Three Solutions to a Simple Commons Problem

Hervé Moulin [1]

We compare the equity and incentive properties of three efficient solutions to a simple problem of cooperative production with binary demands for a homogeneous service, when marginal cost is either monotonically increasing or monotonically decreasing.

The solutions are the familiar competitive equilibrium with equal incomes, the Shapley value of the stand alone cooperative game, and the virtual price solution, applying the egalitarian equivalence idea to this particular model.

Keywords: Cooperative production, No-envy, Shapley value, Tragedy of the commons

JEL Classification : C71, D63, D78

I. Introduction

The "problem of the commons" is a time-honored conundrum of economic theory. When a production technology is the common property of its potential users, how should we organize its exploitation so as to achieve simultaneously i) an efficient utilization of the resources (Pareto optimality), ii) an equitable distribution of the surplus (fairness) and iii) simple decentralized choices for the potential users compatible with dispersed information about private

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characteristics (incentive compatibility)?

Familiar examples of commons include pastures, fishing grounds and other exhaustible resources (Dasgupta and Heal 1979).

In the last decade, congested networks have been successfully analyzed as commons where each user requests a service and waiting until service is completed is costly to the users (see Demers *et al.* (1990); Shenker (1990); Moulin and Shenker (1999); and Crès and Moulin (1999) and references therein).

At the heart of the commons problem is the fact that the returns of the technology vary with the intensity of usage. For instance, average waiting time for a homogeneous service by single server (who serves one user per unit of time) increases linearly in the number of jobs: each new user increases the average cost to all other users. Consequently, in the *free access* regime where everyone chooses freely to request service or not and costs are shared equally, the commons will be inefficiently overutilized, a central difficulty known as the *tragedy* of the commons (Hardin 1968).

In this paper, we look at the simplest conceivable model of the commons, where each potential user wants one unit of a homogeneous service, users only differ by their willingness to pay for service, and total cost varies either as a supermodular (convex) or a submodular (concave) function of total demand. A good supermodular example is the queuing problem just mentioned (each user may or may not need one job; the server processes one job per unit of time; see Crès and Moulin (1999)). A good submodular example is the connection to a cable network (each users may be connected or not; the costs of connecting one more user to the network decreases with the number of existing connections; see Moulin and Shenker (1999)).

We explore three simple solutions, namely three mechanisms defining the property rights of the various agents (those who end up using the commons as well as those who don't) to the overall surplus. Each solution is first best efficient and proposes a different distribution of the surplus across agents; solutions also vary with respect to their incentives properties. We submit that the comparison of these three solutions articulates the central difficulties of the commons problem. Which solution is a more satisfactory answer to what concrete example of the commons problem? The empirical work on the commons gives no simple answer to this question, yet it demonstrates the relevance of our axiomatic approach to the actual institutions governing the commons: see the discussion of the monitoring costs in Orström (1991).

The three solutions are inspired respectively by the three most important ideas of the recent literature on axiomatic fair division. One solution applies the Shapley value (Shapley 1953) to the stand alone cooperative game; another is the familiar competitive equilibrium with equal incomes; and the third solution applies the idea egalitarian equivalence (see Thomson and Varian (1985)).

II. The Model

There is a large population of potential users of the commons, and each agent is describes by his willingness to pay for the (homogeneous) service produced by the commons. We set $d(p)$ to be the number of agents willing to pay at least p for the service. We make the usual microeconomic assumptions on the demand function d: it is decreasing continuous and differentiable, from $p=0$ to $\bar{p}$ for which $d(\bar{p})=0$.

The production technology is described by the marginal costs function $mc(q)$. We shall only discuss the two special cases of increasing marginal costs (*imc*: the function mc is continuous and increasing) and of decreasing marginal costs (*dmc*: the function mc is continuous and decreasing). The more complicated case of mixed returns (where mc is not monotonic) has not received any attention in the mechanism design literature.

We start with the familiar tragedy of the commons, namely the utilization of the technology in the *free access* regime. Every agent decides freely and independently to demand one unit of service; if q is the number of *active* users (*i.e.*, those who request service), each active users pays the average costs $ac(q)$ (inactive users pay nothing).

If the marginal cost is constant at every level of production (so that $ac(q)=mc(q)=\gamma$ for all q) the free access equilibrium is fully efficient and there is no tragedy. As shown on Figure 1, the equilibrium production level is $q_e=d(\gamma)$, and this is the efficient level as well. All efficient agents (*i.e.*, those with willingness to pay $p\geq\gamma$) and only chose are active, and agent p receives a surplus $p-\gamma$.

