Structurally Stable Nash Equilibria in Pure Strategies for a Model of Monopolistic Competition

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A model of monopolistically competitive industry is formulated using the theory of large games. We show that an equilibrium exists for the game and that equilibrium correspondence is upper hemi-continuous. The model's implications are discussed, especially on existence and characteristics of structurally stable equilibria and on the relationship to Kumar and Satterthwaite's (1985) model.

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I. Introduction

Game theory is now widely employed in various fields of economics. The most commonly used form of games is a finite-player non-cooperative game, for which Nash (1950) provided the celebrated solution concept of an equilibrium.

A 'large game' in this paper refers to a specific sort of non-cooperative game, namely a non-cooperative game that has an infinite number of players. Although game theory is usually most

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effective for situations involving a few players (*e.g.*, oligopoly), a game with many players is not necessarily an anomaly, nor a mere mathematical curiosity. Early proponents of game theory were apparently aware of usefulness or even necessity of large games in order for game theory to reach the status of a comprehensive mathematical framework for social science. (See for example the introductory section in Khan-Sun (2001) where they quote pioneering game theorists' remarks to such effects).

This paper considers a model of monopolistic competition using the theory of large games. One of the first meaningful results of this theory was obtained by Schmeidler (1973), who showed that in a game where there are infinitely many non-cooperative players whose actions affect each other's payoffs by means of a social average, a Nash equilibrium consisting of *pure* strategies exists. A force of this result is that the existence of pure strategy equilibria is not generally guaranteed for finite-player games ('matching pennies' is a well-known counterexample).

While mathematical and conceptual issues in the theory of large games have been explored extensively (see Khan-Sun (2001) for a recent survey), this paper presents a very simple model utilizing some basic techniques from that field.

Our model of monopolistic competition is inspired by a model originally proposed by Kumar and Satterthwaite (1985). In hindsight, their model was naturally suited for a large game formulation, but the theory of large games had not sufficiently matured at the time.

In the model, a large number of firms with an identical profit function compete with each other. Each firm ignores impact of its own action on the market, and is affected by other firms' actions through a vector of aggregate statistics. In other words, the profit function takes as its argument a firm's own action and a summary statistic of (almost) all firms. Because of the largeness, the summary statistic is not affected by changes in action of a single firm.

Kumar and Satterthwaite's main contribution was that in such a game the number (or the dimension) of the aggregate statistic can effectively restrict the number of viable equilibria. More precisely, the number of distinct structurally stable equilibria (which will be elaborated on later) can be bounded by the dimension of the aggregate statistic. This has interesting implications for the extent of product or strategy differentiation in the monopolistic competition model.

In Section II, we formulate the model in terms of large games and easily prove that there exists a Nash equilibrium. In Section III, we first prove that equilibrium correspondence is upper hemi-continuous. Then we study the relationship between continuity notions of equilibrium correspondence and structural stability of equilibria. We show that there exists a structurally stable equilibrium. In Section IV, we discuss further technical issues and describe Kumar and Satterthwaite's (1985) model, explaining how introducing an infinite number improved the model.

II. Model and Existence Results

A. Model of Monopolistic Competition: A Special Case

Let the set of firms I be a nonatomic measure space with measure μ.[1] We may simply think of I as [0,1] interval with Lebesgue measure in order to grasp the arguments. This formulation leads to the interpretation that each firm is negligible.

Let the firms have a common set of available actions X—a compact, convex subset of $\mathbb{R}^L$. For example, if a firm's strategic actions encompass "price," "advertising," and "R&D," then we can take L to be 3.

A strategy profile is a measurable function s: $I \rightarrow X$. This function tells us what action each firm chooses. Call the set of all strategy profiles S. Then the set of "averages" of actions of all firms is $\Delta = \{ \int_I s(i) : s \in S \}$.

In order to put firms in an *a priori* identical position,[2] let the (identical) profit function take the form of π. $X \times \Delta \rightarrow \mathbb{R}$. Information is aggregated through a specific form of statistics, namely the average of the actions. The 'dimension' of aggregate statistic is L because an action is an L-vector.

Notice first that the distribution of the "active" actions in a given

[1] This model was inspired by the finite-player model of Kumar and Satterthwaite (1985). We shall discuss the parallels in the final section.

[2] All firms enjoy an identical position *ex ante*, but may choose different actions in equilibrium. We interpret this exhibition of different equilibrium actions as differentiation. Therefore, as a referee has correctly pointed out, we will be interested in asymmetric equilibria of this model.

strategy profile s is represented by the induced measure $\mu \circ s^{-1}$ on X. An action $x \in X$ is active in the strategy profile s if $(\mu \circ s^{-1})(x) = \mu(s^{-1}(x)) = \mu(\{i \in I{:}s(i) = x\}) > 0$, *i.e.* a non-null subset of firms choose the action x. This induced measure (or the distribution) contains all relevant information for the competing firms.

