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Characterizing non-classical correlations between multiple systems from two points of view: resource cost and additivity

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Characterizing non-classical correlations between multiple systems from two points of view: resource cost and additivity

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Abstract

Non-classical correlations between multiple systems lie at the heart of notable features of quantum systems, especially in application to quantum information processing. However, their structures and properties are not yet well understood because of the complexity of multipartite quantum systems.

In this thesis, we look into non-classical correlations between multiple systems from two points of view, with the aim to deepen the understanding of multipartite non-classical correlation.

In the first point of view, multipartite nonclassical correlations are considered as a resource, which cannot be generated by a restricted set of operations, local operations and classical communication (LOCC). Then, we ask how much amount of multipartite entanglement is required to prepare a given multipartite state by LOCC. In other words, we define the Greenberger-Horne-Zeilinger (GHZ) entanglement cost as a generalization of the entanglement cost to the multipartite setting. We provide a LOCC procedure for preparing an arbitrary pure multipartite state from GHZ states, and show that the conversion rate of this procedure is given by multipartite quantum discord of the state to be prepared, which means that multipartite quantum discord is an upper bound for the GHZ entanglement cost.

In the second point of view, bipartite correlations of pairs of subsystems are summed to yield the multipartite correlation of the total system. In this sense, we seek to find general relations between multipartite correlations and bipartite correlations of pairs of subsystems. We provide a general concept of the additivity relations, which is derived in terms of

total correlation, and examine them for entanglement and discord.

keywords: entanglement, discord, nonclassical correlation, multipartite system

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Chapter 1

Introduction

The development of quantum mechanics opened a new perspective on the notion of information, and it led to the emergence of quantum information science. Its applications in computation, cryptography, metrology, and several fields showed significant advantages over the classical case. It was shown that Shor's algorithm [1] can find the prime factors of an integer in polynomial time on a quantum computer, whereas the best known classical algorithm takes exponential time. In cryptography, unconditionally secure communication was shown to be possible by using quantum key distribution protocols [2]. It is also known that employing quantum systems in metrology can give better precision than the same measurement in the classical framework [3].

What makes quantum information processing powerful? The answer is often found in *nonclassical correlation*. We mean by the nonclassical correlation the unique types of correlations between quantum systems that cannot be accounted for by any classical description. The nonclassical correlation is also where notable features of quantum systems comes out as noted in the famous Einstien-Podolsky-Roseny (EPR) thought

experiment [4]. For these reasons, understanding the non-classical correlation has been one of the most important issues in quantum information theory.

A famous kind of non-classical correlations is entanglement. The non-classical property of entanglement was early recognized in the EPR thought experiment in 1935. Later, the role of entanglement in quantum information processing was revealed with discoveries of quantum teleportation [5], quantum dense coding [6], and quantum cryptography [7]. For instance, one entangled bit is required to transfer the information of one qubit in the quantum teleportation protocol. Entanglement is now generally considered a key resource for quantum information processing [3, 5, 6, 8]. In particular, it has been shown that entanglement is a necessary condition for providing exponential speed-up in quantum computation [8].

The primary step in the study of entanglement is to quantify the amount of entanglement in an arbitrary state. A prominent way of the entanglement quantification is based on the property that entanglement cannot be generated by local operations and classical communication (LOCC). In the light of this, one may take the single state as the standard unit to measure the amount of entanglement in an arbitrary state. More precisely, the entanglement cost of a state is defined as the number of singlets that is required to prepare a single copy of the state by LOCC. Similarly, the distillable entanglement of a state is the number of singlets that can be distilled from a copy of the state. On the other hand, the axiomatic approach has also provided entanglement measures, such as the relative entropy of entanglement, the geometric entanglement, etc.

On the other hand, in the early 2000's, the information-theoretic approach to nonclassical correlations led to the introduction of a nonclassi-

cal correlation, quantum discord, whose concept is beyond entanglement. The notion of quantum discord was first introduced in 2001 by H. Ollivier and W. H. Zurek [9] and independently by L. Henderson and V. Vedral [10]. Since then, several other versions of quantum discord have been proposed, including the relative entropy of quantum discord and the geometric discord. Quantum discord initially received a greate attention in mixed-state computation. It has been shown that there is a nontrivial speedup over classical in deterministic quantum computation with one qubit (DQC1) with zero entanglement but with nonzero discord [11]. For this reason, quantum discord was proposed as a figure of merit in DQC1 model. Recent studies of discord has been focused on finding its roles and operational meanings in some information tasks. Several operational meanings have been found in quantum state merging [12, 13], locking of classical correlations [14], super dense coding [15], quantum key distribution [16], and encoding information by coherent interactions [17].

Nonclassical correlation is ultimately required to be studied in the multipartite setting. Nonclassical correlation between multiple systems has a more complex structure and its own unique property. There exist different kinds of multipartite entanglement when classified by the convertibleness under stochastic local operations and classical communication (SLOCC). There also exist genuine multipartite entangelmet [18] and the monogamy of entanglemetn [19] which have no counterpart in the case of classical correlation. However, analyzing multipartite states is difficult in general due to the absence of Schmidt decomposition. GHZ states enable nonlocality test without inequalities.

This thesis is based on my two Phd studies on non-classical correlations between multiple systems:

1. Seungho Yang and Hyunseok Jeong, Physical Review A **92**, 022322

(2015).

2. Seungho Yang, Hyunseok Jeong, and Wonmin Son, Physical Review A **87**, 052114 (2013).

In general, this thesis aims to deepen the understanding of non-classical correlations between multiple systems. In the first part, we try to quantify multipartite entanglement from the resoruce-theroy point of view as in the entanglement cost of bipartite systesms. Hence, this can be seen as a generalization of the entanglement cost to the multipartite case. We also find a connection of the multipartite entanglement cost to multipartite discord.

In the second part, we look into the relation between the multipartite correlation of of the total system and bipartite correlstions of subsystems. We specify four additivity relations that give an instight to which extent the multipartite correlation contains bipartite correlations of subsystems. We examine whether such relations are generally satisfied for different measures of nonclassical correltions, entanglement and discord.

Chapter 2

Background

2.1 Classical information

Information theory was developed by Claude E. Shannon in 1948 with the aim to quantify information and find fundamental limits on information-processing tasks, i.e., data compression and reliable communication. The theory of information is closely related to the description of physical states as information must be stored in a physical system. Therefore, classical information theory is based on the classical description as in the two states of a bit. In this section, we briefly introduce the core results of classical information theory as a preliminary step and consider the theory of information within the framework of quantum physics in the next section.

2.1.1 Shannon entropy and mutual information

Information theory begins with defining entropy as the expectation value of information contained in each message. Let us consider a random variable X, which can have one of the values from 1 to N with probability

distribution $p(x)$. The Shannon entropy of random variable X is defined as

$$\mathcal{H}(X) = -\sum_{x=1}^{N} p(x) \log_2 p(x). \tag{2.1}$$

The Shannon entropy has the maximum value of $\log_2 N$ when the probability distribution is uniform.

The relative entropy quantifies the difference between two probability distributions $p(x)$ and $q(x)$. It is given as

$$D(p(x)||q(x)) = \sum -p(x) \left(\log_2 q(x) + \log_2 \frac{p(x)}{q(x)} \right). \tag{2.2}$$

For two random variables X and Y representing channel's input and output, the correlation between X and Y is given by the mutual information as

$$\mathcal{H}(X : Y) = \mathcal{H}(X) + \mathcal{H}(Y) - \mathcal{H}(X, Y). \tag{2.3}$$

In terms of the relative entropy, the mutual information is written as $H(X : Y) = D(p(x, y)||p(x)p(y))$.

We introduce some useful inequalities involving entropic quantities. The Shannon entropy is subadditive, which means that $\mathcal{H}(X) + \mathcal{H}(Y) \geqslant \mathcal{H}(X, Y)$. From the subadditivity, we see that $\mathcal{H}(X|Y) = \mathcal{H}(X, Y) - \mathcal{H}(Y) \geqslant 0$ and $\mathcal{H}(X : Y) \leqslant \mathcal{H}(X)$. A stronger version of the subadditivity is the strong subadditivity, which states that

$$\mathcal{H}(X, Y) + \mathcal{H}(Y, Z) \geqslant \mathcal{H}(X, Y, Z) + \mathcal{H}(Y). \tag{2.4}$$

In terms of the mutual information, the strong subadditivity can be rewritten as

$$\mathcal{H}(X : YZ) \geqslant \mathcal{H}(X : Y). \tag{2.5}$$

2.1.2 Classical data compression

The Shannon entropy represents the information capacity of random variable X. The information stored in X can be compressed into $\mathcal{H}(X) + \epsilon$ bits with a negligible loss of information; in other words, X can be retrieved from the $\mathcal{H}(X) + \epsilon$ bits with a negligible error. Theoretical background of the compression is provided by what is called the asymptotic equipartition propertiest (AEP). For convenience, we just introduce and use the AEP without proof. Let us consider a sequence of random variables, $X_1, X_2, \ldots, X_n$, which are independent and identically distributed (i.i.d.) according to $p(x)$. We denote the sequence by X^n. The typical set is then defined as follows.

Definition 2.1. (typicality) For an i.i.d. $X_1, X_2, \ldots, X_n$ with $p(x)$, the typical set is the set of sequences $(x_1, x_2, \ldots, x_n) \in \chi^n$ that satisfies

$$2^{-n(\mathcal{H}(X)+\epsilon)} \leqslant p(x_1, x_2, \ldots, x_n) \leqslant 2^{-n(\mathcal{H}(X)-\epsilon)} \tag{2.6}$$

The AEP states that, for a typical set A_ϵ, it holds that

1. $\left|-\frac{1}{n}\log_2 p(x_1, x_2, \ldots, x_n) - \mathcal{H}(x)\right| \leqslant \epsilon$ for $(x_1, x_2, \ldots, x_n) \in A_\epsilon^{(n)}$.
2. $\Pr[(x_1, x_2, \ldots, x_n) \in A_\epsilon^{(n)}] > 1 - \epsilon$ for sufficiently large n.
3. $|A_\epsilon^{(n)}| \leqslant 2^{n(\mathcal{H}(X)+\epsilon)}$, where $|A_\epsilon^{(n)}|$ denotes the size of the typical set $A_\epsilon^{(n)}$.
4. $|A_\epsilon^{(n)}| \geqslant (1-\epsilon)2^{n(\mathcal{H}(X)+\epsilon)}$ for sufficiently large n.

The second property implies that the typical set contains the most of probability, and the third property implies that the size of the typical set is about $\mathcal{H}(X)$. Collectively, it means that X^n can be losslessly compressed into about $n\mathcal{H}(X)$ bits.

The compression described so far corresponds to the noiseless-channel coding, where the encoded bits are sent over a noiseless channel. In the case of noisy channel, the mutual information between input and output bits of the channel plays an important role. Let us consider a channel with a conditional probability distribution $p(y|x)$ where x and y denotes inputs and outputs of the channel, respectively. The famous Shannon's noisy-channel coding theorem states that the mutual information (when maximized over $p(x)$) represents the channel capacity, that is, the maximum amount of information that can be transmitted over a given channel.

2.2 Quantum information

The way of describing the physical states of quantum systems is fundamentally different from that of classical systems, so the concept of quantum information should be newly introduced to employ the quantum systems in information processing. In this section, we introduce the definition of quantum entropy and explain how quantum information can be compressed as in the classical case.

2.2.1 Quantum entropy and correlation

The von Neunmann entropy of a density operator ρ is given by the formula

$$\mathcal{S}(\rho) = -\operatorname{Tr}(\rho \log_2 \rho). \tag{2.7}$$

When the density operator ρ has eigenvalues λ_x, the von Neumann entropy is reduced to

$$\mathcal{S}(\rho) = -\sum_x \lambda_x \log_2 \lambda_x, \tag{2.8}$$

from which we see that the von Nemann entropy of a system is at most $\log_2 d$, where d is the dimension of the system.

The quantum relative entropy between two quantum states ρ and σ is defined as

$$S(\rho||\sigma) = -\operatorname{Tr}(\rho \log_2 \sigma + \rho \log_2 \rho). \tag{2.9}$$

As in the classical case, the quantum relative entropy is always nonnegative and is equal to zero if and only if $\rho = \sigma$.

For a state ρ_{AB} of a bipartite system AB, we denote by ρ_B the reduced density operator for the system B. Also, we use the following notation for the partial trace,

$$\operatorname{Tr}_A(\rho_{AB}) = \sum_{x_A} \langle x_A|\rho_{AB}|x_A\rangle = \rho_B, \tag{2.10}$$

where $\{x_A\}$ is a basis for the system A.

Quantum mutual information is defined as

$$\mathcal{I}(\rho_{AB}) = \mathcal{S}(\rho_A) + \mathcal{S}(\rho_B) - \mathcal{S}(\rho_{AB}), \tag{2.11}$$

and in terms of the relative entropy, it is given as $S(\rho_{AB}||\rho_A \otimes \rho_B)$.

The Ulhman fidelity [20] is a measure of the closeness of two quantum states. For two quantum states ρ and σ the fidlity is defined as

$$F(\rho, \sigma) = \operatorname{Tr} \sqrt{\rho^{\frac{1}{2}} \sigma \rho^{\frac{1}{2}}}. \tag{2.12}$$

One useful property of the fidelity that it is concave in both arguments,

$$F(\sum_i p_i \rho_i, \sum_i q_i \sigma_i) \geqslant \sum_i \sqrt{p_i q_i} F(\rho_i, \sigma_i). \tag{2.13}$$

2.2.2 Quantum data compression

As discussed in Section 2.1.1, the Shannon entropy represents the capacity of classical information. We now show that the von Neuman entropy has the same role for quantum information.

Suppose that a quantum state ρ has an orthonormal decomposition in a basis $\{|x_i\rangle\}$,

$$\rho = \sum_x p(x)|x\rangle\langle x|. \tag{2.14}$$

The state of m identical copies is given as,

$$\rho^{\otimes m} = \sum_{x_1,x_2,\ldots,x_m} p(x_1)p(x_2)\ldots p(x_m)|x_1,x_2,\ldots,x_m\rangle\langle x_1,x_2,\ldots,x_m|. \tag{2.15}$$

As in the classical case, a typical set $\mathcal{A}_\epsilon$ can be defined to be the set of sequences $x_1, x_2, \ldots, x_m$ that satisfy

$$\left| -\frac{1}{n}\log_2\left(p(x_1)p(x_2)\ldots p(x_n)\right) - S(\rho)\right| \leqslant \epsilon. \tag{2.16}$$

In terms of the typical sequences, a typical subspace is defined as

$$\Pi(m,\epsilon) = \sum_{(x_1,x_2,\ldots,x_m)\in\mathcal{A}_\epsilon} \left| -\frac{1}{n}\log_2\left(p(x_1)p(x_2)\ldots p(x_m)\right) - S(\rho)\right| \leqslant \epsilon. \tag{2.17}$$

The quantum version of AEP states that, for a typical subspace $\Pi(m,\epsilon)$, it holds that

1. For any $\epsilon > 0$ and $\delta > 0$, it holds that $\mathrm{Tr}(\Pi(n,\epsilon)\rho^{\otimes m}) \geqslant 1-\delta$ for sufficiently large m.

2. For any $\epsilon > 0$ and $\delta > 0$, the dimension of the typical subspace $V_T(m,\epsilon)$ is bounded as $(1-\delta)2^{m(\mathcal{S}(\rho)-\epsilon)} \leqslant |V_T(m,\epsilon)| \leqslant 2^{m(\mathcal{S}(\rho)+\epsilon)}$

for sufficiently large m.

3. Assume that $P(m)$ is a projector onto any subspace of $\mathscr{H}^{\otimes m}$ and its dimension $|P(m)|$ is at most 2^{mR}, where $R < \mathcal{S}(\rho)$ is fixed. Then for any $\delta > 0$, it holds that $\mathrm{Tr}(S(m)\rho^{\otimes m}) \leqslant \delta$ for sufficiently large m.

The compression is done by a quantum operation that transforms $\rho^{\otimes m}$ into

$$\Lambda(\rho^{\otimes m}) = \Pi(m,\epsilon)\rho^{\otimes m}\Pi(m,\epsilon) + p\ |\bar{0}\rangle\langle\bar{0}|, \tag{2.18}$$

where $p = 1 - \mathrm{Tr}(\Pi(m,\epsilon)\rho)$ and $|\bar{0}\rangle$ is a state chosen from the typical subspace $V_T(m,\epsilon)$ (it is not important what the state is).

The decompression is simply done by the identity operation on $V_T(m,\epsilon)$. The fidelity between $\rho^{\otimes m}$ and the decompressed state $\Lambda(\rho^{\otimes m})$ is

$$\begin{aligned} F(\rho^{\otimes m}, \Lambda(\rho^{\otimes m})) &\geqslant F(\rho^{\otimes m}, (1-p)^{-1}\ \Pi(m,\epsilon)\rho^{\otimes m}) \\ &= \sqrt{1-p} \\ &\geqslant \sqrt{1-\delta}. \end{aligned} \tag{2.19}$$

for any nonzero δ. The first inequality follows from the concavity of the fidelity, and we used the AEP for the third inequality. In addition, we see that the dimension of the support of the compressed state $\Lambda(\rho^{\otimes m})$ is at most $2^{mS((\rho)+\epsilon)}$ from the AEP, so the quantum information in $\rho^{\otimes m}$ can be compressed into $m(S(\rho)+\epsilon)$ qubits with a negligible error.

