Master's Thesis

Performance of Gaussian Belief Propagation Decoder in Polar Coding in AWGN Channel

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Abstract

Since Arikan introduced in 2008, Polar Code has become one of the hot topics in channel code field. The best advantages of polar code are a low complexity algorithm for encoding as well as decoding and capacity achievement in binary input discrete memoryless channel (B-DMC). However, the decoding technique in Polar Codes is the controversial topic, in which the Successive Cancellation (SC) decoder in the Arikan's paper only provided for the BEC channel. Nowadays, a lot of researchers are discussing about decoding in Polar Code to adapt for others channel, especially for continuous channel such as AWGN channel. In this case, Belief Propagation decoder from Low-Density Parity Check code (LDPC) code and List Success Cancellation decoder are considered as the potential decoders which provide the higher performance of Bit Error Rate (BER) not only BEC channel, also AWGN channel.

In other ways, some authors take great effort to expand the block-length for Polar Codes from 2^n to ℓ^n with $\ell > 2$ by figure out the characteristic of generator matrices of polar codes (from the transform matrices). From these properties, we apply a generator matrix for block-length $N = 3^n$, which are used as transform matrices for Polar code systems. There exist more than ten candidates of such size 3×3 kernel matrix. In

previous researches, the authors only apply SC decoder for polar code block length $N = 3^n$ in AWGN channel. However, not all of generator matrices in polar code block length $N = 3^n$ achieved the good performance with SC Decoder. In this study, I propose the Gaussian Belief Propagation Decoder which owns some advantages adapting for polar codes of block length $N = 3^n$. This work improves the performance of BER for any cases which obtained the limited results when using SC decoder.

In this thesis, I present the Gaussian Belief Propagation Decoder in Polar Code with the comparison to Belief Propagation decoder and SC decoder. In addition, the results will be the BER performance of Gaussian BP decoder and the capacity of system for each of generator matrices with block length $N = 3^n$ following varies of code rate and SNR.

Key words: Channel coding, Polar Codes, Gaussian Belief Propagation decoder, Belief Propagation decoder.

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Abbreviation

PC	Polar code
BER	Bit Error Rate
Gauss BP	Gaussian Belief Propagation
LDPC	Low Density Parity Check
B-DMC	Binary Discrete Memory less Channel
BEC	Binary Erasure Channel
SC decoder	Success Cancellation decoder
BP decoder	Belief Propagation decoder
List SC decoder	List Successive Cancellation decoder
AWGN	Additive white Gaussian noise

Chapter 1.

Introduction

The information theory topic is developed by Claude E. Shannon (October 1948) and its applications are widely expanded in everywhere; considering as statistical inference, natural language processing, cryptography, neurobiology, the evolution and function of molecular codes, model selection in ecology, thermal physics, quantum computing, plagiarism detection and other forms of data analysis. Besides, the fundamental of information theory was also applied in channel coding - data encoding and error correction techniques.

In data communication, a requirement of Bit Error Rate (BER) is around from 10^{-6} to 10^{-9} which values were achieved by employing channel coding such as using redundancy into the transmission. However, there are a trade of between performance and capacity. Fortunately, Claude E. Shannon said that the data could be transmitted without error as long as the bit rate is smaller than the channel capacity. The authors provided (infinitely long) random codes achieve capacity to absence errors, but these codes could not be used in practice. After more than 50 years, there are many codes achieving almost error-free communications with rate close to channel capacity. For example, Turbo Codes or Low Density Parity Check (LDPC) approached the Shannon limit less than 1 dB [1].

Polar coding, invented by Arikan (2009), was also a channel coding technique to achieve capacity of binary symmetric channel with low encoding and decoding complexity [2]. This method achieved the symmetric capacity (the capacity of the channel with the same probabilities for the inputs) - *I(W)* - of any Binary Discrete Memory less Channel (B-DMC) such as Binary Erasure Channel (BEC). The origin of polar code bases on the Channel Polarization and Success Cancellation (SC) decoder. In Arikan paper [2], the code length of polar code was $N = 2^n$ with generator matrix began from transform matrix $G = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and the performance of BER is based on which types of used decoder. Three common decoders for polar codes are Successive Cancellation (SC) decoder, Belief Propagation (BP) decoder and List Successive Cancellation (List SC) decoder. Among of them, List SC decoder provided the best performance for polar code but its complexity was so high if compared to BP decoder and SC decoder. In addition, the performance of BER using BP decoder was better than using SC decoder but BP decoder has higher complexity than SC decoder for the case $N = 2^n$ of polar code.

However, if the code length of polar code changed from $N = 2^n$ to $N = \ell^n$ (with $\ell \geq 2$), the question was how we design the generator matrix for polar code with block-length $N = \ell^n$ (with $\ell \geq 2$). To solve this, authors [3] expanded code size from 2^n to ℓ^n (with $\ell \geq 2$) and changed the generator matrices in polar code, too (the code size and the number of generator matrices increase simultaneously) [3]. The authors introduced the properties of matrices which defined generator matrices for polar code and the effect of generator matrices to channel polarization in polar code. Another one specified with case $\ell = 3$ and showed that the performance of BER in Polar code with each generator matrix was not the same [4]. Each generator matrix provided differently performance of polar code in AWGN using SC decoder. Some matrices showed the good performance with SC decoder as $G_{427} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ and $G_{463} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ or some matrices indicated the worse performance as $G_{623} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. The next question was how we enhance the performance of BER of polar code with generator matrices which show the bad performance when apply SC decoder. In the original case, BP and List decoder can provide better performance than SC decoder. We also apply List SC decoder and BP decoder in case generator matrices of $N = 3^n$ providing bad performance. However, List SC decoder ($O(LN \log N)$) shows significant complexity if compared with SC decoder ($O(N \log N)$). It seems impossible if we reduce complexity of system when using List SC decoder. Besides, BP decoder with complexity $O(N \log^2 N)$ also provides the performance better than SC decoder. If we apply BP decoder, we could apply the trellis graphs for polar code. The trellis graphs of BP decoder in polar codes base on the generator matrices in polar code. So, each generator matrix will own trellis graph to decode data. With $\ell = 3$, we have 12 transform matrices for generator matrices, and then we need design 12 trellis graphs for each matrix. For each trellis graph, we design equations and formulas of BP for each decoder algorithm. If we increase ℓ, the number of generator matrices increase simultaneously. So, the number of trellis graph as well as algorithm we need to design for BP decoder is very huge. Therefore, applying BP for this case is not optimal solution.

Instead, I provide Gaussian Belief Propagation decoder to solve this problem. It is an efficient distributed iterative algorithm for solving systems of linear equations Ax = b (one of the most fundamental problems in algebra), with countless applications in the mathematical sciences and engineering. In Gaussian BP decoder, the complexity of Gaussian BP decoder is lower than BP decoder. Furthermore, in Gaussian BP decoder, the design the trellis graph for each matrix is not necessary, we only base on the information of generator matrix to figure out estimation of receiving signal.