Suppose next that the marginal costs function is increasing, as on Figure 2. The free access equilibrium output is q_{fa}, at the intersection of the ac and d curves (*i.e.*, $d(ac(q_{fa}))=q_{fa}$), larger than

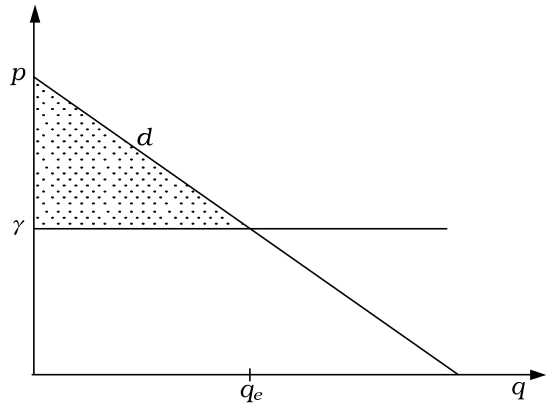

FIGURE 1

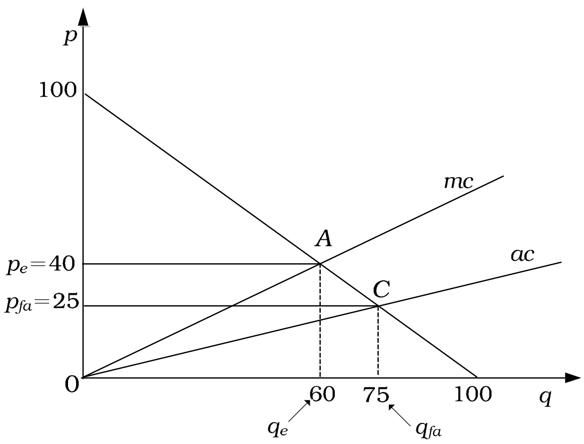

FIGURE 2

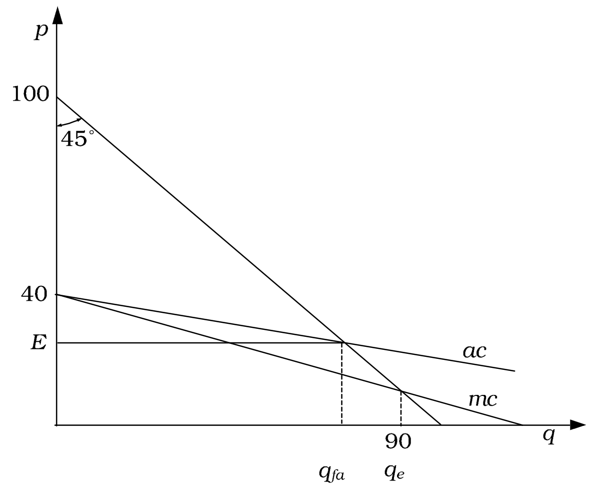

FIGURE 3

the efficient outputs level q_e, at the intersection of mc and d (*i.e.*, $d(mc(q_e))=q_e$). Free access entails overutilization of the commons. A Pareto improving move above the free access equilibrium requires that all agents between $p_{fa}=ac(q_{fa})$ and $p_e=mc(q_e)$ do not request service, and receive instead a cash compensation of at least $p-p_{fa}$ for agent p.[1]

Turning to the case of a decreasing marginal cost function, illustrated on Figure 3, we see that the free access equilibrium entails a production level q_{fa} smaller than the efficient level q_e. The tragedy here is the underutilization of the common property resources. A Pareto improving move from the free access equilibrium requires each agent p between p_e and p_{fa} to become active, and pay at most $p-p_e$ for service.[2]

[1]The virtual price solution discussed in the next section is a simple way to compute a profile of such compensations.

[2]The virtual price solution makes them pay exactly this amount: see Section IV.

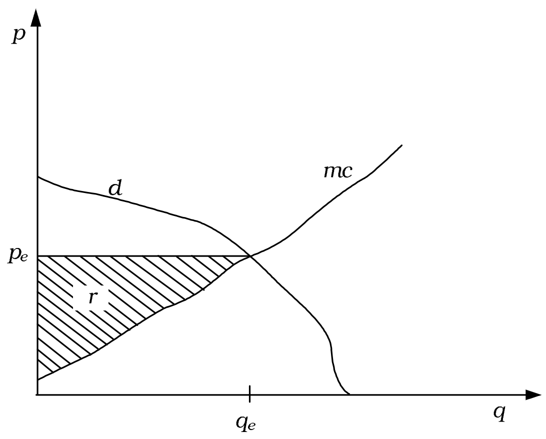

FIGURE 4

III. Increasing Marginal Costs

Our first efficient solution views the technology as the single asset of a firm of which all potential users (all agents described by the demand function) own an equal share. It is the competitive equilibrium with equal incomes (CEEI) familiar to the fair division literature (Young 1994; and Moulin 1995). The firm charges the competitive price p_e for service, so that the buyers are exactly the efficient agents. The firm's competitive profit is $r = p_e \cdot q_e - C(q_e)$, depicted on Figure 4. It is equally split among all shareholders, *i.e.*, all potential users of the commons. Thus the surplus $\sigma_{ce}(p)$ received by agent p is:

$$\sigma_{ce}(p) = (p - p_e)_+ + \frac{r}{d(0)} \tag{1}$$

Note an alternative definition of the CEEI solution, as the only efficient allocation where no agent is envious, namely no agent prefers another agent's allocation to her own.