2.3 Quantum entanglement

The term "entanglement" was coined by Schrödinger in 1935 after observing a peculiar correlation between two quantum systems. For a $2 \otimes 2$-

dimensional bipartite system, a typical example of an entangled state is the singlet state,

$$|\Phi\rangle = \sqrt{1/2}(|01\rangle - |10\rangle).$$

The state of each subsystem is the completely mixed state, $1/2(|0\rangle\langle 0| + |1\rangle\langle 1|)$, which has the maximum uncertainty. When we project one system into a state by a local measurement, the other system is also projected into the same state after the measurement; it applies to any setting of the measurement, which cannot be seen between classical objects. For this reason, we say that this kind of correlations is non-classical.

More generally, we say that a bipartite pure state ψ_{AB} is entangled if it cannot be written as a product of the states of subsystems, $|\psi\rangle_{AB} = |\phi\rangle_A \otimes |\chi\rangle_B$, for which the non-classical aspect appears. For mixed states, classical mixtures of product states are considered to contain no entanglement because classical mixing generally generate only classical correlation. Formally, a bipartite state is separable if it can be written in the form of

$$\rho_{AB} = \sum_j p_j \rho_A^j \otimes \rho_B^j. \tag{2.20}$$

2.3.1 Entanglement from the resource point of view

Here, we introduce a meaningful and important way of entanglement quantification that can make the entanglement an operational quantity but not just a mathematically defined concept. This approach is based on the idea that entanglement is a resource shared between distant parties that cannot be generated by a restriced set of operation, local operations and classical communication (LOCC). More specifically, a scale of entanglement is created by considering asymptotic conversion (i.e., conversion

of m copies of a state into n copies of another state for large n and m) under LOCC. Entanglement is then quantified as the optimal rate of the asymptotic conversion, m/n, from a resource state to the given state (or vice versa). Bennett *et. al.* showed that, in the bipartite case, the asymptotic conversion between any pure states is reversible [21], so when quantifying entanglement of bipartite systems, any choice of a pure entangled state as a resource state results in an equivalent quantification. The singlet state is a natural choice for a resource state, and the corresponding conversion rate is called the entanglement cost (and it is called the distillable entanglement for the reverse direction) [18, 21–25]. The entanglement cost of preparing a bipartite pure state has been shown to equal the entropy of entanglement or, equivalently, the von Neumann entropy of one of the subsystems [21]. In what follows, we introduce the LOCC and then explain how entanglement is quantified from the resource-theory point of view.

Local operations and classical communication

Under the LOCC restriction, each party can apply any operation (including measurement) on their own systems and classically communicate the results to each other. For the quantum operations, we use the operator-sum representation as below.

Definition 2.2. A quantum operation Λ with operation elements $\{M_i\}_{i=1}^N$ is defined as

$$\rho \to \Lambda(\rho) = \sum_{i=1}^{N} M_i \rho M_i^\dagger \tag{2.21}$$

Let us assume a bipartite system AB. Then, an LOCC procedure

begins with a local operation by one party, say, Alice, on system A;

$$\begin{aligned}\rho \to (\Lambda \otimes \mathbb{1})(\rho) &= \sum_{i=1}^{N} \mathrm{Tr}((M_i \otimes \mathbb{1})\rho(M_i^\dagger \otimes \mathbb{1}))\frac{(M_i \otimes \mathbb{1})\rho(M_i^\dagger \otimes \mathbb{1})}{\mathrm{Tr}((M_i \otimes \mathbb{1})\rho(M_i^\dagger \otimes \mathbb{1}))} \\ &= \sum_{i=1}^{N} p_i \sigma_i.\end{aligned}$$

where we defined $p_i = \mathrm{Tr}((M_i \otimes \mathbb{1})\rho(M_i^\dagger \otimes \mathbb{1}))$ and $\sigma_i = p_i^{-1}(M_i \otimes \mathbb{1})\rho(M_i^\dagger \otimes \mathbb{1})$. Next, Alice sends the result i to Bob, and Bob accordingly apply a local quantum operation Γ_i on his system B;

$$\sum_{i=1}^{N} p_i \sigma_i \to \sum_{i=1}^{N} p_i (\mathbb{1} \otimes \Gamma_i)(\sigma_i).$$

Under the LOCC restriction, one can only convert a product state $|0\rangle_A \otimes |0\rangle_B$ to a separable state. Conversely, any separable state can be prepared by LOCC. Hence, the set of separable states is the criteria of zero entanglement.

A similar but different class of operations to LOCC is the set of sparable operations, which is defined as

$$\rho_{AB} \to \sum_i (M_i \otimes N_i)\rho_{AB}(M_i^\dagger \otimes N_i^\dagger) \tag{2.22}$$

with $\sum_i M_i^\dagger M_i = \mathbb{1}_A$ and $\sum_i N_i^\dagger N_i = \mathbb{1}_B$. It can be easily checked that any LOCC operation is a separable operation. However, the converse does not hold. A consequence of the gap between separable operations and LOCC is what is called "nonlocality without entanglement" [26]. Unfortunately, there is no known mathematically simple representation of LOCC at the time of writing, which is why the analysis of LOCC is difficult in general.

Quantum teleportation

Quantum teleporation protocol exploits entanglement to send an unknown quantum state to a distant location. Assume two distant parties, Alice and Bob, who share one singlet. Alice wants to teleport to Bob's location the state of her qubit,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{2.23}$$

with $|\alpha|^2 + |\beta|^2 = 1$.

The quantum circuit for quantum teleportaion is shown in Fig. 2.3.1. Each half of the singlet is in possession of Alice and Bob (which are labeled "A" and "B", respectively), so the initial state is

$$|\psi\rangle_A \otimes |\Phi\rangle_{A'B'} = \frac{1}{\sqrt{2}}\left[\alpha|0\rangle_A(|01\rangle - |10\rangle)_{A'B} + \beta|1\rangle_A(|01\rangle - |10\rangle)_{A'B}\right]. \tag{2.24}$$

Two qubits in Alice's location (i.e., the qubit to teleport and half of the singlet) enter into a C-NOT gate, which changes the state into

$$\frac{1}{\sqrt{2}}\left[\alpha|0\rangle_A(|01\rangle - |10\rangle)_{A'B} + \beta|1\rangle(|11\rangle - |00\rangle)_{A'B}\right]. \tag{2.25}$$

Next the Hadamard operation is applied on A, which results in the state

$$\frac{\alpha}{2}(|0\rangle + |1\rangle)_A(|01\rangle - |10\rangle)_{A'B} + \frac{\beta}{2}(|0\rangle - |1\rangle)_A(|11\rangle - |00\rangle)_{A'B}. \tag{2.26}$$

For the sake of clarity, we regroup the parties as

$$\begin{aligned}&\frac{1}{2}|00\rangle_{AA'}(-\beta|0\rangle + \alpha|1\rangle)_B + \frac{1}{2}|01\rangle_{AA'}(-\alpha|0\rangle + \beta|1\rangle)_B \\ &+\frac{1}{2}|10\rangle_{AA'}(\beta|0\rangle + \alpha|1\rangle)_B + \frac{1}{2}|11\rangle_{AA'}(-\alpha|0\rangle - \beta|1\rangle)_B.\end{aligned} \tag{2.27}$$

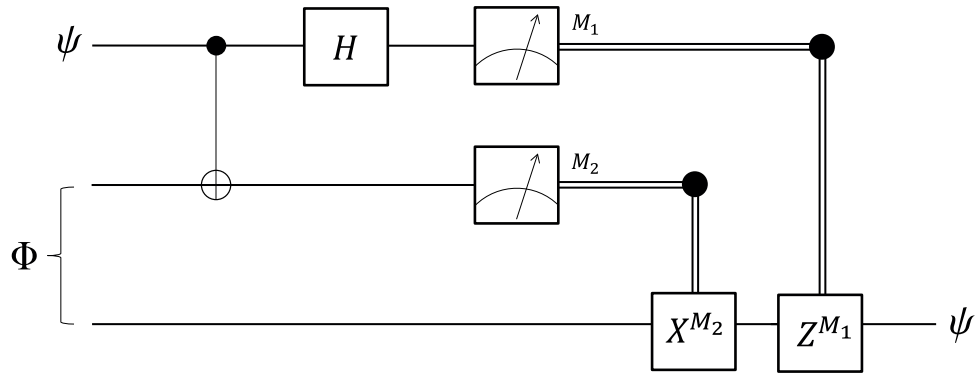


Figure 2.1: Quantum circuit for teleportation, with the first line being the qubit to be sent, the second and third lines each being half of the singlet in possession of Alice and Bob, respectively.

A measurement on AA' in the computational basis is followed, and the measurement outcome is sent to Bob via a classical channel. We see that the post-measurement state of B for the outcome 11 is equal to ψ (up to the global phase), but the post-measurement states for the other outcomes have some errors. Bob can correct the errors by applying X and Z operations, depending the measurement outcomes as

$$\begin{aligned} 00 &\to ZX, \\ 01 &\to Z, \\ 10 &\to X, \\ 11 &\to I. \end{aligned} \tag{2.28}$$

It finally creates the state $\alpha|0\rangle + \beta|1\rangle$ at Bob's location.

We note that the singlet is destroyed on completion of the protocol; in other word, quantum teleportation consumes one singlet to send the information of a qubit. This implies that entanglement is a resource for quantum communication.

Entanglement cost and distillable entanglement

We address one prominent way of quantifying entanglement, based on the resource theory. Here, entanglement is elevated to a valuable resource if possible operations are restricted to LOCC. In other words, one cannot generate entanglement by using only LOCC. Based on this, the scale of entanglement can be created as follows. We take the singlet as the standard unit of entanglement. Then, the entanglement cost of a state ψ is defined as the minimum number of singles needed to prepare a single copy of ψ (in average) by LOCC. Equivalently, the entanglement cost is the optimal rate of the LOCC conversion

$$\Phi^{\otimes nR} \xleftrightarrow{LOCC} \rho^{\otimes n}. \tag{2.29}$$

Formal definition of the entanglement cost is

$$E_c = \inf \Big\{ R : \exists \mathcal{L} : \lim_{n\to\infty} F\big(\mathcal{L}(\Phi^{\otimes nR}), \rho^{\otimes n}\big) \to 1 \Big\} \tag{2.30}$$

Conversely, the distillable entanglement of a bipartite state ρ is defined as the optimal rate of the conversion

$$E_d = \inf \Big\{ R : \exists \mathcal{L} : \lim_{n\to\infty} F\big(\mathcal{L}(\rho^{\otimes n}), \Phi^{\otimes nR})\big) \to 1 \Big\} \tag{2.31}$$

It is evident from the definition that the entanglement cost and the distillable entanglement are non-increasing under LOCC and $E_c(\Phi) = 1$.

The entanglement cost is always larger than or equal to the distillable entanglement. This follows from the fact that LOCC transformation cannot generate more singlets. Let us say that one converts nE_c singlets into n copies of a state ρ, and convert it again into nE_d singlets. They should finally end up with a fewer singlets, so we have $E_d(\rho) \leqslant E_c(\rho)$.

A bipartite pure state in an $n \otimes n$-dimensional Hilbert space is represented in a basis $\{|i\rangle \otimes |j\rangle\}$ as

$$|\psi\rangle = \sum_{i,j=0}^{d-1} a_{ij}|i\rangle \otimes |j\rangle. \tag{2.32}$$

Now, let us consider the entanglement cost and the distillable entanglement of the pure state ψ. We consider an $l \times l$ matrix A with the elements $(A)_{i,j} = a_{ij}$. Using the singular value decomposition, we have $A = UDV$ with a diagonal matrix D and unitary matrices U and V. Two unitary operators U and V can be correspondingly defined as $U|i\rangle = \sum_k (U)_{k,i}|k\rangle$ and $V|i\rangle = \sum_l (V)_{l,i}|l\rangle$. Then, we see that the state ψ has a simple representation in the basis $\{(U|i\rangle) \otimes (V|j\rangle)\}$,

$$|\psi\rangle = \sum_{j=0}^{d-1} d_j \, (U|j\rangle) \otimes (V|j\rangle) = (U \otimes V) \sum_{j=0}^{d-1} d_j |j\rangle \otimes V|j\rangle, \tag{2.33}$$

which is the so-called Schmidt basis.

Each of the subsystems has the same amount of von Neumann entropy,

$$S(\rho_A) = S(\rho_B) = -\sum_j |d_j|^2 \log_2 |d_j|^2. \tag{2.34}$$

Let us define $|\psi'\rangle = (U^\dagger \otimes V^\dagger)|\psi\rangle$. The m-copy of ψ is then written as

$$|\psi'^{\otimes m}\rangle = \sum_{j_1,\ldots,j_m=0}^{d-1} d_{j_1,\ldots,j_m} |j_1, \ldots, j_m\rangle \otimes |j_1, \ldots, j_m\rangle. \tag{2.35}$$

We now apply the AEP to the sequences $j_1, \ldots, j_m$. They are independent and identically distributed variable, so a typical set can be defined

as the set of sequences $j_1, \ldots, j_m$ satisfying

$$2^{-n(S(\rho_A)+\epsilon)} \leqslant |d_{j_1,\ldots,j_m}|^2 \leqslant 2^{n(S(\rho_A)-\epsilon)}. \tag{2.36}$$

We denote the typical set by A_ϵ and the corresponding projector by Π_{A_ϵ} which is given by

$$\Pi_{A_\epsilon} = \sum_{(j_1,\ldots,j_m)\in A_\epsilon} |j_1, \ldots, j_m\rangle\langle j_1, \ldots, j_m|. \tag{2.37}$$

When applied to ψ', it gives

$$\begin{aligned} &|\langle\psi'^{\otimes m}|\Pi_{A_\epsilon}|\psi^{\otimes m}\rangle|^{-\frac{1}{2}}\ \Pi_{A_\epsilon}|\psi'^{\otimes m}\rangle \\ &= |\langle\psi'^{\otimes m}|\Pi_{A_\epsilon}|\psi^{\otimes m}\rangle|^{-\frac{1}{2}} \sum_{(j_1,\ldots,j_m)\in\mathcal{A}_\epsilon} d_{j_1,\ldots,j_m}|j_1, \ldots, j_m\rangle \otimes |j_1, \ldots, j_m\rangle. \end{aligned} \tag{2.38}$$

This state can be made arbitrarily close to ψ' by increasing m .

The fidelity between the two states is given by

$$F\left(\psi'^{\otimes m}, \frac{\Pi_{A_\epsilon}\psi'^{\otimes m}\Pi_{A_\epsilon}}{|\langle\psi'^{\otimes m}|\Pi_{A_\epsilon}|\psi'^{\otimes m}\rangle|^{-1}}\right) = |\langle\psi'^{\otimes m}|\Pi_{A_\epsilon}|\psi'^{\otimes m}\rangle|^{\frac{1}{2}} > \sqrt{1-\epsilon}. \tag{2.39}$$

The AEP implies that

$$|\langle\psi'^{\otimes m}|\Pi_{A_\epsilon}|\psi'^{\otimes m}\rangle| > 1-\epsilon, \tag{2.40}$$

so we see that the fidelity between them approaches unity as $m \to \infty$.

The state $|\langle\psi'^{\otimes m}|\Pi_{A_\epsilon}|\psi'^{\otimes m}\rangle|^{-\frac{1}{2}}\ \Pi_{A_\epsilon}|\psi'^{\otimes m}\rangle$ can be created from about

nS copies of singlets, using the teleportation protocol.

$$E_c(\psi) = E_d(\psi) = S(\rho_A). \tag{2.41}$$

2.3.2 Relative entropy of entanglement

The axiomatic approach to the entanglement measures aims to find a function f of bipartite states ρ_{AB} that satisfies some desirable properties for entanglement measures. Typically, the properties include:

- For pure states ψ_{AB}, $f(\psi_{AB}) = S(\rho_A)$ where ρ_A is the reduced density operator of the subsystem A.
- f vanishes for separable states.
- f does not increase under LOCC.
- f is invariant under local unitary operations, i.e., $f(\rho_{AB}) = f((U \otimes V)\rho_{AB}(U^\dagger \otimes V^\dagger))$ for any local unitary operators U and V.

Other properties such as the asymptotic continuity can be included in some context.

One entanglement measure introduced from the axiomatic approach is the relative entropy of entanglement, which is defined as

$$\mathcal{E}_R(\rho_{AB}) = \min_{\sigma \in \mathcal{S}} S(\rho||\sigma), \tag{2.42}$$

where $\mathcal{S}$ denotes the set of separable states.

When dealing with many copies of a state, it is desirable to consider the regularized version of $\mathcal{E}_R$, which is defined by [27]

$$\mathcal{E}_R^\infty(\rho) = \lim_{t\to\infty} \frac{\mathcal{E}_R(\rho^{\otimes t})}{t}. \tag{2.43}$$

From its definition, we see that $\mathcal{E}_R^\infty \leqslant \mathcal{E}_R$.

The relative entropy of entanglement has a notable relation with the entanglement cost and the distillable entanglement. It has been shown that the regularized version of any entanglement measure $\mathcal{E}$ that is non-increasing under LOCC is a lower bound for E_c and an upper found for E_d, provided that $\mathcal{E}^\infty(GHZ) = 1$ and it satisfies the continuity condition, i.e., if $F(\rho^{\otimes n}, \sigma^{\otimes n}) \to 1$ as $n \to \infty$, then $\frac{1}{n}|\mathcal{E}(\rho^{\otimes n}) - \mathcal{E}(\sigma^n)| \to 0$ as $n \to \infty$ [27]. The relative entropy of entanglement defined in Eq. (2.42) satisfies the continuity condition [28] and $\mathcal{E}_R^\infty(GHZ) = 1$ that leads to

$$E_d(\rho) \leqslant \mathcal{E}_R^\infty(\rho) \leqslant E_c(\rho). \tag{2.44}$$

2.4 Quantum discord

Quantum discord captures non-classical correlation in a different way than entanglement does. Quantum discord was first introduced through the information-theoretic approach in 2001 [9,10]. The main idea behind this approach is that a singlet contains one bit of classical correlation as well as one bit of non-classical correlation. Several arguments supporting this idea can be found in Ref. [6, 29–34]. For instance, the singlet state $1/\sqrt{2}(|01\rangle - |10\rangle)$ can be converted by LOCC into the mixed state $1/2(|01\rangle\langle 01| + |10\rangle\langle 10|)$, which has one bit of classical correlation. Because local operation alone cannot generate classical correlation, a singlet may also be assumed to have one bit of classical correlation.

