Competitive Equilibrium with Non-Concavifiable Preferences

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The no free lunch condition is neither necessary nor sufficient for the utility set to be closed and bounded in asset markets where the preferred sets do not have the same recession cone. This paper characterizes the utility set with non-concavifiable preferences and provides the existence of competitive equilibrium when the set of efficient allocations is not necessarily bounded.

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I. Introduction

An economy with asset markets differs from the classical economy in that the former may allow agents to hold an unlimited negative holding of assets. If short sales are not restricted, the consumption sets are not bounded below. In this case, the classical theorems of the existence of equilibria cannot be applied. Beyond the usual assumptions like continuity and convexity of preferences, the literature employs two more assumptions that make the utility set closed and bounded.[1] One assumption concerns restrictions on convex preferences like concavifiability, while the other concerns the no 'free lunches' condition.[2] These are easily characterized with

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the commodity bundles that always yield nonnegative marginal [1] [2]Concavifiable preferences possess the property that the recession cone of each preferred set coincides with the preferred cone.[3] A nonzero element of the preferred cone becomes a free lunch if it does not have a positive price. Werner (1987) demonstrates that the no free lunch condition is necessary and sufficient for the existence of competitive equilibrium under the assumption that all the preferred sets have the same recession cone and no halflines exist in indifference curves. Subsequently, Page and Wooders (1996), Dana, Le Van and Magnien (1999) and Page, Wooders and Monteiro (2000) among others introduce a general notion of free lunch to explain the compactness of the utility set. All of the literature are concerned about the existence of equilibrium with the compact utility set.

No economic rationale, however, is yet established for the uniformity of the recession cones for the preferred sets or the compactness of the utility set when consumptions arise beyond the domain of the nonnegative quantities in the commodity space. If the preferred sets do not have the same recession cone, then the no free lunch condition is neither necessary nor sufficient for the utility set to be closed and bounded. This paper characterizes the utility set with non-concavifiable preferences and provides the existence of competitive equilibrium when the set of efficient allocations is not necessarily bounded.

One difficulty with the non-compact utility set is that the conventional fixed-point arguments cannot apply for the existence of competitive equilibrium. Fixed point arguments involve the boundary condition either on the price simplex or on the utility set. For example, the boundary condition to be checked with the Negishi approach is that each weakly efficient allocation of consumptions on the boundary of the contract curve have a price support which makes each agent rich enough to purchase his

[1]The closedness hypothesis is fulfilled if the set of individually rational tradings is bounded. A trading is individually rational if it weakly Pareto improves the initial allocation.

[2]Preferences are concavifiable if they can be represented by a concave utility function.

[3]The recession cone of a convex set S in a Euclidean space is the maximal of the convex cones C's that satisfy $S+C=S$.

consumption.[4] It is always satisfied with the compact utility set because its boundary arises where the contract curve meets the indifference curves through the initial endowment for all agents. This is not the case, however, with the non-compact utility set. If the contract curve is unbounded, it does not meet the indifference curve through the initial endowment for some agent which leads to the failure of the boundary condition. To avoid the dilemma with the non-compact utility set, we work out 'a boundary' for the contract curve by tearing away its farther part. The problem is how to make the boundary of the truncated contract curve the 'real' boundary of the utility set. At this point we develop a trick to map the truncated utility set homeomorphically into the unit simplex. In particular, there exists a homeomorphism between the two sets. The ordinary boundary condition is imposed on the worked-out boundary through the simplex. We assume that the artificial boundary of the contract curve behaves as if the truncated part were the whole contract curve of the economy. This condition cannot be dispensed with because it turns out to be necessary and sufficient for competitive equilibrium to exist in a two-agent economy.

The result of this paper is useful to the equilibrium asset pricing literature. Connor (1984) provides a unification of the capital asset pricing model (CAPM) into a competitive equilibrium version of the arbitrage pricing theory (APT). Milne (1988) extends the equilibrium arbitrage pricing theory (EAPT) of Connor in several directions. In particular, Milne (1988) allows preferences to be non-concavifiable so that they may not follow the expected utility or the concavity hypothesis. This extension is valuable because it encompasses many interesting cases which cannot be explained in Connor (1984). The prevalence of those asset pricing models in the literature makes it significant to know under what conditions an asset economy has an equilibrium.

In the case where no restriction is imposed on short-selling, the trading opportunities become much larger than otherwise. In this case, we must look at the other side of the coin. Selling short today entails liabilities to repay tomorrow. The larger today's short sales, the greater tomorrow's risk of default. Default or bankruptcy

[4]Strictly speaking, we must say the 'relative boundary' of the contract curve to emphasize the topological condition.

involves financial distress. Since the extent of default is associated with the size and the proportion of short sales in a portfolio, these factors must be taken into account in making portfolio choices. The degree to which such factors influence portfolio choices may depend on the attitude toward taking the risk of financial distress. The more reluctant people are to face the risk of financial distress, the more conservative they are in taking short positions. As illustrated below, such diverse aspects of taking financial risks are not well represented under those restrictions like concavifiability. Furthermore, Malinvaud (1985) and Luenberger (1995) demonstrate that convex preferences already represent 'risk aversion' without restriction on preferences like concavity. Luenberger (1995) shows that preferences are convex if and only if they are risk averse everywhere and provides the risk aversion coefficient with convex preferences that is a generalization of the Arrow-Pratt measure of risk aversion. These results imply that the concavity assumption is restrictive in describing risk-taking behavior.

It is useful to classify commodity bundles according to their desirability. A commodity bundle is called *locally, indifferently, or uniformly desirable* if its marginal utility is nonnegative at some consumptions, at every consumption along an indifference curve, or at every consumption in the consumption set, respectively.[5] A uniformly desirable bundle is a free lunch if its market value is nonnegative. Mathematically speaking, the set of commodity bundles which are indifferently desirable is the recession cone of the preferred set.

The existence of equilibria problem in an asset economy with unrestricted short sales is initially addressed in Hart (1974). In this work, preferences over contingent returns are assumed to follow the expected utility hypothesis. Instead of abstracting from details on stochastic structures of asset returns, Werner (1987) extends the results of Hart (1974) by imposing the restriction that every preferred set has the same recession cone. Though the latter assumption is weaker than the former, they are indistinguishable in the light of the desirability condition because both require the preferred sets to have the same recession cone. As remarked

[5]These definitions are made in an exact way later. If a utility function u is differentiable, the marginal utility of v at x is the directional derivative of u at x, $D_v u(x) = \lim_{t \to 0}(1/t)\{u(x+tv) - u(x)\}$.

earlier, Werner (1987) shows that an economy has equilibria if there exist non-arbitrage prices. This condition is equivalent to the one that the recession cones for all agents are positively semi-independent.[6] Nielsen (1989) allows the preferred sets to have the distinct recession cone and considers a class of economies in which the set of individually rational allocations is bounded but can be unbounded only if indifference curves have lines. Nielsen (1989) fails, however, to take into account the effect of a bundle which is indifferently but not uniformly desirable on the existence of equilibria.

Now, consider an asset economy in which an agent faces a two-period decision making problem as following. In the first period, the agent trades two contingent claims l and h which will deliver money to the asset holder only if the specific event occurs in the second period. Assume that the utility depends upon the amount of money to be delivered in each contingency and that there is no restriction on short selling these assets. Contingencies in the second period consist of two events L and H. Asset l delivers one unit of money if L occurs in the second period and nothing otherwise, while h one unit of money if H occurs and nothing otherwise. Let y_L and y_H denote the numbers of units of contingent money delivered in the second period. Preferences are assumed to be convex and to be represented by $u(y_L, y_H; \psi)$, where ψ denotes the probabilistic belief that L will occur.

Let $x=(x_l, x_h)$ denote the holdings of contingent claims. Since claims are of Arrow-Debreu type, the utility of holding a portfolio x is given as $u(x_l, x_h; \psi)$.[7] Suppose that indifference curves of $u(x_l, x_h; \psi)$ look as in Figure 1. The initial holding w of assets is on the indifference curve I.

Portfolios which consist of long positions in claims are uniformly desirable to the agent because they generate positive income in

[6] A finite set of cones $\{C_i\}$ in a Euclidean space is positively semi-independent if $v_i \in C_i$ and $\sum_i v_i = 0$ implies $v_i = 0$ for all i.

[7] It is possible to give a numerical example. Suppose a parameter u in (0, 1) satisfies a parabolic relation $\{2(0.5+\psi+u)x_l+x_h-\tan(u+0.5)\pi\}\{(1+\psi-u)x_l+x_h-\tan(u+0.5)\pi\}=1$. For a given u, the relation can be considered representing an indifference curve. The indifference curve is an upper part of the parabola with asymptotes $x_h=\tan(u+0.5)\pi-2(0.5+\psi+u)x_l$ and $x_h=\tan(u+0.5)\pi-(1+\psi-u)x_l$. The recession cone corresponding to u is a convex cone generated by two vectors $(-1, 1+2\psi+2u)$ and $(1, -1-\psi+u)$.

