Pricing Call Options under Stochastic Volatilities

Chang Mo Ahn and D. Chinhyung Cho [1]

This paper derives a closed-form solution for the European call option price when the volatility of the underlying stock returns is governed by a diffusion process. The model uses the continuity property of a diffusion process and the martingale approach to valuation of assets under no arbitrage. The pricing formula differs from the Black-Scholes formula in that it needs a volatility adjustment. The volatility movement is allowed to be contemporaneously correlated with the stock price movement.

Keywords: Continuity, Diffusion, Martingale, No arbitrage, Stochastic volatilities

JEL Classification: G12

I. Introduction

The variability of stock price volatility over time has received considerable attention in the literature of option pricing. The volatility of the underlying stock return changes stochastically over time. There is ample evidence for stochastic volatilities of stock returns (see Clark (1973), Blattberg and Gonedes (1974), Epps and Epps (1976), Kon (1984) and French, Schwert, and Stambaugh (1987)). This paper derives a closed-form solution for the European call option price when the volatility of the underlying stock returns is governed by a diffusion process.

Several explanations for stochastic volatility are offered in the

*Professor, College of Business Administration, Sejong University, 98 Kunja-Dong, Kwangjin-Ku, Seoul, Korea 143-747, (Tel) +82-2-3408-3166, (E-mail) cmahn@sejong.ac.kr; Professor, College of Business Administration, Sejong University, 98 Kunja-Dong, Kwangjin-Ku, Seoul, Korea 143-179, (Tel) +82-2-3408-3171, (E-mail) david@sejong.ac.kr, respectively.
[**Seoul Journal of Economics** 2002, Vol. 15, No. 4]

literature. Castanias (1979), Chiras and Manaster (1978) and French and Roll (1986) attribute changes in volatility to arrival of unexpected information, while Schwert (1989) relates volatility changes to the trading in index futures and/or index options, the level of economic activity, leverage and stock trading volume. Geske (1979) provides an economic rationale for an inverse relationship between the stock price and volatility. If the stock is viewed as a call option on the firm's asset value, the volatility of the stock return is a function of the leverage ratio, which in turn changes as the stock price changes. Flood and Hodrick (1986) and West (1988) relate volatility changes to the presence of bubbles and fads.

The Black-Scholes option pricing model yields a time-varying implied volatility of stock return from option prices. Merville and Pieptea (1989) find empirical evidence that the implied volatility follows a mixed mean-reverting diffusion process with noise. The variability of stock-return variance is important in option pricing since option prices are highly sensitive to variance. This observation motivates researchers to examine the option price when the stock has stochastic volatility. In fact, there is a growing body of literature which examines an option price when the volatility of the stock returns randomly changes over time.

There are two main lines of thinking in the literature regarding option pricing with stochastic volatility. The first approach assumes that there exists a deterministic functional relationship between the stock price and volatility. As the stock price follows a diffusion process, the volatility varies also. For example, in the compound option pricing model of Geske (1979) the volatility is a function of the firm's leverage ratio, while in the constant-elasticity-of-variance model of Cox and Ross (1976) the volatility is of the form $\sigma(S_t,t)=\delta S^{\zeta-1}$. The constant-elasticity-of-variance model of Cox and Ross (1976) and the compound option model of Geske (1979) require the stock price and the instantaneous standard deviation to be functionally dependent and thus instantaneously perfectly correlated. Beckers (1980) also obtains empirical results supporting an inverse relationship between stock price and volatility.

The second approach assumes that the volatility process is governed by a diffusion process of its own. This approach is more general than the first in that the assumption of volatility depending on the stock price level is relaxed, and a probabilistic correlation between the two processes is allowed. Research proposing diffusion-type volatility models includes the papers of Eisenberg (1985), Johnson and Shanno (1987), Hull and White (1987), Scott (1987), Wiggins (1987), Chesney and Scott (1989) and Heston (1993). Since it is more general, we adopt the second approach in this paper.

The option pricing problem for assets with stochastic volatilities is complex since there are no tradeable assets perfectly correlated with the stock volatility, and thus volatility risk cannot be hedged away. Hull and White (1987) and Scott (1987) assume that such risk can be diversified away or is un-correlated with aggregate consumption and thus the volatility is not priced by the market. In order to eliminate volatility risk from the pricing equation, Wiggins (1987) assumes the market portfolio to be the underlying asset and the investors to have log utility. Furthermore, Hull and White (1987) and Scott (1987) obtain a series solution by assuming that volatility and the stock price are instantaneously un-correlated. In the absence of a closed-form solution for the option price, Wiggins (1987) numerically solves the fundamental partial differential equation, and Hull and White (1987) and Scott (1987) employ the Monte Carlo approach in determining the distribution of the final stock price and then use the risk-neutral argument. Heston (1993) uses characteristic functions to value options with stochastic volatility. Even though a numerical solution and the Monte Carlo approach are useful to understand the option price, a closed-form solution is always preferred.

We use the continuity property of Wiener processes and the martingale approach to value call options under no arbitrage. By using these properties, we are able to obtain a closed-form solution for a European call option price when the volatility of the underlying stock returns follows a diffusion process. The contribution of this paper is derivation of a simple formula under stochastic volatility which is similar to the Black-Scholes formula. Even though the instantaneous standard deviation of stock return changes over time, the resulting formula requires volatility adjustments in the Black-Scholes formula. In our analysis, the stock price and volatility are allowed to be correlated. Thus we can examine the implication of this correlation for the option price.

Section II develops the option pricing model when stochastic volatility follows a mean-reverting diffusion process. Section III extends the analysis of Section II when a log-normal diffusion process replaces the mean-reverting diffusion process. Finally, Section IV provides a summary and conclusion.

II. The Model

Consider an economy with two traded assets, a stock and a riskless bond. They are continuously traded in perfect markets, where there are no transaction costs, taxes and no short-sale restrictions. Let $\{z_s(t):t\geq 0\}$ and $\{z_\sigma(t):t\geq 0\}$ be standard Wiener processes defined on the probability space (Ω,F,P), where Ω represents states of the world, F is the collection of events and P is a probability measure. F is the smallest σ algebra which contains $\{F_t:t\geq 0\}$ for all t, where F_t is the right continuous and increasing filtration (see Durrett (1984, p. 12) for definitions). The stock does not pay cash dividends. The stock price S and the instantaneous standard deviation satisfy the following stochastic differential equations

$$dS(t) = \mu S(t)dt + \sigma(t)S(t)dz_s(t) \tag{1}$$

$$d\sigma(t) = \kappa[\underline{\sigma} - \sigma(t)]dt + \delta z_\sigma(t) \tag{2}$$

where $\mu, \kappa, \underline{\sigma}$ and $\delta > 0$ are constants and the correlation coefficient between dz_s and dz_σ is ρ. The correlation coefficient ρ is allowed to take any values between -1 and 1. According to (2), the standard deviation of the stock is expected to drift instantaneously towards the long-run average level $\underline{\sigma}$, with a speed of adjustment κ and its instantaneous variance is δ^2. Equation (2) is called an Ornstein-Uhlenbeck process.[1] Scott (1987) uses this process for the stochastic volatility of the stock. The specification with a mean-reverting process is interesting since empirical evidence suggests that the implied volatility follows a mixed mean-reverting diffusion process with noise (see Merville and Pieptea (1989)). The risk-free bond price satisfies

[1]The probability of negative volatility is positive for an Ornstein-Uhlenbeck process, but small for a mean-reverting diffusion process.

$$dB(t)=rB(t)dt \tag{3}$$

where $r>0$ is a constant risk-free rate.

An equivalent probability measure defined on (Ω,F) has the following properties. First, an equivalent probability P^* means that $P^*(X)=0$ if and only if $P(X)=0$ for $X\in F$. In other words, the two probability measures share the same null sets. Second, the Radon-Nikodym derivative $\psi=dP^*/dP$ satisfies $E(\psi^2)<\infty$.

