M.S. THESIS

CONSENSUS OF LINEAR MULTI-AGENT SYSTEMS WITH MULTILAYER NETWORK USING OUTPUT FEEDBACK CONTROLLER

다층구조에서 출력 되먹임 제어기를 이용한 선형 다개체 시스템의 상태일치

BY

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COMPUTER ENGINEERING COLLEGE OF
ENGINEERING SEOUL NATIONAL UNIVERSITY

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이 논문을 공학석사 학위논문으로 제출함

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Abstract

Consensus of Linear Multi-agent Systems with Multilayer Network Using Output Feedback Controller

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Recently, there is a need for consensus of Multi-agent systems (MAS) in various engineering fields such as unmanned vehicles, cluster robot control, and sensor network. This paper is inspired by the disconnected autonomous vehicle and Connected Car. Connected cars not only get information through sensors like ordinary cars, but also receives various information of roads and traffic situation by wireless communication between vehicles or infra to increase safety of unmanned autonomous driving. However, as a common sensors in typical automobiles have limitations in physical situations such as a limitation of a measurement distance, it is common that not all agents can get information even though all the agents have same sensors. If information between vehicles are not obtained, it may cause great danger during autonomous navigation. Therefore, when the sensor can not be measured, information should be exchanged through wireless communication to realize stable unmanned autonomous driving. Therefore, it is necessary to study the control of MAS consensus considering these realistic problem situation.

Many recent studies on consensus of MAS solve the problem assume all sensors or agents have a single communication graph without considering the characteristics of sensors, and it does not take into account realistic limitations. A communication graph is a structure of connectivity between agents. In order to solve the consensus problem in reality, it is necessary to consider the structure and connectivity of the communication graph according to the type of sensors. Therefore, in this thesis, we present how to set up situations that can occur when controlling MAS to achieve consensus by using a multilayer network. By using structure of the multilayer network, it is possible to separate and analyze several graphs having different structures. In this thesis, we classify sensors as sensors that measure relative information and sensors that provide absolute information of agents through wireless communication such as GPS. We refer to them as relative sensors and absolute sensors. Various types of relative sensors have different physical limitations and therefore have different communication graphs depending on the types of sensors. Also, due to these physical limitations, the connectivity of the communication graphs can not be guaranteed either. Therefore, we introduced the concept of multilayer network to analyze several unconnected graphs and showed consensus of MAS by designing a dynamic controller.

In this thesis, general MAS are analyzed using Multilayer network, and consensus problem is solved mathematically by presenting a dynamic controller. The contribution of this thesis can be summarized as follows.

- Typical results of consensus mainly considered one communication graph. However, this paper considers multiple communication graphs that are not guaranteed to be connected.

- In MAS, only two layers are considered in the result of using structure

of multilayer network. In this thesis, under suggested assumption, the number of layers is not limited.

- Dynamic controller using output information is presented for consensus.

Through these theoretical results, we expect to provide theoretical results solve consensus problem more efficiently.

Keywords: multi-agent systems, consensus, multilayer network, output feedback control

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Symbols and Acronyms

Symbols

$\mathbb{R}$	field of real numbers
$\mathbb{R}^n$	Real Euclidean space of dimension n
$\mathbb{R}^{m\times n}$	space of $m \times n$ matrices with real entries
$\mathbb{C}$	field of complex numbers
$\mathbb{C}^{m\times n}$	space of $m \times n$ matrices with complex entries
$\mathbb{C}_{\geq 0}$	closed right-half complex plane
I_n	$n \times n$ identity matrix
A^T	transpose of the matrix A
A^{-1}	inverse of the square matrix A
$A \otimes B$	Kronecker product of matrices A and B
$\|x\|$	Euclidean norm of the vector x
$\|A\|$	induced norm of the matrix A
$\mathrm{col}(x_1, ..., x_k)$	stacking of vectors $x_i \in \mathbb{R}^{n_i}$, $i = 1, 2, ..., k$
A_i	ith row of the matrix A
$[A_1; A_2]$	stacking of row vectors of A_1 and A_2; i.e., $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$
$[A; B]$	stacking of matrices $A \in \mathbb{R}^{n_1\times n_2}$ and $B \in \mathbb{R}^{n_3\times n_2}$

	where $n_i \in \mathbb{R}$
	for all $i = 1, 2, 3$; i.e., $[A_1; ...; A_{n_2}; B_1; ...; B_{n_3}]$
$\lambda(A)$	set of eigenvalues of $A \in \mathbb{R}^{N \times N}$;
	i.e., $\left\{\lambda_i(A) \in \mathbb{C} : i = 1, .., n\right\}$
$\mathrm{Re}(\lambda)$	real part of the complex number λ
$\lambda_i(A)$	ith eigenvalue of $A \in \mathbb{R}^{n \times n}$
$\lambda_{\max}(A)$	largest eigenvalue of positive matrix $A \in \mathbb{R}^{n \times n}$
$\lambda_{\min}(A)$	smallest eigenvalue of positive matrix $A \in \mathbb{R}^{n \times n}$
$\mathrm{diag}(a_1, ..., a_n)$	$n \times n$ diagonal matrix with constants a_1 to a_n
$:=$	defined as
$\forall$	for all
$\square$	designation of the end of theorem, lemma, proof, assumption and the definition

Acronyms

MAS	multi-agent systems

- A square matrix A is said to be Hurwitz matrix if every eigenvalue λ of A has strictly negative real parts, i.e., $\mathrm{Re}(\lambda) < 0$.
- For any state variable $x(t)$, the time symbol t is omitted when there is no confusion.

Chapter 1

Introduction

1.1 Research Background

Given multi-agent systems (MAS), the problems of reaching consensus are to get close to each other among the individuals. They are interconnected with and communicate some information each other in their neighborhood by network topology. Why does the consensus problem among multi-agents attracted interests of researches in recent years? Because it has been demonstrated that the consensus problem has a lot of applications in various areas such as cooperative control of unmanned aerial vehicles or robots, communication among sensor networks, electrical power grid which engineers are interested in, and Flocking of birds and swarming of bees which biologists are interested in. That is, it has been one of major issues of not only control engineering but also other fields. Considering engineering area, there are applications of the problem include cooperative control of vehicles in wireless seonsor networks for distributed consensus [Li15, Olf07a, Olf07b, Ren05b], platooning of vehicles in transportation

systems [Di15, Fus16, Na10, Wa14] and tracking control [Ho06, Ni10, Ren08].

As research on consensus has matured, more complicated and realistic scenarios must be considered and analyzed to address real-world application. Often, the agents in a network may also communicate with different protocols and strategies. Some recent work addresses the problem of achieving consensus in networks with some degree of heterogeneity. The case of heterogeneous networks has been studied, e.g., switching topology [Li10, Ren05c], communication delay [Lee14, Sun09], and both were [Liu11, Mu11, Olf04, Zh09].

From the above research, it is observed that consensus studies focus on either system or network complexity. However, it rarely deals with the situation that overall multi-agent systems has several communication networks, more specifically, when structure of the graph depends on the types of sensors of agents.

In order to analyze the network complexity, the network of the whole system is expressed using properties of multilayer network to be introduced later. There are results which introduces the multilayer network for the platoon formation of vehicles [Fus16, Wa14]. In [Fus16], the states and estimated states of the agents are layered, and each layer is assumed to be connected. In the case of [Wa14], the sensors are connected to each other and the communication link is added incidentally to increase the accuracy for consensus. In [Bur15], it is shown that the PI controller is divided into two layers and the consensus is solved by applying P and I controller to each layer. In [Sor12], each layer showed consensus for nonlinear systems with different connectivity. However, all of the above results suggest only a static controller using states. In [Tun16], They did not show how to control the system, but they have discussed the relationship between observability and connectivity of the system in cases where each graph using output relative information has different connectivity.

The solution to solve the consensus problem in a single layer is also well known as shown above. However, There are not many results for the multilayer network. For multilayer network, either the connectivity is always the same for each layer, or the networks of layers are connected if they are different. Also, there is almost no result of designing a dynamic controller or controlling the system using output information. Thus, the aim of this thesis is to study strategies based on multilayer network, to solve distributed consensus problems. Hence, we will show to solve the consensus problem in heterogeneous networks by using the low gain dynamic controller in [Seo09].

1.2 Outline and Contributions

The following outlines and summarizes the contributions of each chapter.

Chapter 2. Preliminaries and Background

As a preliminary of this thesis, we review graph theory and provide some useful properties of consensus problems. Also, the some basic definitions are introduced for problem setting. Parts of this chapter is to graph theory which are based on [Bol13, Die06, GR01, Gro04, Ren05a] and related to multilayer networks are based on [ABR07, OSM04]. The Contributions are as follows.

- We review basic definitions of graph theory for consensus problems. In addition, we provide some properties including useful coordinate transformation.
- We introduce a notion and some definitions of multilayer network required in this thesis.

Chapter 3. Problem Formulation

We introduce homogeneous Linear dynamic systems and suggest some condi-

tions to solve consensus problems. Then, the dynamic controller is introduced.

Chapter 4. Double Layers Network

This chapter covers the algorithm of consensus using the dynamic controller in Chapter 3 with double layers network. Specifically, we construct layers where connectivity are not guaranteed based on sensors that obtain relative and absolute output information, and solve the consensus problem through the concept of projection layer.

Chapter 5. General Case for Multilayer Network

By introducing some conditions, we introduce the general algorithm of consensus with multilayer network. The following is a list of the contributions of Chapter 5.

- In order to deal with the practical network system, we have set up a problem by introducing multilayer network. Each layer is configured to contain a physical meaning based on types of sensors.
- We also show that the proposed controller can solve the consensus problem for systems considering two or more layers under some conditions. From these results, we show that consensus is achieved for layers that are not guaranteed to be connected.

Chapter 6. Conclusion

We conclude the thesis and suggest future directions of research.

Chapter 2

Preliminaries and Background

This chapter deals with the fundamental knowledge for understanding the thesis. First, in order to solve the consensus problems, we need to consider the communication topology defines that individual dynamical systems interconnected. A link between the individual systems determines whether a system can send or receive the information through the link. Thus, we know how to model the topological structure of the network in characterizing the consensus problems. Graph theory is one of the useful tool for modeling the communication topology in a network of individual systems [Bol13, Die06, GR01, Gro04, Mer95, New00, Ren05a]. It has been proved that the graphy theory is indeed a useful for the consensus problems [FM04, JLM03, Kim12, Kim13, Kim14, NWC10, Tun08a, Tun08b]. In the subsection 2.1 and 2.2, we introduce the basics of graph theory and some useful properties for the study the consensus problems, which serve as the preliminaries of the result.

The graph theory is applied in a general situation, single layer. However, second, to understand this thesis, we introduce some definitions of graph theory

with multilayer network [BBC14, DSC13, KAB14] in the subsection 2.3. In order to analyze the network systems, recently, the concept of multilayer network has been introduced to solve the problem of consensus [Sor12, Wa14].

2.1 Definitions of Graph Theory for Consensus Problems

We summarize the basic definitions from graph theory for studying consensus problems in this section. Among all possible classes of graphs, we introduce the most general one, i.e., the class of weighted directed graphs. A weighted directed graph consists of nodes, edges connecting the nodes, and weights assigned to their corresponding edges. It contains fixed (or time-invariant), undirected graphs which we will consider throughout the thesis.

Definition 2.1.1. (Weighted directed graph). A time-varying weighted directed graph, denoted by $\mathcal{G}(t)$, is 3-tuple $\mathcal{G}(t) = (\mathcal{N}, \mathcal{E}(t), \mathcal{A}(t))$ of node set $\mathcal{N} = 1, 2, ..., N$, edge set $\mathcal{E}(t) \subseteq \mathcal{N} \times \mathcal{N}$, and weighted adjacency matrix $\mathcal{A}(t) = [\alpha_{ij}(t)] \in \mathbb{R}^{n \times n}$ satisfying the following properties.

(a) There are no self-loops, i.e., $(i, j) \notin \mathcal{E}(t)$ and $\alpha_{ii} = 0$ for all $i \in \mathcal{N}$ and $t \geq 0$.
(b) Each element $\alpha_{ij}(t)$ of the weighted adjacency matrix $\mathcal{A}(t)$ is a nonnegative, bounded, and piecewise continuous function of time t.
(c) Each weight $\alpha_{ij}(t)$ is positive at time t if $(j, i) \in \mathcal{E}(t)$ at the time t. Otherwise, $\alpha_{ij}(t) = 0$. □

In Definition 2.1.1, the nodes $i \in \mathcal{N}$ represent the individual systems and the edges $(i, j \in \mathcal{E})(t)$ are modeled on the interconnections between the individual systems. An edge (i, j) is represented by an arrow oriented toward the node j and tailed at the node i. The special classes of graphs can be derived

Figure 2.1 : Fixed weighted directed graph with five nodes.

from Definition 2.1.1. A graph is said to be fixed (or time-invariant) if it is independent on time t. In this case, it is simply denoted by $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$. A graph $\mathcal{G}(t)$ is unweighted if $\alpha_{ij}(t) \in \{0, 1\}$ for all $i, j \in \mathcal{N}$ and for $t \geq 0$. Thus, one can simply write $\mathcal{G}(t) = \{\mathcal{N}, \mathcal{E}(t)\}$. A balance graph is the graph such that $\sum_{j=1}^{N} \alpha_{ji}(t) = \sum_{j=1}^{N} \alpha_{ij}(t)$ for $i \in \mathcal{N}$ and for $t \geq 0$. An undirected graph is always balanced since the adjacency matrix of the undirected graph is symmetric, i.e., $\mathcal{A}^T(t) = \mathcal{A}(t)$. Throughout this thesis, we consider the fixed, weighted, and undirected graph that is denoted by $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, \mathcal{A}\}$. An example of a fixed weighted directed graph with five nodes can be expressed by a simple figure in Figure 2.1.

2.1.1 Graph Connectedness

In order to reach consensus among the individual systems, it is necessary that certain information propagates through the network of all systems and reaches all the systems. This process represents the communication topology. To share the common information, the graph needs to be connected. Thus, we introduce some definitions concerning the connectivity.

Definition 2.1.2. (Neighbors). Let $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, \mathcal{A}\}$ be a fixed weighted undi-

rected graph. For given $i \in \mathcal{N}$, a node $j \in \mathcal{N}$ is called neighbor of the node i if $(j, i) \in \mathcal{E}$. The neighbors of the node i is the set that contains every neighbor of the node i, and denoted by $\mathcal{N}^{(i)}$; i.e., $\mathcal{N}^{(i)} := \{j \in \mathcal{N} : (j, i) \in \mathcal{E}\}$. □

Definition 2.1.3. (Path). Let $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, \mathcal{A}\}$ be a fixed weighted undirected graph. For a given $i, j \in \mathcal{N}$, a path of length l from the node i to the node j is a sequence of nodes of the form $\{i_0, i_1, ..., i_l\}$ such that $i_0 = i, i_l = j, i_k \in \mathcal{N}_{i_{k+1}}$ for $k = 0, ..., l-1$, and i_k's are distinct for all k. □

The existence of a path from the node $i \in \mathcal{N}$ to the node $j \in \mathcal{N}$ in the graph $\mathcal{G}$ implies that the information propagates from the system represented by the node i to the system represented by the node j. Relying on the definition of a path, graph connectedness is now defined as follows [Bol13, Die06, GR01].

Definition 2.1.4. (Connected graph). A graph $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, \mathcal{A}\}$ is said to be connected if there is a path between any two nodes of the graph $\mathcal{G}$, otherwise disconnected. □

For a directed graph, the connected graph in Definition 2.1.4 is also called quasi strongly connected graph. The directed graph is called the connected if it has a directed spanning tree. However, since we consider only undirected graph, we do not explain about it more. To understand the connectivity, we notice that Figure 2.1 (a) is connected but Figure 2.2 (b) is not.

