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Equilibria for nonatomic routing games with heterogeneous players

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ABSTRACT

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This paper considers a problem that describes a stable state of a congested network involving a large number of heterogeneous users. In a congested network, if there is no central authority that regulates the network flows, the behavior pattern of the network flows is best represented by a noncooperative game, and the Nash equilibrium becomes the key solution concept. This paper, by extension, introduces random coefficients to a disutility function of the game, in order to reflect the network users' heterogeneous preferences. The existence and uniqueness of equilibrium in games with these player-specific coefficients are to be examined. The existence is established by the Brouwer fixed point theorem. The result on the uniqueness extends earlier work by Milchtaich [20], and shows the possibility that topological conditions for which a network has a unique equilibrium can be relaxed in the proposed games.

Keywords : Nonatomic routing games, Taste heterogeneity, Existence, Uniqueness property
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1. Introduction

Some kinds of networks demonstrate congestion effects. In traffic and telecommunications networks, for example, the performance of each resource of the networks (e.g., a road, a telecommunication line) is affected by the number of users loaded on it; the more users pass through a certain road, the more time and discomfort the road will exhibit. In such a congested network, if there is no central authority that regulates the network flow, each user tries to choose his or her best strategy at every moment. Since the performance of each resource is determined by the overall users' choices, each user's optimal strategy changes depending on the choices of others.

However, after some while, the system appears stable in the sense of a constant load on each resource. This can be explained by an equilibrium state in which each user had chosen his or her best strategy so that there is no incentive to change the strategy unilaterally. Therefore, a congested network involving self-interested users is best represented by the concept of *Nash equilibrium* in a noncooperative game. A wide range of theoretical research on these sort of networks has been ongoing and referred to as *congestion games*.

A congestion game can be more specified by a graph $G = (V, E)$ with a set of origin-destination (o-d) pairs in $V \times V$ and the number of players for each o-d pair, in which each player chooses a route that connects his or her origin to destination in the graph. This kind of congestion game is called a *routing game.* A routing game often involves a large number of people so that each individual solely has a negligible impact on the performance (e.g., time, discomfort) of each resource. In this case, we say that the routing game has *nonatomic players* or call the game *nonatomic routing game*. In 1952, Wardrop [31] suggested the first principle of route choice that describes a stable state of such a congested network accommodating nonatomic users. The principle states:

> *The journey times in all routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.*

The principle describes the stable state as a situation where no user can change to a less costly route; it can be viewed as an instance of the Nash equilibrium in a game with nonatomic players [12]. It became accepted as a sound and simple behavioral principle to define an equilibrium in nonatomic routing game, and has been referred to as *Wardrop equilibrium*, or *traffic equilibrium.*

It is well known that Nash equilibrium does not in general optimize social welfare [27]. Wardrop equilibrium can also be significantly inefficient even in very simple routing games in the sense that the total sum of users' disutility at equilibrium is arbitrarily larger than its socially optimum outcome of minimum average disutility. The famous examples displaying a stark difference between the Wardrop equilibrium and the social optimum are given in Section 1.2, and it implies that the rational behavior of self-interested players may result in a highly-inefficient use of the network.

For that reason, in a congested network that accommodates selfish-routing users, it is important to reproduce equilibrium precisely in order to evaluate alternative future scenarios, which include projects evaluation, tolls optimization, and demands estimation [8]. Furthermore, recent studies have substantiated the need to take account of users' heterogeneity for a more precise network flow prediction, through stated and revealed preference data [7, 13]. Thus, a study on the existence and uniqueness of equilibrium in the extended model is desired.

A great deal of results on the existence and the uniqueness of equilibria in congestion games are known in the specific case where all users are identically influenced by congestion, i.e., the same disutility function over the population was assumed [6, 10, 30, 28]. However, in reality, each user evaluates each alternatives differently. There have been a few studies reflecting the heterogeneity of users. A line of research in a traffic network specified finite user classes [9, 21], and another line of research considered a player-specific utility function in atomic [18, 11, 2] and nonatomic congestion games [29, 16, 23].

In this paper, we reflect the user's taste heterogeneity by adopting a probability distribution on the coefficients of a disutility function, which represents the way a user weighs each attribute of the function. The vector of random coefficients is referred to as *taste parameters*, and we adopt the notation.

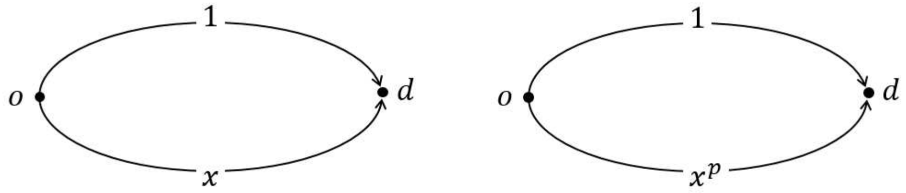
The goal of this paper is to show the existence of equilibrium in the nonatomic routing game with player-specific coefficients, and discuss to what extent of the underlying network topology the uniqueness of equilibrium is guaranteed.