Using the martingale representation theorem (see Kunita and Watanabe (1967) or Durrett (1984, p. 88)) for ψ gives

$$\psi(t)=E(\psi)+\int_0^t \gamma(s)dZ(s) \tag{4}$$

where γ is a two-dimensional vector of F-measurable real functions, and Z is a two-dimensional Wiener process under P, containing z_S and z_σ as elements.
Similarly, γ includes γ_1 and γ_2 as elements. Applying the Girsanov theorem (see Friedman (1975) or Karatzas and Shreve (1988)) gives

$$Z^*(t)=Z(t)-\int_0^t \psi(s)^{-1}<\psi,Z> \tag{5}$$

where Z^* is a two-dimensional Wiener process under the equivalent probability measure P^*, containing z_s^* and z_σ^* as elements, and $<\psi,Z>$ denotes the cross-variation process of ψ and Z.[2]

Using (4) and (5), we can rewrite the stock price process (1) and the volatility process (2) as

$$dS(t)=(\mu-\lambda_S)S(t)dt+\sigma(t)S(t)dz_s^*(t) \tag{6}$$

$$d\sigma(t)=[\kappa(\underline{\sigma}-\sigma)-\lambda_\sigma]dt+\delta dz_\sigma^*(t) \quad \text{or} \quad =\kappa(\overline{\sigma}-\sigma)dt+\delta dz_\sigma^*(t) \tag{7}$$

[2]Let $X=\{X_t,F_t;\ 0\le t\le\infty\}$ be a right-continuous martingale. Then $X^2=\{X_t^2,F_t;\ 0\le t\le\infty\}$ is a non-negative sub-martingale. X^2 has a unique decomposition as the sum of a continuous martingale and a continuous, increasing, integrable process with initial value zero. The latter process is defined as the quadratic variation process and denoted by $<X>$. The cross variation process $<X,Y>$ for the martingales X and Y is defined as

$$<X,Y>=(1/4)[<X+Y>-<X-Y>].$$

See Karatzas and Shreve (1988, p. 31) for details.

where $\lambda_S = -((\gamma_1 + \rho\gamma_2)/\psi)\sigma$ is the risk premium of the stock, $\lambda_\sigma = -((\rho\gamma_1 + \gamma_2)/\psi)\delta$ is the risk premium of volatility and $\overline{\sigma} = \underline{\sigma} - (\lambda_\sigma/\kappa)$. The martingale property of the stock price (as will be shown in (19)) implies that $\mu - \lambda_S - r = 0$. Thus the risk premium of the stock, $\mu - r$, is λ_S. We assume that a martingale process can be made by adjusting λ_S. Since the coefficients such as μ and r are assumed to be constant, λ_S should be constant. As we can see from the definitions of λ_S and λ_σ, ρ and δ affect λ_σ whereas ρ affects λ_S.[3] The drift term can be transformed into a martingale under no arbitrage condition by using the Girsanov theorem and λ_S can be changed through this process.

The processes (6) and (7) have the drift factor adjusted by the risk premia. The stock price process (1) and the volatility process (2) under P are transformed into (6) and (7) under P^* respectively.[4] Note that λ_σ may not be observable in markets since the stock volatility is not a traded asset.

Define $\varepsilon_t^* \sqrt{dt} = dz_S^*(t)$ and $\eta_t^* \sqrt{dt} = dz_\sigma^*(t)$ (or equivalently $\varepsilon_t^* \sqrt{\Delta t} = \Delta z_S^*(t)$ and $\eta_t^* \sqrt{\Delta t} = \Delta z_\sigma^*(t)$ by discretization) where ε_t^* and η_t^* are standard normal variables under P^* occurring at time t. The standard normal variables are serially independent, but they are allowed to be correlated with each other contemporaneously. From the Girsanov theorem, the contemporaneous correlation coefficient between ε^* and η^* is also ρ. The following lemma shows a continuity property of a standard Wiener process which will prove useful in deriving important results.

***Lemma* 1.**
With probability one, $(\Delta z_S^*(i))^2 = (\Delta z_\sigma^*(i))^2 = \Delta t$ and $\Delta z_S^*(i)\Delta z_\sigma^*(i) = \rho\Delta t$ for all i, and $\Delta z_S^*(i)\Delta z_S^*(j) = \Delta z_\sigma^*(i)\Delta z_\sigma^*(j) = \Delta z_S^*(i)\Delta z_\sigma^*(j) = 0$ for $i \neq j$.

Proof: see the Appendix.

[3]Usually, the stock commands a positive risk premium. Under a constant volatility or a zero value of ρ, γ_1 should be negative to provide a positive risk premium λ_S, since σ and ψ are strictly positive. Thus as ρ increases, the risk premium λ_σ of stochastic volatility tends to increase. Similarly, stochastic volatility commands a positive risk premium in a normal situation. As δ increases, the risk premium λ_σ also tends to increase.

[4]Starting from the original system of stochastic differential equations like (1) and (2), Cox, Ingersoll, and Ross (1985) consider the alternative system of equations like (6) and (7).

Let T denote the expiration date of the option. It is assumed that $B(T)$ is unity. Thus we have that $B(0)=e^{-rT}$. The discrete approximations of (3), (6), and (7) are given by

$$\frac{B_i-B_{i-1}}{B_{i-1}}=r\Delta t \tag{3a}$$

$$\frac{S_i-S_{i-1}}{S_{i-1}}=(\mu-\lambda_S)\Delta t+\sigma_{i-1}\Delta z_s^*(i) \tag{6a}$$

$$\sigma_i-\sigma_{i-1}=\kappa(\bar{\sigma}-\sigma_{i-1})\Delta t+\delta dz_\sigma^*(i) \tag{7a}$$

where $\Delta t=T/n$, and i runs from 1 to n. It follows from (3a), (6a) and (7a) that

$$B_n=B_0[1+r\Delta t]^n \tag{8a}$$

$$S_n=S_0\Pi_{k=1}^{n}[1+(\mu-\lambda_S)\Delta t+\sigma_{k-1}\Delta z_s^*(k)] \tag{8b}$$

$$\sigma_{k-1}=\bar{\sigma}+(\sigma_0-\bar{\sigma})(1-\kappa\Delta t)^{k-1}+\delta\textstyle\sum_{i=1}^{k-1}(1-\kappa\Delta t)^{k-1-i}\Delta z_\sigma^*(i). \tag{8c}$$

Substituting σ_{k-1} of (8c) into (8b), we have

$$\begin{aligned}S_n=S_0\Pi_{k=1}^{n}\{&1+(\mu-\lambda_S)\Delta t\\&+[\bar{\sigma}+(\sigma_0-\bar{\sigma})(1-\kappa\Delta t)^{k-1}+\delta\textstyle\sum_{i=1}^{k-1}(1-\kappa\Delta t)^{k-1-i}\Delta z_\sigma^*(i)]\Delta z_s^*(k)\}.\end{aligned} \tag{9}$$

Taking a limit on the both sides of (9) and using Lemma 1 (*i.e.*, $\Delta z_s^*(i)\Delta z_\sigma^*(i)=\rho\Delta t$ for all i, and $\Delta z_s^*(i)\Delta z_s^*(j)=\Delta z_\sigma^*(i)\Delta z_\sigma^*(j)=\Delta z_s^*(i)\Delta z_\sigma^*(j)=0$ for $i\neq j$), we can rewrite (9) as

$$\begin{aligned}\lim_{n\to\infty}S_n=S_0\lim_{n\to\infty}\Pi_{k=1}^{n}\{&1+(\mu-\lambda_S)\Delta t\\&+[\bar{\sigma}+(\sigma_0-\bar{\sigma})(1-\kappa\Delta t)^{k-1}]\Delta z_s^*(k)+o(\Delta t)\}\end{aligned} \tag{10}$$

where $o(\cdot)$ is the asymptotic order symbol defined by $f(\Delta t)=o(\Delta t)$ if $\lim_{\Delta t\to 0}f(\Delta t)/\Delta t=0$.

The dynamics of the stock price have two sources of uncertainty, the stock price uncertainty and the volatility uncertainty. As we see from (10), the continuity property of Wiener processes eliminates the volatility uncertainty. Consequently, the terminal stock price does not exhibit the volatility uncertainty. Using a Taylor series expansion along with Lemma 1, we can rewrite (8a) and (10) as[5]

$$\lim_{n\to\infty} B_n = B_0 \lim_{n\to\infty} \exp\left[\sum_{k=1}^{n} r\Delta t\right] + o(\Delta t) \tag{11a}$$

$$\lim_{n\to\infty} S_n = S_0 \lim_{n\to\infty} \exp\{\sum_{k=1}^{n}\{(\mu - \lambda_S)\Delta t + [\bar{\sigma} + (\sigma_0 - \bar{\sigma})(1-\kappa\Delta t)^{k-1}]\Delta z_s^*(k)$$

$$-(1/2)[\bar{\sigma} + (\sigma_0 - \bar{\sigma})(1-\kappa\Delta t)^{k-1}]^2\Delta t\}\} + o(\Delta t). \tag{11b}$$

Since $\lim_{n\to\infty} S_n = S(T)$ and $\lim_{n\to\infty} B_n = B(T)$, we have that

$$\frac{S(T)}{B(T)} = \frac{S(0)}{B(0)} \lim_{n\to\infty} \exp\{(\mu - \lambda_S - r)T$$

$$+ \sum_{k=1}^{n}\{[\bar{\sigma} + (\sigma_0 - \bar{\sigma})(1-\kappa\Delta t)^{k-1}]\Delta z_s^*(k) \tag{12}$$

$$-\frac{1}{2}[\bar{\sigma} + (\sigma_0 - \bar{\sigma})(1-\kappa\Delta t)^{k-1}]^2\Delta t\}\} + o(\Delta t).$$

Ross (1978) has shown that under no arbitrage, the asset price function is a continuous and bounded linear functional. By the Riesz representation theorem (see Luenberger (1969)), the continuous and linear price functional of an asset can be written as[6]

[5]It is well known that a Taylor series expansion along with Lemma 1 is equations equivalent to Ito's lemma. But a naive application of Ito's lemma before substituting (8c) into (8b) loses information about the distribution of the terminal stock price.