Aggregation of information is said to occur when this distribution is converted into a finite-dimensional vector of statistics, as is the case here: the full information $\mu \circ s^{-1}$ is converted into a single L-dimensional information $s(i)d\mu \in \mathbb{R}^L$.

The (Nash) equilibrium $\int_I$ can be defined in a straightforward manner: A strategy profile s^* is an *equilibrium in pure strategies* if, for almost all $i \in I$, $s^*(i) \in \text{Argmax}_{x' \in X}\, \pi(x', s^*(i)d\mu)$.

Given this setup, we have the following existence theorem. Because the standard existence theorem for large games can be directly applied, we only give a sketch of proof.

Theorem 1. For an industry in which a continuum of identical firms aggregates competitive information into a social average to choose their strategic actions, there exists an equilibrium if π is continuous.

Sketch of Proof: A game can be thought of as a mapping from the space of players I to the space of player characteristics (action sets and payoff functions). The current model is a game where every player is assigned the same action set and payoff function. Hence, the game is a constant-valued mapping, therefore measurable. It easily follows from Theorem 2 in Rath (1992) that there exists a pure strategy equilibrium for this game.

The proof involves constructing a best response correspondence and taking an integral of this correspondence. Showing that the integral (which itself is a correspondence) satisfies the requirements of Kakutani's fixed point theorem completes the proof, because the fixed point is an equilibrium.

B. Model of Monopolistic Competition: The General Case

Although the preceding model uses a standard large game formulation, one important aspect of the model (*i.e.*, aggregation of information) is somewhat limited, so we now extend to a more general model that allows for a broader set of information aggregation.

First, take g. $X \to \mathbb{R}^M$ to be a smooth[3] function that converts each action x (an L-vector) to an M-vector. If M is small, the conversion gives only a crude information on the original action vector, while if M is sufficiently large, it gives a relatively complete description.[4] Since a strategy profile s: $I \to X$ can be thought of as a random variable, $g(s)$ may be considered another random variable (of a different dimension). A vector of summary statistic y on the original information s can be produced by taking the expected value of $g(s)$: $y = \int_I g(s(i))d\mu(i) = \int_X g(x)d(\mu \circ s^{-1})(x)$.[5] Denote the set of such vectors as Y.

Now we let the payoff function take the form π. The *equilibrium in pure strategies* is defined as before as a strategy profile s^*: $I \to X$ such that for almost all $i \in I$, $s^*(i) \in \text{Argmax}_{\,x \in X}\, \pi(x, \int_I g(s^*(i))d\mu)$.

For proving existence of equilibria in this game, Theorem 2 in Rath (1992) cannot be directly applied. We offer an existence theorem and proof here.[6] Although proof is elementary, it is given here for the sake of completeness. While standard proofs work on Δ in our notation (see Theorem 1 above), we need to work on Y. So the proof involves making sure that Y behaves nicely for the purpose.

Intuitively this is easy because Y differs from Δ in only two respects. First, the difference between $\int g(s)$ and $\int s$ should not cause any difficulty because the function g was assumed to be smooth (in fact, measurability should suffice). Second, $\int g(s)$ is in

[3]For proving existence, the function g need only be measurable, but we assume smoothness for later analysis involving structural stability.

[4]All dimensions of the information need not be treated equally. For example, When $L=3$ (price, advertising, R&D), and $M=3$, firms might be looking at average price, average advertising level and average R&D level; or they might be calculating average, variance and the third moment of price only while ignoring information on advertising and R&D altogether.

[5]For example, if $x \in \mathbb{R}$ and $g(x) = (x, x^2)$, then y can give a two-dimensional information of the average and the variance of players' choices. On the other hand, if $M = L$ and g. $X \to \mathbb{R}^M$ is the identity mapping, we recover the special case of the previous subsection.

[6]That the large games can accommodate statistics other than averages was first observed by Rauh (1994). In Rauh (1994), a finite number of *moments*, *i.e.* the expected values of x, $x^2, \ldots, x^M$, are used. The theorem appears as Lemma B in Appendix of Rauh (1997) without proof. However, as is shown here any summary statistic (or any random variable) can be used as long as the finite dimensionality is preserved (*i.e.*, the range of g is finite-dimensional).

$\mathbb{R}^M$ while s is in $\mathbb{R}^L$. This also matters little because as long as dimension is finite, the proof works just as well.

Theorem 2. For an industry in which a continuum of identical firms aggregate the competitive information through an M-dimensional vector of statistics, there exists an equilibrium if π is continuous.