1.1 Motivation and applications

Transportation networks generally accommodate a large number of users, thus nonatomic routing games have been extensively applied to these networks. Recently, Hong et al. [13] conducted an application study on the transit assignment in Seoul metropolitan subway system. From large volumes of smart card data, a distinct difference in passengers' preference was observed. Hence, Hong et al. modelled the system as a nonatomic routing game, and assumed a disutility function which is conditioned on taste parameters. The model considered a Cobb-Douglas disutility function and showed an improvement in passenger flow prediction, which motivated this study for a disutility function in general form.

In addition to transportation networks, nonatomic routing games have been applied to telecommunications network and electric power transmission. Although telecommunications networks are much more centralized than traffic networks, a new concept of networks allows for intelligent routing, thus selfish routing. The Internet provides some example of such an environment [5]. Moreover, electric power transmission has recently emerged as a new application field of nonatomic routing game. In order to avoid high electric power demand peaks, a line of research tries to control the consumer demand pattern [14]. The demand side management of electric power transmission is formulated by a nonatomic routing game, each player of which is a user demanding electric resources during a day.

These aforementioned applications are open to adopt the users' heterogeneity by allowing random coefficients in a disutility function. Al-



(a) Pigou's example (b) A variant of Pigou's example

Figure 1.1: Pigou's example

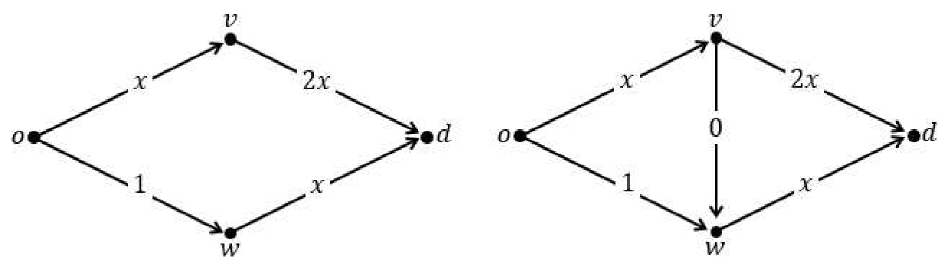
though a disutility function of Cobb-Douglas form showed a good fit in the subway network, with regard to telecommunications network and electric power transmission, a different disutility functional form might be required. Hence, the existence and uniqueness results on a nonatomic routing game with a disutility function in general form, conditioned on taste parameters, can be applied to these applications as well.

1.2 Examples of selfish routing

The examples given in this section demonstrate that the rational behaviors of self-interested users may result in a highly-inefficient use of the network.

Example 1.2.1 [Pigou's example and its variant] Consider a network with a single o-d of demand 1, which is connected by two parallel links like in Figure 1.1 (a). The upper link has a fixed cost $c_1(x) = 1$ independent of the amount of flow on the link x, while the lower link has $c_2(x) = x$.

Let f be the flow on the lower link, then $1-f$ pass through the upper link. The social optimum cost is attained when $(1-f) \times 1 + f \times f$ is minimized, i.e., when $f = 1/2$ with the total sum of the costs of $3/4$.



(a) Initial network (b) Extended network

Figure 1.2: A variant of Braess' Paradox

However, observe that, at the socially optimum state, the user on the upper link can do better if he or she reroute to the lower link. The kind of rerouting continues until the upper link cost is no costly than that of lower link. Therefore the steady state is attained when all users had chosen the lower link, and the total sum of the costs at the equilibrium is 1. In this case, the ratio between the equilibrium cost and the social optimum cost is 4/3.

Moreover, Roughgarden [26] showed that if the lower link cost function is replaced by $c_2(x) = x^p$, where p is some constant then the ratio goes infinity as p becomes larger.

Example 1.2.2 [Braess' Paradox]

Consider a simple network given in Figure 1.2 (a) with a single o-d pair of demand 1. Let f_1 be the amount of flow on the upper route, and f_2 be that on the lower route. Note that the social optimum cost is attained when $f_1 = 3/8$ and $f_2 = 5/8$ with $c_1(f) = 9/8$, $c_2(f) = 13/8$, and the total cost of 1.4375. On the other hand, the system arrives at an equilibrium when both f_1 and f_2 are 0.5, with the same route cost of 3/2 on each route.

Suppose we add a new link of cost 0 which connects from v to w like in Figure 1.2 (b). Let f_3 be the amount of flow on the newly formed route. Then the social optimum is attained when $f_1 = 4/14, f_2 = 5/14, f_3 = 5/14$ with the decreased total cost of 1.393. However, the system arrives at equilibrium when $f_1 = 1/3, f_3 = 2/3$ with the same route cost of 5/3. Note that every user attains increased journey time in the extended network, which displays a counterintuitive result. It demonstrates that a seemingly improvement of a traffic network can rather aggravate the traffic condition.

2. Literature review

Previous literature on the existence and uniqueness of equilibrium in congestion games can be classified by two major models: an atomic version and a nonatomic version. The two models look similar, but they have quite different results.

