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On the matrix sequence $\{\Gamma(A^m)\}_{m=1}^{\infty}$ for a Boolean matrix A whose digraph is linearly connected

(Bool 행렬 A의 그래프가 선형연결 그래프일 때, 행렬열 $\{\Gamma(A^m)\}_{m=1}^{\infty}$에 대한 연구)

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On the matrix sequence $\{\Gamma(A^m)\}_{m=1}^{\infty}$ for a Boolean matrix A whose digraph is linearly connected

A dissertation
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Abstract

In this thesis, we extend the results given by Park *et al.* [12] by studying the convergence of the matrix sequence $\{\Gamma(A^m)\}_{m=1}^{\infty}$ for a matrix $A \in \mathcal{B}_n$ the digraph of which is linearly connected with an arbitrary number of strong components. In the process for generalization, we concretize ideas behind their arguments. We completely characterize A for which $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges. Then we find its limit when all of the irreducible diagonal blocks are of order at least two. We go further to characterize A for which the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ is a J block diagonal matrix. All of these results are derived by studying the m-step competition graph of the digraph of A.

Key words: irreducible Boolean $(0,1)$-matrices; powers of Boolean $(0,1)$-matrices; linearly connected digraphs; index of imprimitivity; m-step competition graphs; graph sequence; powers of digraphs.
Student Number: 2012-21424

Remark: The text of this thesis is a reprint of the material as it appears in *Linear Algebra Appl.* (2014). The coauthor listed in this publication directed and supervised research which forms the basis for the thesis.

Contents

Chapter 1

Introduction

1.1 Preliminaries

In this section, we introduce some basic notions in combinatorial matrix theory, which shall be used in this thesis. We assume that we are already familiar with the basic notions in graph theory such as graphs, digraphs, neighbor, degree, adjacency matrix, connectedness and components.

Let $\mathcal{B} = \{0, 1\}$ denote the two-element Boolean algebra with addition ($+$) and multiplication ($\cdot$) defined by

$$\begin{array}{cccc} 0+0=0, & 0+1=1, & 1+0=1, & 1+1=1, \\ 0\cdot 0=0, & 0\cdot 1=0, & 1\cdot 0=0, & 1\cdot 1=1. \end{array}$$

We denote by $\mathcal{B}_n$ the set of all $n \times n$ matrices over $\mathcal{B}$. We define a matrix operator $\Gamma : \mathcal{B}_n \to \mathcal{B}_n$ by $\Gamma(A) = (\gamma_{ij})$ where

$$\gamma_{ij} = \begin{cases} 0 & \text{if } i = j; \\ 0 & \text{if } i \neq j \text{ and the inner product of row } i \text{ and row } j \text{ of } A \text{ is } 0; \\ 1 & \text{if } i \neq j \text{ and the inner product of row } i \text{ and row } j \text{ of } A \text{ is not } 0. \end{cases}$$

Given a matrix $A \in \mathcal{B}_n$, there exists a unique digraph whose adjacency matrix is A. We call such a digraph the *digraph of* A and denote it by $D(A)$.

The greatest common divisor of all lengths of directed cycles in a nontrivial digraph D is called the *index of imprimitivity of* D. A digraph D is said to be *primitive* if D is strongly connected and has the index of imprimitivity 1. Let A be a matrix in $\mathcal{B}_n$. We call the index of imprimitivity of $D(A)$ the *index of imprimitivity of* A. If $D(A)$ is primitive, then we say that A is *primitive*. If $D(A)$ is strongly connected, then we say A is *irreducible*. For undefined terms in this paper, one may refer to [1].

Given a digraph D, the *competition graph* $C(D)$ of D has the same vertex set as D and has an edge between vertices u and v if and only if there exists a common prey of u and v in D. If (u, v) is an arc of a digraph D, then we call v a *prey* of u (in D) and call u a *predator* of v (in D). A graph G is called the *row graph* of a matrix M if the rows of M are the vertices of G, and two vertices are adjacent in G if and only if their corresponding rows have a nonzero entry in the same column of M. This notion was studied by Greenberg *et al.* [6]. As noted in [6], the competition graph of a digraph D is the row graph of its adjacency matrix. Thus it can easily be checked that the adjacency matrix of the competition graph of a digraph D is $\Gamma(A)$ where A is the adjacency matrix of D.

The notion of competition graph is due to Cohen [5] and has arisen from ecology. Competition graphs also have applications in coding, radio transmission, and modeling of complex economic systems. (See [14] and [15] for a summary of these applications.)

It is well-known that for an irreducible matrix A in $\mathcal{B}_n$, the matrix sequence $\{A^m\}_{m=1}^{\infty}$ converges if and only if A is primitive. Yet, a matrix sequence $\{\Gamma(A^m)\}_{m=1}^{\infty}$ might converge even if the matrix A is not primitive. For example, the mth power of the matrix A given in Figure 1.1 does not converge as m increases since it is not primitive. However, the sequence $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges to A' since $\Gamma(A^m) = A'$

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad A^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad A^4 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \cdots$$

$$A' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Figure 1.1: An example given by Park *et al.* [12]. We note that A, A^2, A^3 are all distinct and $A^4 = A$. Thus $\{A^m\}_{m=1}^{\infty}$ does not converge. However $\Gamma(A^m) = A'$ for each positive integer m.

for any positive integer m.

Park *et al.* [12] showed that $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges as m increases for any irreducible Boolean matrix A and its limit is a block diagonal matrix each of whose blocks consists of all 1s except in the main diagonal and all 0s in the main diagonal up to conjugation by simultaneous permutation of rows and columns. They also completely characterized a matrix $A \in \mathcal{B}_n$ whose digraph has exactly two strongly connected components (in short, strong components) and for which $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges, and found the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ in terms of its digraph when it converges. They derived these facts in terms of the competition graph of the digraph of A.

Given a digraph D and a positive integer m, a vertex y is an m*-step prey* of a vertex x if and only if there exists a directed walk from x to y of length m. Given a digraph D and a positive integer m, the digraph D^m has the same vertex set as D and has an arc (u, v) if and only if v is an m-step prey of u. It is well-known that a digraph D is primitive if and only if D^m equals the digraph which has all possible arcs for any $m \geq N$ for some positive integer N (we call the smallest such integer N the *exponent* of D). Motivated by this, Park *et al.* [12] introduced the notion of convergence of $\{G_n\}_{n=1}^{\infty}$ as follows: A graph sequence $\{G_n\}_{n=1}^{\infty}$ (resp. digraph sequence) *converges* if there exists a positive integer N such that G_n is equal to G_N for any $n \geq N$. They called the graph G_N the *limit* of the graph sequence (resp. digraph sequence). Then they translated their goals described above into competition graph version and showed that $\{C(D^m)\}_{m=1}^{\infty}$ converges to a graph with only complete components as m increases if D is strongly connected, completely characterized a digraph D with exactly two strong components for which $\{C(D^m)\}_{m=1}^{\infty}$ converges, and found the limit of $\{C(D^m)\}_{m=1}^{\infty}$ when $\{C(D^m)\}_{m=1}^{\infty}$ converges.

Given a positive integer m, the *m-step competition graph* of a digraph D, denoted by $C^m(D)$, has the same vertex set as D and has an edge between vertices u and v if and only if there exists an m-step common prey of u and v. The notion of m-step competition graph is introduced by Cho *et al.* [4] and one of the important variations (see the survey articles by Kim [10] and Lundgren [13] for the variations which have been defined and studied by many authors since Cohen introduced the notion of competition). Since its introduction, it has been extensively studied (see for example [2, 3, 7–9, 11, 16]). Cho *et al.* [3] showed that for any digraph D and a positive integer m, $C^m(D) = C(D^m)$. Thus the limit of the graph sequence $\{C(D^m)\}_{m=1}^{\infty}$, if it exists, is the same as that of the graph sequence $\{C^m(D)\}_{m=1}^{\infty}$. Consequently studying the graph sequence $\{C(D^m)\}_{m=1}^{\infty}$ is actually studying the sequence of m-step competition graphs of D. In other words, studying the matrix sequence $\{\Gamma(A^m)\}_{m=1}^{\infty}$ is actually studying the graph sequence $\{C^m(D)\}_{m=1}^{\infty}$, which can be seen by the following commutative digram:

$$\begin{array}{ccc} A^m & \xrightarrow{\quad\Gamma\quad} & \Gamma(A^m) \\ \Big\downarrow & \circlearrowright & \Big\downarrow \\ D(A^m) & \xrightarrow[\quad C\quad]{} & C(D^m(A)) \end{array}$$

1.2 A preview of thesis

In this thesis, we extend the results given by Park *et al.* [12] by studying the convergence of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ for a matrix $A \in \mathcal{B}_n$ satisfying

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} & O & O & \cdots & O \\ O & A_{22} & A_{23} & O & \cdots & O \\ O & O & A_{33} & A_{34} & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & O & \cdots & A_{\eta\eta} \end{bmatrix} \tag{1.1}$$

for a permutation matrix P of order n, irreducible matrices A_{11}, A_{22}, $\ldots$, $A_{\eta\eta}$, and nonzero matrices A_{12}, A_{23}, $\ldots$, $A_{\eta-1,\eta}$ such that if a diagonal block A_{ii} is of order at least two and κ_i is the index of imprimitivity of the digraph of A_{ii}, then

$$A_{ii} = \begin{bmatrix} O & A_{12}^{(ii)} & O & \cdots & O & O \\ O & O & A_{23}^{(ii)} & \cdots & O & O \\ O & O & O & \cdots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \cdots & O & A_{\kappa_i-1,\kappa_i}^{(ii)} \\ A_{\kappa_i,1}^{(ii)} & O & O & \cdots & O & O \end{bmatrix}.$$

We completely characterize A for which $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges in Chapter 2. Then, in Chapter 3, we find its limit when each of A_{11}, A_{22}, ..., $A_{\eta\eta}$ is of order at least two. In this case, the convergence of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ is guaranteed by one of their results:

Theorem 1.2.1 (See [12]). *If a digraph D is trivial or each vertex of D has an out-neighbor, then $\{C(D^m)\}_{m=1}^{\infty}$ converges.*

We go further to generalize one of their results to characterize a matrix A with A_{11}, A_{22}, ..., $A_{\eta\eta}$ of order at least two such that the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ is a J block diagonal matrix. Adopting the notion defined by Park *et al.* [12], we mean by a *J block diagonal* (for short *JBD*) matrix a block diagonal matrix each of whose blocks consists of all 1s except in the main diagonal and all 0s in the main diagonal.

