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교육학석사학위논문

A study on chordality of the moral graph of a directed acyclic graph

(사이클을 포함하지 않는 유향그래프가 삼각화된 모랄그래프를 가질 조건에 대한 연구)

2016년 2월

서울대학교 대학원

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이 승 철

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지도교수 김 서 령

이 논문을 교육학석사 학위논문으로 제출함

2015년 12월

서울대학교 대학원

수학교육과

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2015년 12월

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A study on chordality of the moral graph of a directed acyclic graph

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Master of Science in Mathematics Education
to the faculty of the Graduate School of
Seoul National University

by

Seung Chul Lee

Dissertation Director : Professor Suh-Ryung Kim

Department of Mathematics Education
Seoul National University

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Abstract

Determining whether or not a directed acyclic graph has a chordal graph as its moral graph is motivated by the problem related to the propagation of evidence in a Bayesian network. In this thesis, we will give a characterization of the $(2,2)$ digraphs whose moral graphs are chordal. We give two main results. One states that if the underlying graph of a $(2,2)$ digraph D contains a hole of length n for $n \geq 7$, then a moral graph of D is not a chordal graph. The other states that if an underlying graph of a $(2,2)$ digraph D is chordal, then the moral graph of D is also chordal. In addition, we study chordality of moral graphs of $(2,2)$ digraphs whose underlying graphs are bipartite graphs.

Key words: bayesian network; moral graph; chordal graph; triangulation; competition graph; (i,j) digraph

Student Number: 2014-20932

Contents

Chapter 1

Introduction

1.1 Basic graph terminology

Let $G = (V, E)$ be a graph. A *walk* in G is a sequence of (not necessarily distinct) vertices $v_1, v_2, \ldots, v_k \in V$ such that $v_i v_{i+1} \in E$ for $i = 1, 2, \ldots, k-1$. Such a walk W is written as $W = v_1 v_2 \cdots v_k$. If the vertices in a walk are distinct, then the walk is called a *path*. A cycle in G is a path $v_1 v_2 \cdots v_k$ together with the edge $v_k v_1$ where $k \geq 3$.

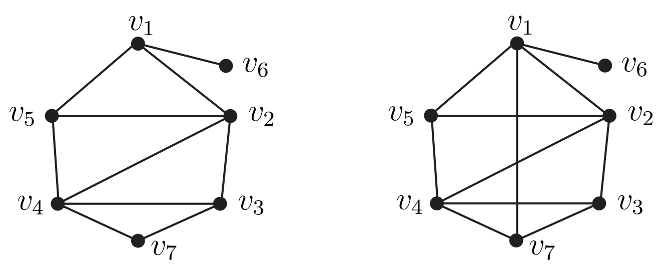
An edge which joins two vertices of a cycle but is not itself an edge of the cycle is a *chord* of that cycle. A chordless cycle in a graph is called a *hole*.

A *complete graph* is a simple graph in which every pair of distinct vertices is joined by exactly one edge. Given an undirected graph G, a vertex v is called *simplicial* if and only if the subgraph of G induced by the vertex set $\{v\} \cup N(v)$ is a complete graph.

A graph G with n vertices is said to have a *perfect elimination ordering* if there is an ordering $(v_1, v_2, \ldots, v_n)$ of vertices of G such that each v_i is simplicial in the subgraph induced by the vertex set $\{v_1, v_2, \ldots, v_n\} - \{v_{i+1}, \ldots, v_n\} = \{v_1, v_2, \ldots, v_i\}$

for $1 \leq i \leq n$.

A graph G is *triangulated* (or *chordal*) if every cycle in G of length greater than 3 has a chord, namely, an edge joining two nonconsecutive vertices on the cycle. For example, a graph G_1 in Figure 1.1(a) is a chordal graph since every cycle in G of length greater than 3 has a chord. A graph G_2 in Figure 1.1(b) is not a chordal graph since a cycle $v_1 v_2 v_3 v_7 v_1$ of length 4 has no chord. Rose [2] has shown that a graph has a perfect elimination ordering if and only if it is a chordal graph.



(a) G_1 (b) G_2

Figure 1.1: Two graphs G_1, G_2

A graph $G = (V, E)$ is an interval graph if we can assign to each x in V a real interval $J(x)$ so that whenever $x \neq y$, $xy \in E$ if and only if $J(x) \cap J(y) \neq \emptyset$. Hajaos [3] has shown that every interval graph is chordal.

1.2 Moral graphs

Given a directed acyclic graph D, the *underlying graph* of directed acyclic graph D, denoted by $U(D)$, is the graph having vertex set $V(D)$ and an edge xy if there is an arc (x, y) or (y, x) for some vertices x, y in D.

The *moral graph* of directed acyclic graph D, denoted by $M(D)$, is the graph with vertex set $V(D)$ and edge set $E(U(D)) \cup \{uv \mid (u, w) \in A(D), (v, w) \in A(D)$ for some $w \in V(D)\}$.

We say that an edge in $M(D)$ is an *induced edge* if it belongs to $C(D)$ but not to $U(D)$. For an induced edge xy in the moral graph $M(D)$, there is a common out-neighbor of x and y by definition. We call such a vertex a *vertex inducing the edge* xy and say that xy *is induced by* v.

For example, given a directed acyclic graph D in Figure 1.2(a), the underlying graph of D is $U(D)$ in Figure 1.2(b), and the moral graph of D is $M(D)$ in Figure 1.2(c). In the moral graph $M(D)$, v_1v_7, v_7v_8, v_2v_5 are induced edges of $M(D)$ and v_6, v_9, v_3 are the vertices inducing v_1v_7, v_7v_8, v_2v_5, respectively.

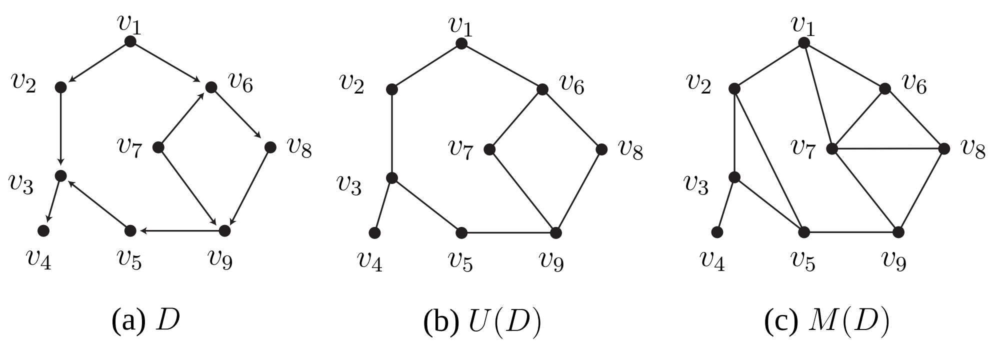


Figure 1.2: (a) A directed acyclic graph D (b) The underlying graph $U(D)$ of D (c) The moral graph $M(D)$ of D

1.3 Bayesian networks

A study on chordality of the moral graph of a directed acyclic graph is motivated by the problem related to the propagation of evidence in a Bayesian network. A *Bayesian network*, coined by Judea Pearl in 1985, consists of a set of nodes and a set of arcs which together constitute a directed acyclic graph (refer to [4]). In a Bayesian network, the nodes represent random variables, all of which, in general, have a finite set of states. The arcs indicate the existence of direct causal connections between the linked variables, and the strengths of these connections are expressed in terms of conditional probabilities. Bayesian networks became extremely popular models in the last decade. They have been used for applications in various areas, such as machine learning, text mining, natural language processing, speech recognition, signal processing, bioinformatics, error-control codes, medical diagnosis, weather forecasting, and cellular networks.

One of the best-known problems, in the context of Bayesian networks, is related to the propagation of evidence. It consists of the assignment of probabilities to the values of the rest of the variables, once the value of some variables are known. Cooper [6] demonstrated that this problem is NP-hard. Most noteworthy algorithms for this solution are the methods of Pearl [7], Shachter [8] and Lauritzen and Spiegelhalter [9]. The first step of this algorithm consists of the process constructing the moral graph of the network structure. The second step of the algorithm is the so-called triangulation of the moral graph, selectively add edges to the moral graph to form a triangulated graph. The remaining steps of the algorithm are irrelavant to the interest of this paper and we skip describing them.

In brief, triangulation of the moral graph plays an important role in algorithm for propagation of evidence in a Bayesian network. In this context, we thought that studying chordality of the moral graph of a Bayesian network is meaningful. As it the only fact used in this study that Bayesian network is a directed acyclic graph, we deal with directed acyclic graphs rather than Beyesian networks to be more general in this thesis.

1.4 Competition graphs and (i, j) digraphs

The *competition graph* of directed acyclic graph D, denoted by $C(D)$, is the graph having vertex set $V(D)$ and an edge vw if there are arcs (v, x) and (w, x) for some vertex x in D. Competition graphs is introduced by Cohen [10] to study ecosystems. Let D be a digraph represents a food web whose vertices are species in the ecosystem, with an arc $(x, y) \in A(D)$ if and only if x preys on y. Then there is an edge xy in $C(D)$ if and only if x and y have a common prey in the food web.