2.4.1 Definition

We first introduce the definition of classical correlation. The clasiscal correlation of a bipartite state ρ_{AB} is defined as

$$\mathcal{C}^{\leftarrow}(\rho_{AB}) = \mathcal{S}(\rho_A) - \min_{\{M_B^j\}} \sum_j p_j \mathcal{S}(\sigma_A^j) \tag{2.45}$$

where

$$\sigma_A^j = \mathrm{Tr}_B \left[\frac{(\mathbb{1}_A \otimes {M^j}_B^\dagger M_B^j)\rho_{AB}}{\mathrm{Tr}((\mathbb{1}_A \otimes {M^j}_B^\dagger M_B^j)\rho_{AB})} \right]. \tag{2.46}$$

The minimization in the definition is over all POVM (positive-operator valued measure) measurements with operation elements.

Now, quantum discord is defined as the diffrenece between the quantum mutal information $\mathcal{T}(\rho_{AB})$ and the classical correlation as

$$\mathcal{D}^{\leftarrow}(\rho_{AB}) = \mathcal{T}(\rho_{AB}) - \mathcal{C}^{\leftarrow}(\rho_{AB}), \tag{2.47}$$

The quantum mutual informatiton is known to capture total correlation that includes the quantum part and the classical part of correlation [29, 30]. Note that the minimization in the definition (2.45) is meant to extract every correlation that is classical. Hence, we see that quantum discord is what is left after subatracting the every classical correlation from the total correlation. Quantum discord is nonnegative and it can be shown using the strong subadditivity of von Neumann entropy, see Ref. [35] for the proof.

The minimization in Eq. (2.45) makes quantum discord difficult to be analytically computed in general. On the other hnad, there is a useful relation called the Koashi-Winter relation that allows one to compute quantum discord for rank-2 states. For a tripartite systesm ABC, it is

given as

$$\mathcal{C}^{\leftarrow}(\rho_{AB}) + E_f(\rho_{AC}) = \mathcal{S}(\rho_A), \tag{2.48}$$

where E_f is the entanglement of formation defined as

$$E_f(\rho_{AC}) = \min\{\sum_i p_i S\left(\mathrm{Tr}_B(\psi^i_{AC})\right) : \rho_{AC} = \sum_i p_i \psi^i_{AC}\}. \tag{2.49}$$

Because the analytical expression of the entanglement of formation has been given for two-qubit states [36], $\mathcal{C}^{\leftarrow}(\rho_{AB})$ can be obtaiend for rank-2 ρ_{AB}. The entanglement of formation E_f for $2 \otimes 2$-dimensional states is given by the follwing formula

$$E_f(\rho_{AC}) = \mathcal{H}_2\left(\frac{1+\sqrt{1-[\mathrm{Conc}(\rho_{AC})]^2}}{2}\right), \tag{2.50}$$

where $\mathcal{H}_2(x) = -x\log_2\log_2 x - (1-x)\log_2(1-x)$ and

$$\mathrm{Conc}(\rho_{AC}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \tag{2.51}$$

is the concurrence. Here λ_i's are the eigenvalues of $\sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ in decreasing order, where $\tilde{\rho}$ is the spin-filpped state, i.e., $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho(\sigma_y \otimes \sigma_y)$ with

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2.52}$$

We note that quantum discord $\mathcal{D}^{\leftarrow}(\rho_{AB})$ can increase by a local operation on B. For example, let us consider the state $1/2(|00\rangle\langle 00| + |11\rangle\langle 11|)$, which apparently has no discord. We consider a local operation on B with elements $\{M_0 = |0\rangle\langle 0| + |+\rangle\langle 1|,\ M_1 = |+\rangle\langle 1|\}$ that changes the state into $1/2(|00\rangle\langle 00| + |1+\rangle\langle 1+|)$. Quantum discord of this state is approximately 0.202 as shown in Example 2.3.

From the definition (2.61), it can be easily seen that $\mathcal{D}^{\leftarrow}(\rho_{AB}) = 0$ if and only if ρ_{AB} can be written in the form

$$\rho_{AB} = \sum_j p_j \sigma_A^j \otimes |j\rangle_B\langle j|. \tag{2.53}$$

Example 2.3. [37] We compute quantum discord $\mathcal{D}^{\leftarrow}$ for a bipartite state $\rho_{AB} = 1/2\,(|00\rangle\langle 00| + |1+\rangle\langle 1+|)$, using the Koashi-Winter relation (2.48). We consider the purification ψ_{ABC} of ρ_{AB},

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{2}}\left(|000\rangle_{ABC} + |1+1\rangle_{ABC}\right). \tag{2.54}$$

The entanglemtn of formation of ρ_{AB} and the totcal correaltion are calculated stratightfowardly as

$$\begin{aligned} E_f(\rho_{AC}) &= \mathcal{H}_2\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right), \\ \mathcal{T}(\rho_{AB}) &= \mathcal{H}_2\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right), \end{aligned} \tag{2.55}$$

where $\mathcal{H}_2(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the binary entropy.

Applying the Koashi-winter relation (2.48) gives $\mathcal{C}^{\leftarrow}(\rho_{AB}) = 1 - \mathcal{H}_2\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)$, from which we finally obtain

$$\mathcal{D}(\rho_{AC}) = 2\mathcal{H}_2\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right) - 1 \approx 0.202. \tag{2.56}$$

Example 2.4. We compute quantum discord $\mathcal{D}^{\leftarrow}$ for a state of $4 \otimes 2$-dimensional system, $\rho_{AB} = 1/4(|00\rangle\langle 00| + |11\rangle\langle 11| + |2+\rangle\langle 2+| + |3-\rangle\langle 3-|)$. Because the maximization in Eq. (2.45) can be achieved by rank-1 projectors, we set ${M^j}_B^{\dagger} M_B^j = a_j|\phi_j\rangle\langle\phi_j|$ with $a_j > 0$. Then, we obtain classical

correlation by simply appling Eq. (2.45),

$$\mathcal{C}^{\leftarrow}(\rho_{AB}) = 1 + \max_{\{a_j|\phi_j\rangle\langle\phi_j|\}} \Bigg[\sum_j \frac{a_j}{8} |\langle\phi_j|0\rangle|^2 \log_2 \frac{|\langle\phi_j|0\rangle|^2}{2} + \frac{a_j}{8} |\langle\phi_j|1\rangle|^2 \log_2 \frac{|\langle\phi_j|1\rangle|^2}{2} + \frac{a_j}{8} |\langle\phi_j|+\rangle|^2 \log_2 \frac{|\langle\phi_j|+\rangle|^2}{2} + \frac{a_j}{8} |\langle\phi_j|-\rangle|^2 \log_2 \frac{|\langle\phi_j|-\rangle|^2}{2} \Bigg]. \tag{2.57}$$

Note that $\sum_j a_j/4 = 1$, so the terms insdie the maximum is a convex combination. Therefore, we obtain an upper bound by maximizing over only one term,

$$\mathcal{C}^{\leftarrow}(\rho_{AB}) \leqslant 1 + \frac{1}{2} \max_{\{|\phi\rangle\langle\phi|\}} \Bigg[|\langle\phi|0\rangle|^2 \log_2 |\langle\phi|0\rangle|^2 + |\langle\phi|1\rangle|^2 \log_2 |\langle\phi|1\rangle|^2 + |\langle\phi|+\rangle|^2 \log_2 |\langle\phi|+\rangle|^2 + |\langle\phi|-\rangle|^2 \log_2 |\langle\phi|-\rangle|^2 \Bigg]. \tag{2.58}$$

We now apply the entropic uncertainty relation [38, 39],

$$\mathcal{H}(C) + \mathcal{H}(D) \geqslant -2 \log 2 f(C, D) \tag{2.59}$$

with $C = \sum_c c|c\rangle\langle c|$, $D = \sum_d d|d\rangle\langle d|$ and $f(C, D) = \max_{c,d} |\langle c|d\rangle|$. Then, it follows that $\mathcal{C}^{\leftarrow}(\rho_{AB}) \leqslant 0.5$. The upperbound can acutally be obtained if we choose $|\phi\rangle = 0$. The quantum mutual information is 1, so we finally have $\mathcal{D}^{\leftarrow}(\rho_{AB}) = 0.5$.

2.4.2 Relative entropy of discord

If a state ρ_{AB} has vanishing $\mathcal{D}^{\leftarrow}$ and $\mathcal{D}^{\rightarrow}$, it shoud be written as

$$\rho_{AB} = \sum_{j,k} p_j |j\rangle_A\langle j| \otimes |k\rangle_B\langle k|, \tag{2.60}$$

and we call the states of this form "classically correlated states". We denote the set of classically correlated states by $\mathfrak{C}$.

Similarly to the definition of the relative entropy of entanglement, the relative entropy of discord [34, 40–45] is defined by simply replacing the set of separable states to the set of classically correlated states as

$$\mathcal{D}(\rho) = \min_{\sigma\in\mathfrak{C}} S(\rho||\sigma) \tag{2.61}$$

This definition of discord is also naturally generalziaed to multipartite systems $P_1 \dots P_n$ as we defined the classical correlated states as $\sum_{\vec{k}} p_{\vec{k}} |\vec{k}\rangle\langle\vec{k}|$ where $|\vec{k}\rangle = |k_1\rangle \otimes |k_2\rangle \cdots \otimes |k_n\rangle$ and $|k_l\rangle$ is a local basis at l-th site.

Similarly as the relative entropy of entanglement, the regularized version of the relative entropy of discord is given by

$$\mathcal{D}_R^{\infty}(\rho) = \lim_{t\to\infty} \frac{\mathcal{D}_R(\rho^{\otimes t})}{t}, \tag{2.62}$$

and it follows from the definitions $\mathcal{D}_R^{\infty} \leqslant \mathcal{D}_R$. We note that $\mathcal{D}_R = \mathcal{E}_R$ for any bipartite pure state, but they are generally not equal for k-partite pure states with $k > 2$.

2.5 Quantum correlations in terms of the relative entropy

Although the quantum relative entropy is not a true distance function in the sense that it is not symmetric (i.e., $S(\rho||\sigma) \neq S(\sigma||\rho)$) and does not satisfy the triangle inequalit in general, let us treat it as a psedo-distance. Total correaltion, quantum entanglemtn, quantum discord, and classical correltion can be represneted in terms the relative-entropy distance as we define the set of states having no correlation of each kind.

Let us assume a multipartite syettem $P_1 \dots P_n$. A multipartite state with no total correlation (i.e., no correlation at all) has a product form, $\rho_1 \otimes \rho_2 \otimes ... \otimes \rho_n$ where ρ_i is a density operator of the system P_i. We call this kind of states "product states" and denote their set by π. Then, the total correlation is represented by

$$\mathcal{T}(\rho) = \min_{\sigma \in \pi} S(\rho||\sigma). \tag{2.63}$$

After a simple algebra, it can be proved that the closest product state to a state $\rho_{1,\dots,n}$ the product of reduced states ρ_i's, i.e., $\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_n$ [34]. Therefore, for an arbitrary multipartite state, Eq. (2.63) is reduced to $\mathcal{T}(\rho) = S(\rho||\rho_1 \otimes \rho_2 \otimes ... \otimes \rho_n) = \sum_i S(\rho_i) - S(\rho)$ [34]. For bpartite systems, this quantity is also known as the quantum mutual information [29,30,46].

Similarly, for n-party quantum system, we say a multipartite state is separable if it can be written in the form $\sum_i p_i \rho_1^i \otimes \dots \otimes \rho_n^i$ where ρ_m^i is the state of P_m. The relative entropy of entanglement is defined as [47–49]

$$\mathcal{E}(\rho) = \min_{\sigma \in \mathcal{S}} S(\rho||\sigma) \tag{2.64}$$

where $\mathcal{S}$ is the set of separable states. The measure of entanglement is

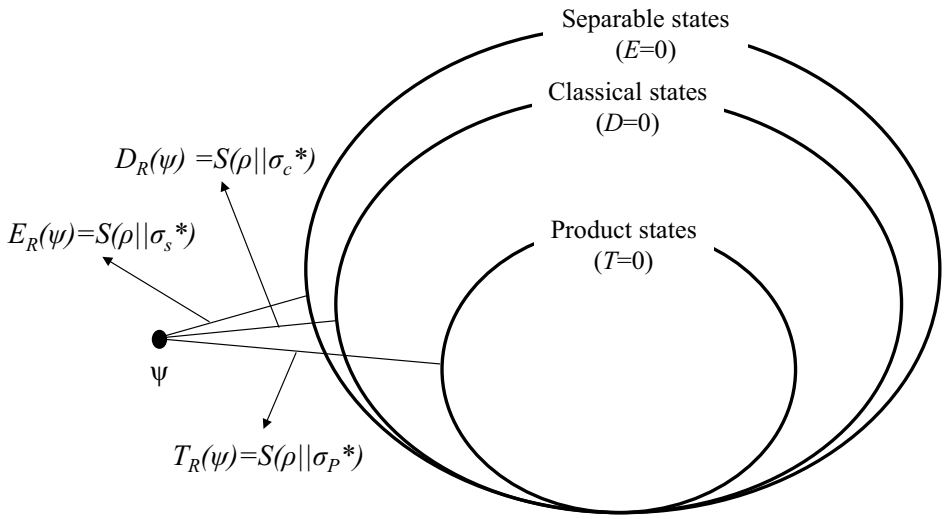


Figure 2.2: Quantum correlations as the relative-entropy distance from each set of free states

naturally applied to multipartite systems as the criterion of separable states has been well-defined for a multipartite system.

As it was seen from their definition, the total mutual information, the relative entropy of entanglement, and the relative entropy of discord are multipartite measures of correlations in a unified view of correlations. Those measures have the clear relation $\mathcal{T} \geq \mathcal{D} \geq \mathcal{E}$ because $\pi \subset \mathcal{C} \subset \mathcal{S}$, and they also enable us to compare the three criteria of correlation, product states, classically correlated states, and separable states.

Example 2.5. We compute quantum discord $\mathcal{D}_R$ of the state $\rho_{AB} = 1/2(|00\rangle\langle 00|+|1+\rangle\langle 1+|$ and the tripartite discord $\mathcal{D}_R$ of In order to calculate $\mathcal{D}_R$, we consider a tripartite state $\rho = p|000\rangle\langle 000|+(1-p)|1+1\rangle\langle 1+1|$. We assume that taking minimization over all local base of B is enough to obtain $\mathcal{D}(A : B : C)$ of the state. Consider a measure of discord $\mathcal{D}^*(\rho_{ABC}) = \min_{\sigma\in\mathcal{C}^*} S(\rho||\sigma)$ where $\mathcal{C}^*$ is the set of all quantum states in the form $\sigma = \sum p_i|i\rangle_B\langle i| \otimes \sigma^i_{AC}$ where $\{|i\rangle_B\}$ is a local basis of

B, and σ^i_{AC}'s are arbitrary quantum states of AC. Apparently, $\mathcal{D}^* \leqslant \mathcal{D}$. It can be shown that the minimum is obtained for a state of the form $\sigma = \sum |i\rangle\langle i| \otimes \langle i|\rho|i\rangle$, as $\mathcal{D}$ can be reduced to the expression (4.8). On the other hand, in the case of our target state, σ is also a classically correlated state for every choosing of the local basis of B. Therefore $D^* = D$ in this case. Now, local base of B can be parameterized as Eq. (4.15), and simple differentiations find two local minima of $S(\rho||\sigma)$ at $\sigma^* = p|000\rangle\langle 000| + \frac{(1-p)}{2}(|101\rangle\langle 101| + |111\rangle\langle 111|)$ and $\sigma^{**} = \frac{p}{2}(|0{+}0\rangle\langle 0{+}0|) + |0{-}0\rangle\langle 0{-}0|) + (1-p)|1{+}1\rangle\langle 1{+}1|$. Consequently, $\mathcal{D}(A:B:C) = \mathcal{D}(A:B) = \min\{p, (1-p)\}$, and then discord of the state $|\psi\rangle = \sqrt{p}|000\rangle + \sqrt{1-p}|1{+}1\rangle$ is computed to $\mathcal{D}(A:B:C) = -p\log_2 p - (1-p)\log_2(1-p) + \min\{p, (1-p)\}$ by applying (4.9).

Chapter 3

Quantum discord as an upper bound for the GHZ entanglement cost

3.1 Introduction

Entanglement of multipartite systems has a more complicated structure than that of bipartite systems. In the bipartite case, any entangled state are asymptotically equivalent, i.e., the asymptotic conversion is reversible, Choice of different resource states result in the equivalent quantification. In the multipartite setting, however, there exist asymptotically inequivalent multipartite states such as the Greenberger-Horne-Zeilinger (GHZ) state and the W state. Therefore, one may choose a resource state and quantify the corresponding multipartite entanglement from the perspective of the resource theory as in the bipartite case, although different choices of the resource state may give rise to independent

quantifications [23]. [1]

However, this way of quantification still remains incomplete in general due to some obstacles. Unlike bipartite states, there is no Schmidt decomposition for multipartite states, which enables bipartite states to be written in a single form. In addition, it is demanding to devise a LOCC procedure for the transformation between multipartite states because of the difficulty of analyzing LOCC [51]. In he case of bipartite pure states, only one round of LOCC is required to describe any LOCC procedure by virtue of the Schmidt decomposition [52].