We derive our results by studying the convergence of $\{C(D^m)\}_{m=1}^{\infty}$ for the digraph D of a matrix A satisfying (1.1). The digraph D has the property that it is weakly connected and has strong components D_1, ..., D_η such that there is an arc going from D_i to D_j for some distinct $i, j \in \{1, \ldots, \eta\}$ only if $j = i + 1$. A digraph with this property shall be said to be *linearly* connected. We note that a weakly connected digraph with two strong components is linearly connected.

Given a linearly connected digraph D with η strong components, unless otherwise stated, we mean by D_1, ..., D_η the strong components of D such that there is an arc going from D_i to D_j for some distinct $i, j \in \{1, \ldots, \eta\}$ only if $j = i + 1$ and by $D_{i,i+1}$ the subdigraph of D induced by $V(D_i) \cup V(D_{i+1})$ for each $i = 1$, ..., $\eta - 1$. We denote by $\kappa(D_i)$ (κ_i for short) the index of imprimitivity of D_i and the sets of imprimitivity of D_i by $U_1^{(i)}$, $U_2^{(i)}$, ..., $U_{\kappa_i}^{(i)}$ for $i = 1$, ..., η.

In this paper, all the graphs and digraphs are assumed to be simple.

Chapter 2

Convergence of $\{\Gamma(A^m)\}_{m=1}^{\infty}$

In this chapter, we completely characterize a matrix $A \in \mathcal{B}_n$ in the form given in (1.1) whose digraph is linearly connected and for which $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges.

We denote by $\ell(W)$ the length of a walk W in a graph or digraph.

Lemma 2.0.2. *Let D be a linearly connected digraph with η strong components. Suppose that a vertex x has an m-step prey in a nontrivial component D_i for some positive integer m and $i \in \{1, 2, \dots, \eta\}$. Then x has an m'-step prey in D_i for all $m' \geq m$. Furthermore, every vertex in D_i is a k-step prey of x for some positive integer k.*

Proof. Let $z \in V(D_i)$ be an m-step prey of x. Take an integer m' satisfying $m' \geq m$. Since D_i is nontrivial and strongly connected, there exists a closed directed walk of a positive length containing all the vertices in D_i. By traversing such a walk, we may find a vertex z' in D_i such that there exists a directed (z, z')-walk of length $m' - m$ in D_i. Then z' is an m'-step prey of x.

Take any vertex $w \in V(D_i)$. Since D_i is strongly connected, there exists a directed (z, w)-walk W. Since z is an m-step prey of x, w is a k-step prey of x for

$k = m + \ell(W)$. □

The following corollary is immediately true by the above theorem.

Corollary 2.0.3. *Let D be a linearly connected digraph with η strong components. Suppose that any two vertices x and y have an m-step common prey belonging to a nontrivial strong component for some positive integer m. Then there exists a positive integer N such that x and y are adjacent in $C(D^m)$ for any integer $m \geq N$.*

Corollary 2.0.3 implies that, when two vertices x and y have an α-step common prey z in a nontrivial strong component of a linearly connected digraph D for a positive integer α, the adjacency of x and y in the limit of $\{C(D^m)\}_m^\infty$ is determined by not the value of α but the fact that x and y have a 'step common prey' z in a nontrivial strong component. In this context, we sometimes omit 'α' and just say that x and y have a 'step common prey'.

The following lemma shall be frequently quoted in the rest of this paper.

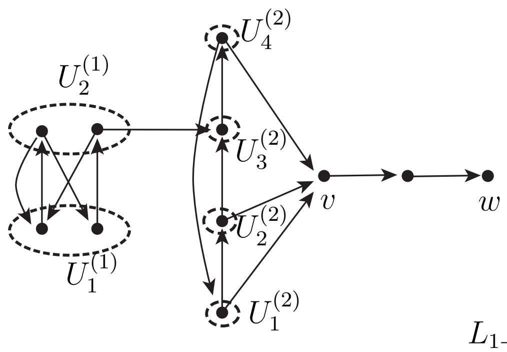
Lemma 2.0.4 (Lemma 3.4.3, [1])**.** *Let D be a nontrivial strongly connected digraph, and U_1, U_2, ..., $U_{\kappa(D)}$ be the sets of imprimitivity of D. Then there exists a positive integer N such that if x and y are vertices belonging respectively to U_i and U_j, then there are directed (x, y)-walks of every length $j - i + t\kappa(D)$ with $t \geq N$.*

For a digraph D and a vertex v of D, $N_D^-(v)$ denotes the set of all in-neighbors of v.

Given a linearly connected digraph D, suppose that D_p is a nontrivial strong component while D_{p+1} is a trivial component consisting of vertex v. Let

$$\Lambda(D) := \left\{ i \mid U_i^{(p)} \cap N_D^-(v) \neq \emptyset \right\} = \{k_1, \dots, k_s\}.$$

Then, by Lemma 2.0.4, it is easy to check that for each $j \in \mathbb{Z}_{\kappa_p}$ and $x \in U_j^{(p)}$, there is a positive integer N_p such that there exist (x, v)-walks of lengths $(k_1 - j) + 1 + t\kappa_p$,



$L_{1\to v} = \{0, 1, 2\}$ $L_{2\to v} = \{0, 1, 3\}$
$L_{3\to v} = \{0, 2, 3\}$ $L_{4\to v} = \{1, 2, 3\}$

D

Figure 2.1: It is easy to see that $(L_{i\to v} \cap L_{j\to v}) \cup ((1 + L_{i\to v}) \cap (1 + L_{j\to v})) \cup ((2 + L_{i\to v}) \cap (2 + L_{j\to v})) = \{0, 1, 2, 3\}$ for any i, j, $1 \le i < j \le 4$. Therefore $\{C(D^m)\}_{m=1}^{\infty}$ converges by Theorem 2.0.5. However, if $D' = D - w$, then $(L_{1\to v} \cap L_{2\to v}) \cup ((1 + L_{1\to v}) \cap (1 + L_{2\to v})) = \{0, 1, 2\}$ and $\{C(D'^m)\}_{m=1}^{\infty}$ diverges by the theorem.

$\ldots$, $(k_s - j) + 1 + t\kappa_p$ for every $t \ge N_p$. We put

$$L_{j\to v} := \{(k_r - j) + 1 \pmod{\kappa_p} \mid r = 1, 2, \ldots, s\}.$$

(See Figure 2.1 for an illustration.) In general, for a nonnegative integer i, we set

$$i + L_{j\to v} := \{(k_r - j) + i + 1 \pmod{\kappa_p} \mid r = 1, 2, \ldots, s\}$$

where $L_{j\to v} = 0 + L_{j\to v}$. Obviously, if D_{p+1}, $\ldots$, $D_{p+\zeta}$ are trivial components of D, then, for $i = 1$, $\ldots$, ζ,

$l \in (i + L_{j_1\to v}) \cap (i + L_{j_2\to v})$ if and only if the vertex of D_{p+i+1} is an $(l + t\kappa_p)$-step common prey of any vertex in $U_{j_1}^{(p)}$ and any vertex in $U_{j_2}^{(p)}$ for any t greater than or equal to some positive integer N_i. $(*)$

Theorem 2.0.5. *Let D be a linearly connected digraph with exactly η strong components. Then $\{C(D^m)\}_{m=1}^{\infty}$ converges if and only if one of the following is true:*

(i) For each $i = 1, \ldots, \eta$, D_i is trivial.

(ii) D_η is nontrivial.

(iii) D_η is trivial whereas there is a nontrivial strong component in D, and if p is the largest index for which D_p is a nontrivial strong component, then $\bigcup_{i=0}^{\eta-p-1}((i + L_{j_1 \to v}) \cap (i + L_{j_2 \to v})) = \emptyset$ or $\mathbb{Z}_{\kappa_p}$ for any j_1, j_2 in $\mathbb{Z}_{\kappa_p}$ where $V(D_{p+1}) = \{v\}$ and $\Lambda(D) = \left\{ i \mid U_i^{(p)} \cap N_D^-(v) \neq \emptyset \right\} = \{k_1, \ldots, k_s\}$ for some integer s, $1 \leq s \leq \kappa_p$.

Proof. We show the 'if' part first. If (i) is true, then $C(D^m)$ is an edgeless graph with the vertex set $V(D)$ for any positive integer m and so $\{C(D^m)\}_{m=1}^{\infty}$ converges. If (ii) is true, then $\{C(D^m)\}_{m=1}^{\infty}$ converges by Theorem 1.2.1.