Proof: Let B: $I \times Y \Rightarrow X$ be the best response correspondence.[7] In other words, $B(i, y) = \text{Argmax}_{x \in X} \pi(x, y)$ where $y = \int_I g(s(i)) d\mu(i)$ for some strategy profile $s(\cdot)$.[8]

Now take $\gamma(B(i, \cdot)) = \{g(x) : x \in B(i, \cdot)\}$, which is a mapping from $Y (\subset \mathbb{R}^M)$ to $\mathbb{R}^M$. Next, define another correspondence $\Gamma : Y \Rightarrow Y$ as follows: $\Gamma(y) = \int \gamma(B(y))$. We need to verify the following. (It helps to see Γ as a smooth transform of the best response correspondence B.)

Claim 1. Y is nonempty, convex and compact.
Claim 2. Γ is nonempty-valued.
Claim 3. Γ convex-valued.
Claim 4. Γ has a closed graph.

Claim 1: That Y is nonempty is obvious. Because X is convex, $Y = g(X)$ (Lyapunov's theorem, see Hildenbrand, 1974, p. 62). So Y is convex and compact.

Claim 2: This is equivalent to showing that there exists a measurable selection for Γ. This follows from Aumann's measurable selection theorem. (Aumann, 1965, Theorem 2)

Claim 3: This again follows from Lyapunov's theorem. (Aumann, 1965, Theorem 1)

Claim 4: This follows from Aumann's version of Fatou's lemma (Aumann, 1965) because best response correspondence is upper hemicontinuous. See also Aumann (1976).

By the claims, Γ fulfills the requirements for Kakutani's fixed point theorem. Hence, there exists a fixed point $y*$, *i.e.* $y* \in \Gamma(y*)$. Then a strategy profile $s*$ such that $\int g(s*(i)) d\mu(i) = y*$ is an

[7] To distinguish a correspondence from a function, we use the notation $\Rightarrow$ instead of $\rightarrow$.

[8] Because payoff function π is common across all players, the index i is superfluous but it helps to avoid confusion in doing integrals.

equilibrium.

This result is false for the case when the number of firms is finite (that is, for the original model by Kumar and Satterthwaite (1985)). Kumar and Satterthwaite (1985) had to invoke an additional assumption[9] on the functional form of the profit function to ensure the existence of equilibria.

IV. Continuity of Equilibrium Correspondence

In this section, we will examine continuity properties of Nash equilibrium correspondence (a mapping that assigns Nash equilibria to each game). We will define upper and lower hemicontinuity properties and prove that our equilibrium correspondence is upper hemicontinuous. This has implications for structurally stable equilibria, a notion used by Kumar and Satterthwaite (1985).

A. Definitions of Continuity Concepts

Let us first recall standard definitions of continuity concepts for a correspondence (*i.e.* a set-valued mapping). The following definitions can be found in Green and Heller (1981) and Hildenbrand (1974). Relatively complete definitions are given here for readers' convenience. Consider an arbitrary correspondence $\psi: X \Rightarrow Y$, where X and Y are topological spaces.[10]

Definition 1. A correspondence ψ is *upper hemi-continuous at* $x \in X$ if:
(1) $\psi(x) \neq \varnothing$ and
(2) for every open neighborhood U of $\psi(x)$, there exists a neighborhood V of x such that $\psi(x) \subset U$ for all $x \in V$.

ψ is *upper hemi-continuous* if it is upper hemi-continuous at all

[9]Specifically, $\pi(x, y) = \pi_0(x) + \sum_{j=1}^{M} y_j \pi_j(x)$. This assumption requires the profit function to be linear — the profit function can be broken into $M+1$ components and expressed as an inner product between a vector of statistics $(1, y_1, \cdots, y_M)$ and a vector of functions $(\pi_0, \pi_1, \cdots, \pi_M)$.

[10]Because X is a compact subset of $\mathbb{R}^L$, it follows that any measurable function $f: I \rightarrow X$ is also integrable, *i.e.* $\int |f| d\mu < \infty$. Moreover, it is also true that $f \in Lp(\mu)$ for any $1 \leq p \leq \infty$.

$x \in X$. Upper hemi-continuity is also characterized by either of following two equivalent conditions:
(1) $\{x \in X: \psi(x) \subset W\}$ is open for any open subset W of Y; or
(2) $\{x \in X: \psi(x) \cap F = \varnothing\}$ is closed for any closed subset F of Y.

Definition 2. A correspondence ψ is *lower hemi-continuous at* $x \in X$ if:
(1) $\psi(x) \neq \varnothing$ and
(2) for every open subset W of Y such that $W \cap \psi(x) \neq \varnothing$, there exists a neighborhood V of x such that $\psi(x') \cap W = \varnothing$ for all $x' \in V$.

ψ is *lower hemi-continuous* if it is lower hemi-continuous at all $x \in X$. Lower hemi-continuity is also characterized by either of the following two equivalent conditions:
(1) $\{x \in X: \psi(x) \subset F\}$ is closed for any closed subset F of Y; or
(2) $\{x \in X: \psi(x) \cap W \neq \varnothing\}$ is open for any open subset W of Y.