2.1.2 Laplacian Matrix and Its Properties

In the field of consensus, the each system is coupled with the others. Therefore we express these couplings as the graph theory. From this subsection, we only consider the fixed weighted undirected graph $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, \mathcal{A}\}$. Then, the

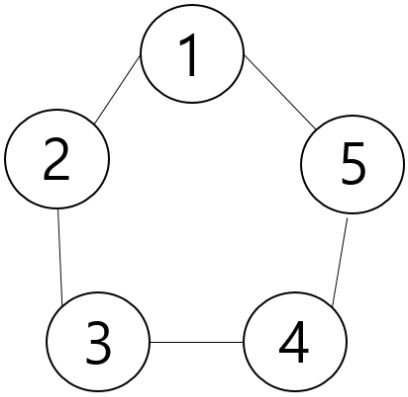

Figure 2.2 : Schematic illustration of the undirected graph.

consensus algorithm [ABR07, OSM04] can be expressed as

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} (x_j(t) - x_i(t)), \ i \in \mathcal{N}, x_i \in \mathbb{R}$$

where x_i is the state of ith agent. The collective dynamics of the group of agents is written as

$$\dot{x}(t) = -\mathcal{L}x(t) \tag{2.1.1}$$

where $x(t)$:=col$(x_1(t), ..., x_N(t))$ and $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$ is a matrix, called Laplacian matrix.

Definition 2.1.5. (Laplacain matrix). Given a graph $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, \mathcal{A}\}$, the matrix

$$\mathcal{L} := \mathcal{D} - \mathcal{A}$$

is called the Laplacian matrix of the graph $\mathcal{G}$, where $\mathcal{D}$:=diag$(\mathcal{A}1_N)$. 1_N is the $N \times 1$ column vector comprising all ones. □

Since we consider the undirected graph, The Laplacian matrix $\mathcal{L}$ is symmetric. The Laplacain matrix can be defined by element-wise such as

$$l_{ij} := \begin{cases} \sum_{k=1}^{N} \alpha_{ik}, & j = i, \\ -\alpha_{ji}, & j \neq i. \end{cases}$$

Since the Laplacian matrix $\mathcal{L}$ can uniquely determines the adjacency matrix $\mathcal{A}$, which characterize the graph $\mathcal{G}$, and therefore characterizes the communication topology. Hence, in order to understand the behavior of interconnection among the systems, we focus the Laplacian matrix $\mathcal{L}$.

Example 1. Consider the graph depicted in Figure 2.2. Then using definitions 2.1.2, 2.1.3 and 2.1.4 we obtain the adjacency, degree and Laplacian matrices, which are given by

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \mathcal{D} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \mathcal{L} = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 2 & 1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}.$$

2.2 Useful Properties for Consensus

This section contains some useful properties and well-known results for solving the consensus problems.

Lemma 2.2.1. Let $v \in \mathbb{R}^n$ where $v = [v_1, ..., v_n]^T$. Let $\mathcal{L} \in \mathbb{R}^{n \times n}$ denote the Laplacian matrix of graph $\mathcal{G}$ and $\mathcal{E}$ denote the set of edges in $\mathcal{G}$. Then $x^T \mathcal{L} v = \sum_{(i,j) \in \mathcal{E}} (v_j - v_i)^2$. □

proof. By using Definition 2.1.5, we have,

$$v^T \mathcal{L} v = v^T (\mathcal{D} - \mathcal{A}) v = v^T \mathcal{D} v - v^T \mathcal{A} v.$$

Expanding this out, we get

$$\sum_i \mathcal{D}_{ii} v_i^2 - [v_1, ..., v_n](\alpha_{ij})[v_1, ..., v_n]^T$$

where the ith diagonal element of $\mathcal{D}$ is $\mathcal{D}_{ii}$ and the ith row and jth column element of adjacency matrix $\mathcal{A}$ is represented as α_{ij}. Note that $\alpha_{ij} = 1$ if there is an edge between vertex i and j and $\alpha = 0$, otherwise. Thus, using the property of adjacency matrix, we obtain that

$$[v_1, ..., v_n](\alpha_{ij})[v_1, ..., v_n]^T = \sum_{(i,j) \in \mathcal{E}} \alpha_{ij} v_i v_j = \sum_{(i,j) \in \mathcal{E}} 2 v_i v_j.$$

Hence, by above property, we finally have that

$$\begin{aligned} v^T \mathcal{L} v &= \sum_i \mathcal{D}_{ii} v_i^2 - \sum_{(i,j) \in \mathcal{E}} 2 v_i v_j = \sum_{(i,j) \in \mathcal{E}} (v_i^2 - 2 v_i v_j + v_j^2) \\ &= \sum_{(i,j) \in \mathcal{E}} (v_j - v_i)^2. \end{aligned}$$

□

Theorem 2.2.1. Let $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, \mathcal{A}\}$ be a graph and $\mathcal{L}$ be Laplacian matrix of $\mathcal{G}$, respectively. Then, $\lambda_2(\mathcal{L}) > 0$ if and only if $\mathcal{G}$ is connected. □

proof. ($\Rightarrow$) First, we show that $\lambda_2 = 0$ if $\mathcal{G}$ is disconnected. If $\mathcal{G}$ is disconnected, $\mathcal{G}$ is consist of the disjoint of graphs, $\mathcal{G}_1$ and $\mathcal{G}_2$. Then, the Laplacian matrix $\mathcal{L}$ of $\mathcal{G}$ denotes as

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_{\mathcal{G}_1} & 0 \\ 0 & \mathcal{L}_{\mathcal{G}_1} \end{bmatrix}.$$

where $\mathcal{L}_{\mathcal{G}_1}$ and $\mathcal{L}_{\mathcal{G}_2}$ are denoted as the Laplacian matrix of the graph $\mathcal{G}_1$ and $\mathcal{G}_1$, respectively. By using the property which the Laplacian matrix $\mathcal{L}$ always contains a zero eigenvalue with its corresponding eigenvector 1_N, we obtain

$$\begin{bmatrix} \mathcal{L}_{\mathcal{G}_1} & 0 \\ 0 & \mathcal{L}_{\mathcal{G}_1} \end{bmatrix} \begin{bmatrix} 1_{N_1} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} \mathcal{L}_{\mathcal{G}_1} & 0 \\ 0 & \mathcal{L}_{\mathcal{G}_1} \end{bmatrix} \begin{bmatrix} 0 \\ 1_{N_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where N_1 :=dim($\mathcal{L}_{\mathcal{G}_1}$) and N_2 :=dim($\mathcal{L}_{\mathcal{G}_2}$). There are eigenvectors, col$(1_{N_1}, 0)$ and col$(0, 1_{N_2})$, if $\mathcal{L}$ with its corresponding eigenvalue zero. Thus, it means that at least two eigenvalues are zero.

($\Leftarrow$) If the graph is connected, there exists the eigenvector $v \in \mathbb{R}^N$ such as $\mathcal{L}v = 0$. Then, by Lemma 2.2.1, we have

$$v^T \mathcal{L} v = \sum_{(i,j)\in\mathcal{E}} (v_j - v_i)^2 = 0$$

It is implies that $v_j = v_i$ for each $(i, j) \in \mathcal{E}$. Thus, the vector 1_N spans the kernel of the Laplacian matrix. Since the Laplacian matrix is symmetric, and diagonalizable, its algebraic and geometric multiplicity are same. □

Lemma 2.2.2. (Schur decomposition). For a matrix $A \in \mathbb{C}^{n\times n}$, there exists a unitary matrix U (i.e., U^{-1} is conjugate transpose U^*, $U^{-1} = U^*$) such that $A = UTU^{-1}$, where T is an upper triangular matrix. Since T is similar to A, its eigenvalues are same as A. It implies that those eigenvalues are the diagonal entries of T. □

Theorem 2.2.2. (Transformation of the Laplacian matrix). Let $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, \mathcal{A}\}$ be a connected graph and $\mathcal{L}$ be Laplacian matrix of $\mathcal{G}$. Then, there exists a nonsingular matrix U such that

$$W\mathcal{L}W^T = \begin{bmatrix} 0 & 0 \\ 0 & \Lambda \end{bmatrix},$$

where Λ =diag$(\lambda_2(\mathcal{L}), ..., \lambda_N(\mathcal{L}))$. The matrices U and U^{-1} are expressed as

$$W = \begin{bmatrix} \frac{1}{N} 1_N^T \\ R^T \end{bmatrix}, \text{ and } W^{-1} = \begin{bmatrix} 1_N^T & Q \end{bmatrix},$$

where R and Q are real matrices of size $N \times (N-1)$ such that $R^T R = \frac{1}{N} I_{N-1}$, $Q^T Q = N I_{N-1}$, $R^T 1_N = 0$, $Q^T 1_N = 0$, and $R^T Q = I_{N-1}$. □

proof. The Laplacian matrix $\mathcal{L}$ of the connected graph $\mathcal{G}$ is symmetric and has zero eigenvalue which is simple. By Lemma 2.2.1, the Laplacian matrix can be expressed as $\mathcal{L} = UTU^T$ where U is a unitary matrix and T is an upper triangualr matrix. Since the Laplacian matrix $\mathcal{L}$ is symmetric, one sees that $\mathcal{L} = \mathcal{L}^T$ implying that $UTU^T = \mathcal{L}^T = U^T T^T U$. Thus, $T = T^T$ means that T is diagonal matrix, T =diag$(0, \Lambda)$. By using the property of the Laplacian matrix that all rows of the Laplacian sum up to zero, we can express the first row of the orthogonal matrix U as $\frac{1}{\sqrt{N}} 1_N^T$. We define new transformation matrix $W := \frac{1}{\sqrt{N}} U$. Then,

$$W = \begin{bmatrix} \frac{1}{N} 1_N^T \\ R^T \end{bmatrix}, \text{ and } W^{-1} = \begin{bmatrix} 1_N^T & Q \end{bmatrix}, \tag{2.2.1}$$

where R and Q are real matrices of size $N \times (N-1)$ such that $R^T R = \frac{1}{N} I_{N-1}$, $Q^T Q = N I_{N-1}$, $R^T 1_N = 0$, $Q^T 1_N = 0$, and $R^T Q = I_{N-1}$. □

By the coordinate change of the Theorem 2.2.2 in (2.2.1)

$$\xi = \begin{bmatrix} \xi_1 \\ \tilde{\xi} \end{bmatrix} = Wx = \begin{bmatrix} \frac{1}{N} 1_N^T \\ R^T \end{bmatrix} x \tag{2.2.2}$$

where $\tilde{\xi} = [\xi_2, ..., \xi_N]^T$. The overall system (2.1.1) is transformed as

$$\begin{aligned} \xi_1 &= 0 \\ \tilde{\xi} &= -\Lambda \tilde{\xi} \end{aligned} \tag{2.2.3}$$

by using properties of Theorem 2.2.2 that $1_N\mathcal{L} = 0$ and $R^T\mathcal{L}Q = \Lambda$. The overall transformed system (2.2.3) implies that $\xi_1(t) = \dfrac{1}{N}\sum_{i=1}^{N} x_i(0)$ and $\lim_{t\to\infty}\tilde{\xi} = 0$.

2.3 Multilayer Network

In single layer network (general network), a graph is represented $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, \mathcal{A}\}$ as it is seen in Chapter 2. To represent network with multi layers or various types of edges, we consider structures that have layers, in addition to nodes and edges. These networks are called multilayer or multiplex networks. Throughout these thesis, we call these systems multilayer networks for no confusion.

Multilayer networks are a collection of networks called layers which may interact with each other, and have been proposed as an effective modeling approach for representing and investigating several problems in many real and man-made networks [BBC14, DSC13, KAB14]. We allow each node to belong to any subset of the layers, and we are able to consider edges that encompass pairwise connections between all possible combinations of nodes and layers. (One can further generalize this framework to consider hyperedges that connect more than two nodes.) That is, a node i in a layer α can be connected to any node j in any layer β.

Definition 2.3.1. (Graph of multi layers). A graph of layer, is a pair $\mathcal{M} = (\mathcal{G}, \mathcal{B})$ where $\mathcal{G}$ is the set of $\mathcal{M}$ graphs $\mathcal{G} := \{\mathcal{G}_1, ..., \mathcal{G}_M\}$ (called layers of $\mathcal{M}$), and $\mathcal{B} := \{\mathcal{G}_{\alpha\beta} : \alpha, \beta \in \{1, ..., M\}, \alpha \neq \beta\}$ is the set of edges representing interconnections between nodes of different layers. $\square$

We introduce a classical example in [Bur15] are transportation networks where two nodes (e.g cities) can be connected by rail and/or road and/or plane, as can be seen in Figure 2.3. In Figure 2.3, we compose the two layers on the

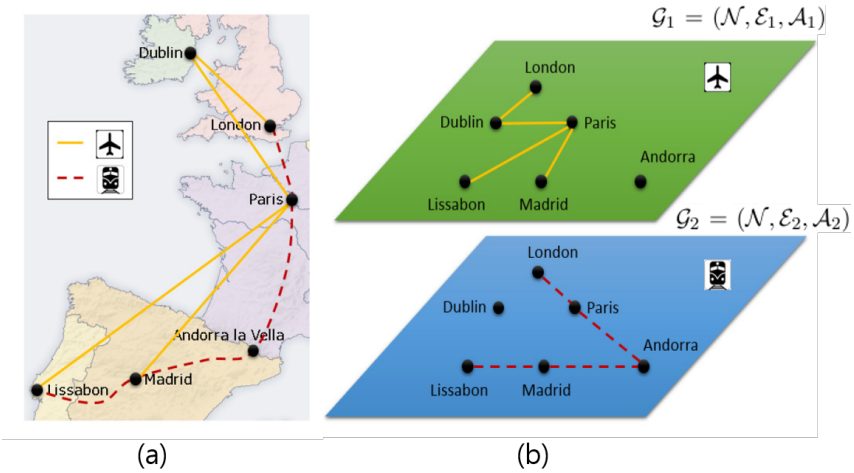

Figure 2.3 : (a) Network of cities interconnected by rail and plane. (b) Multilayer network, with two independent layers (no interconnections between layers, i.e., $\mathcal{B}$ is an empty set) ASM:connectivity of multi graphs the same set of vertexes.

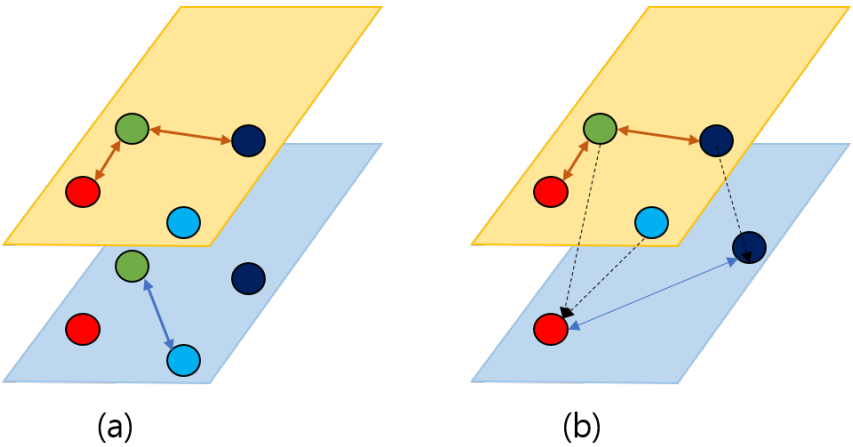

Figure 2.4 : (a) Each layer has same node set but not edge sets without the interconnection between nodes of different layers. (b) Each layer has different node and edge sets with the interconnection between nodes of different layers.

basis of methods of the transportation, the same node sets, cities. We consider the structures of multilayer networks in Figure 2.4. The interconnection between nodes of different layers can be seen in Figure 2.4(b). Hence, like Figure 2.3 and Figure 2.4(a), we consider that all layers include the same node set but not edge sets without the interconnection between nodes of different layers. We deal with that how to construct the multi layers in Chapter 3, more detail.

Definition 2.3.2. (The adjacency matrix of multilayer). The adjacency matrix of each layer G_m will be denoted by $\mathcal{A}_m = [\alpha_{ij}^m] \in \mathbb{R}^{N\times N}$, where

$$\alpha_{ij}^m = \begin{cases} 1, & \text{if } (i_k, i_j) \in \mathcal{E}_m \\ 0, & \text{otherwise,} \end{cases}$$

for $1 \leq j, k \leq N$ and $1 \leq m \leq M$. The interlayer adjacency matrix corresponding to $\mathcal{E}_{\alpha\beta}$ is the matrix $\mathcal{A}_{\alpha\beta} = [\alpha^{\alpha\beta}] \in \mathbb{R}^{N\times N}$ given by

$$\alpha_{ij}^{\alpha\beta} = \begin{cases} 1, & \text{if } (i_k, i_j) \in \mathcal{E}^{[\alpha\beta]} \\ 0, & \text{otherwise.} \end{cases}$$

□

Throughout these thesis, we consider the $\mathcal{B}$ is the empty set. That is, we do not consider the interconnection between nodes od different layers. Thus, we presents the graph of mth layer as $\mathcal{G}_m = (\mathcal{N}, \mathcal{E}_m, \mathcal{A}_m)$.

