a *highest weight vector*. For a highest weight module V with highest weight λ, we have (i) $V = U(\mathfrak{g}^-) \cdot v_\lambda$, (ii) $V = \bigoplus_{\mu\leq\lambda} V_\mu$, $V_\lambda = \mathbb{C}v_\lambda$, and (iii) $\dim V_\mu < \infty$ for all $\mu \leq \lambda$. Clearly, V is in category $\mathcal{O}$. A $\mathfrak{g}$-module $M(\lambda)$ with highest weight λ is called a *Verma module* if every $\mathfrak{g}$-module with highest weight λ is a quotient of $M(\lambda)$. The Verma module $M(\lambda)$ contains a unique maximal submodule $J(\lambda)$. Therefore the quotient $V(\lambda) = M(\lambda)/J(\lambda)$ is irreducible.

Take a linear functional $\rho \in \mathfrak{h}^*$ satisfying $\rho(h_i) = \frac{1}{2}a_{ii}$ for all $i \in I$. Such a linear functional is called a *Weyl vector* of $\mathfrak{g}$. Note that $(\rho|\alpha_i) = \rho(s_i h_i) = \frac{1}{2}(\alpha_i|\alpha_i)$ $(i \in I)$.

Let P^+ be the set of all linear functionals $\lambda \in \mathfrak{h}^*$ satisfying

$$\begin{cases} \lambda(h_i) \in \mathbb{Z}_{\geq 0} & \text{for all } i \in I^{\text{re}}, \\ \lambda(h_i) \in 2\mathbb{Z}_{\geq 0} & \text{for all } i \in I^{\text{re}} \cap I^{\text{odd}}, \\ \lambda(h_i) \geq 0 & \text{for all } i \in I^{\text{im}}. \end{cases}$$

The elements of P^+ are called the *dominant integral weights*. For a dominant integral weight $\lambda \in P^+$, let

$$\begin{aligned}\Phi^+(\lambda) = \{\beta = \sum_{i\in I^{\mathrm{im}}} k_i\alpha_i \in Q^+ |\ (\lambda|\alpha_i) = 0 \ \text{ for } k_i \geq 1,\\ (\alpha_i|\alpha_j) = 0 \ \text{ for } k_i, k_j \geq 1,\ i \neq j,\ \ (\alpha_i|\alpha_i) = 0 \ \text{ for } k_i \geq 2\}.\end{aligned} \tag{4.2}$$

For such an element $\beta \in \Phi^+(\lambda)$, we denote $|\beta| = \sum_{i\in I^{\mathrm{im}}} k_i$ and

$$\varepsilon(\beta) = \prod_{i\in I^{\mathrm{im}}\cap I^{\mathrm{even}}} \binom{m_i}{k_i} \prod_{j\in I^{\mathrm{im}}\cap I^{\mathrm{odd}}} \binom{m_j + k_j - 1}{k_j}.$$

For each $\mu \leq \lambda$, we define $\deg\mu = \begin{cases} \overline{0} & \text{if } \lambda-\mu \text{ is even,} \\ \overline{1} \text{ if } \lambda-\mu & \text{is odd.}\end{cases}$ Then the supercharacters of the highest weight modules are given in the following proposition.

Proposition 4.1. ([21], [23])
(a) *For any $\lambda \in \mathfrak{h}^*$, we have*

$$\operatorname{sch} M(\lambda) = \frac{E^\lambda}{\prod_{\alpha\in Q^-} (1-E^\alpha)^{\operatorname{sdim}\mathfrak{g}_\alpha}}.$$

(b) *For any $\lambda \in P^+$, we have*

$$\operatorname{sch} V(\lambda) = \frac{\sum_{\substack{w\in W\\ \beta\in\Phi^+(\lambda)}} \varepsilon_\lambda(w,\beta) E^{w(\lambda+\rho-\beta)-\rho}}{\prod_{\alpha\in\Phi^-} (1-E^\alpha)^{\operatorname{sdim}\mathfrak{g}_\alpha}}, \tag{4.3}$$

where $\varepsilon_\lambda(w,\beta) = (-1)^{l(w)+|\beta|}(-1)^{\deg(w(\lambda+\rho-\beta)-\rho)}\varepsilon(\beta)$.

In particular, if $\lambda = 0$, we obtain the denominator identity:

$$\prod_{\alpha\in\Phi^-} (1-E^\alpha)^{\operatorname{sdim}\mathfrak{g}_\alpha} = 1 + \sum_{\beta\in\Phi^-} \zeta(\beta) E^\beta, \tag{4.4}$$

where $\zeta(\beta) = \sum_{\substack{w\in W\\ \gamma\in\Phi^+(0)\\ \beta=w(\rho-\gamma)-\rho}} \varepsilon_0(w,\gamma).$ □