Definition 3. A correspondence ψ is *continuous* if it is both upper hemi-continuous and lower hemi-continuous.

B. Upper Hemi-continuity of Equilibrium Correspondence

In this subsection, we will study the properties of Nash equilibrium correspondence. Nash equilibrium correspondence (or simply equilibrium correspondence) is $\psi: \Pi \Rightarrow S$, where Π is the space of all possible profit functions (identified with the space of games because in our model every firm has the same profit function), S is the space of strategy profiles and $\psi(\pi)$ is the set of Nash equilibria of the game $\pi \in \Pi$. We give the sup-norm topology on Π and the essential sup norm topology on S. (See the next subsection for more on the use of these topologies).

We shall first prove upper hemi-continuity of Nash equilibrium correspondence $\psi: \Pi \Rightarrow S$, which we will later show is equivalent to the existence of a structurally stable equilibrium. Any strategy profile in S also belongs to $L_\infty(\mu)$.[11] Because we need to integrate a finite dimensional statistic of the strategy profile, putting the essential sup norm topology (*i.e.* viewing the functions as an

[11]Because X is a compact subset of $\mathbb{R}^L$, it follows that any measurable function $f: I \to X$ is also integrable, *i.e.* $\int |f| d\mu < \infty$. Moreover, it is also true that $f \in L_p(\mu)$ for any $1 \leq p \leq \infty$.

element of $L_\infty(\mu)$) will be more than enough for the purpose.

We first define a notion that is weaker than upper hemi-continuity. Again the definitions and the related facts can be found in Green and Heller (1981) and Hildenbrand (1974). First, the *graph* of a correspondence $\psi:X\Rightarrow Y$ is the set $G(\psi)=\{(x,y):y\in\psi(x)\}$.

Definition 4. A correspondence $\psi:X\Rightarrow Y$ is *closed at* x_0 if for every sequence $(x_n,y_n)\in G(\psi)$ such that $(x_n,y_n)\to(x_0,y_0)$ it is true $(x_0,y_0)\in G(\psi)$ so that if $y_n\in(x_n)$ and $x_n\to x_0$ and $y_n\to y_0$, then $y_0\in\psi(x_0)$.

A correspondence is *closed* if it is closed at all $x\in X$.

The closedness implies (in fact, is equivalent to) upper hemi-continuity if two conditions are met: (1) the range of ψ is compact and (2) $\psi(\pi)$ is a closed set for each π. Therefore, in order to prove upper hemi-continuity, we may proceed by first showing closedness of equilibrium correspondence and then showing compactness of the range of equilibrium correspondence.

Theorem 3. Nash equilibrium correspondence $\psi:\Pi\Rightarrow S$ is upper hemi-continuous.

We shall prove the theorem in two steps.

Claim 1. *The Nash equilibrium correspondence $\psi:\Pi\Rightarrow S$ is closed.*

Proof: Choose a convergent sequence (π_n,s_n) such that $s_n\in\psi(\pi)$ and denote the limit by (π_*,s_*). To prove the claim, we have to show that $s_*\in\psi(\pi_*)$.

First note that $g(s_n)\to g(s_*)$. As s_n converges to s_* almost everywhere, $g(s_n)=g\circ s_n$ also converges to $g(s_*)$ almost everywhere. Then $g(s_n)$ converges to $g(s_*)$ in distribution. See Hildenbrand (1974).

By assumption, we have, for each n, $\pi_n(s_n(i), g(s_n))\geq\pi_n(x, g(s_n))$ for all $x\in X$. As each π_n is (jointly) continuous and π_n converges to π_* in the sup-norm topology, we see $\pi_*(s_*(i), g(s_*))\geq\pi_*(x, g(s_*))$ for all $x\in X$. Hence $s_*\in\psi(\pi_*)$. ■

Claim 2. *S is compact.*

Proof: First, note that the set of all mappings from I to X can be thought of as the product space X^I, where the number of products equals the cardinality of the set I. Because X is a compact subset o by Tychonoff's theorem, X^I is compact in the product topology, whic turn can be viewed as the point-wise convergence topology on the set of functions. Then obviously, X^I is also compact in the almost everywhere convergence topology. Now consider S, the space of measurable functions from I to X with the essential sup norm topology. This set is certainly a subset of X^I. The almost everywhere limit of a sequence of integrable functions is measurable (see, *e.g.*, Phillips, 1984, p. 204), therefore the set S is closed. This establishes that the set S is compact. ■

C. *Continuity of Equilibrium Correspondence and Structural Stability of Equilibria*

Kumar and Satterthwaite (1985) define the notion of structural stability of an equilibrium and characterize structurally stable equilibria in terms of the dimension of the statistics vector that aggregates the information. This result has an interesting implication on the amount of variation (or differentiation) among the actions chosen by firms in equilibrium. In this section, we relate the notion of structural stability to continuity of Nash equilibrium correspondence. Let us first briefly sketch Kumar and Satterthwaite's results.

a) Sketch of Kumar and Satterthwaite's Stability Results

Let $C^\infty_{XY}=\{h:X\times Y\to\mathbb{R} : h\in C^\infty\}$ be the set of smooth real-valued functions defined on $X\times Y$. (The assumption of smoothness is used significantly here.)