The CEEI solution awards a share of surplus to every agent, no matter how low his willingness to pay for the service. An agent p

such that $p<mc(0)$ cannot get any surplus from using the commons, even if he is the sole user – and pays the lowest conceivable cost. Such an agent receives a rent from his share of ownership, and this rent is independent of his willingness to pay. This may or may not be normatively desirable. One important consequence is that membership to the set of potential users must be carefully monitored: declaring oneself a potential user is profitable, irrespective of one's ability to use the commons.

On the other hand, the CEEI solution is fully incentive compatible among a given set of potential users. Because every single agent is negligible with respect to the overall population, misreporting one's willingness to pay is useless. No matter how dispersed the information about the demand (each agent knows her own parameter p; other agents may or may not share this information), the *direct revelation* mechanism induces every participant to reveal p truthfully, and the CEEI solution is faultlessly implemented. This mechanism simply asks everyone to bid a willingness to pay p, computes the corresponding competitive output, price and profit, and implements the corresponding CEEI allocation.

The *virtual price* (VP) solution divides total efficient surplus σ_e in such a way that every agent is indifferent between his actual allocation and the opportunity to buy service at the (common) virtual price p_v. Thus agent p must receive a surplus share $\sigma_v(p) = (p-p_v)_+$, and p_v is determined by the "surplus-balance" condition, namely the total surplus distributed equals the available efficient surplus σ_e:

$$\int_0^{\bar{p}} (p-p_v)_+ \cdot |\dot{d}|(p)dp = \sigma_e = r + \int_{p_e}^{\bar{p}} d(p)dp \Leftrightarrow \int_{p_v}^{p_e} d(p)dp = r \qquad (2)$$

On Figure 5, p_v is characterized by the equality of the areas covered by the two shaded triangles.

The actual VP allocation makes all efficient agents p, $p \geq p_e$, pay p_v for service, while every agent p, $p_v \leq p \leq p_e$, receives a personalized cash transfer $(p-p_v)$ and agents below p_v receive nothing. In particular, all agents below $mc(0)$ receive no cash,[3] which reflects the reasonable principle that benefit from the commons is derived from (potential) usage, and that an agent who "has no business

[3]The inequality $mc(0)<p_v$ holds true, as discussed four paragraphs below.

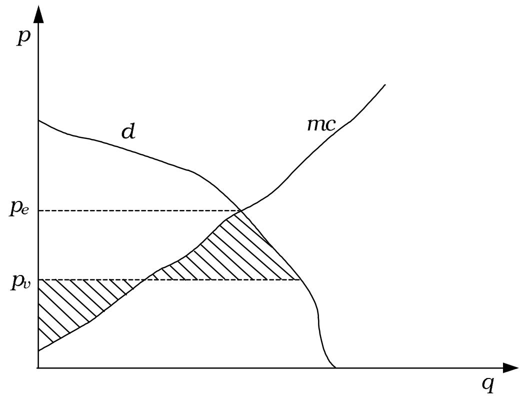

FIGURE 5

using this technology" should not receive any rent. In particular, an open membership policy is feasible, because only those agents who can actually increase total surplus will show up.

The downside of the VP solution has a normative and an incentive component.

On the normative side, all agents between $mc(0)$ and p_v receive no share of the overall surplus. These agents refrain from using the commons for the sake of efficiency, but they are willing to use it if it is not too crowded. They reduce crowding by stepping aside, hence have a legitimate claim to some share or surplus. Both the CEEI and the Shapley solution below respond to this claim, but the VP solution does not.

On the incentive side, the personalized transfers to the agents between p_v and p_e rule out decentralization by direct revelation (in contrast to the successful direct implementation of the CEEI solution). Indeed the agents just below p_e (the barely inefficient agents) receive the largest cash transfer $p_e - p_v$, and other inefficient agents will increase their own payment by reporting a willingness to pay just below p_e. This applies to all inefficient agents, including those below $mc(0)$. In the equilibrium of the direct revelation game corresponding to the VP solution, the efficient agents report some

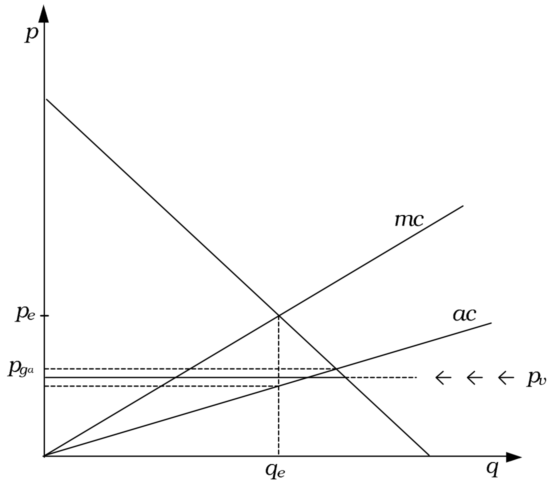

FIGURE 6

$p \geq p_e$ and all inefficient agents report just below p_e, so they all get an identical cash transfer. It is easy to check on Figure 5 that the corresponding outcome is precisely the CEEI solution of the true economy.