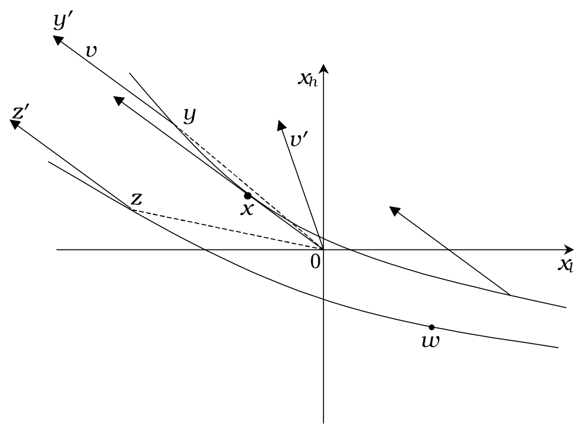

FIGURE 1

each state. A portfolio whose positions are short in one asset and long in the other asset can be indifferently desirable whenever the ratio of the short position to the long one is modest. For example, the bundle v' is indifferently desirable because the marginal utility is positive along I'. In contrast, the marginal utility of v is positive at every holding along I but no longer along I'. Equivalently, $u(w+\lambda v;\ \psi)$ is initially increasing but eventually decreasing in $\lambda \geq 0$.

Specifically, the marginal utility of v is negative at the holdings along I' to the left of x. This is because its addition to such holdings brings about the larger short position in l, as well as its higher proportion to the long position in h (Note that x has the same direction as v or their ratios are the same between a short position in l and a long position in h). For example, a portfolio $y'=y+v$ is worse than y in the sense that 'short' and the ratio of 'short' to 'long' at y' are greater than at y. To take a more general case, choose a portfolio t satisfying $u(t;\ \psi)>u(y;\ \psi)$. Then there is a sufficiently large number λ that makes both the short and relative positions of $t+\lambda v$ so undue that $u(t+\lambda v;\ \psi)<u(y;\ \psi)$ obtains. On the other hand, any units of v are indifferently desirable at every holding along I because they contribute to improving the relative position. For example, the relative position of $z'=z+v$ is better than that of z.

What matters here is the agent's attitude toward short sales. The agent might suffer financial trouble, institutional disadvantages or personal disgraces due to a bankruptcy which could result from an undue imbalance between a short and a long position in his portfolio. Thus his willingness to take a portfolio with mixed positions may be influenced by the absolute size of 'short' as well as the relative magnitude of 'short' to 'long.'

In summary, the portfolio v is indifferently desirable along I while it is not along I'. In other words, it is a direction of recession of the set of consumptions which are preferred to w but not of the set of consumptions which are preferred to x. Therefore the preferred sets of $u(x_l, x_h; \psi)$ do not have the same recession cone. This implies that the utility function $u(x_l, x_h; \psi)$ is not concave (For such results, see Rockafella (1970)).

An economy under study is described in Section II. Section III is devoted to presenting the basic framework of Milne's EAPT model and an example in which the set of weakly efficient allocations is unbounded. Three types of desirability are characterized in Section IV. In Section V, we extend the closedness hypothesis to the case that the utility set is not closed. Near-boundary conditions are formalized in Section VI. The existence of equilibria is shown in Section VII. Section VIII is devoted to the cases which satisfy the near-boundary condition.

The following mathematical notations will be used. By 'int C' and 'cl C', we denote the interior and the closure of a set C in the subspace of a Euclidean space, respectively (the subspace must be clear from the context).

II. The Model

We consider an economy in which l commodities including assets and real goods are traded by m consumers indexed by i. Let I denote a set of m consumers $(1,\cdots,m)$. The consumption set of consumer i is a subset X_i of R^l. The consumer has a preference relation $\geq_i$ over X_i with the endowment w_i in X_i. We make the following basic assumptions about a consumer's characteristics.

(A1) X_i is nonempty, closed, convex and $\geq_i$ is continuous, complete on X_i.

(A2) $\geq_i$ is locally non-satiable and for x and x' in X_i, $x >_i x'$ implies $\alpha x+(1-\alpha)x' >_i x'$ for all $\alpha \in (0, 1)$.[8]

Conditions (A1) and (A2) cover the case of no restrictions on short sales of securities or contingent claims as well as the cases with differentially restricted participation in asset markets. It is well-known that under (A1) and (A2), $\geq_i$ is represented by a continuous and quasiconcave utility function u_i. The function u_i is normalized with $u_i(w_i)=0$ for each i. Let $\bar{u}_i=\sup_{x\in X_i}u_i(x)$. Since a monotonic transformation preserves the preference ordering, we may assume that the range of u_i is in a finite open interval (a, b) for some numbers a and b with $a<0<b$. Then u_i is in (a, b) for every $i\in I$.

An allocation is an m-tuple $x=(x_1,\cdots,x_m)$ with each x_i in X_i. The initial allocation is denoted by $w=(w_1,\cdots,w_m)$. An allocation x is attainable if $\sum_{i=1}^m x_i=\sum_{i=1}^m w_i$ and individually rational if it is attainable, and $u_i(x_i)\geq u_i(w_i)$ for every $i\in I$. Let Q denote the set of individually rational allocations. A consumption y is Pareto attainable for consumer i if there is an allocation x in Q whose ith component is y. By Q_i we denote the set of Pareto attainable consumptions. This set is the projection of Q onto X_i. Set $\hat{u}_i=\sup_{x\in Q_i}u_i(x)$ for each $i\in I$. Let G denote the set of allocations in Q which are weakly efficient, and G_i the projection of G onto X_i. Each point in G_i is called a Pareto consumption.

Define a mapping $U : \Pi X_i \rightarrow R_+^m$ by $U(x)=(u_i(x_i),\cdots,u_m(x_m))$. A point ν in R_+^m is a utility allocation if there is some $y\in X_i$ with $\nu_i\leq u_i(y)$ for every $i\in I$. Let T be a set of utility allocations in R_+^m. Then a utility allocation $\nu\in T$ is weakly efficient (efficient) relative to T if there is no point ν' in T such that $\nu'\gg\nu$ ($\nu'>\nu$, respectively).[9] Let $O(T)(o(T))$ denote a set of utility allocations which are weakly efficient (efficient, resp.) relative to T. A utility allocation ν is attainable if there is some $x\in Q$ with $\nu\leq U(x)$. Let W denote the set of attainable utility allocations. For a utility allocation ν, let $Q(\nu)$ denote a set $\{x\in Q \mid U(x)\geq\nu\}$. By definition, $Q(\nu)$ is not empty if ν is attainable. A price system p in R^l is said to support a utility allocation ν if $p(\sum_{i=1}^m x_i-w)\geq 0$ for each allocation $x\in Q(\nu)$.

[8]Instead of local non-satiation, what is really needed is that a satiation does not occur in the set of attainable allocations or that the set of satiations are unbounded as in Werner (1987).

[9]Let v and v' be vectors in R^m. Then $v\geq v'$ implies $v_i\geq v'_i$ for all $i\in I$; $v>v'$ implies $v\geq v'$ and $v\neq v'$; and $v\gg v'$ implies $v_i>v_i$ for all $i\in I$.

III. An EAPT Example

This section presents an EAPT model of Milne (1988) as a motivation for the current research into a more general theorem on the existence of equilibria than available in the literature. Milne's model is modified into the setting of Section II. A consumer $i \in I$ trades a finite number of assets, which pay contingent returns in the end of the period. Each consumer has preferences over contingent monetary returns, which are represented by a utility function g_i. Let V be a topological vector space. There are finitely many assets indexed by $j = 1, \cdots, l$ which deliver monetary returns $Z_j \in V$.[10] Define a linear mapping Z: $R^l \to V$ by $Z(a) = \sum a_j Z_j$ for a point $a \in R^l$. Tradeable portfolios of each consumer i are restricted to X_i. He obtains a utility $g_i(Z(x))$ from holding a portfolio $x \in X_i$. Assume that the function g_i: $Z(X_i) \to R$ is continuous and quasiconcave. Define the induced utility function over assets u_i: $X_i \to R$ by $u_i(x) = g_i(Z(x))$. Then the function u_i satisfies (A2).

To illustrate an example in which the contract curve is unbounded, assume there are two agents and two assets whose returns are linearly independent. Suppose that the indifference curves and the contract curve of the induced preferences look as in Figure 2, where the Edgeworth's box occupies the whole plane.