Definition 5. The function $h_\lambda\in C^\infty_{XY}$ is a *homotopic perturbation* of a function $h\in C^\infty_{XY}$ if and only if $h_\lambda=h+\lambda f$ for some $f\in C^\infty_{XY}$ and a scalar λ.

A homotopic perturbation is close to the original function in all orders of derivatives. Because firms in our model have a common profit function, we can identify a game with the profit function. Hence a perturbation of a game, equivalently a perturbation of a profit function π can be defined appropriately.

An equilibrium is *structurally stable* if a small perturbation of the game has an equilibrium that is a small perturbation of the equilibrium of the unperturbed game. The following definition has adapted from Kumar and Satterthwaite (1985) to our current settin infinite number of players.

Definition 6. An equilibrium strategy profile $s{:}I\to X$ of a game π is *structurally stable* if for all sufficiently small homotopic perturbation π_λ of π, an equilibrium $s(\lambda)$ exists which is close to s.

The 'closeness' between strategy profiles needs to be defined for the above definition to be meaningful. We choose to work with the topology generated by essential supremum norm on the space of measurable functions. The essential sup norm is given by

$$||s|| = \text{ess sup}_{\ i\in I}|s(i)| = \inf_{\ Z\subset I,\ \mu(Z)=0}\{\sup_{\ i\in I-Z}|s(i)|\}$$

We can define the distance between two functions using this norm and this distance function in turn generates a topology. Two functions that have zero distance have the same value almost everywhere on the domain.

Convergence of measurable functions in this topology (almost everywhere convergence) implies convergence in distribution (See Hildenbrand, 1974, pp. 46, 47 and 51, especially statements (21) and (39) there). This means that while Kumar and Satterthwaite talk about closeness between equilibrium configurations (that is, distribution of chosen actions in equilibrium, $\mu \circ s^{-1}$ in our notation, (q_k,x_k)'s in their notation), we can safely focus on closeness between strategy profiles as it takes care of the closeness between the distributions.

Kumar and Satterthwaite present two theorems. The first theorem shows that a "regular" equilibrium with more than $M+1$ distinct actions, where M is the dimension of the statistics vector, is structurally unstable. The second theorem shows that a "regular" equilibrium that has the "non-singularity" property and not more than $M+1$ distinct actions is structurally stable. (See Kumar and Satterthwaite (1985) for precise statements including regularity and non-singularity).

We can interpret these results as follows. Firms are in an *ex ante* identical position. In a symmetric equilibrium, all firms would take the same action exhibiting no differentiation.[12] But in an asymmetric equilibrium, it is in their best interests given others' choices for firms to choose different actions. Such a differentiation allows us to label this model as monopolistically competitive. More importantly, the extent of the differentiation is not unbounded.

Given the limitations of information processing, they can exhibit differentiation so far as their sophistication of information aggregation allows. If firms only know or care about industry average of actions (so $M=1$), it is likely that in equilibrium they will divide into two groups (for example, high pricing firms and low pricing firms) or will choose the same action. A third group choosing a third action may appear, but this configuration is structurally unstable, meaning that a small perturbation of payoff structure will destroy the equilibrium.

The proofs of these theorems, which use tools from differential topology, rely on the fact that there are only finitely many active actions in any equilibrium. This fact does not have to be brought out for the setting considered by Kumar and Satterthwaite because there are a finite number of firms. In our formulation we cannot *a priori* assume that there are a finite number of active actions.

We note, however, first that a measurable function can be approximated by a step function which does have a finite image. Hence, an equilibrium can be generally approximated by a strategy profile with a finite number of actions. Because when the number of actions is greater than $M+1$ the equilibrium is structurally unstable, we may intuitively infer that in the limit when the number of actions is infinite, the equilibrium must be structurally unstable. This conclusion is not guaranteed of course; it is well-known that the limit phenomenon can be quite different from an approximating phenomenon.

Until we obtain a definitive characterization of an equilibrium with infinitely many actions, we may accept Kumar and Satterthwaite's results in our setting on practical basis. It would be of interest to construct concrete examples dealing with this matter. We now turn our attention to another issue, *i.e.* whether a structurally stable equilibrium exists in general, independent of the dimension of the statistics.

b) Structural Stability of Equilibria and Continuity of Equilibrium

[12]So a symmetric equilibrium will represent a competitive market. Because we are interested in monopolistic competition, we focus on asymmetric equilibria. I thank a referee for emphasizing this point.