In this thesis, based on the advantages of Gaussian Belief Propagation decoder, I propose applying it for the specific case when the code length of Polar Code is $N = 3^n$ in AWGN channel. In the next chapter, a brief of introduction about Polar Code and Polar Code with code length $N = 3^n$ will be presented. In chapter 3, I present the method and algorithm of Gaussian BP decoder and compare to BP decoder algorithm in polar code. Chapter 4 will show the results of the performance of Gaussian BP decoder in Polar code and the final chapter (chapter 5) is designed for thesis conclusions.

Chapter 2.

Polar coding: A review

In this chapter, we discuss about polar codes introduced by Arikan [2], the characteristic of matrices to become generator matrices in Polar Code and generator matrices for case $N = 3^n$ in polar code.

2.1 Polar coding for channel coding

Polar codes, introduced by Arıkan in [2], are linear codes which provably achieve the capacity of symmetric B-DMC's with the encoding/decoding complexity of the codes is $O(N \log N)$. We will introduce Original Polar code in [2] through communication over a B-DMC (W) channel.

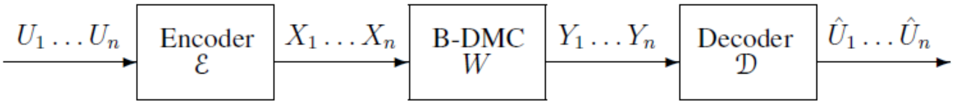

Figure 1. Communication over a B-DMC W

a. Polar Encoder

Polar codes is the linear coding applying the transform $F_2^{\otimes n}$ giving a $2^n \times 2^n$ matrix with the n^{th} Kronecker power of transform matrix $F_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ to the block $N = 2^n$ bits U with generator matrix $G_N = B_N F_2^{\otimes n}$ (B_N is a bit-reversal permutation matrix defined in [2],). Besides, we define code rate $R = K / N$ and the information set I with $I \subset N$ where $K = |I|$, so a source binary vector u_0^{N-1} consisting of K information bits and $N - K$ frozen bits can be mapped a codeword x_0^{N-1}. Codeword will be present by $x_0^{N-1} = u_0^{N-1} G_N$.

b. Chanel Polarization

Channel polarization is an operation which produces N channels $\{W_N^{(i)} : 1 \le i \le N\}$ from N independent copies of a B-DMC W such that the new parallel channels are polarized in the sense that their mutual information is either close to 0 (completely noisy channels) or close to 1 (perfectly noiseless channels). Channel Polarization includes two phases: Channel Combining and Channel Splitting.

- Channel Combining:

Consider from the transform matrix $F_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, the corresponding channel configuration is drawn in figure 2 by combining two independent copies of W.

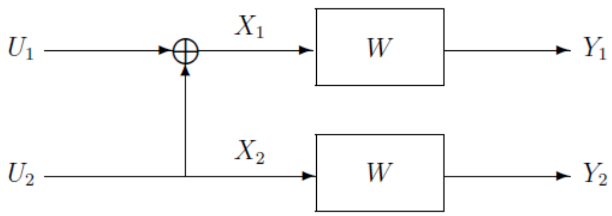

Figure 2. Basic channel transformations

As the basic channel transformations, W^+ and W^- can be defined by the following transition probabilities

$$P_{W^-}(y_1 y_2 \mid u_1) = \sum_{u_2 \in \chi} \frac{1}{2} P(y_1 \mid u_1 \oplus u_2) P(y_2 \mid u_2)$$

$$P_{W^-}(y_1 y_2 u_1 \mid u_2) = \frac{1}{2} P(y_1 \mid u_1 \oplus u_2) P(y_2 \mid u_2)$$

And the evolution of $Z(\mathrm{W})$ satisfies

$$Z(W^-) \leq 2Z(W) - Z(W)^2$$

$$Z(W^+) = Z(W)^2$$

Where equality holds in the first line when W is a binary erasure channel. After combining N independent copies of W in to a channel W_N, the next and final step of channel polarization is to split W_N back into a set of N binary-input channels. Thus, the transition probabilities for W_N channel will be defined

$$W_N^{(i)}(y_1^N, u_1^{i-1} \mid u_i) = \sum_{u_{i+1}^N \in \chi^{N-i}} \frac{1}{2^{N-1}} W_N(y_1^N \mid u_1^N)$$

The channel $W_N^{(i)}$ exhibit a polarization effect in the sense that the fraction of indices i for which the Symmetric Capacity $I(W_N^{(i)})$ is inside the interval $(\delta, 1-\delta)$ goes to zero as N goes to infinity for any fixed $\delta > 0$.

- Channel Splitting:

Considering the channel combining obtained in Fig. 2 the mutual information of channel W_2 can be splitted into two parts using the chain rule of the mutual information.

$$I(u_1^2, y_1^2) = I(u_1, y_1^2) + I(u_1, y_1^2 \mid u_1)$$

And transition probabilities will be presented as Fig 3.

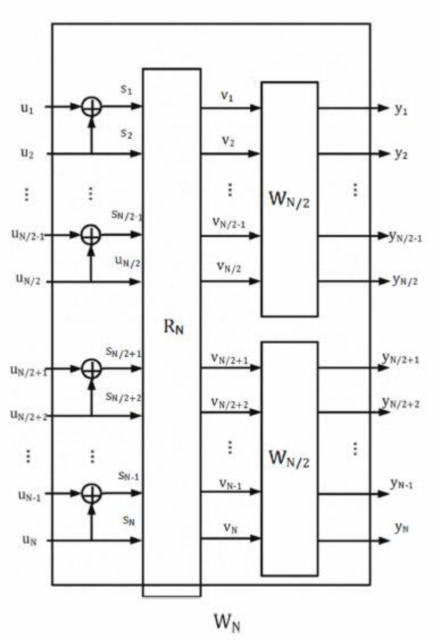

Figure 3. A recursive construction of Wn

c. Polar Decoder

In this part, I will introduce a basis algorithm for polar code decoder which was presented in Arikan [2]. Given an (N, K, A, u_{A^c}) G_N - coset code, we will use a SC decoder that generates its decision $\tilde{u}_1^N$ by computing

$$P_e(N, K, A, u_{A^c}) \le \sum_{i \in A} Z(W_N^{(i)})$$

And the decision of $\tilde{u}_1^N$ will be followed

$$\tilde{u}_1^N = \begin{cases} u_i & \text{if } i \in A^c \\ h_i(y_1^N, \tilde{u}_1^{i-1}) & \text{if } i \in A \end{cases}$$

The decision function $h_i(y_1^N, \tilde{u}_1^{i-1})$ will be defined by the equation

$$h_i(y_1^N, \tilde{u}_1^{i-1}) = \begin{cases} 0, & if \ \dfrac{W_N^{(i)}(y_1^N, \tilde{u}_1^{i-1} \mid 0)}{W_N^{(i)}(y_1^N, \tilde{u}_1^{i-1} \mid 1)} \geq 1 \\ 1, & otherwise \end{cases}$$

2.2 Generator matrices in polar coding

In the previous part, polar code mentioned only for case of $N = 2^n$ and generator matrix will design basing on transform matrix $F_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. However, authors in [3] can expand polar code for case of $N = \ell^n$ with $\ell \geq 2$ by analyzing the properties of polarization matrices to find generator matrix for case of $N = \ell^n$. Generator matrices are presented in Fig.4.