The directions v and $-v$ represent a desired portfolio along the indifference curve I for agent 1 and along II for agent 2 through the initial allocation of claims e, respectively. The picture can be drawn in the following way. First, the contract curve is not empty. For example, the point a is in the contract curve. Second, the vector v is the direction of the line that intersects only once any indifference curve for agent 1 above I, implying that v is a uniformly desirable portfolio for agent 1. On the other hand, $-v$ is indifferently desirable along II but not uniformly desirable for agent 2 because it is a direction of the recession of the preferred set with II but not with the indifference curve through b. As demonstrated later, these conditions require that the contract curve is unbounded.

The reason that the contract curve fails to be bounded in this example is the presence of the portfolio v.[11] In the case with the

[10] An 'insurable' economy with infinitely many assets whose payoffs take the form of a factor model with finite factors is reduced to a finite economy. For details, see Connor (1984).

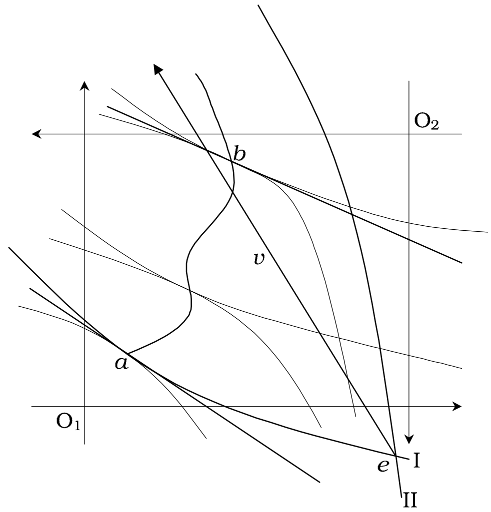

FIGURE 2

compact utility set, the fact that weakly efficient allocations on its boundary have a price support leads to the boundary condition necessary for the Negishi approach to work. The truism, however, does not hold here because the utility set is not closed in R^2_+.

Nevertheless, geometric intuition ensures that an equilibrium exists between *a* and *b* on the contract curve in Figure 2; push a line along the contract curve from *a* to *b* while keeping it tangent to the indifference curves through an efficient allocation, then the line must cross *e* to reach *b*; since this pushing process is continuous, there must be an efficient allocation at which the tangent line goes through *e*. It is worthwhile to note that each agent's initial endowment is located above the tangent budget line through *a* and *b*, respectively. In other words, the boundary condition is realized 'near the boundary.' This idea will be formalized in the subsequent section.

[11]If *v* were not to be indifferently desirable for any agent, the set of individually rational allocations would be compact. For details, see Nielsen (1989).

IV. Unbounded Contract Curves

The types of desirability can be classified by using a direction of recession of a set which is defined in Rockafellar (1970). A vector $v \in R^l$ is a direction of recession of a convex set C in R^l if $x+\lambda v \in C$ for all $x \in C$ and $\lambda \geq 0$. The set of all directions of recession of C is the recession cone of C. The cone is convex, and closed if C is closed. It is well-known that v in R^l is a direction of recession of C if and only if $v = \lim \lambda_n x_n$ for some $\{x_n\}$ in C and $\{\lambda_n\}$ in R_+ with $\lambda_n \to 0$ (For more discussions, see the Appendix). Let $X_i(z)$ or $\tilde{X}_i(u)$ denote the preferred set $\{z' \in X_i \mid u_i(z') \geq u = u_i(z)\}$ and $\tilde{X}_i^0(u)$ a set $\{z' \in X_i \mid u_i(z') \geq u\}$. By $A_i(z)$ ($\tilde{A}_i(u)$), we denote the recession cone of $X_i(z)$ ($\tilde{X}_i(u)$, respectively). We formalize the notions of desirability as follows

***Definition* 1**: A commodity bundle $v \in R^l$ is <u>locally desirable</u> (in short, l-desirable) at some $x \in X_i$ if $u_i(x+v) \geq u_i(x)$, <u>indifferently desirable</u> (i-desirable) along u if $u_i(x+v) \geq u$ for all x satisfying $u_i(x) = u$, and <u>uniformly desirable</u> (u-desirable) if $u_i(x+v) \geq u_i(x)$ for all $x \in X_i$.

Equivalently, v is in $\tilde{A}_i(u)$ if and only if it is i-desirable along u. It is also clear that v is in $A_i(x)$ for every $x \in X_i$ if and only if it is u-desirable. Let A_i denote the collection of commodity bundles which are u-desirable for the consumer i, called the preferred cone. Obviously, it is the intersection of the recession cones $A_i(x)$ over all $x \in X_i$. It is well-known that if u_i is concave, $A_i(x)$'s coincide with A_i for all $x \in X_i$. That is, every preferred set of u_i has the recession cone A_i (This result is stated in Appendix). We assume

(A3) The set of cones $\{A_i\}$ is positively semi-independent.

This condition is closely related to the absence of free lunch. If indifference curves do not contain a halfline for all agents, (A3) is necessary to preclude free lunch. Suppose there is a set $\{v_i\}$, not all zero, that satisfies $v_i \in A_i$ for each $i \in I$ and $\sum_{i \in A_i} v_i = 0$. For any nonzero price system $p \in R^l$, we see $\sum_{i \in A_i} p v_i = 0$, implying $p v_h \leq 0$ and $v_h \neq 0$ for some h. Since indifference curves have no halfline, the marginal utility of v_h at all consumptions is positive. Therefore v_h is a free lunch for the agent h. The condition (A3) is necessary for the existence of efficient allocations and competitive equilibrium.

However, it is far from being a sufficient condition for their existence. It is easy to take a two-agent economy that satisfies (A3) but allows no efficient allocation. The following proposition shows that if G is not empty, it is unbounded as in the example of Figure 2.

***Proposition* 1**
Suppose that there exists $\{v_i\}$ with $v_i \in A_i(w_i)$ such that $\sum v_i = 0$ and $v_h \in A_h \setminus \{0\}$ for some h in I. Assume indifference curves of u_h have no halflines. Then the set G is either empty or closed and unbounded.

Proof: Assume $G \neq \emptyset$. Let $\{x_n\}$ be a sequence in G that converges to a point x. Suppose that x is not weakly efficient. Then there exists an allocation x' in Q such that $U(x') \gg U(x)$. For sufficiently large n, we see $U(x') \gg U(x_n)$, which is impossible. Therefore G is closed.

Suppose G is bounded. It is well-known that if G is closed and bounded, so is the set W. Let W_h denote the projection of W onto the hth axis of R^m. It follows that W_h is closed and bounded in R^m. Let ν_h be the maximal element of W_h. Pick $\nu \in W$ whose hth coordinate is ν_h. Since ν is attainable, there exists an allocation $\bar{x}$ in Q with $\nu \leq U(\bar{x})$. On the other hand, an allocation $x+v$ is attainable. This implies $U(\bar{x}+v) \in W$. Since indifference curves of u_h have no halflines, we see $u_h(\bar{x}_h + v_h) > u_h(\bar{x}_h) \geq \nu_h$. It contradicts the fact that ν_h is maximal in W_h. We conclude that G is closed and unbounded.

Q.E.D

If (A3) holds, however, the economy cannot attain a welfare arbitrarily close to $\bar{u}$.

***Lemma* 1**: The utility allocation $\bar{u}$ is not in the closure of W under (A3).

Proof: To the contrary, suppose $\bar{u}$ is in the closure of W. Then there exists a sequence $\{u_n\}$ of utility vectors in W that converges to $\bar{u}$. For each n, we choose an allocation $x_n \in Q$ that satisfies $u_n \leq U(x_n)$. By definition of $\bar{u}$, we see that $\bar{u} = \lim_{n\to\infty} U(x_n)$. Set $I(\{x_n\}) = \{i \in I : \| x_{in} \| \to \infty\}$. We claim that $I(\{X_n\}) \neq \emptyset$. Suppose not. Then each $\{X_{in}\}$ is bounded. Without loss of generality, we may assume that $\{X_n\}$ converges to an allocation x. Then we see $\bar{u} \leq U(x)$, which contradicts (A2).

Let $\alpha_n = 1/\sum \| x_{in} \|$. Since $\sum \| \alpha_n x_{in} \| = 1$ for every n, each $\{\alpha_n x_{in}\}$ is bounded so that it has a subsequence converging to a point v_i. The fact that $\sum \| x_i \| = 1$ implies $v_h \neq 0$ for some $h \in I(\{x_n\})$. On the other hand, the attainability condition yields $\sum \alpha_n x_{in} = \alpha_n \sum w_i \to 0$ or $\sum v_i = 0$.

