Comparative Statics under Uncertainty for a Decision Model with Two Choice Variables

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This paper considers a class of monotone comparative statics problems that satisfies two distinct kinds of constraints on the decision model, where a single decision-maker has two-dimensional choice set. By using a general model we can present a set of sufficient conditions for the first choice variable to be decreased but the second choice to be increased in response to three types of changes in randomness.

Keywords: Two-dimensional choice set, General model, Changes in randomness

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I. Introduction

It is an important issue to investigate the comparative statics for decision models with two choice variables and one random parameter. In one choice variable model the variable is usually decreased when an increase in risk occurs. On the other hand, in two-choice variable model one decision variable exposed to randomness is still decreased but the other can be increased. The issue of a corner solution is more interesting in two-choice variable model than in one choice variable model. Assuming a corner

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solution in one choice variable model does not allow comparative statics to be conducted for small changes because the only choice variable is determined by the constraint. However, in two-choice variable model, because there are two choice variables and one of them can be an interior solution, comparative statics analysis can still be carried out. Choi, Kim, and Snow (2000), who considered both interior and corner solutions, confirmed the intuition that a decision-maker curtails the first activity, but initiates or expands the second activity in response to an increase in risk associated with the first only if the marginal value of the second activity increases.

Even though two-choice variable model involving randomness encompasses many important decision models in economics, the extension of choice variables from one to two has been very limited. An obstacle to extension originates from the problems of dealing with two first-order conditions that do not arise in one choice variable model. Several decision models with two choice variables have been examined by Batra and Ullah (1974), Feder (1977), Feder, Just, and Schmitz (1977), and Katz, Paroush, and Kahana (1982), but only in the context of specific decision models and only with less general changes in the random variable.

We consider a general decision model including the existing two-choice variable models as special cases, and derive the comparative statics results concerning the effects of three general types of changes in randomness: a simple increase in risk, a relatively strong increase in risk, and a general First Degree Stochastic Dominant (FSD) improvement. By using a general model we can also present a set of sufficient conditions needed for the second choice variable to increase in response to an increase in risk.

The remainder of the paper is organized as follows. In section II, we set out a general decision model with two-choice variables and present two distinct types of constraints, the proportional condition and the independence condition, which make the model tractable. Also included in this section is a discussion of three types of changes in the randomness we consider. Section III derives the comparative statics results, and extends or improves the results of previous works in the literature. Conclusions are presented in the final section.

II. The Model

We consider a general decision model in which utility depends on one outcome variable, z, which in turn, depends on one random variable, $\tilde{x}$, two choice variables, α and δ, and a set of nonrandom parameters, λ. The decision-maker is assumed to choose α and δ to maximize

$$H = Eu\{z(\tilde{x}, \alpha, \delta, \lambda)\} = \int_0^B u\{z(\tilde{x}, \alpha, \delta, \lambda)\} dF(x), \tag{1}$$

where $F(x)$ is a cumulative distribution function (CDF) of the random variable $\tilde{x}$ with support in the interval $[0, B]$. The utility function $u(z)$ is assumed to be three times differentiable with $u' > 0$ and $u'' < 0$; thus, the decision-maker is a strict risk averter. The function $z(\tilde{x}, \alpha, \delta, \lambda)$ is assumed three times differentiable with $z_{\alpha\alpha} < 0$, $z_{\delta\delta} < 0$, and $z_{\alpha\alpha} z_{\delta\delta} - z_{\alpha\delta}^2 > 0$. This condition on z, combined with $u'' < 0$, ensures that the second-order condition for the maximization problem is satisfied. To simplify the discussion, we will consider only the case where $z_x \geq 0$. This assumption, combined with $u'(z) > 0$, indicates that higher values of the random variable are preferred to lower values. Focusing on interior solutions to the problem, it is assumed that both $z_\alpha = 0$ and $z_\delta = 0$ are satisfied for some finite α and δ for all $x \in [0, B]$.