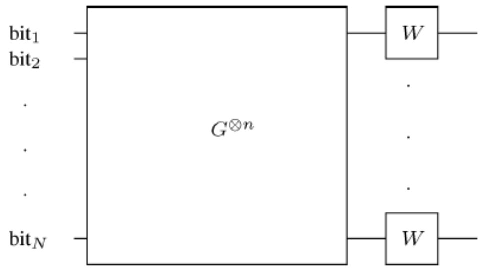

Figure 4. The transform $G^{\otimes n}$ is applied and the resulting vector is transmitted through the channel W

They introduced Channel Transformation for Polarizing Matrices lemma as lemma to defined characteristics of polarizing matrix for any B-DMC *W*.

Lemma 1: For any B-DMC W [3]

i. If G is not upper triangular, then there exists an i for which $W_G^{(i)} \equiv \tilde{W}^{\otimes k}$ or $W_G^{(i)} \equiv W \otimes \tilde{W}^{\otimes k-1}$ for some $k \geq 2$,i.e., *G* is polarzing.

ii. If G is upper triangular, then $W_G^{(i)} \equiv W$ or $W_G^{(i)} \equiv \tilde{W}$ for all $1 \leq i \leq \ell$., *G* is polarizing.

From lemma 1, the first characteristic of a matrix becoming transform matrix for polar code. So, from [3], if G is polarizing, then for any $\delta > 0$

$$\lim_{n \to \infty} \frac{\left| \left\{ i \in \left\{ 1, \dots, \ell^n \right\} : I(W_{G^{\otimes n}}^{(i)}) \in (\delta, 1 - \delta) \right\} \right|}{\ell^n} = 0$$

$$\lim_{n \to \infty} \frac{\left| \left\{ i \in \left\{ 1, \dots, \ell^n \right\} : Z(W_{G^{\otimes n}}^{(i)}) \in (\delta, 1 - \delta) \right\} \right|}{\ell^n} = 0$$

Besides that generator matrix for polar code need to be not upper triangle, [3] shows that that matrices have to be an invertible matrix with 1s on the diagonal to adapt for permutation property. Because of an invertible - matrix corresponding to a permutation, it is always possible to permute its rows and columns to obtain a non-upper triangular matrix with an all-1 diagonal [3].

Furthermore, in [3], authors defined rate of polarization, also called exponent. This is a parameter of generator matrices in polar code to show a meaningful performance measure of polar codes under successive cancellation decoding. Rate of polarization use Partial Distance to calculate by equation.

$$E(G) = \frac{1}{\ell}\sum_{i=1}^{\ell} \log_{\ell} D_i$$

With $E(G)$ is rate of polarization and D_i is calculated by

$$D_i = d_H(g_i, \langle g_{i+1}, \ldots, g_{\ell} \rangle), i = 1, \ldots, \ell - 1$$
$$D_{\ell} = d_H(g_{\ell}, 0).$$

where matrix $G = \left[g_1^T, \ldots, g_{\ell}^T \right]^T$

For example: rate of polarization of matrix $F = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is $E(G) = \frac{1}{2}$. We have

$$E(G) = \frac{1}{2}1\left(\log_2 D_1 + \log_2 D_2\right) = \frac{1}{2}\left(\log_2 1 + \log_2 2\right) = \frac{1}{2}$$

2.3 Generator matrices in case $N = 3^n$

Basing on result in [3], the authors in [4] using all properties about polarization matrix apply for specific case $\ell = 3$. With case $\ell = 3$ -block-length $N = 3^n$, the transform matrices in this case to apply for polar code. In case $\ell = 3$, there are 16 matrices having properties of polarization, so [4] discuss how to choose a good G_3 in around more 16 matrices.

Firstly, we will divide matrices G_3 in three group. Group 1 includes matrices have three "1" in the last row, the group 2 is group of two "1" in the last row and the last group just has one "1" in the last row. Each group has the same way to design $\left\{Z\left(W_N^{(i)}\right)\right\}$.

The second concerned thing is how to design the reliability $\left\{Z\left(W_N^{(i)}\right)\right\}$ for each group. Each group will be concerned and show up the reliability $\left\{Z\left(W_N^{(i)}\right)\right\}$ for each group. For group three "1", we have four matrices have enough properties of polarization. These are in the figure 5:

$$G_3 427 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} G_3 467 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$G_3 627 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} G_3 637 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Figure 5. Matrices in group 1

And the reliability $\left\{ Z\left(W_N^{(i)}\right) \right\}$ of group is defined in [4] following this equation:

$$\begin{aligned} Z(W_{3N}^{(3i-2)}) &= 2Z - Z^2 \\ Z(W_{3N}^{(3i-1)}) &= Z^2 + Z - Z^3 \\ Z(W_{3N}^{(3i)}) &= Z^3 \end{aligned}$$

For the group two "1", we also have 4 matrices G_{623}, G_{463}, G_{425} and G_{475}, however, in this case, this group owning two formulas for the reliability $\left\{ Z\left(W_N^{(i)}\right) \right\}$. Those are:

$$\begin{aligned} Z(W_{3N}^{(3i-2)}) &= Z \\ Z(W_{3N}^{(3i-1)}) &= 2Z - Z^2 \\ Z(W_{3N}^{(3i)}) &= Z^2 \end{aligned} \quad \text{and} \quad \begin{aligned} Z(W_{3N}^{(3i-2)}) &= Z^3 - 3Z^2 + 3Z \\ Z(W_{3N}^{(3i-1)}) &= Z(2Z - Z^2) \\ Z(W_{3N}^{(3i)}) &= Z^2 \end{aligned}$$

$$G_3 461 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} G_3 471 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$G_3 561 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} G_3 571 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Figure 6. Matrices includes one "1"

Fig. 6 demonstrates matrix only including one “1” in the last row and the reliability $\left\{ Z\left(W_N^{(i)}\right)\right\}$ is

$$Z(W_{3N}^{(3i-2)}) = Z$$
$Z(W_{3N}^{(3i-1)}) = 2Z - Z^2$, the same with the group 2.
$$Z(W_{3N}^{(3i)}) = Z^2$$

And the table 1 is the comparison rate of polarization of matrices in three group.