2.1 The existence of equilibrium

Rosenthal [25] first introduced the concept of congestion games that involve identical atomic players, and proved its existence of pure-strategy Nash equilibrium by showing that every congestion game is a potential game. Monderer and Shapley [19] showed the converse that they are the only atomic games for which an exact potential function exists. Thus, if users are non-identical, the existence of equilibrium is no more guaranteed. Moreover, Ackermann and Skopalik [3] showed it is NP-hard to decide whether an instance of congestion games involving non-identical users has a pure Nash equilibrium. Therefore, a line of research has found subclasses of atomic congestion games for which the existence of a pure Nash equilibrium is guaranteed. Milchtaich [18] considered *singleton congestion games* with player-specific disutility functions, each strategy of which is composed of exactly one resource, and showed that every such game has at least one Nash equilibrium. Ackermann et al. [2] extended the singleton congestion games to *matroid congestion games*, where each player's strategy space is the basis of a matroid, and showed these games always possess a Nash equilibrium.

The existence of pure strategy Nash equilibrium in nonatomic games with player-specific disutility function has been proved by Schmeidler [29], Mas-colell [16], and Rath [23]. In the context of nonatomic routing games, however, a great many of research have considered the case of identical users [6, 10, 30, 28]. A line of research specified finite user classes to consider non-identical users, and showed the existence using variational inequalities [9, 21]. However, the existence of equilibrium in nonatomic routing games with player-specific coefficients has not been proved explicitly. Although Milchtaich [19] proved the existence of equilibria in *nonatomic crowding games* with player-specific disutility function, crowding games are more restrictive than routing games in the assumption that it does not allow overlapping strategies.

This paper shows the existence of equilibrium in nonatomic routing games with a disutility function of random parameters.

2.2 The uniqueness of equilibrium

In atomic congestion games, even if we assume identical users, the congestion effect leads to distinct equilibria. On the contrary, in nonatomic congestion games involving identical users, all users have the same disutility at an equilibrium, and moreover, they do so even in distinct equilibria [1]. However, it does not hold anymore when users are differently affected by congestion [9, 20]. Various conditions have been found that ensure the uniqueness in nonatomic case with heterogeneous users. For instance, a unique equilibrium flow is guaranteed when the users' disutility functions on links are identical up to additive constants, i.e., for each pair of users i and j and each link a the difference $|u_a^i(x) - u_a^j(x)|$ is a constant that does not depend on x [4].

Another approach for the uniqueness has examined the network topology for which a unique equilibrium exists given any assignment of strictly increasing and continuous cost functions. Milchtaich [19] and Konishi [15] showed that, in a two-terminal network with parallel routes, the equilibrium flows are unique for any assignment of strictly increasing and continuous cost functions. Such networks are said to have the *uniqueness property*. Milchtaich [20] continued the work on the uniqueness property and fully characterized topological conditions for which a two-terminal network possesses the uniqueness property. Orda, Rom, and Shimkin [22] and Richman and Shimkin [24] obtained the similar results on atomic games. Meunier [17] extended the work of Milchtaich [20] to a network with multiple o-d pairs, and obtained the topology characterization on ring-structured networks that possess the uniqueness property.

This paper examines that the topological condition for the uniqueness property can be relaxed by assuming a common disutility functional form with varying coefficients.

3. The model

We consider a directed network $G = (V, E)$ with a set of m origin-destination (o-d) pairs in $V \times V$, which is indexed by $\{1, 2, \cdots, m\}$. For each o-d pair $i = 1, \cdots, m$, a demand d_i should be routed. Without a loss of generality, assume the demands are normalized so that the sum is equal to 1 by dividing each demand by the total demand.

Let $\gamma \in \Gamma \subseteq \mathbb{R}^l$ be a vector of taste parameters, where Γ is a closed subset of $\mathbb{R}^l$. Let μ be a probability measure on $\mathcal{B}(\Gamma)$, then $(\Gamma, \mathcal{B}(\Gamma), \mu)$ is a probability space that represents the distribution of users' taste parameters.

Let $\mathcal{R}_i$ be a finite set of possible routes in graph G which connect the o-d pair i, and $\mathcal{R}$ denotes $\cup_{i=1}^{m} \mathcal{R}_i$. Let n be the cardinality of $\mathcal{R}$. Define $\mathcal{N}$ to be the set of unit vectors $\{e_1, \cdots, e_n\}$ in $\mathbb{R}^n$, each of which is an indicator vector of a route in $\mathcal{R}$.

A feasible route flow is a n-dimensional vector $x \in \mathbb{R}^n_+$ such that it satisfies the demand constraints, i.e., $\sum_{e_j \in \mathcal{N}_k} x_j = d_k$, for each $k = 1, \cdots, m$. Let X be the set of all feasible route flows. Note that X is a non-empty, compact, and convex subset of some Euclidean space $\mathbb{R}^n$.

Furthermore, for each route $e_j \in \mathcal{N}$, define a nonnegative and non-decreasing route disutility function $u_j(\cdot, \cdot) : \Gamma \times X \to \mathbb{R}$, which is linear in $\gamma \in \Gamma$ and continuous on X. $u_j(\gamma, x)$ describes the disutility of route $e_j \in \mathcal{N}$ when a player with taste parameters γ traverses the route under the route flow x. Define $u(\cdot, \cdot) : \Gamma \times X \to \mathbb{R}^n$ to be $(u_1(\cdot, \cdot), \cdots, u_n(\cdot, \cdot))$.

Thus, the proposed model is defined by a 5-tuple $(G, d, \mathcal{R}, (\Gamma, \mathcal{B}(\Gamma), \mu), u)$.