Definition 2.3.3. (Projection graph of multilayers [Cri12]). Given two graphs ASM:connectivity of multi graphs the same set of nodes, $\mathcal{G}_1 = (\mathcal{N}, \mathcal{E}_1, \mathcal{A}_1)$ and $\mathcal{G}_2 = (\mathcal{N}, \mathcal{E}_2, \mathcal{A}_2)$, the projection graph is defined as $(\mathcal{G}_p) := (\mathcal{N}, \mathcal{E}_p, \mathcal{A}_p)$ with associated adjacency matrix $\mathcal{A}_p := \mathcal{A}_1 \cup \mathcal{A}_2$ and the set of edges $\mathcal{E}_p := \mathcal{E}_1 \cup \mathcal{E}_2$.
□

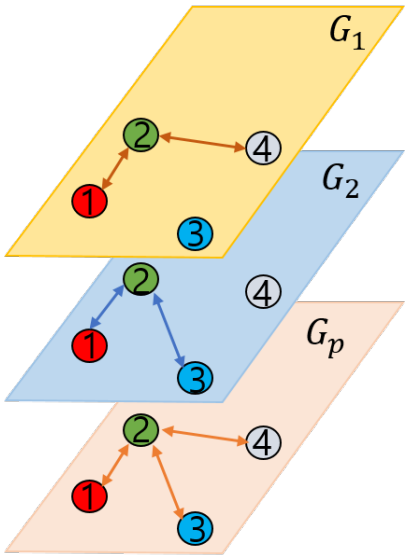

Figure 2.5 : There exists an common edge of $\mathcal{G}_1$ and $\mathcal{G}_2$. However, projection graph $\mathcal{G}_p$ is weighted graph.

Example 2. Consider the graph depicted in Figure 2.5. We obtain the adjacency matrices of graph $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_p$ which are given by

$$\mathcal{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \mathcal{A}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathcal{A}_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

There exists an common edge of $\mathcal{G}_1$ and $\mathcal{G}_2$; i.e., $(1,2) \in \mathcal{E}_1, \mathcal{E}_2$. If there exist common edges, the adjacency matrix of projection graph represented as $\mathcal{A}_p = \mathcal{A}_1 \cup \mathcal{A}_2$ can be written as $\mathcal{A}_p = \mathcal{A}_1 + \mathcal{A}_2$, algebraically. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ are Laplacain matrices of graph of $\mathcal{G}_1$ and $\mathcal{G}_2$. Then, Laplacian matrix $\mathcal{L}_p$ of projection graph can be expressed as $\mathcal{L}_p := \mathcal{L}_1 + \mathcal{L}_2$.

Chapter 3

Problem Formulation

In this chapter, we deal with the output consensus of homogeneous multi-agent systems with multilayer network. Since the consensus problem has a variety of application in many areas, for example, sensor and communication network, and unmanned aerial vehicles, the consensus problem among multi-agent systems have been heavily investigated.

$$\dot{x}_i = Ax_i + Bu_i$$
$$y_i = Cx_i$$

A ith agent dynamic system is represented as above. We consider the matrix C of the system as sensors. That is, the form of matrix C is related to the types of sensors. This is a generally conceivable assumption in reality. There are two main types of sensors; One type of sensors measure the absolute output value of agents, such as GPS. Another type of sensors measure the output information relative to other agents. That is, if jth agent is in the neighborhood of ith

agent such as $j \in \mathcal{N}_i$, these sensors measure $y_j - y_i$. If sensors that measure the absolute and relative output information are represented as $C_1 = [1\ 0]$ and $C_2 = [1\ 1]$, respectively, the ith output value of the sensors that measure absolute and the relative information are not the same as y_i because the sensor type is different. Thus, for consensus using output information, as mentioned earlier, it is necessary to distinguish whether it is an absolute output information or not.

To consider more practical consensus problem, solving the consensus problem of multi-agent systems is required to use multilayer property of network [BBC14, KAB14]. In particular, we take notice of the consensus problem concerning vehicles. The output information of a large number and kind of sensors is used for consensus. In most previous consensus results, each graph of sensor network is assumed to be identical. In [Fus16], two layers are considered in the vehicle consensus problem. However, the authors consider the multilayer network related to the state and the estimated state. However, each of these layers considers the multilayer system associated with the state and the estimated state rather than the output information. In [Tun14], the authors does not mention the network of multilayer, but introduces a similar concept to apply it, and presents a static controller using the output information.

Therefore, in this chapter, the concept of multilayer network is introduced to represent sensors measuring the absolute and relative output information. Even though each sensor is not connected to each other, under the assumption that the projection graph of the two types of sensors is connected, the state of system can be estimated and used to produce relative output value for output consensus. In addition, we show how to solve the output consensus problems by using the low gain controller presented in [Seo09].

3.1 Homogeneous Linear Dynamic Systems

In this section, we address the consensus problem of homogeneous MAS evolving in fixed communication networks. We consider N homogeneous linear dynamic systems given as

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Bu_i(t),\ i \in \mathcal{N} := \{1, 2, \cdots, N\} \\ \begin{bmatrix} y_{\Delta,i} \\ y_{+,i} \end{bmatrix} &= \begin{bmatrix} C_\Delta \\ C_+ \end{bmatrix} x_i(t) \\ = \begin{bmatrix} y^1_{\Delta,i} \\ \vdots \\ y^\gamma_{\Delta,i} \\ y_{+,i} \end{bmatrix} &:= \begin{bmatrix} C^1_\Delta \\ \vdots \\ C^\gamma_\Delta \\ C_+ \end{bmatrix} x_i(t) \end{aligned} \tag{3.1.1}$$

where N is the number of agents, $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_{\Delta,i} \in \mathbb{R}^\gamma$, and $y_{+,i} \in \mathbb{R}^p$ are the state, the input, and the relative and absolute output of the ith agent, respectively. Matrices A, B, C_Δ and C_+ are constant matrices.

Remark 3.1.1. Let sensors measuring the relative and the absolute output information be the *relative and the absolute sensors*, *respectively*. The matrices C_Δ and C_+ are associated with the relative and the absolute sensors. Absolute sensor information is exchanged through wireless communication network. Therefore, although absolute sensor types vary, the absolute sensor graph is the same as the graph of wireless communication network. However, since the relative sensor is directly measured, the graph differs depending on the types of sensors. So we consider γ graphs for the γ types of relative sensors. □

Definition 3.1.1. (Consensus). The consensus problem is solved if there exists a signal $\rho(t) \in R^n$ such that for any initial value $x_i(0)$, $\lim_{t\to\infty} |x_i(t) - \rho(t)| = 0, \forall i = 1, \cdots, N$. □

Definition 3.1.1 says that the consensus is reached asymptotically if the states of the N individual systems (3.1.1) are asymptotically close to each other.

Assumption 3.1.1. All the eigenvalues of the matrix A lie on the imaginary axis. $\square$

When the matrix A is Hurwitz, all state trajectories converge to zero asymptotically without input u_i. Thus, we are only interested in the matrix A whose eigenvalues lie on the imaginary axis.

Assumption 3.1.2. We assume that both (C_Δ, A, B) and (C_+, A, B) are stabilizable and detectable. $\square$

3.2 Graph Theoretical Conditions

In Chapter 2, we discussed a tool for modeling the communication topology is an undirected graph and define the communication topology with multilayer network. Now, we define the graph of sensors depending on kinds of sensors. Thus, the undirected graphs of the absolute sensors are denoted by $\mathcal{G}_+ := (\mathcal{N}, \mathcal{E}_+, \mathcal{A}_+)$, where $\mathcal{N} = \{1, 2, ..., N\}$ is a set of nodes, $\mathcal{E}_+ \subseteq \mathcal{N} \times \mathcal{N}$ is an edge set of pairs of nodes, and the adjacency matrix $\mathcal{A}_+ = [\alpha_{+,ij}] \in \mathbb{R}^{N\times N}$. $\alpha_{+,ij} = 1$ if $(i, j) \in \mathcal{E}_+$, otherwise $\alpha_{+,ij} = 0$. The Laplacian matrix of the relative layer $\mathcal{L}_+ = [l_{+,ij}] \in \mathbb{R}^{N\times N}$ of $\mathcal{G}_+$ is defined as $l_{+,ii} := \sum_{j\neq i} \alpha_{+,ij}$ and $l_{+,ij} := -\alpha_{+,ij}$ for all $i \neq j$.

Similar to above, the undirected graph of the relative sensors can be defined. However, considering that γ kinds of the relative sensors, the undirected graph of the relative sensors is defined as $\mathcal{G}^k_\Delta := (\mathcal{N}, \mathcal{E}^k_\Delta, \mathcal{A}^k_\Delta)$, for $k = 1, ..., \gamma$, where $\mathcal{N} = \{1, 2, ..., N\}$ is a set of nodes, $\mathcal{E}^k_\Delta \subseteq \mathcal{N} \times \mathcal{N}$ is an edge set of pairs of nodes, and the adjacency matrix $\mathcal{A}^k_\Delta = [\alpha^k_{\Delta,ij}] \in \mathbb{R}^{N\times N}$. $\alpha^k_{\Delta,ij} = 1$ if $(i, j) \in \mathcal{E}^k_\Delta$

, otherwise $\alpha_{ij}^k = 0$. For $k = 1, \cdots, \gamma$, the Laplacian matrix of the relative layer $\mathcal{L}_\Delta^k = [l_{\Delta,ij}^k] \in \mathbb{R}^{N\times N}$ of $\mathcal{G}_\Delta^k$ is defined as $l_{\Delta,ii}^k := \sum_{j\neq i} \alpha_{\Delta,ij}^k$ and $l_{\Delta,ij}^k := -\alpha_{\Delta,ij}^k$ for all $i \neq j$. Let layer of graph $\mathcal{G}_\Delta^k$ be the k absolute layer for $k = 1, \cdots, \gamma$.

Assumption 3.2.1. (Connectivity of multi graphs). Given graphs $\mathcal{G}_+$ and $\mathcal{G}_\Delta^k$ sharing the same set of nodes, the projection graph given by

$$\mathcal{G}_p^k := \mathcal{G}_+ \cup \mathcal{G}_\Delta^k, \forall k = 1, ..., \gamma \tag{3.2.1}$$

is connected. □

Remark 3.2.1. Assumption 3.2.1 implies that although the graph $\mathcal{G}_\Delta^k$ or $\mathcal{G}_+$ are disconnected for $k = 1 \cdots, \gamma$, the projection graph is connected. By the Definition 2.3.3, the projection graph $\mathcal{G}_p^k = (\mathcal{N}, \mathcal{E}_p^k, \mathcal{A}_p^k)$ implies that $\mathcal{A}_p^k = \mathcal{A}_\Delta^k \cup \mathcal{A}_+^k$ and $\mathcal{E}_p^k = \mathcal{E}_\Delta^k \cup \mathcal{E}_+$, for $k = 1, \cdots, \gamma$. Let the projection graph $\mathcal{G}_p^k$ be k projection graph, for $k = 1, \cdots, \gamma$. The Laplacian matrix $\mathcal{L}_p^k := [l_{p,ij}^k] \in \mathcal{N} \times \mathcal{N}$ of the projection graph $\mathcal{G}_p^k$ is defined as $l_{p,ii}^k := \sum_{j\neq i} \alpha_{p,ij}^k$ and $l_{p,ij}^k := -\alpha_{p,ij}^k$ for all $i \neq j$. Let the layer of k projection graph be k projection layer, $\forall k = 1, \cdots, \gamma$. □

3.3 Design of Luenberger Observer and Dynamic Controller

In this section, we design the Luenberger observer and dynamic controller of the ith agent. Under Assumption 3.1.2, we design a Luenberger observer of ith agent as

$$\begin{aligned} \dot{\hat{x}}_i(t) &= A\hat{x}_i(t) + L(y_{+,i} - \hat{y}_{+,i}) + Bu_i(t) \\ &= A\hat{x}_i(t) + LC_+(x_i - \hat{x}_i) + Bu_i(t), \ i \in \mathcal{N} \end{aligned} \tag{3.3.1}$$

where $\hat{y}_{+,i} := C_+\hat{x}_i \in R^n$ is the estimated absolute output informaion of the ith agent and matrix L satisfies $(A - LC_+)$ is Hurwitz.

Remark 3.3.1. The states of the ith agent can not be estimated using the relative sensor value. Therefore, the state is estimated using the absolute sensor value. Thus, we design the Luenberger observer by using information of the absolute sensors. □

The dynamic controller $\mathcal{K}(t)$ of the ith agent is represented as

$$\begin{aligned} \dot{\zeta}_i &= (A + KC_\Delta - BB^T P(\epsilon))\zeta_i - K(z_i + \hat{z}_i) \\ u_i &= B^T P(\epsilon)\zeta_i \end{aligned} \tag{3.3.2}$$

where matrix K is selected such that $(A + KC_\Delta)$ is Hurwitz and $P(\epsilon) = P(\epsilon)^T > 0$ is the unique solution of $A^T P + AP - \tau PBB^T P + \epsilon I = 0$ with positive constant τ and ϵ. The ith agent collects the relative output information $z_i(t)$ and estimated relative output information $\hat{z}_i(t)$, using the absolute output relative information. More details are introduced in Chapter 4 and 5.

Remark 3.3.2. When we design the controller 3.3.2, low gain controller in [Seo09] is used. Thus, we introduce how to choose the constant ϵ to design controller in Theorem 4.2.1 and Theorem 5.3.1. In these theorems, we solve the consensus problem using this controller with multilayer network. □

Chapter 4

Double Layers Network

In this chapter, we suggest how to solve the consensus problem for the double layers with multilayer network. That is, we consider one absolute and relative layer, i,e., $\gamma = 1$. Thus, in this chapter, for $\gamma = 1$, we replace the relative matrix C_{Δ}^{1} with C_{Δ}, the relative graph $\mathcal{G}_{\Delta}^{1}$ with $\mathcal{G}_{\Delta}$, and the Laplacian matrix of the relative graph $\mathcal{L}_{p}^{1}$ with $\mathcal{L}_{p}$.

4.1 Relative Output Information

In Chapter 3, we have already introduced the relative output information $z_i(t)$ and estimated relative output information $\hat{z}_i(t)$. In this section, we deal with the relative output information $z_i(t)$ and estimated relative output information

$\hat{z}_i(t)$, respectively, given by

$$\begin{aligned} z_i(t) &:= \sum_{j=1}^{N} a_{\Delta,ij}(y_{\Delta,j} - y_{\Delta,i}) = -\sum_{j=1}^{N} l_{\Delta,ij} y_{\Delta,j} \\ \hat{z}_i(t) &:= \sum_{j=1}^{N} a_{+,ij}(\hat{y}_j - \hat{y}_i) = -\sum_{j=1}^{N} l_{+,ij} \hat{y}_j. \end{aligned} \tag{4.1.1}$$

where estimated output $\hat{y}_i := C_\Delta \hat{x}_i$. The estimated state $\hat{x}$ can be obtained by (3.3.1).

