# ccreative <br> <br> commons 

 <br> <br> commons}
$\begin{array}{lllllllllll}\text { C } & \mathrm{O} & \mathrm{M} & \mathrm{M} & \mathrm{O} & \mathrm{N} & \mathrm{S} & \mathrm{D} & \mathrm{E} & \mathrm{E} & \mathrm{D}\end{array}$

저작자표시-비영리-변경금지 2.0 대한민국
이용자는 아래의 조건을 따르는 경우에 한하여 자유롭게

- 이 저작물을 복제, 배포, 전송, 전시, 공연 및 방송할 수 있습니다.

다음과 같은 조건을 따라야 합니다:


저작자표시. 귀하는 원저작자를 표시하여야 합니다.

비영리. 귀하는 이 저작물을 영리 목적으로 이용할 수 없습니다.

- 귀하는, 이 저작물의 재이용이나 배포의 경우, 이 저작물에 적용된 이용허락조건 을 명확하게 나타내어야 합니다.
- 저작권자로부터 별도의 허가를 받으면 이러한 조건들은 적용되지 않습니다.

저작권법에 따른 이용자의 권리는 위의 내용에 의하여 영향을 받지 않습니다.

이것은 이용허락규약(Legal Code)을 이해하기 숩게 요약한 것입니다.

$$
\text { Disclaimer } \square
$$

## c)Collection

교육학석사학위논문

# Numerical solutions 

## for the golf swing

## 2016년 2월

서울대학교 대학원
수학교육과
임 소 연

# Numerical solutions for the golf swing 

$$
\begin{gathered}
\text { 지도교수 정 상 권 } \\
\text { 이 논문을 교육학석사 학위논문으로 제출함 } \\
\text { 2015년 12월 } \\
\text { 서울대학교 대학원 } \\
\text { 수학교육과 } \\
\text { 임 소 연 }
\end{gathered}
$$

임소연의 교육학석사 학위논문을 인준함
2015년 12월

| 위 원 장 조 한 혁 (인) <br> 부 위 원 장 최 영 기 (인) <br> 위 원 정 상 권 | (인) |
| :--- | :--- | :--- | :--- | :--- | :--- |

# Numerical solutions for the golf swing 

A dissertation submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics Education to the faculty of the Graduate School of Seoul National University

by

Soyon Yim

Dissertation Director : Professor Sangkwon Chung

Department of Mathematics Education<br>Seoul National University

February 2016


#### Abstract

We are going to analyze mathematically the downswing of the golf using the double pendulum model and Lagrangian function. We will show existence for expressed as a system of differential equations of the arm and the club. In order to obtain the clubhead speed, we calaulate the angular speed of the arm and the club by RungeKutta method. Using the calcualted value of the clubhead speed, we can also obtain a projection angle of the maximum projectile range of the ball.


Key words: the double pendulum model; Lagrangian function; existence for differential equations; Runge-Kutta method; quadratic drag.
Student Number: 2012-21422

## Contents

Abstract ..... i
1 Introduction ..... 1
2 The dynamics of the golf swing ..... 5
3 Analysis of the golf swing ..... 9
4 Existence for Differential Equations ..... 13
5 Numerical computation ..... 17
6 Maximizing the projectile range of golf ..... 23
7 Concluding Remarks ..... 28
Abstract (in Korean) ..... 32

## Chapter 1

## Introduction

It is important for athletes looking for a way to improve their records. For a typical example, the research finding ways to increase the distance of the ski jump is the one of them. In order to improve the flying distance of the ski jump, a player has to find ways to increase lift and to decrease air resistance. To find ways in the ski jump, we have to model the dynamics of the ski jump and solve the model equation. In general, the model equation is expressed as a nonlinear system of differential equations. Thus it is almost impossible to find the solution of nonlinear systems of differential equations. We try to find approximate solutions for the systems using a computer.

Like ski jump, the flying distance of the golf ball are affected by the flying speed of a ball, the angle of fire and the drag resistance[8]. Since the flying speed of the ball is proportional to the speed of the clubhead and the angle of fire is primarily dependent on the loft of the clubhead, we hope maximize the clubhead speed to obtain the maximum projecitle range.

In earier studies of the golf swing, the dynamics was just to analyze the pictures
to understand the swing process, which was merely to list the common features of a professional golfer's swing[4]. But since golf swing motions arise in the threedimensional space, it is difficult to analyze the golf's swing. Further, it is also complicate to measure the fact in the sports environment. Hence many people started to interprete the golf swing motions as the double pendulum. The Figure 1.1 shows the structure of the double pendulum. Two levers move as they are connected by the hub. The upper lever $L_{1}$ is rotated about the hub and the lower lever $L_{2}$ is rotated about $m_{1}$. The point $m_{2}$ can move freely.

The upper lever $L_{1}$ move, from a backswing top to the impact point. Applying a double pendulum model for a golf swing, a hub is a left shoulder joint, the upper lever $L_{1}$ is a left arm, $m_{1}$ is a wrist, the lower lever $L_{2}$ is a golf club, $m_{2}$ is a clubhead and $m_{3}$ is a ball.


Figure 1.1: the structure of the double pendulum

Williams[11], Daish[3], and Jorgensen[7] have derived the system of equations of motion under the double pendulum model using the Lagrangian method. Williams derived the following system of differential equation;

$$
\begin{align*}
\frac{X}{m}= & \ddot{x}=-b \dot{\theta}^{2} \cos \theta-b \ddot{\theta} \sin \theta-a \dot{\theta}^{2} \cos \left(\theta+\psi_{0}-\pi\right) \\
& -a \ddot{\theta} \sin \left(\theta+\psi_{0}-\pi\right)  \tag{1.1}\\
\frac{Y}{m}= & \ddot{y}=-b \dot{\theta}^{2} \sin \theta+b \ddot{\theta} \cos \theta-a \dot{\theta}^{2} \sin \left(\theta+\psi_{0}-\pi\right) \\
& +a \ddot{\theta} \cos \left(\theta+\psi_{0}-\pi\right)  \tag{1.2}\\
\frac{C}{m}= & -a b \dot{\theta}^{2} \sin \psi_{0}-a b \ddot{\theta} \cos \psi_{0}+a^{2} \ddot{\theta} \tag{1.3}
\end{align*}
$$

where $(x, y)$ is the position of the clubhead in the swing plane, $\psi$ is the angle between arm and the club, $\theta$ is the angle between arm and the horizontal plane. C is the wrist moment with a constant transverse shear force in the shaft as seen the following Figure 1.2.