Given these assumptions, the first- and second-order conditions for a maximum of (1) can be written as:

$$H_1(\alpha, \delta, \lambda) = \frac{\partial Eu(z)}{\partial \alpha} = Eu'(z) z_\alpha = \int_0^B u'(z) z_\alpha dF(x) = 0 \tag{2}$$

$$H_2(\alpha, \delta, \lambda) = \frac{\partial Eu(z)}{\partial \delta} = Eu'(z) z_\delta = \int_0^B u'(z) z_\delta dF(x) = 0 \tag{3}$$

and

$$H_{11} = E\{u'(z) z_{\alpha\alpha} + u''(z) z_\alpha^2\} < 0 \tag{4}$$

$$H_{22} = E\{u'(z) z_{\delta\delta} + u''(z) z_\delta^2\} < 0 \tag{5}$$

$$H_{12} = E\{u'(z)z_{\alpha\delta} + u''(z)z_{\alpha}z_{\delta}\} \tag{6}$$

$$H = H_{11}H_{22} - H_{12}^{2} > 0. \tag{7}$$

The comparative statics questions addressed here are how the optimal values of α and δ change when random variable $\tilde{x}$ undergoes a simple increase in risk, a relatively strong increase in risk, or a FSD improvement in $\tilde{x}$. The three types of changes in randomness described below can be defined by changing the CDF for $\tilde{x}$ or transforming $\tilde{x}$ deterministically. Meyer and Ormiston (1989) introduce a simple increase in risk (Definition 1.1) as a type of Rothschild-Stiglitz increases in risk introduced by Rothschild and Stiglitz (1970), and Ormiston (1992) defines a simple FSD transformation (Definition 1.3) as a class of FSD shifts. Black and Bulkley (1989) introduce a relatively strong increase in risk (Definition 1.2), which is defined by imposing a monotonicity restriction on the likelihood ratio between a pair of probability densities.

***Definition* 1.1**

The deterministic transformation $t(x)$ represents a simple increase in risk for a random variable given by $F(x)$ if the function $k(x) \equiv t(x) - x$ satisfies

(a) $\int_0^B k(x)dF(x) = 0$,

(b) $\int_0^s k(x)dF(x) \leq 0$ for all $s \in [0, B]$,

(c) $k'(x) \geq 0$.

Numerical example 1.1:

Consider the following discrete random variable $\tilde{x}$ with Prob$(x = -1) = 1/2$ and Prob$(x = 1) = 1/2$. The transformation $t(x) = 2x$ is a simple increase in risk because $k(x) = 2x - x$ satisfies three conditions given in definition 1.1;

(a) $\int_0^B (2x - x)dF(x) = (-1)\frac{1}{2} + (1)\frac{1}{2} = 0$,

(b) $\int_0^s k(x)dF(x) = (-1)\frac{1}{2} \leq 0$ for $s = -1$,

(c) $k'(x)=1\geq 0$.

Meyer and Ormiston (1989) show that if the function $k(x)$ satisfies the first two conditions, then it reduces expected utility for all risk averse decision-makers. Thus the transformation can be interpreted as an increase in risk in the Rothschild and Stiglitz (R-S) sense. The third property, $k'(x)\geq 0$, is the added condition which identifies this particular type of risk increases, and allows general statements to be made concerning the effect of a simple increase in risk on the choice variable selected by a group of decision-makers. The simple increase in risk generalizes the mean-preserving linear transformation introduced by Sandmo (1971) given by $k(x)=(\gamma-1)(\tilde{x}-\bar{x})$ and $k'(x)=(\gamma-1)\geq 0$, where $\bar{x}$ is the mean of $\tilde{x}$ and the nonrandom parameter γ is greater than or equal to one.

***Definition* 1.2**

$G(x)$ represents a relatively strong increase in risk compared with $F(x)$ if

(a) $\int_a^f \{G(x)-F(x)\}dx=0$,

(b) For all points in the interval $[c, d]$, $f(x)\geq g(x)$ and for all points outside this interval $f(x)\leq g(x)$ where $a\leq b\leq c\leq d\leq e\leq f$, with $[a, f]$ being the supports of x under $G(x)$, and [b, e] being the supports under $F(x)$,

(c) $f(x)/g(x)$ is nondecreasing in the interval $[b, c)$,

(d) $f(x)/g(x)$ is nonincreasing in the interval $(d, e]$.