Let J be a finite subset of I^{re}, and let $\Phi_J = \Phi \cap (\sum_{j\in J} \mathbb{Z}\alpha_j)$, $\Phi_J^{\pm} = \Phi^{\pm} \cap \Phi_J$ and $\Phi^{\pm}(J) = \Phi^{\pm} \setminus \Phi_J^{\pm}$. We also denote $Q_J = Q \cap (\sum_{j\in J} \mathbb{Z}\alpha_j)$, $Q_J^{\pm} = Q^{\pm} \cap Q_J$ and $Q^{\pm}(J) = Q^{\pm} \setminus Q_J^{\pm}$. Let $\mathfrak{g}_0^{(J)} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Phi_J} \mathfrak{g}_\alpha\right)$, and $\mathfrak{g}_{\pm}^{(J)} = \bigoplus_{\alpha\in\Phi^{\pm}(J)} \mathfrak{g}_\alpha$. Then we have the *triangular decomposition*:

$$\mathfrak{g} = \mathfrak{g}_-^{(J)} \oplus \mathfrak{g}_0^{(J)} \oplus \mathfrak{g}_+^{(J)},$$

where $\mathfrak{g}_0^{(J)}$ is the Kac-Moody superalgebra (with an extended Cartan subalgebra) associated with the generalized Cartan matrix $A_J = (a_{ij})_{i,j\in J}$ of charge $m_J = (m_i | i \in J)$ colored by $I_J^{\mathrm{odd}} = I^{\mathrm{odd}} \cap J$. Let W_J be the subgroup of W generated by the simple reflections r_j with $j \in J$, and let $W(J) = \{w \in W | \Phi_w \subset \Phi^+(J)\}$, where $\Phi_w = \{\alpha \in \Phi^+ | w^{-1}\alpha < 0\}$. Then $W(J)$ is the set of right coset representatives of W_J in W. In the following proposition, we recall the denominator identity for the Lie superalgebra $\mathfrak{g}_-^{(J)}$.

Proposition 4.2. ([13],[18]) *Let $J \subset I^{\mathrm{re}}$ and let $\mathfrak{g}_-^{(J)} = \bigoplus_{\alpha\in\Phi^-(J)} \mathfrak{g}_\alpha$ be the subalgebra of the generalized Kac-Moody superalgebra $\mathfrak{g}$ defined as above. Then we have*

$$\prod_{\alpha\in\Phi^-(J)} (1 - E^{\alpha})^{\operatorname{sdim} \mathfrak{g}_\alpha} = \sum_{\substack{w\in W(J)\\ \beta\in\Phi^+(0)}} (-1)^{l(w)+|\beta|} \varepsilon(\beta) \operatorname{sch} V_J(w(\rho - \beta) - \rho), \tag{4.5}$$

where $V_J(\mu)$ is the irreducible highest weight module over $\mathfrak{g}_0^{(J)}$ with highest weight μ.

□

From now on, we will use the notation

$$H(\mathfrak{g}_-^{(J)}) = \bigoplus_{k=0}^{\infty} (-1)^k H_k(\mathfrak{g}_-^{(J)}),$$

where

$$H_k(\mathfrak{g}_-^{(J)}) = \sum_{\substack{w\in W(J)\\ \beta\in\Phi^+(0)\\ l(w)+|\beta|=k}} V_J(w(\rho - \beta) - \rho)^{\varepsilon(\beta)}.$$

Then (4.5) can be also written as

$$\prod_{\alpha\in\Phi^-(J)} (1-E^{\alpha})^{\operatorname{sdim}\mathfrak{g}_\alpha} = \operatorname{sch} H(\mathfrak{g}_-^{(J)}).$$

The index set I is assumed to be finite so that we can be guaranteed the existence of dual bases for the nondegenerate bilinear form $(\ |\ )$ on $\mathfrak{h}^*$. Later, we will explain how we can deal with the case when I is infinite (see the remark after Theorem 4.8).

Let

$$C = \sum_{\gamma\in Q^-(J)} C(\gamma)E^{\gamma} \quad \text{and} \quad C^* = \sum_{\gamma\in Q^-(J)} C^*(\gamma)E^{\gamma}$$

be the formal power series whose coefficients are given by

$$\begin{aligned} C(\gamma) &= \sum_{d|\gamma} \frac{1}{d}\operatorname{sdim}\mathfrak{g}_{\frac{\gamma}{d}}, \\ C^*(\gamma) &= (\gamma|\gamma)C(\gamma) - \sum_{\substack{\gamma=\gamma'+\gamma'' \\ \gamma',\gamma''\in Q^-(J)}} (\gamma'|\gamma'')C(\gamma')C(\gamma''). \end{aligned}$$

Then by Theorem 3.4, we obtain the following Peterson-type recursive root multiplicity formula for generalized Kac-Moody superalgebras.

THEOREM 4.3.
(a) *Let*

$$\begin{aligned} D^{(J)} &= \prod_{\alpha\in\Phi^-(J)} (1-E^{\alpha})^{\operatorname{sdim}\mathfrak{g}_\alpha} \\ &= \sum_{\substack{w\in W(J) \\ \beta\in\Phi^+(0)}} (-1)^{l(w)+|\beta|}\varepsilon(\beta)\operatorname{sch} V_J(w(\rho-\beta)-\rho), \end{aligned}$$

where $V_J(\mu)$ is the irreducible highest weight module over $\mathfrak{g}_0^{(J)}$ with highest weight μ. Then for any $\mu\in\mathfrak{h}^*$, the following differential equations hold:

$$\nabla_\mu(D^{(J)}) = -\nabla_\mu(C)D^{(J)}, \qquad \Delta(D^{(J)}) = -C^*D^{(J)}.$$

(b) *For any $\mu \in \mathfrak{h}^*$ and any $\gamma \in Q^-(J)$, we have*

$$(4.6)\qquad \begin{aligned} &(\mu|\gamma)C(\gamma) + \sum_{\substack{\gamma=\gamma'+\gamma'' \\ \gamma',\gamma''\in Q^-(J)}} (\mu|\gamma')C(\gamma')\zeta(\gamma'') = -(\mu|\gamma)\zeta(\gamma), \\ &C^*(\gamma) + \sum_{\substack{\gamma=\gamma'+\gamma'' \\ \gamma',\gamma''\in Q^-(J)}} C^*(\gamma')\zeta(\gamma'') = -(\gamma|\gamma)\zeta(\gamma), \end{aligned}$$

where $\zeta(\gamma) = \sum_{\substack{w\in W(J) \\ \beta\in\Phi^+(0)}} (-1)^{l(w)+|\beta|}\varepsilon(\beta)\,\mathrm{sdim}\, V_J(w(\rho-\beta)-\rho)_\gamma.$ ∎