Figure 1.2: the plane $x y$ represents hands-clubhead movement

Daish[3] derived the system of motion in angles, which is

$$
\begin{align*}
& A \ddot{\theta}+B \ddot{\phi} \cos (\phi-\theta)-B \ddot{\phi}^{2} \sin (\phi-\theta)=-\tau_{0}+\tau_{h}  \tag{1.4}\\
& B \ddot{\theta} \cos (\phi-\theta)+B \dot{\theta}^{2} \sin (\phi-\theta)+C \ddot{\phi}=-\tau_{h} \tag{1.5}
\end{align*}
$$

where $\phi$ and $\theta$ are the angular positions of the arm and the club shaft with respect to the vertical line to the ground, $\tau_{h}$ is the torque exerted by the hands and wrists, $\tau_{0}$ is the torque exerted by golfer's torso.

Jorgensen[7] has obtained the Lagrangian form as

$$
\begin{align*}
T_{a}= & \ddot{\theta}\left(J+M R^{2}\right)+\ddot{\psi} R S \cos (\psi-\theta) \\
& -\dot{\psi}^{2} R S \sin (\psi-\theta)+g(G+M R) \cos \theta,  \tag{1.6}\\
T_{c}= & \ddot{\psi} I+\ddot{\theta} R S \cos (\psi-\theta)+\dot{\theta}^{2} R S \sin (\psi-\theta) \\
& +g S \cos \psi, \tag{1.7}
\end{align*}
$$

which will be discussed later in detail.

Of course, we may think of the golf swing as a triple pendulum motion consisting of torso, forearm, and the golf club. But if we neglect the effect of torso, we may think the golf swing as a double pendulum motion. The double pendulum model transfers the energy according to the laws of conservation of energy and angular momentum. The club transfers the energy between the arms and the golf ball[10]. The potential energy is generated by the club and the arm falling in the gravitational field, the energy is emanated from muscles and the ball gets the energy by collison. Thus it is necessary to get the maximum energy of the clubhead at the moment of impact. Hence in order to obtain the maximum energy of the club, we have to maximize the clubhead speed.

## Chapter 2

## The dynamics of the golf swing

Since we are accustomed to consider the golf swing in the three dimensional space, we may express the position of clubhead as $(x(t), y(t), z(t))$. But we may think that the golf swing arises in a circular plane. In this case, we may express the motion of golf swing in the two dimensional space, which is much simpler than that in the three dimensional Euclidean space. This is the basic idea of the Lagrangian method.

In this chapter, we recall Lagrangian function of golf swing following Jorgensen[7]. As in Jorgensen, we consider the golf swing process as a double pendulum model depicted in Figure 2.1.

In the above figure,
$A B$ : the arm pivoted at $A$,
$B C$ : the golf club connected to the arm at $B$,
$M_{i}$ : the mass at the distance $l_{i}$ from $A$,
$M_{j}$ : the mass at the distance $l_{j}$ from $B$,


Figure 2.1: Golf swing: double pendulum model
$\theta$ : the counterclockwise angle determined by the arm $A B$ from the horizontal axis, $\psi$ : the counterclockwise angle determined by $A B$ and $B C$,
$M$ : the total mass of clubhead,
$\alpha$ : the counterclockwise angle from the at the top of backswing, $\beta$ : the counterclockwise angle from the club position at $90^{\circ}$ to the arm.

Lagrangian Function is attained by the potential energy and the kinetic energy. In order to calculate the kinetic energy, let as consider Figure 2.2. In Figure 2.2, since

$$
\begin{equation*}
\overrightarrow{A B}=(R \cos \theta, R \sin \theta) \quad \text { and } \quad \overrightarrow{B M_{j}}=\left(l_{j} \cos \psi, l_{j} \sin \psi\right) \tag{2.1}
\end{equation*}
$$

we obtain the angular momentum $\vec{w}$ of the club relative to A as

$$
\begin{equation*}
\vec{w}=M_{j}\left(\dot{\theta}(R \cos \theta, R \sin \theta)+\dot{\psi}\left(l_{j} \cos \psi, l_{j} \sin \psi\right)\right) \tag{2.2}
\end{equation*}
$$

Thus the kinetic energy $K_{e}$ is expressed

$$
\begin{align*}
K_{e} & =\sum\left(\frac{\left(M_{i} l_{i} \dot{\theta}\right)^{2}}{2 M_{i}}+\frac{\|\vec{w}\|^{2}}{2 M_{j}}\right) \\
& =\sum\left\{\frac{1}{2} M_{i} l_{i}^{2} \dot{\theta}^{2}+\frac{1}{2} M_{j} R^{2} \dot{\theta}^{2}+\frac{1}{2} M_{j} l_{j}^{2} \dot{\psi}^{2}+M_{j} l_{j} R \cos (\psi-\theta) \dot{\theta} \dot{\psi}\right\} \\
& =\frac{1}{2}\left[\left(J+M R^{2}\right) \dot{\theta}^{2}+I \dot{\psi}^{2}\right]+\dot{\theta} \dot{\psi} R S \cos (\psi-\theta), \tag{2.3}
\end{align*}
$$

where g is the gravitational acceleration,

$$
\begin{aligned}
& J=\sum M_{i} l_{i}^{2} \text { is the moment of inertia of the arm, } \\
& G=\sum M_{i} l_{i} \text { is the moment of the arm, } \\
& I=\sum M_{j} l_{j}^{2} \text { is the moment of inertia of the club, } \\
& S=\sum M_{j} l_{j} \text { is the moment of the club, } \\
& M \text { is the mass of the club, } \\
& R=\overline{A B} .
\end{aligned}
$$