Cohen [10–12] observed empirically that most competition graphs of directed acyclic graphs representing food webs are interval graphs. However, Roberts [14] asked if Cohen's observation is due to an property of the construction. He showed that it isn't by showing that every graph can be made into a competition graph by adding sufficiently many isolated vertices. He then asked for a characterization of competition graph of directed acyclic graphs and, for a characterizing of a directed acyclic graphs whose competition graphs are interval graphs.

For the first problem, Early results were obtained by Roberts [14] and by Opsut [16]. Opsut [16] showed that recognition of competition graphs is an NP-complete problem. However, useful characterization were subsequently obtained by Dutton and Brigham [17], Ludgren and Maybee [18] and Roberts and Steif [15].

For the second problem, unfortunately, Stief [19] showed that there is no forbidden subgraph characterization of directed acyclic graph whose competition graphs are interval. This means that general problem of characterizing directed acyclic graphs whose competition graphs are interval graphs is not easy, which led researches to seek another ways to explain Cohen's observation.

Hefner et al. [20] approached it under certain assumption about directed acyclic graphs D. The assumption which limits the number of predators or preys of a species seems reasonable since the total number of arcs per species in a food web is actually quite small in an *average sense*, i.e., it is about 2 (see [13]). Hefner et al. [20] introduced the notion of (i, j) digraphs. A directed acyclic graph D is said to be an (i, j) ***digraph*** and $(\bar{i}, \bar{j})$ ***digraph*** if for every vertex $x \in D$, $d^-_D(x) \leq i$ and $d^+_D(x) \leq j$, and $d^-_D(x) = i$ and $d^+_D(x) = j$, respectively.

Assuming that each indegree and outdegree is bounded by two, they studied to characterize $(2, 2)$ digraphs whose competition graphs are interval. In contrast to

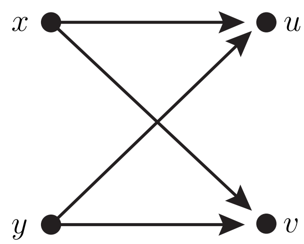


Figure 1.3: $P(2,2)$

Steif's observation, they obtained the following result in the case of $(2,2)$ digraphs:

Theorem 1.4.1 ([20])**.** *There is a forbidden subgraph characterization of* $(2,2)$ *digraphs whose competition graphs are interval.*

To be more specific, they introduced the notion of irredundant. If a digraph D has no subgraph $P(2,2)$ in Figure 1.3, D is said to be *irredundant*. They then characterized $(2,2)$ digraphs whose competition graphs are interval as follows:

Theorem 1.4.2 ([20])**.** *Suppose* D *is a* $(2,2)$ *digraph. Then* D *is an interval digraph if and only if every* $(2,2)$ *irredundant subgraph of* D *with at least one arc contains one of the three digraphs* S,T, *and* U *in Figure 1.4 as a generated subgraph.*

Inspired by the work done by Hefner et al. [20], in this thesis, we take a look at structures of moral graphs of $(2,2)$ digraphs, hoping to provide insights for the further research.

1.5 A preview of thesis

In Chapter 2, we discuss some fundamental properties of chordal graph which are prerequisites for this study. In Chapter 3, we will give a characterization of the

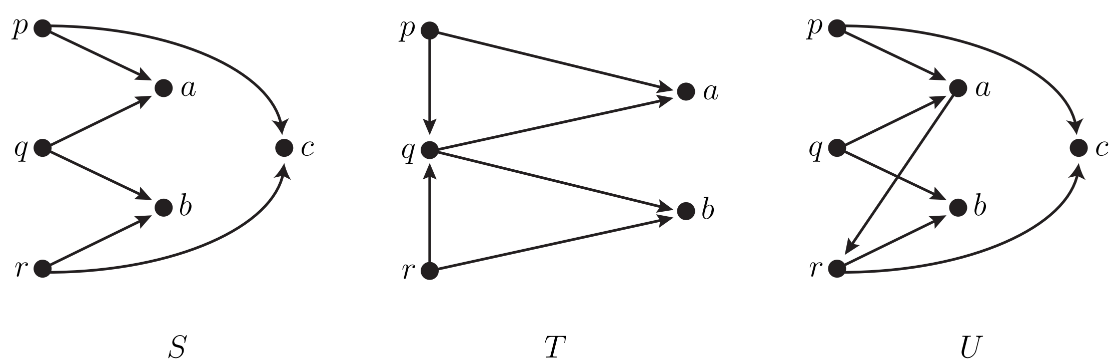


Figure 1.4: Three graphs S, T, U

$(2, 2)$ digraphs whose moral graphs are chordal. We give two main theorems. One states that if the underlying graph of a $(2, 2)$ digraph D contains a hole of length n for $n \geq 7$, then a moral graph of D is not a chordal graph. The other states that if an underlying graph of a $(2, 2)$ digraph D is chordal, then the moral graph of D is also chordal. In addition, we will give a characterization of bipartite $(2, 2)$ digraphs whose moral graphs are chordal.

Chapter 2

Properties of chordal graph

In this chapter, we derive some properties of chordal graphs which are to be used to characterize the $(2, 2)$ digraphs whose moral graphs are chordal.

Proposition 2.0.1. *Any induced subgraph of a chordal graph is also a chordal graph.*

Proposition 2.0.2. *If a chordal graph G contains an n-cycle C_n as a subgraph for $n \geq 3$, then, for any edge xy on C_n, there exists a vertex $v \in V(C_n) - \{x, y\}$ that is adjacent to both x and y.*

Proof. Let G be a chordal graph. To reach a contradiction, suppose that G contains a n-cycle that does not satisfy the property given in the proposition statement, which is to be called Property A for convenience. We take a cycle of the shortest length among such cycles. Let $C_k := v_0v_1 \dots v_{k-1}v_0$ be the such a cycle. Since C_3 satisfies Property A, $k \geq 4$. Then Since G is chordal, there exists a chord v_iv_j for $i < j$. Now we consider two cycles, one of which is a (v_i, v_j)-section of C_k together with the edge v_iv_j and the other of which is the other (v_i, v_j)-section of C_k together with the edge v_iv_j. Both of them have lengths shorter than k, so each edge on these cycles satisfies Property A. This implies that each edge on C_k satisfies Property A and we reach a contradiction. □

Proposition 2.0.3. *If a chordal graph G contains an n-cycle C_n as a subgraph for $n \geq 4$, then $|E(G)| \geq 2n - 3$.*

Proposition 2.0.4. *A hamiltonian chordal graph G with five vertices contains a vertex of degree 4.*

Proof. Let $C = v_1 \cdots v_5 v_1$ be the hamilton cycle of G. To the contrary, suppose that $d(v) \leq 3$ for every $v \in V(G)$. Then

$$2|E(G)| = \sum d(v) \leq 3 \times 5 = 15.$$

Therefore $|E(G)| \leq 15/2$, however, since $|E(G)|$ is an integer, $|E(G)| \leq 7$. Thus the number of chords of C is at most two. Then G is isomorphic to one of the graphs given in Figure 2.1.

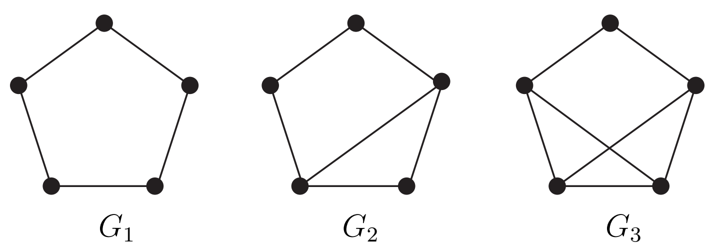


Figure 2.1: Three possible graphs for G

This is a contradiction since none of G_1, G_2, G_3 is a chordal graph. □

Given a chordal graph G that contains C_n as a subgraph for $n \geq 5$, we call a vertex v on C_n a *vertex opposite to a chord* if there is a chord joining the two neighbors of v that lie on C_n (for example, v_5 in Figure 2.2 is a vertex opposite to a chord).

Proposition 2.0.5. *If G is a hamiltonian chordal graph with n-vertices for $n \geq 5$, then there exist at least two nonconsecutive vertices on the cycle each of which is opposite to a chord.*

Proof. We prove by induction on n. First consider the case $n = 5$, which is a basis step. Then, by Proposition 2.0.4, G has a vertex of degree 4, say v_1. Then the two vertices labeled as v_2 and v_5 in Figure 2.2 are vertices as desired.