Take $J = \emptyset$ and let $\rho \in \mathfrak{h}^*$ be a Weyl vector of $\mathfrak{g}$. Then $W(J) = W$ and by definition of $\Phi^+(0)$ and W-invariance of $(\ |\ )$, we have

$$(\Delta + 2\nabla_\rho)(D^{(\emptyset)}) = (\Delta + 2\nabla_\rho)\Big(\sum_{\substack{w\in W \\ \beta\in\Phi^+(0)}} (-1)^{l(w)+|\beta|}\varepsilon(\beta)E^{w(\rho-\beta)-\rho}\Big) = 0.$$

Hence we get

$$(\Delta + 2\nabla_\rho)(D^{(\emptyset)}) = -(C^* + 2\nabla_\rho(C))D^{(\emptyset)} = 0,$$

which implies $C^* + 2\nabla_\rho(C) = 0$. Therefore we obtain *Peterson's root multiplicity formula* extended to generalized Kac-Moody superalgebras.

PROPOSITION 4.4. ([17]) *For any $\gamma \in Q^-$, we have*

$$(4.7)\qquad (\gamma+2\rho|\gamma)C(\gamma) = \sum_{\substack{\gamma=\gamma'+\gamma'' \\ \gamma',\gamma''\in Q^-}} (\gamma'|\gamma'')C(\gamma')C(\gamma''),$$

where $C(\gamma) = \sum_{d|\gamma} \frac{1}{d}\,\mathrm{sdim}\,\mathfrak{g}_{\frac{\gamma}{d}}.$ ∎

Remark. To calculate the root multiplicities recursively, we would like to make sure that $(\gamma|\gamma+2\rho) \neq 0$. If $\gamma \in W\cdot(-2\alpha_i)$ for some $i \in I^{\mathrm{re}} \cap I^{\mathrm{odd}}$, it may occur that $(\gamma|\gamma+2\rho) = 0$, but we already know that $\mathrm{sdim}\,\mathfrak{g}_\gamma = 1$. If $\gamma \notin W\cdot(-2\alpha_i)$, then one can show that $(\gamma|\gamma+2\rho) \leq 0$ and that the equality holds if and only if $\gamma = -\alpha_i$ for some $i \in I$. That is, if γ is not a simple root and $\gamma \notin W\cdot(-2\alpha_i)$ for all $i \in I^{\mathrm{re}} \cap I^{\mathrm{odd}}$, then we have $(\gamma|\gamma+2\rho) < 0$.

For the other extreme, take $J = I^{\mathrm{re}}$. Then $W(J) = \{1\}$ and we get

$$\zeta(\gamma) = \sum_{\beta \in \Phi^+(0)} (-1)^{|\beta|} \varepsilon(\beta) \operatorname{sdim} V_J(-\beta)_\gamma,$$

which is independent of the Weyl group W. For example, if $a_{ij} \neq 0$ for all $i, j \in I^{\mathrm{im}}$, then every element $\beta \in \Phi^+(0)$ has the form $\beta = 0$ or $\beta = \alpha_i (i \in I^{\mathrm{im}})$. Hence we get

$$\zeta(\gamma) = \begin{cases} 1 & \text{if } \gamma = 0, \\ -\sum_{i \in I^{\mathrm{im}}} m_i \operatorname{sdim} V_J(-\alpha_i)_\gamma & \text{otherwise.} \end{cases}$$

Therefore, if the weight multiplicities of irreducible highest weight $\mathfrak{g}_0^{(J)}$-modules are explicitly known, then the Peterson-type root multiplicity formula (4.6) gives a very efficient recursive formula for the root multiplicities of $\mathfrak{g}$.

Finally, by the same technique that was used in deriving our Peterson-type root multiplicity formula, we can derive a *Freundenthal-type weight multiplicity formula* for highest weight modules over generalized Kac-Moody superalgebras.

THEOREM 4.5. *Let V be a highest weight module over a generalized Kac-Moody superalgebra with highest weight $\Lambda \in \mathfrak{h}^*$. Then, for any $\lambda \leq \Lambda$, we have*

$$(4.8) \qquad \begin{aligned} &(|\Lambda + \rho|^2 - |\lambda + \rho|^2) \operatorname{sdim} V_\lambda \\ &= 2 \sum_{\substack{\alpha \in \Phi^+ \\ j \geq 1}} (\lambda + j\alpha | \alpha)(\operatorname{sdim} \mathfrak{g}_\alpha)(\operatorname{sdim} V_{\lambda + j\alpha}) \\ &= 2 \sum_{\beta \in Q^+} (\lambda + \beta | \beta) C(\beta) \operatorname{sdim} V_{\lambda + \beta}. \end{aligned}$$

Proof. Recall that ([11])

$$\operatorname{sch} V = \sum_{\substack{\lambda \leq \Lambda \\ |\lambda + \rho|^2 = |\Lambda + \rho|^2}} c_\lambda \operatorname{sch} M(\lambda), \text{ where } c_\lambda \in \mathbb{Z}, c_\Lambda = 1.$$