4. The existence of equilibria

We define a couple of more notions. For a feasible route flow $x \in X$ and each route $e_j \in \mathcal{N}$, define the set of taste parameters $\Gamma_j(x) \subseteq \Gamma$ such that $\gamma \in \Gamma_j(x)$ if and only if e_j is the unique route of minimum disutility for a player with the taste parameter γ under the flow x.

$$\Gamma_j(x) := \{(\alpha, \beta) \in \Gamma : u_j(\gamma, x) < u_i(\gamma, x), \forall e_i \in \mathcal{N}\}. \tag{4.1}$$

Definition 4.0.1 A *best response* $r(x)$ to a route flow x is a feasible route flow $x' \in X$ where

$$x'_j = \int_{\Gamma_j(x)} d\mu, \text{ for each } e_j \in \mathcal{N}, \tag{4.2}$$

Since the disutility is linear in the taste parameters, $\Gamma_j(x)$ is the intersection of Γ and the open half-spaces of $\{(\alpha, \beta) : u_j(\gamma, x) < u_i(\gamma, x)\}$, $\forall e_i \in \mathcal{N}$. Moreover, note that the boundaries of $\Gamma_j(x)$, which denote the set of taste parameters that have multiple routes of minimum disutility, has a measure of zero.

We define an equilibrium by a stable route flow.

Definition 4.0.2 We say a feasible flow $x \in X$ is an equilibrium if it is a best response to itself, i.e., $x \in r(x)$.

The definition is useful since in reality, it is too costly to keep track of each nonatomic player's whereabouts, and it is the stable route flow with which the system administrator concerns. Moreover, even if the flow may alter, with the unchanged route flow, the users will settle into their optimal routes in the next iteration, due to the best response dynamics.

The existence of equilibrium can be established through the Brouwer fixed point theorem.

Theorem 4.0.3 ***Brouwer*** *Every continuous function from a convex compact subset of a Euclidean space to itself has a fixed point.*

Theorem 4.0.4 *An equilibrium x^* exists, i.e., $x^* \in r(x^*)$.*

Proof Since $u_j(\gamma, \cdot)$ is continuous in X for each $e_j \in \mathcal{N}$, x'_j is continuous in X. Therefore, $r(x)$ is a continuous map from the set of feasible flow X, a convex and compact set, to itself. By the Brouwer fixed point theorem, r has a fixed point. $\square$

Remark 4.0.5 Motivated by Hong et al. [13], this paper extends the model proposed in [13] by introducing extended range of disutility function. The proof of existence is based on the proof in the paper.

4.1 Example

Hong et al. [13] proposed a Cobb-Douglas disutility function in logarithm form, which is conditioned on taste parameters, to predict passenger flows in a subway network. The proposed function is as follow: when a user with taste parameter $\gamma = (\gamma_1, \gamma_2, \gamma_3 = 1 - \gamma_1 - \gamma_2)$ chooses route $e_j \in \mathcal{N}$ under the system state of x, his or her diutility is given by

$$u_j(\gamma, x) = \gamma_1 \ln s_j + \gamma_2 \ln t_j + (1 - \gamma_1 - \gamma_2) \ln l_j(x), \qquad (4.3)$$

where s_j is the attribute for the number of transfers of route e_j, t_j is for the in-vehicle time of route e_j, and $l_j(x)$ is for average passenger load of route e_j when the current route flow is x and $\gamma \geqslant 0$.

Since trains in subway networks run on the timetable, the in-vehicle time is not significantly affected by congestion level. Therefore, Hong et al. took account of the congestion effect explicitly in the disutility function through average passenger load. The congestion level on each route is calculated by its average load on the route:

$$l_j(x) = 1/|n_j| \sum_{e_k \in \mathcal{N}} \delta_{j,k} x_k,$$

where n_j is the number of links on route e_j and $\delta_{j,k}$ is the number of overlapping links between e_j and e_k.

Corollary 4.1.1 *A nonatomic routing game with the disutility function (4.3) has at least one equilibrium.*

Proof. By Theorem 4.0.4, it suffices to check if $u_j(\gamma, \cdot)$ is continuous on X for all $\gamma \in \Gamma$ and $e_j \in N$. For $\epsilon > 0$, consider the set

$$|u_j(\gamma, x) - u_j(\gamma, x')| = |\gamma_3(\ln l_j(x) - \ln l_j(x'))| < \epsilon. \tag{4.4}$$

Then, $|\ln \delta \cdot x - \ln \delta \cdot x'| < \epsilon/\gamma_3$. Since $\ln(\cdot)$ is continuous on $(0, \infty)$, for any $\epsilon' > 0$, there exists $\eta > 0$ such that $|\ln \delta \cdot x - \ln \delta \cdot x'| < \epsilon'$, whenever $|\delta \cdot x - \delta \cdot x'| < \eta$. By the Cauchy-Schwarz inequality, $\|x - x'\|_2 < \eta/\|\delta\|_2$ implies (4.4). □

5. The uniqueness property

5.1 Milchtaich's result

We continue to define equilibrium according to Definition **??** so that a unique equilibrium means a unique system state in a steady state, and furthermore, a unique equilibrium cost.