[6]The stochastic Euler equation derived by an equilibrium asset pricing model (see Ahn and Thompson (1988)) is

$$E_0[mR] = 1$$

where m is the marginal rate of substitution and R is unity plus the rate of return on an asset. Thus R is $\pi(T)/\pi(0)$. Substituting R into the above equation yields

$$E_0[m\pi(T)] = \pi(0).$$

$$\pi(0) = E_0[m\pi(T)] = \int m\pi(T)dP \tag{13}$$

where m and $\pi(T)$ are F_T-measurable functions and E_0 denotes an expectation operator with respect to P based on the information available at time zero. In (13), $\pi(T)$ is the payoff from the asset at time T and m is called the Riesz payoff which turns out to be the marginal rate of substitution for inter-temporal asset pricing models. If equation (13) is applied to a default-free discount bond, $B(0)$, which has the payoff of \$1 at time T, then we have that

$$B(0) = E_0[m] = \int m dP. \tag{14}$$

Using an equivalent probability measure P^*, equation (14) can be rewritten as

$$B(0) = \int m \frac{dP}{dP^*} dP^*. \tag{15}$$

Since P^* is a probability measure (*i.e.*, $\int dP^* = 1$), equation (15) holds if $m = B(0)(dP^*/dP) = B(0)\psi$ where ψ is the Radon-Nikodym derivative.[7] Substituting m into (13) gives

$$\pi(0) = B(0)\int \pi(T)dP^* = B(0)E_0^*[\pi(T)]. \tag{16}$$

Thus equation (13) also holds in an equilibrium model.

[7]By using an inter-temporal general equilibrium model, Cox, Ingersoll, and Ross (1985, eq. 39) show that

$$J_W(W(s),Y(s),s)/J_W(W(0),Y(0),0)$$
$$= \exp\left[-\int_0^s r(W(u),Y(u),u)du + \int_0^s(-\chi'\Sigma)dw(u) - (1/2)\int_0^s |\chi'\Sigma|^2 du\right]$$

where $J_W(W(s),Y(s),s)/J_W(W(0),Y(0),0)$ is the marginal rate of substitution, exp $[-\int_0^s r(W(u),Y(u),u)du$ is the current bond price and $\exp[\int_0^s(-\chi'\Sigma)dw(u) - (1/2)\int_0^s |\chi'\Sigma|^2 du]$ is the Radon-Nikodym derivative.

Thus, in terms of our notation, the above equation can be rewritten as

$$m = B(0)(dP^*/dP).$$

Equation (16) implies that the asset price is the discounted expected value of the asset with respect to P^*. The pricing function is consistent with risk neutrality under the equivalent probability measure P^*. Since $B(T)$ is unity equation, (16) can be rewritten as

$$\frac{\pi(0)}{B(0)}=E_0^*\left[\frac{\pi(T)}{B(T)}\right]. \tag{17}$$

It follows that under no arbitrage, the ratio of the asset price to the bond price, $\pi(t)/B(t)$ for $t\geq 0$, is a P^*-martingale (see Harrison and Kreps (1979) and Harrison and Pliska (1981)). Using a general equilibrium inter-temporal asset pricing model, Cox, Ingersoll, and Ross (1985) also show that this ratio is a P^*-martingale.
We need to mention that market incompleteness problem should be solved before using martingale method under stochastic volatility model.

By taking an expectation with respect to P^* on the both sides of (12), we have that

$$E_0^*\left[\frac{S(T)}{B(T)}\right]=\frac{S(0)}{B(0)}\ \exp\ [(\mu-\lambda_S-r)T]. \tag{18}$$

Since the ratio of the stock price to the bond price is a P^*-martingale as in (17), we should have that[8]

$$\mu-\lambda_S-r=0. \tag{19}$$

From the continuity of the exponential and logarithmic functions and (19), equation (12) can be written as

$$\ln\frac{S(T)}{B(T)}=\ln\frac{S(0)}{B(0)}+\lim_{n\to\infty}\sum_{k=1}^{n}\{[\bar{\sigma}+(\sigma_0-\bar{\sigma})(1-\kappa\Delta t)^{k-1}]\Delta z_s^*(k)$$
$$-\frac{1}{2}[\bar{\sigma}+(\sigma_0-\bar{\sigma})(1-\kappa\Delta t)^{k-1}]^2\Delta t\}+o(\Delta t). \tag{20}$$

[8]Martingale property under no arbitrage provides $\mu-\lambda_S-r=0$.

It follows from a Taylor series expansion and a binomial expansion that $(1-\kappa\Delta t)^{k-1}$ becomes $\exp[-\kappa(k-1)\Delta t]+o(\Delta t)$ in a limit.[9] Thus, equation (20) can be rewritten as

$$\ln\frac{S(T)}{B(T)}=\ln\frac{S(0)}{B(0)}+\lim_{n\to\infty}\sum_{k=1}^{n}\{\{\bar{\sigma}+(\sigma_0-\bar{\sigma})\exp[-\kappa(k-1)\Delta t]\}\Delta z_s^*(k)$$

$$-\frac{1}{2}\{\bar{\sigma}+(\sigma_0-\bar{\sigma})\exp[-\kappa(k-1)\Delta t]^2\Delta t\}+o(\Delta t) \tag{21a}$$

or equivalently

$$\ln\frac{S(T)}{B(T)}=\ln\frac{S(0)}{B(0)}+\int_0^T[\bar{\sigma}+(\sigma_0-\bar{\sigma})e^{-\kappa T}]dz_s^*$$

$$-\frac{1}{2}\int_0^T[\bar{\sigma}+(\sigma_0-\bar{\sigma})e^{-\kappa T}]^2dt. \tag{21b}$$

Thus, $\ln(S(T)/B(T))$ is normally distributed with a mean of

$$E_0^*[\ln\frac{S(T)}{B(T)}]=\ln\frac{S(0)}{B(0)}-\frac{1}{2}[\bar{\sigma}^2T+2\bar{\sigma}(\sigma_0-\bar{\sigma})\frac{1-e^{-\kappa T}}{\kappa}+(\sigma_0-\bar{\sigma})^2\frac{1-e^{-2\kappa T}}{2\kappa}] \tag{22a}$$

and a variance of

[9]Without using expansions, alternatively, we can take a direct sum on terms in (20) and then take a limit. For example,

$$(\sigma_0-\bar{\sigma})^2\lim_{n\to\infty}\sum_{k=1}^{n}(1-\kappa\Delta t)^{2(k-1)}\Delta t=(\sigma_0-\bar{\sigma})^2\lim_{n\to\infty}((1-(1-\kappa\Delta t)^{2n})/(1-(1-\kappa\Delta t)^2))\Delta t$$

$$=(\sigma_0-\bar{\sigma})^2\lim_{n\to\infty}((1-(1-\kappa\Delta t)^{2n})/(2\kappa-\kappa^2\Delta t))=(\sigma_0-\bar{\sigma})^2\lim_{n\to\infty}((1-(1-\kappa\Delta t)^{2n})/2\kappa)$$

$$=(\sigma_0-\bar{\sigma})^2((1-e^{-2\kappa T})/2\kappa)$$

which is the last term of (22b). Thus, the both approaches provide the same final result.

$$Var^*[\ln\frac{S(T)}{B(T)}] = \bar{\sigma}^2 T + 2\,\bar{\sigma}(\sigma_0 - \bar{\sigma})\frac{1-e^{-\kappa T}}{\kappa} + (\sigma_0 - \bar{\sigma})^2\frac{1-e^{-2\kappa T}}{2\kappa} \quad (22b)$$

where Var^* denotes a variance operator based on P^*.

From equation (16), the European call option price is the discounted expected payoff of the option at the expiration date with respect to P^*. Thus the European call option price is given by

$$C(0) = B(0)E_0^*[Max(0, S(T) - X)] \quad (23)$$

where X is the exercise price of the option. The European call option price is given by the following theorem.