In this chapter, we consider the quantification of multipartite entanglement from the resource-theory point of view, with the GHZ state being the resource state. For a k-partite system, GHZ state is given as

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes k} + |1\rangle^{\otimes k}), \tag{3.1}$$

with $|0\rangle$ and $|1\rangle$ denoting orthogonal basis states for local subsystems.

Achievable rates (i.e., upper bounds for the optimal rate) are hardly known for any resource state, except in the case where the singlets shared among the multiple parties are used as resource [53]. The LOCC conversion between multipartite states has been little studied especially in the asymptotic limit, while there have been studies on the stochastic LOCC conversion [54, 55] and the conversion under asymptotically non-entangling operations [56, 57] in the asymptotic limit.

We present a LOCC procedure, $\mathcal{L}_D$, for the asymptotic conversion

[1] As an alternative approach, it has been suggested to consider a minimal reversible entanglement generating set [50], which is a set of a minimum number of states allowing asymptotically reversible conversion to any multipartite pure state. However, it remains as an open problem to find a such set.

from GHZ states to a multipartite pure state ψ, namely,

$$GHZ^{\otimes nR_D} \xrightarrow{\mathcal{L}_D} \psi^{\otimes n}$$

for sufficiently large n. We find that the conversion rate of this procedure, R_D, is given by the multipartite discord of state ψ, captured by the relative entropy of discord. This implies that the optimal rate is upper bounded by the relative entropy of discord. Our study further provides an operational interpretation of multipartite discord for pure states, namely, the consumption rate of GHZ entanglement in the devised procedure. In the multipartite setting, quantum discord of pure states is distinct from entanglement (although they are equal for bipartite pure states), but its roles and meanings are not well understood compared to those of bipartite discord in several information tasks [11–13, 17, 58, 59].

3.2 Preliminaries

We formally define the GHZ entanglement cost of ρ of a k-partite systems as follows.

$$E_c(\rho) = \inf\left\{R : \exists \mathcal{L} \in \text{LOCC s. t. } F\left(\mathcal{L}(GHZ^{\otimes nR}), \rho^{\otimes n}\right) \to \text{ as } n \to \infty\right\}. \tag{3.2}$$

Simply speaking, if $GHZ^{\otimes m}$ is asymptotically transformed into $\psi^{\otimes n}$, the GHZ entanglement cost of ψ is m/n. Here, the formal definition of the asymptotic transformation is as follows. We say that a LOCC procedure, denoted by $\mathcal{L}$, asymptotically prepare ψ from GHZ states at rate R if $F(\psi^{\otimes n}, \mathcal{L}(GHZ^{\otimes m})) \to 1$ and $m/n \to R$ as $n \to \infty$, where F is the fidelity in Eq. (2.12). Conversely, the distillable GHZ entanglement is

defined as

$$E_d(\rho) = \inf \left\{ R : \exists \mathcal{L} \in \text{LOCC s. t. } F\left(\mathcal{L}(\rho^{\otimes n}), GHZ^{\otimes nR}\right) \to \text{ as } n \to \infty \right\}. \tag{3.3}$$

Note that we use the same notation as the entanglement cost and the entanglement distillation. For $k = 2$, they reduce to the entanglement cost and the distillable entanglement of bipartite systems.

The result of Ref. [27], Eq. (2.44), can be straightforwardly generalized to multipartite systems. One can easily show that

$$E_d(\rho) \leqslant \mathcal{E}_R^\infty(\rho) \leqslant E_c(\rho). \tag{3.4}$$

The proof for the second inequality can be found in Theorem 3.18.

It is useful to note the following expression of $\mathcal{D}_R$ for the derivation of the main result of this chapter. It has been shown that one can always find a separable basis $\{|x_1, \ldots, x_k\rangle\}$ such that the complete set of projectors $\{\Pi_j\} = \{|x_1, \ldots, x_k\rangle\langle x_1, \ldots, x_k|\}$ satisfies $\mathcal{D}_R(\rho) = S(\rho || \sum_j \Pi_j \rho \Pi_j)$ [34]. We then obtain

$$\mathcal{D}_R(\psi) = \min_{\{\Pi_j\}} \Big[- \sum \operatorname{tr}(\Pi_j \psi) \log_2 \operatorname{tr}(\Pi_j \psi) \Big] \tag{3.5}$$

for a pure state ψ.

As we mentioned earlier, the W states and the GHZ states are asymptotically inequivalent. This can be shown as follows. First, for the conversion $GHZ^{\otimes m} \to W^{\otimes n}$, we have $\mathcal{E}_R^\infty(GHZ^{\otimes m}) = m$ and $\mathcal{E}_R^\infty(W) = 2\log_2 3 - 2 \approx 1.170$ [60]. From the non-increasing property of the relative entropy of entanglement, it follows that

$$\lim_{\substack{n\to\infty \\ m\to\infty}} R_{GHZ\to W} \geqslant \mathcal{E}_R^\infty(W) = 2\log_2 3 - 2 \approx 1.170.$$

Conversely, for the conversion $W^{\otimes n} \to GHZ^{\otimes m}$, we consider the entanglement between the bipartition $AB : C$. In the same manner, it follows that

$$\lim_{\substack{n\to\infty \\ m\to\infty}} R_{W\to GHZ} \leqslant S(\mathrm{tr}_{BC}(W)) = -\frac{1}{3}\log_2\frac{1}{3} - \frac{2}{3}\log_2\frac{2}{3} \approx 0.918,$$

so we see that $R_{GHZ\to W} > R_{W\to GHZ}$.

3.3 LOCC preparation from GHZ states

We now provide the second main result of this thesis as follows. First, we introduce a pure state Ψ that approximates $\psi^{\otimes n}$. It is based on the asymptotic equipartition property (AEP) [61] which will be briefly explained. Next, we present a LOCC procedure that prepares Ψ from a certain number of copies of a GHZ state, say m copies. Finally, we prove that $F(\Psi, \psi^{\otimes n}) \to 1$ and $m/n \to \mathcal{D}_R^\infty(\psi)$ for $n \to \infty$, so they collectively verifies that the LOCC procedure asymptotically prepare ψ at rate $\mathcal{D}_R^\infty(\psi)$.

3.3.1 Introduction of an approximate state

Here, we introduce a pure state Ψ that approximates $\psi^{\otimes n}$ (i.e., $F(\psi^{\otimes n}, \Psi)$ $\to 1$ for $n \to \infty$). For simplicity, we only consider tripartite systems, but the generalization to any k-partite system is straightforward.

We begin with summarizing the AEP [61]. Consider independent and identically distributed random variables $X^{(1)}, \ldots, X^{(l)}$. Each of the variables has the same probability distribution $p(x)$ and the Shannon entropy H. We define a typical set $\mathcal{A}_\epsilon$ to be a set of sequences $\{x^l = (x^{(1)}, \ldots, x^{(l)})\}$ which satisfy $2^{-l(H+\epsilon)} \leqslant p(x^l) \leqslant 2^{-l(H-\epsilon)}$. Then, the AEP

states that the typical sequences $\{x^l\}$ contain most of the probability, and the size of the typical set, $|\mathcal{A}_\epsilon|$, is about 2^H. The AEP is summarized as follows. For any $\epsilon > 0$,

$$\begin{aligned} \Pr\left[x^l \in \mathcal{A}_\epsilon\right] &> 1-\epsilon, \\ |\mathcal{A}_\epsilon| &> (1-\epsilon)2^{l(H-\epsilon)}, \\ |\mathcal{A}_\epsilon| &\leqslant 2^{l(H+\epsilon)} \end{aligned} \tag{3.6}$$

for sufficiently large l.

Assume that we want asymptotically prepare a tripartite state ψ of an arbitrary dimension. We then consider the state $\phi = \psi^{\otimes t}$ prepared in a tripartite system $\mathbf{P} = \{P_1, P_2, P_3\}$ (i.e., if ψ is a $2 \otimes 2 \otimes 2$ dimensional state, $\dim(P_1) = \dim(P_2) = \dim(P_3) = 2^t$). This state can be written in a separable basis $\{|x_1, \ldots, x_k\rangle\}$ for the system $\mathbf{Q}$ as

$$|\phi\rangle = |\psi\rangle^{\otimes t} = \sum_{x_1,x_2,x_3} C(x_1,x_2,x_3)|x_1,x_2,x_3\rangle \tag{3.7}$$

with some coefficients $C(x_1, x_2, x_3)$. We note that $\{|x_j\rangle\}$'s can be any orthonormal bases for each subsystem, and we do not choose any particular separable basis at this point. The coefficients $|C(x_1, x_2, x_3)|^2$ can be considered a joint probability distribution of random variables X_1, X_2, and X_3, so we set $p(x_1, x_2, x_3) = |C(x_1, x_2, x_3)|^2$ and

$$H = -\sum |C(x_1,x_2,x_3)|^2 \log_2 |C(x_1,x_2,x_3)|^2. \tag{3.8}$$

We may consider independent and identically distributed random variables $X_j^{(1)}, \ldots, X_j^{(l)}$ for j =1, 2, and 3. A typical set is correspondingly

defined as

$$\mathcal{A}_\epsilon = \Big\{(x_1^l, x_2^l, x_3^l) : \big|\log_2 p(x_1^l, x_2^l, x_3^l) - H\big| \leqslant \epsilon\Big\}. \tag{3.9}$$

Then, $n = t\ l$ copies of ψ, which is prepared in a system $\mathbf{Q} = \{Q_1, Q_2, Q_3\} = \{P_1^{\otimes l}, P_2^{\otimes l}, P_3^{\otimes l}\}$, is represented using the typical set $\mathcal{A}_\epsilon$ and its complement $\mathcal{A}_\epsilon^c$ as

$$\begin{aligned} |\psi\rangle_{\mathbf{Q}}^{\otimes n} = & \sum_{(x_1^l, x_2^l, x_3^l)\in\mathcal{A}_\epsilon} C(x_1^l, x_2^l, x_3^l)|x_1^l, x_2^l, x_3^l\rangle \\ & + \sum_{(x_1^l, x_2^l, x_3^l)\in\mathcal{A}_\epsilon^c} C(x_1^l, x_2^l, x_3^l)|x_1^l, x_2^l, x_3^l\rangle. \end{aligned} \tag{3.10}$$

The approximated state Ψ is defined to consist of only the terms corresponding to the typical sequences, so

$$|\Psi\rangle_{\mathbf{Q}} = N_\epsilon^{-\frac{1}{2}} \sum_{(x_1^l, x_2^l, x_3^l)\in\mathcal{A}_\epsilon} C(x_1^l, x_2^l, x_3^l)|x_1^l, x_2^l, x_3^l\rangle \tag{3.11}$$

where $N_\epsilon = \sum_{(x_1^l, x_2^l, x_3^l)\in\mathcal{A}_\epsilon} |C(x_1^l, x_2^l, x_3^l)|^2$ is a normalization factor. As the typical set contains most of the probability, we see that the fidelity between Ψ and $\psi^{\otimes n}$ approaches one as $n \to \infty$. In addition, it follows from the AEP that the number of terms in the expansion of Ψ, $|\mathcal{A}_\epsilon|$, is approximately 2^{lH}.

At this point, it is important to note the relation of the approximated state to the previously presented method in Ref. [21] for bipartite asymptotic preparation. In this method, one party, say the first party, prepares $\psi_{Q_1 Q_1'}^{\otimes n}$ at their site and send the compressed Q_1'-part (the compression uses the AEP) to the second party by using the quantum teleporation. At the end of the procedure, the two parties share an approximated

state of the same form as Ψ. However, this method using the quantum teleportation can only be applied [50] to multipartite states which have Schmidt decompositions, e.g., $\alpha|000\rangle + \beta 111\rangle$. On the other hand, our LOCC procedure directly converts singlets into the state Ψ. It enables us to consider the approximated state Ψ in the multipartite setting, which can actually be created by LOCC as shown in the next subsection.

3.3.2 Preparation of the approximate state

We now present the LOCC procedure $\mathcal{L}_D$ that converts $\log_2 |\mathcal{A}_\epsilon|$ copies of a GHZ state into Ψ, so $\mathcal{L}_D(GHZ^{\otimes \log_2 |\mathcal{A}_\epsilon|}) = \Psi$. We also only consider tripartite systems, and the generalization to any k-partite system is straightforward. For convenience of mathematical descriptions, we change the notation as follows. For x_1^l, x_2^l, and x_3^l such that $(x_1^l, x_2^l, x_3^l) \in \mathcal{A}_\epsilon$, consider the following sets: $\{(x_1^l, x_2^l, x_3^l)\}$, $\{x_1^l\}$, $\{x_2^l\}$, and $\{x_3^l\}$, whose sizes are $|\mathcal{A}_\epsilon|$, α, β and γ, respectively. We replace (x_1^l, x_2^l, x_3^l) with y where $y \in \{0, 1, \ldots, |\mathcal{A}_\epsilon| - 1\}$. Similarly, we do the same for other sets as

$$\begin{aligned} \{x_1^l\} &\to \{f : f \in 0, \ldots, \alpha - 1\}, \\ \{x_2^l\} &\to \{g : g \in 0, \ldots, \beta - 1\}, \\ \{x_3^l\} &\to \{h : h \in 0, \ldots, \gamma - 1\}. \end{aligned} \tag{3.12}$$

Because f, g, and h are completely determined by y, we denote them by $f(y)$, $g(y)$, and $h(y)$. We can then rewrite the state Ψ of the system $\mathbf{Q}$ as

$$|\Psi\rangle_{\mathbf{Q}} = N_\epsilon^{-\frac{1}{2}} \sum_{y=0}^{|\mathcal{A}_\epsilon|-1} C(y) |f(y), g(y), h(y)\rangle_{\mathbf{Q}}. \tag{3.13}$$

The state Ψ can be obtained from $m = \lceil \log_2 |\mathcal{A}_\epsilon| \rceil$ copies of a GHZ

state ($\lceil x \rceil$ is the smallest integer not less than x) by following four steps of LOCC. Assume that m copies of a GHZ state are prepared in an ancillary system $\mathbf{Q}' = \{Q'_1, Q'_2, Q'_3\}$, where $\dim(Q_1) = \dim(Q_2) = \dim(Q_3) = |\mathcal{A}_\epsilon|$. We can rewrite $GHZ^{\otimes m}$ as

$$\begin{aligned} |GHZ\rangle_{\mathbf{Q}'}^{\otimes m} &= 2^{-\frac{m}{2}} \sum_{z_j=0,1} |z_1 \cdots z_m\rangle |z_1 \cdots z_m\rangle |z_1 \cdots z_m\rangle \\ &= 2^{-\frac{m}{2}} \sum_{y=0}^{2^m-1} |y, y, y\rangle_{\mathbf{Q}'}, \end{aligned}$$

where y is the decimal representation of binary strings $z_1 \cdots z_m$. Because $|\mathcal{A}_\epsilon| \leqslant 2^m$, we may discard the terms other than those with $0 \leqslant y \leqslant |\mathcal{A}_\epsilon| - 1$ (by a simple local operation), so we have

$$|\mathcal{A}_\epsilon|^{-\frac{1}{2}} \sum_{y=0}^{|\mathcal{A}_\epsilon|-1} |y, y, y\rangle_{\mathbf{Q}'}.$$

The first step is to change the coefficients from $2^{-\frac{m}{2}}$ to $N_\epsilon^{-1/2} C(y)$ as

$$|\mathcal{A}_\epsilon|^{-\frac{1}{2}} \sum_{y=0}^{|\mathcal{A}_\epsilon|-1} |y, y, y\rangle_{\mathbf{Q}'} \xrightarrow{\text{Step1}} N_\epsilon^{-\frac{1}{2}} \sum_{y=0}^{|\mathcal{A}_\epsilon|-1} C(y) |y, y, y\rangle_{\mathbf{Q}'}.$$

This can be done by local operations of any party. Consider a local measurement described by the measurement operators

$$\left\{ M_j = N_\epsilon^{-\frac{1}{2}} \sum_{y=0}^{|\mathcal{A}_\epsilon|-1} D(y \oplus j) |y\rangle\langle y|, \ 0 \leqslant j \leqslant |\mathcal{A}_\epsilon| - 1 \right\} \tag{3.14}$$

where $\oplus$ denotes addition modulo $|\mathcal{A}_\epsilon|$. It is easy to check that they satisfy the completeness relation $\sum_j {M^\dagger}_j M_j = I$. After the measurement, if the outcome of the measurement is j, each of the parties applies a unitary

operation U: $|y\rangle \to |y \oplus j\rangle$ to complete the first step. The second step is to create the state $|f(y), h(y), g(y)\rangle_{\mathbf{Q}}$ in the system $\mathbf{Q}$, which is initially prepared in $|000\rangle_{\mathbf{Q}}$.

$$\xrightarrow{\text{Step2}} N_\epsilon^{-\frac{1}{2}} \sum_{y=0}^{|\mathcal{A}_\epsilon|-1} C(y)|y, y, y\rangle_{\mathbf{Q}'} \otimes |f(y), h(y), g(y)\rangle_{\mathbf{Q}}.$$

This is achieved by a local unitary operation on Q_1Q_1' that transforms $|y\rangle_{Q_1'} \otimes |0\rangle_{Q_1}$ into $|y\rangle_{Q_1'} \otimes |f(y)\rangle_{Q_1}$, and similar local unitary operations on Q_2Q_2' and Q_3Q_3'.