Types of matrices	**Matrices with three 1s in last row**				**Matrices with two 1s in last row**				**Matrices with one 1s in last row**			
Characteristic Of matrices.	**G427**	G467	**G627**	G637	G623	G463	G425	G475	G461	G471	G561	G571
	1 0 0 **0 1 0** **1 1 1**	1 0 0 1 1 0 1 1 1	**1 1 0** **0 1 0** **1 1 1**	1 1 0 0 1 1 1 1 1	1 1 0 0 1 0 0 1 1	1 0 0 1 1 0 0 1 1	1 0 0 0 1 0 1 0 1	1 0 0 1 1 0 1 0 1	1 0 0 1 1 0 0 0 1	1 0 0 1 1 1 0 0 1	1 0 1 1 1 0 0 0 1	1 0 1 1 1 1 0 0 1
Exponent	**0.753**	0.5436	**0.5436**	0.333	0.4206	0.4206	0.4206	0.4206	0.2103	0.2103	0.2103	0.2103

Table 1: Rate of Polarization for block length $N = 3^n$

Summary, the reliability $\left\{ Z\left(W_N^{(i)}\right)\right\}$ for case $N = 3^n$ has three type:

Type 1: $\begin{array}{l} Z(W_{3N}^{(3i-2)}) = 2Z - Z^2 \\ Z(W_{3N}^{(3i-1)}) = Z^2 + Z - Z^3 \\ Z(W_{3N}^{(3i)}) = Z^3 \end{array}$ for group matrices has three "1" in last row. Fig 7 and Fig 8 is the simulation of reliability of type 1

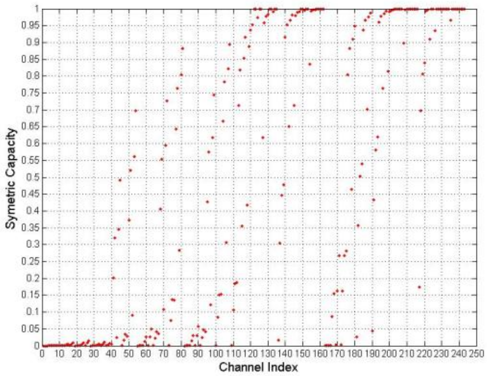

Figure 7. Type 1 of reliability for BEC channel

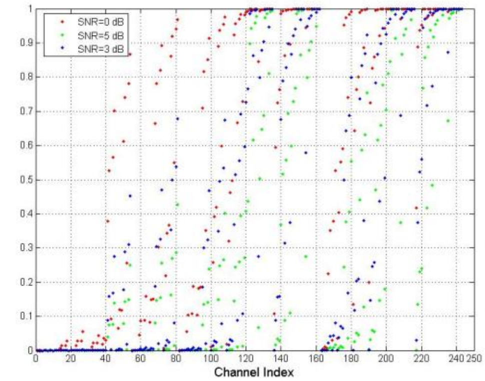

Figure 8. Type 1 of reliability for AWGN channel

Type 2: $$\begin{aligned} Z(W_{3N}^{(3i-2)}) &= Z^3 - 3Z^2 + 3Z \\ Z(W_{3N}^{(3i-1)}) &= Z(2Z - Z^2) \\ Z(W_{3N}^{(3i)}) &= Z^2 \end{aligned}$$ for only group has two "1" in the last row which simulated in Fig.9,10.

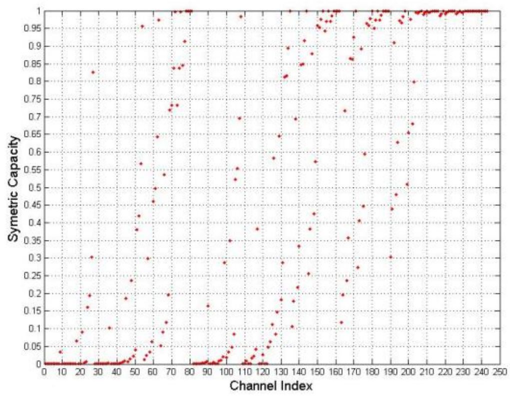

Figure 9. Type 2 of reliability for BEC channel

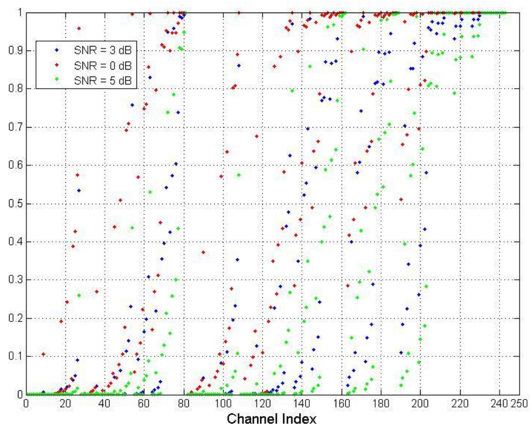

Figure 10. Type 2 of reliability for AWGN channel

Type 3: $$Z(W_{3N}^{(3i-2)}) = Z$$
$$Z(W_{3N}^{(3i-1)}) = 2Z - Z^2$$ for group matrices has two and one "1"
$$Z(W_{3N}^{(3i)}) = Z^2$$

in the last row.

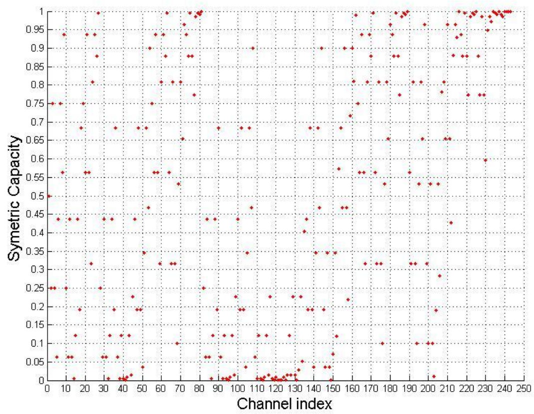

Figure 11. Type3 of reliability for BEC channel

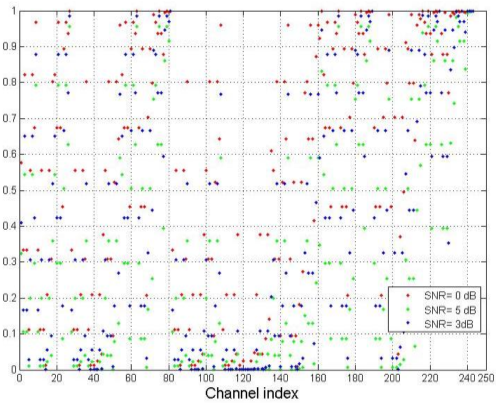

Figure 12. Type 3 of reliability for AWGN channel

From the simulation of each reliability of types, the polarization for each type is not equal. If type 1shows the best polarization which index almost divide in two areas, the type 3 show the worst polarization which capacity of each index is random without any rules.

The author in [4] using SC decoder to evaluate the performance of BER each matrices basing the reliability parameter in AWGN channel. The result will be present in the next part.

Chapter 3.