Remark 4.1.1. Since the relative sensor graph $\mathcal{G}_\Delta$ is not connected, it is necessary to use wireless communication to help the unconnected agents exchange information. After estimating the states of agents by using the absolute sensors in the Luenberger observer, the estimated relative output information is exchanged through wireless communication. □

Define the error state $e_i := x_i - \hat{x}_i$, for all $i \in \mathcal{N}$. The error dynamics are represented as

$$\dot{e}_i = (A - LC_+)e_i \quad \text{for all } i = 1, ..., N. \tag{4.1.2}$$

Lemma 4.1.1. By Assumption 3.2.1 and (4.1.2), the input u_i fed back to the ith agent is given by

$$u_i(t) = -\mathcal{K}(t) \sum_{j=1}^{N} (l_{p,ij} C_\Delta x_j - l_{+,ij} C_\Delta e_j) \tag{4.1.3}$$

where Laplacian matrix $\mathcal{L}_p := [l_{p,ij}]$ of projection graph $\mathcal{G}_p = \mathcal{G}_\Delta \cup \mathcal{G}_+$. □

proof. Using the dynamic controller $\mathcal{K}(t)$ of the ith agent (3.3.2), the input u_i

is written as

$$
\begin{aligned}
u_i(t) &= \mathcal{K}(t)\{z_i(t) + \hat{z}_i(t)\} \\
&= \mathcal{K}(t)\{\sum_{j=1}^{N} a_{\Delta,ij}(y_{\Delta,j} - y_{\Delta,i}) + \sum_{j=1}^{N} a_{+,ij}(\hat{y}_j - \hat{y}_i)\} \\
&= -\mathcal{K}(t)\sum_{j=1}^{N}(l_{\Delta,ij}y_{\Delta,j} + l_{+,ij}\hat{y}_j) \\
&= -\mathcal{K}(t)\sum_{j=1}^{N}(l_{\Delta,ij}C_\Delta x_j + l_{+,ij}C_\Delta \hat{x}_j).
\end{aligned}
$$

By using error state in (4.1.2), the input u_i is represented as

$$
\begin{aligned}
u_i(t) &= -\mathcal{K}(t)\sum_{j=1}^{N}((l_{\Delta,ij} + l_{ij}^{+})C_\Delta x_j - l_{+,ij}C_\Delta e_j) \\
&= -\mathcal{K}(t)\sum_{j=1}^{N}(l_{p,ij}C_\Delta x_j - l_{+,ij}C_\Delta e_j).
\end{aligned}
$$

When the consensus problem is solved, the output relative information $z_i(t) + \hat{z}(t)$ and input signals $u_i(t)$ converge zero. □

Let the new variables χ_i :=col(x_i, ζ_i) and $\bar{e}_i$ =col$(e_i, 0)$. For $i = 1, \cdots, N$, the dynamic system of the ith agent is written as

$$
\begin{aligned}
\dot{\chi}_i &= \begin{bmatrix} A & BB^TP \\ 0 & A + KC_\Delta - BB^TP \end{bmatrix}\chi_i + \begin{bmatrix} 0 \\ -K \end{bmatrix}(z_i + \hat{z}_i) \\
&=: \bar{A}\chi_i + \bar{B}(z_i + \hat{z}_i) \\
\bar{y}_i &= \begin{bmatrix} C_\Delta & 0 \end{bmatrix}\chi_i =: \bar{C}\chi_i \\
\dot{\bar{e}}_i &= \begin{bmatrix} A - LC_+ & 0 \\ 0 & 0 \end{bmatrix}\bar{e}_i =: \bar{D}\bar{e}_i.
\end{aligned}
\tag{4.1.4}
$$

Lemma 4.1.2. Define the new variables $\bar{\chi}$, $\bar{y}$, and $\bar{e}$ are the stacked state vectors, defined as $\bar{\chi}$:=col$(\chi_1, \cdots, \chi_N)$, $\bar{y}$:=col$(\bar{y}_1, \cdots, \bar{y}_N)$ and $\bar{e}$:=col$(\bar{e}_1, \cdots, \bar{e}_N)$.

Let us consider a coordinate change $\mathrm{col}(\xi_1, \tilde{\psi}) := Z\bar{\chi}$ where Z is a nonsingular matrix. Then, the closed loop system can be written as

$$\begin{aligned}
\dot{\xi}_1 &= \bar{A}\xi_1 \\
\dot{\tilde{\psi}} &= \left[\begin{array}{c|c|c} \tilde{A}_2 & \cdots & 0 \\ \hline \vdots & \ddots & \vdots \\ \hline 0 & \cdots & \tilde{A}_N \end{array}\right] \tilde{\psi} + T(R^T\mathcal{L}_+ \otimes \bar{B}\bar{C})\bar{e} \\
&=: A_g\tilde{\psi} + B_g\bar{e} \\
\dot{\bar{e}}(t) &= (I_N \otimes \bar{D})\bar{e}(t)
\end{aligned} \tag{4.1.5}$$

where $\tilde{A}_i = T_i(\bar{A} - \lambda_i(\mathcal{L}_p) \otimes \bar{B}\bar{C})T_i^{-1}$ and $T := \left[\begin{array}{c|c|c} T_2 & \cdots & 0 \\ \hline \vdots & \ddots & \vdots \\ \hline 0 & \cdots & T_N \end{array}\right] \in \mathbb{R}^{2n(N-1)\times 2n(N-1)}$

such as $T_i := \begin{bmatrix} I_n & 0 \\ I_n & \dfrac{1}{\lambda_i(\mathcal{L}_p)} I_n \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$. □

proof. The N systems (4.1.4) are collected and put together into one equation, the global system is obtained; that is, using the Kronecker product, the whole network is compactly written as

$$\begin{aligned}
\dot{\bar{\chi}}(t) &= [(I_N \otimes \bar{A}) - (I_N \otimes \bar{B})(\mathcal{L}_p \otimes I_{2n})(I_N \otimes \bar{C})]\bar{\chi}(t) \\
&\quad + (I_N \otimes \bar{B})(\mathcal{L}_+ \otimes I_{2n})(I_N \otimes \bar{C})\bar{e}(t) \\
&= (I_N \otimes \bar{A} - \mathcal{L}_p \otimes \bar{B}\bar{C})\bar{\chi}(t) + (\mathcal{L}_+ \otimes \bar{B}\bar{C})\bar{e}(t) \\
\dot{\bar{e}}(t) &= (I_N \otimes \bar{D})\bar{e}(t).
\end{aligned} \tag{4.1.6}$$

Note that the overall representation (4.1.6) has a benefit that it clearly shows that the local interaction structure from the graph $\mathcal{L}_p$, as well as the location representation (4.1.4) of the individual system. Under Assumption 3.2.1, graph of projection layer $\mathcal{L}_p$ is connected. However graph $\mathcal{L}_+$ does not have to. By

the coordinate transformation in Theorem 2.2.2,

$$\xi := \begin{bmatrix} \xi_1 \\ \tilde{\xi} \end{bmatrix} = (W \otimes I_{2n})\bar{\chi} = (\begin{bmatrix} \frac{1}{N}1_N^T \\ R^T \end{bmatrix} \otimes I_{2n})\bar{\chi}$$

where $\tilde{\xi}$:=col$(\xi_2, ..., \xi_N)$, the transformed closed loop system can be written as

$$\begin{aligned} \dot{\xi}_1 &= \bar{A}\xi_1 \\ \dot{\tilde{\xi}} &= [I_{N-1} \otimes \bar{A} - \Lambda_p \otimes \bar{B}\bar{C}]\tilde{\xi} + (R^T\mathcal{L}_+ \otimes \bar{B}\bar{C})\bar{e}(t) \\ \dot{\bar{e}}(t) &= (I_N \otimes \bar{D})\bar{e}(t) \end{aligned} \tag{4.1.7}$$

where Λ_p is defined as Λ_p =diag$(\lambda_2(\mathcal{L}_p), ..., \lambda_N(\mathcal{L}_p))$.

We consider a nonsingular matrix $T := \left[\begin{array}{c|c|c} T_2 & \cdots & 0 \\ \hline \vdots & \ddots & \vdots \\ \hline 0 & \cdots & T_N \end{array}\right] \in \mathbb{R}^{2n(N-1)\times 2n(N-1)}$ where $T_i := \begin{bmatrix} I_n & 0 \\ I_n & \frac{1}{\lambda_i(\mathcal{L}_p)}I_n \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$. Let us consider a coordinate change $\psi := T\tilde{\xi}$ where $\tilde{\psi}$:=col$(\psi_2, ..., \psi_N)$. Then we obtain system (4.1.5). □

To solve the consensus problem, we show that $\tilde{\xi} \to 0$ as $t \to \infty$ in following theorem.

4.2 Design of Controller with Double Layers

Theorem 4.2.1. (Design controller). Under assumptions, consider Luenberger observer (3.3.1) and controller (3.3.2), in which $\tau = \min_{i=1,\cdots,N} \lambda_i(\mathcal{L}_p)$. Then there exists a positive constant $\bar{\epsilon}$ such that, for each $0 < \epsilon \leq \bar{\epsilon}$, the consensus problem is solved. □

proof. Note that $\lim_{\epsilon\to 0} P(\epsilon) = 0$. To show that the origin of system (4.1.5) is

asymptotically stable for sufficiently small ϵ, take a Lyapunov function

$$V(\tilde{\psi}, \bar{e}) = \tilde{\psi}^T (I_{N-1} \otimes \tilde{P})\tilde{\psi} + \bar{e}^T (I_N \otimes \tilde{F})\bar{e}$$

where $\tilde{P} := \begin{bmatrix} P(\epsilon) & 0 \\ 0 & \lambda_{\max}^{\frac{1}{2}}(P(\epsilon))Q \end{bmatrix}$ such that $Q = Q^T > 0$ is the solution of $(A + KC_\Delta)^T Q + Q(A + KC_\Delta) = -I$ and $\tilde{F} := \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}$ satisfies $(A + LC_+)^T F + F(A + LC_+) = -\nu I$ such as a positive constant ν satisfies that $\nu > \alpha |B_g^T|^2 |Q|^2$ where a positive constant $\alpha > \max(\frac{1}{2}, \frac{\lambda_{\max}(P)}{\epsilon})$. The time derivative of Lyapunov function V along (4.1.5) becomes

$$\begin{aligned}
\dot{V} &= \dot{\tilde{\psi}}_i^T (I_{N-1} \otimes \tilde{P})\tilde{\psi} + \tilde{\psi}^T (I_{N-1} \otimes \tilde{P})\dot{\tilde{\psi}} + \dot{\bar{e}}^T (I_N \otimes \tilde{F})\bar{e} + \bar{e}^T (I_N \otimes \tilde{F})\dot{\bar{e}} \\
&= (A_g \tilde{\psi} + B_g \bar{e})^T (I_{N-1} \otimes \tilde{P})\tilde{\psi} + \tilde{\psi}^T (I_{N-1} \otimes \tilde{P})(A_g \tilde{\psi} + B_g \bar{e}) \\
&\quad + \bar{e}^T (I_N \otimes \bar{D}^T \tilde{F})\bar{e} + \bar{e}^T (I_N \otimes \tilde{F}\bar{D})\bar{e} \\
&= \tilde{\psi}^T [A_g^T (I_{N-1} \otimes \tilde{P}) + (I_{N-1} \otimes \tilde{P})A_g]\tilde{\psi} + 2\bar{e}^T B_g^T (I_{N-1} \otimes \tilde{P})\tilde{\psi} \\
&\quad + \bar{e}^T \left(I_N \otimes \begin{bmatrix} -\nu I_n & 0 \\ 0 & 0 \end{bmatrix} \right) \bar{e}.
\end{aligned}$$

To avoid the complexity of proof, the time derivative of Lyapunov function V is divided into $\dot{V}_1 := \tilde{\psi}^T [A_g^T (I_{N-1} \otimes \tilde{P}) + (I_{N-1} \otimes \tilde{P})A_g]\tilde{\psi}$ and $\dot{V}_2 := 2\bar{e}^T B_g^T (I_{N-1} \otimes \tilde{P})\tilde{\psi} + \bar{e}^T \left(I_N \otimes \begin{bmatrix} -\nu I_n & 0 \\ 0 & 0 \end{bmatrix} \right) \bar{e}$. We consider the time derivative of Lyapunov function $V_i^{(1)}$ related to the ith agent, for $i = 2, ..., N$, such as $V_1 := [V_{2,1}, ..., V_{N,1}]^T \in \mathbb{R}^{N-1}$. In addition, since $\psi_i \in \mathbb{R}^{2n}$, we define new state $\psi_{i1}, \psi_{i2} \in \mathbb{R}^n$ as $\psi_i :=\mathrm{col}(\psi_{i1}, \psi_{i2})$. Throughout this proof, the ith eigenvalue of Laplacian $\mathcal{L}_p$, $\lambda_i(\mathcal{L}_p)$ is denoted by λ_i. Since $\bar{e}_i =\mathrm{col}(e_i, 0)$, we define the state $\tilde{e} :=\mathrm{col}(e_1, e_2, ..., e_N)$. By Appendix A and B, the time derivative of Lyapunov

function $V_{i,1}$ and V_2 is obtained as

$$\dot{V}_1 \leq -\epsilon \sum_{i=2}^{N} |\psi_{i1}|^2 - \lambda_{\max}^{\frac{1}{2}}(P)(1-\delta(\epsilon)) \sum_{i=2}^{N} |\psi_{i2}|^2, \ \forall i = 2, \cdots, N, \tag{4.2.1}$$

$$\dot{V}_2 \leq (\alpha |B_g^T|^2 |Q|^2| - \nu)|\tilde{e}|^2 + \frac{\lambda_{\max}(P)}{\alpha} |\tilde{\psi}|^2 \tag{4.2.2}$$

where $\delta(\epsilon) := 2\frac{|\lambda_i|^2}{\tau}|B^T|^2 \lambda_{\max}^{\frac{3}{2}}(P) + 2\lambda_{\max}^{\frac{1}{2}}(P)\frac{|\lambda_i - 1|^2}{\tau}|B^T Q|$ $+2\lambda_{\max}(P)|\lambda_i - 1||QBB^T|$. Therefore, the time derivative of Lyapunov function V using (4.2.1) and (4.2.2), becomes

$$\begin{aligned}
\dot{V} &= \dot{V}_1 + \dot{V}_2 \\
&\leq -\left(\epsilon - \frac{\lambda_{\max}(P)}{\alpha}\right) \sum_{i=2}^{N} |\psi_{i1}|^2 - \lambda_{\max}^{\frac{1}{2}}(P) \sum_{i=2}^{N} \left(1 - \delta(\epsilon) - \frac{\lambda_{\max}^{\frac{1}{2}}(P)}{\alpha}\right) |\psi_{i2}|^2 \\
&\quad - (\nu - \alpha |B_g^T|^2 |Q|^2|)|\tilde{e}|^2.
\end{aligned} \tag{4.2.3}$$

From the proof, take a sufficiently small ϵ to satisfy $\delta(\epsilon) < \frac{1}{2}$ and select a positive constant α such as $\alpha > \max\left(\frac{1}{2}, \frac{\lambda_{\max}(P)}{\epsilon}\right)$. Then, a positive constant ν can be chosen such that $\nu > \alpha |B_g^T|^2 |Q|^2$. It is clear that the value of $\bar{\epsilon}$ becomes smaller as $\max_{i=2,\ldots,N} |\lambda_i|$ gets larger. That is, the proposed controller is based on the low gain. While a high gain controller may destroy stability of systems, the low gain approach is a safe way to solve consensus problems. $\square$

Chapter 5

General Case for Multilayer Network

In Chapter 4, we set the problem by considering double layers, and solved the consensus problem. However, in reality, a graph of each relative sensor may be different depending on the characteristics of sensors. Although there are many types of absolute sensors, a single graph is considered. Because these sensors are, in fact, supposed to exchange information with the wireless network. However, since the relative sensors measure the relative output information directly, the graphs may be different depending on kinds of sensors. Thus, in this chapter, we try to solve the consensus problem with multilayer network by using the controller (3.3.2).

5.1 Additional Assumption

Assumption 5.1.1. (Modal form). We assume that the matrix A is represented as modal form. $\square$

By Assumption 3.1.1, all poles of matrix A replace on the imaginary axis. Thus, we consider that the matrix A is in modal form. It follows that the matrix A in modal form is skew symmetric.