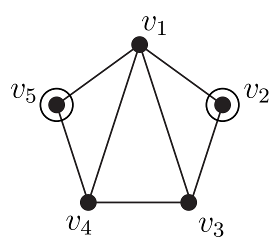


Figure 2.2: A labeled 5-cycle with the vertex v_1 of degree 4

Suppose that the statement holds for $n \leq k$ ($k \geq 5$). Let G be a chordal graph with $k + 1$ vertices that contains a $k + 1$-cycle C_{k+1} as a subgraph. Let $C_{k+1} = v_1v_2 \cdots v_{k+1}v_1$. For the edge v_1v_2 on C_{k+1}, there exists a vertex on C_{k+1}, say v_m, which is adjacent to v_1 and v_2 by Proposition 2.0.2. We consider the two cases: (i) $m \in \{3, k + 1\}$; (ii) $m \in \{4, \ldots, k\}$.

We consider the case $m \in \{3, k + 1\}$. Suppose that $m = 3$. Then v_2 is a vertex opposite to a chord. See Figure 2.4(a) for an illustration. Let H be the graph obtained by deleting v_2 from G. Then H is a chordal graph with k vertices that includes a k-cycle $v_1v_3v_4 \cdots v_{k+1}v_1$ as a subgraph. By the induction hypothesis, there exist two nonconsecutive vertices x, y in H each of which is opposite to a

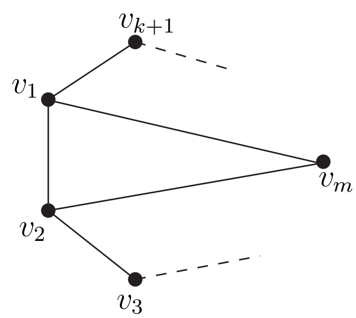


Figure 2.3: A vertex v_m which is adjacent to v_1 and v_2

chord of the k-cycle. Since v_1 and v_3 are adjacent in H, $\{x, y\} \neq \{v_1, v_3\}$. Without loss of generality, we may assume that $x \neq v_1$ and $x \neq v_3$. Then x and v_2 are not obviously consecutive on C_{k+1}. Since the chord of the k-cycle to which x is opposite is a chord of C_{k+1}, v_2 and x are desired vertices. A similar argument may be applied to showing the existence of two nonconsecutive vertices each of which is opposite to a chord in the case $m = k + 1$.

Now we consider the case $m \in \{4, \dots, k\}$. Let H_1, H_2 be the subgraphs of G induced by v_1 together with the vertices on the (v_m, v_{k+1})-section of C_{k+1} that does not contain v_1 and (v_2, v_m)-section of C_{k+1} that does not include v_1, respectively. See Figure 2.4(b) for an illustration. Then H_1 and H_2 are chordal graphs with $k-m$ vertices and $m-1$ vertices, respectively, and contain a $(k-m)$-cycle, say C_{k-m}, and an $(m-1)$-cycle, say C_{m-1}, respectively. By the induction hypothesis, there exist two nonconsecutive vertices w, z in H_1 each of which is opposite to a chord of the C_{k-m}. Since v_1 and v_m are consecutive on the C_{k-m}, $\{w, z\} \neq \{v_1, v_m\}$. Without loss of generality, we may assume that $w \neq v_1$ and $w \neq v_m$. Similarly, one can show that there is a vertex x not equal to v_2 or v_m that is opposite to a chord of the C_{m-1}. Then, it is obvious that w and x are not consecutive on the C_{k+1}. Since the chords of the C_{k-m}, C_{m-1} to which w, x are opposite, respectively, are chords of C_{k+1}, we have shown that w and x are desired vertices.

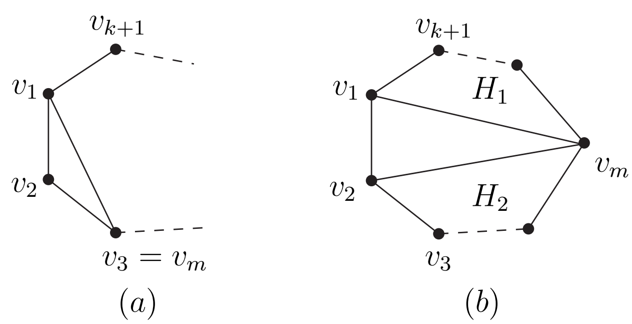


Figure 2.4: A vertex v_m which is adjacent to v_1 and v_2

□

We call the configuration given in Figure 2.5 a *W-configuration*. Then the following proposition is true.

Proposition 2.0.6. *Suppose that G is a hamiltonian chordal graph with 7 vertices such that $d(v) \leq 4$ for every $v \in V(G)$. Then G contains a W-configuration.*

Proof. Let $C = v_1 \cdots v_7 v_1$ be the Hamilton cycle of G. Since G is a chordal graph with 7 vertices, $|E(G)| \geq 11$ by Proposition 2.0.3. Then

$$\sum_{v \in V(G)} d(v) = 2|E(G)| \geq 2 \times 11 = 22$$

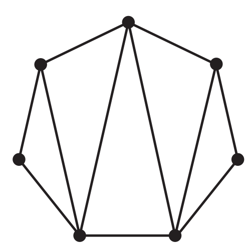


Figure 2.5: W-configuration

Therefore there exists a vertex of degree 4 by the Pigeon-Hole Principle. Without loss of generality, we may assume that v_1 is a vertex of degree 4. Then, by symmetry, $\{v_5, v_6\}$, $\{v_4, v_6\}$, $\{v_3, v_6\}$, $\{v_4, v_5\}$ are all the possible pairs of neighbors of v_1 other than v_2 and v_7. See Figure 2.6 for an illustration.

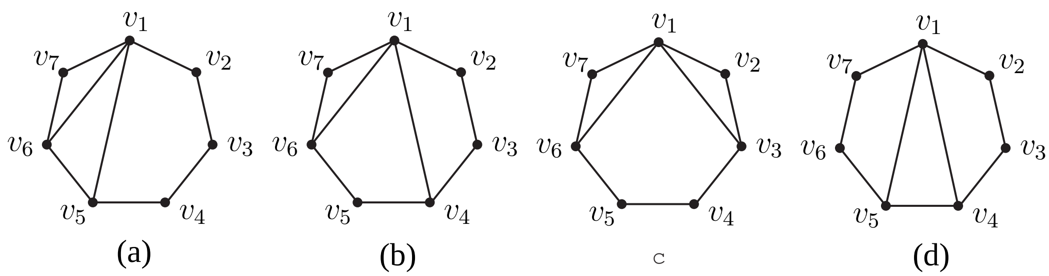


Figure 2.6: The possible neighbors of v_1

First, we consider the case (a). Then G has a 5-cycle $C_5 := v_1 \cdots v_5 v_1$ as a subgraph and the subgraph H_1 induced by $V(C_5)$ is a hamiltonian chordal graph. Thus, by Proposition 2.0.4, H_1 has a vertex u with $d_{H_1}(u) = 4$. Since $d_G(v) \leq 4$ for every $v \in V(G)$, u must be v_2. Then $v_2v_5, v_2v_4 \in E(G)$, which forms a W-configuration.

Second, we consider the case (b). Then, for the cycle $v_1v_4v_5v_6v_1$ and the edge v_4v_5, v_1 or v_6 is a vertex that is adjacent to both v_4 and v_5 by proposition 2.0.2. We note that v_1 is adjacent to four vertices other than v_5. Since $d_G(v) \leq 4$ for every $v \in G$, v_1 cannot be joined to v_5 and so v_6 is a vertex that is adjacent to both v_4 and v_5. Similarly, one can show that, for the cycle $v_1v_4v_3v_2v_1$ and the edge v_2v_3, v_4 is adjacent to v_2 and v_3. Then $d_G(v_4) \geq 5$, a contradiction.

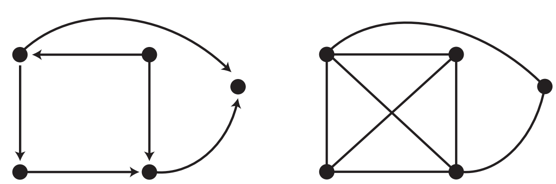
Third, we consider the case (c). Then G has a 5-cycle $v_1v_3v_4v_5v_6v_1$ as a subgraph and, by Proposition 2.0.4, the subgraph H_2 induced $\{v_1, v_3, v_4, v_5, v_6\}$ has a vertex of degree 4 all of whose neighbors are in H_2. Now we reach a contradiction to the hypothesis that $d_G(v) \leq 4$ for every $v \in V(G)$ as follows: if $d_{H_2}(v_1) = 4$, then $d_G(v_1) \geq 6$; if $d_{H_2}(v_4) = 4$ or $d_{H_2}(v_5) = 4$, then $d_G(v_1) \geq 5$; if $d_{H_2}(v_3) = 4$ or $d_{H_2}(v_6) = 4$, then $d_G(v_3) \geq 5$ or $d_G(v_6) \geq 5$, respectively.