An interesting observation, illustrated on Figure 6, is that the virtual price p_v is always smaller than the free access equilibrium price, and larger than the average cost at the competitive level:

$$ac(q_e) < p_v < p_{fa} = ac(q_{fa}) < p_e \tag{3}$$

These inequalities follow by comparing on Figure 6 the areas of the two shaded triangles (Figure 5) when $p = ac(q_e)$ and $p = p_{fa}$ respectively.

In the free access equilibrium outcome, the surplus awarded to agent p is $(p - p_{fa})_+$ and (3) implies $(p - p_{fa})_+ \leq (p - p_v)_+$. Therefore the VP allocation is Pareto superior to the free access equilibrium, and the free access mechanism "subimplements" the VP solution. That is, in those configurations of d and mc where the free access equilibrium does not lose much surplus (moderate tragedy), it offers a good approximation of VP, with the same distributive implications.

A proper implementation of the VP solution requires a mechanism more complex than the direct revelation one, as well as some specific assumptions about the dispersion of information among the agents (See Moulin (1995, Chapter 6)).

The third and last solution applies to our commons problem the familiar Shapley value of cooperative games with transferable utility. For each subset S of the potential users, we write $v(S)$ for the *stand alone surplus* of S, namely the efficient surplus of the commons problem with the same technology but only S for potential users. Thus for $S=N$ containing all agents accounted for in d, we have $v(N)=\sigma_e$. For $S=\{p\}$ containing a single agent willing to pay p for service, we have $v(S)=(p-mc(0))_+$.[4]

The Shapley value assigns to agent p a share of surplus $\sigma_s(p)$ computed as the expected value of $v(S\cup\{p\})-v(S)$, when all potential users are randomly ordered – with uniform probability on all orderings, and S is the random set of agents preceding p in the ordering. The function σ_s is fairly easy to compute, thanks to our assumption of a continuum of agents, and to the law of large numbers:

$$\sigma_s(p)=(p-p_e)_+ + \theta(p), \text{ where } \theta(p)=\int_{mc(0)}^{\min\{p,p_e\}}\frac{mc^{-1}}{d}(t)dt, \text{ for all } p\geq mc(0) \quad (4)$$

The derivation of this formula is explained in the Appendix. Figure 7 shows a slice $[t,t+dt]$ of the competitive profit r, for $mc(0)\leq t\leq p_e$. This slice is equally divided among all agents willing to pay t or more: hence each gets a share $(mc^{-1}/d)(t)$. For $t\geq p_e$, the slice of σ_e-r is equally divided among all agents not smaller than t; summing up between p_e and $\bar{p}$ yields a share $p-p_e$ for agent p.

The Shapley solution charges the same price for service to each efficient agent, namely $p_e-\theta(p_e)$. It gives a cash transfer $\theta(p)$ to an inefficient yet potentially active agent p, $mc(0)\leq p\leq p_e$. Agents p, $p\leq mc(0)$, receive nothing: these agents are not active even if they are standing alone. Observe that the cash transfer to an inefficient agent p decreases when p decreases, and vanishes for $p=mc(0)$. Thus every potentially active agent ($p\geq mc(0)$) receives a positive

[4]We identify $mc(0)=mc(1)$ in view of the large number of agents.

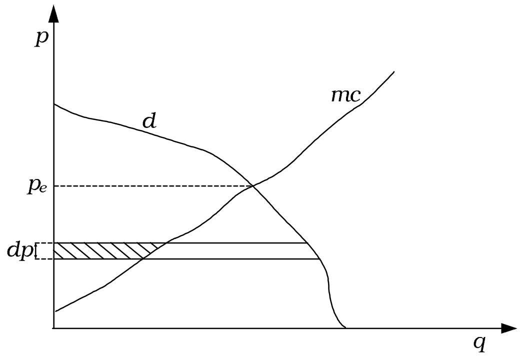

FIGURE 7

share of surplus (like in CEEI, but unlike in VP), and only the potentially active do (unlike in CEEI).

That the Shapley solution strikes an appealing compromise between VP and CEEI is also clear when we compare the three surplus distribution functions σ_{ce} (formula (1)), σ_s (formula (4)) and $\sigma_v(p)=(p-p_v)_+$. Figure 8 depicts these three functions in the numerical example of Figure 2. The following facts hold true for any (decreasing) demand and (increasing) marginal cost functions:

efficient agents prefer *VP* to Shapley, and the latter to CEEI:

$$p \geq p_e \implies \sigma_v(p) > \sigma_s(p) > \sigma_{ce}(p) \tag{5}$$

barely potentially active agents prefer CEEI to Shapley, and the latter to *VP*:

$$p = mc(0) + \varepsilon \implies \sigma_{ce}(p) > \sigma_s(p) > \sigma_v(p) \tag{6}$$

inactive agents prefer CEEI, and are indifferent between Shapley and VP:

$$p \leq mc(0) \implies \sigma_{ce}(p) > \sigma_s(p) = \sigma_v(p) = 0 \tag{7}$$

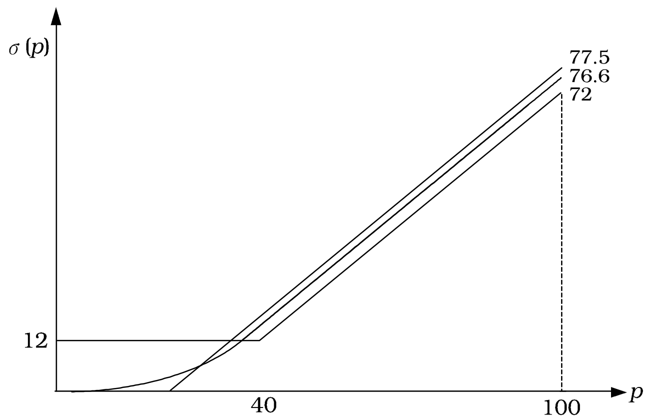

FIGURE 8

Another general fact is that any two of our three surplus functions cross exactly once: the supporters of one or the other method partition the demand around a critical threshold. This implies for instance that voting by majority to select one of the three methods will never generate a cycle.

The proof of the claims is given in the Appendix.

The Shapley solution distributes personalized cash payment to the inefficient yet potentially active agents. This create the same incentives to misreports in the direct revelation game as for the VP solution. Each potential user reports p, and the Shapley allocation is computed and implemented on the basis of the reported values: the inefficient agents will report that they are barely inefficient in order to maximize their cash reward. The equilibrium of the reporting game, once again, is just the (true) CEEI allocation.

On the other hand, the Shapley solution is "sub-implemented" by a very simple mechanism called the *random priority* game (Crès and Moulin 1999). An ordering of all potential users is randomly selected (with uniform probability on all orderings) and the agents are successively offered to buy (one unit of) service at the "current" marginal cost: when the turn of agent p comes, if exactly q agents before her did buy service, she is offered price $mc(q)$. Naturally, she accepts if and only if $p > mc(q)$.

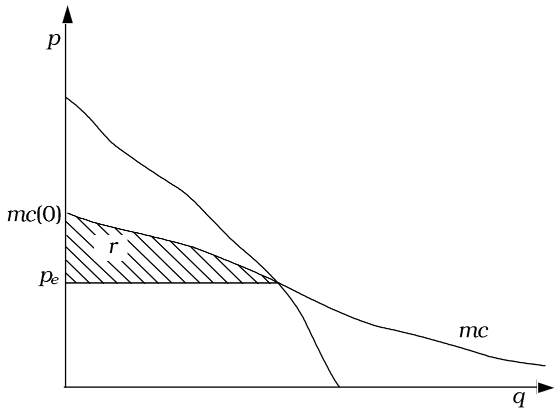

FIGURE 9

Thanks to the law of large numbers, it is not difficult to compute the probability that agent p does buy and her expected payment. The formula for the resulting (expected) surplus is given in the Appendix. The interesting property is that the random priority equilibrium allocation is Pareto inferior to the Shapley solution. When the efficiency loss of the former allocation is small, this means that it is a good approximation of the latter.

The numerical example of Figure 2 illustrates the two subimplementation properties. The free access equilibrium outcome overproduces by 25% (as $q_{fa}=75$ and $q_e=60$) but the surplus loss is only 6.25% as $\sigma_e=3{,}000$ and $\sigma_{fa}=2{,}812.5$.

The random priority equilibrium overproduces by 21.6% ($q_{rp}=$ 73.0) and its surplus loss is only 8%.

IV. Decreasing Marginal Costs

Under decreasing marginal costs, when the service is offered at the competitive price p_e (defined by $mc(d(p_e))=p_e$; see Figure 9), the revenue collected $p_e \cdot q_e$ falls short of the actual cost. The competitive deficit $\gamma'=C(q_e)-p_e \cdot q_e$ is depicted as the shaded area on Figure 9.

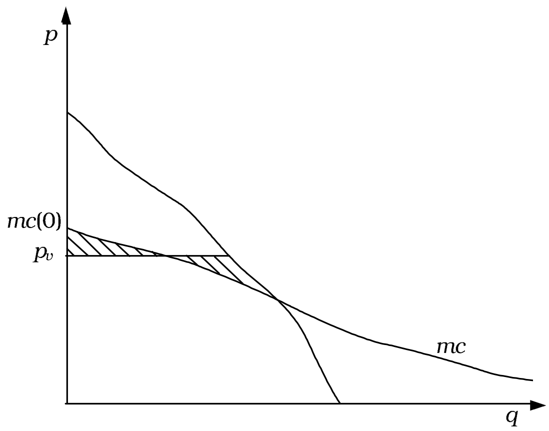

FIGURE 10

The CEEI solution divide γ' equally among all potential users, so that the surplus of agents p is $\sigma_{ce}(p)=(p-p_e)_+-(\gamma'/d(0))$. Thus all agents pay the tax $\gamma'/d(0)$, and service is offered for the additional fee p_e. This is the "public service" solution, where all agents are forced to subsidize the firm in order that it charges the price generating the efficient demand.