Assumption 5.1.2. We consider the matrix B such that $BB^T > 0$. □

Lemma 5.1.1. (Upper bound of the largest eigenvalue of matrix P [Inequality (25)[Kwon96]]). The matrix P is the solution of the Riccati equation such as $A^TP + AP - \tau PBB^TP + \epsilon I = 0$. Under Assumption 5.1.1 and 5.1.2,

$$\lambda_{max}(P) \leq \frac{\sqrt{\epsilon}}{\tau^{\frac{1}{2}}\lambda_{min}^{\frac{1}{4}}(BB^T)}.$$

□

Proof. By result of [Kwon96], the largest eigenvalue of matrix P is satisfied as

$$\lambda_{max}(P) \leq \frac{-\lambda_{max}\left(\frac{(A+A^T)}{2}\right) + \sqrt{\lambda_{max}^2\left(\frac{(A+A^T)}{2}\right) + \lambda_{min}(BB^T)\epsilon}}{\lambda_{min}(BB^T)}$$

where matrix P is the solution of the Riccati equation such as $A^TP + AP - PBB^TP + \epsilon I = 0$. Thus, we consider the positive constant τ and the fact that $\lambda_{max}\left(\frac{(A+A^T)}{2}\right) = 0$ under Assumption 5.1.1. Then Lemma is obtained. □

Assumption 5.1.3. (Laplacian matrix commute). Each Laplacian matrix L_p^k of projection graph $\mathcal{G}_p^k$ is assumed to be commute, i.e., $L_p^kL_p^m = L_p^mL_p^k$ where $k, m = 1, ..., \gamma$. □

Lemma 5.1.2. (Simultaneously diagonalizable [Theorem 1.3.2,[Hor12]]). Let C be a set such as $\{\mathcal{L}_p^k | L_p^kL_p^m = L_p^mL_p^k, \forall k, m = 1, ..., \gamma\}$. A set D is defined such that for all Laplacian matrix $\mathcal{L}_p^k \in D, \forall k = 1, \cdots, \gamma$, there is a single nonsingular T satisfying that $T^{-1}\mathcal{L}_p^kT$ are simultaneously diagonalizable. Then for any given $\mathcal{L}_p^k \in C$ and any given ordering $\lambda_1, \cdots, \lambda_N$ of the eigenvalues of $\mathcal{L}_p^k$, there is a nonsingular matrix T such that $T^{-1}\mathcal{L}_p^kT$ =diag$(\lambda_1, \cdots, \lambda_N)$ and $T^{-1}\mathcal{L}_m^kT$ is diagonal for every $\mathcal{L}_p^k \in C$. □

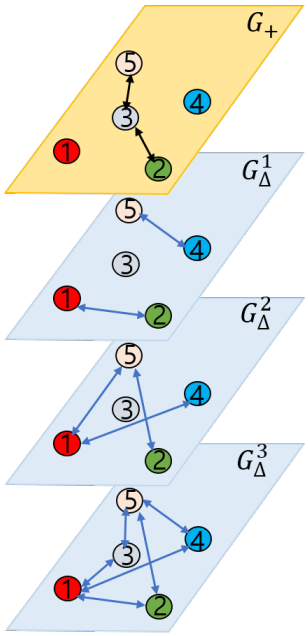

Figure 5.1 : Schematic illustration of the 4 multi layers. Three Laplacian matrices of projection graphs commute

This lemma implies that if any Laplacian matrices commute, then there exists a nonsingular matrix such that it simultaneously diagonalizes matrices.

Example 3. Under Assumption 3.2.1 and 5.1.3, consider the multi graphs depicted in Figure 5.1. The projection graph is represented as $\mathcal{G}_p^i = \mathcal{G}_+ \cup \mathcal{G}_p^i$ for all $i = 1, 2, 3$. For all $i = 1, 2, 3$, the Laplacian matrix $\mathcal{L}_p^i$ of each projection

graph $\mathcal{G}_p^i$, is given by

$$\mathcal{L}_p^1 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \mathcal{L}_p^2 = \begin{bmatrix} 2 & 0 & 0 & -1 & -1 \\ 0 & 2 & -1 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{bmatrix},$$

$$\mathcal{L}_p^3 = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & 0 & -1 & 3 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix}.$$

we can confirm commutativity of Laplacian matrix such as $\mathcal{L}_p^k \mathcal{L}_p^m = \mathcal{L}_p^m \mathcal{L}_p^k$, for all $m, k = 1, 2, 3$.

5.2 Relative output information

In Chapter 4, we already introduce the relative output information. However, since we consider multi graphs of matrix C_Δ^k for all $k = 1, \cdots, \gamma$, the structure of relative output information is different from (4.1.1).

The relative output information and estimated output information are rep-

resented as

$$
\begin{aligned}
z_i(t) &= \begin{bmatrix} \sum_{j=1}^{N} a^1_{\Delta,ij}(y^1_{\Delta,j} - y^1_{\Delta,i}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{j=1}^{N} a^\gamma_{\Delta,ij}(y^\gamma_{\Delta,j} - y^\gamma_{\Delta,ij}) \end{bmatrix} \\
&= -\begin{bmatrix} \sum_{j=1}^{N} l^1_{\Delta,ij} y^1_{\Delta,j} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{j=1}^{N} l^\gamma_{\Delta,ij} y^\gamma_{\Delta,j} \end{bmatrix} \\
\text{and } \hat{z}_i(t) &= \begin{bmatrix} \sum_{j=1}^{N} a_{+,ij}(\hat{y}^1_j - \hat{y}^1_i) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{j=1}^{N} a_{+,ij}(\hat{y}^\gamma_j - \hat{y}^\gamma_i) \end{bmatrix} \\
&= -\begin{bmatrix} \sum_{j=1}^{N} l_{+,ij} \hat{y}^1_j & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{j=1}^{N} l_{+,ij} \hat{y}_{\gamma,ij} \end{bmatrix}
\end{aligned}
\tag{5.2.1}
$$

where $y^k_{\Delta,i} := C^k_\Delta x_i$ and $\hat{y}_{k,i} := C^k_\Delta \hat{x}_i$ are defined as relative and estimated output value of ith agent of k the layer for all $k = 1, 2, ..., \gamma$ and $i = 1, \cdots, N$. Thus, that filtered information is fed back to the ith agent by

$$
\begin{aligned}
u_i(t) &= \mathcal{K}(t)(z_i(t) + \hat{z}_i(t)) \\
&= -\mathcal{K}(t) \begin{bmatrix} \sum_{j=1}^{N} (l^1_{\Delta,ij} y^{(1)}_j + l_{+,ij} \hat{y}^1_j) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{j=1}^{N} (l^\gamma_{\Delta,ij} y^\gamma_j + l_{+,ij} \hat{y}^\gamma_j) \end{bmatrix}.
\end{aligned}
\tag{5.2.2}
$$

Since the matrix $C_\Delta = [C^{(1)}_\Delta; ...; C^{(\gamma)}_\Delta]$ and the error state $e_i := x_i - \hat{x}_i$, $\forall i \in \mathcal{N}$, the sum of relative output information and estimated relative output

information is written as

$$
\begin{aligned}
& z_i(t) + \hat{z}_i(t) \\
&= -\begin{bmatrix} \sum_{j=1}^{N}(l^1_{\Delta,ij}C^{(1)}_{\Delta}x_j + l_{+,ij}C^1_{\Delta}\hat{x}_j) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{j=1}^{N}(l^1_{\Delta,ij}C^{\gamma}_{\Delta}x_j + l_{+,ij}C^{(\gamma)}_{\Delta}\hat{x}_j) \end{bmatrix} \\
&= -\begin{bmatrix} \sum_{j=1}^{N}(l^1_{\Delta,ij} + l_{+,ij}C^1_{\Delta}x_j & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{j=1}^{N}(l^{\gamma}_{\Delta,ij} + l_{+,ij}C^{\gamma}_{\Delta}x_j \end{bmatrix} \\
&\quad + \begin{bmatrix} \sum_{j=1}^{N} l_{+,ij}C^1_{\Delta}e_j & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{j=1}^{N} l_{+,ij}C^{\gamma}_{\Delta}e_j \end{bmatrix} \\
&= -\begin{bmatrix} \sum_{j=1}^{N} l^1_{p,ij}C^1_{\Delta}x_j & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{j=1}^{N} l^{\gamma}_{p,ij}C^{\gamma}_{\Delta}x_j \end{bmatrix} \\
&\quad + \begin{bmatrix} \sum_{j=1}^{N} l_{+,ij}C^1_{\Delta}e_j & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{j=1}^{N} l_{+,ij}C^{\gamma}_{\Delta}e_j \end{bmatrix}
\end{aligned}
\tag{5.2.3}
$$

where Laplacian matrix $\mathcal{L}^k_p := [l^k_{p,ij}]$ of projection graph $\mathcal{G}^k_p = \mathcal{G}_+ \cup \mathcal{G}^k_\Delta$ by Assumption 3.2.1, $\forall k = 1, ..., \gamma$.

The overall system is represented as (4.1.4), where new variables $\chi_i :=\text{col}(x_i, \zeta_i)$ and $\bar{e}_i =\text{col}(e_i, 0)$. Define the new variables $\bar{\chi}$, $\bar{y}$, and $\bar{e}$ are the stacked state vectors, defined as $\bar{\chi} :=\text{col}(\chi_1, \cdots, \chi_N)$, $\bar{y} :=\text{col}(\bar{y}_1, \cdots, \bar{y}_N)$ and $\bar{e} :=\text{col}(\bar{e}_1, \cdots, \bar{e}_N)$ However, considering the multi graphs, the overall system is compactly repre-

sented as

$$
\begin{aligned}
\dot{\bar{\chi}}(t) &= \Big[(I_N \otimes \bar{A}) - (I_N \otimes \bar{B})(\mathcal{L}_p^1 \otimes M_1 + \cdots + \mathcal{L}_p^\gamma \otimes M_\gamma)(I_N \otimes \bar{C})\Big]\,\bar{\chi}(t) \\
&\quad + \Big[(I_N \otimes \bar{B})(\mathcal{L}_+ \otimes I_\gamma)(I_N \otimes \bar{C})\Big]\,\bar{e}(t) \\
&= \Big[(I_N \otimes \bar{A}) - (\mathcal{L}_p^1 \otimes \bar{B}M_1\bar{C}) - \cdots - (\mathcal{L}_p^\gamma \otimes \bar{B}M_\gamma\bar{C})\Big]\,\bar{\chi}(t) + (\mathcal{L}_+ \otimes \bar{B}\bar{C})\bar{e}(t) \\
\dot{\bar{e}}(t) &= (I_N \otimes \bar{D})\bar{e}(t)
\end{aligned}
\tag{5.2.4}
$$

where the new variables $\bar{\chi}$, $\bar{y}$, and $\bar{e}$ are the stacked state vectors, defined as $\bar{\chi}$:=col$(\chi_1, \cdots, \chi_N)$, $\bar{y}$:=col$(\bar{y}_1, \cdots, \bar{y}_N)$ and $\bar{e}$:=col$(\bar{e}_1, \cdots, \bar{e}_N)$. Define the square matrix $M_k \in \mathbb{R}^{\gamma\times\gamma}$ that its element in kth row and column is 1 and others are 0, for all $k = 1, ..., \gamma$.

For example, $M_1 = \begin{bmatrix} 1 & \cdots & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$.

The following Example 4 explains the process of expressing the overall systems (5.2.4) differently from (4.1.6) since we consider three or more layers simultaneously.

Example 4. Consider the multi graphs depicted in Figure 5.2. It is easy to see that the number of layers is two except for the layer related to the sensor that measures the absolute value, so that the relative layer degree is 2. That is, it is implies that relative matrix is represented as $C_\Delta = [C_\Delta^{(1)}; C_\Delta^{(2)}]$ in this example. When the Laplacian matrix $\mathcal{L}_p^i := [\bar{l}_{ij}^{(i)}]$ of each projection graph $\mathcal{G}_p^i := \mathcal{G}_+ \cup \mathcal{G}_\Delta^i$

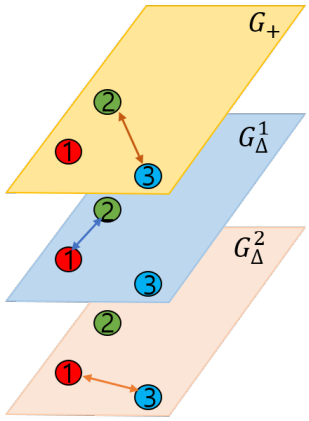

Figure 5.2 : Schematic illustration of three individual agents with multilayer network.

for $i = 1, 2$, is written as

$$\mathcal{L}_p^1 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \ \mathcal{L}_p^2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \text{ and } \mathcal{L}_+ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

the ith system (4.1.4) is written as

$$\begin{aligned}
\dot{\chi}_i &= \begin{bmatrix} A & BC_\zeta \\ 0 & A_\zeta \end{bmatrix} \chi_i + \begin{bmatrix} 0 \\ B_\zeta \end{bmatrix} (z_i + \hat{z}_i) \\
&=: \bar{A}\chi_i - \bar{B} \begin{bmatrix} \sum_{j=1}^N \bar{l}_{ij}^{(1)} C_\Delta^{(1)} x_j & 0 \\ 0 & \sum_{j=1}^N \bar{l}_{ij}^{(2)} C_\Delta^{(2)} x_j \end{bmatrix} \\
&\quad + \bar{B} \begin{bmatrix} \sum_{j=1}^N l_{ij}^{(+)} C_\Delta^{(1)} e_j & 0 \\ 0 & \sum_{j=1}^N l_{ij}^{(+)} C_\Delta^{(2)} e_j \end{bmatrix} \\
\bar{y}_i &= \begin{bmatrix} C_\Delta & 0 \end{bmatrix} \chi_i =: \bar{C}\chi_i \\
\bar{e}_i &= \begin{bmatrix} A - LC_+ & 0 \\ 0 & 0 \end{bmatrix} \bar{e}_i =: \bar{D}\bar{e}_i.
\end{aligned}$$

If so, the whole network is written as

$$\dot{\bar{\chi}}(t) = [(I_2 \otimes \bar{A}) - (I_2 \otimes \bar{B})\tilde{\mathcal{L}}_p(I_2 \otimes \begin{bmatrix} C_\Delta^{(1)} \\ C_\Delta^{(2)} \end{bmatrix})]\bar{\chi}(t)$$

$$+ (I_N \otimes \bar{B})\tilde{\mathcal{L}}_+(I_N \otimes \begin{bmatrix} C_\Delta^{(1)} \\ C_\Delta^{(2)} \end{bmatrix})\bar{e}(t)$$

$$\dot{\bar{e}}(t) = (I_N \otimes \bar{D})\bar{e}(t)$$

where $\tilde{\mathcal{L}}_p$, and $\tilde{\mathcal{L}}_+ \in \mathbb{R}^{6\times 6}$ are matrices represented as

$$\tilde{\mathcal{L}}_p = \left[\begin{array}{c|c|c|c|c|c} 1 & & -1 & & 0 & \\ \hline & 1 & & 0 & & -1 \\ \hline -1 & & 2 & & -1 & \\ \hline & 0 & & 1 & & -1 \\ \hline 0 & & -1 & & 1 & \\ \hline & -1 & & -1 & & 2 \end{array}\right], \text{ and } \tilde{\mathcal{L}}_+ = \left[\begin{array}{c|c|c|c|c|c} 0 & & 0 & & 0 & \\ \hline & 0 & & 1 & & -1 \\ \hline 0 & & -1 & & 1 & \\ \hline & 0 & & 0 & & 0 \\ \hline 0 & & 1 & & -1 & \\ \hline & 0 & & -1 & & 1 \end{array}\right].$$

The blanks of matrices above are filled with zeros. When considering multiple Laplacian matrices simultaneously, it can be easily assumed that the Laplacian matrix will be diagonalized. However, since the elements of Laplacian matrix must be filled by the rows of matrix C_Δ, the elements of the Laplacian matrix are scattered as above. For output consensus, we can confirm that the Laplacian matrix of overall system is not simply diagonal. The Laplacian matrix of the overall system considering multi graphs appear as an intersection of the elements of each Laplacian matrix. Through the matrices $M_1 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $M_1 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 2}$ are defined in (5.2.4), the above whole network is

expressed as

$$\dot{\bar{\chi}}(t) = [(I_2 \otimes \bar{A}) - (I_2 \otimes \bar{B})(\mathcal{L}_p^1 \otimes M_1)(I_2 \otimes \bar{C}) - (I_2 \otimes \bar{B})(\mathcal{L}_p^2 \otimes M_2)(I_2 \otimes \bar{C})]\bar{\chi}(t)$$
$$+ (I_N \otimes \bar{B})(\mathcal{L}_+ \otimes I_2)(I_N \otimes \bar{C})\bar{e}(t)$$
$$\dot{\bar{e}}(t) = (I_N \otimes \bar{D})\bar{e}(t).$$

Notice that the matrices $\tilde{\mathcal{L}}_p$ and $\tilde{\mathcal{L}}_+$ can be expressed as $\tilde{\mathcal{L}}_p = \mathcal{L}_p^1 \otimes M_1 + \mathcal{L}_p^2 \otimes M_2$ and $\tilde{\mathcal{L}}_+ = \mathcal{L}_+ \otimes M_1 + \mathcal{L}_+ \otimes M_2 = \mathcal{L}_+ \otimes (M_1 + M_2) = \mathcal{L}_+ \otimes I_2$. □