Numerical example 1.2:

Consider the following two random variables with probability densities $f(x)$ and $g(x)$, respectively; $f(x)=x+1$ for $-1\leq x\leq 0$ and $-x+1$ for $0\leq x\leq 1$, $g(x)=(1/8)x+(1/4)$ for $-2\leq x\leq 0$ and $-(1/8)x+(1/4)$ for $0\leq x\leq 2$. It is easy to show that $g(x)$ and $f(x)$ satisfies the four conditions given in definition 1.2:

(a) $\int_{-2}^{2}[G(x)-F(x)]dx=\int_{-2}^{2}[\{(1/4)x+(1/2)\}-\{(1/2)x+(1/2)\}]dx=0$,

(b) $f(x)\geq g(x)$ for $\forall x\in[c, d]=[-(6/7), (6/7)]$ and $f(x)\leq g(x)$ for all points outside the interval $[-(6/7), (6/7)]$,

(c) $[f(x)/g(x)]' = (1/8) \geq 0$ where $f(x) = x+1$ and $g(x) = (1/8)x + (1/4)$ for $[b,\ c] = [-1,\ -(6/7)]$,

(d) $[f(x)/g(x)]' = -(1/8) \leq 0$ for $[d,\ e] = [(6/7),\ 1]$.

Conditions (a) and (b) are sufficient for $G(x)$ to represent a R-S increase in risk, and also to represent a strong increase in risk. It is the case where $b=c$ and $d=e$ that is considered by Meyer and Ormiston (1985). The relatively strong or strong increase in risk also includes a global increase in risk introduced by Kraus (1979) as a special case. This is an increase in risk from an initial nonrandom situation, where $x = \bar{x}$, to a situation when $\tilde{x}$ is random with mean $\bar{x}$.

***Definition* 1.3**

The deterministic transformation $t(x)$ represents a FSD improvement in $\tilde{x}$ if and only if $k(x) \equiv t(x) - x \geq 0$ for all x in $[0,\ B]$.

Numerical example 1.3:

Consider a discrete random variable $\tilde{x}$ with Prob$(x=-1) = 1/2$ and Prob$(x=1) = 1/2$. The transformation $t(x) = e^x$ is a FSD improvement because $k(x) = e^x - x \geq 0$ for -1 and 1, respectively.

Sandmo (1971) considers a special case of the FSD improvement in $\tilde{x}$, where $k(x) = \theta > 0$ and $k'(x) = 0$, where θ is a nonnegative constant. That is, the FSD transformation used by Sandmo is linear in $\tilde{x}$. Note that the existing two choice decision models only consider quite restrictive changes in $\tilde{x}$; that is, an increase in risk represented by a mean-preserving linear transformation, a global increase in risk, and a FSD improvement represented by an increase in θ.

In its general form, a decision model with two choice variables requires a fairly rigid structure for the comparative statics to be determined. To see this, suppose that a relatively strong increase in risk occurs. From (2) and (3), the following comparative statics are given concerning the effect on α and δ of this increase in risk:

$$\frac{\partial \alpha}{\partial (r.s.)} = \frac{1}{H}\left[-\frac{\partial H_1}{\partial (r.s.)} H_{22} + \frac{\partial H_2}{\partial (r.s.)} H_{12} \right] \tag{8}$$

$$\frac{\partial \delta}{\partial (r.s.)} = \frac{1}{H}\left[-\frac{\partial H_2}{\partial (r.s.)} H_{11} + \frac{\partial H_1}{\partial (r.s.)} H_{12} \right], \tag{9}$$

where $\partial H_1/\partial(r.s.)$ and $\partial\alpha/\partial(r.s.)$ represent the effect of a relatively strong increase in risk on H_1 and α, respectively. Notice that even if the signs of $\partial H_1/\partial(r.s.)$ and $\partial H_2/\partial(r.s.)$ are known, this is not sufficient for deriving determinate comparative statics because H_{12} can be positive or negative. For example, suppose that $\partial H_1/\partial(r.s.) < 0$ and $\partial H_2/\partial(r.s.) < 0$. Then, we have unambiguous comparative statics results if $H_{12} > 0$.

To formulate a general two-choice decision model, we look at the specific models in the literature to ask what structure of the decision model makes determinate comparative statics possible. We identify two types of constraints that are essential in predicting the impact of changes in the random parameter and confirm that they hold in many economic decision models. The set of restrictions requires that the partial derivative of the outcome variable with respect to the additional choice variable be either independent of $\tilde{x}$ or proportional to the partial derivative of the outcome variable with respect to the one choice variable at a rate that is independent of $\tilde{x}$. Quite interestingly, these constraints have been fundamental to the decision model even not involving randomness, have been used before, and yet have not been used to full advantage in the analysis of economic models involving randomness.