Figure 2.2: Diagram of arm and club

On the other hands, the pontential energy $P_{e}$ can be put in the form

$$
P_{e}=\sum\left(g M_{i} l_{i} \sin \theta+g M R \sin \theta+g M_{j} l_{j} \sin \psi\right)
$$

$$
\begin{equation*}
=g[(G+M R) \sin \theta+S \sin \psi] . \tag{2.4}
\end{equation*}
$$

Therefore the function of the golf swing expressed as

$$
\begin{align*}
L=\frac{1}{2}[(J & \left.\left.+M R^{2}\right) \dot{\theta}^{2}+I \dot{\psi}^{2}\right]+\dot{\theta} \dot{\psi} R S \cos (\psi-\theta) \\
& -g[(G+M R) \sin \theta+S \sin \psi] \tag{2.5}
\end{align*}
$$

Using the Lagrangian equation from the Lagrangian function, the torques $T_{a}$ and $T_{c}$ applied to the arm and club, respectively, one obtained as

$$
\begin{align*}
T_{a}= & \frac{d}{d t}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta} \\
= & \ddot{\theta}\left(J+M R^{2}\right)+\ddot{\psi} R S \cos (\psi-\theta) \\
& -\dot{\psi}^{2} R S \sin (\psi-\theta)+g(G+M R) \cos \theta,  \tag{2.6}\\
T_{c}= & \frac{d}{d t}\left(\frac{\partial L}{\partial \dot{\psi}}\right)-\frac{\partial L}{\partial \psi} \\
= & \ddot{\psi} I+\ddot{\theta} R S \cos (\psi-\theta)+\dot{\theta}^{2} R S \sin (\psi-\theta) \\
& +g S \cos \psi \tag{2.7}
\end{align*}
$$

and

$$
\begin{align*}
T_{s} & =T_{a}+T_{c} \\
& =\ddot{\theta}\left[\left(J+M R^{2}\right)+R S \cos (\psi-\theta)\right]+\ddot{\psi}[I+R S \cos (\psi-\theta)] \\
& -\left(\dot{\psi}^{2}-\dot{\theta}^{2}\right) R S \sin (\psi-\theta)+g[(G+M R) \cos \theta+S \cos \psi] . \tag{2.8}
\end{align*}
$$

## Chapter 3

## Analysis of the golf swing

Since the gravitational torque on the club is somewhat smaller, we may neglect it. In this case, the torques $T_{a}$ and $T_{c}$ become

$$
\begin{align*}
T_{c}= & \ddot{\psi} I+\ddot{\theta} R S \cos (\psi-\theta)+\dot{\theta}^{2} R S \sin (\psi-\theta)  \tag{3.1}\\
T_{s}= & \ddot{\theta}\left[\left(J+M R^{2}\right)+R S \cos (\psi-\theta)\right]+\ddot{\psi}[I+R S \cos (\psi-\theta)] \\
& -\left(\dot{\psi}^{2}-\dot{\theta}^{2}\right) R S \sin (\psi-\theta) \tag{3.2}
\end{align*}
$$

Dividing (3.1) and (3.2) by $T_{s}$ and replacing $\left(1 / T_{s}\right)\left(d^{2} \theta / d t^{2}\right)$ by $\frac{d^{2} \theta}{d z^{2}}$, we obtain

$$
\begin{align*}
1= & \left.J+M R^{2}+R S \cos (\psi-\theta)\right] \frac{d^{2} \theta}{d z^{2}}+[I+R S \cos (\psi-\theta)] \frac{d^{2} \psi}{d z^{2}} \\
& -\left[\left(\frac{d \psi}{d z}\right)^{2}-\left(\frac{d \theta}{d z}\right)^{2}\right] R S \sin (\psi-\theta)  \tag{3.3}\\
\frac{T_{c}}{T_{s}}= & I \frac{d^{2} \psi}{d z^{2}}+\frac{d^{2} \theta}{d z^{2}} R S \cos (\psi-\theta)+\left(\frac{d \theta}{d z}\right)^{2} R S \sin (\psi-\theta) \tag{3.4}
\end{align*}
$$

where $z=\left(\sqrt{T_{s}}\right) t$ is a new parameter.

Let the angle $\alpha$ be measured counterclockwise from the position at the top of the backswing and the angle $\beta$ be measured counterclockwise from the cocked position
at $90^{\circ}$ to the arms. That is, the angle $\beta$ measures the angle when the wrist has been uncocked. Using the angles $\alpha$ and $\beta$, we can express $(3.3)$ and $(3,4)$ as

$$
\begin{align*}
1= & \left(J+I+M R^{2}+2 R S \sin \beta\right) \ddot{\alpha}+(I+R S \sin \beta) \ddot{\beta} \\
& {\left[(\dot{\beta}+\dot{\alpha})^{2}-\dot{\alpha}^{2}\right] R S \cos \beta, }  \tag{3.5}\\
T_{c} / T_{s}= & I \ddot{\beta}+(I+R S \sin \beta) \ddot{\alpha}-\dot{\alpha}^{2} R S \cos \beta, \tag{3.6}
\end{align*}
$$

where a dotted letter refers to differentiation with respect to $z$.

Jorgensen[7] assumed that the torque $T_{s}$ is constant for the swings under consideration. His assumption help making a first approximation to what actually happens, althogh it is impossibile to swing a club with $T_{s}$ constant. Having cocked his wrist, a golfer starts golf swing at the top of the backswing with $\alpha=\dot{\alpha}=0$. If we assume the angle between the club and his arm is $90^{\circ}(\beta=0)$, (3.6) becomes

$$
\begin{equation*}
T_{c}=T_{s} I(\ddot{\alpha}+\ddot{\beta}) \tag{3.7}
\end{equation*}
$$

When a golfer swings the club keeping wrist cocked for part of the swing, maintaining $\beta=\dot{\beta}=\ddot{\beta}=0$, (3.6) becomes

$$
\begin{equation*}
T_{c}=T_{s}\left(I \ddot{\alpha}-\dot{\alpha}^{2} R S\right) \tag{3.8}
\end{equation*}
$$

and (3.5) becomes

$$
\ddot{\alpha}=\frac{1}{\left(J+I+M R^{2}\right)} .
$$

Since $\ddot{\alpha}$ is constant, $\dot{\alpha}^{2}=2 \alpha \ddot{\alpha}$. Thus (3.6) becomes

$$
\begin{equation*}
\frac{T_{c}}{T_{s}}=\frac{I-2 \alpha R S}{J+I+M R^{2}} . \tag{3.9}
\end{equation*}
$$

If $\alpha=0$, then

$$
T_{c}=\frac{T_{s} I}{\left(J+I+M R^{2}\right)}>0
$$

Thus $T_{c}$ decreases linearly with $\alpha$ and becomes zero for $\alpha=\frac{I}{2 R S}$.