The third step is to disentangle Q_2' and Q_3' as

$$\xrightarrow{\text{Step3}} N_\epsilon^{-\frac{1}{2}} \sum_{y=0}^{|\mathcal{A}_\epsilon|-1} C(y)|y\rangle_{Q_1'} \otimes |f(y), g(y), h(y)\rangle_{\mathbf{Q}}. \tag{3.15}$$

In oder to perform this step, the second and third parties perform local measurements on their systems using the measurement operators

$$\{M_j = |\mathcal{A}_\epsilon|^{-1} J|j\rangle\langle j|J^\dagger\},$$

where J is a complex Hadamard operation, defined as $\langle y|J|y'\rangle = \exp[2\pi i \cdot yy'/|\mathcal{A}_\epsilon|]$. Depending on the measurement outcomes, the first party can perform a phase-shifting operation to complete the third step.

The final step is to disentangle the ancillary system Q_1' from $\mathbf{Q}$ to obtain Ψ_Q as

$$\xrightarrow{\text{Step4}} N_\epsilon^{-\frac{1}{2}} \sum_{y=0}^{|\mathcal{A}_\epsilon|-1} C(y)|f(y), g(y), h(y)\rangle_{\mathbf{Q}} = |\Psi\rangle_{\mathbf{Q}}. \tag{3.16}$$

To address the procedure for this step, we introduce a variable $K(y)$,

which is defined as

$$K(y) = \beta\gamma f(y) + \gamma g(y) + h(y). \tag{3.17}$$

Then, we can consider a linear isometry transformation V on the system Q_1' which transforms $|y\rangle$ into $|K(y)\rangle$. This requires additional ancillary qubits in Q_1' so that the dimension of Q_1' equals $K(|\mathcal{A}_\epsilon|-1)+1$. Applying V to the state in Eq. (3.15) gives

$$\sum_{y=0}^{|\mathcal{A}_\epsilon|-1} C(y)|K(y)\rangle_{Q_1'} \otimes |f(y), g(y), h(y)\rangle_{\mathbf{Q}}.$$

Next, the first party performs a measurement on the ancillary system Q_1' using the operators

$$\Big\{ M_j : M_j = \frac{|0\rangle\langle j|J^\dagger}{[K(|\mathcal{A}_\epsilon|-1)+1]^{\frac{1}{2}}},\ 0 \leqslant j \leqslant K(|\mathcal{A}_\epsilon|-1) \Big\},$$

where $\tilde{J}$ is a complex Hadamard operation defined as

$$\langle y|\tilde{J}|y'\rangle = \exp\left[2\pi i \cdot yy' \cdot [K(|\mathcal{A}_\epsilon|-1)+1]^{-1}\right]$$

for $0 \leqslant y,\ y' \leqslant K(|\mathcal{A}_\epsilon|-1)+1$. The completeness relation, $\sum_j M_j^\dagger M_j = I$, can be checked from the orthogonality of the Hadamard operation, $\tilde{J}\tilde{J}^\dagger = [K(|\mathcal{A}_\epsilon|-1)+1]\cdot I$. If the measurement outcome is j, the resulting state is

$$\begin{aligned} &|0\rangle_{Q_1'} \sum_{y=0}^{|\mathcal{A}_\epsilon|-1} C(y)\langle j|J^\dagger|K(y)\rangle \otimes |f(y), g(y), h(y)\rangle_{\mathbf{Q}} \\ =&|0\rangle_{Q_1'} \sum_{y=0}^{|\mathcal{A}_\epsilon|-1} C(y)e^{-2\pi i\cdot jK(y)} \otimes |f(y), g(y), h(y)\rangle_{\mathbf{Q}}. \end{aligned}$$

Finally, local phase-shifting operations can remove the phase $\exp[-2\pi i \cdot jK(y)]$ by using Eq. (3.17). Consider a phase-shifting operation by the first party that transforms

$$|f(y)\rangle \to \exp[2\pi i \cdot j\beta\gamma f(y)]|f(y)\rangle.$$

Similarly, consider phase-shifting operations by the second and the third parties that transform

$$|g(y)\rangle \to \exp[2\pi i \cdot j\gamma f(y)]|g(y)\rangle,$$
$$|h(y)\rangle \to \exp[2\pi i \cdot jh(y)]|f(y)\rangle.$$

Applying those operations completes the fourth step.

3.3.3 Asymptotic preparation from GHZ states

We denote by $\mathcal{L}_D$ the LOCC procedure described in the previous subsection. We have shown that it prepares Ψ from $GHZ^{\otimes \lceil \log_2 |\mathcal{A}_\epsilon| \rceil}$, i.e., $\mathcal{L}_D(GHZ^{\otimes \lceil \log_2 |\mathcal{A}_\epsilon| \rceil}) = \Psi$. In practice, it can asymptotically prepares ψ from GHZ states at rate $\mathcal{D}_R^\infty$. This can be explained as follows. First, the AEP implies that the fidelity $F(\Psi, \psi^{\otimes n})$ approaches one as $n \to \infty$, so $\mathcal{L}_D$ asymptotically prepares $\psi^{\otimes n}$. Second, the conversion rate, which is given by $n^{-1}\lceil \log_2 |\mathcal{A}_\epsilon| \rceil$, can be reduced down to $\mathcal{D}_R^\infty(\psi)$. Recall that the typical set in Eq. (3.7) depends on the separable basis $\{x_1, x_2, x_3\}$, which we have not specified yet. We now choose the separable basis so that it gives the minimum size of the typical set. Let us denote the minimum size by $\log_2 |\mathcal{A}_\epsilon^*|$ and denote the corresponding Shannon entropy by H^*. It then follows from the AEP that $\log_2 |\mathcal{A}_\epsilon^*| \approx lH^*$. In addition, a comparison between Eqs. (3.5) and (3.8) leads to $H^* = \mathcal{D}_R(\psi^{\otimes t})$. Putting these

together, we have $n^{-1}\lceil \log_2 |\mathcal{A}_\epsilon^*| \rceil = t^{-1}l^{-1}\lceil \log_2 |\mathcal{A}_\epsilon^*| \rceil \approx t^{-1}\mathcal{D}_R(\psi^{\otimes t})$. Finally, from the definition of $\mathcal{D}_R^\infty$, the conversion rate is approximately found to be $\mathcal{D}_R^\infty(\psi)$ for large t. We thus reach the following theorem.

Theorem 3.1. *The LOCC procedure $\mathcal{L}_D$ asymptotically prepares a multipartite pure state ψ from GHZ states at rate $\mathcal{D}_R^\infty(\psi)$. Namely, for any $\epsilon > 0$, $\delta > 0$,*

$$\begin{aligned} F\Big[\mathcal{L}_D(GHZ^{\otimes \lceil \log_2 |\mathcal{A}_\epsilon^*| \rceil}), \psi^{\otimes n}\Big] &> 1-\epsilon, \\ \Big| n^{-1}\lceil \log_2 |\mathcal{A}_\epsilon^*| \rceil - \mathcal{D}_R^\infty(\psi)\Big| &< \delta. \end{aligned} \tag{3.18}$$

for sufficiently large n.

proof— The LOCC procedure, $\mathcal{L}_D$, works for any separable basis $\{|x_1, x_2, x_3\rangle\}$ in Eq. (3.7). In a given separable basis, the Shannon entropy of the random variables X_1, X_2, and X_3 is given as

$$H = -\sum |C(x_1, x_2, x_3)|^2 \log_2 |C(x_1, x_2, x_3)|^2,$$

where $|C(x_1, x_2, x_3)|^2 = \mathrm{tr}(|x_1, x_2, x_3\rangle\langle x_1, x_2, x_3|\psi^{\otimes t})$. We note again that the typical set $\mathcal{A}_\epsilon$ depends on the separable basis and so does the Shannon entropy. We can make H equal $\mathcal{D}_R(\psi^{\otimes t})$ by choosing a suitable separable basis. Using the expression of $\mathcal{D}_R$ in Eq. (3.5), we can write

$$\begin{aligned} \mathcal{D}_R(\psi^{\otimes t}) = \min_{\{|x_1,x_2,x_3\rangle\langle x_1,x_2,x_3|\}} \Big[-\sum \mathrm{tr}(|x_1, x_2, x_3\rangle\langle x_1, x_2, x_3|\psi^{\otimes t}) \\ \times \log_2 \mathrm{tr}(|x_1, x_2, x_3\rangle\langle x_1, x_2, x_3|\psi^{\otimes t})\Big]. \end{aligned}$$

Therefore, by choosing a separable basis that attains the minimum in the above equation, we have $H = \mathcal{D}_R(\psi^{\otimes t})$. For clarity, we add a superscript

* to the corresponding typical set and Shannon entropy as $\mathcal{A}_\epsilon^*$ and H^*. Then, it follows from the AEP that, for any $\epsilon' > 0$,

$$\begin{aligned} N_{\epsilon'} &> (1-\epsilon'), \\ (1-\epsilon')2^{l(H^*-\epsilon')} < |\mathcal{A}_{\epsilon'}^*| &\leqslant 2^{l(H^*+\epsilon')} \end{aligned} \tag{3.19}$$

for sufficiently large l. The fidelity between $\psi^{\otimes n}$ in Eq. (3.10) and $\Psi = \mathcal{L}_D(GHZ^{\otimes \lceil \log |\mathcal{A}_{\epsilon'}^*| \rceil})$ in Eq. (3.11) is given as

$$F(\Psi, \psi^{\otimes n}) = \sum_{(x_1^l, x_2^l, x_3^l) \in \mathcal{A}_\epsilon} \frac{|C(x_1^l, x_2^l, x_3^l)|^2}{N_{\epsilon'}^{\frac{1}{2}}} = N_{\epsilon'}^{\frac{1}{2}}$$

In addition, applying the first inequality in Eq. (3.19) gives $N_{\epsilon'}^{\frac{1}{2}} > (1-\epsilon')^{\frac{1}{2}}$. By choosing ϵ' such that $\epsilon' < 1-(1-\epsilon)^2$ for any ϵ, we obtain

$$F[\mathcal{L}_D(GHZ^{\otimes \lceil \log |\mathcal{A}_{\epsilon'}^*| \rceil}), \psi^{\otimes n}] > 1-\epsilon$$

for any ϵ.

As the LOCC procedure asymptotically prepares ψ^n from $GHZ^{\lceil \log_2 |\mathcal{A}_\epsilon^*| \rceil}$, the conversion rate is given by $n^{-1} \log_2 |\mathcal{A}^*|$ with $n = t \cdot l$ (the ceiling function $\lceil\ \rceil$ can be ignored for large n). Using the second inequality in Eq. (3.19) and $H^* = \mathcal{D}_R(\psi^{\otimes t})$, it is straightforward to show that

$$\left| \frac{\log_2 |\mathcal{A}_{\epsilon'}^*|}{l} - \mathcal{D}_R(\psi^{\otimes t}) \right| < \epsilon' - \frac{\log_2(1-\epsilon')}{l}.$$

By dividing the above equation by t and choosing ϵ' such that $\epsilon' - l^{-1}\log_2(1-\epsilon') < \delta \cdot t$ (there always exists such $\epsilon' > 0$ for any $l,\ t,\ \delta > 0$.), we have

$$\left| \frac{\log_2 |\mathcal{A}_{\epsilon'}^*|}{n} - \frac{\mathcal{D}_R(\psi^{\otimes t})}{t} \right| < \delta.$$

Finally, $\mathcal{D}_R(\psi^{\otimes t}) \cdot t^{-1}$ approaches $\mathcal{D}_R^\infty(\psi)$ as $t \to \infty$ by the definition of $\mathcal{D}_R^\infty$, and it completes the proof of the theorem. ■

3.4 Applications and examples

In this section, we present upper and lower bounds for the GHZ entanglement cost (equivalently, the optimal rate) and examine them for several examples. We also compare the LOCC procedure $\mathcal{L}_D$ with another procedure that uses singlets as resource. Finally, we discuss the application of $\mathcal{L}_D$ to general mixed states.

3.4.1 Upper and lower bounds for the GHZ entanglement cost

The existence of $\mathcal{L}_D$ implies that $\mathcal{D}_R^\infty$ is an upper bound for E_c, as summarized in the following theorem.

Theorem 3.2. *The relative entropy of entanglement and the relative entropy of discord are lower and upper bounds for the GHZ entanglement cost, respectively.*

$$\mathcal{E}_R^\infty(\psi) \leqslant E_c(\psi) \leqslant \mathcal{D}_R^\infty(\psi).$$

proof— The upper bound is the direct consequence of the existence of $\mathcal{L}_D$. The lower bound can be shown as follows. From the non-increasing property of an entanglement measure $\mathcal{E}$ under LOCC, we see that

$$\mathcal{E}(GHZ^{\otimes nE_c}) = nR^* \geqslant \mathcal{E}(\mathcal{L}(GHZ^{\otimes nE_c})$$

From the continuity, we have $\mathcal{E}(\mathcal{L}(GHZ^{\otimes nE_c}) \geqslant \mathcal{E}(\psi^{\otimes n}) - n\epsilon$.

$$E_c \geqslant \frac{\mathcal{E}(\psi^{\otimes n})}{n} - \epsilon$$

■

Example 3.3. For generalized GHZ states, $\sqrt{p}|000\rangle + \sqrt{1-p}|111\rangle$, the two bounds coincide, so that our procedure is optimal and $E_c = \mathcal{D}_R^\infty = -p\log_2 p - (1-p)\log_2(1-p)$. In addition, it is known that GHZ states can be distilled from the generalized GHZ states at the same rate [50].

We note that the GHZ entanglement cost can be either greater or less than 1 as $\mathcal{E}_R^\infty$ and $\mathcal{D}_R^\infty$ can be so.

Example 3.4. For the state $|\psi\rangle = (|000\rangle + |+11\rangle)/\sqrt{2}$ with $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, The relative entropy of entanglement and discord is given as $\mathcal{E}_R^\infty = 1$ and $\mathcal{D}_R = 1.5$ [62], so $1 \leqslant E_c \leqslant 1.5$.

Example 3.5. For the W state, $|W\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$, $\mathcal{E}_R^\infty = 2\log_2 3 - 2 \approx 1.170$ [60] and $\mathcal{D}_R = \log_2 3 \approx 1.585$ so that $1.170 < E_c < 1.585$.

3.4.2 Comparison with preparation of states from singlets

For comparison, we consider a LOCC procedure that prepares a multipartite pure state $\psi_{\mathbf{Q}}^{\otimes n}$ from the singlets shared among $\mathbf{Q}$ [53]. In this procedure, a single party, say the first party, locally prepares the pure state $\psi_{\mathbf{Q}}^{\otimes n}$ and uses quantum teleportation [5] and data compression [63] to distribute the state. For the tripartite case, it requires nS_2 singlets between Q_1-Q_2 and nS_3 singlets between Q_1-Q_3, where S_i denotes the von

Neumann entropy of the i-th subsystem. Therefore, it requires $n(S_2+S_3)$ singlets in total. Considering all the permutations of Q_1, Q_2 and Q_3, the consumption rate of singlets is

$$R_T = S_1 + S_2 + S_2 - \max\{S_1, S_2, S_3\}.$$

Since a singlet is obtainable from a single copy of a GHZ state, this procedure can also be achieved by consuming GHZ entanglement at rate R_T. We have no proof for $\mathcal{D}_R^\infty \leqslant R_T$ for general states. However, it has been shown in Ref. [62] that $\mathcal{D}_R \leqslant R_T$ (so $\mathcal{D}_R^\infty \leqslant R_T$) for a few kinds of three-qubit states including the generalized W states $\alpha|001\rangle + \beta|010\rangle + \gamma|100\rangle$, and generalized GHZ states $\alpha|000\rangle + \beta|111\rangle$. For instance, for the W state, $\mathcal{D}_R \approx 1.585$ and $R_T \approx 1.837$. We also realize that $\mathcal{D}_R^\infty = \frac{1}{k-1} R_T$ for the k-qubit generalized GHZ states, $\alpha|0^{\otimes k}\rangle + \beta|1^{\otimes k}\rangle$, where we have used $R_T = \left[\sum_{i=1}^k S_i\right] - \max\{S_1, \ldots, S_k\}$. In the case of the k-qubit W state $(|00\cdots1\rangle + |00\cdots10\rangle + \cdots + |100\cdots0\rangle)/\sqrt{k}$, one can check that $\mathcal{D}_R \leqslant R_T$ for any k.

3.4.3 Application of $\mathcal{L}_D$ to general mixed states

Before presenting how to prepare general mixed states by using $\mathcal{L}_D$, we note that the relation $E_C \leqslant \mathcal{D}_R^\infty$ does not hold for general mixed states. This can be shown by taking an counter example of a bipartite state as in Example. 3.6.

For a general mixed state σ, our procedure can be applied to prepare the pure states ψ_i's such that $\sigma = \sum_i p_i \psi_i$. Then, the mixed state σ can be obtained by classically mixing them. This technique has already been used to generalize the entanglement cost of bipartite pure states to bipartite mixed states [18]. In our case, the rate of the GHZ entanglement

consumption is given by

$$R_D(\rho) = \lim_{t\to\infty} \frac{r_D(\rho^{\otimes t})}{t}, \tag{3.20}$$

where

$$r_D(\sigma) = \inf\big\{ \sum_i p_i \mathcal{D}_R(\psi_i) : \sigma = \sum_i p_i \psi_i \big\}. \tag{3.21}$$

We see that the rate R_D vanishes for fully separable states. For bipartite systems, discord $\mathcal{D}_R(\psi_i)$ in Eq. (3.21) is equal to the entropy of entanglement, so $R_D(\rho)$ is reduced to the regularized version of the entanglement of formation [22]. It is known that the regularized entanglement of formation is equal to the entanglement cost for bipartite states [25].