Now we suppose that (iii) is true. Take two vertices x and y of D. If x and y do not have an α-step common prey for any positive integer α, then x and y are not adjacent in $C(D^\alpha)$ for any positive integer α. Now consider the case where x and y have a step common prey. If x or y is a vertex of a trivial component appearing after D_p, then x and y cannot have an α-step common prey for any positive integer α. Thus x and y belong to components appearing before D_{p+1}. If they have a step common prey in D_q for $q \leq p$, then they have a step common prey in D_p and so there exists an integer M such that x and y are adjacent in $C(D^m)$ for any integer $m \geq M$ by Corollary 2.0.3. Suppose that x and y have a step common prey only in a trivial component appearing after the component D_p and let w be a step common prey of x and y. Then there exist a directed (x, w)-walk W_1 and a directed (y, w)-walk W_2 of the same length. Deleting from W_1 and W_2 the vertices in trivial components appearing after D_p, we obtain a directed (x, w_1)-walk and a directed (y, w_2)-walk of the same length where $w_1 \in U_{j_1}^{(p)}$ and $w_2 \in U_{j_2}^{(p)}$ for some $j_1, j_2 \in \mathbb{Z}_{\kappa_p}$. Then w_1 and w_2 have a 1-step common prey v and so $1 \in L_{j_1 \to v} \cap L_{j_2 \to v}$. Therefore $L_{j_1 \to v} \cap L_{j_2 \to v} \neq \emptyset$. Then, as we assumed that (iii) is true, $\bigcup_{i=0}^{\eta-p-1}((i + L_{j_1 \to v}) \cap (i + L_{j_2 \to v})) = \mathbb{Z}_{\kappa_p}$. By $(*)$, there exists a positive integer N_{κ_p} such that x and y have an $(l + t\kappa_p)$-step common prey for any $l \in \mathbb{Z}_{\kappa_p}$ and $t \geq N_{\kappa_p}$, which implies that x and y are adjacent in $C(D^\alpha)$ for any integer α greater than or equal to $N_{\kappa_p}\kappa_p$.

We show the 'only if' part by verifying the contrapositive. Suppose that D_η is trivial whereas there is a nontrivial strong component in D and that for the largest index p such that D_p is a nontrivial strong component, $\emptyset \subsetneq \bigcup_{i=0}^{\eta-p-1}((i + L_{j_1 \to v}) \cap (i + L_{j_2 \to v})) \subsetneq \mathbb{Z}_{\kappa_p}$ for some $j_1, j_2 \in \mathbb{Z}_{\kappa_p}$. Then $(i_1 + L_{j_1 \to v}) \cap (i_1 + L_{j_2 \to v}) \neq \emptyset$ for some $i_1 \in \{0, \ldots, \eta - p - 1\}$. Therefore, by $(*)$, for some $l \in \mathbb{Z}_{\kappa_p}$, the vertex of D_{p+i_1+1} is a common prey of any vertex in $U_{j_1}^{(p)}$ and any vertex in $U_{j_2}^{(p)}$ in $D^{l+t\kappa_p}$ for any integer t greater than or equal to some positive integer N_{i_1}. Thus every vertex in $U_{j_1}^{(p)}$ and every vertex in $U_{j_2}^{(p)}$ are adjacent in $C(D^{l+t\kappa_p})$ for any integer $t \geq N_{i_1}$. On the other hand, since $\bigcup_{i=0}^{\eta-p-1}((i + L_{j_1 \to v}) \cap (i + L_{j_2 \to v})) \neq \mathbb{Z}_{\kappa_p}$, there is an element l' in $\mathbb{Z}_{\kappa_p}$ such that $l' \notin (i + L_{j_1 \to v}) \cap (i + L_{j_2 \to v})$ for each $i = 0, \ldots, \eta - p - 1$. Then, by $(*)$, for any $i = 0, \ldots, \eta - p - 1$ and for any positive integer N, there exists an integer $t \geq N$ such that the vertex of D_{p+i+1} is not an $(l' + t\kappa_p)$-step common prey of some vertex in $U_{j_1}^{(p)}$ and some vertex in $U_{j_2}^{(p)}$. That is, for any positive integer N, there exists an integer $\beta \geq N$ such that some vertex in $U_{j_1}^{(p)}$ and some vertex in $U_{j_2}^{(p)}$ are not adjacent in $C(D^{l'+\beta\kappa_p})$. However, we have shown the existence of N_{i_1} such that every vertex in $U_{j_1}^{(p)}$ and every vertex in $U_{j_2}^{(p)}$ are adjacent in $C(D^{l+t\kappa_p})$ for any integer $t \geq N_{i_1}$. Hence we can conclude that $\{C(D^m)\}_{m=1}^{\infty}$ diverges. □

For a matrix A in the form given in (1.1), $p, q = 1, \ldots, \eta$; $i = 1, \ldots, \kappa_p$; $j = 1$,

..., κ_{p+1}, we let

> $A_{ij}^{(pq)}$ denote the submatrix of PAP^T induced by the rows of PAP^T intersecting $A_{i,i+1}^{(pp)}$ and the columns of PAP^T intersecting $A_{j-1,j}^{(qq)}$, where $A_{\kappa_p,\kappa_p+1}^{(pp)}$ and $A_{0,1}^{(qq)}$ are identified with $A_{\kappa_p,1}^{(pp)}$ and $A_{\kappa_q,1}^{(qq)}$, resepctively. (§)

We note that $A_{ij}^{(pq)}$ is a zero matrix if $|p-q| \geq 2$.

Suppose that, for $p \leq \eta - 1$, A_{pp} is the last diagonal block of order at least two in PAP^T. Then the set

$$\Lambda(D) = \left\{ i \mid U_i^{(p)} \cap N_D^-(v) \neq \emptyset \right\}$$

used in Theorem 2.0.5 corresponds to the set

$$\Lambda(A) := \{ i \mid \text{The column of } PAP^T \text{ intersecting the trivial block } A_{p+1,p+1} \text{ and a row of } PAP^T \text{ intersecting } A_{i,i+1}^{(pp)} \text{ meet at } 1. \}$$

Furthermore $L_{j \to v}$ used in the same theorem corresponds to

$$L_{j \to (p+1)} := \{ (k_r - j) + 1 \pmod{\kappa_p} \mid r = 1, 2, \ldots, s \}$$

where $\Lambda(A) = \{k_1, k_2, \ldots, k_s\}$.

Now we are ready to translate Theorem 2.0.5 into matrix version.

Corollary 2.0.6. *Suppose that $A \in \mathcal{B}_n$ is a matrix in the form given in (1.1). Then $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges if and only if one of the following holds:*

(i) For each $i = 1, \ldots, \eta$, A_{ii} is of order one.

(ii) $A_{\eta\eta}$ is of order at least two.

(iii) $A_{\eta\eta}$ is of order one whereas there is a diagonal block of order at least two of PAP^T, and if p is the largest index such that A_{pp} is of order at least two, then $\bigcup_{i=0}^{\eta-p-1} ((i + L_{j_1 \to (p+1)}) \cap (i + L_{j_2 \to (p+1)})) = \emptyset$ or $\mathbb{Z}_{\kappa_p}$ for any j_1, j_2 in $\mathbb{Z}_{\kappa_p}$ where $\Lambda(A) = \{k_1, \ldots, k_s\}$ for some integer s, $1 \leq s \leq \kappa_p$.

Chapter 3

The limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$

3.1 The limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$

In this section, we find the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ for a matrix in the form given in (1.1) when A_{11}, A_{22}, …, $A_{\eta\eta}$ are of order at least two. The convergence of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ for such a matrix A is guaranteed by Theorem 1.2.1 or Theorem 2.0.5.

To characterize the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ for a matrix $A \in \mathcal{B}_n$ whose digraph has exactly two strong components and for which $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges, Park *et al.* [12] introduced the notion B_D for a weakly connected digraph D with exactly two strong components D_1 and D_2 where D_2 is nontrivial.

Definition 3.1.1 ([12]). *We take a weakly connected digraph D with exactly two strong components D_1 and D_2 where D_2 is nontrivial. Let $I(D) = \{(k,l) \mid (x,y) \in A(D)$ for some $x \in U_k^{(1)}, y \in U_l^{(2)}\}$. Let $B_D = \left(\mathbb{Z}_{\kappa(D_1)}, \mathbb{Z}_{\kappa(D_2)}\right)$ be the bipartite graph defined as follows. If D_1 is nontrivial, then B_D has an edge (i,j) if and only if $i \equiv k+1+p \pmod{\kappa(D_1)}$ and $j \equiv l+p \pmod{\kappa(D_2)}$ for some $(k,l) \in I(D)$ and some integer p. If D_1 is trivial, then B_D has an edge (i,j) if and only if $j \equiv l-1$*

$\pmod{\kappa(D_2)}$ *for some* $(1,l) \in I(D)$*, which is obtained by substituting* $p = -1$ *and* $k(D_1) = 1$ *in the nontrivial case.*

They described the limit of $\{C(D^m)\}_{m=1}^{\infty}$ using a notion of 'expansion' of B_D.