Theorem 1. The European call option price is given by

$$C(0) = SN(d_1) - X\,exp[-rT]N(d_2) \quad (24)$$

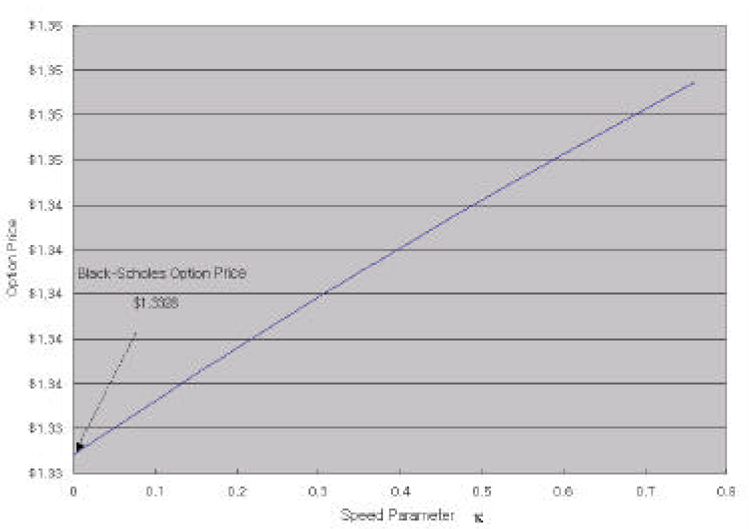
where $S \equiv S(0)$, $N(\cdot)$ denotes a standard cumulative distribution function and

$$d_1 = \frac{\ln(S/X) + rT + \frac{1}{2}\sigma_T^2}{\sigma_T} \quad (24a)$$

$$d_2 = \frac{\ln(S/X) + rT - \frac{1}{2}\sigma_T^2}{\sigma_T} \quad (24b)$$

$$\sigma_T^2 = \bar{\sigma}^2 T + 2\,\bar{\sigma}(\sigma_0 - \bar{\sigma})\frac{1-e^{-\kappa T}}{\kappa} + (\sigma_0 - \bar{\sigma})^2\frac{1-e^{-2\kappa T}}{2\kappa} \quad (24c)$$

$$\bar{\sigma} = \underline{\sigma} - \frac{\lambda_\sigma}{\kappa}. \quad (24d)$$



The following parameter values are used to obtain option prices: S=\$45, X=\$45, r=0.05 and T=time to expiration=0.5

FIGURE 1

CALL OPTION PRICES UNDER STOSHASTIC VOLATILITY

Proof: *see the Appendix.*

If the stock volatility becomes constant, we have that $\kappa=\delta=0$. It follows from $\lambda_\sigma=-((\rho\gamma_1+\gamma_2)/\psi)\delta$ in (7) that $\delta=0$ implies $\lambda_\sigma=0$. It follows from (24d) that $\overline{\sigma}=\underline{\sigma}$. Applying L'Hospital's rule gives that $\lim_{\kappa\to0}((1-e^{-\kappa T})/\kappa)=\lim_{\kappa\to0}Te^{-\kappa T}=T$ and $\lim_{\kappa\to0}((1-e^{-2\kappa T})/2\kappa)=\lim_{\kappa\to0}Te^{-2\kappa T}=T$. It follows from (24c) that $\sigma^2_T=\overline{\sigma}^2T+2\overline{\sigma}(\sigma_0-\overline{\sigma})T+(\sigma_0-\overline{\sigma})^2T=[\overline{\sigma}+(\sigma_0-\overline{\sigma})]^2T=\sigma^2_0T$.

The Black-Scholes option pricing formula is obtained. Our formula (24) includes the Black-Scholes formula as a special case. Option prices are obtained with S=\$45, X=\$45, r=0.05, T=0.5 and various κ and are plotted in Figure 1. Note that the Black-Scholes option price is the option price under stochastic volatilty with $\kappa=0$.

If the speed parameter κ is zero but volatility is stochastic ($\delta\neq0$), then the formula (24) needs a change in (24c) such that[10]

$$\sigma^2_T = \sigma^2_0 T - \lambda_\sigma \sigma_0 T^2 + \lambda^2_\sigma T^3/3. \tag{25}$$

The main difference between our formula (24) and the Black-Scholes formula is in the specification of the variance of the underlying stock return. Using a Taylor series expansion on (24c), we have that

$$\sigma^2_T = \bar{\sigma}^2 T + 2\bar{\sigma}((\sigma_0 - \bar{\sigma})(T - \frac{\kappa T^2}{2} + \frac{\kappa^2 T^3}{6} - \frac{\kappa^3 T^4}{24} \cdots$$
$$+ (\sigma_0 - \bar{\sigma})^2 (T - \kappa T^2 + \frac{2\kappa^2 T^3}{3} - \frac{\kappa^3 T^4}{3} + \cdots). \tag{26}$$

The difference between our formula and the Black-Scholes formula comes from the second and third terms of the right-hand side of (26). It becomes apparent that for short-term-maturity options, the Black-Scholes formula provides very similar option prices to one obtained from (24) since terms with high orders of T (time to expiration) have negligible effects, as seen in (26). Intuitively, the effect of stochastic volatility is small because the volatility does not change much for a short-term maturity. The pricing formula of Theorem 1 describes the volatility adjustment to be made in the Black-Scholes model when volatility is stochastic.

Even though risk-neutral pricing has been used in the previous literature, the martingale property of the asset price has not been extensively exploited so far. As shown by equation (17), a general form of risk-neutral pricing arises. Thus the option price is the discounted expected option value at the expiration date under P^* (the risk-neutral or equivalent probability measure).[11] Results

[10] In this case, the counterparts of (21a) and (21b) are

$$\ln(S(T)/B(T)) = \ln(S(0)/B(0)) + \lim_{n\to\infty}\sum_{k=1}^{n}\{[\sigma_0 - (k-1)\lambda_\sigma \Delta t]\Delta z_s^*(k)$$
$$-(1/2)[\sigma_0 - (k-1)\lambda_\sigma \Delta t]^2 \Delta t\} + o(\Delta t)$$

or equivalently

$$\ln\ (S(T)/B(T)) = \ln\ (S(0)/B(0)) + \int_0^T (\sigma_0 - \lambda_\sigma t) dz_s^* - (1/2)\int_0^T (\sigma_0 - \lambda_\sigma t)^2 dt.$$

[11] The original form of risk neutral pricing arises from the fact that the

presented here extend the work of Hull and White (1987), Scott (1987) and Wiggins (1987) by deriving a closed-form solution for the option price under stochastic volatility. This is possible by utilizing the martingale and continuity properties. The martingale property of the ratio of the stock price to the bond price has been utilized as in equation (19). The continuity property of a Wiener process has that with probability 1, $(\Delta z_s^*(i))^2 = (\Delta z_\sigma^*(i))^2 = \Delta t$ and $\Delta z_s^*(i)\,\Delta z_\sigma^*(i) = \rho\Delta t$ for all i, and $\Delta z_s^*(i)\,\Delta z_s^*(j) = \Delta z_\sigma^*(i)\,\Delta z_\sigma^*(j) = \Delta z_s^*(i)$ $\Delta z_\sigma^*(j) = 0$ for $i \neq j$ (Lemma 1). This property eliminates the volatility uncertainty for the terminal stock price. Consequently, the terminal stock price follows a log-normal distribution under the equivalent probability measure.

Comparative statics are performed by differentiating (24) with respect to each parameter:

$$\frac{\partial C}{\partial S} = N(d_1) > 0 \tag{27a}$$

$$\frac{\partial C}{\partial T} = Xre^{-rT}N(d_2) + \frac{Xe^{-rT}}{2\,\sigma_T\sqrt{2\,\pi}}exp(-\frac{d_2{}^2}{2})[\,\bar{\sigma} + (\,\sigma_0 - \bar{\sigma})e^{-\kappa T}]^2 > 0 \tag{27b}$$

$$\frac{\partial C}{\partial X} = -e^{-rT}N(d_2) < 0 \tag{27c}$$

$$\frac{\partial C}{\partial r} = TXe^{-rT}N(d_2) > 0 \tag{27d}$$

$$\frac{\partial C}{\partial\,\sigma_0} = \frac{Xe^{-rT}}{2\,\sigma_T\sqrt{2\,\pi}}exp\left(-\frac{d_2{}^2}{2}\right)[\,\sigma_0(1-e^{-2\kappa T}) + \bar{\sigma}(1-e^{-\kappa T})^2]/\,\kappa > 0 \tag{27e}$$

$$\frac{\partial C}{\partial\kappa} = \frac{Xe^{-rT}}{2\,\sigma_T\sqrt{2\,\pi}}exp\left(-\frac{d_2{}^2}{2}\right)\Bigg[\frac{2\,\bar{\sigma}\,\lambda_\sigma T}{\kappa^2} + 2(\,\sigma_0 - 2\,\bar{\sigma})\,\lambda_\sigma\frac{1-e^{-\kappa T}}{\kappa^3}$$

Black-Scholes partial differential equation is independent of risk preferences (see Cox and Ross (1976)). Hull and White (1987) do not impose the martingale property on the ratio of the terminal stock price to the terminal bond price based on the information at time zero (see their equation (11)).

$$-2(\sigma_0-\bar{\sigma})\,\lambda_\sigma\,\frac{1-e^{-2\kappa T}}{2\kappa^3}+2\bar{\sigma}(\sigma_0-\bar{\sigma})\frac{\kappa T e^{-\kappa T}+e^{-\kappa T}-1}{\kappa^2} \tag{27f}$$

$$+(\sigma_0-\bar{\sigma})^2\frac{2\kappa T e^{-2\kappa T}+e^{-2\kappa T}-1}{2\kappa^2}\Bigg]$$

$$\frac{\partial C}{\partial \underline{\sigma}}=\frac{Xe^{-rT}}{2\sigma_T\sqrt{2\pi}}exp\left(-\frac{{d_2}^2}{2}\right)\Bigg[\bar{\sigma}T(\sigma_0-2\bar{\sigma})\frac{1-e^{-\kappa T}}{\kappa}$$

$$-(\sigma_0-\bar{\sigma})\frac{1-e^{-2\kappa T}}{2\kappa}\Bigg] \tag{27g}$$

$$\frac{\partial C}{\partial \lambda_\sigma}=\frac{Xe^{-rT}}{\sigma_T\sqrt{2\pi}}exp\left(-\frac{{d_2}^2}{2}\right)\Bigg[-\frac{\bar{\sigma}T}{\kappa}-(\sigma_0-2\bar{\sigma})\frac{1-e^{-\kappa T}}{\kappa^2}$$

$$-(\sigma_0-\bar{\sigma})\frac{1-e^{-2\kappa T}}{2\kappa^2}\Bigg]. \tag{27h}$$

The above comparative statics are straightforward. The call option price increases with the current stock price, the risk-free interest rate, the time to the expiration date and the initial volatility of stock return.