Example 3.6. Consider the following state,

$$(1-2p)|\Phi^+\rangle\langle\Phi^+| + p\big(|00\rangle\langle 00| + |11\rangle\langle 11|\big) \tag{3.22}$$

with $|\Phi^+\rangle = \sqrt{1/2}(|00\rangle + |11\rangle)$ and $0 \leqslant p \leqslant 1/2$. The entanglement cost of the state is $E_C = H_2(1/2 + \sqrt{p(1-p)})$ [64] and the relative entropy of discord is $\mathcal{D}_R^\infty = 1 - H_2(p)$, where $H_2(x) = -x\log_2 x - (1-x)\log_2(1-x)$. We plot E_C and $\mathcal{D}_R^\infty$ in Fig. 3.1, and it shows that that $E_C > \mathcal{D}_R^\infty$, equivalently, $H_2(1/2 + \sqrt{p(1-p)}) \geqslant 1 - H_2(p)$ for any $0 < p < 1/2$.

Example 3.7. We provide an example of LOCC procedure that converts two copies of the W state to GHZ state with probability 4/9. For convenience, we consider a unitarily equivalent state,

$$|W'\rangle = \frac{1}{\sqrt{3}}\left(|000\rangle + |011\rangle + |101\rangle\right). \tag{3.23}$$

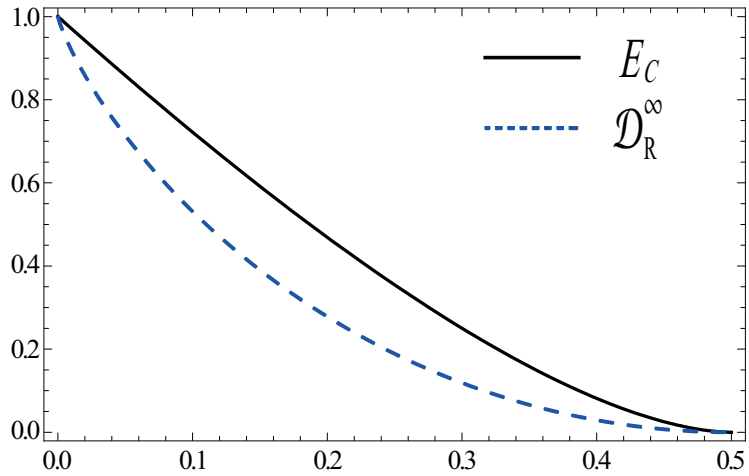


Figure 3.1: Comparison of E_C and $\mathcal{D}_R^\infty$ for the state (3.22)

Its two-copy state is represented as

$$|W'\rangle^{\otimes 2} = \frac{1}{3}\left(|00,00,00\rangle + |01,10,11\rangle\right) + \frac{1}{3}(|00,01,01\rangle + |01,00,01\rangle + |00,11,11\rangle + |00,10,10\rangle + |11,00,11\rangle + |01,00,10\rangle + |10,01,11\rangle). \tag{3.24}$$

Consider a POVM measurement on system A with operation elements

$$M_A^0 = |00\rangle\langle 00| + |01\rangle\langle 01|, \qquad M_1 = 1 - M_0. \tag{3.25}$$

When the outcome is 0, it transforms the state into

$$M_A^0|W'\rangle^{\otimes 2} = \frac{1}{3}\left(|00,00,00\rangle + |01,10,11\rangle\right) + \frac{1}{3}(|00,01,01\rangle + |01,00,01\rangle + |00,11,11\rangle + |00,10,10\rangle). \tag{3.26}$$

Next, consider POVM measurements each on B and C with operation

elements

$$\begin{aligned} M_B^0 &= |00\rangle\langle 00| + |10\rangle + \langle 10|, \;\; M_B^1 = 1 - M_B^0, \\ M_C^0 &= |00\rangle\langle 00| + |11\rangle\langle 11|, \;\; M_C^1 = 1 - M_C^0 \end{aligned} \tag{3.27}$$

We see that the resulting state for the outcome 000 is

$$M_C^0 M_B^0 M_A^0 |W'\rangle^{\otimes 2} = \frac{1}{3}(|00,00,00\rangle + |01,10,11\rangle) = \sqrt{\frac{2}{9}}|GHZ\rangle \tag{3.28}$$

We discard the other outcomes, so the probability of success is 4/9.

3.4.4 Discussion

We also note that the GHZ entanglement cost and the distillable GHZ entanglement are not consistent in measuring the amounts of multipartite entanglement. In other words, there is a case that $E_c(\rho) > E_c(\sigma)$ but $E_d(\rho) < E_d(\sigma)$ for two multipartite states ρ and σ. For instance, for the GHZ state and the W state, $E_c(GHZ) = 1$ and $\log_2(9/4) \leqslant E_c(W) \leqslant \log_2 3$ as we will show later, so $E_c(GHZ) \leqslant E_c(W)$. However, in terms of the distillable GHZ entanglement, $E_d \leqslant H_2(1/3) \approx 0.92$. We have studied about four additivity relations of the measures of multipartite correlations which are based on the quantum relative entropy. The relations were motivated by the definition of the total mutual information and the strong subadditivity of von Neumann entropy. They specify multipartite correlations in terms of bipartite correlations in subsystems, so they give insight of to what extent multipartite correlations contain bipartite correlations of subsystems. argued that discord have the additivity relations for pure states, but it is not always true for mixed states.

Chapter 4

Additivity relations of quantum correlations

In this chapter, we study the characterization of non-classical correlations between multiple systems from a different perspective than the resource-theory point of view. Here, we look into the relations between multipartite correlations of the total system and bipartite correlations between individual subsystems.

4.1 Introduction

Non-classical correlations have properties proper to multipartite systems. An example is the genuine multipartite entanglement. Let us consider the GHZ state, $1/\sqrt{2}(|000\rangle + |111\rangle)$. It is evident that there exists multipartite non-classical correlation as shown in the GHZ experiment [65, 66]. However, the state of any bipartite subsystem is $1/2(|00\rangle\langle 00| + |11\rangle\langle 11|)$, so no entanglement exists between any two of the subsystems. In contrast, in the case correlation between three classical random variables X, Y,

and Z, there should be no correlation at all, i.e., $p(x, y, z) = p(x)p(y)p(z)$ if there is no correlation between any two random variables. Another example is the monogamy of entanglement [19], which states that, for three parties A, B, and C, the entanglement between A and B restricts the entanglement between A and C. This property can not also be found in classical correlation.

In light of those observations, it may be meaningful to study the relation between multipartite correlation and bipartite correlations between subsystems. From the hierarchical perspective, measures of correlations can be classified into three parts: total correlations, quantum correlations, classical correlations [30] as discussed in Sec. 2.5. Because total correlation has a simple representation in terms of von Neumann entropy, we first examine the relations for total correlation. We specify four relations, which we call the additivity relations, of total correlation. Then, we examine whether similar type of correlation can be derived for the case of entanglement and discord. In Section 4.2.2 and Section 4.2.2, it is shown that the additivity relations are satisfied in many cases for entanglement and discord. As specific examples, we will consider the generalized GHZ-states, a variant of the generalized GHZ-states, the generalized W-states, and a few mixed states in Section 4.3.

4.2 General concepts

In this section, we discuss the additivity relation of multipartite correlation. We derive four different additivity relations using the total correlation (2.63). At first, we discuss how a multipartite correlation in general can be related with sets of bipartite correlations. This leads to three different types of inequalities and one equality. It is our main question

whether such inequalities and equality are satisfied for the case of non-classical correlations.

4.2.1 Additivity of the total correlation

Total correlation counts all the correlation existing in a multipartite system. The structure of non-classical correlations between multiple systems is complicated in general, but total correlation itself is decomposed into the bipartite correlations between subsystems. In other words, the total correlation of a tripartite system ABC can in principle be decomposed into two different types of bipartite correlations as

$$\mathcal{T}(A:B:C) = \mathcal{T}(A:B) + \mathcal{T}(AB:C) \tag{4.1}$$

where the equality holds for any permutation of A, B, and C. The equation shows how the tripartite correlation is quantified in terms of bipartite correlations. If we consider only the subsystem AB among the total system, there is only correlation between A and B. In addition, if a system C is joined to the subsystem AB, it is natural to understand that the total amount of correlation is increased and the amount corresponds to the correlation existing between the subsystem AB together with the correlation between AB and the additional system C. The additional correlation is taken by the term $\mathcal{T}(AB:C)$ while it is still characterized by the bipartite correlation between two different subsystems, AB and C.

The relation between the multipartite correlation and the bipartite correlations is well summarized in the famous strong subadditivity relation of Shannon entropy $\mathcal{H}(AB) + \mathcal{H}(BC) \geqslant \mathcal{H}(B) + \mathcal{H}(ABC)$ where $\mathcal{H}(\cdot)$ is the Shannon entropy of the system "$\cdot$". The inequality is relatively

easy to prove, while the quantum version of the strong subadditivity with von Neumann entropy is difficult. The strong subadditivity of von Neumann entropy $S(\rho) = -\mathrm{Tr}[\rho \log \rho]$ [67] refers to the following inequality for tripartite quantum state ρ_{ABC}.

$$\mathcal{S}(AB) + \mathcal{S}(BC) \geqslant \mathcal{S}(ABC) + \mathcal{S}(B)$$

where $\mathcal{S}(\cdot)$ is the von Neumann entropy of the system "$\cdot$". The strong subadditivity is widely used in quantum information theory [68, 69]. The proof of the inequality can be found in [67, 70].

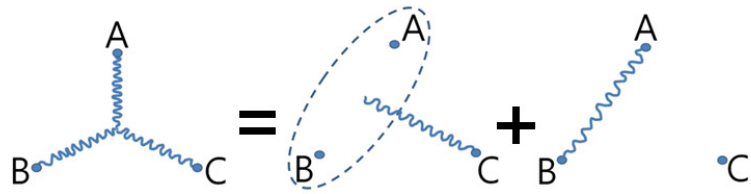
Using the strong subadditivity of von Neumann entropy, the bounds on the total correlation are obtained as

$$\mathcal{T}(AB:C) \geqslant \mathcal{T}(A:C), \tag{4.2}$$

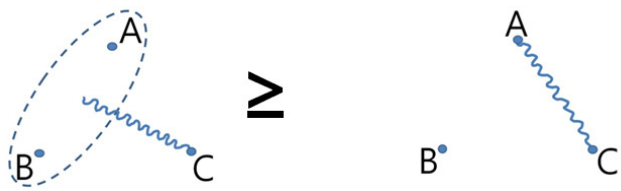
$$\mathcal{T}(A:B:C) \geqslant \mathcal{T}(A:B) + \mathcal{T}(A:C), \tag{4.3}$$

$$\mathcal{T}(A:B:C) \leqslant \mathcal{T}(BC:A) + \mathcal{T}(AC:B). \tag{4.4}$$

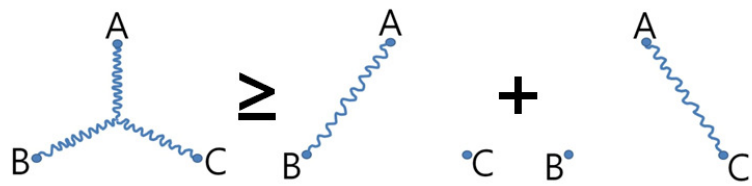
From the definition, it is straightforward to show that the three inequalities above are equivalent to the strong subadditivity of von Neumann entropy. The strong subadditivity frequently has been represented in the form of Eq. (4.2) [68], but it can also be represented in the forms of Eqs. (4.3) and (4.4) by using the tripartite total correlation. We can give intuitive interpretations of the three inequalities. The first inequality states that the correlation between AB and C contains the correlation between A and C. The correlation between A and B is missing in the right-hand side. The second inequality is also intuitively clear because the left-hand side of the inequality counts all the correlations among A, B, and C, so it is greater than or equal to the correlations between AB



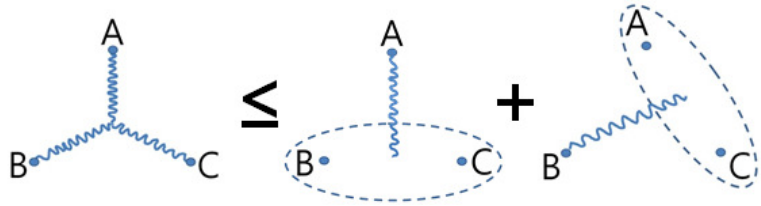
(a) Decomposition of tripartite correlation.



(b) Lower bound for bipartition.



(c) Lower bound in tripartite correlation.



(d) Upper bound in tripartite correlation.

Figure 4.1: Conceptual diagrams for the relations between the correlations in a three-party system ABC. Sub-figures (a) to (d) correspond to Eqs. (4.1) to (4.4) respectively.

and AC only. In that case, it can be said that the correlation between B and C has not been counted. In the third inequality, both two terms of the right-hand side contain the correlation between A and C (i.e. double counting), so it is less than or equal to the tripartite correlation. Those relations between the correlations in a tripartite system ABC are represented in the conceptual diagrams in Fig. 4.1.

4.2.2 Additivity of the relative entropy of entanglement

The total information covers all the correlations in both a classical part and a non-classical part. One may wonder that there exist the consistent inequalities as (4.1), (4.2), (4.3) and (4.4) for the measures of non-classical correlations. A natural candidate for non-classical correlation is quantum entanglement. There are several measures of entanglement which have been studied extensively so far [18]. However, our question requires a measure which can be naturally generalized to multipartite systems and which is consistent to the non-classical part of total information in Eq. (2.63). A proper measure in that purpose is the relative entropy of entanglement which is defined in Eq. (2.64). The relative entropy of entanglement is always less than or equal to the total information and is naturally generalized to multipartite systems.

For the relative entropy of entanglement, there exists an inequality which corresponds to Eq. (4.1) of the total information. The authors of [71] first showed that the following inequality holds for any tripartite pure system ABC.

$$\mathcal{E}(A:B:C) \geqslant \mathcal{E}(A:B) + \mathcal{E}(AB:C). \tag{4.5}$$

They derived it by using the inequality $S(\sigma_{ABC}||\rho_{ABC}) - S(\sigma_{AB}||\rho_{AB}) \geqslant S(\sigma_{AB}) - S(\sigma_{ABC})$ [72], which holds for any state ρ_{ABC} and for any triseparable state σ_{ABC}. Unlike the equation (4.1) of the total correlation, Eq. (4.5) is an inequality. It means that the tripartite entanglement contains entanglement between A, B and AB, C but they are not exactly same.

Also, the following inequality holds for any tripartite system ABC:

$$\mathcal{E}(AB : C) \geqslant \mathcal{E}(A : C). \tag{4.6}$$

It can be derived directly from the monotonicity of the relative entropy under partial trace, $S(\rho^A||\sigma^A) \leqslant S(\rho^{AB}||\sigma^{AB})$ [68].

Combining Eqs. (4.5) and (4.6) gives the following inequalities which hold for any tripartite pure system ABC:

$$\mathcal{E}(A : B : C) \geqslant \mathcal{E}(A : B) + \mathcal{E}(A : C). \tag{4.7}$$

At the same time, a symmetrized version of the inequality can be found as $\mathcal{E}(A : B : C) \geqslant \frac{2}{3}\big[\mathcal{E}(A : B) + \mathcal{E}(B : C) + \mathcal{E}(A : C)\big]$ by taking the average of Eq. (4.7) over all permutations of A, B, and C.

The inequality (4.7) corresponds to Eq. (4.3) of the total mutual correlation. It is a weaker inequality than the monogamy inequality of entanglement $\mathcal{E}(A : BC) \geqslant \mathcal{E}(A : B) + \mathcal{E}(A : C)$, which is not known if it holds for the entanglement measure. It is not an unique property of nonclassical correlations. We can see the analogy in the classical correlations between classical random variables. For three classical random variables X, Y and Z, $H(X : Y : Z) \geqslant H(X : Y) + H(X : Z)$ where $H(X : Y : Z) = H(X) + H(Y) + H(Z) - H(X, Y, Z)$ is a multivariate mutual information.

There is also the same form of upper bound for the regularized relative entropy of entanglement $\mathcal{E}^{\infty}(\rho) = \lim_{n\to\infty} \frac{1}{n}\mathcal{E}(\rho^{\otimes n})$,

$$\mathcal{E}^{\infty}(A:B:C) \leqslant \mathcal{E}^{\infty}(BC:A) + \mathcal{E}^{\infty}(AC:B),$$

which holds for any tripartite pure system. The inequality is also proven in [71], and it follows from the fact the $\mathcal{E}$ is an entanglement monotone.

Therefore, it is possible to conclude that similar kinds of additivity relation with total correlation are satisfied for the case of entanglement. Although it is found that they do not coincide sharply, they asymptotically behave in the same way.

4.2.3 Additivity of the relative entropy of discord

Now we consider the quantum discord in Eq. ((2.61)) as another candidate of non-classical correlations. It is well-known that the criterion of non-zero quantum discord is not identical to the criterion of non-zero entanglement. Unlike entanglement, quantum discord is defined to be non-zero even when a state is separable. The difference makes it non-trivial to consider whether there exist the consistent additivity relations as Eqs. (4.1) to (4.4) for quantum discord in multipartite systems. We take the relative entropy of discord in Eq. ((2.61)) as quantum discord measure. The relative entropy of discord differs from the relative entropy of entanglement in the set of states whose minimization should be taken. It enable us to genuinely compare the difference in the separability criterion and the classicality criterion.