Definition 3.1.2 ([12])**.** *Given a bipartite graph* $B = (X, Y)$*, we construct a supergraph of* B *as follows. We write each edge of* B *in the arc form* (x, y) *to make clear that* $x \in X$ *and* $y \in Y$*. Then we replace each vertex* v *with a complete graph* G_v *(of any size) so that* G_v *and* G_w *are vertex-disjoint if* $v \neq w$*, and join each vertex of* G_x *and each vertex of* G_y *whenever either* (x, y) *is an edge of* B *or there exists* $z \in Y$ *such that* (x, z) *and* (y, z) *are edges of* B*. We say that the resulting graph is an* expansion of B.

Theorem 3.1.3 (See [12])**.** *Let* D *be a weakly connected digraph with two strong components* D_1 *and* D_2 *such that no arc goes from* D_2 *to* D_1*,* D_2 *is nontrivial, and* $\{C(D^m)\}_{m=1}^{\infty}$ *converges. Then the limit of* $\{C(D^m)\}_{m=1}^{\infty}$ *is an expansion of the bipartite graph* B_D *defined in Definition 3.1.1.*

Given a linearly connected digraph D with η strong components D_1, …, D_η, we recall that $D_{p,p+1}$ denotes the subdigraph of D induced by $V(D_p) \cup V(D_{p+1})$ for each $p = 1$, …, $\eta - 1$. Noting that $D_{p,p+1}$ is a weakly connected digraph with two strong components, we extend Definition 3.1.1 as follows to take care of the general case given in (1.1).

Definition 3.1.4. *Let* D *be a linearly connected digraph with* η *nontrivial strong components and let*

$$I(D_{p,p+1}) := \{(k,l) \mid (x,y) \in A(D) \text{ for some } x \in U_k^{(p)} \text{ and } y \in U_l^{(p+1)}\}.$$

We define the competition skeleton graph *(CS-graph for short) of* D *as the* η*-partite graph* $PT_D^{(\eta)} = (V_1, V_2, \ldots, V_\eta)$ *with* $V_i = \mathbb{Z}_{\kappa_i}, 1 \leq i \leq \eta$ *and an edge* (i,j) *if*

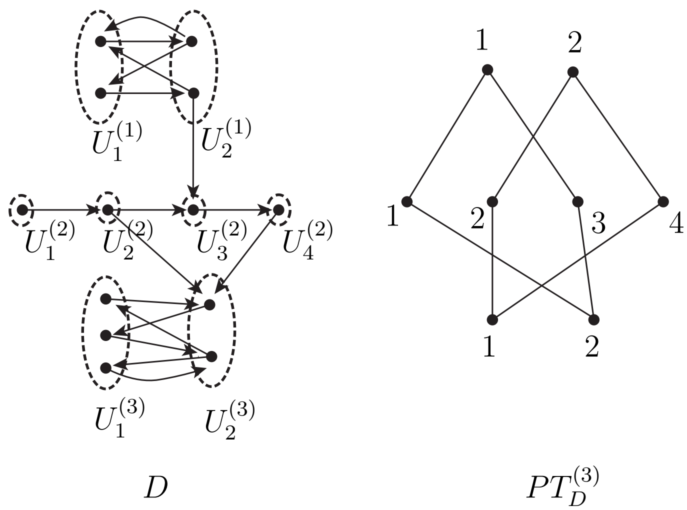


Figure 3.1: A strongly connected digraph D and its CS-graph $PT_D^{(3)}$.

and only if for $p \in \{1, \ldots, \eta - 1\}$, $(i, j) \in E(B_{D_{p,p+1}})$. *(See Figure 3.1 for an illustration.)*

In the following, for convenience sake, we write $B_{p,p+1}$ for $B_{D_{p,p+1}}$. If $\{i, j\}$ is an edge of $B_{p,p+1}$ where $i \in \mathbb{Z}_{\kappa_p}$ and $j \in \mathbb{Z}_{\kappa_{p+1}}$, we denote it by (i, j) instead of $\{i, j\}$.

We state a few useful lemmas.

Lemma 3.1.5. *Let D be a weakly connected digraph with exactly two strong components D_1 and D_2 both of which are nontrivial. Then the following are equivalent:*

(a) There exists a directed (u, v)-walk of length $2s\kappa_1\kappa_2$ for some vertices $u \in U_i^{(1)}$ and $v \in U_j^{(2)}$ and for some positive integer s.

(b) For any vertices $x \in U_i^{(1)}$ and $y \in U_j^{(2)}$, there exists a positive integer $N(x, y)$ such that, for any $t \geq N(x, y)$, there exists a directed (x, y)-walk of length $2t\kappa_1\kappa_2$.

Proof. To show (a) implies (b), suppose that there exists a directed (u, v)-walk of length $2s\kappa_1\kappa_2$ for some vertices $u \in U_i^{(1)}$ and $v \in U_j^{(2)}$ and for some positive integer s. Take any vertices $x \in U_i^{(1)}$ and $y \in U_j^{(2)}$. Then, by Lemma 2.0.4, there exist positive integers N_1 and N_2 such that there are directed (x, u)-walks in D_1 of every length $s_1\kappa_1$ with $s_1 \geq N_1$ and there are directed (v, y)-walks in D_2 of every length $s_2\kappa_2$ with $s_2 \geq N_2$. Let $N = s + N_1 + N_2$. Fix any $t \geq N$ and choose two positive integers $s_1 \geq N_1, s_2 \geq N_2$ satisfying $t = s + s_1 + s_2$. Then there exist a directed (x, u)-walk Q of length $2s_1\kappa_1\kappa_2$ and a directed (v, y)-walk R of length $2s_2\kappa_1\kappa_2$. Thus we obtain a directed (x, y)-walk QPR of length $2(s+s_1+s_2)\kappa_1\kappa_2 = 2t\kappa_1\kappa_2$.

The statement (a) immediately follows from (b) by taking any vertices $u \in U_i^{(1)}$ and $v \in U_j^{(2)}$ and $s = N(u, v)$. □

By Lemma 3.1.5, the following two lemmas are logically equivalent:

Lemma 3.1.6 (See [12]). *Let D be a weakly connected digraph with exactly two strong components D_1 and D_2 both of which are nontrivial. Then (i, j) is an edge of B_D if and only if there exists a directed (u, v)-walk of length $2s\kappa_1\kappa_2$ for some vertices $u \in U_i^{(1)}$ and $v \in U_j^{(2)}$ and for some positive integer s.*

Lemma 3.1.7. *Let D be a weakly connected digraph with exactly two strong components D_1 and D_2 both of which are nontrivial. Then (i, j) is an edge of B_D if and only if, for any vertices $x \in U_i^{(1)}$ and $y \in U_j^{(2)}$, there exists a positive integer $N(x, y)$ such that, for any $t \geq N(x, y)$, there exists a directed (x, y)-walk of length $2t\kappa_1\kappa_2$.*

We shall generalize Lemma 3.1.6. To do so, we need the following lemma.

Lemma 3.1.8. *Let D be a linearly connected digraph with only nontrivial strong components as many as $\eta \geq 3$. Let $x \in U_i^{(p)}$ and $z \in U_k^{(q)}$ for positive integers*

p and q satisfying $p+2 \leq q \leq \eta$. If there exists a directed (x,z)-walk of length $s\kappa_{p+1}\kappa_{p+2}\cdots\kappa_q$ for some positive integer s, then, for some $j \in \mathbb{Z}_{\kappa_{p+1}}$ and for some $y \in U_j^{(p+1)}$, there exist

(i) a positive integer K and a directed (x,y)-walk of length $k\kappa_{p+1}$ for each integer $k \geq K$, in particular, a directed (x,y)-walk of length $2K\kappa_p\kappa_{p+1}$;

(ii) a positive integer K' and a directed (y,z)-walk of length $k'\kappa_{p+1}$ for each integer $k' \geq K'$, in particular, directed (y,z)-walks of lengths $2K'\kappa_{p+1}\kappa_{p+2}$ and $K'\kappa_{p+1}\kappa_{p+2}\cdots\kappa_q$, respectively.

Proof. Let W be a directed (x,z)-walk of length $s\kappa_{p+1}\kappa_{p+2}\cdots\kappa_q$. Since D is linearly connected, there is a vertex y_1 on W belonging to D_{p+1}. Then $y_1 \in U_{j_1}^{(p+1)}$ for some $j_1 \in \mathbb{Z}_{\kappa_{p+1}}$. We denote by W_1 and W_2 the (x,y_1)-section of W and (y_1,z)-section of W, respectively, and then, by j the element in $\mathbb{Z}_{\kappa_{p+1}}$ satisfying

$$j \equiv j_1 - \ell(W_1) \pmod{\kappa_{p+1}},$$

or

$$\ell(W_1) = j_1 - j + t\kappa_{p+1} \tag{3.1}$$

for some integer t.

Now we take a vertex in $U_j^{(p+1)}$ and call it y. By Lemma 2.0.4, there exists $L \in \mathbb{N}$ such that, for any integer $k \geq L$, there exists a directed (y_1,y)-walk of $(j-j_1)+k\kappa_{p+1}$ and so a directed (x,y)-walk of length $\ell(W_1)+(j-j_1)+k\kappa_{p+1}$, which equals $(t+k)\kappa_{p+1}$ by (3.1). We let $K = \max\{L, L+t\}$ and (i) follows.