Lemma 5.2.1. (Closed systems with double layers network). Under the Lemma 5.1.2 and Theorem 2.2.2, Let us consider a coordinate change $\mathrm{col}(\xi_1, \tilde{\psi}) := Z\bar{\chi}$ where Z is a nonsingular matrix. Then, the closed loop system can be written as

$$\begin{aligned}
\dot{\xi}_1 &= \bar{A}\xi_1 \\
\dot{\tilde{\psi}} &= \left[\begin{array}{c|c|c} \tilde{A}_2 & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & \tilde{A}_N \end{array}\right] \tilde{\psi} + T(R^T \mathcal{L}_+ \otimes \bar{B}\bar{C})\bar{e} \\
&=: A_g \tilde{\psi} + B_g \bar{e} \\
\dot{\bar{e}}(t) &= (I_N \otimes \bar{D})\bar{e}(t)
\end{aligned} \tag{5.2.5}$$

where matrix $\tilde{A}_i = T_i(\bar{A} - (\lambda_i^1 \otimes \bar{B}M_1\bar{C}) \cdots - (\lambda_i^\gamma \otimes \bar{B}M_\gamma\bar{C}))T_i^{-1}$. We consider a nonsingular matrix $T := \left[\begin{array}{c|c|c} T_2 & \cdots & 0 \\ \hline \vdots & \ddots & \vdots \\ \hline 0 & \cdots & T_N \end{array}\right] \in \mathbb{R}^{2n(N-1)\times 2n(N-1)}$ where $T_i := \begin{bmatrix} I & 0 \\ I & \frac{1}{\pi_i} I \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$, in which, $\pi_i := \max_{k=1,\ldots,\gamma}(\lambda_i(\mathcal{L}_p^k))$. Let us consider a coordinate change $\psi := T\tilde{\xi}$ where $\tilde{\psi} := \mathrm{col}(\psi_2, ..., \psi_N)$. Then we obtain system

(5.2.5). □

proof. Under Assumption 3.2.1, graph of projection layer $\mathcal{L}_p^k$ is connected for all $k = 1, \cdots, \gamma$. However graph $\mathcal{L}_+$ does not have to be connected. By the coordinate transformation in Theorem 2.2.2 and Lemma 5.1.2,

$$\begin{aligned}
\dot{\xi}_1 &= \bar{A}\xi_1 \\
\dot{\tilde{\xi}} &= [I_{N-1} \otimes \bar{A} - (\Lambda^1 \otimes \bar{B}M_1\bar{C}) \cdots - (\Lambda^\gamma \otimes \bar{B}M_\gamma\bar{C})]\tilde{\xi} + (R^T\mathcal{L}_+ \otimes \bar{B}\bar{C})\bar{e}(t) \\
\dot{\bar{e}}(t) &= (I_N \otimes \bar{D})\bar{e}(t)
\end{aligned} \tag{5.2.6}$$

where Λ^k is defined as $\Lambda^k = diag(\lambda_2(\mathcal{L}_p^k), ..., \lambda_N(\mathcal{L}_p^k))$ for all $k = 1, ..., \gamma$. We consider a nonsingular matrix $T := \left[\begin{array}{c|c|c} T_2 & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & T_N \end{array}\right] \in \mathbb{R}^{2n(N-1)\times 2n(N-1)}$ where $T_i := \begin{bmatrix} I & 0 \\ I & \frac{1}{\pi_i}I \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$, in which, $\pi_i := \max_{k=1,\ldots,\gamma}(\lambda_i(\mathcal{L}_p^k))$. Let us consider a coordinate change $\psi := T\tilde{\xi}$ where $\tilde{\psi}$:=col$(\psi_2, ..., \psi_N)$. Then we obtain system (5.2.5). □

5.3 Design of Controller with multilayer network

Theorem 5.3.1. (Design controller). Under assumptions in Chapter 4 and 3.1.1, consider Luenberger observer (3.3.1) and controller (3.3.2), in which $\tau := min_{i,k}\lambda_i(\mathcal{L}_p^k)$. Then there exists a positive constant $\bar{\epsilon}$ such that, for each $0 < \epsilon \leq \bar{\epsilon}$, the consensus problem is solved. □

proof. This proof is almost same as the proof of Theorem 4.2.1. Thus, only the difference is described in detail. Take a Lyapunov function

$$V(\tilde{\psi}, \bar{e}) = \tilde{\psi}^T(I_{N-1} \otimes \tilde{P})\tilde{\psi} + \bar{e}^T(I_N \otimes \tilde{F})\bar{e}$$

where $\tilde{P} := \begin{bmatrix} P(\epsilon) & 0 \\ 0 & \lambda_{\max}^{\frac{1}{2}}(P(\epsilon))Q \end{bmatrix}$ such that $Q = Q^T > 0$ is the solution of $(A + KC_\Delta)^T Q + Q(A + KC_\Delta) = -\mu I$ and $\tilde{F} := \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}$ such that $(A + LC_+)^T F + F(A + LC_+) = -\nu I$. The only difference between proof of Theorem 4.2.1 and this proof is that the the right-hand side of Lyapunov equation of matrix Q sets to $-\mu I$ instead of $-I$.

The time derivative of Lyapunov function V becomes

$$\begin{aligned}
\dot{V} &= \dot{\tilde{\psi}}_i^T (I_{N-1} \otimes \tilde{P})\tilde{\psi} + \tilde{\psi}^T (I_{N-1} \otimes \tilde{P})\dot{\tilde{\psi}} + \dot{\bar{e}}^T (I_{N-1} \otimes \tilde{F})\bar{e} + \bar{e}^T (I_N \otimes \tilde{F})\dot{\bar{e}} \\
&= \tilde{\psi}^T [A_g^T (I_{N-1} \otimes \tilde{P}) + (I_{N-1} \otimes \tilde{P})A_g]\tilde{\psi} + 2\bar{e}^T B_g (I_{N-1} \otimes \tilde{P})\tilde{\psi} \\
&\quad + \bar{e}^T \left(I_{N-1} \otimes \begin{bmatrix} -\nu I_n & 0 \\ 0 & 0 \end{bmatrix} \right) \bar{e}.
\end{aligned}$$

The matrix $\tilde{A}_i$ in closed systems can be obtained by Appendix C. The time derivative of Lyapunov function $V := V_2 + V_3$ is divided into $\dot{V}_3 := \tilde{\psi}^T [A_g^T (I_{N-1} \otimes \tilde{P}) + (I_{N-1} \otimes \tilde{P})A_g]\tilde{\psi}$ and $\dot{V}_2 := 2\bar{e}^T B_g^T (I_{N-1} \otimes \tilde{P})\tilde{\psi} + \bar{e}^T \left(I_{N-1} \otimes \begin{bmatrix} -\nu I_n & 0 \\ 0 & 0 \end{bmatrix} \right) \bar{e}$. To obtain V_3, we consider the Lyapunov function $V_{i,3}$ related ith agent, for $i = 2, ..., N$, such as $V_3 := [V_{2,3}, \cdots, V_{N,3}]^T$. In addition, since the $\psi_i \in \mathbb{R}^{2n}$, we define new state $\psi_{i1}, \psi_{i2} \in \mathbb{R}^n$ such as $\psi_i :=$col(ψ_{i1}, ψ_{i2}).

Define $Y_k := (\lambda_i^k - \lambda_i^1) K M_k C_\Delta$ for all $k = 2, ..., \gamma$. The time derivative of Lyapunov function $V_{3,i}$ is defined as $\dot{V}_{3,i} := \dot{V}_{1,i} + \dot{V}_{4,i}$,in which $\dot{V}_{1,i}$ in proof of

Theorem 4.2.1, obtained that

$$\begin{aligned}\dot{V}_{1,i} &= -\epsilon\psi_{i1}^T\psi_{i1} + \tau\psi_{i1}^T PBB^T P\psi_{i1} - 2\pi_i\psi_{i1}^T PBB^T P\psi_{i1} \\ &\quad + (\pi_i\psi_{i1}^T PBB^T P\psi_{i2} + \pi_i\psi_{i2}^T PBB^T P\psi_{i1}) - \mu\lambda_{\max}^{\frac{1}{2}}(P)\psi_{i2}^T\psi_{i2} \\ &\quad - \lambda_{\max}^{\frac{1}{2}}(P)(\pi_i - 1)[\psi_{i2}^T QBB^T P\psi_{i1} + \psi_{i1}^T PBB^T Q\psi_{i2}] \\ &\quad + \lambda_{\max}^{\frac{1}{2}}(P)(\pi_i - 1)[\psi_{i2}^T QBB^T P\psi_{i2} + \psi_{i2}^T PBB^T Q\psi_{i2}]\end{aligned}$$

and $\dot{V}_{4,i} = \dfrac{\lambda_{\max}^{\frac{1}{2}}(P)}{\pi_i}[\psi_{i1}^T(Y_2^T Q+, \cdots, +Y_\gamma^T Q)\psi_{i2} + \psi_{i2}^T(QY_2+, \cdots, +QY_\gamma)\psi_{i1}]$.

Using the Young's inequality to $\dot{V}_{4,i}$, for constant $\beta > 0$, we obtain that

$$\begin{aligned}|\dot{V}_{4,i}| &\le |\frac{2\lambda_{\max}^{\frac{1}{2}}(P)}{\pi_i}\psi_{i1}^T(Y_2^T Q+, \cdots, +Y_\gamma^T Q)\psi_{i2}| \\ &\le \frac{\beta}{\pi_i^2}\lambda_{\max}(P)|Y_2^T+, \cdots, +Y_\gamma^T|^2|\psi_{i1}|^2|Q|^2 + \frac{1}{\beta}|\psi_{i2}|^2.\end{aligned}$$

With (4.2.3) in proof of Theorem 4.2.1, $\dot{V}$ is written as

$$\begin{aligned}\dot{V} \le &-\left(\epsilon - \frac{\lambda_{\max}(P)}{\alpha} - \frac{\beta}{\tau^2}\lambda_{\max}(P)|Y_2^T+, \cdots, +Y_\gamma^T|^2|Q|^2\right)\sum_{i=2}^N |\psi_{i1}|^2 \\ &- \lambda_{\max}^{\frac{1}{2}}(P)\sum_{i=2}^N\left(\mu - \delta(\epsilon) - \frac{\lambda_{\max}^{\frac{1}{2}}(P)}{\alpha} - \frac{1}{\beta}\right)|\psi_{i2}|^2 - (\nu - \alpha|B_g^T|^2|Q|^2|)|\tilde{e}|^2\end{aligned}$$

where $\delta(\epsilon) := 2\dfrac{|\pi_i|^2}{\tau}|B^T|^2\lambda_{\max}^{\frac{3}{2}}(P) + 2\lambda_{\max}^{\frac{1}{2}}(P)\dfrac{|\pi_i - 1|^2}{\tau}|B^T Q|$
$+ 2\lambda_{\max}(P)|\pi_i - 1||QBB^T|$.

Note that the solution of Lyapunov equation $(A+KC_\Delta)^T Q + Q(A+KC_\Delta) = -\mu I$ is written as $Q = -\mu\int_0^\infty \exp\{A + KC_\Delta)\tau\}\exp\{(A + KC_\Delta)^T\tau\}d\tau$. Define $\tilde{Q} := |\int_0^\infty \exp\{A + KC_\Delta)\tau\}\exp\{(A + KC_\Delta)^T\tau\}d\tau|$. Norm of matrix Q written as $|Q|$ is such as $|Q| \le \mu\tilde{Q}$. The positive constant μ is defined as $\mu := \dfrac{2}{\beta}$. Refer to Appendix D for details of the proof. From the proof, take a sufficiently small ϵ to satisfy that

$$\epsilon^{\frac{1}{4}} < \frac{\tau^{\frac{17}{4}}\lambda_{min}^{\frac{5}{4}}(BB^T)}{64|\pi_i|^2\tilde{Q}^2|B^T|^2|Y_2^T+, \cdots, +Y_\gamma^T|^2}$$

and $\lambda_{\max}^{\frac{1}{2}}(P)\dfrac{|\pi_i - 1|^2}{\tau}|B^T| + \lambda_{\max}(P)|\pi_i - 1||BB^T| < \dfrac{1}{8\tilde{Q}}$. Choose a positive constant β such that $\dfrac{1}{\beta} < \dfrac{\epsilon\tau^2}{8\lambda_{\max}(P)\tilde{Q}|Y_2^T+, \cdots, +Y_\gamma^T|^2}$. Then, select a positive constant α such that $\dfrac{1}{\alpha} < \max\left(\dfrac{\epsilon}{2\lambda_{\max}(P)}, \dfrac{1}{4\beta\lambda_{max}^{\frac{1}{2}}(P)}\right)$. A positive constant ν can be chosen such that $\nu > \alpha|B_g^T|^2|Q|^2$. Under these conditions of constants and ϵ, we solve consensus problems. □

The proof of Theorem 5.3.1 is confirmed to be almost similar to the proof of Theorem 4.2.1. The reason for this is that there is a nonsingular matrix that simultaneously diagonalizes all Laplacian matrices of projection graphs under Assumption 5.1.3. To alleviate Assumption 5.1.3 a little further, we propose the following.

Remark 5.3.1. (alleviate Assumption 5.1.3 [Sor12]). When Laplacian matrices of projection graphs satisfy Assumption 5.1.3 or the following, for all $k = 1, ..., \gamma$,

$$\mathcal{L}_p^k = \begin{bmatrix} a_1^k - \bar{a}^k & a_2^k & \cdots & a_{N-1}^k & a_N^k \\ a_1^k & a_2^k - \bar{a}^k & \cdots & a_{N-1}^k & a_N^k \\ & & \ddots & & \\ a_1^k & a_2^k & \cdots & a_{N-1}^k - \bar{a}^k & a_N^k \\ a_1^k & a_2^k & \cdots & a_{N-1}^k & a_N^k - \bar{a}^k \end{bmatrix}, \tag{5.3.1}$$

where $\bar{a}^k = \sum_{j=1}^N a_j^k$, then theorem 5.3.1 is satisfied. □

If Laplacian matrices of projection graphs look like (5.3.1), it will always have N-1 identical eigenvalues and simple zero eigenvalue. Then, it is easy to find a nonsingular matrix that it diagonalizes all the Laplacian matrices of projection graphs such as Assumption 5.1.3 or (5.3.1). However, most of these matrices appear in the directed graph. In an undirected graph, only fully connected

graph in which every pair of distinct vertices is connected by a unique edge. So we do not prove separately in directed graphs. However, since the low gain controller presented in the Theorem 5.3.1 holds for directed graphs, it can be proved similarly Theorem 5.3.1.

5.4 Illustrative Examples

5.4.1 Example 1. Double Layers

In order to demonstrate the result given in Theorem 4.2.1, consider the individual system given by

$$\dot{x}_i(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} x_i(t) + \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} u_i(t), \ i \in \mathcal{N} =: \{1, ..., 5\}$$

$$y_i(t) = \begin{bmatrix} C_+ \\ \hline C_\Delta \end{bmatrix} x_i(t) = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{array}\right] x_i(t) =: \left[\begin{array}{c} 0 \\ 0 \\ \hline y_{i1} \\ y_{i2} \\ y_{i3} \\ y_{i4} \end{array}\right] \tag{5.4.1}$$

where $x_i \in R^4$, $u_i \in \mathbb{R}^4$, and $y_i \in \mathbb{R}^6$. It is seen that Assumption 3.1.1 holds since the eigenvalues of A are $\{0, 0, j, -j\}$. Suppose that the communication network is given by Figure 5.3. In Figure 5.3, dash and solid curves imply the network of absolute, and relative sensors, respectively. The network of each layers is represented by

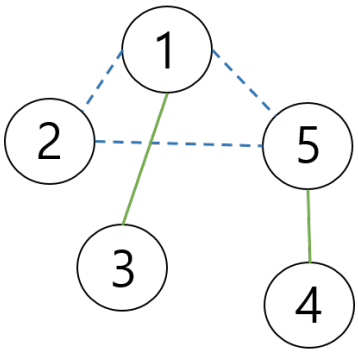

Figure 5.3 : An example of undirected network topology.

$$\mathcal{L}_+ = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{bmatrix}, \ \mathcal{L}_\Delta = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix},$$

$$\text{and } \mathcal{L}_p = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{bmatrix}.$$

The eigenvalues of $\mathcal{L}_p$ are $\{0, 0.70, 1.39, 3.62, 4.30\}$. That is, Assumption 3.2.1 holds with topology.