First, we show that there is the consistent lower bound on the tripartite relative entropy of discord. The minimization in $\mathcal{D}(\rho) = \min_{\sigma\in\mathcal{C}} S(\rho||\sigma)$

can be reduced to a minimization over set of local bases $\{|\vec{k}\rangle\}$ [34]

$$\mathcal{D}(\rho) = \min_{\{|\vec{k}\rangle\}} \Lambda_\rho(\{|\vec{k}\rangle\}) - S(\rho) \tag{4.8}$$

where $\Lambda_\rho(\{|\vec{k}\rangle\}) = -\sum_{\vec{k}} \langle\vec{k}|\rho|\vec{k}\rangle \log\langle\vec{k}|\rho|\vec{k}\rangle$. For the matter of convenience, we write $A:B:C$ instead of ρ_{ABC} from now on. A:B:C denotes a tripartition of a system ABC.

Theorem 4.1. *The following inequality holds for any tripartite pure system ABC.*

$$\mathcal{D}(A:B:C) \geqslant \mathcal{D}(A:B) + \mathcal{D}(AB:C). \tag{4.9}$$

proof — For any pure state, bipartite discord is equivalent to the entropy of entanglement, so $\mathcal{D}(AB:C) = S(AB) = S(C)$. Then the inequality is equivalent to $\min \Lambda_{A:B:C}(\{|\vec{k}\rangle\}) \geqslant \min \Lambda_{A:B}(\{|\vec{k}\rangle\})$. Let $\{|\vec{k}^*\rangle\}$ be a basis which minimizes $\Lambda_{A:B:C}(\{|\vec{k}\rangle\})$, and consider the set of local measurement operators $\{|\vec{k}^*\rangle\langle\vec{k}^*|\}$. The outcomes of the local measurements can be treated as three classical random variables, which are denoted X, Y and Z. They correspond to the outcomes of the local measurements on A, B and C respectively. For any classical random variables X, Y and Z, Shannon entropies satisfy $H(X,Y,Z) \geqslant H(X,Y)$. Using this inequality, $\min \Lambda_{A:B:C}(\{|\vec{k}\rangle\}) = H(X,Y,Z) \geqslant H(X,Y) \geqslant \min \Lambda_{A:B}(\{|\vec{k}\rangle\})$, so the proof is completed. ■

We also found the analogous inequality for discord as the inequality (4.2) of the total correlation and the inequality (4.6) of the relative entropy of entanglement.

Theorem 4.2. *The following inequality holds for any tripartite pure sys-*

tem ABC.

$$\mathcal{D}(AB:C) \geqslant \frac{1}{2}\Big[\mathcal{D}(A:C)+\mathcal{D}(B:C)\Big], \tag{4.10}$$

and its proof can be given in the below.

proof — The inequality (4.10) can be easily proven from just $\mathcal{T} \geqslant \mathcal{D}$. Given a pure system ABC, the bipartite discord $\mathcal{D}(AB:C)$ is equivalent to the entropy of entanglement $S(C)$, and $S(AC) = S(B)$ and $S(BC) = S(A)$. Therefore, $S(C) = \frac{1}{2}[\mathcal{T}(A:C)+\mathcal{T}(B:C)] \geqslant \frac{1}{2}[\mathcal{D}(A:C)+\mathcal{D}(B:C)]$ from $\mathcal{T} \geqslant \mathcal{D}$. ■

Corollary. Consider a tripartite pure system ABC whose parties are ordered so $\mathcal{D}(A:B) \geqslant \mathcal{D}(B:C) \geqslant \mathcal{D}(A:C)$. Then the following inequalities hold.

$$\mathcal{D}(AB:C) \geqslant \mathcal{D}(A:C), \tag{4.11}$$

$$\mathcal{D}(A:B:C) \geqslant \mathcal{D}(A:B)+\mathcal{D}(A:C), \tag{4.12}$$

$$\mathcal{D}(BC:A)+\mathcal{D}(AC:B) \geqslant \mathcal{D}(A:B)+\mathcal{D}(A:C). \tag{4.13}$$

proof — They directly follow by Eqs. (4.9), (4.10), and the assumption $D(A:B) \geqslant D(B:C) \geqslant D(A:C)$. On the other hand, by taking the average of (4.12) over all permutations of three parties, one can obtain $\mathcal{D}(A:B:C) \geqslant \frac{2}{3}[\mathcal{D}(A:B)+\mathcal{D}(B:C)+\mathcal{D}(A:C)]$.

In order that Eqs. (4.11), (4.12), and (4.13) are satisfied for any permutation of A, B, and C, one needs to show that

$$\mathcal{D}(AB:C) \geqslant \max\{\mathcal{D}(B:C), \mathcal{D}(A:C)\} \tag{4.14}$$

for any tripartite pure system ABC. Its analytical proof is not found. However, we conjecture with numerical evidence that it holds for any

three-qubit pure state. For pure states, $\mathcal{D}(AB:C) = S(C)$ and $\mathcal{D}(A:C) = \min \Lambda_{A:C}(|\vec{k}\rangle) - S(B)$. The main task of the numerical computations is the minimization of $\Lambda_{A:C}(|\vec{k}\rangle)$ over all local bases. In general, there are several local minima in $\Lambda(|\vec{k}\rangle)$, so the results of the numerical minimizations can not guarantee that it always finds the global minimum; it just provides an upper bound. Fortunately, however, the part consisting the minimization is on the right-hand side of the inequality (4.14), so we can provide numerical evidence for the inequality by testing numerous samples. We compare the both sides in Eq. (4.14) for 10^6 random samples of pure three-qubit states. No quantum state has been found violating Eq. (4.14). The result is given in Fig. 4.2. The random samples are generated from the parameterization [73] of three-qubit states $|\psi\rangle = \lambda_0|000\rangle + e^{i\phi}\lambda_1|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle$. To minimize Eq. (4.8), we parameterized an arbitrary local basis $\{|0'\rangle, |1'\rangle\}$ as

$$\begin{aligned} |0\rangle &= \sqrt{1-t^2}|0'\rangle + t|1'\rangle \\ |1\rangle &= e^{i\phi}(t|0'\rangle - \sqrt{1-t^2}|1'\rangle), \end{aligned} \tag{4.15}$$

where $0 \leqslant e^{i\phi} \leqslant 2\pi$ and $0 \leqslant t \leqslant 1$. In fact, 4 real parameters are required to cover all bases, but it can be reduced to 2 as we only consider $\Lambda_\rho(\{|\vec{k}\rangle\})$. The number of parameters is two per each qubit, so computing $\min \Lambda_{A:C}(\{|\vec{k}\rangle\})$ requires numerical minimization over 4 parameters.

The inequality (4.11) is suggestive of the monogamy inequality [74–76]. We note that they look similar, but are different. The monogamy of entanglement is an unique property of quantum states, and it is not found in correlations of classical variables. In contrast, Eq. (4.13) also holds for the mutual information of classical variables. The inequality (4.13) is studied for the global discord in Ref. [77], and it is shown that

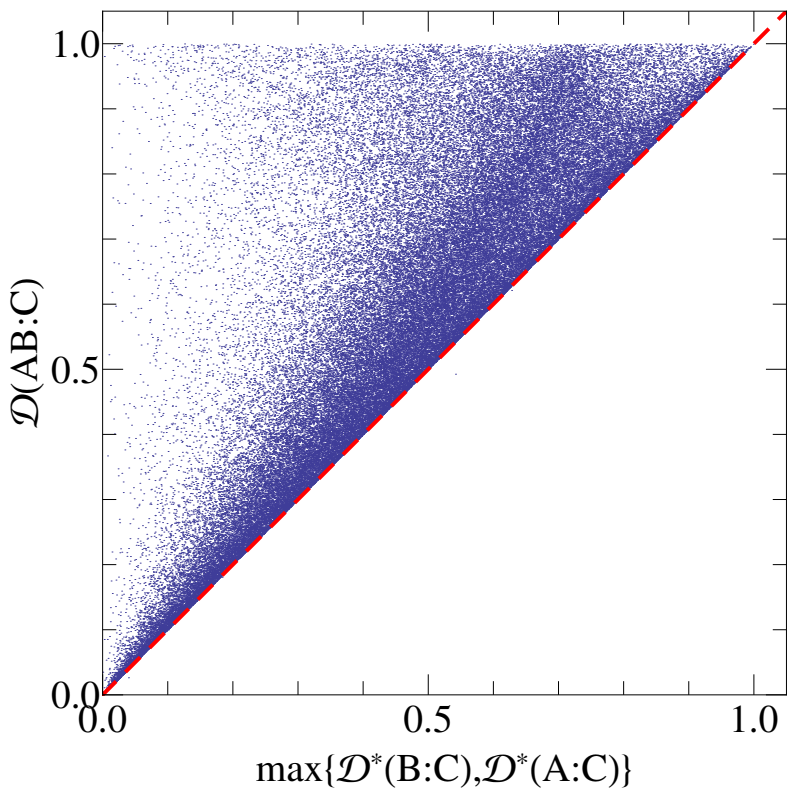


Figure 4.2: Numerical tests of the inequality (4.14) for 10^6 random samples of three-qubit pure states. No sample was found violating. $\mathcal{D}^*$ is approximated discord obtained from numerical minimization. Rigorously speaking, $\mathcal{D}^*$ is equal to or greater than $\mathcal{D}$.

Eq. (4.11) is a sufficient condition for Eq. (4.13).

For general mixed states, we could not find any example violating Eqs. (4.11), (4.12), or (4.13). However, we note that they can be violated by the other permutations of A, B, and C. Bipartite discord of some mixed states can increase after discarding a subsystem, so Eq. (4.14) dose not hold for general mixed states. Consider the state $\rho = p|000\rangle\langle 000| + (1-p)|1{+}1\rangle\langle 1{+}1|$ where $|+\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$. For the state, $\mathcal{D}(BC : A) = 0$ and $\mathcal{D}(A : C) > 0$, so $\mathcal{D}(BC : A) \ngeqslant \mathcal{D}(A : C)$ for any measure of discord. In our measure of discord, $\mathcal{D}(BC : A) = 0$ and $\mathcal{D}(A : C) = \mathcal{D}(A : B : C) = \min\{p, (1-p)\}$. The details of the calculation ca be found in Example. 2.5. In contrast, we mentioned that the relative entropy of entanglement always satisfies $\mathcal{E}(AB : C) \geqslant \mathcal{E}(B : C)$. The difference of entanglement and discord occurs from the criteria of classically correlated states and separable states. If ρ_{ABC} is separable in the bipartition AB and C, then ρ_{AC} and ρ_{BC} are also separable. However, ρ_{AC} and ρ_{BC} can be not classically correlated although ρ_{ABC} is classically correlated in the bipartition AB and C.

On the other hand, we leave a unanswered question of whether the inequality $\mathcal{D}(A : B : C) \leqslant \mathcal{D}(BC : A) + \mathcal{D}(AC : B)$ holds for tripartite systems ABC whose parties are ordered so $\mathcal{D}(A : B) \geqslant \mathcal{D}(B : C) \geqslant \mathcal{D}(A : C)$. There is an example violating it for the other permutations of A, B, and C. For the state $\rho = p|000\rangle\langle 000| + (1-p)|1{+}1\rangle\langle 1{+}1|$, $\mathcal{D}(AB : C) + \mathcal{D}(BC : A) = 0$, but $\mathcal{D}(A : B : C) = \min\{p, (1-p)\}$.

We have discussed about the consistent additivity relations for the multipartite measures of total correlations and non-classical correlations. It would also be worth to consider the relations for multipartite measures of classical correlations. In terms of the quantum relative entropy,

a measure of classical correlations is defined [34] as

$$\mathcal{C}(\rho) = \min_{\sigma \in \pi} S(\chi_\rho || \sigma). \tag{4.16}$$

Here, χ_ρ is the closest classically correlated state to ρ. It is not known whether the measure of classical correlations also has the consistent additivity relations as the measures of total correlations and non-classical correlations. The part is left as an open question. On the other hand, it is evident that for the classically correlated states, $\rho \in \mathcal{C}$, those measures also follow the relations (4.1), (4.2), (4.3), and (4.4) of the total correlation because the classical correlation is just the total correlation.

4.3 Examples

In the previous section, we have shown that the additivity relations from Eq. (4.1) to Eq. (4.4) are also satisfied by entanglement and discord (for discord, Eq. (4.4) was left unsolved, and only Eq. (4.13) is proved) with help of numerical analysis, especially when the total system is in a three-qubit pure state. We note that the equality (4.1) for the total correlation becomes an inequality (*i.e.*, '=' is replaced with '$\geqslant$') when applied to entanglement and discord. In subsection A, we now present several examples of three-qubit pure states in which all of multipartite entanglement, discord and classical correlations satisfy all the the additivity relations from (4.2) to (4.4) as well as the inequality (4.9). In the following section, we have tried to identify the validity of additivity relation when the total system is in a mixed state. It was possible to find specific cases where the additivity relation is not satisfied.

Example 4.3. We first consider the generalized GHZ state of the form

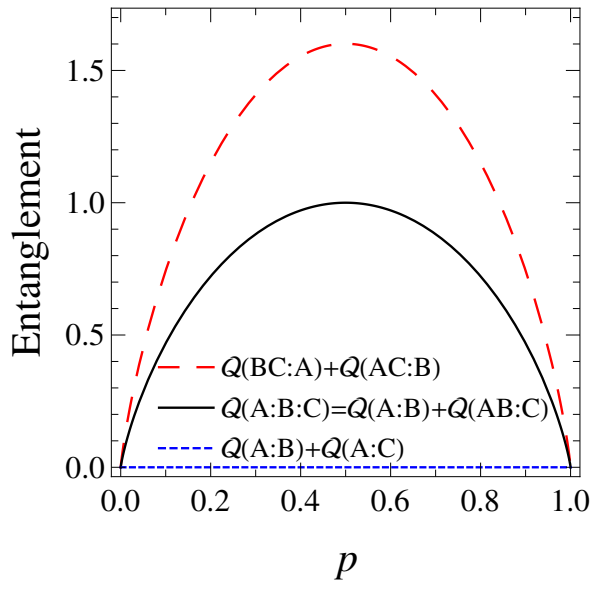
$$|GHZ\rangle = \alpha|000\rangle + \beta|111\rangle). \tag{4.17}$$

with $|\alpha|^2 + |\beta|^2 = 1$. If we let $\mathcal{Q}$ represent either $\mathcal{E}$ or $\mathcal{D}$, $\mathcal{Q}(A : B : C) = \mathcal{Q}(AB : C) = -|\alpha|^2 \log |\alpha|^2 - |\beta|^2 \log |\beta|^2$ and $\mathcal{Q}(A : B) = 0$ for any permutation of A, B, and C, so that all the additivity relations from (4.1) to (4.4) are satisfied. All the additivity relations are then satisfied for any permutation. In addition, we can calculate classical correlations $\mathcal{C}$ in Eq. (4.16). Using the fact that the closest product state to a given state is the product of its reduced states [34], we find $\mathcal{C}(A : B : C)/2 = \mathcal{C}(AB : C) = \mathcal{C}(A : B) = -|\alpha|^2 \log |\alpha|^2 - |\beta|^2 \log |\beta|^2$. Again, all the additivity relations are met.

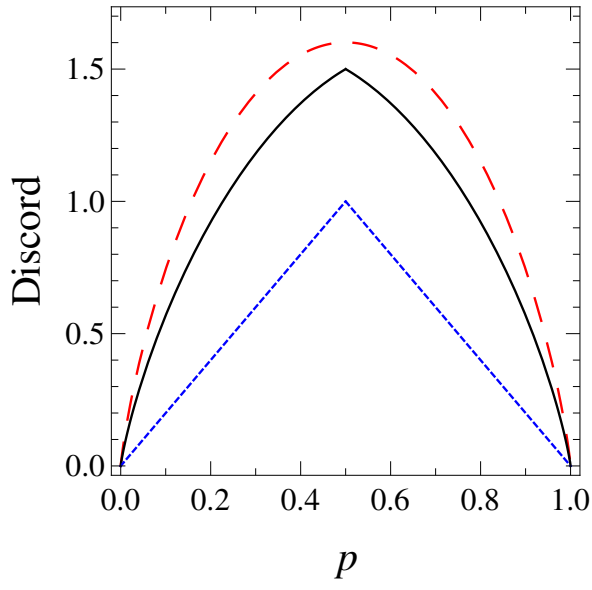
Example 4.4. We consider the state W

$$(\sqrt{p}|000\rangle + \sqrt{1-p}|+11\rangle), \tag{4.18}$$

which is obtained by applying an appropriate non-local unitary operation to the GHZ state. Unlike the GHZ state, this state has nonzero discord $\mathcal{D}(A : B)$ and $\mathcal{D}(A : C)$. There is no entanglement in any bipartite subsystem of $|\phi\rangle$. We can obtain a representation for $\mathcal{E}(A : B : C)$ by using inequality (4.5). Let us consider a tripartite separable state $\sigma = p|000\rangle\langle 000| + (1-p)|+11\rangle\langle +11|$, and it is clear that $\mathcal{E}(A : B : C) \leqslant S(|\phi\rangle\langle\phi||\sigma) = -p \log p - (1-p)\log(1-p)$. However, it is also true that $\mathcal{E}(A : B) + \mathcal{E}(AB : C) = -p \log p - (1-p)\log(1-p)$, so $\mathcal{E}(A : B : C) = \mathcal{E}(A : B) + \mathcal{E}(AB : C) = -p \log p - (1-p)\log(1-p)$ from Eq. (4.5). Bipartite entanglement is easily found as $\mathcal{E}(AC : B) = -p \log p - (1-p)\log(1-p)$ and $\mathcal{E}(BC : A) = -\lambda_+ \log \lambda_+ - \lambda_- \log \lambda_-$



(a)



(b)

Figure 4.3: Additivity relations of entanglement (left) and discord (right) for the states $|\phi\rangle = \sqrt{p}|000\rangle + \sqrt{(1-p)}|\pm 11\rangle$, plotted versus p. Symbol $\mathcal{Q}$ is used to represent $\mathcal{E}$ or $\mathcal{D}$.

where $\lambda_\pm = 1/2 \pm \sqrt{1/4 - p(1-p)/2}$. Now one can check that all the additivity relations are satisfied. For the other permutations of A, B and C, the results are the same except for a change of '$=$' in (4.1) to '$\geqslant$'. We plot $\mathcal{E}(A:B:C)$, $\mathcal{E}(BC:A)+\mathcal{E}(AC:B)$ and $\mathcal{E}(A:B)+\mathcal{E}(A:C)$ in Fig. 4.3(a).