Now show that the vector v_h is in A_h. Pick any $x \in X_h$. Then there exists a number N such that $u_h(x_{hn}) \geq u_h(x)$ for every $n \geq N$. For every n with $\alpha_n \leq 1$, we have $u_h((1-\alpha_n)x + \alpha_n x_{hn}) \geq u_h(x)$. Since u_h is continuous, letting $n \to \infty$ yields a relation $u_h(x+v_h) \geq u_h(x)$. Thus the vector v_h is in A_h. The fact that $\sum v_i = 0$ contradicts (A3). Therefore u is not in the closure of W.

Q.E.D

Let P denote a set $\{p \in R^l \mid \| p \| = 1$ and $pv \geq 0$ for all $v \in \sum A_i\}$. A price system in P gives a non-negative value to every commodity bundle which is u-desirable to an agent. The relative interior of P consists of prices that do not admit free lunches. For a point $\nu \in W$, we denote by $P(\nu)$ a set of p's in P supporting ν. The following proposition shows that $P(\nu)$ is not empty for a point ν in $O(W)$.

***Proposition* 2**

A utility allocation $\nu \in O(W)$ is supported by a price system $p \in P$. In particular, any $x \in Q(\nu)$ satisfies $px_i \leq p\tilde{X}_i(\nu_i)$ for every $i \in I$.

Proof: We claim that $\sum_{i=1}^m w_i \notin \sum_{i=1}^m \tilde{X}_i^{\circ}(\nu_i)$. Otherwise, there exists an allocation $z \in Q$ satisfying $U(z) \gg \nu$. This contradicts the weak efficiency of ν. By the separating hyperplane theorem, there exists p' in $R^l \setminus \{0\}$ such that $p' \sum_{i=1}^m w_i \leq p' \sum_{i=1}^m \tilde{X}_i^{\circ}(\nu_i)$. Set $p = p' / \| p' \|$. It follows from (A2) that $\sum_{i=1}^m \tilde{X}_i(\nu_i) \subset$ the closure of $\sum_{i=1}^m \tilde{X}_i^{\circ}(\nu_i)$. This leads to a relation $p \sum_{i=1}^m w_i \leq p \sum_{i=1}^m \tilde{X}_i(\nu_i)$. For any $x \in Q(\nu)$, we can infer that $p \sum_{i=1}^m w_i \leq p \sum_{i=1}^m (x_i + A_i)$ and $p \sum_{i=1}^m w_i = p \sum_{i=1}^m x_i \leq p(\tilde{X}_h(\nu_h) + \sum_{i \neq h} x_i)$ for every $h \in I$. We conclude that $p \sum_{i=1}^m A_i \geq 0$ and $p\tilde{X}_h(\nu_h) \geq px_h$.

Q.E.D

V. Compact Truncations of the Utility Set

To ensure the existence of equilibria, general equilibrium models have adopted explicitly or implicitly the hypothesis that the utility set is closed and bounded. This hypothesis is inappropriate to the present setting because the utility frontier may not be closed as shown in Section III. To have its non-compact version, we need the following assumption.

(A4) Suppose that there is $\{v_i\}$, not all zero, with $v_i \in A_i(\omega_i$ $\sum_{i=1}^{m}$ $v_i = 0$. Then for each $\nu \in O(\mathrm{cl}\ W)$ with $\nu \ll \bar{u}$, there exists $h \in I$ such that $v_h \notin \tilde{A}_h(\nu_h)$.

This condition states that $\{\tilde{A}_i(\nu_i)\}$ is positively semi-independent for all ν which is efficient relative to cl W. Note that utility allocations in the frontier of cl W need not be attainable. For some $\varepsilon \gg 0$ in R^m, set $W(\varepsilon) = \{\nu \in W : \nu \leq \bar{u} - \varepsilon\}$. If indifference curves have no halflines for all $i \in I$, (A4) is necessary for utility allocations in $W(\varepsilon)$ to be attainable for each $\varepsilon \gg 0$. Suppose that (A4) does not hold. Pick $\{v_i\}$, not all zero, with $v_i \in A_i(\omega_i)$ and $\sum v_i = 0$, and a point ν in $o(W(\varepsilon))$. Let x be an allocation satisfying $U(x) = \nu$. Clearly an allocation $x + v$ is attainable. Since no indifference curves contain a half-line for every i, we must have $u_i(x_i + v_i) > u_i(x_i)$ for the non-zero v_i. This contradicts the efficiency of x. The following proposition demonstrates the sufficiency.

***Proposition* 3**

For every $\varepsilon \gg 0$ in R^m, $W(\varepsilon)$ is closed in R^m_+ under (A4).

Proof: Suppose not. Then for some $\varepsilon \gg 0$, there is ν in cl $W(\varepsilon) \setminus W(\varepsilon)$. Pick a sequence $\{\nu_n\}$ in $W(\varepsilon)$ such that $\nu_n \rightarrow \nu$. Then for each n, there exists x_n in Q that satisfies $\nu_n \leq U(x_n)$. Suppose that $\{x_n\}$ has a subsequence that converges to a point x in Q. Then we see that $\nu \leq U(x)$ and therefore, $\nu \in W(\varepsilon)$, which is impossible. Thus we must have $\| x_{in} \| \rightarrow \infty$ for some $i \in I$. Set $I(\{x_n\}) = \{i \in I : \| x_{in} \| \rightarrow \infty\}$.

Let $\gamma_n = 1/\sum \| x_{in} \|$ for sufficiently large n. Then we obtain

$$\sum_{i=1}^{m} \gamma_n \| x_{in} \| = 1 \text{ for every } n \text{ and}$$
$$\sum_{i=1}^{m} \gamma_n x_{in} = \sum_{i=1}^{m} \gamma_n \omega_i \rightarrow 0.$$

Since $\{\gamma_n x_{in}\}$ is bounded for each i, it has a subsequence which converges to a point v_i. The set of vectors $\{v_i\}$ satisfies the relations $\sum \| v_i \| = 1$ and $\sum v_i = 0$. Then there exists $h \in I(\{x_n\})$ with $v_h \neq 0$.

We claim that $v_h \in \tilde{A}_h(\nu_h)$. We can choose $z \in X_h$ with $u_h(z) \geq \nu_h$ because $\nu_h < \bar{u}_h$. For a given number $\lambda \geq 0$, choose N such that $\lambda \gamma_n < 1$ for all $n \geq N$. Then it follows from (A2) that for every $n \geq N$,

$$\lim_{n\to\infty} u_h(\lambda\gamma_n x_{hn} + (1-\lambda\gamma_n)z) \geq \min\{u_h(x_{hn}),\ u_h(z)\}.$$

Since u_h is continuous, we obtain $u_h(z+\lambda v_h) \geq \nu_h$, which implies that $v_h \in \tilde{A}_h(\nu_h)$. This contradicts (A4).

Q.E.D

Suppose $\hat{u} \ll \bar{u}$. Then the set W is a subset of $W(\varepsilon)$ for some $\varepsilon \gg 0$. It follows from Proposition 3 that W is closed under (A4).

***Corollary* 1**
If $\hat{u} \ll \bar{u}$, W is closed under (A4). If indifference curves have no halflines for all $i \in I$, (A4) is necessary and sufficient for W to be closed.

VI. Near-Boundary Conditions

This section is devoted to formalizing the idea of the near-boundary condition illustrated in Section III. We take a closed subset of the utility frontier $O(W)$ by truncating W with a closed rectangle such that its boundary is sufficiently near the boundary of $O(W)$. Then we proceed to impose the desired requirement on the boundary of the truncated utility frontier. The idea is simple as revealed in Figure 3 but the formal description involves more or less complex procedures.

For a point $\varepsilon \gg 0$ in R^m, define a set $K(\varepsilon) = \{\nu \in R^m_+ \mid \nu_i \leq \bar{u}_i - \varepsilon_i$ for every $i \in I\}$ and set $W_\varepsilon = O(W(\varepsilon)) \cap O(W)$, By Proposition 3, W_ε is closed. Note that if $\hat{u} \ll \bar{u} - \varepsilon$, then $W_\varepsilon = O(W)$.

Let Δ denote the set $\{s \in R^m_+ \mid \sum_{i=1}^m s_i = 1\}$ and for each i, Δ_i the set $\{s \in \Delta \mid s_i = 0\}$. For each $s \in \Delta$, set $\tau(s) = \max(\alpha > 0 \mid \alpha s \in \mathrm{cl}\ W)$ and $f(s) = \tau(s)s$, and $\tau_\varepsilon(s) = \max\{\alpha > 0 \mid \alpha s \in K(\varepsilon)\}$ and $d_\varepsilon(s) = \tau_\varepsilon(s)s$. The following lemma is immediate from Moore (1975).