Gaussian BP decoder in polar coding

In the previous part, we define transform matrix for block length $N = 3^n$ and the reliability parameters for that case. In this part-the main part of this thesis, I propose Gaussian BP decoder for polar code with block length $N = 3^n$. In this part, before presenting Gaussian BP decoder, I will introduce BP decoder to compare with Gaussian BP decoder.

3.1Belief Propagation Decoding in Polar code

Beginning from block length $N = 2^n$, the BP decoder introduced firstly for polar code in [5] and [6]. BP decoder is the kind of decoding algorithm from low-density parity-check (LDPC). Arikan compared performance of SC decoder and BP decoder in [5] and show that BP decoder provides performance better than SC decoder for polar code in BEC channel. In [6], BP decoder still is better than SC decoder in AWGN channel. Figure 7 [7] shows performance of BP decoder, SC decoder and List SC decoder in B-AWGN channel.

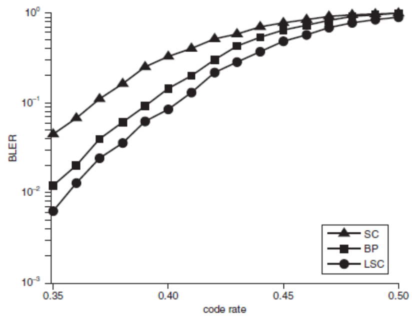

Figure 13. The performance in B-AWGN

The rule to design BP decoder in polar code is based on trellis graph. The trellis graph in polar code is based on encoding graph of polar code. The encoding graph for block-length $N = 2^n$ is sampled in the Fig.8 [8] with $n = 3$.

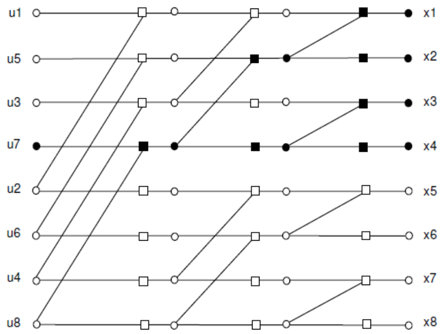

Figure 14. The encoding graph for N=8

We can consider the SC decoding as a special BP scheduling over the standard factor graph of polar codes in a lemma of [7]. This lemma said that using SC decoding for the bit U_i is equivalent to applying BP with the knowledge of $U_o,\ldots,U_{i-1}$ and all other bits unknown (and a uniform prior on them) and the frozen bits belong $U_i,\ldots,U_{N-1}$ then it is in general strictly better than a SC decoder. BP decoder uses a little different encoder than SC decoder. It means that the information bits will be chosen from the Reed Muller (RM) rule [8]. Polar codes differ from RM codes only in the choice of generator vectors. The method of finding the frozen and information bits of RM is based on minimum distance $d_{\min}$ to decision.

So, applying BP algorithm for block-length $N = 3^n$ will be the same method. For this case, besides choosing frozen set by RM algorithm, the trellis graph will design basing on the generator matrices. Therefore, design BP decoder for block length $N = 3^n$ is not same for each generator matrices. I can cover all matrices for this case. Therefore, for each group of generator matrices of $N = 3^n$, I will present BP decoder method for them.

I consider three matrices in each group with $n = 2$ (block length $N = 9$), including:

$$G_{463} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad G_{461} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad G_{627} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Beginning with matrix G_{463} of group 2, firstly, we consider the diagram of operator matrix with input and output in figure 15

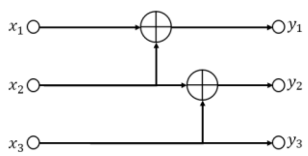

Figure 15. Diagram for G_{463}

So, from diagram in Fig.15, the decoding graph is designed with x_i^{N-1} is decoding bits and $\widehat{u}_i^{N-1}$ is estimation bits as in Fig.16

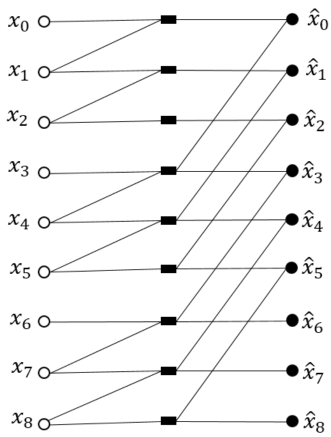

Figure 16. Decoding Graph for G_{463}

Basing on the decoding of Polar code, the formulas of BP decoder will be built from that. With graph of matrices G_{463} we have the formulas for each node as:

$$p^t(n, 3i-2) = f_1(p^{t-1}(n-1, 3i-2), p^{t-1}(n-1, 3i-1))$$
$$p^t(n, 3i-1) = f_2(p^{t-1}(n-1, 3i-1), p^{t-1}(n-1, 3i))$$
$$p^t(n, 3i) = f_3(p^{t-1}(n-1, 3i-1), p^{t-1}(n-1, 3i))$$

With f_1, f_2, f_3 base on the probability message passed node of pervious iteration. The decision will be decided in the last iteration with this equation:

$$\hat{u}_i = \begin{cases} 1 & if \;\; p_i^{last\ t} < 1 \\ 0 & else \end{cases}$$

We can apply the same way with matrices in group 3 and 1. The Fig 17 and 18 is graph for the matrix G_{461} of group 3 and Fig. 19 and 20 is for matrix G_{627} of group 1

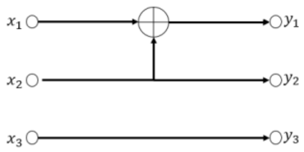

Figure 17. Diagram for G_{461}

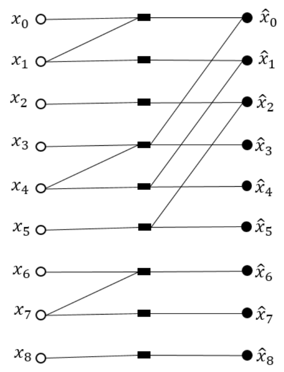

Figure 18. Decoding Graph for G_{461}

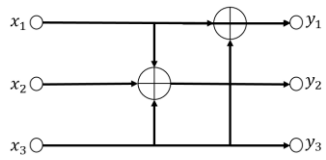

Figure 19. Diagram for G_{627}

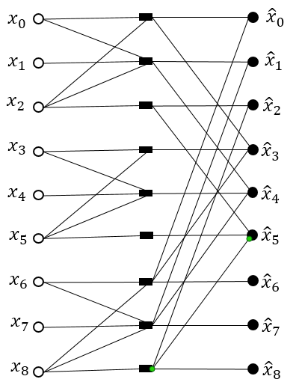

Figure 20. Decoding Graph for G_{627}

3.2 Gaussian Belief Propagation Decoding in Polar code

Gaussian Belief Propagation Decoding is method for linear system $Ax = b$, with A is generator matrix for encoding, x is input data vector and b is encoding vector. This method will show a unique solution if and only if matrix A is full ranks. From the lemma 2, the generator matrices of polar code always are full ranks.