Taking $K = \begin{bmatrix} -0.6 & -0.2 & 0.4 & 0 \\ -0.4 & -0.8 & -0.4 & 0 \\ -1.1 & 0.8 & -1.1 & -0.5 \\ 0.8 & 0.6 & 0.8 & -2 \end{bmatrix}$ to take the eigenvalues of $(A + KC_\Delta)$ as $\{-1, -2, -3, -4\}$. We take $\epsilon = 0.01$. The stable controller $\mathcal{K}(s)$ be-

comes

$$\dot{\zeta}_i = \begin{bmatrix} -1.20 & -0.60 & -0.12 & -0.10 \\ -0.17 & -2.57 & -0.12 & -0.027 \\ -0.08 & -0.29 & -3.20 & -0.05 \\ -0.11 & -0.31 & -0.10 & -4.13 \end{bmatrix} \zeta_i + \begin{bmatrix} 0.6 & 0.2 & -0.4 & 0 \\ 0.4 & 0.8 & 0.4 & 0 \\ 1.1 & -0.8 & 1.1 & 0.5 \\ -0.8 & -0.6 & -0.8 & 2 \end{bmatrix} (z_i + \hat{z}_i)$$

$$u_i = \begin{bmatrix} 0.0846 & 0.2874 & 0.0164 & -0.0284 \\ 0.0263 & 0.0299 & 0.0013 & 0.0779 \\ -0.0028 & -0.0136 & -0.0885 & 0.0015 \\ 0.0805 & 0.2795 & 0.1034 & 0.0549 \end{bmatrix} \zeta_i.$$

See Figure 5.4 for the simulation result. By generating estimated output relative information using the Luenberger observers, the plots (a),(b),(c) and (d) are represented as the consensus trajectories of output relative information, y_{i1}, y_{i2}, y_{i3} and y_{i4}, respectively. Although the systems are unstable, the overall systems are stable with the dynamic controller.

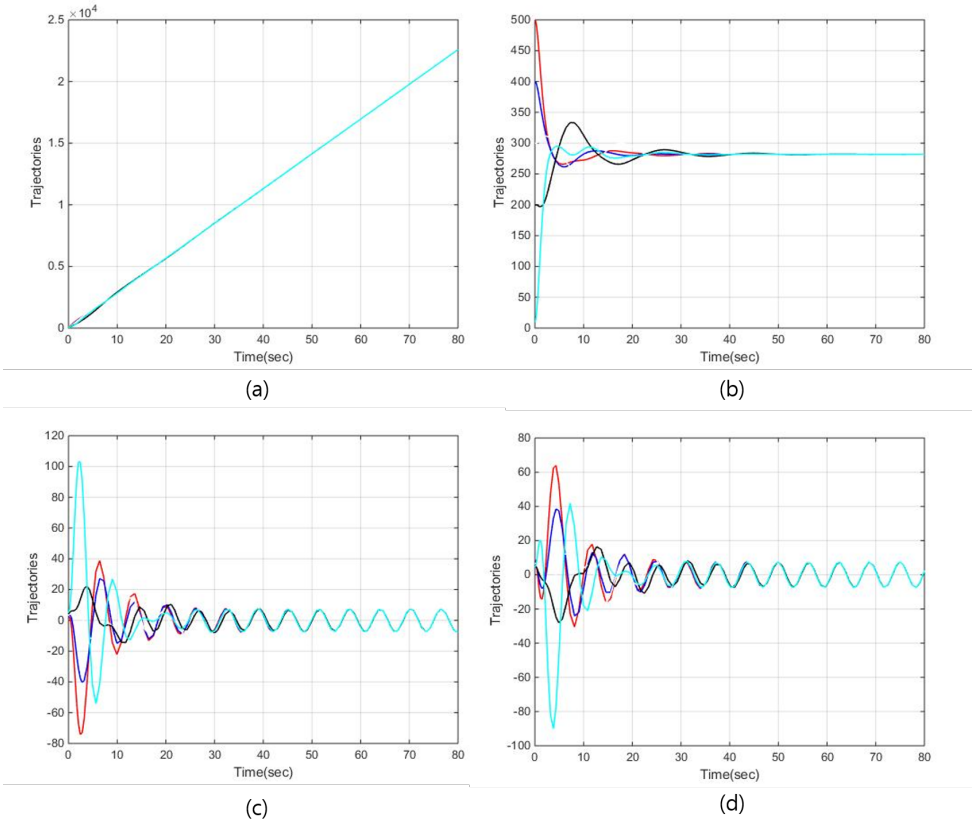

Figure 5.4 : Output trajectories of five individual agents over 80 seconds.

5.4.2 Example 2. Multi Layers

In order to demonstrate the result given in Theorem 5.3.1, consider the individual system (5.4.1) of illustrative Example 1. Consider three commutable Laplacian matrices introduced in example 3 as Laplacian matrices of projection graphs. The edges common to all three graphs are regarded as the sensor network that obtains the absolute output information. It is represented as Laplacian matrix $\mathcal{L}_+$. For the graphs with the remaining edges, the Laplacian matrices of graphs $\mathcal{L}_\Delta^1$, $\mathcal{L}_\Delta^2$, and $\mathcal{L}_\Delta^1$ are as follows. That is, the Laplacian matrices of projection graphs is written as $\mathcal{L}_p^1 = \mathcal{L}_+ \cup \mathcal{L}_\Delta^1$, $\mathcal{L}_p^2 = \mathcal{L}_+ \cup \mathcal{L}_\Delta^2$ and $\mathcal{L}_p^3 = \mathcal{L}_+ \cup \mathcal{L}_\Delta^3$ in Example 3. Refer to Figure 3.1 to help you understand this multilayer network. The Laplacian matrices are given by

$$\mathcal{L}_+ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \mathcal{L}_{\Delta 1} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix},$$

$$\mathcal{L}_{\Delta 2} = \begin{bmatrix} 2 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{bmatrix}, \text{ and } \mathcal{L}_{\Delta 3} = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 2 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix}.$$

The eigenvalues of $\mathcal{L}_p^1$, $\mathcal{L}_p^2$, and $\mathcal{L}_p^3$ are $\{0, 0.38, 1.38, 2.62, 3.62\}$, $\{0.0.38, 1.38, 2.62, 3.62\}$, and $\{0, 3, 3, 5, 5\}$. That is, Assumption 3.2.1 holds with topology. We select the constant τ that meets the conditions of Theorem 5.3.1 such as $\tau = 0.38$. Tak-

ing $K = \begin{bmatrix} -0.6 & -0.2 & 0.4 & 0 \\ -0.4 & -0.8 & -0.4 & 0 \\ -1.1 & 0.8 & -1.1 & -0.5 \\ 0.8 & 0.6 & 0.8 & -2.0 \end{bmatrix}$ to take the eigenvalues of $(A + KC_\Delta)$ as $\{-1, -2, -3, -4\}$. We take $\epsilon = 0.01$. The stable controller $\mathcal{K}(s)$ becomes

$$\dot{\zeta}_i = \begin{bmatrix} -1.26 & -0.96 & -0.16 & -0.16 \\ -0.22 & -2.91 & -0.16 & -0.05 \\ -0.11 & -0.47 & -3.26 & -0.08 \\ -0.14 & -0.50 & -0.14 & -4.12 \end{bmatrix} \zeta_i + \begin{bmatrix} 0. & 0.2 & -0.4 & 0 \\ 0.4 & 0.8 & 0.4 & 0 \\ 1.1 & -0.8 & 1.1 & 0.5 \\ -0.8 & -0.6 & -0.8 & 2 \end{bmatrix} (z_i + \hat{z}_i)$$

$$u_i = \begin{bmatrix} 0.1138 & 0.4577 & 0.0207 & -0.0316 \\ 0.0326 & 0.0496 & 0.0020 & 0.1072 \\ -0.0034 & -0.0173 & -0.1181 & 0.0014 \\ 0.1103 & 0.4503 & 0.1375 & 0.0812 \end{bmatrix} \zeta_i.$$

See Figure 5.5 for the simulation result. We confirm that the consensus problem is solved.

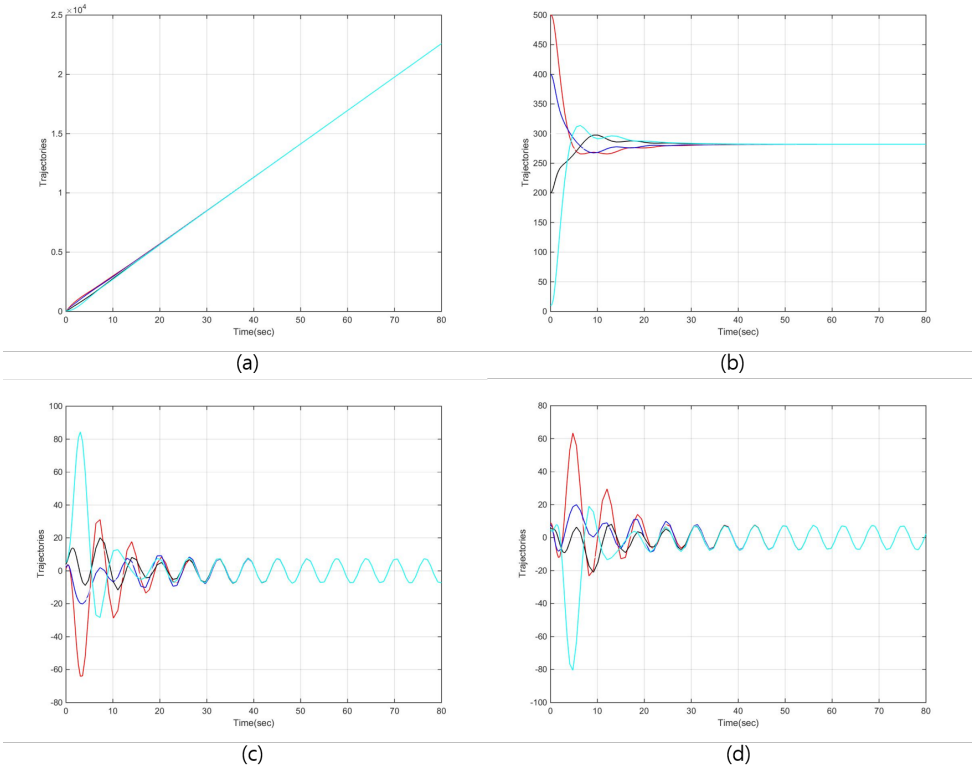

Figure 5.5 : Output trajectories of five individual agents over 80 seconds.

Chapter 6

Conclusion

6.1 Summary and Discussion

We discussed distributed dynamic controller strategy to control networks of MAS so as to achieve consensus. This chapter summarizes the whole contents of the thesis that have been addressed so far, and provides some future works.

Throughout the thesis, we have considered the consensus problem of MAS, which has inherent complexities of networks compared to controlling systems in a single layer. In order to cope with those complexities, we first have presented some definitions and results of graph theory in Chapter 2. However, generally known graph theory is not sufficient to deal with complexities rises in multilayer consensus problem. Thus, some definitions of multilayer network are introduced to solve complexities in Chapter 2.

In order to solve the complexities of network, we have set up the problem with multilayer network considering the characteristics of sensors. We show how to solve the consensus using the dynamic controller using output information

in a different situation from other results of consensus. Also, using concept of projection of graphs, we could solve the problem even if some sensors are not connected. Considering two or more layers, we introduced the structural problems caused by using output information in multilayer network, and presented the conditions for solving them.

6.2 Future Work

Some open problems are left for further study.

- The consensus problem in multilayer network has rarely been addressed, and the result of using output information [Tun16] and the result of designing dynamic controller are few. Also, the concept of projection graphs is rarely introduced to reduce the conditions of connectivity. However, as mentioned above, the structural problems that occur when using the output information in the multilayer network are introduced, and the conditions for solving the problems are presented. The commutativity of Laplacian matrices, which is the condition presented , is a strong condition to apply. In situations where three or more layers are considered, this condition should be relaxed.

- Since the main theorem is a sufficient condition in Chapter 3, it needs to study the necessary condition to find satisfy the necessary and sufficient conditions for consensus with multilayer network.

Appendix

Appendix A

Given the stable controller in Theorem 4.2.1,

$$\tilde{A}_i = \begin{bmatrix} A - \lambda_i BB^T P & \lambda_i BB^T P \\ (1-\lambda_i)BB^T P & A + KC_\Delta + (\lambda_i - 1)BB^T P \end{bmatrix}.$$

The time derivative of Lyapunov function $V_{i,1}$ is obtained as

$$\begin{aligned}
\dot{V}_{i,1} &= \psi_i^T \tilde{A}_i^T \tilde{P} \psi_i + \psi_i^T \tilde{P} \tilde{A}_i \psi_i \\
&= \begin{bmatrix} \psi_{i1}^T \\ \psi_{i2}^T \end{bmatrix} \left[\begin{array}{c:c} A^T P - \lambda_i PBB^T P & \lambda_{\max}^{\frac{1}{2}}(P)(1-\lambda_i)PBB^T Q \\ \hdashline \lambda_i PBB^T P & \begin{array}{c} \lambda_{\max}^{\frac{1}{2}}(P)[(A+KC_\Delta)^T Q \\ +(\lambda_i - 1)PBB^T Q] \end{array} \end{array}\right] \begin{bmatrix} \psi_{i1} & \psi_{i2} \end{bmatrix} \\
&\quad + \begin{bmatrix} \psi_{i1}^T \\ \psi_{i2}^T \end{bmatrix} \left[\begin{array}{c:c} PA - \lambda_i PBB^T P & \lambda_i PBB^T P \\ \hdashline \lambda_{\max}^{\frac{1}{2}}(P)(1-\lambda_i)QBB^T P & \begin{array}{c} \lambda_{\max}^{\frac{1}{2}}(P)[Q(A+KC_\Delta) \\ +(\lambda_i - 1)QBB^T P] \end{array} \end{array}\right] \begin{bmatrix} \psi_{i1} & \psi_{i2} \end{bmatrix} \\
&= -\epsilon \psi_{i1}^T \psi_{i1} + \tau \psi_{i1}^T PBB^T P \psi_{i1} - 2\lambda_i \psi_{i1}^T PBB^T P \psi_{i1} \\
&\quad + (\lambda_i \psi_{i1}^T PBB^T P \psi_{i2} + \lambda_i \psi_{i2}^T PBB^T P \psi_{i1}) - \lambda_{\max}^{\frac{1}{2}}(P) \psi_{i2}^2 \psi_{i2} \\
&\quad - \lambda_{\max}^{\frac{1}{2}}(P)(\lambda_i - 1)[\psi_{i2}^T QBB^T P \psi_{i1} + \psi_{i1}^T PBB^T Q \psi_{i2}] \\
&\quad + \lambda_{\max}^{\frac{1}{2}}(P)(\lambda_i - 1)[\psi_{i2}^T QBB^T P \psi_{i2} + \psi_{i2}^T PBB^T Q \psi_{i2}].
\end{aligned}$$