Milchtaich [19] and Konishi [15] showed that a congestion game in two-terminal networks with parallel routes has a unique equilibrium for any assignment of strictly increasing and continuous link-based cost functions. Such networks in which a unique equilibrium is guaranteed are said to possess the uniqueness property.

By extension, Milchtaich [20] showed all five networks in Figure 5.1, as well as all networks obtained by connecting several of them in series have the uniqueness property. Furthermore, he examined that these are essentially the only two-terminal networks with the uniqueness property.

He showed that any two-terminal network that cannot be obtained by connecting several of the networks in Figure 5.1 in series has one of networks in Figure 5.2 *embedded in the wide sense* in it.

He defined the notion of being embedded in a wide sense as follows:

Definition 5.1.1 A network G' is *embedded in the wide sense* in a network G'' if G'' can be obtained from G' by applying the following operations any number of times in any order:

1. The *subdivision of an edge* with a new vertex in the middle.

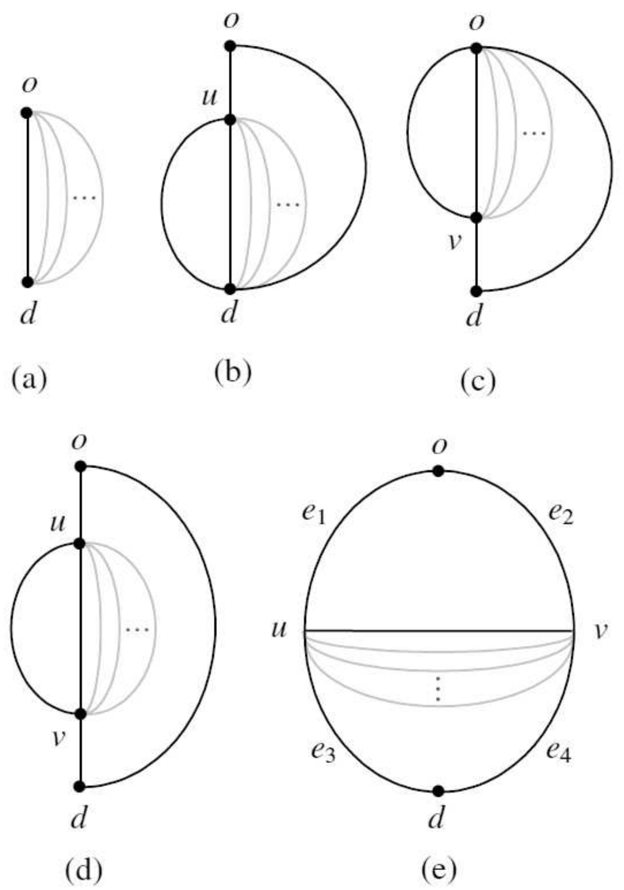


Figure 5.1: Networks with the uniqueness property [19]

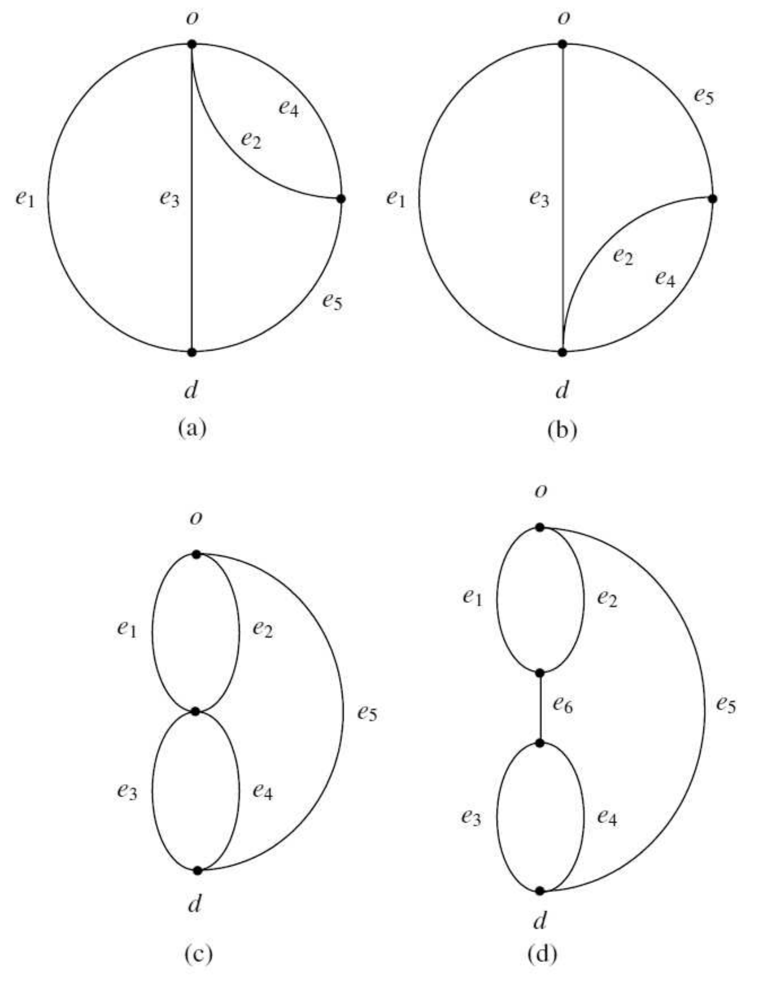


Figure 5.2: The forbidden networks [19]