Finally, we consider the case (d). For the cycle $v_1v_5v_6v_7v_1$ and the edge v_1v_5, v_6 or v_7 is a vertex that is adjacent to both v_1 and v_5 by proposition 2.0.2. Since $d_G(v_1) \leq 4$, v_7 is the vertex that is adjacent to both v_1 and v_5. For the cycle $v_1v_2v_3v_4v_1$ and the edge v_1v_4, v_2 is the vertex that is adjacent to both v_1 and v_4. Then $v_5v_7, v_2v_4 \in E(G)$, which induces a W-configuration. $\square$

Chapter 3

A characterization of chordality for the moral graph of $(2,2)$ digraph

We can construct a $(2,2)$ digraph whose moral graph contains K_4 as a subgraph. A graph in Figure 3.1(b) is the moral graph of a $(2,2)$ digraph D in Figure 3.1(a). However, there is no $(2,2)$ digraph whose moral graph contains K_n for $n \geq 5$.



(a) (b)

Figure 3.1: (a) A $(2,2)$ digraph D (b) The moral graph of D contains K_4

Theorem 3.0.7. *For any $(2,2)$ digraph D, the moral graph of D is K_n-free for any $n \geq 5$.*

Proof. First, we show that, for any $(2,2)$ digraph D, the moral graph of D is K_5-

free. To the contrary, suppose that there is a $(2, 2)$ digraph D whose moral graph has K_5 as a subgraph. Let $v_1, v_2, \ldots, v_5$ be the vertices of K_5. Let D_1 be the subdigraph of D induced by $\{v_1, \ldots, v_5\}$. Since D_1 is acyclic, there is a vertex of indegree 0 in D_1. Without loss of generality, we may assume that v_1 is such a vertex. Now consider the edges v_1v_2,v_1v_3,v_1v_4,v_1v_5 in $M(D)$. At most two of them are induced edges by Proposition 3.0.9. Therefore, at least two of them belong to $U(D)$. By the way, they must be arcs outgoing from v_1 since v_1 has indegree 0 in D_1. Since v_1 has outdegree at most two in D, v_1 has outdegree exactly two in D and so exactly two of v_1v_2,v_1v_3,v_1v_4,v_1v_5 belong to $U(D)$. Consequently, exactly two edges among v_1v_2,v_1v_3,v_1v_4,v_1v_5 are induced edges.

We may assume that v_1v_2, v_1v_3 are induced edges which is incident to v_1, that is, v_4,v_5 are the out-neighbors of v_1 in D (see Figure 3.2(a) as an illustration). Since a vertex inducing v_1v_2 is an out-neighbor of v_1, v_1v_2 is induced by v_4 or v_5. Without loss of generality, we may assume that v_4 is a common out-neighbor of v_1, v_2 and v_5 is a common out-neighbor of v_1, v_3 (see Figure 3.2(b) as an illustration). Then, since both v_4 and v_5 have indegree two, the edge v_4v_5 cannot belong to $U(D)$ and so is an induced edge in $M(D)$. Since (v_1, v_4),(v_2, v_4),(v_1, v_5),(v_3, v_5) are arcs of D, none of v_1,v_2,v_3 is a common out-neighbor of v_4 and v_5 (see Figure 3.2(c) as an illustration). Therefore, the edge v_4v_5 is induced by a vertex w_1 distinct from v_1, v_2, v_3.

We show that at least one of v_3v_4, v_2v_5 is an induced edge. To the contrary, suppose that both v_3v_4 and v_2v_5 are not induced edges. Then v_2, v_3 are out-neighbors of v_5, v_4, respectively since v_1 and v_2 are in-neighbors of v_4, and v_1 and v_3 are in-neighbors of v_5, filling indegree, respectively. Then $v_2 \to v_4 \to v_3 \to v_5 \to v_2$ is a directed cycle, which contradicts the hypothesis that D is acyclic. Without loss of generality, we may assume that v_3v_4 is an induced edge. Since (v_1, v_4),(v_2, v_4),(v_1, v_5),

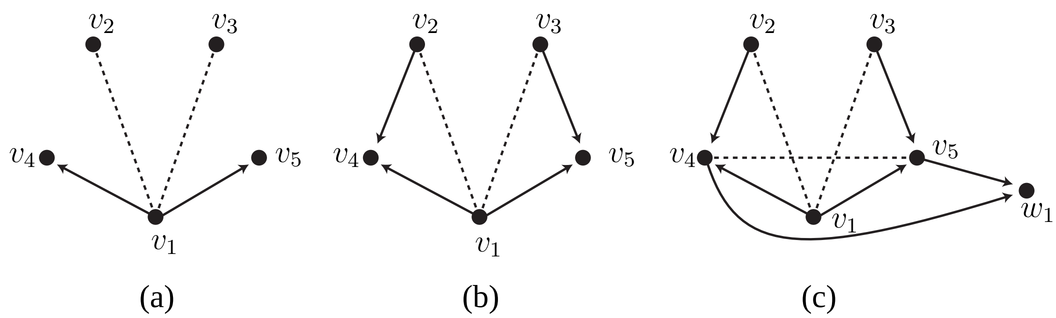


Figure 3.2: (a) The edges incident to v_1 in K_5 (b) The vertices v_4, v_5 inducing edges v_1v_2, v_1v_3 (c) The vertex w_1 inducing v_4v_5

and (v_3, v_5) are arcs of D, none of v_1,v_2,v_5 is a common out-neighbor of v_3 and v_4 (see Figure 3.3(a) as an illustration). Therefore, the edge v_3v_4 is induced by a vertex w_2 distinct from v_1, v_2, v_5. Recall that v_3v_4, v_3v_1 are induced edges which are incident to v_3. Since v_3 cannot be incident to three induced edges by Proposition 3.0.9, the edge v_2v_3 is in $U(D)$. Then the vertex v_3 is an out-neighbor of v_2 since v_5 and w_2 are out-neighbors of v_3, filling outdegree (see Figure 3.3(b) as an illustration).

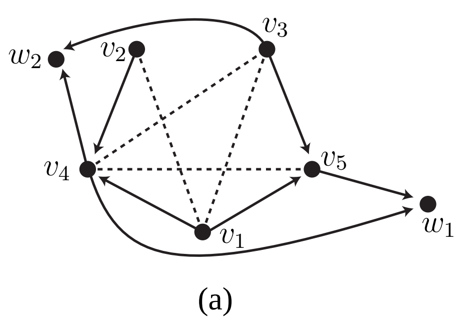


Figure 3.3: (a) The vertex w_2 inducing an edge v_3v_4 in K_5 (b) The edge v_2v_3 in D_1

Finally we take a look at the edge v_2v_5. Since the indegree of v_5 is two that is achieved by v_1 and v_5, v_2 cannot be an in-neighbor of v_5. If v_2 is an out-neighbor of v_5, then it results in the directed cycle $v_2 \to v_3 \to v_5 \to v_2$. Therefore v_2v_5 is an induced edge. Since v_3 and v_4 are the out-neighbor of v_2, v_3 or v_4 is a vertex inducing v_2v_5. Since (v_3, v_5) is an arc in D, v_3 cannot induce the edge v_2v_5. Since the edges v_4v_5 is an induced edge, v_4 cannot induce the edge v_2v_5 either and we reach a contradiction. Since K_5 is a subgraph of K_n for $n \geq 5$, the moral graph of D is K_n-free for $n \geq 5$. $\square$

Theorem 3.0.7 tells us that if the underlying graph of a $(2,2)$ digraph has a hole with a sufficiently large length, then it gives rise to a hole in its moral graph as there are not enough chords to fill in. This motivates us to find the length of shortest one among such holes in the underlying graph of $(2,2)$ digraph.

Proposition 3.0.8. *In a $(2,2)$ digraph D, there is no vertex that induces more than one induced edge in the moral graph of D.*

Proof. To the contrary, suppose that there exists a vertex x inducing two different induced edges yz, uw in the moral graph of D for some $y, z, u, w \in V(D)$. Then $N_D^-(x) = \{y, z, u, w\}$. Since $yz \neq uw$, $|\{y, z, u, w\}| \geq 3$. Therefore $d_D^-(x) \geq 3$ which contradicts the hypothesis that D is a $(2,2)$ digraph. $\square$

Proposition 3.0.9. *Given a $(2,2)$ digraph D, there is no vertex that is incident to three induced edges in the moral graph of D.*

Proof. To the contrary, suppose that there exists a vertex x incident to three induced edges xy, xz, xw for some distinct vertices y, z, w in $M(D)$. Let u_1, u_2, u_3 be vertices in D inducing xy, xz, xw, respectively. Then $\{u_1, u_2, u_3\} \subset N_D^+(x)$ and, by Proposition 3.0.8, u_1, u_2, u_3 are all distinct vertices. Thus $d_D^+(x) \geq 3$ which contradicts the hypothesis that D is a $(2,2)$ digraph. $\square$

Theorem 3.0.10. *If the underlying graph of a* $(2,2)$ *digraph* D *contains a hole of length* n *for* $n \geq 7$*, then a moral graph of* D *is not a chordal graph.*

Proof. Let $H_1 = v_1v_2\cdots v_nv_1$ be a hole in the the underlying graph of D and M_1 be the subgraph of $M(D)$ induced by $\{v_1, \ldots, v_n\}$. Then, no edge on H_1 is an induced edge and the edges of M_1 not on H_1 are induced edges in $M(D)$. To the contrary, suppose that $M(D)$ is a chordal graph. Then M_1 is also a chordal graph by Proposition 2.0.1.