By using the Young's inequality,
$|\lambda_i \psi_{i1}^T PBB^T P\psi_{i2}| \leq \frac{\tau}{4}\psi_{i1}^T PBB^T P\psi_{i1} + \frac{|\lambda_i|^2}{\tau}|B^T|^2\lambda_{\max}^2(P)|\psi_{i2}|^2$,
$\lambda_{\max}^{\frac{1}{2}}(P)|(\lambda_i - 1)|\psi_{i2}^T QBB^T P\psi_{i1}^T|$
$\leq \frac{\tau}{4}\psi_{i1}^T PBB^T P\psi_{i1} + \lambda_{\max}(P)\frac{|\lambda_i - 1|^2}{\tau}|B^T Q|^2|\psi_{i2}|^2$,
and $\lambda_{\max}^{\frac{1}{2}}(P)|(\lambda_i - 1)\psi_{i2}^T QBB^T P\psi_{i2}| \leq \lambda_{\max}^{\frac{3}{2}}(P)|\lambda_i - 1||QBB^T||\psi_{i2}|^2$,
it is seen that

$$\begin{aligned}\dot{V}_{i,1} &\leq -\epsilon|\psi_{i1}|^2 - \lambda_{\max}^{\frac{1}{2}}(P)|\psi_{i2}|^2 + 2\frac{|\lambda_i|^2}{\tau}|B^T|^2\lambda_{\max}^2(P)|\psi_{i2}|^2 \\ &\quad + 2\lambda_{\max}(P)\frac{|\lambda_i - 1|^2}{\tau}|B^T Q||\psi_{i2}|^2 + 2\lambda_{\max}^{\frac{3}{2}}(P)|\lambda_i - 1||QBB^T||\psi_{i2}|^2 \\ &\leq -\epsilon|\psi_{i1}|^2 - \lambda_{\max}^{\frac{1}{2}}(P)(1 - \delta(\epsilon))|\psi_{i2}|^2\end{aligned}$$

where $\delta(\epsilon) := -2\frac{|\lambda_i|^2}{\tau}|B^T|^2\lambda_{\max}^{\frac{3}{2}}(P) - 2\lambda_{\max}^{\frac{1}{2}}(P)\frac{|\lambda_i - 1|^2}{\tau}|B^T Q|$
$- 2\lambda_{\max}(P)|\lambda_i - 1||QBB^T|$. Thus, $\dot{V}_1$ that is represented as
$\dot{V}_1 = \sum_{i=2}^N \dot{V}_{i,1}$ is given by

$$\dot{V}_1 \leq -\epsilon\sum_{i=2}^N |\psi_{i1}|^2 - \lambda_{\max}^{\frac{1}{2}}(P)\sum_{i=2}^N (1 - \delta(\epsilon))|\psi_{i2}|^2.$$

Appendix B

The time derivative of Lyapunov function V_2 becomes

$$\begin{aligned}\dot{V}_2 &= 2\bar{e}^T B_g^T (I_{N-1} \otimes \tilde{P})\tilde{\psi} + \bar{e}^T (I_N \otimes \begin{bmatrix} -\nu & 0 \\ 0 & 0 \end{bmatrix})\bar{e} \\ &\leq |2\bar{e}^T B_g^T (I_{N-1} \otimes \tilde{P})\tilde{\psi}| - \nu|\tilde{e}|^2.\end{aligned}$$

By using the Young's inequality, for a positive constant α,

$$\begin{aligned}|2\bar{e}^T B_g^T (I_{N-1} \otimes \tilde{P})\tilde{\psi}| &\leq \alpha|B_g^T|^2|Q|^2||\bar{e}|^2 + \frac{\lambda_{\max}(P)}{\alpha}|\tilde{\psi}|^2 \\ &= \alpha|B_g^T|^2|Q|^2||\tilde{e}|^2 + \frac{\lambda_{\max}(P)}{\alpha}|\tilde{\psi}|^2,\end{aligned}$$

It is seen that

$$\dot{V}_2 \leq (\alpha|B_g^T|^2|Q|^2| - \nu)|\tilde{e}|^2 + \frac{\lambda_{\max}(P)}{\alpha}|\tilde{\psi}|^2.$$

Appendix C

Given the stable controller in Theorem 5.3.1, the matrix $\tilde{A}_i$ is represented as

$$\tilde{A}_i = \left[\begin{array}{c:c} A - \pi_i BB^T P & \pi_i BB^T P \\ \hdashline \begin{array}{c} (1-\pi_i)BB^T P - KC_\Delta \\ +\dfrac{1}{\pi_i}(\lambda_i^1 KM_1 C_\Delta + \cdots + \lambda_i^\gamma KM_\gamma C_\Delta) \end{array} & A + KC_\Delta + (\pi_i - 1)BB^T P \end{array}\right].$$

Using the facts that $\lambda_i^1 KM_1 C_\Delta + \lambda_i^1 KM_2 C_\Delta +, \cdots, \lambda_i^1 KM_\gamma C_\Delta = \lambda_i^1 KC_\Delta$, then we obtain that

$$\tilde{A}_i = \left[\begin{array}{c:c} A - \pi_i BB^T P & \pi_i BB^T P \\ \hdashline \begin{array}{c} (1-\pi_i)BB^T P - KC_\Delta \\ +\dfrac{1}{\pi_i}(\lambda_i^1 KC_\Delta + \cdots + (\lambda_i^\gamma - \lambda_i^1)KM_\gamma C_\Delta) \end{array} & A + KC_\Delta + (\pi_i - 1)BB^T P \end{array}\right].$$

Let λ_i^1 as the largest eigenvalue of ith agent among relative layers, i.e., $\pi_i = \lambda_i^1$. Then, the matrix $\tilde{A}_i$ is written as

$$\tilde{A}_i = \left[\begin{array}{c:c} A - \pi_i BB^T P & \pi_i BB^T P \\ \hdashline \begin{array}{c} (1-\pi_i)BB^T P \\ +\dfrac{1}{\pi_i}(\lambda_i^2 - \lambda_i^1)KM_2 C_\Delta \\ \cdots + \dfrac{1}{\pi_i}(\lambda_i^\gamma - \lambda_i^1)KM_\gamma C_\Delta \end{array} & A + KC_\Delta + (\pi_i - 1)BB^T P \end{array}\right].$$

Appendix D

$$\dot{V} \leq -\left(\epsilon - \frac{\lambda_{\max}(P)}{\alpha} - \frac{\beta}{\tau^2}\lambda_{\max}(P)|Y_2^T+,\cdots,+Y_\gamma^T|^2|Q|^2\right)\sum_{i=2}^{N}|\psi_{i1}|^2$$
$$-\lambda_{\max}^{\frac{1}{2}}(P)\sum_{i=2}^{N}\left(\mu - \delta(\epsilon) - \frac{\lambda_{\max}^{\frac{1}{2}}(P)}{\alpha} - \frac{1}{\beta}\right)|\psi_{i2}|^2 - (\nu - \alpha|B_g^T|^2|Q|^2|)|\tilde{e}|^2$$

where $\delta(\epsilon) := 2\frac{|\pi_i|^2}{\tau}|B^T|^2\lambda_{\max}^{\frac{3}{2}}(P) + 2\lambda_{\max}^{\frac{1}{2}}(P)\frac{|\pi_i - 1|^2}{\tau}|B^TQ|$
$+ 2\lambda_{\max}(P)|\pi_i - 1||QBB^T|$.

Using fact that $|Q| \leq \mu\tilde{Q}$ and $\mu = \frac{2}{\beta}$, $\dot{V}$ can be represented as

$$\dot{V} \leq -\left(\epsilon - \frac{\lambda_{\max}(P)}{\alpha} - \frac{4}{\beta\tau^2}\lambda_{\max}(P)|Y_2^T+,\cdots,+Y_\gamma^T|^2\tilde{Q}^2\right)\sum_{i=2}^{N}|\psi_{i1}|^2$$
$$-\lambda_{\max}^{\frac{1}{2}}(P)\sum_{i=2}^{N}\left(\frac{1}{\beta} - \delta(\epsilon) - \frac{\lambda_{\max}^{\frac{1}{2}}(P)}{\alpha}\right)|\psi_{i2}|^2 - (\nu - \alpha\mu^2|B_g^T|^2\tilde{Q}^2)|\tilde{e}|^2$$

where $\delta(\epsilon) := 2\frac{|\pi_i|^2}{\tau}|B^T|^2\lambda_{\max}^{\frac{3}{2}}(P) + \frac{4}{\beta}\lambda_{\max}^{\frac{1}{2}}(P)\frac{|\pi_i - 1|^2}{\tau}|B^T|\tilde{Q}$
$+ \frac{4}{\beta}\lambda_{\max}(P)|\pi_i - 1|BB^T|\tilde{Q}$.

Let $S_1 := \epsilon - \frac{\lambda_{\max}(P)}{\alpha} - \frac{4}{\beta\tau^2}\lambda_{\max}(P)|Y_2^T+,\cdots,+Y_\gamma^T|^2\tilde{Q}^2$, $S_2 := \frac{1}{\beta} - \delta(\epsilon) - \frac{\lambda_{\max}^{\frac{1}{2}}(P)}{\alpha}$, and $S_3 := \nu - \alpha|B_g^T|^2\tilde{Q}^2$. To show that $\dot{V} < 0$, we confirm that $S_i > 0$ for all $i = 1, 2, 3$.

For $S_1 > 0$, $\frac{\lambda_{\max}(P)}{\alpha} < \frac{\epsilon}{2}$ and $\frac{4}{\beta\tau^2}\lambda_{\max}(P)|Y_2^T+,\cdots,+Y_\gamma^T|^2\tilde{Q}^2 < \frac{\epsilon}{2}$. Thus, positive constants β and α satisfy that $\frac{1}{\beta} < \frac{\epsilon\tau^2}{8\lambda_{\max}(P)|Y_2^T+,\cdots,+Y_\gamma^T|^2\tilde{Q}^2}$ and $\frac{1}{\alpha} < \frac{\epsilon}{2\lambda_{\max}(P)}$, respectively. For $S_2 > 0$, $2\frac{|\pi_i|^2}{\tau}|B^T|^2\lambda_{\max}^{\frac{3}{2}}(P) < \frac{1}{4\beta}$, $\frac{\lambda_{\max}^{\frac{1}{2}}(P)}{\alpha} < \frac{1}{4\beta}$, and $\lambda_{\max}^{\frac{1}{2}}(P)\frac{|\pi_i - 1|^2}{\tau}|B^T|\tilde{Q} + \lambda_{\max}(P)|\pi_i - 1|BB^T|\tilde{Q} < \frac{1}{8}$.

The first term can be written that

$$\frac{|\pi_i|^2}{\tau}|B^T|^2\lambda_{\max}^{\frac{3}{2}}(P) < \frac{1}{8\beta} < \frac{\epsilon\tau^2}{64\lambda_{\max}(P)|Y_2^T+,\cdots,+Y_\gamma^T|^2\tilde{Q}^2}.$$

That is, first term is represented as

$$\lambda_{\max}^{\frac{5}{2}}(P) < \frac{\epsilon\tau^3}{64|\pi_i|^2|B^T|^2|Y_2^T+,\cdots,+Y_\gamma^T|^2\tilde{Q}^2}.$$

By Lemma 5.1.1, we obtain that $\lambda_{max}(P) \leq \dfrac{\epsilon^{\frac{1}{2}}}{\tau^{\frac{1}{2}}\lambda_{min}^{\frac{1}{2}}(BB^T)}$. Thus, for first term is positive, we obtain that

$$\epsilon^{\frac{1}{4}} < \frac{\tau^{\frac{17}{4}}\lambda_{min}^{\frac{5}{4}}(BB^T)}{64|\pi_i|^2\tilde{Q}^2|B^T|^2|Y_2^T+,\cdots,+Y_\gamma^T|^2}.$$

From second and third terms, we obtain easily that $\dfrac{1}{\alpha} < \dfrac{1}{4\beta\lambda_{\max}^{\frac{1}{2}}(P)}$, and $\lambda_{\max}^{\frac{1}{2}}(P)\dfrac{|\pi_i-1|^2}{\tau}|B^T| + \lambda_{\max}(P)|\pi_i-1|BB^T| < \dfrac{1}{8\tilde{Q}}$. That is, we can take a sufficiently small ϵ to satisfy that

$$\epsilon^{\frac{1}{4}} < \frac{\tau^{\frac{17}{4}}\lambda_{min}^{\frac{5}{4}}(BB^T)}{64|\pi_i|^2\tilde{Q}^2|B^T|^2|Y_2^T+,\cdots,+Y_\gamma^T|^2}$$

and $\lambda_{\max}^{\frac{1}{2}}(P)\dfrac{|\pi_i-1|^2}{\tau}|B^T| + \lambda_{\max}(P)|\pi_i-1|BB^T| < \dfrac{1}{8\tilde{Q}}$. After constant ϵ is chosen, constant β can be selected. After constant ϵ, and β are selected, constant α is chosen such that $\dfrac{1}{\alpha} < \max\left(\dfrac{\epsilon}{2\lambda_{\max}(P)}, \dfrac{1}{4\beta\lambda_{\max}^{\frac{1}{2}}(P)}\right)$. Then, for $S_3 > 0$, constant ν can be chosen such that $\nu > \alpha|B_g^T|^2|Q|^2$.

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국문초록

CONSENSUS OF LINEAR MULTI-AGENT SYSTEMS WITH MULTILAYER NETWORK USING OUTPUT FEEDBACK CONTROLLER

다층구조에서 출력 되먹임 제어기를 이용한 선형 다개체 시스템의 상태일치

최근 무인 자동차, 군집 로봇제어, 센서 네트워크 등 다양한 공학분야에서 다개체 시스템의 상태일치(consensus)의 필요성이 대두되고 있다. 본 논문은 무인주행자율 자동차인 커넥티드 카에서 영감을 얻었다. 커넥티드 카는 기존의 자동차처럼 센서를 통해 정보를 얻을 뿐 아니라 차량, 또는 인프라간의 무선통신으로 도로에 대한 정보, 교통상황 등 다양한 정보를 주고받아 무인자율주행에 안전성을 높인다. 하지만, 일반자동차에 흔히 볼 수 있는 센서가 측정거리의 한계 등 물리적인 상황에서 제약을 갖고 있듯이, 모든 개체들이 동일한 센서를 가지고 있음에도 불구하고 상황에 따라 모든 개체가 센서의 값을 측정하지 못하는 상황이 발생한다. 이렇게 차량 간 정보를 충분히 얻지 못하는 상황은 무인자율주행에서 큰 위험성을 초래한다. 따라서 센서로 측정하지 못할 때에는 무선통신을 통해 정보를 주고받아 안정적인 무인자율주행을 구현하여야 한다. 그러므로 현실적인 문제상황을 고려한 다개체 시스템 일치에 대한 제어 연구가 필요하다. 최근 다개체 시스템의 상태일치에 대한 많은 연구들은 센서의 특성을 고려하지 않고 모든 센서 또는 개체가 하나의 통신 그래프를 갖고 있다고 가정한 상태로 문제를 해결하고 있으나, 이는 현실적인 한계 상황을 고려하지 않는 것이다. 통신 그래프(communication graph)란 개체간의 연결성의 구조를 말하며, 현실에서 다개체 시스템 상태일치 문제를 풀기 위해서는 센서의 종류에 따른 통신 그래프의 구조성과 연결 여부까

지 고려하여야 한다. 따라서 본 논문에서는 현실에서 다개체 시스템의 상태일치 제어를 할 때 발생할 수 있는 상황을 다층구조(multilayer network)를 이용하여 설정하고 이를 해결하는 방법을 이론적으로 제시한다. 다층구조를 이용하면 구조가 서로 다른 여러 개의 그래프를 분리하여 해석할 수 있다. 본 논문에서는 센서의 종류를 상대적인 거리, 속도 등을 직접 측정을 하는 센서와 GPS처럼 절대적인 개체의 정보를 무선통신을 통하여 제공하는 센서로 구분하고, 이를 각각 상대적인 센서와 절대적인 센서로 칭한다. 다양한 종류의 상대적인 센서는 각기 다른 물리적인 한계성을 갖게 되며 이로 인하여 센서의 종류에 따라 각각 다른 통신 그래프를 갖게 된다. 또한 이러한 물리적인 한계로 인하여 통신 그래프의 연결성(connectivity) 또한 보장할 수 없다. 따라서 연결되지 않은 여러 개의 그래프를 해석하기 위하여 다층구조 개념을 도입하고, 동적 제어기를 제시하여 상태일치를 보였다. 본 논문에서는 일반적인 다개체 시스템을 다층구조로 해석하였고, 동적 제어기를 제시하여 상태일치를 수학적으로 보였다. 논문의 기여하는 점은 다음과 같이 정리할 수 있다.

- 다개체 시스템의 상태일치의 기존의 결과들은 주로 하나의 통신 그래프를 고려하였다. 그러나 본 눈문에서는 연결성이 보장되지 않은 많은 통신 그래프를 고려한다.
- 다개체 시스템에서 다층구조를 이용한 결과 중 2개의 층(layer)만을 고려한 결과가 많다. 본 논문에서는 제시한 가정 하에서, 층의 개수의 제약을 받지 않는다.
- 출력정보를 이용한 동적 제어기를 제시하여 상태일치를 보였다.

이러한 이론적인 결과를 통하여, 실제 다개체 시스템에서 발생할 수 있는 문제점들을 효과적으로 해결 할 수 있을 것으로 기대한다.

주요어: 다개체 시스템, 상태일치, 다층구조, 출력제어이론
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