For convenience, set

$$A = \operatorname{sch} V,$$
$$B = \textstyle\prod_{\alpha\in\Phi^-}(1-E^{\alpha})^{\operatorname{sdim}\mathfrak{g}_\alpha},$$
$$D = \sum_{\substack{\lambda\le\Lambda\\ |\lambda+\rho|^2=|\Lambda+\rho|^2}} c_\lambda E^{\lambda}.$$

Then by Proposition 4.1 (a), we have $AB = D$. Applying the Laplacian and the ρ-directional derivative yields

$$(\Delta+2\nabla_\rho)(AB)$$
$$= A(\Delta+2\nabla_\rho)(B) + B(\Delta+2\nabla_\rho)(A) + 2AB\sum_i \partial_i(\log A)\partial^i(\log B).$$

On the other hand, since $|\lambda+\rho|^2 = |\Lambda+\rho|^2$, we get

$$(\Delta+2\nabla_\rho)(D) = (\Lambda|\Lambda+2\rho)D.$$

Recall that $(\Delta+2\nabla_\rho)(B) = 0$. Hence by combining these equations, we conclude

$$(\Delta+2\nabla_\rho)(A) + 2\sum_i \partial_i(A)\partial^i(\log B) = (\Lambda|\Lambda+2\rho)A.$$

By comparing the coefficients of E^{λ}, we obtain the desired result. ∎

Remark.

(a) By taking a look at the formula (4.8), a natural question arises: what would happen if $|\lambda+\rho|^2 = |\Lambda+\rho|^2$? If $\Lambda \in P^+$, we can prove that $|\lambda+\rho|^2 \le |\Lambda+\rho|^2$ for all weights $\lambda\le\Lambda$ and that the equality holds if and only if $\lambda = \Lambda$.

(b) In [17], the Freudenthal-type weight multiplicity formula was derived for generalized Kac-Moody superalgebras using the *Casimir operator*.

(c) Our results in this section also hold when I is infinite. If $\beta = \sum k_j\alpha_j \in Q^-$, let $J = \{j\in I | k_j \ne 0\}$ and we can apply our theorems to the generalized Kac-Moody superalgebra $\mathfrak{g}(A_J, \underline{m}_J, I_J^{\mathrm{odd}})$ which corresponds to the finite index set J.

§5. Examples and tables

In this section, we illustrate how to apply our Peterson-type root multiplicity formula with the examples of rank 2 generalized Kac-Moody superalgebras and Monstrous Lie superalgebras. We also give an example of computing the weight multiplicities for irreducible highest weight modules over rank 2 generalized Kac-Moody superalgebras. At the end of this section, we present some tables of root and weight multiplicities for these algebras and modules.

EXAMPLE 5.1. Let $A = (a)$ be a rank 1 Borcherds-Cartan matrix of charge $\underline{m} = (r)$ with $I^{\mathrm{odd}} = I$, and let $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, I^{\mathrm{odd}})$ be the associated generalized Kac-Moody superalgebra. Here, we have $a = 2$, $a = 0$, or $a < 0$, and $r \in \mathbb{Z}_{>0}$. We denote by α the only simple root of $\mathfrak{g}$.

If $a = 2$, then $\mathfrak{g}$ is the 5-dimensional ortho-symplectic Lie superalgebra $osp(1, 2)$ and we have

$$\operatorname{sdim} \mathfrak{g}_{n\alpha} = \begin{cases} -1 & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ 0 & \text{if } n \geq 3. \end{cases}$$

If $a = 0$, then $\mathfrak{g}$ is the Heisenberg Lie superalgebra $sl(1, 2r)$ and we have

$$\operatorname{sdim} \mathfrak{g}_{n\alpha} = \begin{cases} -r & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

If $a < 0$, then the simple root α is odd and imaginary with multiplicity $r \geq 1$. We identify the root lattice with $\mathbb{Z}$ by setting $\alpha = 1$ and choose $\mu = \frac{1}{a}$ so that $(\mu|1) = 1$.

For $n > 1$, the formula (4.6) yields

$$-nC(-n) = \sum_{k=1}^{n-1} kC(-k)\zeta(-n+k) + n\zeta(-n).$$

Since $\zeta(-1) = r$ and $\zeta(-n) = 0$ for $n \geq 2$, we get $nC(-n) = -r(n-1)C(-n+1)$, which gives

$$C(-n) = C(n) = \frac{1}{n}(-r)^n. \tag{5.1}$$

Hence we obtain

$$\operatorname{sdim} \mathfrak{g}_{n\alpha} = \frac{1}{n}(-r)^n - \sum_{\substack{d>1 \\ d|n}} \frac{1}{d} \operatorname{sdim} \mathfrak{g}_{\frac{n}{d}\alpha}.$$

If we apply the Möbius inversion to (5.1), we get the *Witt formula* for free Lie superalgebras:

$$\operatorname{sdim} \mathfrak{g}_{n\alpha} = \frac{1}{n} \sum_{d|n} \mu(d)(-r)^{\frac{n}{d}}.$$

In particular, the subalgebra $\mathfrak{g}_+ = \bigoplus_{n\geq 1} \mathfrak{g}_{n\alpha}$ (resp. $\mathfrak{g}_- = \bigoplus_{n\geq 1} \mathfrak{g}_{-n\alpha}$) is the free Lie superalgebra generated by the subspace $\mathfrak{g}_\alpha$ (resp. $\mathfrak{g}_{-\alpha}$).