Lemma 2: The function $d_\varepsilon : \Delta \to O(K(\varepsilon))$ is homeomorphic. And if $w \notin O(W)$, the function $f : \Delta \to O(\mathrm{cl}\ W)$ is also homeomorphic.

Set $\Delta_\varepsilon = f^{-1}(W_\varepsilon)$ and let f_ε denote the restriction of f to Δ_ε. By Lemma 2, f_ε is a homeomorphism between Δ_ε and W_ε. Let $d_{2\varepsilon}$ denote the restriction of d_ε to Δ_ε. Set $T_\varepsilon = d_{2\varepsilon} \circ f_\varepsilon^{-1}(W_\varepsilon)$. By Lemma 2, $d_{2\varepsilon}$ is a homeomorphism between Δ_ε and T_ε. One of the

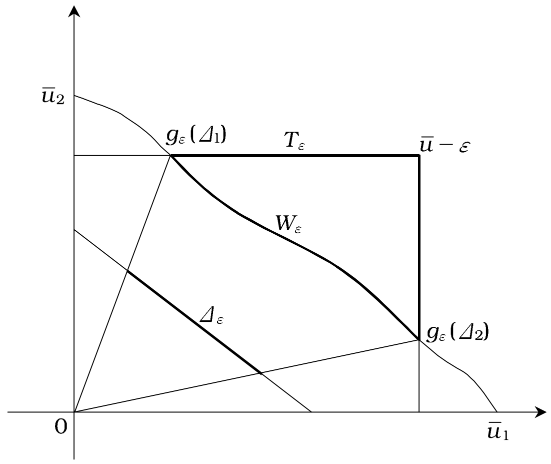

FIGURE 3

difficulties with this section is to show that there is a homeomorphism between T_ε and $O(K(\varepsilon))$. The homeomorphism will be constructed to serve well our goal of obtaining the near-boundary condition.

By Lemma 1, for sufficiently small $\varepsilon \gg 0$, we see $\bar{u}-\varepsilon \notin \mathrm{cl}\, W$. From now on, we implicitly assume that this condition is satisfied with the vector ε whenever it is used later. Let $B(\bar{u}-\varepsilon, r_\varepsilon)$ denote an open ball centered at $\bar{u}-\varepsilon$ with radius r_ε that does not intersect cl$W$. Clearly, $K(\varepsilon)\setminus W$ has a nonempty interior in R^m for a small vector $\varepsilon \gg 0$. It is clear that $\bar{u}-\varepsilon$ is in the relative interior of T_ε.[12] Let t be a point in $O(K(\varepsilon))\setminus\{\bar{u}-\varepsilon\}$. Since T_ε is compact, $\{v\in R^m \mid v=\alpha(t-(\bar{u}-\varepsilon))+\bar{u}-\varepsilon$ for some $\alpha\geq 0\}$ always intersects the boundary of T_ε. For each $t\in O(K(\varepsilon))\setminus\{\bar{u}-\varepsilon\}$, define numbers

$$\delta(t)=\max\{\alpha\in R_+ \mid \alpha(t-(\bar{u}-\varepsilon))+\bar{u}-\varepsilon\in T_\varepsilon\} \text{ and}$$
$$\delta_0(t)=\max\{\alpha\in R_+ \mid \alpha(t-(\bar{u}-\varepsilon))+\bar{u}-\varepsilon\in O(K(\varepsilon))\}$$

These numbers are well-defined because T_ε and $O(K(\varepsilon))$ are compact. Clearly, for each $t\in O(K(\varepsilon))\setminus\{\bar{u}-\varepsilon\}$, we have

[12]The set T_ε is considered a subspace of the subspace $O(K(\varepsilon))$ of R^m.

$$\delta(t)(t-(\bar{u}-\varepsilon))+\bar{u}-\varepsilon \notin T_\varepsilon \setminus B(\bar{u}-\varepsilon,\ r_\varepsilon).$$

The following lemma shows the continuity of functions δ and δ_0.

Lemma 3: The functions δ and δ_0 from $O(K(\varepsilon))\setminus\{\bar{u}-\varepsilon\}$ to R_+ are continuous.

The proof of Lemma 3 is relegated to the Appendix. The lemma enables us to show that T_ε and $O(K(\varepsilon))$ are homeomorphic. Define a mapping $h_\varepsilon : T_\varepsilon \to O(K(\varepsilon))$ by

$$h_\varepsilon(t)=\begin{cases}\dfrac{\delta_0(t)}{\delta(t)}(t-(\bar{u}-\varepsilon))+\bar{u}-\varepsilon \text{ if } t\neq\bar{u}-\varepsilon\\ \bar{u}-\varepsilon \text{ if } t=\bar{u}-\varepsilon\end{cases}$$

The inverse mapping h_ε^{-1} takes the form of

$$h_\varepsilon^{-1}(t)=\begin{cases}\dfrac{\delta_0(t)}{\delta(t)}(t-(\bar{u}-\varepsilon))+\bar{u}-\varepsilon \text{ if } t\neq\bar{u}-\varepsilon\\ \bar{u}-\varepsilon \text{ if } t=\bar{u}-\varepsilon\end{cases}$$

The following lemma shows that both h_ε and h_ε^{-1} are continuous.

Lemma 4: The function h_ε and h_ε^{-1} are continuous.

Proof: By Lemma 3, we have only to show the continuity of h_ε and h_ε^{-1} at $t=\bar{u}-\varepsilon$. Recall that $\delta_0(t)/\delta(t)\geq 1$ for all $t\in O(K(\varepsilon))\setminus\{\bar{u}-\varepsilon\}$. Thus they are continuous at $\bar{u}-\varepsilon$ if $\lim_{t\to\bar{u}-\varepsilon}\delta_0(t)/\delta(t)$ is finite.

Suppose that $\delta_0(t_n)/\delta(t_n)\to\infty$ as $t_n\to\bar{u}-\varepsilon$. For each n, set $v_n=\delta_0(t_n)(t_n-(\bar{u}-\varepsilon))+u-\bar{\varepsilon}$ and $w_n=\delta(t_n)(t_n-(\bar{u}-\varepsilon))+\bar{u}-\varepsilon$. By definition each w_n is in T_ε. Since $v_n\in O(K(\varepsilon))$, $\{v_n\}$ is bounded. It follows that $w_n-(\bar{u}-\varepsilon)=(\delta(t_n)/\delta_0(t_n))(v_n-(\bar{u}-\varepsilon))\to 0$, which contradicts the fact that $w_n\notin T_\varepsilon\setminus B(\bar{u}-\varepsilon,\ r_\varepsilon)$.

Q.E.D

Clearly, h_ε and h_ε^{-1} are bijective. Therefore h_ε is a homeomorphism between T_ε and $O(K(\varepsilon))$.

Define a function $g_\varepsilon : \varDelta \to W_\varepsilon$ by $g_\varepsilon(s)=f_\varepsilon\circ d_{2\varepsilon}^{-1}\circ h_\varepsilon^{-1}\circ d_\varepsilon(s)$;

$$\varDelta \xrightarrow{d_\varepsilon} O(K(\varepsilon)) \xrightarrow{h_\varepsilon^{-1}} T_\varepsilon \xrightarrow{d_{2\varepsilon}^{-1}} \varDelta_\varepsilon \xrightarrow{f_\varepsilon} W_\varepsilon$$

The function g_ε is a homeomorphism between Δ and W_ε. Note that $g_\varepsilon = f$ if W is closed and $W \subset K(\varepsilon)$ for some $\varepsilon \gg 0$. Since g_ε and f are homeomorphisms, $\cup_{i=1}^m g_\varepsilon(\Delta_i)$ and $\cup_{i=1}^m f(\Delta_i)$ form the relative boundary of W_ε and $O(\text{cl } W)$, respectively. For each i, $g_\varepsilon(\Delta_i)$ comes closer to $f(\Delta_i)$ as ε goes to zero. The set $g_\varepsilon(\Delta_i)$ is identified as a component of the near-boundary set of $O(\text{cl } W)$. We impose the boundary condition on $g_\varepsilon(\Delta_i)$ as following.

(A5) There is a point ε^* in R^m_{++} such that for each $\nu \in g_{\varepsilon^*}(\Delta_i)$ and each $p \in P(\nu)$, (i) either all $x \in Q(\nu)$ satisfy $pw_i \geq px_i$ for all $i \in I$ or (ii) all $x \in Q(\nu)$ satisfy $pw_i \leq px_i$ for all $i \in I$.