In our model, the volatility movement and the stock price movement are allowed to be correlated with each other. By the visual inspection of (24) we can see that the correlation between them has an effect on the call option price only through the risk premium of stochastic volatility λ_σ. Using a Taylor series expansion, we can rewrite (27h) as

$$\frac{\partial C}{\partial \lambda_\sigma}=\frac{Xe^{-rT}}{\sigma_T\sqrt{2\pi}}exp\left(-\frac{{d_2}^2}{2}\right)\Bigg[\sigma_0\left(-\frac{T^2}{2}+\frac{\kappa T^3}{2}-\frac{7\kappa^2T^4}{24}+\frac{\kappa^3T^5}{8}-\frac{31\kappa^4T^6}{720}+\cdots\right)$$

$$+\bar{\sigma}\left(-\frac{\kappa T^3}{3}+\frac{\kappa^2T^4}{4}-\frac{7\kappa^3T^5}{60}+\frac{\kappa^4T^6}{24}-\cdots\right)\Bigg]. \tag{27i}$$

It follows from (27i) that for short-term-maturity options, as the risk premium λ_σ of stochastic volatility increases, the call option price decreases. As discussed in Footnote 3, as the correlation coefficient ρ increases, the risk premium of stochastic volatility λ_σ tends to increase. Thus, as ρ increases, the call option price decreases for short-term-maturity options.

Similarly, the diffusion coefficient δ of stochastic volatility affects the call option price only through the risk premium λ_σ of stochastic volatility. As discussed in Footnote 3, as δ increases, λ_σ tends to increase. Thus, as δ increases, the call option price decreases for short-term-maturity options.

Formula (24) includes the risk premium λ_σ of stochastic volatility as a parameter. The premium is unobservable since the volatility is not a traded asset. Earlier papers assume that volatility risk is not priced by the markets, *i.e.*, $\lambda_\sigma=0$. For instance, Hull and White (1987) assume that volatility is un-correlated with aggregate consumption. Scott (1987) assumes that volatility risk can be diversified away and that volatility and stock returns are un-correlated. Wiggins (1987) assumes that the market portfolio is the underlying asset and investors have logarithmic utility. Assuming that λ_σ is zero for any of the above arguments, we can test our formula with market data by additionally estimating the speed parameter κ and the long-run mean value $\underline{\sigma}$.

If λ_σ is assumed to be zero, following Hull and White (1987), Scott (1987) and Wiggins (1987), then the correlation coefficient ρ and the diffusion coefficient δ do not affect the call option price at all. Thus the economic implication of non-priced volatility risk is that the call option price is not affected by the correlation between the stock price movement and the volatility movement and the standard deviation of stochastic volatility.

III. Lognormal Diffusion Volatility

In this section, we extend the analysis of Section II when a lognormal diffusion process replaces a mean-reverting diffusion process (2) for the volatility of the stock. We specifically focus on the differences from Section II.

Assume that the volatility of the stock follows a lognormal diffusion process given by

$$d\sigma(t) = \alpha\sigma(t)dt + \delta\sigma(t)dz_\sigma(t) \tag{28}$$

where α and $\delta > 0$ are constants and the correlation coefficient between dz_s and dz_σ is ρ.

By transforming the stock price process and the volatility process into those under an equivalent probability measure P^*, we have that

$$dS(t) = (\mu - \lambda_S)S(t)dt + \sigma(t)S(t)dz_s^*(t) \tag{6}$$

$$d\sigma(t) = (\alpha - \lambda_\sigma)\sigma(t)dt + \delta\sigma(t)dz_\sigma^*(t) \tag{29}$$

where $\lambda_\sigma = -((\rho\gamma_1 + \gamma_2)/\psi)\delta$ is the risk premium of volatility. The discrete approximations of (6) and (29) are given by

$$\frac{S_i - S_{i-1}}{S_{i-1}} = (\mu - \lambda_S)\Delta t + \sigma_{i-1}\Delta z_s^*(i) \tag{6a}$$

$$\frac{\sigma_i - \sigma_{i-1}}{\sigma_{i-1}} = (\alpha - \lambda_\sigma)\Delta t + \delta\Delta z_\sigma^*(i) \tag{29a}$$

where $\Delta t = T/n$, i runs from 1 to n. It follows from (6a) and (29a) that

$$S_n = S_0 \Pi_{k=1}^{n}[1 + (\mu - \lambda_S)\Delta t + \sigma_{k-1}\Delta z_s^*(k)] \tag{30a}$$

$$\sigma_{k-1} = \sigma_0 \Pi_{i=1}^{k-1}[1 + (\alpha - \lambda_\sigma)\Delta t + \delta\Delta z_\sigma^*(i)]. \tag{30b}$$

Substituting σ_k of (30b) into (30a), we have

$$S_n = S_0 \Pi_{i=1}^{n}\{1 + (\mu - \lambda_S)\Delta t + \sigma_0 \Pi_{i=1}^{k-1}[1 + (\alpha - \lambda_\sigma)\Delta t + \delta\Delta z_\sigma^*(i)]\Delta z_s^*(k)\}. \tag{31}$$

Taking a limit on the both sides of (31) and using Lemma 1 (*i.e.*, $\Delta z_\sigma^*(i)\Delta z_s^*(k) = \rho\Delta t$ for $i = k$, and $\Delta z_s^*(i)\Delta z_s^*(k) = \Delta z_\sigma^*(i)\Delta z_s^*(k) = 0$ for $i \neq k$), we can rewrite (31) as[12]

$$\lim_{n\to\infty} S_n = S_0 \lim_{n\to\infty} \Pi_{k=1}^{n}[1 + (\mu - \lambda_S)\Delta t + \sigma_0[1 + (\alpha - \lambda_\sigma)\Delta t]^{k-1}\Delta z_s^*(k) + o(\Delta t)]. \tag{32}$$

There are two sources of uncertainty, the stock price uncertainty and the volatility uncertainty. The volatility uncertainty has been eliminated by the continuity property of Wiener processes. Consequently, the terminal stock price follows a log-normal distribution under the equivalent probability measure. Using a Taylor series[12]

$$\lim_{n\to\infty} S_n = S_0 \lim_{n\to\infty} exp\{\textstyle\sum_{k=1}^{n}\{(\mu-\lambda_S)\Delta t+\sigma_0[1+(\alpha-\lambda_\sigma)\Delta t]^{k-1}\Delta z_S^*(k)$$
$$-(\sigma_0{}^2/2)[1+(\alpha-\lambda_\sigma)\Delta t]^{2(k-1)}\Delta t\}\}+o(\Delta t). \tag{33}$$

Since $\lim_{n\to\infty} S_n = S(T)$ and $\lim_{n\to\infty} B_n = B(T)$, by using (11a) and (33), we have that

$$\frac{S(T)}{B(T)} = \frac{S(0)}{B(0)} \lim_{n\to\infty} exp\{(\mu-\lambda_S-r)T+\textstyle\sum_{k=1}^{n}\{\sigma_0[1+(\alpha-\lambda_\sigma)\Delta t]^{k-1}\Delta z_S^*(k)$$
$$-(\sigma_0{}^2/2)[1+(\alpha-\lambda_\sigma)\Delta t]^{2(k-1)}\Delta t\}\}+o(\Delta t). \tag{34}$$

By taking an expectation with respect to P^* on the both sides of (34), we have that

$$E_0^*\left[\frac{S(T)}{B(T)}\right] = \frac{S(0)}{B(0)}\, exp[(\mu-\lambda_S-r)T]. \tag{35}$$

Since the ratio of the stock price to the bond price is a P^*-martingale as in (17), we should have that[13]

$$exp[(\mu-\lambda_S-r)T]=1. \tag{36}$$

By the continuity of the exponential and logarithmic functions and

[12] It follows from Lemma 1 that

$$\lim_{n\to\infty} \Pi_{k=1}^{n} \Pi_{i=1}^{k-1}[1+(\alpha-\lambda_\sigma)\Delta t+\delta\Delta z_\sigma^*(i)]\Delta z_S^*(k)$$
$$=\lim_{n\to\infty}\Pi_{k=1}^{n}\{[1+(\alpha-\lambda_\sigma)\Delta t]^{k-1}\Delta z_S^*(k)+o(\Delta t)].$$

[13] The same result has been obtained in section II with the mean-verting diffusion process of stochastic volatility.