One constraint is the proportional condition $z_\alpha = (z_{\alpha x}/z_{\delta x})z_\delta$, where the proportionality factor $z_{\alpha x}/z_{\delta x}$ does not depend on $\tilde{x}$. This condition is required to make the effect of a change in $\tilde{x}$ on H_1 proportional to on H_2, and is met in specific models investigated by Batra and Ullah (1974), Feder (1977), and Feder, Just, and Schmitz (1977).

Another constraint is the independence condition $z_{\delta x} = 0$. The condition implies that z_δ does not depend on the random variable and thus one of the two first-order conditions $H_2 = 0$ remains unchanged regardless of changes in risk. The condition is satisfied in Katz, Paroush, and Kahana (1982) and Choi, Kim, and Snow (2000). In order to make the two-choice variable model tractable, we shall assume either the proportional or the independence condition.

III. The Comparative Statics Analysis

Even though there are two first-order conditions, (2) and (3), in the model, notice that if it satisfies one of two types of constraints, looking at either of them individually is the same as looking at a decision model with one-choice variable. Therefore, the results of previous studies based on the case of one-choice variable can be used. We deal with a simple increase in risk here, and the other two changes in randomness (relatively a strong increase in risk and a FSD improvement) in Appendix A and B. Let $H_1(\alpha, \delta, \lambda, \theta)$ denote the derivative with respect to the choice variable α of expected utility when the random variable is transformed according to $t(x)=x+\theta k(x)$, where $0\leq\theta\leq 1$; that is, $H_1(\alpha, \delta, \lambda, \theta)=\int_0^B U'[z\{x+\theta k(x), \alpha, \delta, \lambda\}]z_\alpha(\{x+\theta k(x), \alpha, \delta, \lambda\})dF(x)$. Note that $H_1(\alpha, \delta, \lambda, 0)=0$ for the initial optimal values of α and δ.

A. The Proportional Condition: $z_\alpha=(z_{\alpha x}/z_{\delta x})z_\delta$

***Corollary* 1.1**

$\partial H_1/\partial\theta|_{\theta=0}<0$ when the random variable is transformed according to $t(x)=x+\theta k(x)$ if

(a) $u(z)$ displays decreasing absolute risk aversion (DARA),

(b) $z_x\geq 0$, $z_{xx}\leq 0$, $z_{\alpha x}\geq 0$, and $z_{\alpha xx}\leq 0$,

(c) $t(x)$ represents a simple increase in risk.

Proof: The proof is given in Meyer and Ormiston (1989) and is simply sketched here. For the initial optimal values of α and δ,

$$\left.\frac{\partial H_1}{\partial\theta}\right|_{\theta=0}=\int_0^B\{u'(z)z_{\alpha x}+u''(z)z_\alpha z_x\}k(x)dF(x)$$

$$=\int_0^B u'(z)z_{\alpha x}k(x)dF(x)+\int_0^B u''(z)z_\alpha z_x k(x)dF(x).$$

They show that under the conditions of the corollary, $\partial H_1/\partial\theta|_{\theta=0}<0$.

Q.E.D.

***Proposition* 1.1**

An economic agent choosing α and δ to maximize $\int_0^B u\{z(\tilde{x}, \alpha, \delta, \lambda)\}dF(x)$ will decrease the optimal values of α and δ when the random variable is transformed according to $t(x)=x+\theta k(x)$ if

(a) $u(z)$ displays DARA,

(b) $z_x \geq 0$, $z_{xx}=0$, $z_{\alpha x}$, $z_{\delta x} \geq 0$,

(c) $z_\alpha = (z_{\alpha x}/z_{\delta x})z_\delta$ at $H_1=0$ and $H_2=0$ and $z_{\alpha\delta}>0$,

(d) $t(x)$ represents a simple increase in risk.

Proof: If $z_\alpha = (z_{\alpha x}/z_{\delta x})z_\delta$ is a characteristic of the model, and $z_{\alpha x}$ and $z_{\delta x}$ do not depend on the random variable, then $H_1 = (z_{\alpha x}/z_{\delta x})H_2$ at both $H_1=0$ and $H_2=0$. Therefore, $\partial H_2/\partial\theta = (z_{\delta x}/z_{\alpha x})\,\partial H_1/\partial\theta$. By using Cramer's rule,

$$\left.\frac{\partial\alpha}{\partial\theta}\right|_{\theta=0} = \frac{1}{H}\left[-\frac{\partial H_1}{\partial\theta}\,H_{22} + \frac{\partial H_2}{\partial\theta}\,H_{12}\right]$$

$$= -\frac{1}{H}\left(\frac{\partial H_1}{\partial\theta}\right)\left[\,H_{22} - \frac{z_{\delta x}}{z_{\alpha x}}\,H_{12}\right].$$