EXAMPLE 5.2. Let $I = \{0, 1\}$ be the index set and let $A = \begin{pmatrix} 2 & -a \\ -b & -c \end{pmatrix}$ $(a, b, c \in \mathbb{Z}_{\geq 0})$ be a Borcherds-Cartan matrix of charge $\underline{m} = (1, r)$. Take $I^{\text{odd}} = \{1\}$ and let $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, I^{\text{odd}})$ be the associated generalized Kac-Moody superalgebra. We identify $\alpha = m\alpha_0 + n\alpha_1 \in Q$ with $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ and choose $\mu \in \mathfrak{h}^*$ such that $(\mu|(m, n)) = m$. We would like to apply our Peterson-type root multiplicity formula (4.6) with $J = \{0\}$. In this case, the subalgebra $\mathfrak{g}_0^{(J)}$ is isomorphic to $sl(2, \mathbb{C}) + \mathfrak{h}$.

(a) If $a, b, c > 0$, then

$$\begin{aligned} &H_0(\mathfrak{g}_-^{(J)}) = \mathbb{C}, \\ &H_1(\mathfrak{g}_-^{(J)}) = V_J(-\alpha_1)^{\oplus r}, \\ &H_k(\mathfrak{g}_-^{(J)}) = 0 \text{ for } k \geq 2, \end{aligned}$$

where $V_J(-\alpha_1)^{\oplus r}$ is the r-copies of $(a+1)$-dimensional irreducible $sl(2, \mathbb{C})$-module, and we have

$$\zeta(-m, -n) = \operatorname{sdim} H(\mathfrak{g}_-^{(J)})_{(-m,-n)} = \begin{cases} 1 & \text{if } (m, n) = (0, 0), \\ r & \text{if } 0 \leq m \leq a, n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we obtain

$$\operatorname{sdim} \mathfrak{g}_{(m,n)} = C(m, n) - \sum_{\substack{d>1 \\ d|(m,n)}} \frac{1}{d} \operatorname{sdim} \mathfrak{g}_{(\frac{m}{d}, \frac{n}{d})},$$

where $C(m,n)$ are determined recursively by

$$C(m,n)=\begin{cases} -r & \text{if } 1\le m\le a, n=1,\\ -\dfrac{r}{m}\displaystyle\sum_{k=1}^{m} kC(k,n-1) & \text{if } 1\le m\le a, n\ge 2,\\ -\dfrac{r}{m}\displaystyle\sum_{k=m-a}^{m} kC(k,n-1) & \text{if } m>a, n\ge 2,\\ 0 & \text{otherwise.}\end{cases}$$

(b) If $a,b>0$ and $c=0$, then $H_k(\mathfrak{g}_-^{(J)})=V_J(-k\alpha_1)^{\oplus r}$, and hence we have

$$\begin{aligned}\zeta(-m,-n)&=\operatorname{sdim} H(\mathfrak{g}_-^{(J)})_{(-m,-n)}\\ &=\begin{cases} 1 & \text{if } (m,n)=(0,0),\\ \begin{pmatrix} r+n-1\\ n\end{pmatrix} & \text{if } 0\le m\le na, n\ge 1,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Therefore we obtain

$$\operatorname{sdim}\mathfrak{g}_{(m,n)}=C(m,n)-\sum_{\substack{d>1\\ d|(m,n)}}\frac{1}{d}\operatorname{sdim}\mathfrak{g}_{(\frac{m}{d},\frac{n}{d})},$$

where $C(m,n)$ are determined recursively by

$$C(m,n)=\begin{cases} -r & \text{if } 1\le m\le a, n=1,\\ -\dfrac{1}{m}\displaystyle\sum_{\substack{1\le l\le n-1\\ \sigma(l)\le k\le m}} k\begin{pmatrix} r+n-l-1\\ r-1\end{pmatrix}C(k,l)-\begin{pmatrix} r=n-1\\ r-1\end{pmatrix} & \\ & \text{if } 1\le m\le na, n\ge 2,\\ -\dfrac{1}{m}\displaystyle\sum_{\substack{1\le l\le n-1\\ \sigma(l)\le k\le m}} k\begin{pmatrix} r+n-l-1\\ r-1\end{pmatrix}C(k,l) & \\ & \text{if } m>na, n\ge 2,\\ 0 & \text{otherwise.}\end{cases}$$

Here $\sigma(l) = \max\{1, m - a(n-l)\}$.

We present the root multiplicity tables for these generalized Kac-Moody superalgebras in Table 5.1 − Table 5.5.

EXAMPLE 5.3. Let $\mathfrak{g}$ be the rank 2 generalized Kac-Moody superalgebra considered in Example 5.2 and let $V(\Lambda)$ be the irreducible highest weight $\mathfrak{g}$-module with highest weight Λ, where Λ is defined by $\Lambda(h_i) = \lambda_i$ $(i = 0, 1)$. Set

$$W(m,n) = \operatorname{sdim} V(\Lambda)_{\Lambda - m\alpha_0 - n\alpha_1} \quad (m, n \in \mathbb{Z}_{\geq 0}).$$

Then by the Freudenthal-type weight multiplicity formula (4.8), we obtain

$$W(m,n) = \frac{N(m,n)}{D(m,n)},$$

where

$$\begin{aligned} N(m,n) = 2 \sum_{\substack{0 \leq k \leq m \\ 0 \leq l \leq n}} &(bk\lambda_0 + al\lambda_1 - b(m-k)(2k - al) + a(n-l)(bk + cl)) \\ &\times C(k,l) W(m-k, n-l), \\ D(m,n) = 2(bm\lambda_0 + an\lambda_1) &- (2bm^2 - 2abmn - acn^2) + 2bm - acn, \end{aligned}$$

and $C(k,l)$ are determined by the recursive formulas in Example 5.2.