Lemma 2:

If matrix G_ℓ is full rank, the Kronecker product of G_ℓ being $G_\ell^{\otimes n}$ is full rank, too.

Prof: $G_\ell^{\otimes n}$ is full rank if and only if rank of $G_\ell^{\otimes n}$ is total n sum of rank of sub matrices G_ℓ

$$G_\ell^{\otimes n} = \underbrace{G_\ell \otimes G_\ell \otimes G_\ell \ldots \otimes G_\ell}_{n} = \prod_n G_\ell$$

$$rank(G_\ell^{\otimes n}) = rank(\underbrace{G_\ell \otimes G_\ell \otimes G_\ell \ldots \otimes G_\ell}_{n}) = rank(\prod_n G_\ell)$$
$$= \underbrace{rank(G_\ell) \otimes rank(G_\ell) \ldots \otimes rank(G_\ell)}_{n} = n \times rank(G_\ell)$$

So, if we apply the Gaussian BP decoder for polar code, we only get a unique solution decoding vector. The Gaussian BP decoding based on the graph ζ and the associated joint Gaussian probability density function $p(x) \sim N(\mu = A^{-1}b, A^{-1})$. So the target solution $x^* = A^{-1}b$ is equal to $\mu = A^{-1}b$ which is the mean vector of the distribution $p(x)$ basing on μ_i - the marginal mean and P_i inverse variance (sometimes called the precision).

Gaussian BP is a special case of continuous BP, where the underlying distribution is Gaussian. The Gaussian BP update rules by substituting Gaussian distributions into the continuous BP update equations below:

$$m_{ij}(x_j) \propto \int_{x_i} \psi_{ij}(x_i, x_j)\phi_i(x_i) \prod_{k \in N(i) \setminus j} m_{ki}(x_i) dx_i$$

With m_{ij} is the message send from node *i* to node *j*.

$$p(x_i) = \alpha \phi_i(x_i) \prod_{k \in N(i)} m_{ki}(x_i) dx_i$$

From the graph in Fig.20, the messages relevant for the computation of message m_{ij} is shown. So, from the graphical model in fig.20 and equations above, to find the mean μ_i to decision the decoding bits, we have to calculate μ_i from P_i :

$$P_i = \overbrace{P_{ii}}^{\phi_i(x_x)} + \sum_{k \in N(i)} \overbrace{P_{ki}}^{m_{ki}(x_i)}$$

$$\mu_i = P_i^{-1}(\overbrace{P_{ii}\mu_{ii}}^{\phi_i(x_i)} + \sum_{k \in N(i)} \overbrace{P_{ki}\mu_{ki}}^{m_{ki}(x_i)})$$

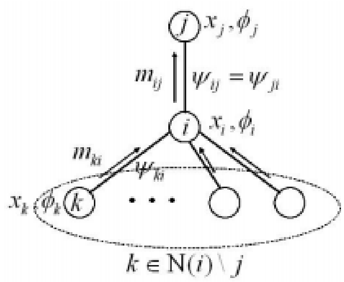

Figure 21. Graphical model: The neighborhood of node i

And the figure 22 is the algorithm to implement the Gaussian BP decoder.

1. *Initialize:*
   - ✓ Set the neighborhood $\mathrm{N}(i)$ to include $\forall k \neq i$ such that $A_{ki} \neq 0$.
   - ✓ Fix the scalars $P_{ii} = A_{ii}$ and $\mu_{ii} = b_i / A_{ii}$, $\forall i$.
   - ✓ Set the initial $i \to \mathrm{N}(i)$ broadcast messages $\tilde{P}_i = 0$ and $\tilde{\mu}_i = 0$.
   - ✓ Set the initial $k \to i, k \in \mathrm{N}(i)$ internal scalars $P_{ki} = 0$ and $\mu_{ki} = 0$.
   - ✓ Set a convergence threshold ϵ.
2. *Iterate:*
   - ✓ Broadcast the aggregated sum messages $\tilde{P}_i = P_{ii} + \sum_{k \in \mathrm{N}(i)} P_{ki}$, $\tilde{\mu}_i = \tilde{P}_i^{-1}(P_{ii}\mu_{ii} + \sum_{k \in \mathrm{N}(i)} P_{ki}\mu_{ki})$, $\forall i$ (under chosen scheduling).
   - ✓ Compute the $i \to j, i \in \mathrm{N}(j)$ internal scalars $P_{ij} = -A_{ij}^2 / (\tilde{P}_i - P_{ji})$, $\mu_{ij} = (\tilde{P}_i \tilde{\mu}_i - P_{ji}\mu_{ji}) / A_{ij}$.
3. *Check:*
   - ✓ If the internal scalars P_{ij} and μ_{ij} did not converge (w.r.t. ϵ), return to Step 2.
   - ✓ Else, continue to Step 4.
4. *Infer:*
   - ✓ Compute the marginal means $\mu_i = (P_{ii}\mu_{ii} + \sum_{k \in \mathrm{N}(i)} P_{ki}\mu_{ki}) / (P_{ii} + \sum_{k \in \mathrm{N}(i)} P_{ki}) = \tilde{\mu}_i$, $\forall i$.
   - (✓ Optionally compute the marginal precisions $P_i = P_{ii} + \sum_{k \in \mathrm{N}(i)} P_{ki} = \tilde{P}_i$)
5. *Solve:*
   - ✓ Find the solution $x_i^* = \mu_i$, $\forall i$.

Figure 22. Algorithm for Gaussian BP decoder

To apply in polar code, the matrix is needed to use as information for Gaussian BP decoder is the Kronecker product of F_ℓ (following the encoder of polar code). Besides, the method to find the information bits index and frozen bits index bases on the RM rules which method use the maximum weight (minimum Hamming distance) of each row of matrix F_ℓ .

The results of the performance of BER in Polar code using Gaussian BP decoder in AWGN channel will be shown in the next part.

Chapter 4.

Simulation

In this chapter, the result of Gaussian BP decoder method in polar code will be presented with results of some matrices in polar code with block length $N = 3^n$. This part includes two sections: section 1 is system model and section 2 focuses to the results of system.

4.1 System model

The system model of thesis is demonstrated in the fig below:

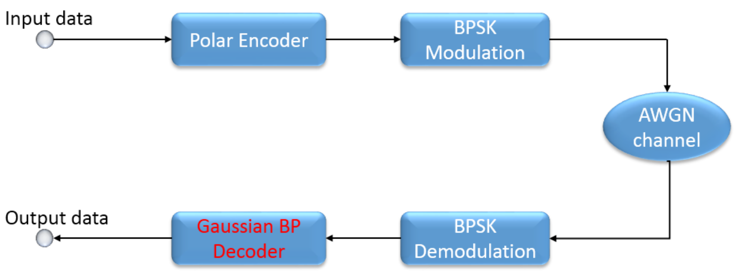

Figure 23. Gaussian BP decoder in Polar code in AWGN channel

Input data become the encoding data after encoding by Polar codes with RM rule to find the information bits set and frozen bits set and it modulated by Binary Phase Shift Keying (BPSK). The encoding data is transmitted through AWGN channel and goes to the receiver. The receiving signal after BPSK demodulation, I use the Gaussian BP decoder method as method I mention in pervious to find the estimation bits.