In the case of the relative entropy of discord, one can show that $\mathcal{D}(A:B:C) = -p\log p - (1-p)\log(1-p) + \min\{p,(1-p)\}$ and $\mathcal{D}(A:B) = \mathcal{D}(A:C) = \min\{p,(1-p)\}$, see Example. 2.5 for the calculation. We also simply obtain $\mathcal{D}(BC:A) = -\lambda_+\log\lambda_+ - \lambda_-\log\lambda_-$ and $\mathcal{D}(AC:B) = -p\log p - (1-p)\log(1-p)$, and find that the all the additivity relations are satisfied. For the other permutations, again, the same results are obtained except for the change from '$=$' to '$\geqslant$' in Eq. (4.1). The results are plotted in Fig. 4.3(b). In addition, we examine the additivity relations for the measure of classical correlations. The classical correlations are obtained as

$$\begin{aligned}
\mathcal{C}(A:B:C) =& -\frac{q}{2}\log\frac{q}{2} - (1-\frac{q}{2})\log(1-\frac{q}{2}) \\
& - q\log q - (1-q)\log(1-q) - q, \\
\mathcal{C}(A:B) =& -\frac{q}{2}\log\frac{q}{2} - (1-\frac{q}{2})\log(1-\frac{q}{2}) - q, \\
\mathcal{C}(AB:C) =& -q\log q - (1-q)\log(1-q), \\
\mathcal{C}(BC:A) =& -\lambda_+\log\lambda_+ - \lambda_-\log\lambda_-,
\end{aligned}$$

where $q = \min\{p, 1-p\}$, and all the additivity relations are met.

Example 4.5. We now consider a W-state

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \tag{4.19}$$

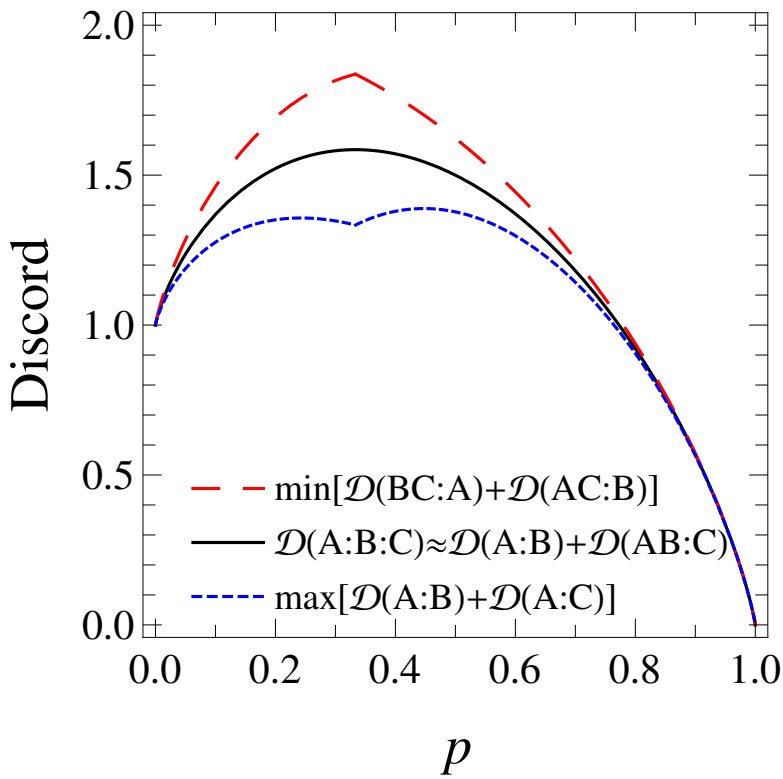


Figure 4.4: Additivity relations of discord for the generalized W-states $\sqrt{p}|001\rangle + \sqrt{\frac{1-p}{2}}\big(|010\rangle + |100\rangle\big)$, plotted versus p. Here, the min (max) stands for taking the minimum (maximum) over all permutations of A,B, and C.

with which it is known that $\mathcal{E}(BC:A) = \mathcal{E}(AC:B) = \log 3 - 2/3 \approx 0.92$, $\mathcal{E}(A:B:C) = 2\log 3 - 2 \approx 1.17$ [78], and $\mathcal{E}(A:B) = \mathcal{E}(A:C) = \log 3 - 4/3 \approx 0.25$ [49]. All the additivity relations are then satisfied. In a more general case of

$$\alpha|001\rangle + \beta|010\rangle + \gamma|100\rangle, \tag{4.20}$$

the additivity relations of discord can be examined as follows. First, quantum discord with bipartition AB and C is easily obtained to $\mathcal{D}(AB:C) = -|\alpha|^2 \log|\alpha|^2 - (|\beta|^2 + |\gamma|^2)\log(|\beta|^2 + |\gamma|^2)$ because the total system is pure. $\mathcal{D}(A:C)$ and $\mathcal{D}(A:B:C)$ are upper bounded respectively by $-|\alpha|^2 \log(|\alpha|^2/|\alpha|^2 + |\gamma|^2) - |\gamma|^2 \log(|\gamma|^2/|\alpha|^2 + |\gamma|^2)$ and $-|\alpha|^2 \log|\alpha|^2 - |\beta|^2 \log|\beta|^2 - |\gamma|^2 \log|\gamma|^2$ as we choose the logical basis in the expression (4.8). The inequality (4.2) is then satisfied for discord as

$$\begin{aligned} \mathcal{D}(AB:C) - \mathcal{D}(A:C) \geqslant & -(|\alpha|^2 + |\gamma|^2)\log(|\alpha|^2 + |\gamma|^2) \\ & -(|\beta|^2 + |\gamma|^2)\log(|\beta|^2 + |\gamma|^2) \\ & +|\gamma|^2 \log|\gamma|^2 \geqslant 0. \end{aligned} \tag{4.21}$$

The last inequality holds because for given $|\gamma|$, $-(|\alpha|^2 + |\gamma|^2)\log(|\alpha|^2 + |\gamma|^2) - (|\beta|^2 + |\gamma|^2)\log(|\beta|^2 + |\gamma|^2)$ is minimized at $\alpha = 0$ or $\beta = 0$. The inequality (4.4) is similarly satisfied as

$$\begin{aligned} \mathcal{D}(AB:C) + \mathcal{D}(A:BC) - \mathcal{D}(A:B:C) & \\ \geqslant & -(|\alpha|^2 + |\beta|^2)\log(|\alpha|^2 + |\beta|^2) \\ & -(|\beta|^2 + |\gamma|^2)\log(|\beta|^2 + |\gamma|^2) \\ & +|\beta|^2 \log|\beta|^2 \geqslant 0. \end{aligned} \tag{4.22}$$

The inequalities also hold for the other permutations of ABC, because

the procedure is invariant under permutations. In Fig. 4.4, we plot the numerical result for the special case $|\alpha|^2 = p$ and $|\beta|^2 = |\gamma|^2 = \frac{1-p}{2}$. It shows that $\mathcal{D}(A : B : C) \approx \mathcal{D}(A : B) + \mathcal{D}(AB : C)$, so Eq. (4.1) approximately holds .

4.3.1 Mixed states case

Our arguments about the additvity relations of discord were restricted to pure states. We noted that the additivity relations of discord are violated by some mixed systems whose ordering does not satisfy $\mathcal{D}(A : B) \geqslant \mathcal{D}(B : C) \geqslant \mathcal{D}(A : C)$. In this subsection, we examine a couple of mixed state examples states in order to check whether the additivity relations of discord hold.

Example 4.6. We consider the state

$$\rho = (1-p)|GHZ\rangle\langle GHZ| + p\mathcal{I}/8 \tag{4.23}$$

where $\mathcal{I}$ is the identity matrix. We see that $\mathcal{D}(A : B : C) = \mathcal{D}(BC : A)$ and $\mathcal{D}(A : B) = 0$, so the additivity relations are trivially met.

Example 4.7. For the state

$$\rho = (1-p)|W\rangle\langle W| + p\mathcal{I}/8, \tag{4.24}$$

the additivity relations are also satisfied and it is shown in Fig. 4.5.

Example 4.8. Finally, we consider the state $\rho = (1-p)|W\rangle\langle W| + \frac{p}{2}(|000\rangle\langle 000| + |1{+}1\rangle\langle 1{+}1|)$. It satisfies the ordering requirement, so the relation (4.3) is satisfied for all p, but $\mathcal{D}(A : B : C) \ngeq \mathcal{D}(A : B) + \mathcal{D}(B : C)$ for $p > p^*$ where $p^* \approx 0.8$.

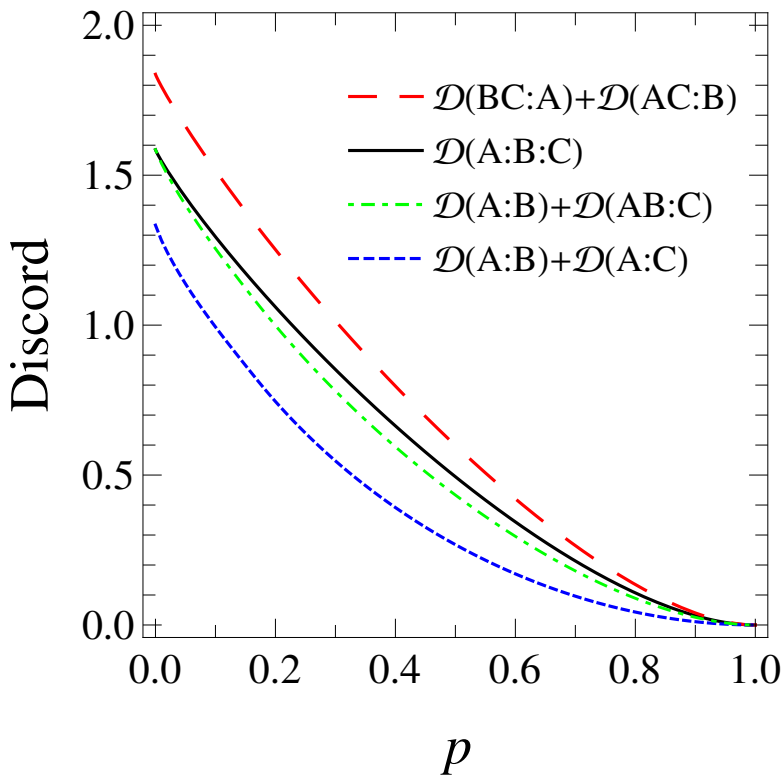


Figure 4.5: Additivity relations of discord for the mixed state $\rho = (1 - p)|W\rangle\langle W| + p\mathcal{I}/8$, plotted versus p.

Let us compute the relative entropy of discord of the state $\rho = p|000\rangle\langle 000| + (1-p)|1{+}1\rangle\langle 1{+}1|$. First, we argue that taking minimization over all local base of B is enough to obtain $\mathcal{D}(A:B:C)$ of the state. Consider a measure of discord $\mathcal{D}^*(\rho_{ABC}) = \min_{\sigma\in\mathcal{C}^*} S(\rho||\sigma)$ where $\mathcal{C}^*$ is the set of all quantum states in the form $\sigma = \sum p_i |i\rangle_B\langle i| \otimes \sigma^i_{AC}$ where $\{|i\rangle_B\}$ is a local basis of B, and σ^i_{AC}'s are arbitrary quantum states of AC. Apparently, $\mathcal{D}^* \leq \mathcal{D}$. It can be shown that the minimum is obtained for a state of the form $\sigma = \sum |i\rangle\langle i| \otimes \langle i|\rho|i\rangle$, as $\mathcal{D}$ can be reduced to the expression (4.8). On the other hand, in the case of our target state, σ is also a classically correlated state for every choosing of the local basis of B. Therefore $D^* = D$ in this case. Now, local base of B can be parameterized as Eq. (4.15), and simple differentiations find two local minima of $S(\rho||\sigma)$ at $\sigma^* = p|000\rangle\langle 000| + \frac{(1-p)}{2}(|101\rangle\langle 101| + |111\rangle\langle 111|)$ and $\sigma^{**} = \frac{p}{2}(|0{+}0\rangle\langle 0{+}0| + |0{-}0\rangle\langle 0{-}0|) + (1-p)|1{+}1\rangle\langle 1{+}1|$. Consequently, $\mathcal{D}(A:B:C) = \mathcal{D}(A:B) = \min\{p, (1-p)\}$, and then discord of the state $|\psi\rangle = \sqrt{p}|000\rangle + \sqrt{1-p}|1{+}1\rangle$ is computed to $\mathcal{D}(A:B:C) = -p\log p - (1-p)\log(1-p) + \min\{p, (1-p)\}$ by applying (4.9).

4.3.2 Conclusion

We have studied about four additivity relations of the measures of multipartite correlations which are based on the quantum relative entropy. The relations were motivated by the definition of the total correlation and the strong subadditivity of von Neumann entropy. They specify multipartite correlations in terms of bipartite correlations in subsystems, so they give insight of to what extent multipartite correlations contain bipartite correlations of subsystems. argued that discord have the additivity relations for pure states, but it is not always true for mixed states.

Chapter 5

Conclusion

The huge gains in processing power over classical information processing is the major motivation of quantum information science. Non-classical correlation is generally considered to be the origin of the power of quantum information processing, and most of circuits for quantum information tasks involve many qubits. For this reason, non-classical correlation ultimately needs to be studied in the multipartite setting. However, many basic questions in multipartite non-classical correlation remain unsolved because of the complex structure of multipartite systems.

In this thesis, we considered the characterization of non-classical correlations between multiple systems from two perspectives. First, we considered the quantification of multipartite entanglement from the resource-theory point of view. GHZ entanglement entanglement was considered as an unit of multipartite entanglement, and we defined the GHZ entanglement cost of preparing an arbitrary multipartite states. We proposed a LOCC procedure that can prepare any multipartite states, and the procedure was shown to consume GHZ entanglement as much as multipartite discord of the state being prepared. The implication of this result

is that GHZ entanglement as much as discord suffices to a multipartite entangled state.

Second, we looked into the relation between the multipartite correlation of the total system and the bipartite correlations of pairs of subsystems. This approach is motivated from the paradigm that bipartite correlations of subsystems are added to yield the multipartite correlation. We specified four additivity relations, which were derived in terms of the total correlation and the examine the additivity relations for entanglement and discord. We found that the additivity relations are satisfied in many cases for entanglement and discord as well, although we took an example of a mixed-state case that violated some of the additivity relations.

This thesis was aimed to deepen the understanding of non-classical correlations between multiple systems. I believe that this thesis helps understand the properties of multipartite nonclassical correlation and technical parts of the result are useful for the research in the future.

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국문초록

여러 시스템 사이의 비고전적인 상관관계는 특히 양자정보처리로의 응용에서 양자 시스템이 갖는 중요한 특징의 근원이다. 하지만 그의 복잡성 때문에, 여러 시스템 사이의 비고전적 상관관계는 아직 많이 알려지지 않았다. 본 박사학위 논문에서는, 여러 시스템 간의 비 고전적인 상관관계에 대한 이해를 깊게 하기 위해, 두 가지 관점에서 들여다본다.

첫 번째 관점에서, 비고전적 상관관계는 국소적 작용 및 고전적인 통신에 의해 생성 될 수없는 자원으로 간주된다. 국소작용과 고전통신만이 가능할 때, 주어진 여러 부분 양자 상태를 생성하기 위해 필요한 여러 부분 양자 얽힘의 양이 얼마인지 알아본다. 본 연구에서는 양자 얽힘 비용을 여러 부분 설정으로 일반화하려한다. 더 구체적으로, 임의의 여러 부분 양자 상태가 주어졌을 때, 본 양자상태를 생성하기 위한 국소작용과 고전통신을 제시한다. 이 때 필요한 여러 부분 양자 얽힘의 양은 여러 부분 양자 얽힘 비용의 상한이 된다.

두 번째 관점에서는, 하부 시스템들 사이의 상관관계가 더해져 전체 시스템의 여러 부분 상관관계를 이룬다고 여긴다. 이러한 맥락에서, 여러 부분 상관관계 및 하부 시스템 간의 비고전적 상관관계 사이의 일반적인 관계식을 찾으려 한다. 본 연구에서는 가산성 관계식을 도입하고 그 기본 개념을 제공한다. 그리고 양자 얽힘과 양자 불일치가 가산성 관계식을 만족시키는지 확인한다.