Correspondence

In the following lemmas, we investigate the relationship between structural stability of equilibria and continuity of Nash equilibrium correspondence. As before, we give the sup-norm topology on Π and the essential sup norm topology on S.[13]

***Lemma* 1.** If $\psi(\pi_0)$, the set of Nash equilibria for the game π_0, consists of structurally stable equilibria only, then equilibrium correspondence ψ is lower hemi-continuous at π_0.

***Proof*:** Fix an open subset $W \subset S$ with diameter D such that $\psi(\pi_0) \cap W \neq \varnothing$. We need to show that there exists a neighborhood of π_0 whose image via ψ intersects with W. Pick an $s_* \in \psi(\pi_0) \cap W$. Let $V(s_*, \varepsilon) = \{\pi \in \Pi$: is a perturbation of π_0 such that there exists an $s \in \psi(\pi)$ with $\|s, s_*\| < \varepsilon$. Let $\varepsilon < D$ and $\psi(\pi)$, where $\pi \in V(s_*, \varepsilon)$, intersects with W. ■

Remark.

Structural stability does *not* imply *upper* hemi-continuity of ψ. To see this, let us try proving it. We might proceed as follows. Fix a neighborhood U of $\psi(\pi_0)$ with diameter D. Then we need to show that there exists a neighborhood of π_0 whose image via ψ is contained in U, or in other words, the distance between any two equilibria is less than D.

Fix $s_* \in \psi(\pi_0)$. Since s_* is structurally stable by assumption, a sufficiently small perturbation of π_0 always has an equilibrium that is arbitrarily close to s_*. Let $V(s_*, \varepsilon)$ be the same as in the above proof. But $\psi(\pi)$ could still have another equilibrium that is far from s_*. Let $V(\varepsilon) = \cap \quad V(s_*, \varepsilon)$. Hence, for all $s_* \in \psi(\pi_0)$, there exists is a $\pi \in V(\varepsilon)$ that has $s \in \psi(\pi)$ within ε of s_*.

This does not prove $V(\varepsilon)$ is the desired neighborhood of π_0. The image of $V(\varepsilon)$ touches every point of $\psi(\pi_0)$ within ε, but $\psi(V(\varepsilon))$ may contain other points far from $\psi(\pi_0)$. ■

The next lemma proves the converse of above Lemma,

[13]A referee thought it confusing to use homotopic perturbation of π in defining structural stability and to use sup-norm topology on Π in defining continuity notions. Homotopic perturbation was employed by Kumar and Satterthwaite (1985) to use techniques from differential topology and requires smoothness, while our definitions of continuity are standard. A homotopic perturbation always lies in a small open neighborhood (in sup-norm topology), while in an open neighborhood, one can always select a homotopic perturbation of a smooth π.

establishing the equivalence of two notions.

***Lemma* 2.**
If equilibrium correspondence ψ is lower hemi-continuous at π_0, then all equilibria of π_0 are structurally stable.

***Proof*:** Fix $s_* \in \psi(\pi_0)$. If we can find a perturbation π of π_0 that has an equilibrium near s_*, then we have shown that s_* is structurally stable. Pick a neighborhood of s_* and call it W (so that $\psi(\pi_0) \cap W \neq \varnothing$). Then, by lower hemi-continuity of ψ at π_0, there exists a neighborhood V of π_0 whose image via ψ intersects with W. Choose a perturbation π of π_0 from this neighborhood V. Then $\psi(\pi) \cap W \neq \varnothing$), which means that there exists an $s \in \psi(\pi) \cap W$. Let the diameter of W be arbitrarily small, then s must be close to s_*.

In hindsight, it is intuitively clear that lower hemi-continuity and structural stability should be related. When the spaces X and Y are both metric, a lower hemi-continuous correspondence $\psi : X \rightrightarrows Y$ has the following property: for every sequence $(x_n) \subset X$ that converges to $x_0 \in X$ and for every $y_0 \in \psi(x_0)$, there exists a sequence $(y_n) \subset Y$ such that $y_n \in \psi(x_n)$ for all n and $y_n \to x_n$. Translated for equilibrium correspondence, this roughly means: for a sequence of games (π_n) that converges to the game π_0 and for every equilibrium s_0 of the game π_0, there exists a sequence of equilibria (s_n) such that s_n is an equilibrium of the game π_n and the sequence of equilibrium converges to the equilibrium s_0. Although stated in terms of sequences, this corresponds to the idea of structural stability.

What is brought out by this line of investigation is the distinction of a game having only stable equilibria and a game having at least one stable equilibrium. The following lemma shows this.