Simplifying the term $[H_{22}-(z_{\delta x}/z_{\alpha x})H_{12}]$ using $z_\alpha = (z_{\alpha x}/z_{\delta x})z_\delta$, $[H_{22}-(z_{\delta x}/z_{\alpha x})H_{12}] = E(1/z_{\alpha x})u'(z)(z_{\alpha x}z_{\delta\delta} - z_{\delta x}z_{\alpha\delta})$. Substituting this into $\partial\alpha/\partial\theta|_{\theta=0}$, $\partial\alpha/\partial\theta|_{\theta=0} = -(1/H)(\partial H_1/\partial\theta)E(1/z_{\alpha x})u'(z)(z_{\alpha x}z_{\delta\delta} - z_{\delta x}z_{\alpha\delta})$. By corollary 1.1, $\partial H_1/\partial\theta|_{\theta=0}<0$. Thus, under the conditions of the proposition, $\partial\alpha/\partial\theta|_{\theta=0}<0$. By using a similar procedure, we can show that $\partial\delta/\partial\theta|_{\theta=0} = -(1/H)(\partial H_1/\partial\theta)E(1/z_{\alpha x})u'(z)(z_{\delta x}z_{\alpha\alpha} - z_{\alpha x}z_{\alpha\delta})<0$.

Q.E.D.

Remark 1.1: The proposition implies that if both $z_{\delta x}$ and $z_{\alpha\delta}$ are negative, the first activity α is decreased in response to a simple increase in risk but the second activity δ is increased. In other words, as long as a decrease in α increases the marginal value of the second activity in addition to $z_{\delta x}<0$, δ can be increased as a result of a decrease in α.

Proposition 1.1 gives conditions sufficient to yield determinate comparative statics results concerning the effect on α and δ of a simple increase in risk. Condition (a) restricts the set of decision-makers to those exhibiting DARA. DARA is generally thought to be a reasonable assumption concerning preferences. Condition (b) restricts the model. Note that $z_{xx}<0$ is allowed in decision models with one-choice variable, but in two choice decision model, $z_{xx}=0$ is required to make $(z_{\alpha x}/z_{\delta x})$ nonrandom. Obviously, $z_{\alpha xx}=0$ since $z_{xx}=0$. Condition (c) further restricts the model. The proportional condition, $z_\alpha=(z_{\alpha x}/z_{\delta x})z_\delta$ is assumed to make the model tractable.

The condition, $z_{\alpha\delta}>0$ is also added to allow determinate statements to be made concerning the effect on α and δ of a change in the random parameter. Condition (d) requires that an increase in risk be simple. While the conditions in Proposition 1.1 are rather restrictive, they do include the specific models with two-choice variables presented by Batra and Ullah (1974), Feder (1977), and Feder, Just, and Schmitz (1977) as special cases.

Example 1.1: In the model of Feder (1977), z has a specific form such that $z(\tilde{x}, \alpha, \delta, \lambda)=\tilde{x}f(\alpha, \delta)+g(\alpha, \delta)+\lambda$. The model, however, can be classified as a general decision model since it does include many specific models as special cases. For example, if $g(\alpha, \delta)+\lambda=-\lambda_1\alpha-\lambda_2\delta$, then the model is equivalent to that of Batra and Ullah (1974), where $\tilde{x}$ is the price of output, α and δ are two inputs, λ_i $(i=1, 2)$ represent the prices of the inputs, and f is a production function. The proportional condition, $z_\alpha=(z_{\alpha x}/z_{\delta x})z_\delta$ holds in the model. To see this, look at the first-order conditions: $H_1(\alpha, \delta, \lambda)=Eu'(z)(\tilde{x}f_\alpha+g_\alpha)=0$ and $H_2(\alpha, \delta, \lambda)=Eu'(z)(\tilde{x}f_\delta+g_\delta)=0$. Because $f_\alpha/f_\delta=g_\alpha/g_\delta$, this implies that $(z_{\alpha x}/z_{\delta x})z_\delta=(f_\alpha/f_\delta)(\tilde{x}f_\delta+g_\delta)=\tilde{x}f_\alpha+g_\alpha=z_\alpha$. Obviously here $z_{xx}=0$. Also, it should be noted that even though Feder assumes a general form for the objective function which includes more than one-choice variable, he did not derive comparative statics concerning the effects on the choice variables, α and δ, of changes in the random parameter, except for the one-choice variable case. Feder (1977, p. 509) referred that "it is not possible to determine the direction of impact on the different control variables $\cdots$. Definite results, however, can be obtained for the function f." Here we present simple conditions about the function z which are sufficient for determining the direction of changes in choice variables when the random parameter $\tilde{x}$ undergoes the three types of changes in randomness discussed in Section II.