(36), we can rewrite equation (34) as

$$\ln\frac{S(T)}{B(T)}=\ln\frac{S(0)}{B(0)}+\lim_{n\to\infty}\sum_{k=1}^{n}\{\sigma_0[1+(\alpha-\lambda_\sigma)\Delta t]^{k-1}\Delta z_s^*(k) \\ -(\sigma_0^2/2)[1+(\alpha-\lambda_\sigma)\Delta t]^{2(k-1)}\Delta t+o(\Delta t)\}. \tag{37}$$

It follows from a Taylor series expansion that $[1+(\alpha-\lambda_\sigma)\Delta t]^{k-1}$ becomes $exp[(\alpha-\lambda_\sigma)(k-1)\Delta t]+o(\Delta t)$. Thus, equation (37) can be rewritten as

$$\ln\frac{S(T)}{B(T)}=\ln\frac{S(0)}{B(0)}+\lim_{n\to\infty}\sum_{k=1}^{n}\{\sigma_0\ exp[(\alpha-\lambda_\sigma)(k-1)\Delta t]\Delta z_s^*(k) \\ -(\sigma_0^2/2)\ exp[2(\alpha-\lambda_\sigma)(k-1)\Delta t]\Delta t+o(\Delta t)\} \tag{38a}$$

or equivalently

$$=\ln\frac{S(0)}{B(0)}+\int_0^T\sigma_0\ exp[(\alpha-\lambda_\sigma)t]dz_s^*-(\sigma_0^2/2)\int_0^T exp[2(\alpha-\lambda_\sigma)t]dt. \tag{38b}$$

It follows that $\ln(S(T)/B(T))$ is normally distributed with a mean of

$$E_0^*\left[\ln\frac{S(T)}{B(T)}\right]=\ln\frac{S(0)}{B(0)}-\frac{\sigma_0^2}{4(\alpha-\lambda_\sigma)}\{exp[2(\alpha-\lambda_\sigma)T]-1\} \tag{39a}$$

and a variance of

$$Var^*\left[\ln\frac{S(T)}{B(T)}\right]=\frac{\sigma_0^2}{2(\alpha-\lambda_\sigma)}\{exp[2(\alpha-\lambda_\sigma)T]-1\} \tag{39b}$$

where Var* denotes a variance operator based on P^*. The European call option price can be obtained by using equation (23).

Theorem 2. The European call option price is given by

$$C(0)=SN(d_1)-X\ exp[-rT]N(d_2) \tag{40}$$

where $S \equiv S(0)$, $N(\cdot)$ denotes a standard cumulative distribution function and

$$d_1 = \frac{\ln(S/X) + rT + \frac{1}{2}\sigma_T^2}{\sigma_T} \tag{40a}$$

$$d_2 = \frac{\ln(S/X) + rT - \frac{1}{2}\sigma_T^2}{\sigma_T} \tag{40b}$$

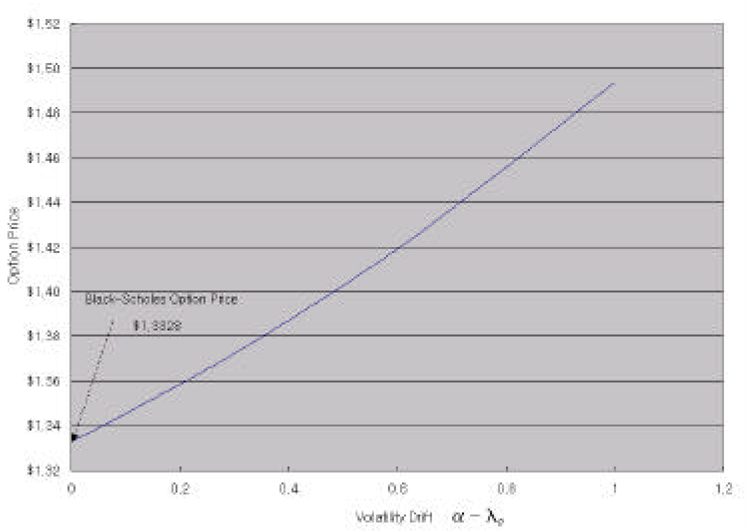
$$\sigma_T^2 = \frac{\sigma_0^2}{2(\alpha - \lambda_\sigma)}\{exp[2(\alpha - \lambda_\sigma)T] - 1\}. \tag{40c}$$

Proof: see the Appendix.

As we can see in Section II, the pricing formula is similar to the Black-Scholes formula except for the volatility adjustment when volatility is stochastic. If the stock volatility becomes constant, we have that $\alpha = \delta = \lambda_\sigma = 0$. Applying L'Hospital's rule gives that $\lim_{(\alpha-\lambda_\sigma)\to 0}(\sigma_0^2/2(\alpha-\lambda_\sigma))\{exp[2(\alpha-\lambda_\sigma)T]-1\} = \lim_{(\alpha-\lambda_\sigma)\to 0}\sigma_0^2 T\, exp[2(\alpha-\lambda_\sigma)T] = \sigma_0^2 T$. It follows from (40c) that $\sigma^2 T$ becomes $\sigma_0^2 T$ with $\alpha = \lambda_\sigma = 0$ and the Black-Scholes option pricing formula is obtained. Our formula (40) includes the Black-Scholes formula as a special case. Option prices are obtained with $S=\$45$, $X=\$45$, $r=0.05$, $T=0.5$ and various $\alpha - \lambda_\sigma$ and are plotted in Figure 2. Note that the Black-Scholes option price is the option price under stochastic volatilty with $\alpha - \lambda_\sigma = 0$.

The main difference between our formula and the Black-Scholes formula is in the specification of the variance of the underlying stock return. Using a Taylor series expansion, we have that

$$\sigma_T^2 = \sigma_0^2[T + (\alpha - \lambda_\sigma)T^2 + \frac{2(\alpha - \lambda_\sigma)^2 T^3}{3} + \frac{(\alpha - \lambda_\sigma)^3 T^4}{3} + \frac{2(\alpha - \lambda_\sigma)^4 T^5}{15} + \cdots]. \tag{41}$$



The following parameter values are used to obtain option prices: S=\$45, X=\$45, r=0.05 and T=time to expiration=0.5

FIGURE 2

CALL OPTION PRICES UNDER STOSHASTIC VOLATILITY

It becomes apparent that for short-term-maturity options, the Black-Scholes formula provides very similar option prices to one obtained from (40) since terms with high orders of T (time to expiration) have negligible effects, as seen in (41). The pricing formula of Theorem 2 describes the volatility adjustment to be made in the Black-Scholes model when volatility is stochastic.

As we discussed in Section II, the call option price (40) increases with the current stock price, the risk-free interest rate, the time to the expiration date and the initial volatility of stock return. The effect of the risk premium λ_σ of stochastic volatility on the call option price is given by

$$\frac{\partial C}{\partial \lambda_\sigma}=\frac{Xe^{-rT}}{2\sigma_T\sqrt{2\pi}}exp\left(-\frac{d_2{}^2}{2}\right)\left\{-\frac{T\,exp[2(\alpha-\lambda_\sigma)T]}{\alpha-\lambda_\sigma}+\frac{exp[2(\alpha-\lambda_\sigma)T]-1}{2}\right\}$$

$$=\frac{Xe^{-rT}}{2\sigma_T\sqrt{2\pi}}exp\left(-\frac{d_2{}^2}{2}\right)\left[-T^2-\frac{4}{3}(\alpha-\lambda_\sigma)T^3-(\alpha-\lambda_\sigma)^2T^4\right. \qquad (42)$$

$$\left.-\frac{8}{15}(\alpha-\lambda_\sigma)^3T^5-\cdots\right].$$

The effect of the risk premium λ_σ of stochastic volatility on the call option price depends on the sign of $\alpha-\lambda_\sigma$. Thus if $\alpha-\lambda_\sigma$ is positive, as λ_σ increases, the call option price decreases. On the other hand, if $\alpha-\lambda_\sigma$ is negative, as λ_σ increases, the call option price may decrease or increase.

From the visual inspection of formula (40), we can easily see that the diffusion coefficient δ of stochastic volatility and the correlation coefficient ρ between the stock price movement and the volatility movement affect the call option price only through the risk premium λ_σ of stochastic volatility. As discussed in Footnote 3, as ρ and δ increase, λ_σ increases. Thus, as ρ and δ increase, the call option price decreases if $\alpha-\lambda_\sigma$ is positive.

If volatility risk is not priced ($\lambda_\sigma=0$), following Hull and White (1987), Scott (1987) and Wiggins (1987), then ρ and δ do not affect the call option price.