As W is decomposed into W_1 and W_2,

$$s\kappa_{p+1}\kappa_{p+2}\cdots\kappa_q = \ell(W_1) + \ell(W_2)$$

By Lemma 2.0.4 again, there exists $L' \in \mathbb{N}$ such that, for each integer $k' \geq L'$, there exists a directed (y, y_1)-walk of length $(j_1 - j) + k'\kappa_{p+1}$ and so a directed (y, z)-walk of length $(j_1 - j) + k'\kappa_{p+1} + \ell(W_2)$. Then for each integer $k' \geq L'$,

$$\begin{aligned}
&(j_1 - j) + k'\kappa_{p+1} + \ell(W_2) \\
&= (j_1 - j) + k'\kappa_{p+1} + (s\kappa_{p+1}\kappa_{p+2} \cdots \kappa_q - \ell(W_1)) \\
&= k'\kappa_{p+1} + (s\kappa_{p+1}\kappa_{p+2} \cdots \kappa_q - t\kappa_{p+1}) \\
&= (k' + s\kappa_{p+2} \cdots \kappa_q - t)\kappa_{p+1}.
\end{aligned}$$

We let $K' = \max\{L', L' + s\kappa_{p+2} \cdots \kappa_q - t\}$, which satisfies (ii). □

Corollary 3.1.9. *Let D be a linearly connected digraph with only nontrivial strong components as many as $\eta \geq 3$ and let $x \in U_i^{(p)}$ and $z \in U_k^{(p+2)}$ for a positive integer p satisfying $p + 2 \leq \eta$. If there exists a directed (x, z)-walk of length $s\kappa_{p+1}\kappa_{p+2}$ for some positive integer s, then there exists $j \in \mathbb{Z}_{\kappa_{p+1}}$ such that $(i, j) \in E(B_{p,p+1})$ and $(j, k) \in E(B_{p+1,p+2})$.*

Proof. By the hypothesis, there exists y in D_{p+1} satisfying (i) and (ii) of Lemma 3.1.8. Then, by Lemma 3.1.6, (i) and (ii) imply $(i, j) \in E(B_{p,p+1})$ and $(j, k) \in E(B_{p+1,p+2})$, respectively, which is the desired conclusion. □

The following lemma is a generalization of Lemma 3.1.6 and Corollary 3.1.9.

Lemma 3.1.10. *Let D be a linearly connected digraph with only nontrivial strong components as many as $\eta \geq 3$. Let $x \in U_i^{(p)}$ and $z \in U_k^{(q)}$ for positive integers p and q satisfying $p + 2 \leq q \leq \eta$. If there exists a directed (x, z)-walk of length $s\kappa_{p+1}\kappa_{p+2} \cdots \kappa_q$ for some positive integer s, then there exists an (i, k)-path in the CS-graph of D defined in Definition 3.1.4.*

Proof. We proceed by induction on $m = q - p + 1$. The statement is true for $m = 3$ by Corollary 3.1.9. Suppose that the statement is true for $m - 1$ ($m \geq 4$). Let W be a directed (x, z)-walk of length $s\kappa_{p+1}\kappa_{p+2} \cdots \kappa_q$ for some $s \in \mathbb{N}$. Then, by Lemma 3.1.8, for some $j \in \mathbb{Z}_{\kappa_{p+1}}$ and for a vertex $y \in U_j^{(p+1)}$, there exist a directed (x, y)-walk of length $2K\kappa_p\kappa_{p+1}$ for some $K \in \mathbb{N}$ and a directed (y, z)-walk of length $K'\kappa_{p+1}\kappa_{p+2} \cdots \kappa_q$ for some $K' \in \mathbb{N}$. Since there exists a directed (x, y)-walk of length $2K\kappa_p\kappa_{p+1}$, by Lemma 3.1.6, $(i, j) \in E(B_{p,p+1}) \subset E(PT_D^{(\eta)})$.

Since there is a directed (y, z)-walk of length $K'\kappa_{p+1}\kappa_{p+2} \cdots \kappa_q$, by the induction hypothesis, there exists a (j, k)-path in $PT_D^{(\eta)}$, which is concatenated together with the edge (i, j) to form an (i, k)-path in $PT_D^{(\eta)}$. □

The next theorem plays a key role in describing the limit of $\{C(D^m)\}_{m=1}^{\infty}$.

Theorem 3.1.11. *Let D be a linearly connected digraph with only nontrivial strong components as many as $\eta \geq 3$. Suppose that $x \in U_i^{(p)}$ and $y \in U_j^{(q)}$ for integers p, q, i, j with $1 \leq p \leq q \leq \eta$, $i \in \mathbb{Z}_{\kappa_p}$, $j \in \mathbb{Z}_{\kappa_q}$, respectively. Then, for any integer $r \geq \max\{p+1, q\}$, x and y have a step common prey in D_r if and only if there exist an (i, k)-path and a (j, k)-path in the CS-graph of D for some $k \in \mathbb{Z}_{\kappa_r}$ satisfying the property that $k = j$ if and only if $q = r$.*

Proof. To show the 'if' part, suppose that for an integer $r \geq \max\{p + 1, q\}$ and an element $k \in \mathbb{Z}_{\kappa_r}$, there exist an (i, k)-path P and a (j, k)-path Q where $k = j$ if and only if $q = r$. Let $i_0 = i$, $j_0 = j$, $i_{\ell(P)} = j_{\ell(Q)} = k$, and P and Q denote directed paths

$$i_0 \to i_1 \to \cdots \to i_{\ell(P)-1} \to i_{\ell(P)} \quad \text{and} \quad j_0 \to j_1 \to \cdots \to j_{\ell(Q)-1} \to j_{\ell(Q)},$$

respectively. Since $r \geq p+1$, by the definition of a CS-graph, we may take a vertex $x_t \in U_{i_t}^{(p+t)}$, especially $x_0 = x$ and, in case $q = r$, $x_{\ell(P)} = y$, for each $t = 0$, 1,

..., $\ell(P)$. We set $z := x_{\ell(P)}$. We will show that z is an m-step common prey of x and y for some $m \in \mathbb{N}$. For simplicity, let $\lambda = \kappa_1\kappa_2\cdots\kappa_\eta$. By Lemma 3.1.7, there exists (a sufficiently large) $N_1 \in \mathbb{N}$ such that, for any $s \geq N_1$, there exists a directed (x_t, x_{t+1})-walk $W_{t+1}(s)$ of length $2s\lambda$ for each $t = 0, 1, \ldots, \ell(P) - 1$. For any integer $s \geq N_1$, the concatenation of $W_1(s)$, ..., $W_{\ell(P)}(s)$ results in a directed (x, z)-walk of length $2s\ell(P)\lambda$. Suppose $r = q$. Then $k = j$ and $z = y$ by the assumption. By Lemma 2.0.4, there exists a positive integer N_2 such that, for any integer $s \geq N_2$, there exists a directed (y, y)-walk of length $2s\ell(P)\lambda$. If we let $m = 2\ell(P)\lambda\max\{N_1, N_2\}$, then y is an m-step common prey of x and y. Now suppose that $r \geq q + 1$. Then we may take a vertex $y_t \in U_{j_t}^{(q+t)}$, especially $y_0 = y$ and $y_{\ell(Q)} = z$, for $t = 0, \ldots, \ell(Q)$. By Lemma 3.1.7, there exists (a sufficiently large) $N_3 \in \mathbb{N}$ such that, for any integer $s \geq N_3$, there exists a directed (y_t, y_{t+1})-walk $W'_{t+1}(s)$ of length $2s\lambda$ for each $t = 0, 1, \ldots, \ell(Q) - 1$. For any integer $s \geq N_3$, the concatenation of $W'_1(s)$, ..., $W'_{\ell(Q)}(s)$ results in a directed (y, z)-walk of length $2s\ell(Q)\lambda$. As there exists a directed (x, z)-walk of length $2s\ell(P)\lambda$ for any integer $s \geq N_1$, z is an m-step common prey of x and y for $m := 2\ell(P)\ell(Q)\lambda\max\{N_1, N_3\}$.

To show the 'only if' part, suppose that x and y have an m-step common prey in D_r for some $r \geq \max\{p+1, q\}$ and $m \in \mathbb{N}$. Now we take $s \in \mathbb{N}$ such that $2s\kappa_{p+1}\kappa_{p+2}\cdots\kappa_q \geq m$. Then, by Lemma 2.0.2, x and y have a $2s\kappa_{p+1}\kappa_{p+2}\cdots\kappa_q$-step common prey z in D_r. Since z belongs to D_r, $z \in U_k^{(r)}$ for some $k \in \mathbb{Z}_{\kappa_r}$. If $r \geq p + 2$, then there is an (i, k)-path in $PT_D^{(\eta)}$ by Lemma 3.1.10 and if $r = p + 1$, then there is an edge (i, k) in $B_{p,p+1}$ by Lemma 3.1.6, which is an (i, k)-path in $PT_D^{(\eta)}$. Thus we have shown that there is an (i, k)-path in $PT_D^{(\eta)}$.

If $r \geq q+2$, then, by Lemma 3.1.10, there is a (j, k)-path in $PT_D^{(\eta)}$. If $r = q+1$, then there is an edge (j, k) in $B_{q,q+1}$ by Lemma 3.1.6, which is a (j, k)-path in

$PT_D^{(\eta)}$.