If there exist a vertex v in M_1 with $d_{M_1}(v) \geq 5$, then v is incident to at least three induced edges since the edges incident to v are induced edges except the two edges on H_1, which contradicts Proposition 3.0.9. Therefore $d_{M_1}(v) \leq 4$ for every $v \in V(M_1)$.

Now we show that M_1 has a W-configuration. Since M_1 is a hamiltonian chordal graph with n vertices and $n \geq 7$, there exists a vertex on M_1 opposite to a chord by Proposition 2.0.5. Without loss of generality, we may assume that v_2 is such a vertex. Let M_2 be the graph obtained by deleting v_2 from M_1. Since v_2 is a vertex opposite to a chord, M_2 is a hamiltonian chordal graph with $n-1$ vertices with the Hamilton cycle $v_1v_3v_4\cdots v_nv_1$. Since $n-1 \geq 6$, we may apply Proposition 2.0.5 again to have a vertex M_2 that is opposite to a chord. We delete one of such vertices from M_2 to obtain a hamiltonian chordal graph with $n-2$ vertices. It is easy to check that the vertex sequence of the newly obtained Hamilton cycle is a subsequence of H_1. We continue this process until we obtain a hamiltonian chordal graph M^* with 7 vertices. Let $v_{n_1}v_{n_2}\cdots v_{n_7}v_{n_1}$ be the vertex sequence of the Hamilton cycle just obtained. Since it is subsequence of H_1, $n_1 < n_2 < \cdots < n_7$. Therefore M^* has a W-configuration by Proposition 2.0.6. Without loss of generality, we may take a W-configuration as given in Figure 3.4(a). Note that an edge $v_mv_k (1 \leq m, k \leq n)$ is on H_1 if and only if $|m-k| = 1$ or $|m-k| = n-1$.

Since $n_1 < n_2 < \cdots < n_7$, the index difference for the end vertices of an edge not on the Hamilton cycle $v_{n_1}v_{n_2}\cdots v_{n_7}v_{n_1}$ is neither 1 nor $n-1$. Thus each edge not on the Hamilton cycle is an induced edge in M^*. Therefore the edges on the W-configuration are induced edges. Now v_{n_5} is incident to v_{n_1}, v_{n_4}, v_{n_6}, v_{n_7}, and $v_{n_5}v_{n_1}$ and $v_{n_5}v_{n_7}$ are induced edges. Thus, by Proposition 3.0.9, $v_{n_4}v_{n_5}$ and $v_{n_5}v_{n_6}$ are edges on H_1. A similar argument can be applied for the vertex v_{n_4} to obtain another edge $v_{n_3}v_{n_4}$ on H_1. See Figure 3.4(b) for an illustration.

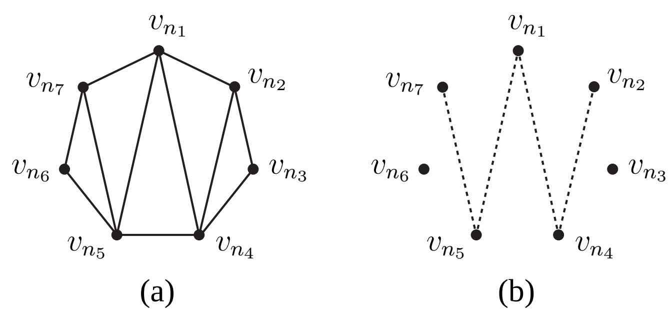
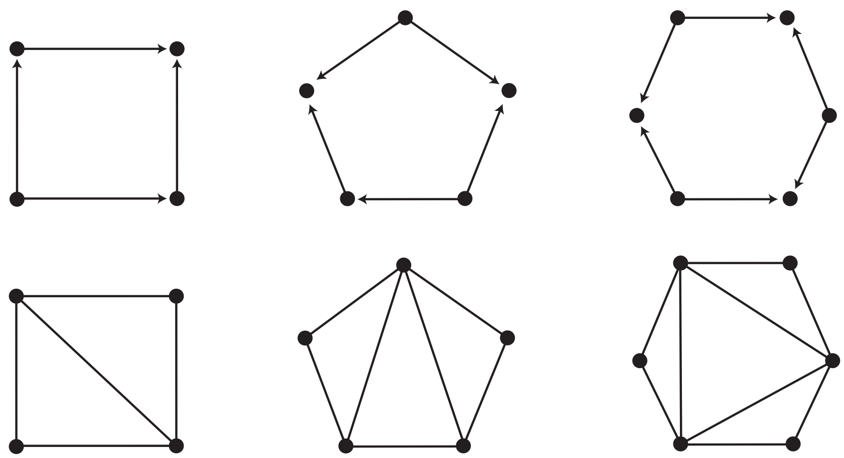


Figure 3.4: (a) A W-configuration of M^* (b) The induced edges in M^*

In the following, we show that $v_{n_4}v_{n_5}$ is not on H_1 to reach a contradiction. The vertex v_{n_4} is not an out-neighbor of v_{n_5} and vice versa in D. To see why, let w_1 and w_2 be vertices in D inducing $v_{n_5}v_{n_1}, v_{n_5}v_{n_7}$, respectively. Then, by Proposition 3.0.8, $w_1 \neq w_2$. Since D is a $(2,2)$ digraph, v_{n_5} has at most two out-neighbors in D and so $N_D^+(v_{n_5}) = \{w_1, w_2\}$. Since $v_{n_1}v_{n_4}$ is an induced edge, v_{n_4} is not an out-neighbor of v_{n_1} in D and so v_{n_4} cannot induce the edge $v_{n_1}v_{n_5}$. Thus $v_{n_4} \neq w_1$. Since H_1 is a hole and $2 \leq |n_4 - n_7| \leq n-2$, v_{n_4} cannot be an out-neighbor of v_{n_7} in D and so v_{n_4} cannot induce the edge $v_{n_5}v_{n_7}$. Thus $v_{n_4} \neq w_2$. Since $N_D^+(v_{n_5}) = \{w_1, w_2\}$, v_{n_4} cannot be an out-neighbor of v_{n_5}. A similar argument can be applied to show that v_{n_5} cannot be an out-neighbor of v_{n_4}. Hence $v_{n_4}v_{n_5}$ cannot be on H_1 and we reach a contradiction. □

We can construct $(2, 2)$ digraph whose underlying graph has a hole of length 4 or 5 or 6 while its moral graph is chordal. The underlying graphs of $(2, 2)$ digraphs D_1, D_2, D_3 in Figure 3.5 has a hole of length $4, 5, 6$, respectively while their moral graphs are all chordal.



(a) (b) (c)

Figure 3.5: (a) $(2, 2)$ digraph D_1 and its moral graph (b) $(2, 2)$ digraph D_2 and its moral graph (c) $(2, 2)$ digraph D_3 and its moral graph

As the vertices of holes in the underlying graphs are joined by edges in the moral graphs, one may expect that the moral graph cannot have a hole longer than the ones in the underlying graph. Contrary to this expectation, the longest hole in the underlying graph of D in Figure 3.6(a) has a length 4 while its moral graph $M(D)$ in Figure 3.6(b) has the hole $H = v_1v_2v_3v_6v_7v_4v_1$ of length 6. However, a $(2, 2)$ digraph lives up to the expectation as long as its underlying graph is chordal. Before we prove it, we need to state the following propositions.

Proposition 3.0.11. *Suppose that the moral graph of a $(2, 2)$ digraph D has a hole H. Then any vertex inducing an edge on H is not on H.*

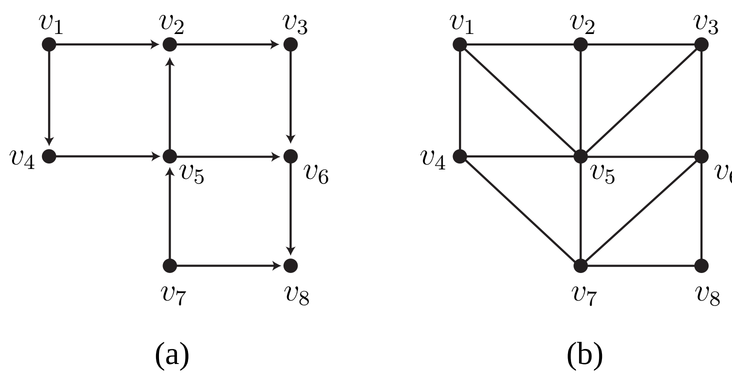


Figure 3.6: (a) An $(2, 2)$ digraph D (b) The moral graph of D

Proof. To the contrary, suppose that there exists a vertex z on H inducing an edge xy on H. Then the edge xz or the edge yz is a chord of H and we reached a contradiction. □

Given a $(2, 2)$ digraph D, suppose that $M(D)$ has a hole H of length n for $n \geq 4$ and H has m induced edges $e_1, e_2, \ldots, e_m$. Let $w_1, w_2, \ldots, w_m$ be vertices inducing $e_1, e_2, \ldots, e_m$, respectively. Then $w_1, w_2, \ldots, w_m \in V(D) - V(H)$ by Proposition 3.0.11. Let L be the subgraph of $U(D)$ induced by $V(H) \cup \{w_1, w_2, \ldots, w_m\}$. Then L is a hamiltonian graph with $n + m$ vertices in $U(D)$. We call L the *subgraph extended from* H. (for example the graph L in Figure 3.7(b) is the subgraph extended from H in Figure 3.7(a).)