In the *imc* case, the CEEI solution gives a rent even to agents who are not potentially active ($p<mc(0)$), which seems excessively generous. In the *dmc* case, on the contrary, the solution is excessively harsh on all inefficient agents. As before, the CEEI solution is incentive compatible in the sense of strategyproofness: in the direct revelation mechanism, no one benefits by reporting untruthfully his or her willingness to pay. However, implementation of the solution requires to carefully monitor the set of agents who share the ownership of the "firm." In the *dmc* case, the issue is to prevent anyone from leaving the scene[5]: the participation of the inefficient agents is not voluntary.

The virtual price solution is defined by precisely the same equation (2), but the virtual price p_v is now larger than the efficient

[5]In the *imc* case, monitoring was necessary to prevent immigration instead of emigration.

price p_e. Figure 10 illustrates the equation:

$$\int_{p_e}^{p_v} d(p)dp = \gamma'$$

by the property that the two shaded triangles have equal area.

The VP allocation gives one unit of output to all efficient agents ($p \geq p_e$). It charges the price $min\{p, p_v\}$ to agent p. That is, all agents $p \geq p_v$ pay p_v, and an agent p, $p_e \leq p \leq p_v$, pays exactly p: he is indifferent between buying service or not, and receives zero surplus. All inefficient agents get no service and pay nothing.

It is normatively appealing to leave the inefficient agents out and to ask no money from them. After all, they are "dummies" (in the terminology of cooperative games), namely their presence never improves the stand alone surplus of any coalition. On the other hand, they are not required to pay anything, which ensures voluntary participation, and allows an open membership policy. One objection to the VP solution is the rough treatment of the "low" efficient agents – those between p_e and p_v. The solution squeezes out all surplus by charging exactly their own valuation, thus making them indifferent between participating or not. Under CEEI, the barely efficient agents net a surplus loss, which is worse.

The incentives properties of the VP solution are worse under *dmc* than under *imc*. The difficulty is the same, namely the personalized payments for the same service. In the direct revelation game, every efficient agent wants to pretend that he is barely efficient so as to pay (almost) p_e. But if all efficient agents pretend to "be" $p_e + \varepsilon$, the VP solution based on these reports is the zero production: no one gets any service and no money changes hand! The strategic instability in the misreporting game is that of a Battle of the Sexes situation. Decentralized behavior can wipe out the cooperative surplus entirely (when too many agents deflate their willingness to pay).

As in the *imc* case, the VP allocation is Pareto superior to the free access equilibrium allocation. This results from the following inequalities, of which the geometric proof is just as easy as in the case of inequalities (3):

$$p_e < ac(q_e) < p_v < p_{fa} = ac(q_{fa}) \tag{8}$$

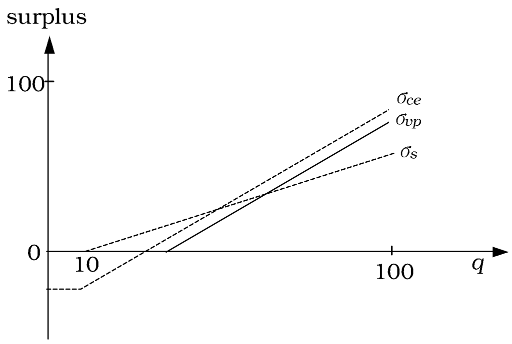

FIGURE 11

The Shapley solution is defined, exactly as in the previous section, by its surplus distribution $\sigma_s(p)$, the Shapley value of the stand alone cooperative game. The explicit formula is as follows:

$$\sigma_s(p)=p-(p_e+\theta'(p)) \text{ where } \theta'(p)=\int_{p_e}^{\min\{p,mc(0)\}}\frac{mc^{-1}}{d}(t)dt, \text{ for all } p\geq p_e$$

$$\sigma_s(p)=0 \text{ for } p\leq p_e. \tag{9}$$

Thus every efficient agent is charged p_e plus the personalized surcharge $\theta'(p)$ for service. An important fact is that θ' increases in p, from $\theta'(p_e)=0$ to $\theta'(p)=\theta'(mc(0))$ for all $p\geq mc(0)$. A second key observation is $p_e+\theta'(p)<p$ for all $p>p_e$ – purchase of service by agent p is voluntary. And finally the surplus function σ_s increases with p, which correctly rewards a high willingness to pay, whose contribution to overall surplus is larger.[6]

Figure 11 depicts the surplus functions of our three solutions CEEI, *VP* and Shapley in the numerical example of Figure 3. Several features of this figure are perfectly general:

[6]The last two observations follow at once from the fact that $mc^{-1}<d$ in the interval where the integral is defined.

agents who are active even when standing alone prefer CEEI:

$$p \geq mc(0) \Rightarrow \sigma_{ce}(p) > \sigma_v(p), \quad \sigma_s(p)$$

barely efficient agents prefer Shapley to *VP*, and the latter to CEEI:

$$p = p_e + \varepsilon \Rightarrow \sigma_s(p) > \sigma_v(p) > \sigma_{ce}(p)$$

inefficient agents are indifferent between Shapley and *VP*, and CEEI is worst:

$$p < p_e \Rightarrow \sigma_s(p) = \sigma_v(p) = 0 > \sigma_{ce}(p)$$

The single crossing property of any two of the three surplus functions is a general fact as well.