The golfer must release the torque applying the club by his wrists at any moment during the downswing. Until then, his wrist has been hindering the uncocking of his wrists. Jorgensen[7] suggested $T_{c}$ drops quickly to zero and stays at that value after the arms have turned through some angle $\alpha=\frac{(N+1) I}{2 R S}$. He called the quantity $N$ the "hindrance parameter" for the swing. $T_{c}$ becomes zero as golfer swing the club with perfectly flexible wrists. Then (3.5) and (3.6) become

$$
\begin{align*}
\ddot{\alpha} & =\frac{1-\ddot{\beta} R S \sin \beta-(\dot{\alpha}+\dot{\beta})^{2} R S \cos \beta}{J+M R^{2}+R S \sin \beta},  \tag{3.10}\\
I \ddot{\beta} & =\dot{\alpha}^{2} R S \cos \beta-\ddot{\alpha} R S \sin \beta-\ddot{\alpha} I . \tag{3.11}
\end{align*}
$$

By adding and subtracting, (3.10) and (3.11) become

$$
\begin{align*}
& \ddot{\alpha}=\frac{I-I(\dot{\alpha}+\dot{\beta})^{2} R S \cos \beta-\dot{\alpha}^{2} R^{2} S^{2} \sin \beta \cos \beta}{I\left(J+M R^{2}\right)-R^{2} S^{2} \sin ^{2} \beta}  \tag{3.12}\\
& \ddot{\beta}=\frac{R S \cos \beta\left\{E(\dot{\alpha}+\dot{\beta})^{2}+\dot{\alpha}^{2} F\right\}-E}{I\left(J+M R^{2}\right)-R^{2} S^{2} \sin ^{2} \beta} \tag{3.13}
\end{align*}
$$

where $E=(I+R S \sin \beta), F=\left(J+M R^{2}+R S \sin \beta\right)$.

If (3.12) and (3.13) have a unique solution on the interval $|z| \leq \frac{\sqrt{T_{s}}}{4}$ since $t \leq \frac{1}{2}$, we can change a sloution of (3.12) and (3.13) to a sloution about $t$. Since $z=\sqrt{T_{s}} t$, we obtain $\frac{d \alpha}{d t}=\sqrt{T_{s}} \frac{d \alpha}{d z}, \frac{d \beta}{d t}=\sqrt{T_{s}} \frac{d \beta}{d z}$. Using a sloution about $t$, we have to calculate the clubhead speed.

To calculate the speed of clubhead Figure 3.1 is represented by a vector as follows:

$$
\begin{equation*}
\overrightarrow{A C}=(R \cos \theta+r \cos \psi, R \sin \theta+r \sin \psi) \tag{3.14}
\end{equation*}
$$



Figure 3.1: Diagram showing speed of clubhead
where $\lambda$ is the angle of backswing top, and $r$ is the length of club $B C$.
Since $\lambda$ is the angle of backswing top,

$$
\theta=\frac{3}{2} \pi+\alpha-\lambda, \quad \psi=\alpha+\beta+\pi-\lambda .
$$

The speed of clubhead $\left\|\overrightarrow{A C}^{\prime}\right\|$ is as follows.

$$
\begin{equation*}
\left\|\overrightarrow{A C}^{\prime}\right\|=\sqrt{r^{2}\left(\frac{d \alpha}{d t}+\frac{d \beta}{d t}\right)^{2}+R^{2}\left(\frac{d \alpha}{d t}\right)^{2}+2 r R \frac{d \alpha}{d t}\left(\frac{d \alpha}{d t}+\frac{d \beta}{d t}\right) q} \tag{3.15}
\end{equation*}
$$

where $q=\sin \theta \sin \psi+\cos \theta \cos \psi$.

## Chapter 4

## Existence for Differential Equations

We express which is (3.12) and (3.13) as a system of first order differential equations.

$$
\begin{align*}
& \dot{\alpha}=x,  \tag{4.1}\\
& \dot{\beta}=y,  \tag{4.2}\\
& \dot{x}=\frac{I-I(x+y)^{2} R S \cos \beta-x^{2} R^{2} S^{2} \sin \beta \cos \beta}{I\left(J+M R^{2}\right)-R^{2} S^{2} \sin ^{2} \beta},  \tag{4.3}\\
& \dot{y}=\frac{R S \cos \beta\left\{E(x+y)^{2}+x^{2} F\right\}-E}{I\left(J+M R^{2}\right)-R^{2} S^{2} \sin ^{2} \beta} \tag{4.4}
\end{align*}
$$

with initial conditions

$$
\begin{aligned}
\alpha(0) & =\frac{(N+1) I}{2 R S} \\
\beta(0) & =0 \\
x(0) & =\sqrt{\frac{(N+1) I}{R S\left(J+I+M R^{2}\right)}}, \\
y(0) & =0
\end{aligned}
$$

We now consider existence of a solution for a system of first order ordinary differential equations. In general, a normal system of first order ordinary differential equations is expressed as

$$
\left\{\begin{align*}
\frac{d x_{1}}{d t} & =X_{1}\left(x_{1}, \ldots, x_{n} ; t\right)  \tag{4.5}\\
& \vdots \\
\frac{d x_{n}}{d t} & =X_{n}\left(x_{1}, \ldots, x_{n} ; t\right)
\end{align*}\right.
$$