IV. Conclusion

We have examined the European call option price when the volatility of the stock returns follows a diffusion process. The continuity property of a diffusion process and the martingale approach are used to obtain a closed-form solution.

Even though the instantaneous standard deviation of stock return is uncertain over time, the resulting pricing formula is straightforward. The volatility adjustment is to be made in the Black-Scholes formula, when volatility is stochastic. The volatility movement is contemporaneously correlated with the stock price movement. The closed-form solution for the option price allows us to find the effect of stochastic volatility on the option price.

We show that the drift factor and the risk premium of stochastic volatility are important in option pricing. The volatility of stock price volatility and the correlation between the volatility movement and the stock price movement affect the option price only through the risk premium of stochastic volatility. Thus they do not affect the option price if volatility risk is not priced by the markets (*i.e.*, the risk premium of stochastic volatility is zero).

Appendix

***Proof of Lemma 1* :**

i) We use E_0^* to denote an expectation operator with respect to P^* based on the information available at time zero. Consider $(\Delta z_s^*(i))^2 = \varepsilon_i^{*2}\Delta t$. Then we have that

$$E_0^*[(\Delta z_s^*(i))^2] = E_0^*[\varepsilon_1^{*2}\Delta t] = \Delta t E_0^*[\varepsilon_1^{*2}] = \Delta t$$

and

$$\begin{aligned}\mathrm{Var}^*[(\Delta z_s^*(i))^2] &= E_0^*[(\Delta z_s^*(i))^4] - \{E_0^*[\Delta z_s^*(i)^2]\}^2 \\ &= E_0^*[\varepsilon_1^{*4}\Delta t^2] - \Delta t^2 = 2\Delta t^2 = \mathrm{o}(\Delta t)\end{aligned}$$

where Var* is a variance operator based on P^* and the asymptotic order symbol $\mathrm{o}(\Delta t)$ is defined by $f(\Delta t) = \mathrm{o}(\Delta t)$ if $\lim_{\Delta t \to 0} f(\Delta t)/\Delta t = 0$. Since $(\Delta z_s^*(i))^2$ is non-stochastic, $(\Delta z_s^*(i))^2 = E_0^*[(\Delta z_s^*(i))^2]$. Thus we have that $(\Delta z_s^*(i))^2 = \Delta t$. Similarly, $(\Delta z_\sigma^*(i))^2 = \Delta t$.

ii) Consider $\Delta z_s^*(i)\Delta z_\sigma^*(i) = \varepsilon_i^* \eta_i^* \Delta t$. Then we have that

$$E_0^*[\Delta z_s^*(i)\Delta z_\sigma^*(i)] = E_0^*[\varepsilon_i^* \eta_i^* \Delta t] = \rho\Delta t$$

and

$$\begin{aligned}\mathrm{Var}^*[\Delta z_s^*(i)\Delta z_\sigma^*(i)] &= E_0^*[(\Delta z_s^*(i)\Delta z_\sigma^*(i))^2] - \{E_0^*[\Delta z_s^*(i)\Delta z_\sigma^*(i)]\}^2 \\ &= E_0^*[\varepsilon_i^{*2} \eta_i^{*2} \Delta t^2] - \rho^2\Delta t^2 = \Delta t^2(1-\rho^2) = \mathrm{o}(\Delta t).\end{aligned}$$

Since $\Delta z_s^*(i)\Delta z_\sigma^*(i)$ is non-stochastic, $\Delta z_s^*(i)\Delta z_\sigma^*(i) = E_0^*[\Delta z_s^*(i)\Delta z_\sigma^*(i)]$. Thus we have $\Delta z_s^*(i)\Delta z_\sigma^*(i) = \rho\Delta t$.

iii) Consider $\Delta z_s^*(i)\Delta z_s^*(j) = \varepsilon_i^* \varepsilon_j^* \Delta t$ for $i<j$. Then, we have that

$$E_0^*[\Delta z_s^*(i)\Delta z_s^*(j)] = E_0^*[\varepsilon_i^* \varepsilon_j^* \Delta t] = E_0^*[\varepsilon_i^* E_{j-1}^*[\varepsilon_j^*]]\Delta t = 0$$

and

$$\text{Var}^*[\Delta z_s^*(i)\Delta z_s^*(j)] = E_0^*[(\Delta z_s^*(i)\Delta z_s^*(j))^2] - \{E_0^*[\Delta z_s^*(i)\Delta z_s^*(j)]\}^2$$

$$= E_0^*[\varepsilon_i^{*2}\varepsilon_j^{*2}\Delta t^2] = \Delta t^2 = \text{o}(\Delta t).$$

Since $\Delta z_s^*(i)\Delta z_s^*(j)$ is non-stochastic, $\Delta z_s^*(i)\Delta z_s^*(j) = E_0^*[\Delta z_s^*(i)\Delta z_s^*(j)]$. Thus we have that $\Delta z_s^*(i)\Delta z_s^*(j) = 0$. We have the same result if $i > j$. Similarly, we obtain that $\Delta z_\sigma^*(i)\Delta z_\sigma^*(j) = 0$ and $\Delta z_s^*(i)\Delta z_\sigma^*(j) = 0$ for $i \neq j$.

Q.E.D.

Proof of Theorem 1 :

Let y denote $\ln(S(T)/B(T)$, μ_T denote $E_0^*[\ln(S(T)/B(T)]$ and σ_T^2 denote $Var^*[\ln(S(T)/B(T)]$. Note that y is normally distributed as shown in (21a) or (21b), $B(T) = 1$ and $B(0) = e^{-rT}$.

The European call option price is given by

$$C = B(0)E_0^*[Max[0, S(T) - X]]$$

$$= B(0)E_0^*[Max[0, \frac{S(T)}{B(T)} - X]]$$

$$= B(0)\frac{1}{\sqrt{2\pi}\,\sigma_T}\int_{-\infty}^{\infty}[Max[0, \frac{S(T)}{B(T)} - X]\ exp[-\frac{(y-\mu_T)^2}{2\sigma_T^2}]dy$$

$$= B(0)\frac{1}{\sqrt{2\pi}\,\sigma_T}\int_{-\infty}^{\infty}[Max[0, e^y - X]\ exp[-\frac{(y-\mu_T)^2}{2\sigma_T^2}]dy$$

$$= B(0)\frac{1}{\sqrt{2\pi}\,\sigma_T}\int_{\ln X}^{\infty} e^y exp[-\frac{(y-\mu_T)^2}{2\sigma_T^2}]dy$$

$$-XB(0)\frac{1}{\sqrt{2\pi}\,\sigma_T}\int_{\ln X}^{\infty} exp[-\frac{(y-\mu_T)^2}{2\sigma_T^2}]dy$$

$$= B(0)\frac{1}{\sqrt{2\pi}\,\sigma_T}\int_{\ln X}^{\infty} exp[-\frac{y^2 - 2\mu_T y - 2\sigma_T^2 y + \mu_T^2}{2\sigma_T^2}]dy$$

$$-XB(0)N(\frac{\mu_T-\ln X}{\sigma_T}),$$ where $N(\cdot)$ is a standard normal distribution

$$=B(0)\frac{1}{\sqrt{2\pi}\ \sigma_T}\int_{\ln X}^{\infty}exp[-\frac{(y-\mu_T-\sigma_T^2)^2+\mu_T^2-(\mu_T+\sigma_T^2)^2}{2\sigma_T^2}]dy$$

$$-XB(0)N(\frac{\mu_T-\ln X}{\sigma_T})$$

$$=B(0)exp[\mu_T+\frac{\sigma_T^2}{2}]\frac{1}{\sqrt{2\pi}\ \sigma_T}\int_{\ln X}^{\infty}exp[-\frac{(y-\mu_T-\sigma_T^2)^2}{2\sigma_T^2}]dy$$

$$-XB(0)N(\frac{\mu_T-\ln X}{\sigma_T})$$

$$=B(0)exp[\mu_T+\frac{\sigma_T^2}{2}]N(\frac{\mu_T-\ln X+\sigma_T^2}{\sigma_T})$$

$$-XB(0)N(\frac{\mu_T-\ln X}{\sigma_T})$$

$$=B(0)exp[\mu_T+\frac{\sigma_T^2}{2}]N(\frac{\mu_T-\ln X+\sigma_T^2}{\sigma_T})-XB(0)N(\frac{\mu_T-\ln X}{\sigma_T})$$

$$=B(0)exp[\ln\frac{S(0)}{B(0)}]N(\frac{\mu_T-\ln X+\sigma_T^2}{\sigma_T})-XB(0)N(\frac{\mu_T-\ln X}{\sigma_T})$$

$$=SN(d_1)-X\,exp[-rT]N(d_2)$$

where $S\equiv S(0)$ and

$$d_1=\frac{\mu_T-\ln X+\sigma_T^2}{\sigma_T}$$

$$= \frac{\ln(S/X)+rT+(1/2)[\bar{\sigma}T+2\bar{\sigma}(\sigma_0-\bar{\sigma})(1-e^{-\kappa T})/\kappa+(\sigma_0-\bar{\sigma})^2(1-e^{-2\kappa T})/2\kappa]}{\sqrt{\bar{\sigma}T+2\bar{\sigma}(\sigma_0-\bar{\sigma})(1-e^{-\kappa T})/\kappa+(\sigma_0-\bar{\sigma})^2(1-e^{-2\kappa T})/2\kappa}}$$

$$d_2 = \frac{\mu_T - \ln X}{\sigma_T}$$

$$= \frac{\ln(S/X)+rT-(1/2)[\bar{\sigma}T+2\bar{\sigma}(\sigma_0-\bar{\sigma})(1-e^{-\kappa T})/\kappa+(\sigma_0-\bar{\sigma})^2(1-e^{-2\kappa T})/2\kappa]}{\sqrt{\bar{\sigma}T+2\bar{\sigma}(\sigma_0-\bar{\sigma})(1-e^{-\kappa T})/\kappa+(\sigma_0-\bar{\sigma})^2(1-e^{-2\kappa T})/2\kappa}}$$

Q.E.D.