4.2 Simulation results

In this part, I will show all result of Gaussian BP decoder in polar code for AWGN channel. Firstly, I will show the simulation results about comparison between SC decoder and Gaussian BP decoder in AWGN channel. The results will be shown in 4 matrices, including: G_{623}, G_{463}, G_{427} and G_{461} which matrices get the best results with SC decoder as in [3].

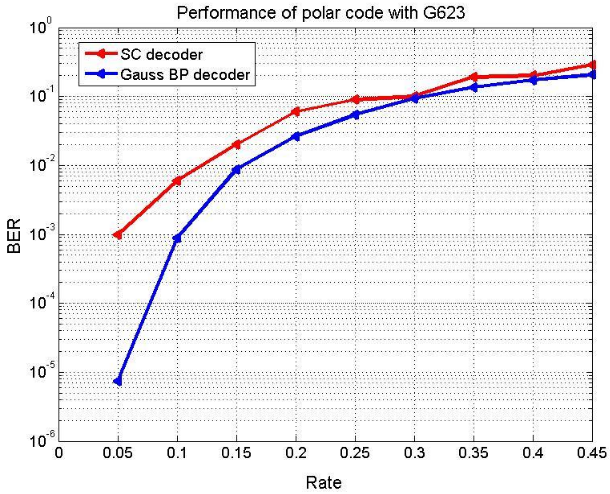

Figure 24. Performance of Gaussian BP decoder in PC for G_{62}

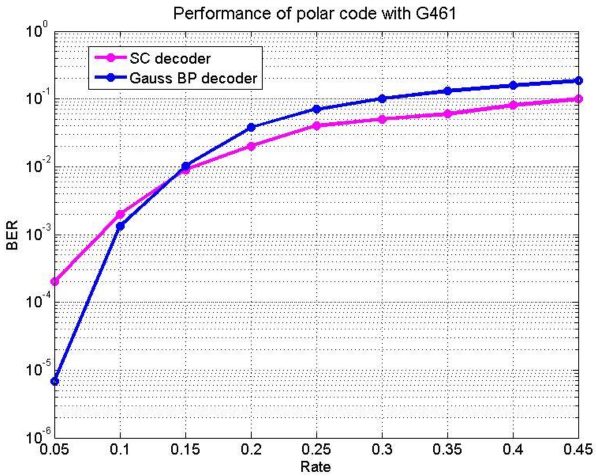

Figure 25. Performance of Gaussian BP decoder in PC for G_{461}

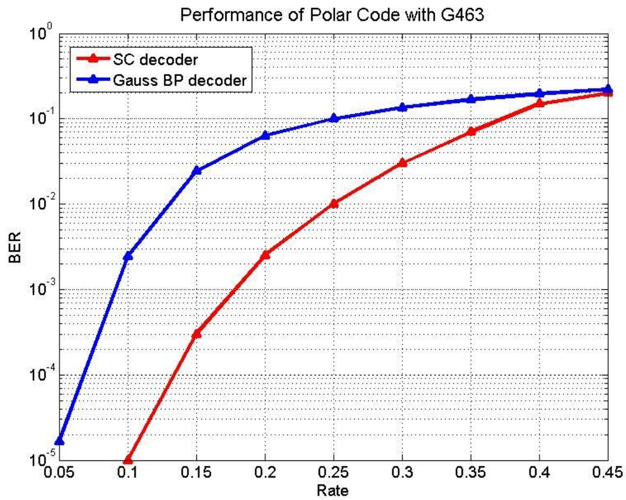

Figure 26. Performance of Gaussian BP decoder in PC for G_{463}

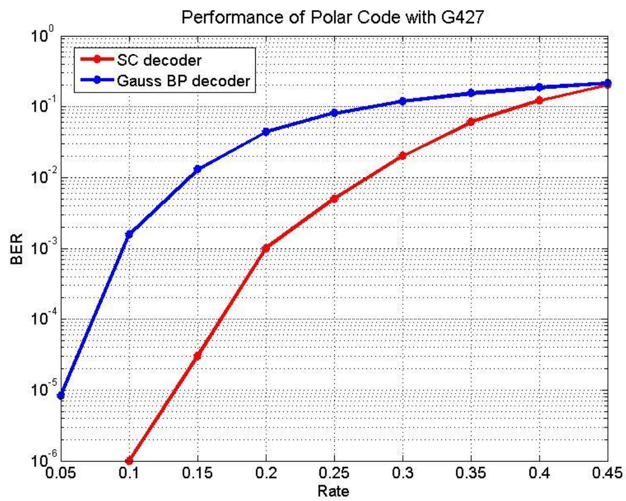

Figure 27. Performance of Gaussian BP decoder in PC for G_{427}

From the simulation result, we see that with matrices G_{623} and G_{461}, Gaussian BP decoder provides the better result than SC decoder. For example, at rate equals 0.15, Gaussian BP can get under 10^{-2} while SC decoder always higher than 10^{-2}. However, Gaussian cannot get the better result than SC decoder in case for G_{463}, G_{427} which matrices get really best results with SC decoder in polar code for AWGN channel.

Next result will be the comparison of the performance of matrices in the same group. In these result, I figure out the matrix can adapt well for polar code in AWGN channel.