By construction, agent i is less favorably treated in the weakly efficient allocation $\nu \in g_\varepsilon(\Delta_i)$ than other agents for sufficiently small ε. This point is clear when W is closed. In this case W coincides with W_ε for a small $\varepsilon \gg 0$. By definition, the ith component ν_i of ν is equal to zero. Let $p \in P(\nu)$. Then (i) of (A5) imposes the condition that agent i be wealthy enough at p to afford his less favorable position at ν.

It is easy to see that (A5) always holds if W is closed. We choose $\varepsilon \gg 0$ such that W coincides with W_ε. Let $\nu \in g_\varepsilon(\Delta_i)$ and $x \in Q(\nu)$. Recall that $\nu_i = 0$. It follows from Proposition 2 that for every $p \in P(\nu)$, $px_i \leq p\bar{X}_i(0)$, which implies $px_i \leq pw_i$.

Figure 2 helps understand the role of (A5) for the existence of equilibrium. Let ν^a and ν^b denote the utility allocation at the points a and b, respectively. Let $p_a \in P(\nu^a)$ and $p_b \in P(\nu^b)$, and $x^a \in Q(\nu^a)$ and $x^b \in Q(\nu^b)$. (A5) trivially holds for agent 1 because $u_1(x_1^a) = u(w_1)$. (A5) holds for agent 2, too because $p_b x_2^b < p_b w_2$. As discussed before, these boundary-like conditions ensure the existence of equilibrium between a and b along the contract curve.

General cases for which (A5) holds will be discussed in the next section. The condition (A5) cannot be dispensed with because (A5) is necessary for the existence of equilibria in a two-agent economy. Suppose it is not the case with $m=2$. Choose a point x in G. First, assume that $p(x_1 - w_1) \geq 0$ for a price system $p \in P(U(x))$. If $p(x_1 - w_1) = 0$, (x, p) is an equilibrium by Walras' law. Thus we must have $p(x_1 - w_1) > 0$. This implies that for each $z \in G \setminus \{x\}$, z_2 satisfies $q(z_2 - w_2) < 0$ where q is a price system supporting $U(z)$; otherwise, there would be some $z' \in G$ with $q(z'_1 - w_1) < 0$ and then, the pushing process as described in Section III would ensure the existence of equilibria between the two points on the contract curve which represent x and z, respectively. This result contradicts (A5). Similar arguments apply to the case that $p(x_1 - w_1) \leq 0$.

VII. The Existence of Competitive Equilibria

A pair (x, p) in $Q \times R^l$ with $p \neq 0$ is a quasi-equilibrium if for every i, $px_i = pw_i$ and $pz \geq px_i$ whenever $z >_i x_i$. The pair (x, p) is an equilibrium if for every i, $pz > px_i$ whenever $z >_i x_i$. We need further assumptions to verify that a quasi-equilibrium of an economy is in fact an equilibrium.

If w is weakly efficient, it is easy to see from Proposition 2 that for some p in $P(0)$, (w, p) is a quasi-equilibrium. Hence we may assume without loss of generality that w is not weakly efficient. The main result of the paper is provided as follows.

Theorem 1 : Under the assumptions (A1)-(A5), the economy has a quasi-equilibrium $(p, x) \in P \times Q$.

The proof of this theorem appears in the Appendix. It is well-known that a quasi-equilibrium becomes an equilibrium under the minimum wealth constraint

(M) Every p in P satisfies $pw_i > \inf pX_i$ for all $i \in I$.

VIII. Examples

This section is devoted to the illustrations for which (A5) holds. The conditions (Al)-(A4) and (M) are assumed to hold throughout this section. For each i, let W_i denote a set $\{\nu \in W \mid \nu_i = 0\}$ and R_i the set of attainable utility allocations for the restricted economy in which the agent i is not allowed to trade so that his consumption is restricted to w_i. It is clear that $R_i \subset W_i$ for every $i \in I$. We make the following definition.

Definition 2: An agent i is conducive (to the economy) if for any $\nu \in R_i$, there is $\nu' \in W_i$ satisfying $\nu'_h > \nu_h$ for each $h \neq i$.

An agent i is conducive if the rest of the economy gets better by making him free to trade. In a special case that $m=2$, every agent is conducive if and only if w is not weakly efficient. Before going to examples, we quote the following lemma from Moore (1975, p. 284).

Lemma 5: Let ν and ν' be points in W satisfying $\nu'_i \geq \nu_i$ for all $i \in I$ and $\nu'_i > \nu_i$ for all i for which $\nu'_i > 0$. Then there exists $z \in Q$ which satisfies $U(z) \gg \nu$.

For a positive vector $\varepsilon \in R^l$, set $\nu_i(\varepsilon) = \max\{a \in R_+ \mid a$ is the ith element of some $\nu \in g_\varepsilon(\Delta_i)\}$ and $\mu_i(\varepsilon) = \min\{a \in R_+ \mid a$ is the ith element of some $\nu \in g_\varepsilon(\Delta_i)\}$. Define a set $G_i(\varepsilon) = \{x \in G \mid U(x) \in g_\varepsilon(\Delta_i)\}$. Take a sequence $\varepsilon_n \to 0$ such that $\nu(\varepsilon_n) \to \bar{\nu}$ and $\mu(\varepsilon_n) \to \bar{\mu}$ for some $\bar{\nu}_i$ and $\bar{\mu}_i$. Set $I_\infty = \{i \in I \mid \bar{\nu}_i > 0\}$. Let $B(x, r)$ ($B(r)$) denote an open ball in R^l centered at x (zero, respectively) with the radius $r>0$.

We assume throughout the following two examples that indifference curves do not contain a line segment in $X_i \cap B(w_i, r)$ for some $r>0$.

Example **1:** Assume that for every i, (i) i is conducive to the economy and (ii) $\bar{\nu}_i = 0$. Then the condition (i) of (A5) is fulfilled.

Proof: First, we show that $w_i \notin G_i$ for each i. To the contrary, suppose that $w_i \in G_i$. Let x be an allocation in G with $x_i = w_i$. Then the utility allocation $U(x)$ is in R_i. Since i is conducive, we can find ν such that $\nu_h > u_h(x_h)$ for every $h \neq i$ and $\nu_i = 0$. By Lemma 5, there is x' in Q which satisfies $U(x') \gg U(x)$. This contradicts the weak efficiency of x.

Since G_i is closed and w_i is not in G_i, there is $r>0$ in R^l such that $B(w_i, r)$ does not intersect G_i. Without loss of generality, we can assume that r is chosen in such a way that indifference curves do not contain a line segment in $X_i \cap B(w_i, r)$. Set $q_n = \inf\{p(w_i - z) \mid z$ is the ith element of some $x \in G(\varepsilon_n)$ and $p \in P$ supports $x\}$. Since P and $G(\varepsilon_n)$ are compact, there exist $p_n \in P$ and $x \in G(\varepsilon_n)$ for each n such that p_n supports x_n and $q_n = p_n(w_i - x_{in})$. Let p be a point in P to which a subsequence of $\{p_n\}$ converges.

Now we show that for some ε_n, (i) of (A5) holds. Consider the case that $\{x_{in}\}$ is bounded. Let x' be the limit point of $\{x_{in}\}$. Then it follows from the condition (ii) that $u_i(x') = u_i(w_i)$. Since u_i is locally non-satiable by (A2) and $u_i(x_{in}) \to u_i(w_i)$, we can pick $v_n \to 0$ such that $u_i(x_{in}) = u_i(w_i + v_n)$. Recalling that p_n supports x_{in}, we have $p_n x_{in}$

$\leq p_n(w_i+v_n)$. Passing to the limit, we see $px' \leq pw_i$. Since $x' \notin B(w;\ r)$, (A2) implies that for a number $\alpha \in (0,\ 1)$, $u_i(x') < u_i(\alpha w_i + (1-\alpha)x')$. By (M), we can choose $z \in X_i$ such that $p(z-w_i) < 0$. Then there is a small number $\beta \in (0,\ 1)$ such that $u_i(x') < u_i((1-\beta)(\alpha w_i + (1-\alpha)x') + \beta z))$. This implies $px' \leq p((1-\beta)(\alpha w_i + (1-\alpha)x') + \beta z))$. Since $pz < pw_i$, we have $px' < p((1-\beta)(\alpha w_i + (1-\alpha)x') + \beta w_i))$ or $px' < pw_i$, implying that lim inf $q_n = p(w_i - x') > 0$.