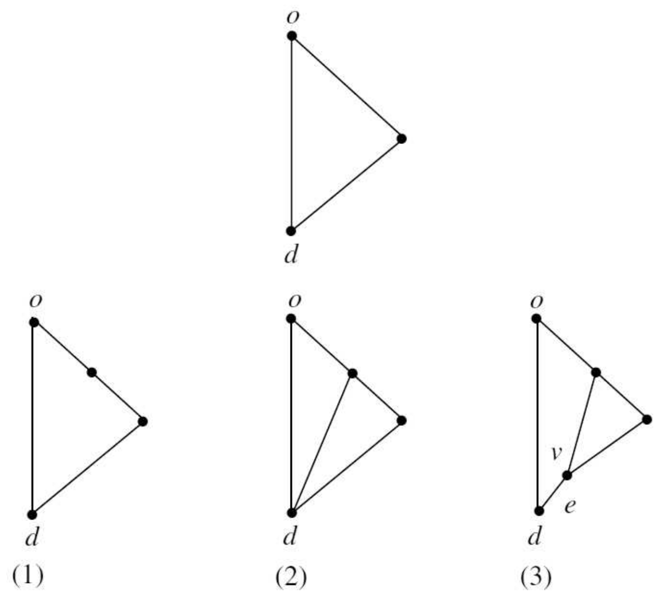


Figure 5.3: Embedding [19]

2. The *addition of a new edge* joining two existing vertices.

3. The *subdivision of a terminal vertex, o or d*: addition of a new vertex v and a new edge e joining the terminal vertex with v, where v replaces the terminal vertex as the end vertex in two or more edges originally incident with the terminal vertex.

For example, a network in the upper side of Figure 5.3 is embedded in each of networks (1), (2), and (3). The network (1) can be obtained by the subdivision of a edge, while the network (2) being obtained from the network (1) through the addition of a new edge. The network (3) is attained by the subdivision of a terminal vertex of the network (2).

He proved that if a network G does not have the uniqueness property, the same is true for every network G' which has G embedded in the wide sense in it.

Therefore, by showing that each of the forbidden networks does not have the uniqueness property, he concluded that any two-terminal network that cannot be obtained by connecting several of the nearly-parallel networks does not have the uniqueness property, and the forbidden networks are the minimal networks for which multiple equilibrium costs are possible.

5.2 Networks with the uniqueness property

In this paper, we restrict the range of disutility functions such that it has a common functional form across the population and is conditioned on player-specific parameters. The goal is to find a relaxed topological condition for which a network has the uniqueness property, with the restriction on disutility functions.

Milchtaich [19, 20] considered additive cost functions such that a route cost is given by the sum of the link costs of which it consists. In order to extend his results, from now on, we deal with additive disutility functions and calculate the cost of a route by the sum of the costs of the links constituting the route. Suppose that the route set $\mathcal{R} \subseteq 2^E$ is indexed with $\{1, \cdots, |\mathcal{R}|\}$. Let $f \in \mathbb{R}^{|E|}$ be a link-flow vector that expresses the amount of flow on each link, i.e., f_l means the flow size on link $l \in E$. Then a link-flow vector can be derived from a system state $x \in X$ such that $f_l = \sum_{j \ni l} x_j$.

Now, we consider the case where the common disutility functional form has one fixed attribute and one variable attribute. The disutility of route j of a player with parameter $\gamma \in [0, 1]$ under a system state x

is given by

$$u_j(\gamma, x) = \gamma a_j + (1-\gamma)c_j(x),$$

where $a_j = \sum_{l \in j} a_l$, a_l is the fixed attribute value of link $l \in E$, and $c_j(x) = \sum_{l \in j} c(f_l)$, $c(f_l)$ is the variable attribute of link $l \in E$. Here $c(\cdot)$ is a strictly increasing continuous function and f is derived from the system state x.

Without loss of generality, assume that there is no pair of routes with the same fixed attribute value, and that routes are indexed with numbers $1, \cdots, n$, in descending order of fixed attribute value, i.e., for $i > j$, $a_i < a_j$.

Lemma 5.2.1 *At equilibrium x', if $x'_k > 0$ then*

$$c_k(x') < c_j(x'), \quad \textit{for all } j > k \textit{ with } x'_j > 0,$$

and the preference space of unit interval $[0,1]$ is partitioned into at most n subintervals $\Gamma_k \subseteq [0,1]$, $k = 1, \cdots, n$, each of which is the set of γ's on route k, and the order of subintervals accords with the index order.

Proof. The first statement holds since in equilibria all used routes have to be Pareto-efficient. Figure 5.4 illustrates disutility value on each route according to γ.