Now suppose that $r = q$. Since z is a step common prey of x and y, there exist a directed (x, z)-walk W_1 and a directed (y, z)-walk W_2 of the same length. Since y and z belong to the same strong component, there exists a directed (z, y)-walk W_3. Then W_1W_3 and W_2W_3 are a directed (x, y)-walk and a directed (y, y)-walk, respectively, of the same length, which implies that y is a step common prey of x and y. In particular, by Lemma 2.0.4, there exists a positive integer N such that, for any $s \geq N$, y is a $2s\kappa_{p+1}\kappa_{p+2}\cdots\kappa_q$-step common prey of x and y. Then there is an (i, j)-path in $PT_D^{(\eta)}$ by Lemma 3.1.6 or Lemma 3.1.10 depending upon whether $q = p + 1$ or $q \geq p + 2$. We take the (j, j)-path for $j \in \mathbb{Z}_{\kappa_q}$, which is trivial, and set $k := j$.

If $k = j$, then $q = r$ as $j \in \mathbb{Z}_{k_q}$ and $k \in \mathbb{Z}_{k_r}$ and so the theorem follows. $\square$

Based on the observations which we have made so far, we generalize the notion of expansion given in Definition 3.1.2 as follows.

Definition 3.1.12. *Let $G = (V_1, \ldots, V_\eta)$ be an η-partite graph without edges between V_i and V_j if $|i - j| \geq 2$. From G, we construct a supergraph of G as follows (see Figure 3.2 for an illustration):*

(Step 1) We replace each vertex v with a complete graph G_v (of any size) so that G_v and G_w are vertex-disjoint if $v \neq w$.

(Step 2) For each $x \in V_i$, $y \in V_j$ where $i \leq j$, we join each vertex of G_x and each vertex of G_y if there is an integer $k \geq j$ satisfying one of the following: (i) $k \geq j + 1$ and there are an (x, z)-path and a (y, z)-path in G for some $z \in V_k$; (ii) $j = k$ and there is an (x, y)-path.

We say that any graph resulting from this construction is an expansion of G.

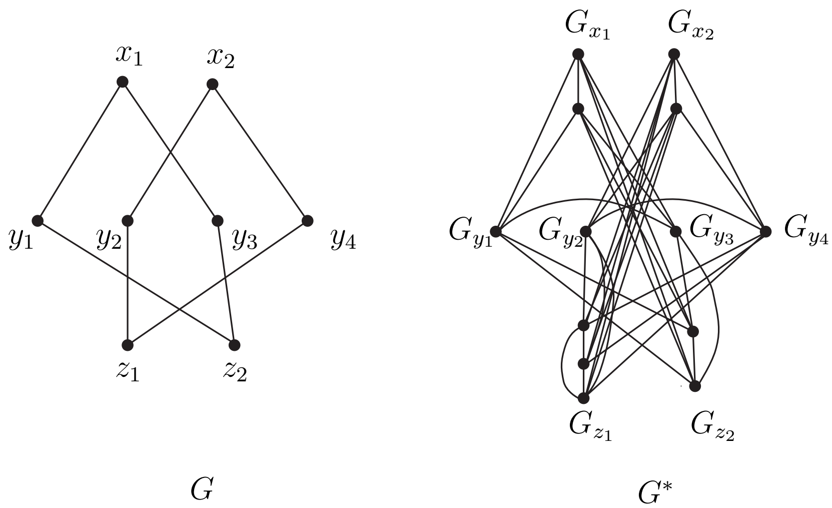


Figure 3.2: A graph G and an expansion G^* of G.

We note that an expansion of an η-partite graph G is uniquely determined by the vertex sets of complete graphs replacing the vertices of G.

In order to describe the limit of the graph sequence $\{C(D^m)\}_{m=1}^{\infty}$ for a linearly connected digraph D with only nontrivial strong components, we need one more lemma in the following.

Lemma 3.1.13 (See [12]). *Let D be a weakly connected digraph with exactly two strong components D_1 and D_2. Then there exists an integer M such that $C(D^m)$ contains complete graphs whose vertex sets are $U_1^{(1)}, \ldots, U_{\kappa(D_1)}^{(1)}, U_1^{(2)}, \ldots, U_{\kappa(D_2)}^{(2)}$, respectively, as subgraphs for $m \geq M$.*

Now we present the theorem describing the limit of the sequence $\{C(D^m)\}_{m=1}^{\infty}$ for a linearly connected digraph D with only nontrivial strong components.

Theorem 3.1.14. *Let D be a linearly connected digraph with only nontrivial strong components as many as $\eta \geq 3$. Then the limit graph of the sequence $\{C(D^m)\}_{m=1}^{\infty}$*

is an expansion of the CS-graph of D*.*

Proof. Denote by G the limit graph of the sequence $\{C(D^m)\}_{m=1}^{\infty}$. We take the expansion G^* of $PT_D^{(\eta)}$ obtained by replacing the vertex i in $\mathbb{Z}_{\kappa_j}$ with the complete graph whose vertex set is the set of imprimitivity $U_l^{(s)}$ of D_s for $l \in \mathbb{Z}_{\kappa_s}$ and $s \in \{1, \ldots, \eta\}$. Obviously $V(G) = V(G^*)$. In the following, we show that $E(G) = E(G^*)$.

Now take two vertices x and y of G. Then $x \in U_i^{(p)}$ and $y \in U_j^{(q)}$ for some integers p, q, i, j satisfying $1 \leq p, q \leq \eta$, $i \in \mathbb{Z}_{\kappa_p}$, $j \in \mathbb{Z}_{\kappa_q}$, respectively. Without loss of generality, we may assume $p \leq q$. First, suppose that x and y are adjacent in G. Then x and y have a step common prey in D_r for some integer $r \geq q$. If $p = r$, then $q = r$ and so, by one of well-known properties of sets of imprimitivity, $i = j$. Then, by the definition of expansion, x and y are adjacent in G^*. Now we assume that $p \leq r + 1$. Then, by Theorem 3.1.11, there exists $k \in \mathbb{Z}_{\kappa_r}$ such that there exist an (i, k)-path and a (j, k)-path in $PT_D^{(\eta)}$. By the definition of expansion, all the vertices in $U_i^{(p)}$ and all the vertices in $U_j^{(q)}$ are adjacent and so x and y are adjacent in G^*. Hence we have shown that $E(G) \subset E(G^*)$.

To show $E(G^*) \subset E(G)$, suppose that vertices u and v are adjacent in G^*. Then, by the definition of expansion, one of the following holds:

(i) u and v belong to the same set of imprimitivity, that is, $\{u, v\} \subset U_l^{(s)}$ for some $s \in \{1, \ldots, \eta\}$ and $l \in \mathbb{Z}_{\kappa_s}$;

(ii) u and v belong to different sets $U_l^{(s)}$ and $U_{l'}^{(s')}$, respectively, of imprimitivity for some integers $1 \leq s \leq s' \leq \eta$, $l \in \mathbb{Z}_{\kappa_s}$, $l' \in \mathbb{Z}_{\kappa_{s'}}$ and either there exist an (l, k)-path and an (l', k)-path in $PT_D^{(\eta)}$ for some integer r, $s' + 1 \leq r \leq \eta$, and $k \in \mathbb{Z}_{\kappa_r}$ or there is an (l, l')-path.

Noting that the m-step competition graph of $D_{i,i+1}$ is the subgraph of the m-step competition graph of D for any $m \in \mathbb{N}$ and $i \in \{1, \ldots, \eta-1\}$, we can conclude that u and v are adjacent in G if (i) is true by Lemma 3.1.13. Consider the case (ii). Then, by Theorem 3.1.11, u and v have a step common prey. Thus, by Corollary 2.0.3, u and v are adjacent in G. $\square$

We consider a matrix A in the form given in (1.1) where A_{ii} has order at least two for each $i = 1, 2, \ldots, \eta$. Then for a block $A^{(p,p+1)}_{i,j}$ of $A_{p,p+1}$ defined in (§) for $p = 1, \ldots, \eta-1$, it is easy to see that $A^{(p,p+1)}_{ij} \neq O$ if and only if $(i,j) \in I(D_{p,p+1})$ where D is the digraph of A. Now Theorem 3.1.14 may be translated into matrix version in the following way:

Corollary 3.1.15. *Let $A \in \mathcal{B}_n$ be a matrix in the form given in (1.1) where A_{ii} has order at least two for each $i = 1, 2, \ldots, \eta$. Then $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges to a (symmetric) matrix A' such that*

$$PA'P^T = \begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdots & C_{1,\eta-1} & C_{1,\eta} \\ C_{21} & C_{22} & C_{23} & \cdots & C_{2,\eta-1} & C_{2,\eta} \\ C_{31} & C_{32} & C_{33} & \cdots & C_{3,\eta-1} & C_{3,\eta} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{\eta-1,1} & C_{\eta-1,2} & C_{\eta-1,3} & \cdots & C_{\eta-1,\eta-1} & C_{\eta-1,\eta} \\ C_{\eta,1} & C_{\eta,2} & C_{\eta,3} & \cdots & C_{\eta,\eta-1} & C_{\eta\eta} \end{bmatrix}$$

where, for each p, $q = 1, \ldots, \eta$, C_{pq} satisfies the following:

(i) The order of C_{pp} is the same as the order of A_{pp};

(ii) $C_{pq} = C^T_{qp}$;

(iii) The block $C^{(pq)}_{ij}$ of C_{pq} has the order the same as $A^{(pq)}_{ij}$ and is located in the position in $PA'P^T$ the same as the position where $A^{(pq)}_{ij}$ is located in PAP^T.