We turn to the case that $\| x_{in} \| \to \infty$. Let v be a point in R^l to which a subsequence of $\{x_{in} / \| x_{in} \|\}$ converges. Since $u_i(x_{in}) \geq u_i(w_i)$ for every n, v is in $A_i(w_i)$. The condition (ii) implies $u_i(x_{in}) \to u_i(w_i)$. Since indifference curves have no line segment around w_i, we see $u_i(w_i+v) > u_i(w_i)$. Then for sufficiently large n, there exists $0 < \gamma < 1$ such that $u_i((1-\gamma)(w_i+v) + \gamma(z+v)) > u_i(x_{in})$. From now on, we assume that n is large enough to ensure it. Since p_n supports x_n, we have $p_n x_{in} \leq p_n(w_i+v+\gamma(z-w_i))$. The fact that $p_n(x_{in} / \| x_{in} \|) \leq p_n(w_i + v + \gamma(z-w_i)) / \| x_{in} \|$ implies $pv \leq 0$. Since $p(z-w_i) < 0$, we see that lim inf $p_n x_{in} \leq$ lim $p_n(w_i + v + \gamma(z-w_i)) < pw_i$ or lim inf $q_n > 0$. Therefore we conclude that $q_N > 0$ for some number N.

Q.E.D

Example 1 is the case where the contract curve or plane meets or asymptotically meet the indifference curve or plane of agent i through the initial endowment, This condition would hold in Figure 2 if the contract curve were to come closer and closer to the indifference curve II and lead to no discrete gap between them in the limit. Otherwise, Figure 2 belongs to the following class of Example 2 which extends Example 1 to the case that the gap between the contract curve and the indifference curve through the initial endowment need not be zero in the limit.

***Example* 2:** Assume that $I_\infty \neq \emptyset$ and an economy satisfies the following conditions; (i) If $i \in I_\infty$, then (a) for any $\nu \in [\overline{\nu_i},\ \overline{\mu_i}]$ and $v \in \tilde{A}_i(\nu)$, a set $\{w_i + \lambda v \mid \lambda \geq 0\}$ intersects $\tilde{X}_i(\nu)$ and (b) for any $\{x_n\}$ in G with $U(x_n) \in g_{\varepsilon n}(\Delta_i)$, we have $\| x_{in} \| \to \infty$, and (ii) if $i \in I \setminus I_\infty$, i is conducive. Then (i) of (A5) is satisfied.

Proof: For any $i \in I \setminus I_\infty$, the same argument applies as in Example 1. In the case that $i \in I_\infty$, we can still proceed in a similar way as in Example 1. Set $q_n = \min\{p_n(w_i - z) \mid z$ is the ith element of some $x \in G(\varepsilon_n)$ and $p \in P$ supports $x\}$. Since P and $G(\varepsilon_n)$ are compact, there exist $p_n \in P$ and $x_n \in G(\varepsilon_n)$ for each n such that $q_n = p_n(w_i - x_{in})$ and p_n supports x_n. Without loss of generality, we will assume that $p_n \to p$, $u_i(x_{in}) \to \hat{\nu}_i$, and $x_{in}/\|x_{in}\| \to v_i$ for some $p \in P$, $\hat{\nu}_i \in R_+$, and $v \in R^l$, respectively. Clearly, $\hat{\nu}_i$ is in $[\overline{\mu}_i, \overline{\nu}_i]$.

To show that (i) of (A5) holds for some ε_n, we have only to prove that lim inf $q_n > 0$. By the same argument as in proving Proposition 3, v_i is in $\tilde{A}_i(\hat{\nu}_i)$. Since $\hat{\nu}_i \in [\overline{\mu}_i, \overline{\nu}_i]$, (i) ensures that there exists $\lambda > 0$ such that $u_i(w_i + \lambda v_i) > \hat{v}_i$. From (M), choose $z \in X_i$ such that $p(z - w_i) < 0$. Then for sufficiently large n, there exists $0 < \gamma < 1$ such that $u_i((1-\gamma)(w_i + \lambda v_i) + \gamma(z + \lambda v_i)) = u_i(w_i + \lambda v_i + \gamma(z - w_i)) > u_i(x_{in})$. From now on, we assume that n is large enough to ensure the inequality. Since p_n supports x_n, we have $p_n x_{in} < p_n(w_i + \lambda v_i + \gamma(z - w_i))$. By the same argument of Example 1, we have $pv_i \leq 0$. It follows that lim inf $p_n x_{in} \leq$ lim inf $p_n(w_i + \lambda v_i + \gamma(z - w_i)) < pw_i$.

Q.E.D

Appendix

Let C be a convex set in R^l. By $\Gamma(C)$, we denote the recession cone of C. The cone $\Gamma(C)$ contains 0 in R^l and for two convex subsets C_1 and C_2 of R^l, $C_1 \subset C_2$ implies $\Gamma(C_1) \subset \Gamma(C_2)$. The following results about the recession cone in R^l are found in Rockafellar (1970).

***Lemma* 1A:** Suppose C is a non-empty closed convex set in R^l. Then $\Gamma(C)$ is closed, and it consists of all possible limits of sequences $\{x_n/\lambda_n\}$ with $x_n \in C$, $\lambda_n > 0$, and $\lambda_n \to \infty$.

Lemma 1A leads to the following corollary.

***Corollary*:** Let C be a non-empty closed convex set in R^l. Then (i) C is bounded if and only if $\Gamma(C) = \{0\}$. (ii) Let v be a vector in E. If $z + \lambda v \in C$ for some $z \in C$ and all $\lambda \geq 0$, then $v \in \Gamma(C)$.

A concave function has a simple property in terms of the recession cones of the level sets.

***Lemma* 2A:** Suppose that $f(x)$ is a concave function which is continuous on a convex subset C of R^l. Then the level set $P(x) = \{y \in C \mid f(y) \geq f(x)\}$ has the same recession cone for all $x \in C$.

***Proop of Lemma* 3:** First, show the continuity of δ. For a point $t \in O(K(\varepsilon)) \setminus \{\bar{u}-\varepsilon\}$, choose $t_n \rightarrow t$. We must prove that $\delta(t_n) \rightarrow \delta(t)$. Let $r(t')$ denote $\delta(t')(t'-(\bar{u}-\varepsilon))+\bar{u}-\varepsilon$ for each $t' \in O(K(\varepsilon))$. Set $v_n = r(t_n)$ for each n, $v=r(t)$, and $v' = \lim v_n = \lim \delta(t_n)(t-(\bar{u}-\varepsilon))+\bar{u}-\varepsilon$. Since T_ε is compact, $v' \in T_\varepsilon$. This implies that $\lim \delta(t_n) \leq \delta(t)$. We have only to show that $\lim \delta(t_n) \geq \delta(t)$. If v' is in the relative boundary of $O(K(\varepsilon))$, there is $i \in I$ that satisfies $v'_i = 0$ or $\lim \delta(t_n) = (\bar{u}-\varepsilon)/((\bar{u}-\varepsilon)-t_i)$. On the other hand, $v_i \geq 0$ produces $\delta(t) \leq (\bar{u}-\varepsilon)/((\bar{u}-\varepsilon)-t_i)$. Then we obtains $\delta(t) \geq \lim \delta(t_n)$. It remains to verify $\delta(t) \geq \lim \delta(t_n)$ in the case that

$$v' \in \text{ int } O(K(\varepsilon)).$$

The verification is conducted in the following three steps.

Step 1: We claim that v' is in $O(W)$. First, show that the point v' is in W. Suppose that $v' \notin W$. Then for sufficiently large n, v_n is in $\text{int } O(K(\varepsilon)) \setminus W$ since v' is in $\text{int } O(K(\varepsilon))$ and W is closed. This implies that $(\delta(t_n)+\eta)(t_n-(\bar{u}-\varepsilon))+\bar{u}-\varepsilon \in \text{int } O(K(\varepsilon)) \setminus W$ holds for some small number $\eta > 0$. Thus the point is in T_ε, which contradicts the maximality of $\delta(t_n)$. Now show that v' is in $O(W)$. Suppose that $v' \in W \setminus O(W)$. Then there is a point $v'' \in W$ satisfying $v'' \gg v'$. Recall that v' is in $\text{int } O(K(\varepsilon))$. This implies $v' \gg 0$. Thus for sufficiently large n, $v'' \gg v' \gg 0$ must be satisfied. Since the point v_n is in $\text{int } W$, we see $v_n \notin T_\varepsilon$, which is impossible. Therefore we conclude that $v' \in O(W)$.