Lemma 3.0.12. *Suppose that the moral graph of a* $(2, 2)$ *digraph* D *has a hole* H*. If* L *is the subgraph extended from* H*, no pair of vertices in* $V(L) - V(H)$ *is adjacent in* D*.*

Proof. To the contrary, suppose that there exist vertices $w_1, w_2 \in V(L) - V(H)$ that are adjacent in D. By the definition of L, each of w_1, w_2 is vertex inducing an edge on H. Thus w_1 and w_2 both have two in-neighbors in D that belong to

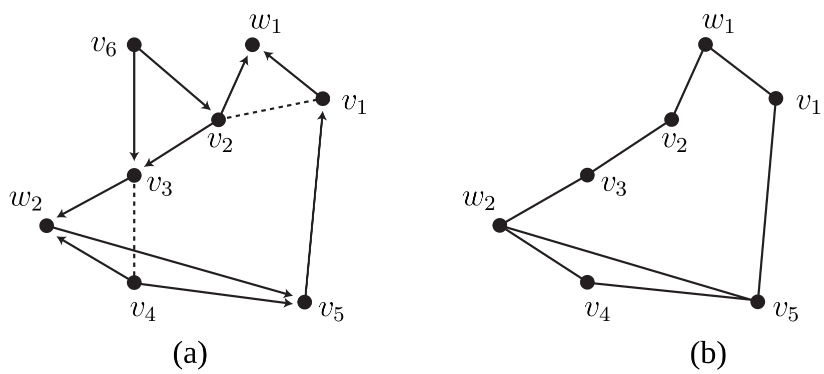


Figure 3.7: (a) A hole $H = v_1 \cdots v_5 v_1$ in the moral graph of $(2, 2)$ digraph D (b) The subgraph extended from H

$V(H)$. Since w_1 and w_2 is adjacent in D, w_1 or w_2 is an in-neighbor of w_2 or w_1, respectively. However, since neither w_1 nor w_2 is on H, one of w_1, w_2 has indegree at least three and we reach a contradiction. □

Lemma 3.0.13. *Suppose that the underlying graph of a $(2, 2)$ digraph D is chordal and the moral graph of D has a hole H. Let L be the subgraph extended from H. If xy is an induced edge on H and w is the vertex inducing xy in L, then there exists a vertex z in H that is adjacent to both x and w in L.*

Proof. Since L is hamiltonian by definition, L has a Hamilton cycle C. Since L is an induced subgraph of $U(D)$ which is chordal, L is chordal by Proposition 2.0.1. By the definition of L, the edge xw is on C. Thus, there exists a vertex $z \in V(C) - \{x, w\}$ that is adjacent to both x and w in D by Proposition 2.0.2. By proposition 3.0.11, w belongs to $V(L) - V(H)$. Since w and z are adjacent in $U(D)$, z belongs to H by Lemma 3.0.12 and this completes the proof. □

Proposition 3.0.14. *Suppose that the moral graph of a $(2, 2)$ digraph D has a hole*

H. *If the underlying graph of* D *is chordal, then* H *does not have consecutive induced edges.*

Proof. To the contrary, suppose that there exist two consecutive induced edges v_1v_2, v_2v_3 on H. Let w_1 and w_2 be vertices inducing v_1v_2 and v_2v_3, respectively (see Figure 3.8 for an illustration). Let L be the subgraph extended by H. Then $w_1, w_2 \in V(L) - V(H)$ by Proposition 3.0.11.

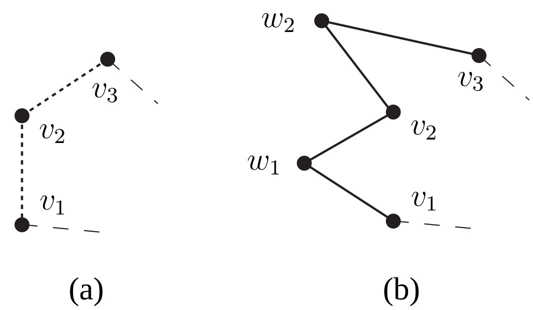


Figure 3.8: (a) A hole H in $M(D)$ (b) Edges $v_1w_1, w_1v_2, v_2w_2, w_2v_3$ on the the cycle C

Now there exists a vertex $y \in V(H)$ that is adjacent to both v_2 and w_1 in L by Lemma 3.0.13. Since y is adjacent to v_2 in D, yv_2 is not an induced edge. However, since v_1v_2 and v_2v_3 are induced edges by our assumption, $y \neq v_1$ and $y \neq v_3$. Since v_1,v_2 and v_3 are consecutive vertices on H, v_2 and y are nonconsecutive on H. Therefore v_2y is a chord of the hole H and we reach a contradiction. □

Theorem 3.0.15. *If an underlying graph of a* $(2, 2)$ *digraph* D *is chordal, then the moral graph of* D *is also chordal.*

Proof. By contradiction. Suppose that the moral graph $M(D)$ of D is not chordal. Then $M(D)$ has a hole $H = v_1 \cdots v_nv_1$ for some integer $n \geq 4$. Then there exists at least one induced edge on H, for otherwise, H would be a hole in $U(D)$, contradicting the hypothesis that $U(D)$ is chordal. Without loss of generality, we may assume that v_1v_2 is an induced edge on H. Then v_2v_3 and v_1v_n on H are not induced edges by Proposition 3.0.14. Now let L be the subgraph extended by H. Let C be a Hamilton cycle of L. Then, by Proposition 3.0.11, v_1v_2 is induced by w_1 for some $w_1 \in V(C) - V(H)$ (see Figure 3.9 for an illustration).

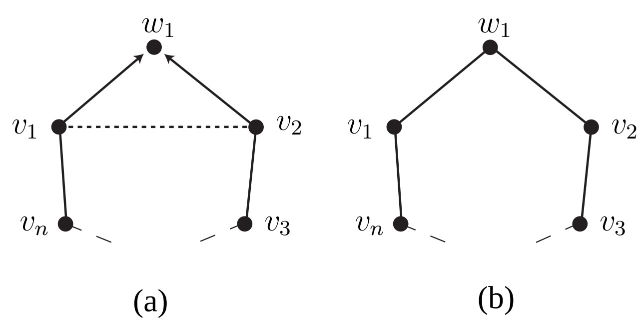


Figure 3.9: (a) A hole $H = v_1 \cdots v_n v_1$ and a vertex w_1 inducing the v_1v_2 in $M(D)$ (b) Edges $v_nv_1, v_1w_1, w_1v_2, v_2v_3$ in the Hamilton cycle C of L

Now there exists a vertex $y \in V(H)$ that is adjacent to both v_1 and w_1 in L by Lemma 3.0.13. For the edge yv_1 on L, $y = v_2$ of $y = v_n$ since $y \in H$ and $H = v_1 \cdots v_n v_1$ is a hole in $M(D)$. However, v_1v_2 is assumed to be an induced edge in $M(D)$, so $y = v_n$. Then the edge w_1y in $U(D)$ is now w_1v_n. Since $v_n \notin N_D^-(w_1) = \{v_1, v_2\}$, w_1 is an in-neighbor of v_n in D. In addition, the edge v_nv_1 in $U(D)$ corresponds to the arc (v_n, v_1) in D, for otherwise, $v_1 \to w_1 \to v_n \to v_1$ is a directed cycle in D which contradicts the acyclicity of D. Thus $N_D^-(v_n) = \{v_1, w_1\}$. By symmetry, one can show that $N_D^-(v_3) = \{v_2, w_1\}$.(see Figure 3.10 for an illustration.)