Turning finally to the implementation of the Shapley solution, we note that the direct revelation game is strategically similar to the revelation game of the virtual price solution. Every efficient agent wishes to report a willingness to pay barely above p_e, so as to pay almost nothing for service. If too many agents do this, the efficient output under the reported demand function falls to zero and all surplus is lost. Therefore the situation is a generalized Battle of the Sexes where efficient agents seek to commit to underreporting so as to prevent other efficient agents from doing the same.

The random priority mechanism is defined as before and still delivers a subimplementation of the Shapley solution. In the numerical example of Figure 3, both the free access and random priority equilibrium outcomes produce 20% less than the efficient output, and collect all but 5% of the overall surplus:

$$\frac{q_{fa}}{q_e} = 80\% \;; \frac{\sigma_{fa}}{\sigma_e} = 96\% \;; \frac{q_{rp}}{q_e} = 79\% \;; \frac{\sigma_{rp}}{\sigma_e} = 95\%$$

These figures are encouraging, however the good performance of the random priority mechanism depends on the fact that a large fraction of the demand lies above the highest marginal cost $mc(0)$. If the highest willingness to pay $\bar{p}$ is below $mc(0)$ (or if only a small fraction of the potential users is above $mc(0)$), the random priority mechanism either is a non starter, as no one is ready to buy the first − most expensive − unit, or it collects a small fraction of the overall surplus.

V. Conclusion

Our three solutions offer three conflicting interpretations of the property rights to the surplus generated by a one input − one output commons. The CEEI solution is captured by the simple tests of No Envy (in combination with efficiency); it distributes the competitive profit (*imc* case) or loss (*dmc* case) uniformly among all agents, efficient and inefficient alike.

The virtual price solution distributes the surplus as if everyone was charged the same price for service, although the actual allocation gives a personalized cash transfer to the barely inefficient agents (*imc* case) or charges a personalized price to the barely efficient ones (*dmc* case). It gives no surplus whatsoever to many potentially active agents (*imc* case) and even to some efficient agents (*dmc* case).

The Shapley solution strikes a reasonable compromise between the above two solutions, giving a positive share of surplus to all potentially active agents (*imc* case) and to all efficient agents (*dmc* case). However, it is the hardest to implement of the three.

Finally we mention two popular equity tests from the fair division literature (see *e.g.*, Moulin (1995) or Young (1994)), the Population Monotonicity (PM) and Resource Monotonicity (RM) properties. PM states that when new agents join the set of potential users, the welfare of no existing agent should increase (*imc* case) or decrease (*dmc* case). RM states that when the technology improves, in the sense that the function mc is replaced by a smaller function mc' ($mc'(q) \leq mc(q)$ for all q), the welfare of no agent should decrease.

It is ease to check that the CEEI Solution fails both tests whereas the VP and Shapley solutions meet both tests.

Appendix

1. Proof of Formula (4)

Fix an agent p and let S be the (random) see of agents preceding p, when all orderings of N are equiprobable. Let λ be the size of relative to N : $\lambda = |S|/d(0)$.

The law of large numbers (see, *e.g.*, Feller (1971)) implies that the demand function generated by the set S of agents is simply λd. We call $p(\lambda)$ the efficient price in the economy (S,mc). Thus $p(\lambda)$ is the unique solution of the following equation with unknown p:

$$mc(\lambda \cdot d(p)) = p \Leftrightarrow \lambda = \frac{mc^{-1}}{d}(p)$$

The marginal surplus contributed by p to coalition S is $v(S \cup \{p\}) - v(S) = (p - p(\lambda))_+$. Therefore the surplus share awarded to agent p by the Shapley solution is:

$$\sigma_s(p) = \int_0^1 (p - p(\lambda))_+ d\lambda$$

Upon setting $f = mc^{-1}/d$ and changing variable λ to t such that $\lambda = f(t)$, the above integral becomes:

$$\sigma_s(p) = \int_{mc(0)}^{p_e} (p-t)_+ \dot{f}(t)dt = \int_{mc(0)}^{p_o} (p-t)\dot{f}(t)dt \text{ where } p_o = \min\{p,\ p_e\}$$

Integrating by parts yields:

$$\sigma_s(p) = [(p-t)f(t)]_{mc(0)}^{p_o} + \int_{mc(0)}^{p_o} f(t)dt$$

which is precisely formula (4).

Note that the proof of formula (9) in the *dmc* case is entirely similar.

2. *Proof of Statements* (5), (6), (7) *and the Single Crossing Property*

Both functions σ_v and σ_{ce} are made of two linear pieces with successive slopes 0 and 1 and their integral over $[0, \bar{p}]$ are equal. The single crossing property follows at once.

From (4) it is clear that the slope $\dot{\sigma}_s(p)$ is increasing, remains between 0 and 1 and equals 1 for $p \geq p_e$. Taking into account the relative position of our three functions at $p = \bar{p}$ and $p = 0$ (properties (5) and (7)), we see that σ_s and σ_v as well as σ_s and σ_{ce}, cross only once.