Because of the first statement in the lemma, for the set of routes with positive flows, the slope of each corresponding line gets smaller as the index increases. Since a function that takes minimum values among a given set of affine functions is concave, the slope of the function should decrease as γ gets larger. Therefore, there are at most n subintervals where each line has the minimum value and the order of subintervals

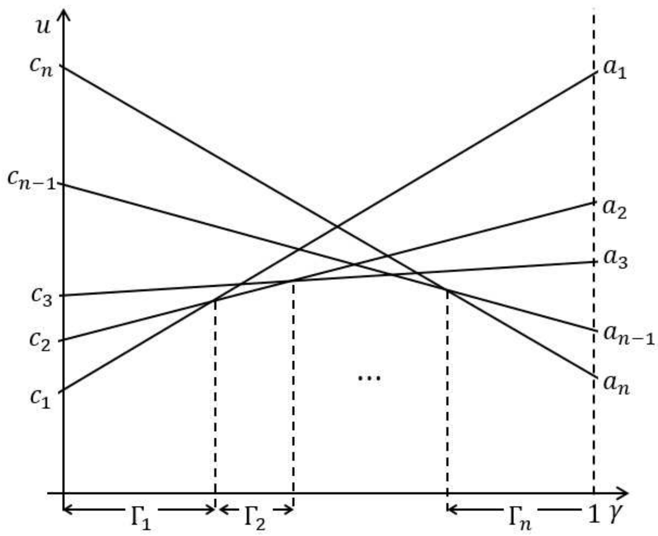


Figure 5.4: Disutility values according to γ

should accords with the index. □

Now assume that an equilibrium point x' to be positive, i.e., $x' > 0$. Then, by Lemma 5.2.1 the unit interval $[0, 1]$ is subdivided by n subintervals, $\Gamma_k = [\gamma_{k-1,k}, \gamma_{k,k+1}]$, $k = 1, \cdots, n$, where $\gamma_{0,1} = 0$ and $\gamma_{n,n+1} = 1$. Define $\gamma_{k,k+1}$ as the splitting point between Γ_k and Γ_{k+1}. In order to determine Γ_k we only need to check $\gamma_{k-1,k}$ and $\gamma_{k,k+1}$.

Let $c_{k,k+1}(x)$ be the difference in the variable attribute of route k and $k+1$ and $a_{k,k+1}$ be the that of the fixed attribute. Note that

$$\gamma_{k,k+1}(x) = \frac{c_{k+1,k}(x)}{a_{k,k+1} + c_{k+1,k}(x)}$$

is an increasing function in $c_{k+1,k}(x)$, since $a_{k,k+1} > 0$.

Using the above observation, we show that the network (a) in figure 5.2 no more has multiple equilibria if we assume the same functional

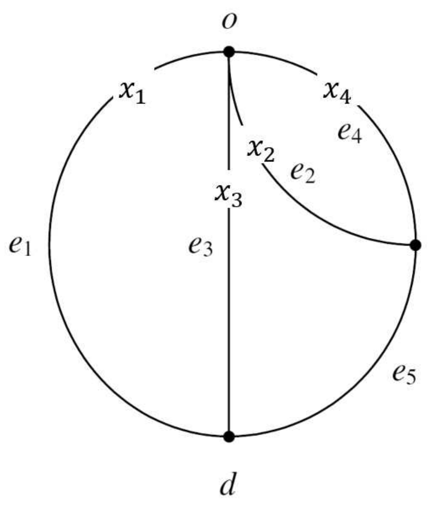


Figure 5.5: Flow on Network (a): case I

form across the population.

Theorem 5.2.2 *Consider a nonatomic routing game specified by any strictly increasing continuous function c for a variable attribute and a probability distribution of taste parameters. The game has a unique equilibrium in network (a) in Figure 5.2 and all network homeomorphic to it.*

Proof. Let x and x' be two distinct equilibria. Observe that only a route k with the largest fixed variable among the overlapping routes can have zero flow at equilibrium. That's because, if an independent route has zero flow at some equilibrium, the amount of flow on the route at any equilibrium has to be zero as well. Therefore, we can exclude the route from the route set, and we have unique equilibrium since the resulting network is nearly parallel. Assume that there is no such an independent route with zero flow. Then, there exist only one equilibrium that has $x_k = 0$; since the resulting network is a nearly parallel network. Therefore, we can assume that $x > 0$ without loss of generality since if there are distinct equilibria, at least one of them should be positive.

Assume $x > 0$ and the routes are indexed in descending order of their fixed attribute values. Let $\Delta x = x' - x$. There are two cases to be checked: a case where route 1 is an independent route that does not have an overlapping route as given in figure 5.5, and another case in which route 1 has an overlapping route as given in 5.6. We first check the case I.

(i) $\Delta x_1 > 0$. Then by the previous lemma, $\gamma_{1,2}$ has to be increased, hence $\Delta c_{2,1} > 0$, i.e., $\Delta c_2 > \Delta c_1 > 0$. In order for c_2 to be increased, one of the following cases holds:

i-a) Both $\Delta x_2, \Delta x_4 > 0$. Then x_3 has to be decreased by the amount of $\Delta(x_1+x_2+x_4)$. Since $\Delta x_2 > 0$ and $\Delta\gamma_{1,2} > 0$, $\gamma_{2,3}$ should increase, i.e., $\Delta c_3 > \Delta c_2 > 0$. Then x_3 has to be increased, which is a contradiction.

i-b) Only $\Delta x_2 > 0$. Then $\Delta x_2 + \Delta x_4/2 > 0$, i.e., $|\Delta x_4| < 2\Delta x_2$. Since $\Delta x_2 > 0$ and $\Delta\gamma_{1,2} > 0, \Delta c_3 > \Delta c_2 > 0$. Then $\Delta x_3 > 0$, meaning $\Delta c_2 < \Delta c_3 < \Delta c_4$, which is a contradiction.

i-c) Only $\Delta x_4 > 0$. Then $1/2\Delta x_4 > |\Delta x_2|$. Since $\Delta x_4 > 0$, $\Delta c_4 - \Delta c_3 < 0$. Therefore $\Delta c_2 < \Delta c_4 < \Delta c_3$ and $\Delta x_2 = -(\Delta x_1 + \Delta x_3 + \Delta x_4)$ a contradiction.