Applying Proposition 1.1, all risk-averse decision-makers exhibiting DARA, when faced with a simple increase in risk, will decrease the optimal values of α and δ if $z_x(=f(\alpha, \delta))\geq 0$, $z_{\alpha x}(=f_\alpha(\alpha, \delta))$, $z_{\delta x}(=f_\delta(\alpha, \delta))\geq 0$, and $z_{\alpha\delta}(=\tilde{x}f_{\alpha\delta}+g_{\alpha\delta})>0$. However, if $z_{\delta x}(=f_\delta(\alpha, \delta))<0$ and $z_{\alpha\delta}(=\tilde{x}f_{\alpha\delta}+g_{\alpha\delta})<0$, δ is increased as a result of a decrease of α.

B. The Independence Condition: $z_{\delta x}=0$

Because the condition, $z_{\delta x}=0$ allows the comparative statics analysis to be simplified, the details are omitted here. The effects on α and δ of a relatively strong increase in risk and a FSD improvement are provided in Appendix B.

***Proposition* 2.1**
An economic agent choosing α and δ to maximize $\int_0^B u(z(\tilde{x}, \alpha, \delta, \lambda))dF(x)$ will decrease the optimal values of α and δ (decrease α and increase δ) when the random variable is transformed according to $t(x)=x+\theta k(x)$ if

(a) $u(z)$ displays DARA,

(b) $z_x \geq 0$, $z_{xx} \leq 0$, $z_{\alpha x} \geq 0$, and $z_{\alpha xx} \leq 0$,

(c) $z_{\delta x}=0$ and $z_{\alpha\delta}>(<)0$,

(d) $t(x)$ represents a simple increase in risk.

Proof: $z_{\delta x}=0$ implies that the condition $H_2=0$ is equivalent to $z_\delta=0$. Thus, $\partial H_2/\partial\theta|_{\theta=0}=0$. Then we have the following comparative statics: $\partial\alpha/\partial\theta|_{\theta=0}=-(1/H)[(\partial H_1/\partial\theta)H_{22}]$, $\partial\delta/\partial\theta|_{\theta=0}=(1/H)[(\partial H_1/\partial\theta)H_{12}]$. Note that $H_{12}=Eu'(z)z_{\alpha\delta}$ since $z_\delta=0$ from $H_2=0$, and $\partial H_1/\partial\theta|_{\theta=0}<0$ by corollary 1.1. Thus, under the conditions of the proposition, $\partial\alpha/\partial\theta|_{\theta=0}<0$ and $\partial\delta/\partial\theta|_{\theta=0}<0$ when $z_{\alpha\delta}>0$, while $\partial\alpha/\partial\theta|_{\theta=0}<0$ and $\partial\delta/\partial\theta|_{\theta=0}>0$ when $z_{\alpha\delta}<0$.

Q.E.D.

***Remark* 2.1**: As long as $z_{\alpha\delta}<0$, the first activity α is decreased in response to a simple increase in risk but the second activity δ is increased.

Proposition 2.1 gives conditions sufficient to yield unambiguous comparative static results concerning the effect on α and δ of a simple increase in risk. Condition (a) requires that preferences exhibit DARA. Condition (b) restricts the model. However, it does not require that z_{xx} must be equal to zero. Condition (c) further restricts the model. Even though the condition makes the model tractable, it is a severe restriction on the decision model. Condition (d) is the same as in Proposition 1.1.