Let us review some notations and facts on vectors and vector-valued functions. For a vector $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right)$, define $\|\mathbf{x}\|=\sqrt{x_{1}^{2}+\cdots+x_{n}^{2}}$. Then the normal system (4.5) can be written as its vector form as in [6]

$$
\frac{d \mathbf{x}}{d t}=\mathbf{X}(\mathbf{x}, t)
$$

Definition 4.1. A vector-valued function $\mathbf{X}(\mathbf{x}, t)$ is said to satisfy a Lipschitz condition in a region $Q$ in $(\mathrm{x}, \mathrm{t})$-space if, for some constant $L$ (called the Lipschitz constant), we have

$$
\begin{equation*}
\|\mathbf{X}(\mathbf{x}, t)-\mathbf{X}(\mathbf{y}, t)\| \leq L\|\mathbf{x}-\mathbf{y}\| \tag{4.6}
\end{equation*}
$$

whenever $(\mathrm{x}, t) \in Q$ and $(\mathbf{y}, t) \in Q$.

It is well known in [6] that the following Lemma 4.2 and Theoreom 4.3 holds.
Lemma 4.2. If $\mathbf{X}(\mathrm{x}, t)$ has continuous partial derivatives on a bounded closed convex domain $D$, then it satisfies a Lipschitz condition in $Q$.

Theorem 4.3. Assume that $\mathbf{X}(\mathrm{x}, t)$ is continuous and satisfies the Lipschitz condition (4.6) on the interval $|t-a| \leq P$ for all $\mathbf{x}, \mathbf{y}$. Then the initial value problem

$$
\begin{equation*}
\frac{d \mathbf{x}}{d t}=\mathbf{X}(\mathbf{x}, t), \quad \mathbf{X}(\mathbf{x}, 0)=\mathbf{X}_{\mathbf{0}} \tag{4.7}
\end{equation*}
$$

has a unique solution on the interval $|t-a| \leq P$.

Let $\mathbf{x}=(\alpha, \beta, x, y)$ and $\mathbf{X}(\mathbf{x}, z)$ is the vector field. Then (4.1) - (4.4) become

$$
\begin{aligned}
& \frac{d \alpha}{d z}=\mathbf{X}_{\mathbf{1}}(\mathbf{x}, z) \\
& \frac{d \beta}{d z}=\mathbf{X}_{\mathbf{2}}(\mathbf{x}, z), \\
& \frac{d x}{d z}=\mathbf{X}_{\mathbf{3}}(\mathbf{x}, z), \\
& \frac{d y}{d z}=\mathbf{X}_{\mathbf{4}}(\mathbf{x}, z) .
\end{aligned}
$$

It is trivial that $\mathbf{X}_{\mathbf{1}}(\mathbf{x}, z)$ and $\mathbf{X}_{\mathbf{2}}(\mathbf{x}, z)$ have continuous partial derivatives on a bounded closed convex domain. So we just have to show $\mathbf{X}_{\mathbf{3}}(\mathbf{x}, z)$ and $\mathbf{X}_{\mathbf{4}}(\mathbf{x}, z)$ have continuous partial derivatives on a bounded closed convex domain. We obtain

$$
\begin{align*}
& \left\|\frac{\partial \mathbf{X}_{\mathbf{3}}(\mathbf{x}, z)}{\partial \beta}\right\| \\
& =\| \frac{\left\{I(x+y)^{2} R S \sin \beta-x^{2} R^{2} S^{2} \cos 2 \beta\right\}\left\{I\left(J+M R^{2}\right)-R^{2} S^{2} \sin \beta\right\}}{\left\{I\left(J+M R^{2}\right)-R^{2} S^{2} \sin \beta\right\}^{2}} \\
& +\frac{R^{2} S^{2} \cos \beta\left\{I-I(x+y)^{2} R S \cos \beta-x^{2} R^{2} S^{2} \sin \beta \cos \beta\right\}}{\left\{I\left(J+M R^{2}\right)-R^{2} S^{2} \sin \beta\right\}^{2}} \| \tag{4.8}
\end{align*}
$$

Since

$$
I=\sum M_{j} l_{j}^{2}, J=\sum M_{i} l_{i}^{2}, S=\sum M_{j} l_{j}, R=\overline{A B}
$$

and $M$ is the mass of the club,

$$
I\left(J+M R^{2}\right)-R^{2} S^{2} \sin \beta \geq I\left(J+M R^{2}\right)-R^{2} S^{2}>0
$$

Hence in (4.8)

$$
\left\{I\left(J+M R^{2}\right)-R^{2} S^{2} \sin \beta\right\}^{2}>\left\{I\left(J+M R^{2}\right)-R^{2} S^{2}\right\}^{2}>0
$$

Therefore $\mathbf{X}_{\mathbf{3}}(\mathrm{x}, z)$ has continuous partial derivatives on a bounded closed convex domain. Similarly, we can prove $\mathbf{X}_{\mathbf{4}}(\mathbf{x}, z)$ has continuous partial derivatives on a
bounded closed convex domain.
By Lemma 4.2 and Theorem 4.3, the initial value problem

$$
\frac{d \mathbf{x}}{d z}=\mathbf{X}(\mathbf{x}, z)
$$

has a unique solution on the interval $|z| \leq \frac{\sqrt{T_{s}}}{4}$.

## Chapter 5

## Numerical computation

Since the system (4.1) - (4.4) is nonlinear, it is not easy to find the analytical solution. Therefore we may have to apply numerical methods in order to obtain approximate solutions. In this chapter, we are going to apply the fourth-order Runge-Kutta method to solve the system.