Furthermore,

$$C_{ij}^{(pq)} = \begin{cases} J^* & \textit{if (P1) or (P2) is satisfied;} \\ O & \textit{otherwise.} \end{cases}$$

(P1) $p = q$ *and* $i = j$*;*

(P2) $p \neq q$ *or* $i \neq j$*, and either*

- *there exist positive integers* t_1 *and* t_2 *satisfying* $p + t_1 = q + t_2$ *such that, for each* $r = 0, \ldots, t_1$ *and* $s = 0, \ldots, t_2$*, there exist some positive integers* i_r*,* j_s *with* $i_0 = i$*,* $j_0 = j$*, and* $i_{t_1} = j_{t_2}$*, some integers* k_r*,* l_r*,* m_s*,* n_s*, and nonnegative integers* a_r*,* b_s *for which* $A_{k_r l_r}^{(p+r,p+r+1)} \neq O$*,* $A_{m_s n_s}^{(q+s,q+s+1)} \neq O$ *and which satisfy* $i_r \equiv k_r + a_r + 1 \pmod{\kappa_{p+r}}$*,* $i_{r+1} \equiv l_r + a_r \pmod{\kappa_{p+r+1}}$*,* $j_s \equiv m_s + b_s + 1 \pmod{\kappa_{q+s}}$*,* $j_{s+1} \equiv n_s + b_s \pmod{\kappa_{q+s+1}}$*, or*
- $p \neq q$ *and, if* $p < q$*, then there exists a positive integer* t *satisfying* $p + t = q$ *such that for each* $r = 0, \ldots, t$*, there exist some positive integer* i_r *(*$i_0 = i$*,* $i_q = j$*), some integers* k_r*,* l_r *and nonnegative integer* a_r *for which* $A_{k_r l_r}^{(p+r,p+r+1)} \neq O$ *and which satisfy* $i_r \equiv k_r + a_r + 1 \pmod{\kappa_{p+r}}$*,* $i_{r+1} \equiv l_r + a_r \pmod{\kappa_{p+r+1}}$*; if* $q < p$*, then* $C_{pq}^T = J^*$*.*

(We mean by O *a zero matrix of any size and by* J^* *a square matrix of any order such that all the diagonal entries are 0 and all the off-diagonal entries are 1.)*

3.2 Limit of a particular form: the disjoint union of complete subgraphs

We observe that the graph G^* in Figure 3.2 is the disjoint union of two complete components $G_{x_1} \cup G_{y_1} \cup G_{y_3} \cup G_{z_2}$ and $G_{x_2} \cup G_{y_2} \cup G_{y_4} \cup G_{z_1}$. Park *et al.* [12] characterized a digraph D for which $\{C(D^m)\}_{m=1}^{\infty}$ converges to the union of complete subgraphs as follows:

Theorem 3.2.1 (See [12])**.** *Let D be a weakly connected digraph with exactly two strong components D_1 and D_2 both of which are nontrivial and without arc from D_2 to D_1. Suppose that $\{C(D^m)\}_{m=1}^{\infty}$ converges to a graph G. Then G is the disjoint union of complete subgraphs if and only if κ_2 divides κ_1 and $i - j \equiv i' - j' \pmod{\kappa_2}$ for any (i,j), $(i',j') \in I(D)$.*

In the rest of this paper, we generalize the above theorem. To do so, we need the following lemma:

Lemma 3.2.2. *Let D be a linearly connected digraph with only nontrivial strong components D_1, D_2, ..., D_η ($\eta \geq 2$) such that there is an arc going from D_i to D_j for some distinct $i, j \in \{1, \ldots, \eta\}$ only if $j = i + 1$. Suppose that, in the CS-graph $PT_D^{(\eta)}$ of D, κ_η divides κ_p and $i - j \equiv i' - j' \pmod{\kappa_\eta}$ for any (i,j), $(i',j') \in I(D_{p,p+1})$ for each $p = 1, \ldots, \eta - 1$. Then there exist an (x,k)-path and a (y,k)-path for some $k \in \mathbb{Z}_{\kappa_\eta}$ in $PT_D^{(\eta)}$ if and only if $x \equiv y \pmod{\kappa_\eta}$ for any x, y in $\mathbb{Z}_{\kappa_r}$ and for any $r = 1, \ldots, \eta$.*

Proof. To show the 'only if' part, suppose that, for some $r \in \{1, \ldots, \eta\}$, there exist an (x,k)-path and a (y,k)-path in $PT_D^{(\eta)}$ for some $k \in \mathbb{Z}_{\kappa_\eta}$ and x, y in $\mathbb{Z}_{\kappa_r}$. If $r = \eta$, then $x \equiv y \pmod{\kappa_\eta}$ and the lemma is trivially true. Now suppose that $r < \eta$. Then there exist an (x,k)-path $x s_{r+1} \cdots s_{\eta-1} k$ and a (y,k)-path $y t_{r+1} \cdots t_{\eta-1} k$

where $\{s_m, t_m\} \subset \mathbb{Z}_{\kappa_m}$ for $m = r+1, \ldots, \eta-1$. By the definition of B_D, for each $m = r, \ldots, \eta-1$,

$$(s_m - s_{m+1}) - (g_m - g_{m+1}) - 1 \in \langle \kappa_m, \kappa_{m+1} \rangle \subset \langle \kappa_\eta \rangle$$

and

$$(t_m - t_{m+1}) - (h_m - h_{m+1}) - 1 \in \langle \kappa_m, \kappa_{m+1} \rangle \subset \langle \kappa_\eta \rangle$$

where $s_r = x$, $t_r = y$, $s_\eta = t_\eta = k$, $(g_m, g_{m+1}) \in I(D_{m,m+1})$, $(h_m, h_{m+1}) \in I(D_{m,m+1})$, $\langle \kappa_\eta \rangle = \{\alpha\kappa_\eta \mid \alpha \in \mathbb{Z}\}$, and $\langle \kappa_m, \kappa_{m+1} \rangle = \{\beta\kappa_m + \gamma\kappa_{m+1} \mid \beta \in \mathbb{Z}, \gamma \in \mathbb{Z}\}$. By the hypothesis, $(g_m - g_{m+1}) - (h_m - h_{m+1}) \in \langle \kappa_\eta \rangle$ and so $(s_m - s_{m+1}) - (t_m - t_{m+1}) \in \langle \kappa_\eta \rangle$ for each $m = r, \ldots, \eta-1$. Therefore, $(x-k) - (y-k) \in \langle \kappa_\eta \rangle$ and this completes the proof of the 'only if' part.

We prove the 'if' part by induction on $\eta - p$ for $p = 1, \ldots, \eta-1$. Suppose that $x \equiv y \pmod{\kappa_\eta}$ for x, y in $\mathbb{Z}_{\kappa_{\eta-1}}$. Since D is linearly connected, there is an element (k, l) in $I(D_{\eta-1,\eta})$. Since x, y, and k belong to $\mathbb{Z}_{\kappa_{\eta-1}}$, $x \equiv k + m + 1 \pmod{\kappa_{\eta-1}}$ and $y \equiv k + m' + 1 \pmod{\kappa_{\eta-1}}$ for some integers m and m'. Then, by definition of B_D, $(x, z) \in B_{\eta-1,\eta}$ and $(x, z') \in B_{\eta-1,\eta}$ for a vertex $z \in \mathbb{Z}_{\kappa_\eta}$ satisfying $z \equiv l + m \pmod{\kappa_\eta}$ and a vertex $z' \in \mathbb{Z}_{\kappa_\eta}$ satisfying $z' \equiv l + m' \pmod{\kappa_\eta}$. Since $\kappa_\eta \mid \kappa_{\eta-1}$, $x \equiv k + m + 1 \pmod{\kappa_\eta}$ and $y \equiv k + m' + 1 \pmod{\kappa_\eta}$. Since $x \equiv y \pmod{\kappa_\eta}$, $m \equiv m' \pmod{\kappa_\eta}$ and so $z \equiv z' \pmod{\kappa_\eta}$. Since both z and z' belong to $\mathbb{Z}_{\kappa_\eta}$, $z = z'$ by the definition of $PT_D^{(\eta)}$ and so the 'if' part is true for $\eta - p = 1$.

Suppose that the 'if' part is true for $\eta - p$ $(p \geq 1)$. Then we assume that $x \equiv y \pmod{\kappa_\eta}$ for x, y in $\mathbb{Z}_{\kappa_{\eta-p-1}}$. By applying a similar argument as above, we may show that there exist vertices w and w' in $\mathbb{Z}_{\kappa_{\eta-p}}$ such that $w \equiv w' \pmod{\kappa_\eta}$ and there are edges (x, w) and (y, w') in $PT_D^{(\eta)}$. Then, by the induction hypothesis, there exist a (w, z)-path P and a (w', z)-path Q for $z \in \mathbb{Z}_{\kappa_\eta}$. Accordingly xP and yQ

are an (x, z)-path and a (y, z)-path, respectively, and this completes the proof of the 'if' part. $\square$

Theorem 3.2.3. *Let D be a linearly connected digraph with only nontrivial strong components D_1, D_2, ..., D_η ($\eta \geq 2$) such that there is an arc going from D_i to D_j for some distinct $i, j \in \{1, \ldots, \eta\}$ only if $j = i + 1$. Suppose that a graph G is the limit of $\{C(D^m)\}_{m=1}^{\infty}$. Then G is the disjoint union of complete subgraphs if and only if κ_η divides κ_p and $i - j \equiv i' - j' \pmod{\kappa_\eta}$ for any (i, j), $(i', j') \in I(D_{p,p+1})$ for each $p = 1$, ..., $\eta - 1$.*

Proof. By induction on the number η of strong components of a linearly connected digraph. By Theorem 3.2.1 and Lemma 3.2.2, the statement is true for $\eta = 2$. Suppose that the statement is true for $\eta - 1$ ($\eta \geq 3$). Let D be a linearly connected digraph with only nontrivial strong components D_1, D_2, ..., D_η such that there is an arc going from D_i to D_j for some distinct $i, j \in \{1, \ldots, \eta\}$ only if $j = i + 1$. We delete D_1 from D to obtain a linearly connected digraph F with $\eta - 1$ strong components D_2, ..., D_η. By the induction hypothesis, $\{C(F^m)\}_{m=1}^{\infty}$ converges to the disjoint union of complete subgraphs if and only if $\kappa(D_\eta)$ divides $\kappa(D_p)$ and $i - j \equiv i' - j' \pmod{\kappa(D_\eta)}$ for any (i, j), $(i', j') \in I(D_{p,p+1})$ for each $p = 2$, ..., $\eta - 1$.