Now we consider the graph L^* obtained by deleting v_1, v_2 from L. Then L^* is hamiltonian and, by Proposition 2.0.1, chordal. For an edge w_1v_3 on L^*, there

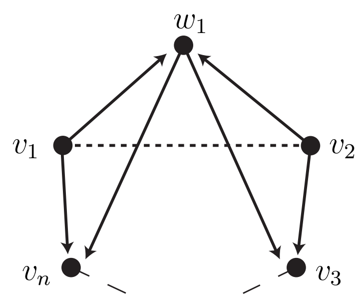


Figure 3.10: The arcs of D corresponding to $v_1v_n, v_1w_1, v_2v_3, v_2w_1, w_1v_n, w_1v_3$ in $U(D)$

exists a vertex $z \in V(L^*) - \{w_1, v_3\}$ that is adjacent to both w_1 and v_3 in D by Proposition 2.0.2. Since zw_1 and zv_3 are edges of L^* and L^* is a subgraph of $U(D)$, they are not induced edges in $M(D)$. Then $z = v_n$ since $z \in N_{U(D)}(w_1) = \{v_1, v_2, v_3, v_n\}$ and $z \notin \{v_1, v_2, v_3\}$. Then the edge zv_3 in $U(D)$ is now v_nv_3. Since zv_3 is an edge in L^*, v_nv_3 is an edge in $U(D)$. Therefore either (v_3, v_n) or (v_n, v_3) is an arc in D. However $v_n \notin \{v_2, w_1\} = N_D^-(v_3)$ and $v_3 \notin \{v_1, w_1\} = N_D^-(v_n)$ which is a contradiction. $\square$

In the rest of this thesis, we study chordality of moral graphs of $(2, 2)$ digraphs whose underlying graphs are bipartite graphs.

Proposition 3.0.16. *Let H be a hole of length greater than three in an underlying graph of a $(2, 2)$ digraph D. Then, in the moral graph of D, its subgraph induced by the vertex of H is not a hole.*

Proof. Let $H = v_1v_2 \cdots v_lv_1$ for some $l \geq 4$. Since D is acyclic, (v_i, v_{i+1}) and (v_{i+2}, v_{i+1}) are arcs of D for some $i \in \{1, \ldots, l\}$ (we reduce the subscripts to modulo l). Then v_i and v_{i+2} are adjacent in $M(D)$, so the subgraph induced by $V(H)$ is not a hole. $\square$

Proposition 3.0.17. *Suppose that the underlying graph of a $(2, 2)$ digraph D is a bipartite graph with a bipartition (X, Y). Given an edge in the moral graph of D, it is an induced edge if and only if its ends are in the same part in the underlying graph of D. Moreover a vertex inducing an induced edge e is in the part different from the one to which the ends of e belong.*

Proof. Let xy be an edge in the moral graph of D. Since x and y belong to in the same part, there is no arc between them in D, which immediately implies that xy is an induced edge. To show the converse by contradiction, suppose there is an induced edge xy with $x \in X$ and $y \in Y$. Let z be the vertex inducing xy. Then zx and zy are edges in $U(D)$. This implies that $z \in Y$ and $z \in X$, which is impossible. Thus the ends of an induced edge belong to the same part. The moreover part is obvious by the statement just proven. □

Theorem 3.0.18. *If the underlying graph of a $(2, 2)$ digraph D is a bipartite graph with a bipartition (X, Y) satisfying $min\{|X|, |Y|\} \leq 2$, then the moral graph of D is chordal.*

Proof. Without loss of generality, we may assume that $|X| \leq |Y|$. If $|X| = 1$, then $U(D)$ is a star with or without isolated vertices and so $M(D)$ is chordal by Theorem 3.0.15. We consider the case $|X| = 2$. Let $V(X) = \{x_1, x_2\}$. To the contrary, suppose that $M(D)$ is not a chordal. Then $M(D)$ has a hole H of length l for $l \geq 4$. Now $|V(H) \cap X| = 0$ or 1 or 2 since $|X| = 2$.

First we consider the case $|V(H) \cap X| = 0$. Then $V(H) \subset Y$. Thus there exist at least four induced edges e_1, e_2, e_3, e_4 on H by proposition 3.0.17. Moreover, by the same proposition, the four vertices w_1, w_2, w_3, w_4 inducing e_1, e_2, e_3, e_4, respectively, belong to X. By Proposition 3.0.8, w_1, w_2, w_3 and w_4 are all distinct, which is a contradiction to the hypothesis $|X| = 2$.

Next, we consider the case $|V(H) \cap X| = 1$. Without loss of generality, we may assume that $x_1 \in V(H) \cap X$. Then $H = x_1 y_2 y_3 \cdots y_l x_1$ for some $y_2, y_3, \ldots, y_l \in Y$. Now $y_2 y_3, y_3 y_4$ are induced edges and the vertices inducing $y_2 y_3, y_3 y_4$ are in X by Proposition 3.0.17. Since $|X| = 2$, the vertices inducing $y_2 y_3, y_3 y_4$ are x_1, x_2 or x_2, x_1, respectively, by Proposition 3.0.8. However x_1 is not adjacent to y_3 in $U(D)$ since x_1 and y_3 are nonconsecutive on the hole H. Thus x_1 cannot be the vertex inducing $y_2 y_3$ or $y_3 y_4$, which is a contradiction.

Finally we consider the case $|V(H) \cap X| = 2$. Then $x_1, x_2 \in V(H)$. Suppose to the contrary that H has an induced edge e. First, assume that the ends of e belong to Y. Then $e = yy'$ for some $y, y' \in Y$. By Proposition 3.0.17, x_1 or x_2 is an vertex inducing $e = yy'$. Without loss of generality, we may assume that x_1 is such a vertex. Then $x_1 y y' x_1$ is a 3-cycle in $M(D)$, so $x_1 y$ or $x_1 y'$ is a chord of H, which is a contradiction. Therefore $e = x_1 x_2$ by Proposition 3.0.17. Since H is a hole, it has two more vertices other than x_1, x_2. Then those vertices belong to Y, so there is an edge on H both of whose ends are in Y. By Proposition 3.0.17, it is an induced edge and we reach a contradiction. Thus there in no induced edge on H. Hence H is a hole in $U(D)$, which contradicts Proposition 3.0.16. $\square$

Chapter 4

Concluding remarks

We have obtained the the following two facts as main results in this thesis.

(i) If the underlying graph of a $(2, 2)$ digraph D contains a hole of length n for $n \geq 7$, then a moral graph of D is not a chordal graph (Theorem 3.0.10).

(ii) If an underlying graph of a $(2, 2)$ digraph D is chordal, then the moral graph of D is also chordal (Theorem 3.0.15).

Unfortunately, these results may not be extended to a general case.

For example, (ii) does not hold for the $(3, 3)$ digraph D given in Figure 4.1 as the underlying graph of D is chordal while the moral graph of D is not chordal.

As a matter of fact, regarding to (i), we may construct a $(2, 3)$ digraph D_n such that the underlying graph of D_n contains a hole of length n while the moral graph of D_n is chordal for every integer $n \geq 7$.

First we construct such a digraph D_{2k+1} for $k \geq 3$ as follows;

(i) $V(D_{2k+1}) = \{x\} \cup \{v_1, v_2, \ldots, v_{2k}\} \cup \{w_1, w_2, \ldots, w_{2k-2}\}$.

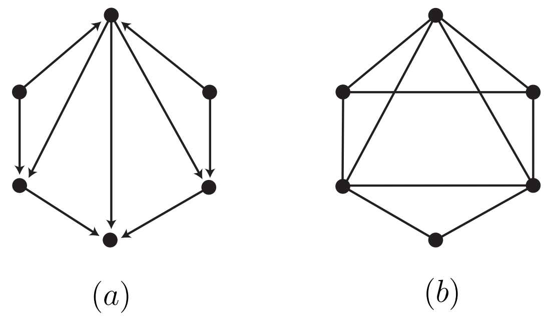


Figure 4.1: (a) A $(3,3)$ digraph D (b) The moral graph of D

(ii) The arcs are the ones on the directed path $x \to v_1 \to v_3 \to \cdots \to v_{2k-1} \to v_{2k}$, $x \to v_2 \to v_4 \to \cdots \to v_{2k}$ together with the arcs (v_1, w_1),(v_{2k-1}, w_{2k-2}) and the arcs $(v_i, w_i), (v_i, w_{i-1})$ for $i \in \{2, 3, \ldots, 2k-2\}$.

See Figure 4.2 and Figure 4.3 as an illustration.