Before we compute approximate solutions, we recall the fourth-order Runge-Kutta method applied to

$$
\frac{d y}{d x}=f(x, y), \quad y\left(x_{0}\right)=y_{0}
$$

We first take uniform step length $h=x_{n+1}-x_{n}$. Then we calculate

$$
\begin{aligned}
& k_{1}=f\left(x_{n}, y_{n}\right), \\
& k_{2}=f\left(x_{n}+\frac{h}{2}, y_{n}+\frac{k_{1}}{2}\right), \\
& k_{3}=f\left(x_{n}+\frac{h}{2}, y_{n}+\frac{k_{2}}{2}\right), \\
& k_{4}=f\left(x_{n}+h, y_{n}+k_{3}\right) .
\end{aligned}
$$

Then the fourth-order Runge-Kutta method, we obtain

$$
\begin{equation*}
y_{n+1}=y_{n}+\frac{h}{6}\left(k_{1}+2 k_{2}+2 k_{3}+k_{4}\right) . \tag{5.1}
\end{equation*}
$$

Similiary, we may apply the fourth-order Runge-Kutta method to the system (4.1) (4.4)

$$
\begin{aligned}
\dot{\alpha} & =f(z, \alpha, \beta, x, y), & & \alpha(0)=\alpha_{0}, \\
\dot{\beta} & =g(z, \alpha, \beta, x, y), & & \beta(0)=\beta_{0}, \\
\dot{x} & =i(z, \alpha, \beta, x, y), & & x(0)=x_{0}, \\
\dot{y} & =j(z, \alpha, \beta, x, y), & & y(0)=y_{0},
\end{aligned}
$$

where $h=z_{n+1}-z_{n}$. The steps are followings.

## Calculate

$$
\begin{aligned}
k_{1} & =f\left(z_{n}, \alpha_{n}, \beta_{n}, x_{n}, y_{n}\right) \\
l_{1} & =g\left(z_{n}, \alpha_{n}, \beta_{n}, x_{n}, y_{n}\right) \\
m_{1} & =i\left(z_{n}, \alpha_{n}, \beta_{n}, x_{n}, y_{n}\right) \\
n_{1} & =j\left(z_{n}, \alpha_{n}, \beta_{n}, x_{n}, y_{n}\right) \\
k_{2} & =f\left(z_{n}+\frac{h}{2}, \alpha_{n}+\frac{k_{1}}{2}, \beta_{n}+\frac{l_{1}}{2}, x_{n}+\frac{m_{1}}{2}, y_{n}+\frac{n_{1}}{2}\right) \\
l_{2} & =g\left(z_{n}+\frac{h}{2}, \alpha_{n}+\frac{k_{1}}{2}, \beta_{n}+\frac{l_{1}}{2}, x_{n}+\frac{m_{1}}{2}, y_{n}+\frac{n_{1}}{2}\right) \\
m_{2} & =i\left(z_{n}+\frac{h}{2}, \alpha_{n}+\frac{k_{1}}{2}, \beta_{n}+\frac{l_{1}}{2}, x_{n}+\frac{m_{1}}{2}, y_{n}+\frac{n_{1}}{2}\right) \\
n_{2} & =j\left(z_{n}+\frac{h}{2}, \alpha_{n}+\frac{k_{1}}{2}, \beta_{n}+\frac{l_{1}}{2}, x_{n}+\frac{m_{1}}{2}, y_{n}+\frac{n_{1}}{2}\right) \\
k_{3} & =f\left(z_{n}+\frac{h}{2}, \alpha_{n}+\frac{k_{2}}{2}, \beta_{n}+\frac{l_{2}}{2}, x_{n}+\frac{m_{2}}{2}, y_{n}+\frac{n_{2}}{2}\right), \\
l_{3} & =g\left(z_{n}+\frac{h}{2}, \alpha_{n}+\frac{k_{2}}{2}, \beta_{n}+\frac{l_{2}}{2}, x_{n}+\frac{m_{2}}{2}, y_{n}+\frac{n_{2}}{2}\right) \\
m_{3} & =i\left(z_{n}+\frac{h}{2}, \alpha_{n}+\frac{k_{2}}{2}, \beta_{n}+\frac{l_{2}}{2}, x_{n}+\frac{m_{2}}{2}, y_{n}+\frac{n_{2}}{2}\right), \\
n_{3} & =j\left(z_{n}+\frac{h}{2}, \alpha_{n}+\frac{k_{2}}{2}, \beta_{n}+\frac{l_{2}}{2}, x_{n}+\frac{m_{2}}{2}, y_{n}+\frac{n_{2}}{2}\right) \\
k_{4} & =f\left(z_{n}+h, \alpha_{n}+k_{3}, \beta_{n}+l_{3}, x_{n}+m_{3}, y_{n}+n_{3}\right) \\
l_{4} & =g\left(z_{n}+h, \alpha_{n}+k_{3}, \beta_{n}+l_{3}, x_{n}+m_{3}, y_{n}+n_{3}\right) \\
m_{4} & =i\left(z_{n}+h, \alpha_{n}+k_{3}, \beta_{n}+l_{3}, x_{n}+m_{3}, y_{n}+n_{3}\right) \\
n_{4} & =j\left(z_{n}+h, \alpha_{n}+k_{3}, \beta_{n}+l_{3}, x_{n}+m_{3}, y_{n}+n_{3}\right) \\
& =1
\end{aligned}
$$

And then

$$
\begin{aligned}
\alpha_{n+1} & =\alpha_{n}+\frac{h}{6}\left(k_{1}+2 k_{2}+2 k_{3}+k_{4}\right) \\
\beta_{n+1} & =\beta_{n}+\frac{h}{6}\left(l_{1}+2 l_{2}+2 l_{3}+l_{4}\right) \\
x_{n+1} & =x_{n}+\frac{h}{6}\left(m_{1}+2 m_{2}+2 m_{3}+m_{4}\right), \\
y_{n+1} & =y_{n}+\frac{h}{6}\left(n_{1}+2 n_{2}+2 n_{3}+n_{4}\right) .
\end{aligned}
$$

Since

$$
\frac{d \alpha}{d t}=\sqrt{T_{s}} \frac{d \alpha}{d z} \quad \text { and } \quad \frac{d \beta}{d t}=\sqrt{T_{s}} \frac{d \beta}{d z}
$$

we can obtain the speed as a function of $t$. Figures 5.1-5.3 are the graph of speed as functions of time for suitably choosen parameters $J=0.25$ slugs $\mathrm{ft}^{2}, I=0.146$ slugs $\mathrm{ft}^{2}, M=0.031$ slugs, $R=2 \mathrm{ft}, S=0.059$ slugs $\mathrm{ft}, T_{s}=103 \mathrm{lb} \mathrm{ft}$. These choosen parameters are obtained by the club of a No. 5 iron as in [7][8].