In Table 5.6 − Table 5.8, we present the weight multiplicity tables for these modules over rank 2 generalized Kac-Moody superalgebra $\mathfrak{g}$.

EXAMPLE 5.4. Let $I = \{-1\} \cup \{1, 2, 3, \cdots\}$ be the index set and let $A = (-(i+j))_{i,j \in I}$ be the Borcherds-Cartan matrix of the Monster Lie algebra ([2]). Consider a normalized q-series $F(q) = \sum_{n=-1}^{\infty} f(n)q^n$ such that $f(-1) = 1$, $f(0) = 0$, and $f(n) \in \mathbb{Z}$ for all $n \geq 1$. We define the charge of the matrix A to be $\underline{m} = (|f(i)| : i \in I)$ and set $I^{\text{even}} = \{i \in I | \ f(i) > 0\}$, $I^{\text{odd}} = \{i \in I | \ f(i) < 0\}$. (We neglect those i's for which $f(i) = 0$.) Then the generalized Kac-Moody superalgebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, I^{\text{odd}})$ is called the *Monstrous Lie superalgebra* associated with the normalized q-series $F(q) = \sum_{n=-1}^{\infty} f(n)q^n$ ([13]).

We identify the simple roots α_i $(i \in I)$ with $(1, i) \in \mathbb{Z} \times \mathbb{Z}$ and define a nondegenerate symmetric bilinear form on $\mathbb{Z} \times \mathbb{Z}$ by $((k,l)|(m,n)) = -(lm + kn)$. Set $\rho = (1, 0)$ so that we have $(\rho|\alpha_i) = -i = \frac{1}{2}(\alpha_i|\alpha_i)$ for all

$i \in I$. We would like to apply our Peterson-type root multiplicity formula (4.6) to the Monstrous Lie superalgebra $\mathfrak{g}$.

Take $J = \{-1\}$ and consider the triangular decomposition

$$\mathfrak{g} = \mathfrak{g}_-^{(J)} \oplus \mathfrak{g}_0^{(J)} \oplus \mathfrak{g}_+^{(J)},$$

where $\mathfrak{g}_0^{(J)} = \langle e_{-1}, f_{-1}, h_{-1} \rangle + \mathfrak{h} \cong sl(2,\mathbb{C}) + \mathfrak{h}$, $\mathfrak{g}_\pm^{(J)} = \bigoplus_{m,n=1}^{\infty} \mathfrak{g}_{(\pm m, \pm n)}$. Since $W(J) = \{1\}$, we have

$$\begin{aligned} H_0(\mathfrak{g}_-^{(J)}) &= \mathbb{C}, \\ H_1(\mathfrak{g}_-^{(J)}) &= \bigoplus_{i=1}^{\infty} V_J(-\alpha_i)^{\oplus |f(i)|}, \\ H_k(\mathfrak{g}_-^{(J)}) &= 0 \quad \text{if } \ k \geq 2, \end{aligned}$$

where $V_J(-\alpha_i)$ is the i-dimensional irreducible $sl(2,\mathbb{C})$-module. Hence the denominator identity for the Lie superalgebra $\mathfrak{g}_-^{(J)}$ is equal to

$$\prod_{m,n=1}^{\infty} (1 - p^m q^n)^{\operatorname{sdim} \mathfrak{g}_{(m,n)}} = 1 - \sum_{i,j=1}^{\infty} f(i+j-1) p^i q^j,$$

where $p = E^{(-1,0)}$, $q = E^{(0,-1)}$. Therefore, we obtain

$$\begin{aligned} \operatorname{sdim} \mathfrak{g}_{(m,n)} = f(m+n-1) &- \sum_{\substack{d>1 \\ d|(m,n)}} \frac{1}{d} \operatorname{sdim} \mathfrak{g}_{(\frac{m}{d}, \frac{n}{d})} \\ &+ \frac{1}{n} \sum_{(m,n)=(k,l)+(s,t)} l \Big(\sum_{d|(k,l)} \frac{1}{d} \operatorname{sdim} \mathfrak{g}_{(\frac{k}{d}, \frac{l}{d})} \Big) f(s+t-1). \end{aligned}$$

In Table 5.9 and Table 5.10, we present the root multiplicity tables for the Monstrous Lie superalgebras associated with the Thompson series T_{2A} and T_{2B}.

Root multiplicity tables of $\mathfrak{g}(A, \underline{m}, I^{\text{odd}})$ in Example 5.2

(Here each entries in the tables represent sdim$\mathfrak{g}_{m\alpha_0+n\alpha_1}$.)