In order to establish property (5), it is cleanly enough to show:

$$p_v \le p_e - \int_0^{p_e} \frac{mc^{-1}}{d} \le p_e - \frac{r}{d(0)}$$

First we prove the right-hand inequality:

$$\text{for all } p,\ \frac{mc^{-1}}{d}(p) \ge \frac{mc^{-1}(p)}{d(0)} \Rightarrow \int_0^{p_e} \frac{mc^{-1}}{d} \ge \frac{1}{d(0)} \int_{p_v}^{p_e} mc^{-1} = \frac{r}{d(0)}$$

Next we use the definition of p_v (equation (2)) to show:

$$\int_0^{p_v} \frac{mc^{-1}}{d} \le \frac{1}{d(p_v)} \int_0^{p_v} mc^{-1} = \frac{1}{d(p_v)} \int_0^{p_e} (d - mc^{-1})$$

This implies:

$$\int_0^{p_e} \frac{mc^{-1}}{d} \le \int_{p_v}^{p_e} \frac{mc^{-1}}{d} + \frac{d - mc^{-1}}{d(p_v)} = \int_{p_v}^{p_e} \left\{ 1 + (d - mc^{-1}) \cdot \left(\frac{1}{d(p_v)} - \frac{1}{d} \right) \right\}$$

Between p_v and p_e, $d \ge mc^{-1}$ and $1/d(p_v) \le 1/d$, therefore the right-hand integral is bounded above by $p_e - p_v$, as was to be proven.

The straightforward proof of properties (6) and (7) is omitted.

3. The Random Priority Equilibrium Allocation

The computation of the equilibrium allocation is adapted from Crès and Moulin (1999).

Given a random ordering of N, and a number λ, $0 \le \lambda \le 1$, we denote by S the set of the first $\lambda d(0)$ agents. By the law of large numbers, the numb $q(\lambda)$ of units purchased by the agents in S, is deterministic. We compute $q(1)$, namely the total output at the equilibrium allocation of our mechanism.

Clearly $q(\lambda)$ is non-decreasing in λ and $q(1) \le mc^{-1}(\bar{p})$, because nobody is willing to pay more than $\bar{p}$ for service. The probability that the first agent after S buys service at price $mc(q(\lambda))$ is $d(mc(q(\lambda)))/d(0)$ therefore $\dot{q}(\lambda) = d(mc(q(\lambda)))$ and

$$\int_0^{q(\lambda)} \frac{1}{d(mc(s))} ds = \lambda \text{ for all } \lambda,\ 0 \le \lambda \le 1$$

Two cases arise:

if $\int_0^{mc^{-1}(\bar{p})} \frac{1}{d(mc)} \leq 1$ then $q(1)$ is the solution of $\int_0^{q(1)} \frac{1}{d(mc)} = 1$

if $\int_0^{mc^{-1}(\bar{p})} \frac{1}{d(mc)} \geq 1$ then $q(1) = mc^{-1}(\bar{p})$

Observe that $q_e < q(1)$: the random priority equilibrium inefficiently overproduces. Indeed $d(mc(q)) > q_e$ for all $q < q_e$, implying $\int_0^{q_e} (1/d(mc)) < 1$, and the claim.

We compute next the equilibrium allocation of agent p. Let λ be the relative size of the random set of agents preceding her. Agent p buys service if $mc(q(\lambda)) \leq p \Leftrightarrow q(\lambda) \leq mc^{-1}(p)$. Therefore the overall probability $\pi(p)$ that our agent is served is:

$$\pi(p) = \int_0^{\min\{mc^{-1}(p),\, q(1)\}} \frac{1}{d(mc)}$$

and her expected payment $\mu(p)$:

$$\mu(p) = \int_0^{\pi(p)} mc(q(\lambda)) d\lambda = \int_0^{\min\{mc^{-1}(p),\, q(1)\}} \frac{mc(s)}{d(mc(s))} ds$$

(where the right-hand equality follows the change of variable $s = q(\lambda)$).

Therefore the surplus $\sigma_{rp}(p)$ awarded to agent p in the random priority equilibrium is:

$$\sigma_{rp}(p) = \int_0^{\min\{mc^{-1}(p),\, q(1)\}} \frac{p - mc(s)}{d(mc(s))} ds$$

Finally we check that the above surplus is not larger than the Shapley surplus $\sigma_s(p)$, which is the announced subimplementation property. Rather than comparing the above integral with that in (4), we consider the surplus share of our agent when the relative size λ of the coalition preceding her is fixed. Her Shapley surplus is $p - p(\lambda)$, where $p(\lambda)$ is defined in the proof of formula (4). Her random priority surplus is $p-mc(q(\lambda))$. We showed above that the random priority equilibrium overproduces: $q(1)>q_e \Leftrightarrow mc(q(1))>p_e$. Applying this to the economy (S,mc) gives $mc(q(\lambda))>p(\lambda)$ and the desired conclusion.

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