Figure 5.1(a) shows the graphs of angular speeds with $\mathrm{N}=0$ and Figure 5.1(b) shows a stroboscopic diagram when the backswing angle is $146^{\circ}$. In these conditions, the clubhead hits the ball when $\beta=82^{\circ}$.
Figure 5.2(a) shows the graphs of angular speeds with $\mathrm{N}=1$ and Figure 5.2(b) shows a stroboscopic diagram when the backswing angle is $146^{\circ}$. In these conditions, the clubhead hits the ball when $\beta=72^{\circ}$.
If $t_{1}$ is the time at $\alpha=\frac{I}{2 R S}, t_{2}$ is the time at $\alpha=\frac{(N+1) I}{2 R S}$ and $t_{3}$ is the time at impact, then $t_{2}-t_{1}$ is time to delay uncocoking of the wrist. When $N=1$, $t_{2}-t_{1}=0.0328$. Figures 5.1 and 5.2 show that $\dot{\beta}$ increases abruptly for larger hindrance when the wrist is relaxed. On the other hand, $\dot{\alpha}$ is not changed significantly by larger hindrance.


Figure 5.1: angular speeds with $\mathrm{N}=0$ and the stroboscopic diagram .



Figure 5.2: angular speeds with $\mathrm{N}=1$ and the stroboscopic diagram.

The clubheadspeeds of a No. 5 iron are shown in Figure 5.3. If we use the prior calculated values of $\alpha, \beta, \dot{\alpha}, \dot{\beta}$, we can obtain the clubheadspeed by (3.15). When the backswing angle is $146^{\circ}$, the clubhead speed at impact with $N=0$ is 167.38 $\mathrm{ft} / \mathrm{sec}$ and the clubhead speed at impact with $N=1$ is $175.7 \mathrm{ft} / \mathrm{sec}$. When $N=1$, a golfer has to delay uncocking of the wrist for 0.0328 seconds and the increaed speed is roughly $8 \mathrm{ft} / \mathrm{sec}$. However it is not easy for amateur golfers to hit the ball as $\beta=72^{\circ}$. If $N=2$ with same conditions, the clubhead speed at impact is 183.69 $\mathrm{ft} / \mathrm{sec}$. But the clubhead hit the ball when $\beta$ is $46^{\circ}$. So it is impossibe to swing like this way. If $N=0.3$ with same conditions, the clubhead speed at impact is 170.15
$\mathrm{ft} / \mathrm{sec}$. And the clubhead hit the ball when $\beta$ is $80^{\circ}$.


Figure 5.3: speed curve of clubhead.

Now we consider the relationship between the backswing angle and the clubhead speed. When the backswing angle is $166^{\circ}$ with $N=1$, the clubhead speed at impact is $176.66 \mathrm{ft} / \mathrm{sec}$ and the clubhead hit the ball with $\beta=90^{\circ}$. This result means that the increased rate of the clubhead speed is only $1 \mathrm{ft} / \mathrm{sec}$ when the backswing angle is increased by $20^{\circ}$. Thus, we know that the increasing of the backswing angle doesn't help to increase the clubhead speed and to hit the ball with precision.

## Chapter 6

## Maximizing the projectile range of golf

Ignoring air resistance, the horizontal range of the projectile has a maximum distance with a firing angle of $45^{\circ}$. Althogh we ignore air resitance, the angle of launch for maximum horizontal travel is in general less than $45^{\circ}$. That reason is the angle of launch for maximum horizontal travel depend on a height $h$ above the ground[1]. Furthermore, maximizing the projectile range of golf is affected by drag. Using the prior calculated clubhead speed, we can maximize the projectile range of golf with the quadratic drag.

The drag force $D$ is known to be

$$
\begin{equation*}
D=\frac{1}{2} C_{D} A \rho v^{2}, \tag{6.1}
\end{equation*}
$$

where $C_{D}$ is the drag coefficient, $A$ is the cross-sectional area of the ball, $\rho$ is the air density and $v$ is the ball speed. For low speeds in air drag force goes as the square
of the speed[5]. For a slowly moving golf ball we have

$$
\begin{equation*}
c v_{t}^{2}=m g \tag{6.2}
\end{equation*}
$$

where $m$ is weight of a golf ball and $c$ is a quadratic drag coefficient. The equation of motion for the case of quadratic drag can be expressed as

$$
\begin{align*}
m \frac{d v_{x}}{d t} & =-c v^{2} \cos \theta  \tag{6.3}\\
m \frac{d v_{y}}{d t} & =-m g-c v^{2} \sin \theta \tag{6.4}
\end{align*}
$$

Thus (6.3) and (6.4) become

$$
\begin{align*}
\frac{d v_{x}}{d t} & =-\frac{c}{m} \sqrt{v_{x}^{2}+v_{y}^{2}} v_{x}  \tag{6.5}\\
\frac{d v_{y}}{d t} & =-g-\frac{c}{m} \sqrt{v_{x}^{2}+v_{y}^{2}} v_{y} . \tag{6.6}
\end{align*}
$$

Williams[12] has found that the drag force was $0.000783 \mathrm{v} l \mathrm{l}$. The weight of a golf ball is 1.62 oz and a linear drag coefficient $c_{1}$ is $0.000783 \mathrm{lb} /(\mathrm{ft} / \mathrm{s})$. Hence

$$
\begin{equation*}
\frac{c_{1}}{m}=\frac{0.000783}{(1.62 / 16) / 32}=0.25 \mathrm{~s}^{-1} \tag{6.7}
\end{equation*}
$$

In the case of a nonspinning golf ball with linear air resistance, the termimal velocity $v_{t}$ is

$$
\begin{equation*}
v_{t}=\frac{m g}{c_{1}}=\frac{32}{0.25}=128 \mathrm{ft} / \mathrm{s} \tag{6.8}
\end{equation*}
$$

To get a hypothetical quadratic drag coefficient we can use the terminal velocity of $128 \mathrm{ft} / \mathrm{s}$ in (6.8), although the well-driven golf ball faces linear drag[5]. In this case, (6.2) becomes

$$
\begin{equation*}
\frac{c}{m}=\frac{g}{v_{t}^{2}}=\frac{32}{128^{2}}=\frac{1}{512} f t^{-1} \tag{6.9}
\end{equation*}
$$

The ball speed $v_{0}$ immediately after impact is

$$
\begin{equation*}
v_{0}=(1+e) \frac{M}{M+m} v_{c}, \tag{6.10}
\end{equation*}
$$

where $v_{c}$ is the clubhead speed at impact, $M$ is the weight of the clubhead and $e$ is a restitution coefficient .