Step 2: Show that for a sufficiently small open ball B in R^m, $(v+B) \cap O(K(\varepsilon)) \subset (v'-R^m_+) \cap O(K(\varepsilon))$. Let $\tilde{v}$ be a point in $(v+B) \cap O(K(\varepsilon))$. Then there is $b \in B$ which satisfies $\tilde{v} = v+b \in O(K(\varepsilon))$. Clearly we have $0 \leq \tilde{v} \leq \bar{u}-\varepsilon$. We can rewrite $\tilde{v}$ as $\tilde{v} = v' - (\delta(t) - \lim \delta(t_n))(\bar{u}-\varepsilon-t)+b$. Set $b' = (\delta(t) - \lim \delta(t_n))(\bar{u}-\varepsilon-t)-b$. We must show that $b' \in R^m_+$. If $t_i = \bar{u}_i - \varepsilon_i$, we see that $b'_i = (\bar{u}_i-\varepsilon_i)-v_i \geq 0$. If $t_i < \bar{u}_i - \varepsilon_i$, we see that $b'_i = (\delta(t) - \lim \delta(t_n))(\bar{u}_i-\varepsilon_i-t_i)-b_i > 0$ since $\delta(t) - \lim \delta(t_n) > 0$ and b_i is a sufficiently small number. Those results imply that $b' \in R^m$.

Step 3: Since $v' \in O(W)$, we have $(v'-R^m_+) \cap O(K(\varepsilon)) \subset W \cap O(K(\varepsilon))$. It follows from Step 2 that $(v+B) \cap O(K(\varepsilon)) \subset W \cap O(K(\varepsilon))$. Since $(v+B) \cap O(K(\varepsilon))$ is an open neighborhood of v in $O(K(\varepsilon))$, the point v is in $\text{int}(W \cap O(K(\varepsilon)))$. This implies that $v \notin O(K(\varepsilon)) \setminus \text{int }(W \cap O(K(\varepsilon)))$. On the other hand, a set $O(K(\varepsilon)) \setminus W$ is a subset of $O(K(\varepsilon)) \setminus \text{int }(W \cap O(K(\varepsilon)))$ which is closed. It follows that $T_\varepsilon \subset O(K(\varepsilon)) \setminus \text{int }(W \cap O(K(\varepsilon)))$. This implies $v \notin T_\varepsilon$, which is impossible. Therefore we conclude that $\delta(t) = \lim \delta(t_n)$.

We remark that the above argument holds for the case that $W \subset$

$K(\varepsilon)$. But $W \subset K(\varepsilon)$ implies $O(K(\varepsilon)) = T_\varepsilon$. Since the function δ_0 coincides with δ in this case, the continuity of δ_0 is immediate.

Q.E.D

***Proof of Theorem* 1:** The argument is an adaptation of Magill (1981) and Brown and Werner's (1995) proof to the case under consideration. For a point $s \in \Delta$, set $\Pi(s) = P(g_{\varepsilon^*}(s))$. Let $p(s)$ denote a point in $\Pi(s)$.

For an allocation $x(s) \in Q(g_{\varepsilon^*}(s))$, define a correspondence $\Phi = (\Phi_1, \cdots, \Phi_m) : \Delta \to R^m$ by

$$\Phi(s) = \{(e_1, \cdots, e_m) \in R^m \mid e_i = p(w_i - x_i(s)),\ (i \in I) \text{ for some } p \in \Pi(s)\}$$

Then $\Phi(s)$ is convex and not empty from Proposition 2. We note that for every $e \in \Phi(s)$, $\sum_{i=1}^m e_i = 0$ since $\sum_{i \in I}(w_i - x_i(s)) = 0$. We consider only the case that (i) of (A5) holds (If (ii) of (A5) holds, we have only to replace e_i by $e'_i = p(x_i(s) - w_i)$ for every $i \in I$).

Step 1: We claim that the range of Φ is bounded. Since each $\nu \in W_{\varepsilon}^*$ satisfies $\nu \ll \bar{u}$, we can pick a point $\hat{x}_i \in X_i$ for every i such that $u_i(\hat{x}_i) > \max\{a \in R_+ \mid a$ is the ith element of some $\nu \in W_{\varepsilon}^*\}$. Then for any s in Δ, we obtain $u_i(\hat{x}_i) > g_i(s)$. It follows from Proposition 2 that $p(s)x_i(s) \leq p(s)\hat{x}_i$ for some $p(s) \in \Pi(s)$, which implies that $\Phi(\Delta)$ is bounded above. On the other hand, $x(s)$ satisfies $\sum_{i=1}^m p(s)x_i(s) = \sum_{i=1}^m p(s)w_i$. Since $p(s)x_i(s)$ is bounded above for all $s \in \Delta$, it implies that $p(s)x_i(s)$ is bounded below as well. Consequently, $\Phi(\Delta)$ is bounded.

Step 2: We will show that Φ is upper hemicontinuous. Since the range of Φ is bounded, we may assume that it is a correspondence from Δ to some compact subset of R^l containing $\Phi(\Delta)$. Let $\{s_n\}$ be a sequence in Δ conversing to a point s. Choose ν_n and ν in W_{ε}^* such that $\nu_n = g_{\varepsilon^*}(s_n)$ and $\nu = g_{\varepsilon^*}(s)$. Since P is compact, there is $p \in P$ such that $p(s_n) \to P$. For some point e in R^m, let $e(s_n) \to e$. We will show that $e = p(w_i - x_i(s))$ and $p \in \Pi(s)$. Recall that $u_i(\hat{x}_i) > u_i(x_i(s)) \geq \nu_i$ for every s. By (A2) we see $u_i(\alpha\hat{x}_i + (1-\alpha)x_i(s)) > u_i(x_i(s))$ for every α in (0, 1) For sufficiently large n, we obtain $u_i(\alpha\hat{x}_i + (1-\alpha)x_i(s)) > \nu_{in}$. It follows from Proposition 2 that $p(s_n)(\alpha\hat{x}_i + (1-\alpha)x_i(s)) \geq p(s_n)x_i(s)$. Passing to the limit, we have $p(\alpha\hat{x}_i + (1-\alpha)x_i(s)) \geq pw_i - e_i$ for all α in (0, 1), which results in an inequality $e_i \geq p(w_i - x_i(s))$. Since $\sum_{i \in I} e_i$, we have $e_i = p(w_i - x_i(s))$ for every i.

It remains to show that $p \in \Pi(s)$. Let x' be an allocation such that $u_i(x'_i) > \nu_i$ for every i. Then for sufficiently large n, we have $u_i(x'_i) > \nu_{in}$.

It follows from Proposition 2 that $p(s_n)(\sum_{i=1}^{m}x'_i - w) \geq 0$. Passing to the limit, we obtain $p(\sum_{i=1}^{m}x'_i - w) \geq 0$. Let x be an allocation which satisfies $u_i(x_i) \geq \nu_i$ for every i. Since preferences are locally non-satiable, we have $p(\sum_{i=1}^{m}x_i - w) \geq 0$. It implies $p \in \Pi(s)$.

Step 3: This step involves an application of Kakutani's fixed point theorem. Let V be a compact and convex set in R^m containing $\Phi(\Delta)$. We define a correspondence $\Psi = (\Psi_1, \cdots, \Psi_m) : \Delta \times V \rightarrow \Delta \times V$ by

$$\Psi(s, t) = \left(\frac{\max(s_1 + t_1, 0)}{\sum_{i=1}^{m}\max(s_i + t_i, 0)}, \cdots, \frac{\max(s_m + t_m, 0)}{\sum_{i=1}^{m}\max(s_i + t_i, 0)}, \Phi(s) \right)$$

Then Ψ is upper hemicontinuous on $\Delta \times V$ and $\Psi(s, t)$ is nonempty, convex and compact for all $(s, t) \in \Delta \times V$. By Kakutani's fixed point theorem, there is a point $(\bar{s}, \bar{t}) \in \Delta \times V$ such that $(\bar{s}, \bar{t}) \in \Psi(\bar{s}, \bar{t})$. Then we see $\bar{t}_i = e_i(\bar{s}) = p(\bar{s})(w_i - x_i(\bar{s}))$ for all $i \in I$. Suppose that $s_i = 0$ for some $i \in I$. Then it follows from (i) of (A5) that $\bar{t}_i \geq 0$. Since $\max(\bar{t}_i, 0) = 0$, we see $\bar{t}_i = 0$. For other i's with $\bar{s}_i > 0$, we have $\max(\bar{s}_i + \bar{t}_i, 0) > 0$, which implies $\max(\bar{s}_i + \bar{t}_i, 0) = \bar{s}_i + \bar{t}_i$. It yields $\sum_{i=1}^{m}\max(\bar{s}_i + \bar{t}_i, 0) = 1$, This results in $\bar{s}_i + \bar{t}_i = \bar{s}_i$, which yields $\bar{t}_i = 0$. Therefore, we see $0 \in \Phi(\bar{s})$. We come to the conclusion that the pair of $x(\bar{s}) \in Q$ and $p(\bar{s}) \in P$ is a quasi-equilibrium of the economy.

Q.E.D

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