***Example* 2.1**: Choi, Kim, and Snow (2000) consider a specific model in which $z(\tilde{x}, \alpha, \delta, \lambda) = \tilde{x}\alpha + \lambda\delta - c(\alpha, \delta)$, where α and δ are nonnegative choice variables, λ is exogenously determined, and the cost function $c(\alpha, \delta)$ is assumed to be continuously differentiable and convex. One example that fits their model is the entrepreneurial competitive firm facing output-price uncertainty $\tilde{x}$ while deciding on supply of the product α and on the supply of a second product δ whose price is λ. Applying Proposition 2.1, all risk-averse decision-makers exhibiting DARA, when faced with a simple increase in risk, will decrease the optimal values of α and δ if $z_{\alpha\delta} = -c_{\alpha\delta} > 0$. However, if $z_{\alpha\delta} = -c_{\alpha\delta} < 0$, then δ is increased as a result of a decrease of α. The intuition behind this result is straightforward: an increase in risk causes the decision-maker to reduce activity α, which reduces the marginal cost of the second activity δ if the two are cost substitutes ($c_{\alpha\delta} > 0$).

IV. Conclusions

This paper examines a general decision model with two-choice variables and one random variable that satisfies two types of constraints on the decision model. One is the proportional condition where the proportionality factor does not depend on random variable. Another is the independence condition requiring that a change in a random variable does not affect one of two first-order conditions. These constraints are rather restrictive but we can derive determinate comparative statics results based on them. In addition, by using a general model we can present a set of sufficient conditions for the first choice variable to be decreased but the second choice to be increased in response to three types of changes in randomness.

Appendix A: The Proportional Condition $z_{\alpha} = (z_{\alpha x}/z_{\delta x})z_{\delta}$

***Corollary* 1.2**

Assume that a relatively strong increase in risk occurs. Then, $\partial H_1/\partial(r.s.) < 0$ if

(a) $u'(z) > 0$ and $u''(z) < 0$,

(b) $z_x \geq 0$, $z_{\alpha x} \geq 0$, and $z_{\alpha xx} \leq 0$.

Remark 1.2: Let $\partial H_1/\partial(r.s.)$ denote the effect on H_1 of a relatively strong increase in risk; that is, $\partial H_1/\partial(r.s.) = \int_0^B [u'(z)z_\alpha d\{G(x) - F(x)\}]$, where $G(x)$ represents a relatively strong increase in risk from $F(x)$. Note that $H_1(\alpha, \delta, \lambda) = \int_0^B u'(z)z_\alpha dF(x) = 0$ for the optimal values of α and δ under $F(x)$.

Proof: The proof is given in Black and Bulkley (1989) and is simply sketched here. Black and Bulkley demonstrate that given the assumptions about $z(\tilde{x}, \alpha, \delta, \lambda)$ and $u(z)$, $\partial H_1/\partial(r.s.) = \int_0^B u'(z)z_\alpha d\{G(x) - F(x)\} < 0$ for the optimal value of α and δ under $F(x)$, where $G(x)$ represents a relatively strong increase in risk from $F(x)$.

Q.E.D.

Proposition 1.2

An economic agent choosing α and δ to maximize $\int_0^B u\{z(\tilde{x}, \alpha, \delta, \lambda)\}dF(x)$ will decrease the optimal values of α and δ if

(a) $u'(z) > 0$ and $u''(z) < 0$,

(b) $z_x \geq 0$, $z_{xx} = 0$, $z_{\alpha x}$, $z_{\delta x} \geq 0$,

(c) $z_\alpha = (z_{\alpha x}/z_{\delta x})z_\delta$ at $H_1 = 0$ and $H_2 = 0$ and $z_{\alpha\delta} > 0$,

(d) $G(x)$ represents a relatively strong increase in risk from $F(x)$.

Proof: By using a similar procedure which is provided for Proposition 1.1,

$$\frac{\partial \alpha}{\partial(r.s.)} = -\frac{1}{H}\left[\frac{\partial H_1}{\partial(r.s.)}\right] E\left(\frac{1}{z_{\alpha x}}\right) u'(z)(z_{\alpha x}z_{\delta\delta} - z_{\delta x}z_{\alpha\delta})$$

$$\frac{\partial \delta}{\partial(r.s.)} = -\frac{1}{H}\left[\frac{\partial H_1}{\partial(r.s.)}\right] E\left(\frac{1}{z_{\alpha x}}\right) u'(z)(z_{\delta x}z_{\alpha\alpha} - z_{\alpha x}z_{\alpha\delta}).$$

By corollary 1.2, $\partial H_1/\partial(r.s.) < 0$. Thus, one can conclude that under the conditions of the proposition, the optimal values of α and δ are decreased.

Q.E.D.