Given $v_{0}$, the clubface and the speed of ball are shown in Figure 6.1.


Figure 6.1: the clubface and velocities of the ball

It follows from (6.10) that

$$
\begin{align*}
& v_{x_{0}}=(1+e) \frac{M}{M+m} v_{c} \cos ^{2} \theta_{0},  \tag{6.11}\\
& v_{y_{0}}=(1+e) \frac{M}{M+m} v_{c} \cos \theta_{0} \sin \theta_{0}, \tag{6.12}
\end{align*}
$$

where $\theta_{0}$ is projection angle from a height $h$ above the ground. Figure 6.2 shows the projectile trajectory launched with initial ball speed $v_{0} \cos \theta_{0}$ and projection angle $\theta_{0}$ from a height $h$ above the ground. Since the restitution coefficient of the golf ball is 0.83 , we may choose $e=0.83$.


Figure 6.2: Projectile trajectory

In Chapter 4, when the backswing angle is $146^{\circ}$ and $N=1$, we obtained velocity clubhead speed $v_{c}=170.15 \mathrm{ft} / \mathrm{s}$ at impact with $\beta=80^{\circ}$.
Thus, by Ruge-Kutta method we can obtain maximum projecitle range with quadratic drag. Figure 6.3 shows a plot of coordinates for projection angle $\theta_{0}$ from $5^{\circ}$ to $50^{\circ}$. We may see that maximum range for angle is obtained roughly $30^{\circ}$. Also, the loft angle of the No. 5 iron that we actually use in general is roughly $28^{\circ}$.


Figure 6.3: Trajectories for various projetion angles with quadratic drag

## Chapter 7

## Concluding Remarks

Computaional results in Chapter 5 show that the wrist action of a golfer affect clubhead speed in the moment of impact. Jorgensen[7] observed that any torque acting to assist the uncocking of the wrist at the beginning of the downswing is not helpful to increase the clubhead speed. From $t=t_{1}$ to $t=t_{2}$, the negative torque of club works by wrist to maintain the original wrist-cock angle $\beta(0)$. In this thesis, we showed that the negative torque of the club help the clubhead speed increase. For a larger hindrance parameter, the clubhead speed increases. But the higher the hindrance parameters, it is more difficult for amateur golfers to hit the ball with precision. According to calculated values of the clubhead speed in Chapter 6, it is necessary to take $\mathrm{N}=0.3$.

Springs and Neal[9] showed that a properly timed wrist torque applied by the golfer to the club's handle can increase clubhead speed. Their simulation results show that muscular wrist torque is desirable for maximizing clubhead speed at impact. Since the golfer hinders the uncocking of the wrist by applying a negative torque during the swing, he can't produce as large a clubhead speed. However, the club hit the
ball freely since the action of the golfer's wrists is entirely passive after the torque of the club decrease through zero[8].

So there have been used to predict the effects of applying a positive negative wrist torque during the golf swing[2,7,8,9,10].

We found that the projecitle range of golf ball with quadratic drag has maximum for a firing angle of roughly $30^{\circ}$ in Chapter 6.

## Bibliography

[1] R. A. Brown, Maximizing of a projectile, The Physics Teacher 30(1992), 344347.
[2] R. Cross, A double pendulum swing experiment: In search of a better bat Am. J. Phys. 73(2005), 330-339.
[3] C. B. Daish, The physics of ball games, English University Press, London(1972)
[4] J. C. Derksen, A new method for continuous recording of trunk postures while playing golf, J. Appl. Biomech. 12(1996), 116-129.
[5] H. Erlichson, Maximum projectile range wth drag and lift with particular application to golf, Am. J. Phys. 51(1983), 357-362.
[6] J. Hu and W. Li, Theory of ordinary differential equations, HKUST(2005), 1-14.
[7] T. Jorgensen, On the dynamics of the swing of a golf swing, Am. J. Phys. 38(1970), 644-651.
[8] T. Jorgensen, The physics of golf, Springer-Verlag(1999), 2nd ed.
[9] E. J. Springs and R. J. Neal, An insight into the importance of wrist torque in driving the golf ball: A simulation study, J. Appl. Biomech. 16(2000), 356366.
[10] R. White, On the efficiency of the golf swing, Am. J. Phys. 74(2006), 10881094.
[11] D. Williams, The dynamics of the golf swing, Quart. J. Mech. Appl. Math. 20(1966), 247-264.
[12] D. Williams, Drag force on a golf ball in fight and its practical significance, Quart. J. Mech. Appl. Math. 12(1959), 387-392.

## 국문초록

이 논문에서는 라그링지안 함수와 이중진자 모델을 이용하여 골프의 다운스윙과정 을 수학적으로 해석하려고 한다. 팔과 클럽의 미분방정식 시스템으로 표현된 해의 존재성을 보일 것이다. 클럽해드의 속력을 구하기 위해 룬지쿠타 방법으로 팔과 클 럽의 각속도를 구한다. 계산된 클럽해드의 속력을 이용해 최대가 되게 하는 발사체 의 범위를 구할 수 있다.

주요어휘: 이중진자 모델, 라그랑지안 함수, 미분방정식의 해의 존재성, 룬지쿠타 방법, 이차항력.

학번: 2012-21422