***Lemma* 3.**
If equilibrium correspondence ψ is upper hemi-continuous at π_0, then π_0 has at least one structurally stable equilibrium.

***Proof*:** Suppose ψ is upper hemi-continuous at π_0. Choose a neighborhood U of $\psi(\pi_0)$. Then, there exists a neighborhood V of π_0 whose image via ψ is contained in U. If we take a sufficiently small perturbation π of π_0 so that $\pi \in V$, then $\psi(\pi) \subset U$, which means that π has an equilibrium arbitrarily close to some points in $\psi(\pi_0)$. Hence, π_0 has at least one structurally stable equilibrium.

Remark.
The above proof does *not* show that *all* points in $\psi(\pi_0)$ are structurally stable. ■

By the lemmas, we know that upper hemi-continuity of ψ implies the existence of a structurally stable equilibrium and lower hemi-continuity of ψ is equivalent to structural stability of all equilibria.

Thus we have shown that the notion of structural stability is closely related with notions of continuity of equilibrium correspondence. Best response correspondences — Nash equilibrium correspondence is a variant — seldom exhibit lower hemi-continuity. See Kim (1997) for an example of demanding conditions for achieving lower hemi-continuity. From our analysis here it is only natural to find lower hemi-continuity to be restrictive; a lower hemi-continuous equilibrium correspondence implies that those equilibria are all structurally stable! Instead, we know equilibrium correspondence is upper hemi-continuous in general (see also Rath, 1996), so we know that we have at least one structurally stable equilibrium in general. We have the following theorem.

***Theorem* 4.** There is at least one structurally stable equilibrium for our model of monopolistic competition.

***Proof*:** Follows from Lemma 3 and Theorem 3. ■

IV. Discussion and Concluding Remarks

A. Lower Hemi-continuity of Nash Equilibrium Correspondence

As shown above, lower hemi-continuity of equilibrium correspondence is equivalent to structural stability of *all* equilibria of games. Hence, a result on lower hemi-continuity would provide a powerful statement on structural stability of equilibria.

But lower hemi-continuity of a correspondence that involves maximization is very difficult to obtain. For this reason, we have elsewhere devised elaborate topologies in a somewhat different setting to guarantee lower hemi-continuity of argmax correspondence in Kim (1997). The techniques developed in Kim (1997) may be of help here, but are not directly applicable because Kim (1997) was concerned with equilibria of a single game and here we are concerned with perturbed games.

To take a different approach, it is possible to utilize a weaker notion of lower hemi-continuity. Housman (1988) considers a model of large games and proves results on continuity of Nash equilibrium correspondence. Specifically, he noted the difficulty of proving lower hemi-continuity and suggested a weaker result, namely using a notion of approximate equilibria and proving a 'near' lower hemi-continuity result. A similar strategy works here, but the implication for the relationship of such weaker notion with structural stability is weaker and more subtle. We leave that exercise to our subsequent work.

B. Economic Implications of the Model

We have avoided discussing in detail the original model of Kumar and Satterthwaite (1985) and its economic implications. Before closing the paper, let us examine the parallels between our model and Kumar and Satterthwaite (1985).

Kumar and Satterthwaite (1985) consider I identical firms, where I is a large finite integer. Each firm chooses an action from a compact convex set $X \subset \mathbb{R}^L$. It is important to note that they allow only pure strategies *a priori*.

Suppose K distinct actions are observed among firms in equilibrium. Then the industry configuration can be described by a vector $z = \{(q_1, x_1), \cdots, (q_K, x_K)\} \in Z \subset [0,1] \times X$, where q_i is the proportion of firms choosing the action $x_i \in X$ (hence, $\sum q_i = 1$).

An industry configuration z is a complete description of the strategic information. Aggregation of information occurs if this information is converted to an M-dimensional vector $y = f(z)$, where $f: Z \to \mathbb{R}^M$ is a smooth function. Smoothness is needed for the stability results, which are then implicitly used for the existence result by Kumar and Satterthwaite. Denote the set of such vectors by Y.

The (common) profit function for the firms is given by a smooth function $\pi.\ X \times Y \to \mathbb{R}$. Hence, the complete description of an industry (or a game) with I firms is given by $\langle X, \pi, f \rangle$.

An industry configuration $z^* = \{(q^*_k, x^*_k)\}_{k=1}^K$ is an *equilibrium* if, for all $k = 1, \cdots, K$, $x^*_k \in \text{Argmax}_{x \in X}\ \pi(x, f(z^*))$. A moment's reflection will show that this is in fact a Nash equilibrium of the game.