Table 5.1 : $r = 1, a = 2, c > 0$

$m\backslash n$	1	2	3	4	5	6	7	8	9	10
1	-1	1	-1	1	-1	1	-1	1	-1	1
2	-1	2	-2	2	-3	4	-4	4	-5	6
3	0	1	-2	4	-6	8	-11	14	-17	21
4	0	1	-2	4	-9	16	-23	32	-46	63
5	0	0	-1	4	-10	21	-38	63	-98	145
6	0	0	0	2	-9	24	-51	96	-172	288
7	0	0	0	1	-6	21	-56	127	-256	474
8	0	0	0	0	-3	16	-51	136	-323	681
9	0	0	0	0	-1	8	-38	127	-348	835
10	0	0	0	0	0	4	-23	96	-323	900

Table 5.2 : $r = 1, a = 3, c > 0$

$m\backslash n$	1	2	3	4	5	6	7	8	9	10
1	-1	1	-1	1	-1	1	-1	1	-1	1
2	-1	2	-2	2	-3	4	-4	4	-5	6
3	-1	2	-3	5	-7	9	-12	15	-18	22
4	0	2	-4	7	-13	21	-29	39	-54	72
5	0	1	-4	10	-20	36	-59	91	-134	190
6	0	1	-3	10	-27	57	-104	176	-288	449
7	0	0	-2	10	-31	76	-161	309	-550	924
8	0	0	-1	7	-31	93	-222	474	-939	1727
9	0	0	0	5	-27	96	-274	666	-1449	2905
10	0	0	0	2	-20	93	-304	836	-2039	4490

Table 5.3 : $r = 2, a = 1, c > 0$

$m\backslash n$	1	2	3	4	5	6	7	8	9	10
1	-2	4	-8	16	-32	64	-128	256	-512	1024
2	0	3	-8	22	-64	164	-384	888	-2048	4624
3	0	0	-2	16	-64	212	-640	1792	-4776	12288
4	0	0	0	3	-32	164	-640	2228	-7168	21536
5	0	0	0	0	-6	64	-384	1792	-7168	25804
6	0	0	0	0	0	11	-128	888	-4776	21536
7	0	0	0	0	0	0	-18	256	-2048	12288
8	0	0	0	0	0	0	0	30	-512	4624
9	0	0	0	0	0	0	0	0	-56	1024
10	0	0	0	0	0	0	0	0	0	105

Table 5.4 : $r = 2, a = 2, c > 0$

$m\backslash n$	1	2	3	4	5	6	7	8	9	10
1	-2	4	-8	16	-32	64	-128	256	-512	1024
2	-2	7	-16	38	-96	228	-512	1144	-2560	5648
3	0	4	-18	64	-192	532	-1408	3584	-8872	21504
4	0	3	-16	73	-288	968	-2944	8492	-23552	63024
5	0	0	-8	64	-326	1344	-4864	16128	-50176	148684
6	0	0	-2	38	-288	1511	-6528	25056	-88400	291936
7	0	0	0	16	-192	1344	-7186	32512	-131072	485376
8	0	0	0	3	-96	968	-6528	35386	-165376	692880
9	0	0	0	0	-32	532	-4864	32512	-178568	855040
10	0	0	0	0	-6	228	-2944	25056	-165376	916949

Table 5.5 : $r = 2, a = 3, c > 0$

$m\backslash n$	1	2	3	4	5	6	7	8	9	10
1	-2	4	-8	16	-32	64	-128	256	-512	1024
2	-2	7	-16	38	-96	228	-512	1144	-2560	5648
3	-2	8	-26	80	-224	596	-1536	3840	-9384	22528
4	0	7	-32	121	-416	1288	-3712	10284	-27648	72240
5	0	4	-32	160	-646	2304	-7552	23296	-68608	194764
6	0	3	-26	172	-864	3595	-13312	45656	-147792	456304
7	0	0	-16	160	-992	4864	-20626	79104	-281600	946176
8	0	0	-8	121	-992	5840	-28416	122274	-480768	1762000
9	0	0	-2	80	-864	6184	-35072	170496	-742448	2974720
10	0	0	0	38	-646	5840	-38912	215216	-1043968	4588149

Weight multiplicity tables of $\mathfrak{g}(A, \underline{m}, I^{\mathrm{odd}})$ in Example 5.3
(Here each entries in the tables represent $W(m, n)$.)

Table 5.6 : $r = 1, a = 2, c > 0, \Lambda(h_0) = 1, \Lambda(h_1) = 0$

$m\backslash n$	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	1	-1	1	-1	1	-1	1	-1	1	-1	1
2	0	-1	2	-3	4	-5	6	-7	8	-9	10
3	0	0	2	-5	9	-14	20	-27	35	-44	54
4	0	0	1	-5	13	-26	45	-71	105	-148	201
5	0	0	0	-3	13	-35	75	-140	238	-378	570
6	0	0	0	-1	9	-35	96	-216	427	-770	1296
7	0	0	0	0	4	-26	96	-267	623	-1288	2436
8	0	0	0	0	1	-14	75	-267	750	-1800	3858
9	0	0	0	0	0	-5	45	-216	750	-2123	5211
10	0	0	0	0	0	-1	20	-140	623	-2123	6046

Table 5.7 : $r = 1, a = 2, c > 0, \Lambda(h_0) = 0, \Lambda(h_1) = 1$

$m\backslash n$	0	1	2	3	4	5	6	7	8	9	10
0	1	-1	1	-1	1	-1	1	-1	1	-1	1
1	0	-1	2	-3	4	-5	6	-7	8	-9	10
2	0	-1	3	-6	10	-15	21	-28	36	-45	55
3	0	0	2	-7	16	-30	50	-77	112	-156	210
4	0	0	1	-6	19	-45	90	-161	266	-414	615
5	0	0	0	-3	16	-51	126	-266	504	-882	1452
6	0	0	0	-1	10	-45	141	-357	784	-1554	2850
7	0	0	0	0	4	-30	126	-393	1016	-2304	4740
8	0	0	0	0	1	-15	90	-357	1107	-2907	6765
9	0	0	0	0	0	-5	50	-266	1016	-3139	8350
10	0	0	0	0	0	-1	21	-161	784	-2907	8953