Proof of Theorem 2 :

Let y denote $\ln(S(T)/B(T))$, μ_T denote $E_0^*[\ln(S(T)/B(T)]$ and σ_T^2 denote $Var^*[\ln(S(T)/B(T)]$. Note that y is normally distributed as shown in (38a) or (38b), $B(T)=1$ and $B(0)=e^{-rT}$. The proof strategy of this theorem closely follows the one of Theorem 1.

The European call option price is given by

$$C = B(0)E_0^*[Max[0, S(T)-X]]$$

$$= B(0)E_0^*[Max[0, \frac{S(T)}{B(T)} - X]]$$

$$= B(0)exp[\ln\frac{S(0)}{B(0)}]N(\frac{\mu_T - \ln X + \sigma_T^2}{\sigma_T}) - XB(0)N(\frac{\mu_T - \ln X}{\sigma_T})$$

$$= SN(d_1) - X\,exp[-rT]N(d_2)$$

where $S \equiv S(0)$ and

$$d_1 = \frac{\mu_T - \ln X + \sigma_T^2}{\sigma_T}$$

$$= \frac{\ln(S/X)+rT+(1/2)\sigma_0^2\{exp[2(\alpha-\lambda_\sigma)T]-1\}/[2(\alpha-\lambda_\sigma)]}{\sqrt{\sigma_0^2\{exp[2(\alpha-\lambda_\sigma)T]-1\}/[2(\alpha-\lambda_\sigma)]}}$$

$$d_2 = \frac{\mu_T - \ln X}{\sigma_T}$$

$$= \frac{\ln(S/X) + rT - (1/2)\,\sigma_0^{\,2}\{exp[2(\alpha - \lambda_\sigma)T] - 1\}/[2(\alpha - \lambda_\sigma)]}{\sqrt{\sigma_0^{\,2}\{exp[2(\alpha - \lambda_\sigma)T] - 1\}/[2(\alpha - \lambda_\sigma)]}}$$

Q.E.D.

(*Received 2 September 2002; Revised 20 Octorber 2003*)

References

Ahn, C., and Thompson, H. "Jump-Diffusion Processes and the Term Structure of Interest Rates." *Journal of Finance* 43 (No. 1 1988): 155-74.

Beckers, S. "The Constant Elasticity of Variance Model and Its Implications for Option Pricing." *Journal of Finance* 35 (No. 3 1980): 661-73.

Black, F., and Scholes, M. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81 (No. 3 1973): 637-59.

Blattberg, R., and Gonedes, N. "A Comparison of the Stable and Student Distributions as Statistical Models for Stock Prices." *Journal of Business* 47 (No. 2 1974): 244-80.

Castanias, R. "Macroinformation and the Variability of Stock Market Prices." *Journal of Finance* 34 (No. 2 1979): 439-50.

Chesney, M., and Scott, L. "Pricing European Currency Options: A Comparison of Modified Black-Scholes Model and a Random Variance Model." *Journal of Financial and Quantitative Analysis* 24 (No. 3 1989): 367-84.

Chiras, D., and Manaster, S. "The Information Content of Option Prices and a Test of Market Efficiency." *Journal of Financial Economics* 6 (Nos. 2-3 1978): 213-34.

Clark, P. "A Subordinated Stochastic Process Model with Finite Variance for Speculative Prices." *Econometrica* 41 (No. 1 1973): 135-55.

Cox, J., and Ross, S. "The Valuation of Options for Alternative Stochastic Processes." *Journal of Financial Economics* 3 (Nos.

1-2 1976): 145-66.

Cox, J., Ingersoll, J., and Ross, S. "Intertemporal General Equilibrium Model of Asset Prices." *Econometrica* 53 (No. 2 1985): 363-84.

Durrett, R. *Brownian Motion and Martingales in Analysis*. California: Wadsworth Advanced Books & Software, Belmont, 1984.

Eisenberg, L. Random Variance Option Pricing and Spread Valuation, Working Paper, University of Pennsylvania, 1985.

Epps, T., and Epps, M. "The Stochastic Dependence of Security Price Changes and Transaction Volumes: Implications for the Mixture-of-Distribution Hypothesis." *Econometrica* 44 (No. 2 1976): 305-21.

Flood, R., and Hodrick, R. "Asset Price Volatility, Bubbles and Process Switching." *Journal of Finance* 41 (No. 4 1986): 831-42.

French, K., and Roll, R. "Stock Return Variances: The Arrival of Information and the Reaction of Traders." *Journal of Financial Economics* 17 (No. 1 1986): 5-26.

French, K., Schwert, W., and Stambaugh, R. "Expected Stock Returns and Volatility." *Journal of Financial Economics* 19 (No. 1 1987): 3-29.

Friedman, A. *Stochastic Differential Equations and Applications*. Academic Press, 1975.

Geske, R. "The Valuation of Corporate Liabilities as Compound Options." *Journal of Financial and Quantitative Analysis* 12 (No. 4 1979): 541-52.

Harrison, J., and Kreps, D. "Martingales and Arbitrage in Multiperiod Securities Markets." *Journal of Economic Theory* 20 (No. 3 1979): 381-408.

Harrison, J., and Pliska, S. "Martingales and Stochastic Integrals in the Theory of Continuous Trading." *Stochastic Processes and Their Applications* 11 (No. 3 1981): 215-60.

Heath, D., and Jarrow, R. "Arbitrage, Continuous Trading, and Margin Requirement." *Journal of Finance* 42 (No. 5 1987): 1129-42.

Heston, S. "A Closed-Form Solutions for Options with Stochastic Volatility with Applications to Bond and Currency Options." *Review of Financial Studies* 6 (No. 2 1993): 325-43.

Hull, J., and White, A. "The Pricing of Options on Assets with

Stochastic Volatilities." *Journal of Finance* 42 (No. 2 1987): 281-300.

Johnson, H., and Shanno, D. "Option Pricing when the Volatility is Changing." *Journal of Financial and Quantitative Analysis* 22 (No. 2 1987): 143-51.

Karatzas, I., and Shreve, S. *Brownian Motion and Stochastic Calculus*. Spinger-Verlag, 1988.

Kon, S. "Models of Stock Returns—A Comparison." *Journal of Finance* 39 (No. 1 1984): 147-66.

Kunita, H., and Watanabe, S. "On Square Integrable Martingales." *Nagoya Mathematics Journal* 30 (1967): 209-45.

Luenberger, D. *Optimization by Vector Space Methods*. Wiley, 1969.

Merville, L., and Pieptea, D. "Stock Price Volatility, Mean-Reverting Diffusion and Noise." *Journal of Financial Economics* 24 (No. 1 1989): 193-214.

Merton, R. "Theory of Rational Option Pricing." *Bell Journal of Economics* and *Management Science* 4 (No. 1 1973): 141-83.

Roll, R. "An Analytical Valuation Formula for Unprotected American Call Options on Stock with Known Dividends." *Journal of Financial Economics* 5 (No. 2 1977): 251-58.

Ross, S. "A Simple Approach to the Valuation of Risky Streams." *Journal of Business* 51 (No. 3 1978): 453-75.

Rubinstein, M. "Nonparametric Tests of Alternative Option Pricing Models Using All Reported Trades and Quotes on the 30 Most Active CBOE Option Classes from August 23, 1976 through August 31, 1978." *Journal of Finance* 40 (No. 2 1985): 455-80.

Schwert, W. "Why Does Stock Market Change over Time?" *Journal of Finance* 44 (No. 5 1989): 1115-54.

Scott, L. "Option Pricing when the Variance Changes Randomly: Theory, Estimation and an Application." *Journal of Financial and Quantitative Analysis* 22 (No. 4 1987): 419-38.

West, K. "Bubbles, Fads and Stock Price Volatility Tests: A Partial Evaluation." *Journal of Finance* 43 (No. 3 1988): 639-56.

Wiggins, J. "Options Values under Stochastic Volatility." *Journal of Financial Economics* 19 (No. 2 1987): 351-372.