By the hypothesis, $\{C(D^m)\}_{m=1}^{\infty}$ converges to G. By Theorem 3.1.14, G is an expansion of $PT_D^{(\eta)}$. Since there is no way to reach from a vertex in D_j to a vertex in D_i for $j > i$, $\{C(F^m)\}_{m=1}^{\infty}$ converges to a graph, say G', which is an expansion of $PT_D^{(\eta)} - \mathbb{Z}_{\kappa_1}$. Therefore $G' = G - V(D_1)$ by the way in which the expansion is constructed in the proof of Theorem 3.1.14.

To show the 'only if' part, suppose that G is the disjoint union of complete subgraphs. Then, since $G' = G - V(D_1)$, G' is the disjoint union complete subgraphs.

Hence, by induction hypothesis, κ_η divides κ_p and $i - j \equiv i' - j' \pmod{\kappa_\eta}$ for any (i, j), $(i', j') \in I(D_{p,p+1})$ for each $p = 2, \ldots, \eta - 1$.

We first make the following observation:

If the complete graph replacing x and the complete graph replacing y are contained in the same component in G for x and y in $\mathbb{Z}_{\kappa_2}$, then $x \equiv y \pmod{\kappa_\eta}$. (†)

To see why, suppose that the complete graph replacing x and the complete graph replacing y are contained in the same component in G for x and y in $\mathbb{Z}_{\kappa_2}$. Then, for some $r \in \{2, \ldots, \eta\}$, there are an (x, k')-path and a (y, k')-path for some $k' \in \mathbb{Z}_{\kappa_r}$ by the definition of expansion. Since D is linearly connected, $PT_D^{(\eta)}$ is connected and so there is a (k', k)-path for some $k \in \mathbb{Z}_{\kappa_\eta}$ in $PT_D^{(\eta)}$. Then, by Lemma 3.2.2, $x \equiv y \pmod{\kappa_\eta}$.

Now we show that $\kappa_\eta \mid \kappa_1$. Since D is linearly connected, there is an edge (a, b) in $B_{1,2}$ and so, by the definition of B_D, for some $(i, j) \in I(D_{1,2})$ and for some integer l,

$$a \equiv i + l + 1 \pmod{\kappa_1}, \quad b \equiv j + l \pmod{\kappa_2}.$$

Since $(i, j) \in I(D_{1,2})$, it is true that $(i + l + \kappa_1 + 1, j + l + \kappa_1)$ is an edge of $B_{1,2}$. Since $i + l + \kappa_1 + 1 \equiv a \pmod{\kappa_1}$, it is true that $(a, j + l + \kappa_1)$ is an edge of $B_{1,2}$. Then, since (a, b) is also an edge of $PT_D^{(\eta)}$, the complete graphs replacing b and $j + l + \kappa_1$ should be contained in the same component in G by the hypothesis. By the observation (†), $b \equiv j + l + \kappa_1 \pmod{\kappa_\eta}$. Since $b \equiv j + l \pmod{\kappa_2}$ and $\kappa_\eta \mid \kappa_2$, we can conclude that $\kappa_\eta \mid \kappa_1$.

Take (i, j), $(i', j') \in I(D_{1,2})$. Without loss of generality, we may assume that $i' > i$. Since $(i, j) \in I(D_{1,2})$, by the definition of B_D, $(i + (i' - i) + 1, j + (i' - i))$ is an edge of $B_{1,2}$ and so is $(i' + 1, j + i' - i)$. In addition, $(i' + 1, j')$ is an edge of $B_{1,2}$ as $(i', j') \in I(D_{1,2})$. Therefore both $(i' + 1, j + i' - i)$ and $(i' + 1, j')$ are edges of

$B_{1,2}$. Then, by the hypothesis, the complete graphs replacing $j+i'-i$ and j' should be contained in the same component in G. By the observation (†), $j+i'-i \equiv j' \pmod{\kappa_\eta}$. Thus $i-j \equiv i'-j' \pmod{\kappa_\eta}$.

To show the 'if' part, suppose that κ_η divides κ_p and $i-j \equiv i'-j' \pmod{\kappa_\eta}$ for any (i,j), $(i',j') \in I(D_{p,p+1})$ for each $p = 1, \ldots, \eta-1$. By the induction hypothesis, G' is the disjoint union of complete subgraphs. Take a vertex $a \in \mathbb{Z}_{\kappa_1}$ of $PT_D^{(\eta)}$. If a has no neighbor in $B_{1,2}$, then its degree is zero and the complete graph representing it in G is a component of G. Suppose that a has a neighbor in $B_{1,2}$. Let (a,b) and (a,c) are edges of $B_{1,2}$. Then, by definition, for some (i,j), $(i',j') \in I(D_{1,2})$ and for some integers l, l',

$$\begin{aligned} a &\equiv i+l+1 \pmod{\kappa_1}, \quad & b &\equiv j+l \pmod{\kappa_2}, \\ a &\equiv i'+l'+1 \pmod{\kappa_1}, \quad & c &\equiv j'+l' \pmod{\kappa_2}. \end{aligned}$$

Since $\kappa_\eta | \kappa_1$,

$$a \equiv i+l+1 \equiv i'+l'+1 \pmod{\kappa_\eta},$$

and so $l-l' \equiv i'-i \pmod{\kappa_\eta}$. Since $\kappa_\eta | \kappa_2$,

$$b-c \equiv (j+l)-(j'+l') \equiv (j-j')+(l-l') \equiv (j-j')+(i'-i) \equiv 0 \pmod{\kappa_\eta}.$$

Thus, by Lemma 3.2.2, there exist a (b,z)-path and a (c,z)-path for some $z \in \mathbb{Z}_{\kappa_\eta}$. By the definition of expansion, the complete graph replacing b and the complete graph replacing c induce the complete subgraph of G. Thus G is the disjoin union of complete subgraphs. □

Theorem 3.2.3 is translated into matrix version as follows:

Corollary 3.2.4. *Suppose that $A \in \mathcal{B}_n$ is a matrix in the form given in (1.1) where the block A_{ii} has order at least two for each $i = 1, \ldots, \eta$. Let A' be the limit of $\{\Gamma(A^m)\}_{m=1}^\infty$. Then the following are equivalent:*

(a) The matrix $PA'P^T$ *is a JBD matrix.*

(b) For each $p = 1, \ldots, \eta - 1$, κ_η *divides* κ_p *and* $i - j \equiv i' - j' \pmod{\kappa_\eta}$ *whenever* $A_{ij}^{(p,p+1)}$ *and* $A_{i'j'}^{(p,p+1)}$ *defined in* $(\S)$ *are nonzero matrices.*

Chapter 4

Conclusions and closing remarks

In this paper, we found the limit of the matrix sequence $\{\Gamma(A^m)\}_{m=1}^{\infty}$ for a matrix A in the form given in (1.1) when all of the diagonal blocks are of order at least two. We know from Theorem 2.0.5 that there are other cases where $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges. We suggest that the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ be computed for A satisfying the condition for each of such cases.

In addition, Theorem 2.1 tells us the convergence of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ only when $D(A)$ is linearly connected. We suggest to find the condition where $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges even if $D(A)$ is not linearly connected.

Finally, for a Boolean matrix A in the form given in (1.1) when all of the diagonal blocks are of order at least two, we can ask that what is the smallest power N such that $\Gamma(A^m)$ equals the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ for any $m \geq N$?

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국문초록

이 논문에서는 성분의 개수가 임의로 주어진 선형연결그래프를 그것의 그래프로 갖는 Bool 행렬 $A \in \mathcal{B}_n$에 대하여 연구함으로써 Park *et al.* [12]의 결과를 확장하였다. 일반화의 과정에서, 그들이 사용한 증명의 핵심적인 아이디어를 구체화하였는데, 먼저 $\{\Gamma(A^m)\}_{m=1}^{\infty}$가 수렴하게 되는 행렬 A의 특징을 완벽하게 분석하였다. 또한 모든 대각블록이 2차 이상의 기약행렬인 A 대하여 $\{\Gamma(A^m)\}_{m=1}^{\infty}$의 극한을 구하였다. 더 나아가, $\{\Gamma(A^m)\}_{m=1}^{\infty}$의 극한이 J 블록대각행렬이 되는 행렬 A를 분석하였다. 이 모든 결과들은 A의 유향그래프의 경쟁그래프를 연구함으로써 얻어졌다.

주요어휘: 기약 Bool (0,1)-행렬, Bool (0,1)-행렬의 거듭제곱, 선형연결 그래프, 비원시성 지수, m-걸음 경쟁그래프, 그래프열, 그래프의 거듭제곱

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