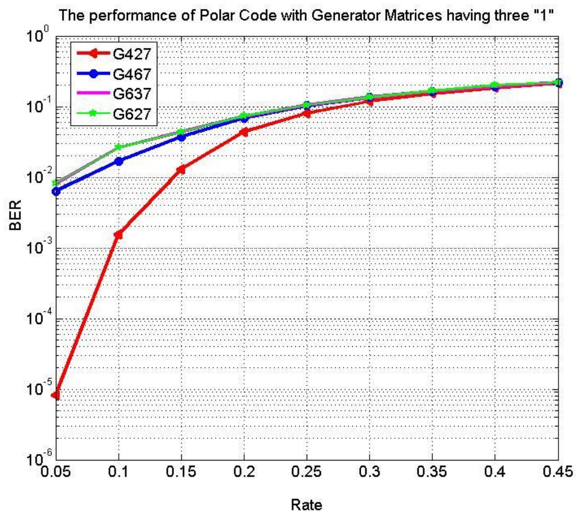

Figure 28. Performance of Gaussian BP decoder for Group 1

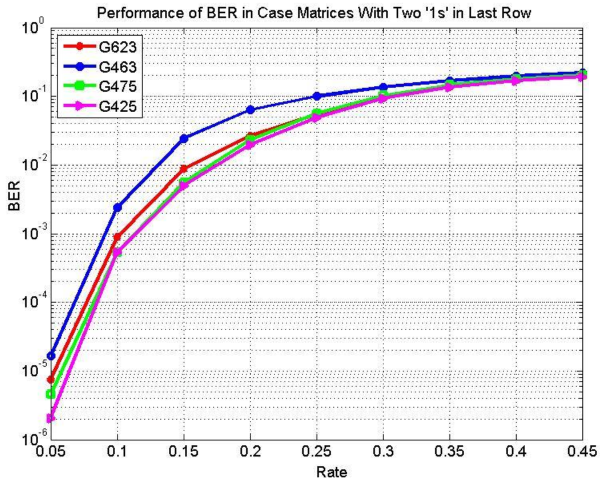

Figure 29. Performance of Gaussian BP decoder for Group 2

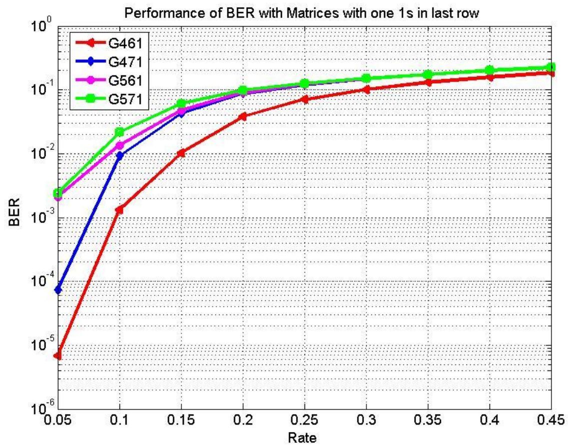

Figure 30. Performance of Gaussian BP decoder for Group 3

From the results above, we easily to know that in group 1, G427 gets the best performance for Gaussian BP decoder. With group

2, G425 and G475 is absolutely close performance and are the best result for group 2. In group 3, G571 is the best with the performance is much better than others.

The last result is the comparison of all matrices in block length $N = 3^n$ with changing SNR. In the result to compare performance of Gaussian BP decoder with changing of SNR, only matrix G_{427} cannot go down even increasing SNR.

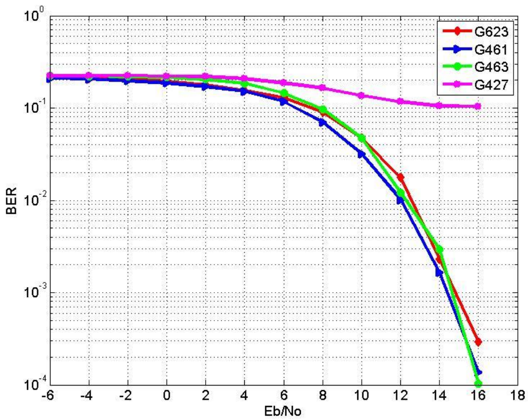

Figure 31. Performance of matrices with SNR

Besides, the capacity of channel in two cases G_{623} and G_{461} are considers in Fig. 22 and Fig.33. Clearly, the higher code rate is, the better capacity the channel gets. As in the part about polar code with block length $N = 3^n$, the type 1 and 2 get the better channel polarization, therefore, the capacity of matrix G_{623} belonging type 2 have the better than the capacity of matrix G_{461} which belonging type 1- the worst polarization of polar code with block length $N = 3^n$. From comparison Fig. 32 and Fig. 33 with same SNR equal 5 dB and same rate be 0.35, the capacity in case of matrix G_{461} only get higher than 0.15 while the capacity of matrix G_{623} is higher than 0.25 (if considering the maximum capacity is 1). In addition, the capacity will increase significantly when increasing SNR of system. With the same rate 0.35, the capacity grows up from 0.05 to 0.15 and to 0.27 when SNR increases from 0 dB to 3 dB and to 5 dB.

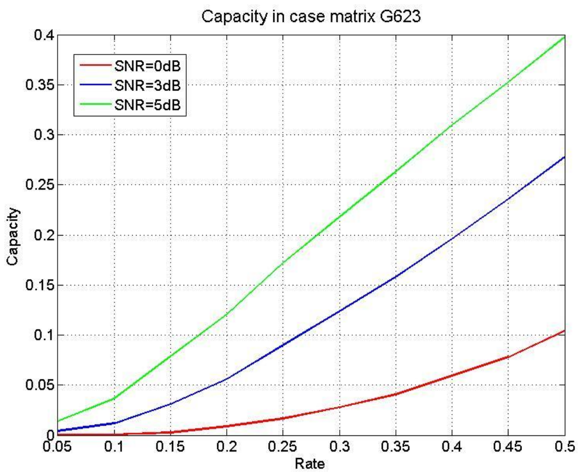

Figure 32: Capacity of matrix G_{623}

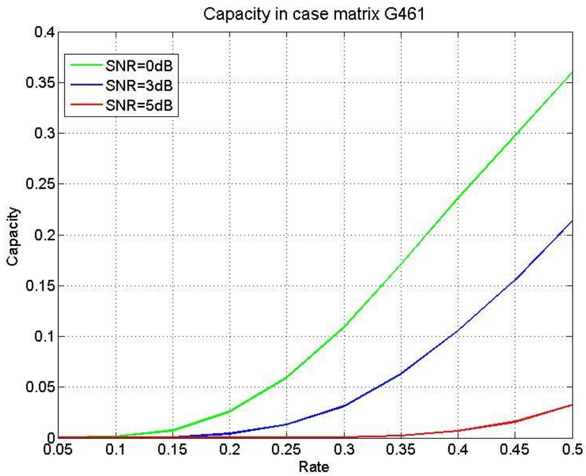

Figure 33: Capacity of matrix G_{461}

Chapter 5.

Conclusion

The decoding method plays an important role in Polar code's achievement in channel coding. There are many researches who decoding in Polar codes beside SC decoder - the decoder in Arikan paper - which provides better performance for polar code such as BP decoder, List SC decoder, even them still work well for polar code with block length $N = 3^n$. However, they own some disadvantage not adapting for this case.

Therefore, in this thesis, I present the new decoding algorithm - Gaussian Belief Propagation decoder - for polar code with block-length $N = 3^n$. If the number of algorithm of BP decoder for polar increase following the number of generator matrices, Gaussian BP decoder provide the technique which use for all matrices in this case and beyond cases with $\ell > 3$. In block-length $N = 3^n$, Gaussian BP decoder can achieve performance for some of matrices, which cannot get the good results with SC decoder in AWGN channel as cases of matrices G_{463} and G_{641}, the performance of theirs is not good when using the SC decoder. In addition, the capacity of polar code system using matrices G_{463} and G_{641} are checked with vary of rate and SNR. Furthermore, the complexity of Gaussian BP decoder is lower than BP decoder and Gaussian can be apply for bigger block-length of polar code.

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