Table 5.8 : $r = 1, a = 2, c > 0, \Lambda(h_0) = 1, \Lambda(h_1) = 1$

$m\backslash n$	0	1	2	3	4	5	6	7	8	9	10
0	1	-1	1	-1	1	-1	1	-1	1	-1	1
1	1	-2	3	-4	5	-6	7	-8	9	-10	11
2	0	-2	5	-9	14	-20	27	-35	44	-54	65
3	0	-1	5	-13	26	-45	71	-105	148	-201	265
4	0	0	3	-13	35	-75	140	-238	378	-570	825
5	0	0	1	-9	35	-96	216	-427	770	-1296	2067
6	0	0	0	-4	26	-96	267	-623	1288	-2436	4302
7	0	0	0	-1	14	-75	267	-750	1800	-3858	7590
8	0	0	0	0	5	-45	216	-750	2123	-5211	11505
9	0	0	0	0	1	-20	140	-623	2123	-6046	15115
10	0	0	0	0	0	-6	71	-427	1800	-6046	17303

Monstrous Lie superalgebra associated with T_{2A}

(Here each entries represent sdim$\mathfrak{g}_{(m,n)}$)

Table 5.9

$m\backslash n$	1	2	3
1	4372	96256	1240002
2	96256	10795008	431529984
3	1240002	431529984	42616961892
4	10698752	10128277504	2125795885056
5	74428120	166564106240	68134255043715
6	431529984	2126227415040	1588198806411264
7	2206741887	22327393665024	29030493318777216
8	10117578752	200750502117376	437155796944945152
9	42616961892	1588198806411264	5614282459787463036
10	166564106240	11283779936849920	63071424165763399680

$m\backslash n$	4	5
1	10698752	74428120
2	10128277504	166564106240
3	2125795885056	68134255043715
4	200750502117376	11283613372743680
5	11283613372743680	1040545340935546700
6	437157922740830208	63071424165763399680
7	12748902531008430080	2776078010473426349320
8	296560986580914798592	94724982482640008642560
9	5729955600122990051328	2622912698569732740150840
10	94724993766253381386240	60941641644938584748902400

$m\backslash n$	6	7
1	431529984	2206741887
2	2126227415040	22327393665024
3	1588198806411264	29030493318777216
4	437157922740830208	12748902531008430080
5	63071424165763399680	2776078010473426349320
6	5729957188321796462592	365905674143822100234240
7	365905674143822100234240	32901400981362568466623924
8	17653726811858612022411264	2182844606054987051800985600
9	676781767952182209443807232	112867667380626713409166646166
10	21390391153642343366410813440	4734083066463091210586418544640

$m\backslash n$	8
1	10117578752
2	200750502117376
3	437155796944945152
4	296560986580914798592
5	94724982482640008642560
6	17653726811858612022411264
7	2182844606054987051800985600
8	194804003233778348867415179264
9	13307182910011920334374580912128
10	726344627267043576138497719009280

$m\backslash n$	9
1	42616961892
2	1588198806411264
3	5614282459787463036
4	5729955600122990051328
5	2622912698569732740150840
6	676781767952182209443807232
7	112867667380626713409166646166
8	13307182910011920334374580912128
9	1180772370567563904378531394829748
10	82539410782781631041963275207495680

$m\backslash n$	10
1	166564106240
2	11283779936849920
3	630714241657633996680
4	94724993766253381386240
5	60941641644938584748902400
6	21390391153642343366410813440
7	4734083066463091210586418544640
8	726344627267043576138497719009280
9	82539410782781631041963275207495680
10	7290799226637980819592664955451443200

Monstrous Lie superalgebra associated with T_{2B}

(Here each entries in the tables represent sdim$\mathfrak{g}_{(m,n)}$)

Table 5.10

$m\backslash n$	1	2	3	4
1	276	-2048	11202	-49152
2	-2048	49152	-614400	5373952
3	11202	-614400	14478180	-216072192
4	-49152	5373952	-216072192	5061476352
5	184024	-37122048	2390434947	-83300614144
6	-614400	216072192	-21301241856	1063005978624
7	1881471	-1102430208	160791890304	-11164248047616
8	-5373952	5061476352	-1063005978624	100372723007488
9	14478180	-21301241856	6300794030460	-794110053826560
10	-37122048	83300614144	-34065932304384	5641848336678912

$m\backslash n$	5	6
1	184024	-614400
2	-37122048	216072192
3	2390434947	-21301241856
4	-83300614144	1063005978624
5	1945403602764	-34065932304384
6	-34065932304384	794110053826560
7	478625723149576	-14515166263443456
8	-5641848336678912	218578429975461888
9	57567784186189368	-2807138079496716288
10	-520271697765971968	31535729115847852032

$m\backslash n$	7	8
1	1881471	-5373952
2	-1102430208	5061476352
3	160791890304	-1063005978624
4	-11164248047616	100372723007488
5	478625723149576	-5641848336678912
6	-14515166263443456	218578429975461888
7	337945040343588276	-6374456847628238848
8	-6374456847628238848	148280443106626633728
9	101150679669913197462	-2864978197116521938944
10	-1388038765923851599872	47362494062244172660736

$m\backslash n$	9	10
1	14478180	-37122048
2	-21301241856	83300614144
3	6300794030460	-34065932304384
4	-794110053826560	5641848336678912
5	57567784186189368	-520271697765971968
6	-2807138079496716288	31535729115847852032
7	101150679669913197462	-1388038765923851599872
8	-2864978197116521938944	47362494062244172660736
9	66600077798590855556532	-1311456320500974276980736
10	-1311456320500974276980736	30470821082605141257437 1

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