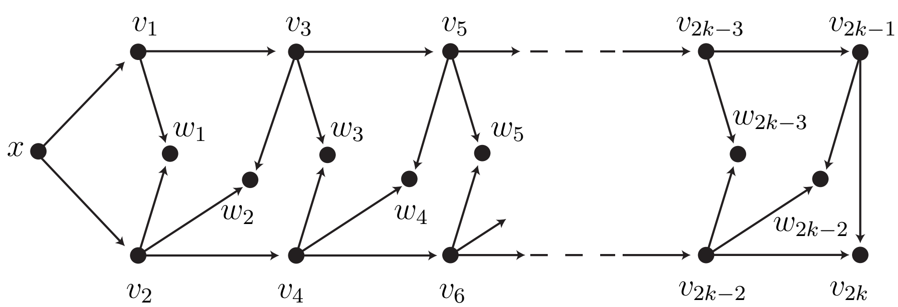


Figure 4.2: A $(2,3)$ digraph D_{2k+1}

It is easy to see that $xv_1v_3 \cdots v_{2k-1}v_{2k}v_{2k-2} \cdots v_6v_4v_2x$ is a hole of length $2k+1$ in the underlying graph of D_{2k+1}. To show that the moral graph of D_{2k+1} is chordal, we claim, by induction on k, that

$$(v_1, x, v_2, w_1, v_3, w_2, v_4, \ldots, v_i, w_{i-1}, v_{i+1}, \ldots, v_{2k-1}, w_{2k-2}, v_{2k})$$

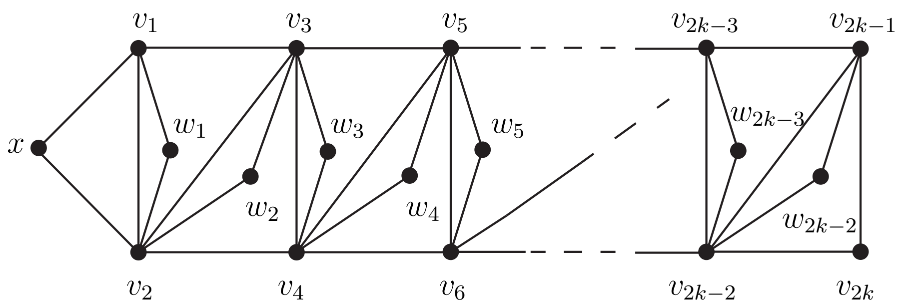


Figure 4.3: The moral graph of D_{2k+1}

is a perfect elimination ordering of the moral graph of D_{2k+1}. First consider the case $k = 3$ to take care of the basis step. Then we can easily check that

$$(v_1, x, v_2, w_1, v_3, w_2, v_4, w_3, v_5, w_4, v_6)$$

is a perfect elimination ordering in the moral graph of D_7 (see Figure 4.4(b) as an illustration).

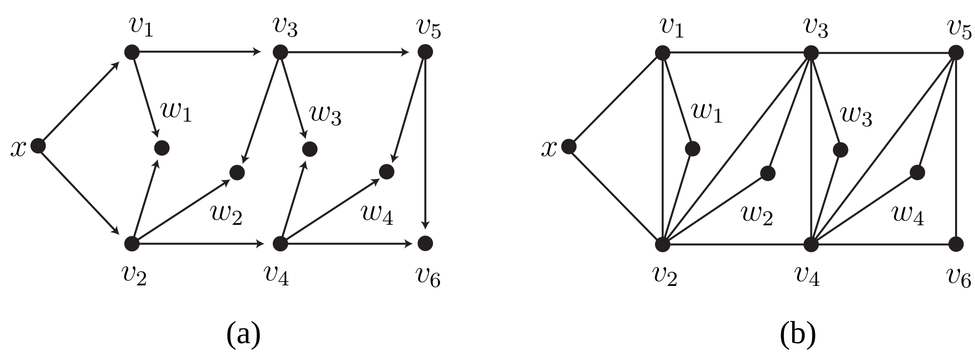


Figure 4.4: (a) A $(2, 3)$ digraph D_7 (b) The moral graph of D_7

Suppose that the statement holds for odd integer less than k for $k \geq 4$. Now we wish to show that

$$(v_1, x, v_2, w_1, v_3, w_2, v_4, \ldots, v_i, w_{i-1}, v_{i+1}, \ldots, v_{2k-3}, w_{2k-4}, v_{2k-2}, w_{2k-3}, v_{2k-1}, w_{2k-2}, v_{2k})$$

is a perfect elimination ordering of the moral graph of D_{2k+1}. It is easy to check that $v_{2k}, w_{2k-2}, w_{2k-3}$ are simplicial vertices of $M(D_{2k+1})$. Moreover the neighbors v_{2k-3} and v_{2k-2} of v_{2k-1} in $M(D_{2k+1}) - \{v_{2k}, w_{2k-2}\}$ are adjacent to form a clique. Deleting v_{2k}, v_{2k-1}, w_{2k-2}, w_{2k-3} from $M(D_{2k+1})$ results in $M(D_{2k-1})$. By the induction hypothesis,

$$(v_1, x, v_2, w_1, v_3, w_2, v_4, \ldots, v_i, w_{i-1}, v_{i+1}, \ldots, v_{2k-3}, w_{2k-4}, v_{2k-2})$$

is a perfect elimination ordering of $M(D_{2k-1})$ and this completes the proof of the claim.

Next, we construct a digraph D_{2k} for $k \geq 4$ as follows:

(i) $V(D_{2k}) = \{x\} \cup \{v_1, v_2, \ldots, v_{2k-1}\} \cup \{w_1, w_2, \ldots, w_{2k-3}\}$.

(ii) The arcs are the ones on the directed path $x \to v_1 \to v_3 \to \cdots \to v_{2k-1} \to v_{2k-2}$, $x \to v_2 \to v_4 \to \cdots \to v_{2k-2}$ together with the arcs (v_1, w_1),(v_{2k-2}, w_{2k-3}) and the arcs $(v_i, w_i), (v_i, w_{i-1})$ for $i \in \{2, 3, \ldots, 2k-3\}$.

See Figure 4.5 and Figure 4.6 as an illustration.

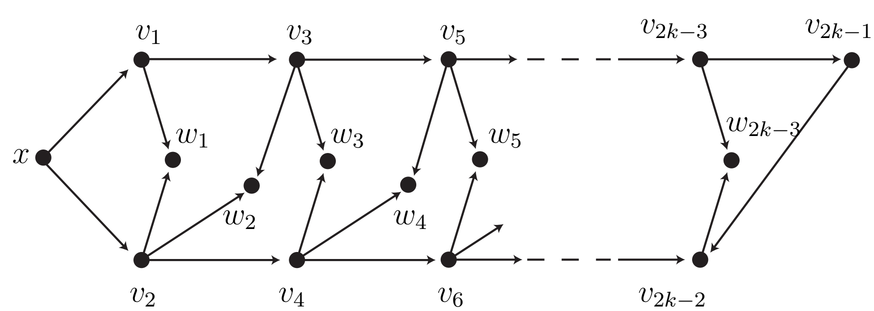


Figure 4.5: A $(2, 3)$ digraph D_{2k}

It is easy to see that $xv_2v_4v_6 \cdots v_{2k-2}v_{2k-1}v_{2k-3} \cdots v_5v_3v_1x$ is a hole of length $2k$ in the underlying graph of D_{2k}. A similar argument may be applied to showing

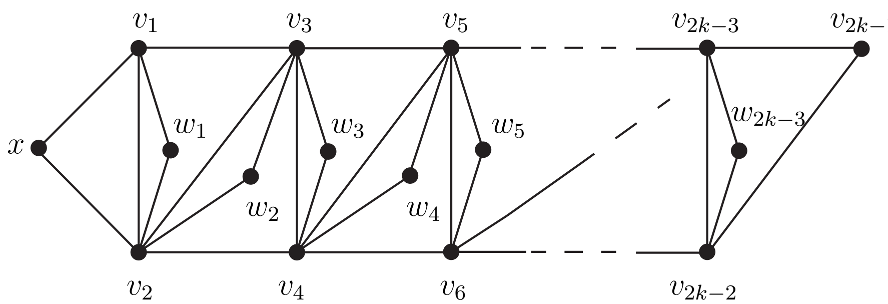


Figure 4.6: The moral graph of D_{2k}

that

$$(v_1, x, v_2, w_1, v_3, w_2, v_4, \ldots, v_i, w_{i-1}, v_{i+1}, \ldots, v_{2k-2}, w_{2k-3}, v_{2k-1})$$

is a perfect elimination ordering in the moral graph of D_{2k}.

For a further study, we suggest finding a sufficient and necessary condition for a $(2, 2)$ digraph having the chordal moral graph. Eventually, we hope to see a characterization of (i, j) digraphs whose moral graphs are chordal for $i \geq 3$ or $j \geq 3$.

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국문초록

사이클을 포함하지 않는 유향그래프가 삼각화된 모랄그래프를 가질 조건에 대한 연구는 베이지안 네트워크의 징후전달과 관련된 문제에서 비롯되었다. 이 논문에서는 삼각화된 모랄그래프를 가지는 $(2,2)$ 유향그래프의 특징을 분석하여 두 개의 주된 결과를 얻었다. 그 중 하나는 $(2,2)$ 유향그래프에서 유향변의 방향을 제거해 만든 그래프가 길이 7 이상인 홀을 가지는 경우, 그것의 모랄그래프는 삼각화된 그래프가 아니라는 것이다. 그리고 나머지 하나는 $(2,2)$ 유향그래프에서 유향변의 방향을 제거해 만든 그래프가 삼각화된 그래프라면 그것의 모랄그래프 역시 삼각화된 그래프라는 것이다. 또한 이분 $(2,2)$ 유향그래프가 삼각화된 모랄그래프를 가질 충분조건을 구하였다.

주요어휘: 베이지안 네트워크, 모랄그래프, 삼각화된 그래프, 삼각화, 경쟁그래프, (i,j) 유향그래프

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