(ii) $\Delta x_1 < 0$. The similar argument holds.

Therefore $\Delta x_1 = 0$. If we subtract x_1 from demand and exclude route 1 from the network, it results in the network with the uniqueness property. So the equilibrium is unique.

Now we check case II in which route 1 overlaps with another route.

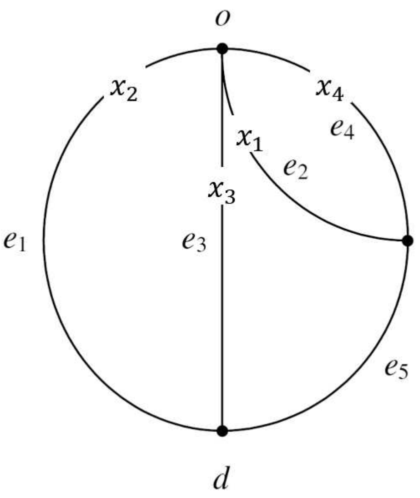


Figure 5.6: Flow on Network (a): case II

(i) $\Delta x_1 > 0$. Then by the previous lemma, $\Delta\gamma_{1,2} > 0$, hence $\Delta c_2 > \Delta c_1$.

i-a) $\Delta c_1 > 0$. Then $\Delta c_2 > 0$, i.e., $x_2 > 0$. Therefore $\Delta c_3 > \Delta c_2 > 0$, meaning $\Delta c_4 > 0$, which is a contradiction.

i-b) $\Delta c_1 < 0$. Then $\Delta x_4 < 0$, and $|\Delta x_4| > 2\Delta x_1$. $\Delta c_3 < \Delta c_4 < 0$, meaning $\Delta x_3 < 0$. $\Delta c_2 < \Delta c_3 < 0$, meaning $\Delta x_2 < 0$. Then $\Delta x_1 = -(\Delta x_2 + \Delta x_3 + \Delta x_4)$, which is a contradiction.

(ii) $\Delta x_1 < 0$. The similar argument holds.

Therefore $\Delta x_1 = 0$. If we subtract x_1 from demand and exclude route 1 from the network, it results in the network with the uniqueness property. So the equilibrium is unique. Milchtaich [20] showed that graph homeomorphism preserves the uniqueness property, all networks homeomorphic to the network Figure 5.2 (a) also possess the uniqueness property. □

6. Conclusion and further research

This paper considered a nonatomic routing game with a disutility function conditioned on player-specific coefficients. The proposed game will be useful as a huge amount of revealed preference (RP) data becomes available, such as smart card data and GPS data. The distribution of taste parameters is expected to be inferred more accurately, and moreover, is likely to be the maximal information about the user's preference heterogeneity.

This paper established the existence of equilibrium in the proposed game by the Brouwer fixed point theorem.

With regard to the uniqueness of equilibrium, this paper considered a disutility function with two attributes, one of which is for fixed attribute and the other is for variable attribute. It is a meaningful result since the two-attributes model is widely used in discrete choice models in the context of willingness-to-pay or value-of-time. A disutility function with more than two attributes, such as the disutility function for a subway network described in Section 4.1, is worthwhile to study.

This paper examines the possibility that the restrictive but reasonable assumption on disutility function, which assumes a common functional form with random coefficients, results in a wider range of networks having the uniqueness property. It is shown that the networks in Figure 5.2 (a) and (b) have the uniqueness property under the assumption. There are a couple of more things in question: to what extent of the networks having Figure 5.2 (a) or (b) embedded in it is the uniqueness property guaranteed? does the network in Figure 5.2 (d) have the uniqueness property?

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초 록

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본 논문은 서로 다른 특성의 수 많은 이용자들을 갖는 혼잡 네트워크의 안정적인 상태를 표현하는 문제를 고려한다. 혼잡 네트워크에서 흐름을 제어하는 중앙 관리자가 없으면, 네트워크 흐름의 양상은 비협조 게임으로 가장 잘 표현이 되고, 내쉬 균형이 주요한 해가 되게 된다. 본 논문에서는 네트워크 이용자들의 다양성을 반영하기 위하여 비효용 함수의 계수들을 확률변수라고 가정한다. 본 논문은 승객마다 서로 다른 계수들을 갖는 게임의 균형의 존재성과 유일성에 대해서 확인한다. 존재성은 Brouwer의 고정점 정리를 통해 보여진다. 유일성에 관해서는 Milchtaich [20]가 제시한 균형의 유일성이 보장되는 네트워크 구조에 관한 조건이 본 논문에 제시된 게임에서는 완화될 수 있음을 보인다.

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