Theorem 5 (Kumar-Satterthwaite) For an industry in which a large finite number I of identical firms aggregate the competitive information through an M-dimensional vector of statistics, there exists an equilibrium if the following functional restrictions hold:

$$\pi(x,y) = \pi_0(x) + \sum_{j=1}^{M} y_j \pi_j(x), \tag{1}$$

$$y = f(z) = \sum_{k=1}^{K} q_k g(x_k), \text{ where } g.\ X \to \mathbb{R}^M. \tag{2}$$

The restrictive nature of the condition (1) is obvious. The profit function is assumed to be linear (to be precise, the profit function can be broken into $M+1$ components and expressed as an inner product between a vector of statistics $(1, y_1, \cdots, y_M)$ and a vector of functions $(\pi_0, \pi_1, \cdots, \pi_M)$). The condition (2) is not as restrictive; it is a requirement that the statistics take the form of the expected value of a random vector g and our model employs similar formulation.

The statistics function $f: Z \to \mathbb{R}^M$, which is actually given a form $\sum q_k g(x_k)$ for Theorem above can be thought of as the expected value of $g(x)$ in our reformulation (*i.e.*, $\int_X g(x) d(\mu \circ s^{-1})(x)$).

In summary, there are I firms, where I is a large but *finite* integer, who choose only pure strategies. The firms have an identical profit function. Each firm ignores the impact of its own action on the market and aggregates the actions chosen by the competitors into a finite-dimensional vector of summary statistics in calculating its profit.

We intend to argue that large games provide a natural framework for Kumar and Satterthwaite's model that resolves some conceptual and technical difficulties.

Take, for example, the assumption that each firm ignores its own impact. For I firms, where I is finite, that a firm would ignore the impact of its own action on the industry is a behavioral assumption imposed by the modeler, which is very poor if I is small and only nearly acceptable if I is large but finite. "For small I this is a poor assumption that should be dropped. If, however, it is dropped, then pure strategy equilibria may fail to exist." (Kumar and Satterthwaite, 1985, p. 44) One of their justifications for this assumption is drawn from Chamberlin's (1933) theory of monopolistic competition. It is interesting that for Chamberlin 'an infinite number of firms' was not a strange concept.[14]

Not only is our infinite-firm formulation compatible with Chamberlin's original theory, but it also strengthens Kumar and Satterthwaite's model in several ways. Note in the above quote that Kumar and Satterthwaite are concerned about the existence of *pure* strategy equilibria.[15] One of the most interesting things about "large games" is the existence of pure strategy equilibria (Schmeidler, 1973, Theorem 2). So it is natural that our reformulation easily admits pure strategy equilibria. Our analysis also highlights the fact that Kumar and Satterthwaite's finite-firm model, with their insistence on pure strategy equilibria, requires a linearity restriction on the form of profit function, which is unnecessary in our case.

Of course, the importance of Kumar and Satterthwaite's work lies elsewhere, that is, in the finding that the structurally stable equilibria have a limited extent of variations in active actions and that this phenomenon occurs because of the aggregation of information by the firms.

Kumar and Satterthwaite's main objective was to show that the aggregation of information (via an M-dimensional vector of statistics) places an upper bound on the number of active actions in a structurally stable equilibrium.

This conclusion basically carries over to our more general setting, with one important provision. Their Theorem 2, which establishes the structural stability of an equilibrium configuration with less than $M+1$ distinct actions, needs minimal modification. The case is somewhat different for their Theorem 1, which shows an equilibrium with more than $M+1$ actions is structurally unstable, the difficulty here being that their argument rests on the assumption that there are a finite number of distinct actions in equilibrium. This assumption is satisfied in their model because there are only a finite (but large) number of firms, while in our case, one cannot exclude in advance the possibility of an infinite number of active actions.

This is a good place to note the important role played by the large but finite number I of Kumar and Satterthwaite. The

[14]In the eighth edition of Chamberlin (1933) published in 1962: p. 7 "infinite...large enough," p. 11n "large," p. 48 "infinite...very large," p. 54 "large enough...infinite," p. 72n "a large number of,"... the list continues.

[15]Also pp. 37 - 38: "Firms are forbidden to employ mixed strategies." (Kumar and Satterthwaite, 1985)

largeness justifies their behavioral assumptions but because I is still finite the existence result requires a linearity assumption. On the other hand, finiteness rules out the possibility of an infinite number of actions in an equilibrium configuration, and is used to establish the stability results.

Finally, our upper hemi-continuity result showed that there always exists a structurally stable equilibrium for the game. Kumar and Satterthwaite (1985) were probably not concerned very much with existence issues. They provide three theorems, first two of which are on structural stability of equilibria without showing existence of such equilibria and the last is on existence of Nash equilibria without mention of stability. As pointed out above, their existence result required a linearity assumption that is unnecessary for their stability results and for existence in general setting as laid out here. So our results in this paper provide a more solid foundation for their findings both in setting a natural framework and ensuring existence of equilibria and structurally stable equilibria.

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