***Corollary* 1.3**

$\partial H_1/\partial\theta|_{\theta=0} > 0$ when the random variable is transformed according to $t(x)=x+\theta k(x)$ if

(a) $u(z)$ displays DARA,

(b) $z_x \geq 0$, $z_{xx} \leq 0$, and $z_{\alpha x} \geq 0$

(c) $t(x)$ represents a FSD improvement in $\tilde{x}$ and $k'(x) \leq 0$.

Proof: The proof is given in Ormiston (1992) and is simply sketched here. For the initial optimal values of α and δ,

$$\left.\frac{\partial H_1}{\partial \theta}\right|_{\theta=0} = \int_0^B \{u'(z)z_{\alpha x} + u''(z)z_\alpha z_x\}k(x)dF(x)$$

$$= \int_0^B \{u'(z)z_{\alpha x}\}k(x)dF(x) + \int_0^B \{-R_A(z)z_x k(x)\}u'(z)z_\alpha dF(x).$$

Ormiston demonstrates that under the conditions of the corollary, $\partial H_1/\partial \theta|_{\theta=0} > 0$. This is a local result in the sense that the result holds only at $\theta=0$ and for small changes in θ.

Q.E.D.

***Proposition* 1.3**

An economic agent choosing α and δ to maximize $\int_0^B u\{z(\tilde{x}, \alpha, \delta, \lambda)\}dF(x)$ will increase the optimal values of α and δ when the random variable is transformed according to $t(x)=x+\theta k(x)$ if

(a) $u(z)$ displays DARA,

(b) $z_x \geq 0$, $z_{xx}=0$, $z_{\alpha x}$, $z_{\delta x} \geq 0$,

(c) $z_\alpha = (z_{\alpha x}/z_{\delta x})z_\delta$ at $H_1=0$ and $H_2=0$ and $z_{\alpha\delta}>0$,

(d) $k(x) \geq 0$ and $k'(x) \leq 0$.

Proof: By using a similar procedure which is provided for Proposition 1.1,

$$\left.\frac{\partial \alpha}{\partial \theta}\right|_{\theta=0} = -\frac{1}{H}\left[\frac{\partial H_1}{\partial \theta}\right] E\left(\frac{1}{z_{\alpha x}}\right) u'(z)(z_{\alpha x}z_{\delta\delta} - z_{\delta x}z_{\alpha\delta}),$$

$$\left.\frac{\partial \delta}{\partial \theta}\right|_{\theta=0} = -\frac{1}{H}\left[\frac{\partial H_1}{\partial \theta}\right] E\left(\frac{1}{z_{\alpha x}}\right) u'(z)(z_{\delta x}z_{\alpha\alpha} - z_{\alpha x}z_{\alpha\delta}) < 0.$$

By corollary 1.3, $\partial H_1/\partial\theta|_{\theta=0}>0$; that is, given the assumptions about $z(\tilde{x}, \alpha, \delta, \lambda)$, $u(z)$, and $k'(x)$, the sign of $H_1(\alpha, \delta, \lambda, \theta)$, evaluated at the initial optimal values of α and δ and $\theta=0$, changes from zero to positive as a result of a FSD improvement.

Q.E.D.

Appendix B: The Independence Condition $z_{\delta x}=0$

***Proposition* 2.2**

An economic agent choosing α and δ to maximize $\int_0^B u\{z(\tilde{x}, \alpha, \delta, \lambda)\}dF(x)$ will decrease the optimal values of α and δ (decrease α and increase δ) if

(a) $u'(z)>0$ and $u''(z)<0$,

(b) $z_x\geq 0$, $z_{\alpha x}\geq 0$, and $z_{\alpha xx}\leq 0$,

(c) $z_{\delta x}=0$ and $z_{\alpha\delta}>(<)0$,

(d) $G(x)$ represents a relatively strong increase in risk from $F(x)$.

***Proposition* 2.3**

An economic agent choosing α and δ to maximize $\int_0^B u\{z(\tilde{x}, \alpha, \delta, \lambda)\}dF(x)$ will increase the optimal values of α and δ (increase α and decrease δ) when the random variable is transformed according to $t(x)=x+\theta k(x)$ if

(a) $u(z)$ displays DARA,

(b) $z_x\geq 0$, $z_{xx}\leq 0$, and $z_{\alpha x}\geq 0$,

(c) $z_{\delta x}=0$ and $z_{\alpha\delta}>(<)0$,

(d) $k(x)\geq 0$ and $k'(x)\leq 0$.

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